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Dedication

All praise to Allah, today we fold waking the nights and the tiredness of the days, and the summary of our journey between the two covers of this humble work.

To the utmost knowledge lighthouse and the chosen Imam, to the master of creation, to our greatest and most honored prophet Mohamed - May peace and grace from Allah be upon him.

To my loving parents, Laid and Zahra whose words of encouragement and push for tenacity ring in my ears, who have never failed to give me financial and moral support, and also give me some courage to stand when I fall and give me spirit when I am hopeless. Allah save them.

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الملخص

يتعلق العمل الحالي بتقدير نظام شبه ماركوف (SMS) بطريقة غير معلميه مع حالات محدودة. نقدم بناء مقدرات النواة لمؤشرات ومقاييس مهمة مختلفة لعملية شبه ماركوف ثم نظهر التقارب القوي والحالة الطبيعية المقاربة.

أولاً، نوفر مقدرات النواة للخصائص الرئيسية لعملية شبه ماركوف ذات الوقت المستمر، مثل أوقات الإقامة المشروطة وغير المشروطة في حالة، ونواة شبه ماركوف، بالإضافة إلى مشتقات الرادون-نيكوديم المرتبطة بها. الهدف الرئيسي هو إنشاء خصائص مقاربة مثل الاتساق القوي الموحد والحالة الطبيعية المقاربة.

ثانيًا، ندرس موثوقية أنظمة شبه ماركوف. نقدم مقدر النواة للموثوقية والقياسات ذات الصلة: معدل الفشل، والتوافر، وندرس الخصائص المقاربة للمقدرات المقترحة.

من أجل إثبات فعالية نتائجنا النظرية، يتم تحقيق كل جزء من خلال مثال رقمي.

الكلمات الرئيسية:

عمليات شبه ماركوف، مقدر النواة، أوقات الإقامة، نواة شبه ماركوف، مصفوفة تجديد ماركوف، مصفوفة انتقال شبه ماركوف، التوافر، الموثوقية، معدل الفشل، الاتساق، الوضع الطبيعي المقارب.

Résumé: Le présent travail porte sur l'estimation d'un système semi-markovien (SMS) à états finis par une méthode non paramétrique. Nous présentons la construction des estimateurs à noyau pour différents indicateurs et mesures qui sont importants pour un processus semi-markovien, puis nous établissons la convergence forte ainsi que la normalité asymptotique de ces estimateurs.

Premièrement, par la méthode d'estimation à noyau, nous construisons des estimateurs des principales caractéristiques d'un processus semi-markovien en temps continu, telles que les temps de séjour conditionnel et inconditionnel, le noyau semi-markovien, ainsi que les premières dérivées des mesures précédentes. L'objectif principal est donc d'établir certaines propriétés asymptotiques des estimateurs construits.

Dans un second temps, nous étudions la fiabilité des systèmes semi-markoviens. Nous introduisons un estimateur à noyau de la fiabilité ainsi que du taux de défaillance et de la disponibilité. Ensuite, nous étudions les propriétés asymptotique des estimateurs proposés.

Afin de prouver l'efficacité de nos résultats théoriques, chaque partie est illustré à travers un exemple numérique.

Mots clés: processus semi-markoviens, estimateur à noyau, temps de séjour, noyau semi-markovien, matrice de renouvellement markovienne, matrice de transition semi-markovienne, disponibilité, fiabilité, taux de défaillance, consistance, normalité asymptotique.

Abstract: The present work concerns the estimation of a finite state semi-Markov system (SMS) by a nonparametric method. We present the construction of kernel estimators for different important indicators and measures of the semi-Markov process, then we prove their strong convergence and asymptotic normality.

Firstly, we provide kernel estimators of the main characteristics of a continuous-time semi-Markov process, like conditional and unconditional sojourn times in a state, semi-Markov kernel, as well as their associated derivatives. The main goal is to establish asymptotic properties as the uniform strong consistency and asymptotic normality.

Secondly, we study the reliability of semi-Markov systems. We introduce a kernel estimator of the reliability and its related measurements, as failure rate and availability. We also study the asymptotic properties of the proposed estimators.

In order to illustrate the quality of our theoretical results, each part is achieved by a numerical example.

Keywords: semi-Markov processes, kernel estimator, sojourn times, semi-Markov kernel, Markov renewal matrix, semi-Markov transition matrix, availability, reliability, failure rate, consistency, asymptotic normality.

List of works

Publications

1. Chafiâa Ayhar, Vlad Stefan Barbu, Fatiha Mokhtari, and Saâdia Rahmani. « On the asymptotic properties of some kernel estimators for continuous-time semi-Markov processes. Journal of Nonparametric Statistics », 34(1), 2022.

2. Chafiâa Ayhar, Fatiha Mokhtari, Vlad Stefan Barbu, and Saâdia Rahmani. « Nonparametric estimators of the reliability and related functions for semi-Markov systems », submitted, 2022.

3. Fatiha Mokhtari, Chafiâa Ayhar, Vlad Stefan Barbu, and Saâdia Rahmani. « Kernel estimators of Markov renewal and Transition Matrix of semi-Markov systems », submitted, 2022.

Communications

1. Participation to Eleventh Meeting of Mathematical Analysis and Applications RAMA11 at the University of Sidi Bel Abbes, Algeria, November 21-24, 2019; https://fse.univ-sba.dz/rama11/. « Estimation of discrete semi-Markov process ».

2. Participation to the Colloque International Modélisation Stochastique et Statistique Modeling (MSS) at the University of USTHB, Algeria, November 24-26, 2019; https://mss2019.usthb.dz/inscription/index.html. « Empirical estimators of continuous semi-Markov processes ».

3. Participation to « StatMod2020-Statistical Modeling with Applications », University of Rouen - Normandy and Institute of Mathematical Statistics and Applied Mathematics Gheorghe Mihoc-Caius Iacob of Romanian Academy; Bucharest, November 6-7, 2020; https://ismma.ro/?page_id=645. « Reliability of Semi-Markov Systems ».

4. Participation to « StatMod2021-Statistical Modeling with Applications », University of Rouen - Normandy, Institute of Mathematical Statistics and Applied Mathematics « Gheorghe Mihoc-Caius Iacob »of Romanian Academy and University of Caen - Normandy; Rouen, December 3-4, 2021; https://statmod2021.sciencesconf.org/. « Numerical solutions of Markov renewal equations for continuous-time semi-Markov processes ».

Participation to schools

Participation to the CIMPA research school: « Stochastic Analysis and Applications », in Saida, Algeria, March 09, 2019.

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Notations

Abbreviations

$a.s$	Almost sure
CLT	Central limit theorem
EMC	Embedded Markov chain
MRP	Markov renewal process
$r.v$	Random variable
SMP	Semi-Markov process
SMC	Semi-Markov chain
$SLLN$	Strong law of large numbers

Sets

D	Failure states of the semi-Markov system
$E = \{1, \ldots, s\}$	Finite state space
U	Operating states of the semi-Markov system
$\mathbb{N}$	Set of natural numbers
$\mathbb{N}^* = \mathbb{N} \setminus \{0\}$	Set of strictly positive natural numbers
$\mathbb{R}_+ = [0, \infty)$	Set of nonnegative real numbers
$\mathbb{R}_+^* = (0, \infty)$	Set of strictly positive real numbers

Convergence

$\xrightarrow{a.s.}$	Almost sure convergence (strong consistency)
$\xrightarrow{\mathcal{D}}$	Convergence in distribution
$\xrightarrow{\mathbb{P}}$	Convergence in probability

Probabilistic

$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space
$\mathbb{P}$	Probability
$\mathbb{P}_i(\cdot)$	Conditional probability $\mathbb{P}(\cdot \vert J_0 = i)$
$\mathbb{E}$	Expectation
$\mathbb{E}_i$	Conditional expectation corresponding to $\mathbb{P}_i$
$Z := (Z_t)_{t\in\mathbb{R}_+}$	Semi-Markov process (SMP)
$(J, S) := (J_n, S_n)_{n\in\mathbb{N}}$	Markov renewal process (MRP)
$J := (J_n)_{n\in\mathbb{N}}$	Visited states, embedded Markov chain (EMC)
$S := (S_n)_{n\in\mathbb{N}}$	Jump times of the semi-Markov process
$X := (X_n)_{n\in\mathbb{N}}$	Sojourn times between successive jumps
$(S_n^i)_{i\in E, n\in\mathbb{N}}$	Renewal process of successive times of visits to state i
$\alpha := (\alpha_i, i \in E)$	Initial law of the SMP and of the EMC
$\nu := (\nu_i, i \in E)$	Stationary law of the EMC
$\pi := (\pi_i, i \in E)$	Stationary law of the SMP
$p := (p_{ij})_{i,j\in \mathrm{E}}$	Transition matrix of the EMC J
$q(\cdot) := (q_{ij}(\cdot), i, j \in \mathrm{E})$	Density of the semi-Markov kernel
$Q(\cdot) := (Q_{ij}(\cdot), i, j \in \mathrm{E})$	Semi-Markov kernel
$f(\cdot) := (f_{ij}(\cdot), i, j \in \mathrm{E})$	Conditional density sojourn time distribution in state i, before visiting state j
$F(\cdot) := F_{ij}(\cdot), i, j \in \mathrm{E}$	Conditional sojourn time distribution in state i, before visiting state j
$\mathbf{F}(\cdot) := F_i(\cdot), i \in \mathrm{E}$	Sojourn time distribution in state i
$\overline{F}(\cdot) := (\overline{F}_i(\cdot), i \in \mathrm{E})$	Survival function in state i
$\mathbf{f}(\cdot) := diag(f_i(\cdot), i \in \mathrm{E})$	Matrix of sojourn time distribution functions
$H(\cdot)$	Kernel function
$K(\cdot)$	Derivative of kernel function H
$h(\cdot)$	Smoothing parameter
$Q^{(n)}$	n-fold Stieltjes convolution of Q
$P(\cdot) := (P_{ij})(\cdot), i, j \in \mathrm{E})$	Transition function of the semi-Markov process Z
$\Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in \mathrm{E})$	Markov renewal function

m_i	Mean sojourn time in state i
m_{ij}	Mean sojourn time in state i, before visiting state j
m	Mean sojourn times of SMP
μ_{ij}	Mean recurrence time from state i to state j, for semi Markov process Z
μ_{ii}	Mean recurrence time in state i for the SMP
μ^*_{ij}	Mean recurrence time from state i to state j, for embedded Markov chain J
$N(T)$	Number of jumps of Z in the time interval $[1, T]$
$N_i(T)$	Number of visits to state i of the SMP, up to time T
$N_{ij}(T)$	Number of transitions from state i to state j of the SMP, up to time T
$N_{ij}(t, T)$	Number of transitions from state i to state j of the SMP, up to time T, with sojourn time in state i inferior or equal to t
$\mathcal{H}(T)$	Sample path of the SMP

Reliability measurements of a semi-Markov system

$A(\cdot)$	Availability
$\lambda(\cdot)$	Failure rate
$R(\cdot)$	Reliability

Various symbols

$*$	Convolution product
$\mathcal{N}(0, \sigma^2)$	Standard normal random variable (mean $\mu = 0$, variance σ^2)
δ_{ij}	Symbol of Kronecker, i.e. $\delta_{ij} = 1$, if $i = j$, $\delta_{ij} = 0$, otherwise

Chapter 1

General introduction

1.1 Literature review

In this dissertation, we introduce a class of stochastic processes known as semi-Markov processes. We are interested in studying semi-Markov processes and the corresponding functions in statistical inference (semi-Markov kernel, sojourn time distribution function, reliability and their measurements, etc.) using nonparametric kernel estimation, which is one of the most important topic in mathematical statistics.

Semi-Markov processes and Markov renewal processes represent an important class of stochastic processes that naturally generalize Markov jump processes and renewal processes. The semi-Markov approach is significantly more flexible for applications than the Markov approach, since the sojourn time in each state of a Markov process is exponentially distributed in continuous time (resp. geometrically distributed in discrete time); this is why the Markov hypothesis is extremely restrictive and limited. Furthermore, a Markov model is distinguished by the absence of memory: if we know the past and present states of a system, the future visited states depend only on the present state and they are independent of anything that has happened in the past. In the semi-Markov case, the sojourn time distribution can be any distribution on $\mathbb{R}^+$ (resp. on $\mathbb{N}$). A semi-Markov process still preserves the Markov property but in a modified and more flexible way. In contrast to the classical Markov property, the memoryless property of a semi-Markov process does not act on the calendar time $(0, 1, \ldots, t, t+1, \ldots)$ but on a random time governed by the jump time process J, $(J_0, J_1, \ldots, J_n, J_{n+1}, \ldots)$. In this way, we obtain the more flexible Markov hypothesis referred before.

A semi-Markov process is built by means of a Markov renewal process, which is defined by a two-dimensional process. The first component of the process represents the successively states of the process, that define a Markov chain, because the transition to the next state does not depend on the history of the process before it enters the current state. The second process describes the time moments when the changes in the states of the process take place.

We are also interested in this thesis to apply our methodology to applications in systems reliability data which are measurements of the time to failure of any particular unit, that means trouble-free performance of the device for a specified amount of time. They are particularly significant in applications and the development of genuine technological systems. Stochastic processes represent the main tool for the reliability theory. Markov processes, semi-Markov processes, and renewal processes have been mainly used for describing the evolution of a system. For such a system, each state can be either up or down, framework that is considered in our work. In many real applications, the finite state space, even the countable one, is insufficient to characterize and predict the reliability of a real-life system. Moreover, when using the Markov assumption, we add a supplementary constraint, since the sojourn times of such a system are exponentially distributed in continuous time (resp. geometrically distributed in the discrete case). As a result, we provide a systematic modeling of reliability measures using semi-Markov processes.

Lévy [78], Smith [116] and Takács [118] introduced the semi-Markov processes independently and almost simultaneously; later, essential developments of this theory were proposed by Pyke [101, 102], Pyke and Schaufele [103, 104], Çinlar [37], [36], Gikhman and Skorokhod [55] and Shurenkov [114], Moore and Pyke [87], Koroluk and Turbin [74, 75], Limnios [83].

The semi-Markov process limit theorem are given by Feller [53], Pyke and Schaufele [103], Yackel [130], Moore and Pyke [87], Grigorescu and Oprisan [59], and Athreya et al. [9], as well as the references therein.

The study of discrete and continuous-time semi-Markov processes, associated probability modeling, and accompanying estimation techniques (mostly in a nonparametric environment) are important from both theoretically and applied point of view. Moore and Pyke [87] and Lagakos et al. [76] were the first to publish works on SMP estimation (empirical estimators and maximum likelihood estimators). Later Gill [56] and Andersen et al. [7] used point process theory to investigate Kaplan-Meier type estimators. The asymptotic local normality was investigated by Akritas and Roussas [2]. Atuncar et al. [10] explored empirical estimators of semi-Markov process sojourn time distributions and proved their consistency and asymptotic normality. Ouhbi and Limnios [93, 98] provided empirical estimators for finite nonlinear functionals of SMPs with various appealing asymptotic properties, as well as associated reliability theory and estimation. Window censored nonparametric SMP estimation was proposed by Alvarez [5], whereas Ouhbi and Limnios [94] were the first to present nonparametric estimators of the Markov renewal matrix; the same authors examined the semi-Markov transition matrix in [96]. In that study, a maximum likelihood estimator of the hazard rate function is developed, and the failure rate of a semi-Markov system is calculated in [95]. In [97] the rate of occurrence of the failure function is investigated. Limnios [79] established the invariance principle for the empirical estimator of semi-Markov kernels. Limnios et al. [84] defined and investigated the asymptotic properties of estimators of the stationary distribution of the embedded chain for a SMP and of the mean sojourn times. We need also to include Limnios and Oprişan pioneer monograph [83].

Recently, there has been a surge in interest in the statistical inference of discrete-time semi-Markov processes. Indeed, Barbu et al. [17] considered a semi-Markov discrete-time framework and provided a computational method to solve the corresponding Markov renewal equation. We refer to Barbu and Limnios [19, 21] and to Trevezas and Limnios [119] for nonparametric estimation in the discrete-time semi-Markov framework. The authors explored empirical estimators applied to nonparametric semi-Markov systems with good asymptotic properties in the first two publications. If the sojourn time in the system's last state is not taken into account, these estimators can be considered approximate maximum likelihood estimators (MLEs). They also showed that these estimators are strongly consistent and each component of the approximate maximum likelihood estimators is asymptotically normal. A generalization of this setting is proposed in the third cited work, where exact MLEs are proposed and asymptotic properties are also investigated. We can cite also Barbu et al. [20, 17] for nonparametric estimation of failure rate functions, reliability and survival analysis of discrete time semi-Markov processes.

Semi-Markov models allow for more flexible sojourn time distributions, making them ideal for applications and real-world systems in a variety of fields. Such investigations are primarily used in reliability, survival analysis, seismology, finance, insurance, climatology, queueing theory, and a variety of other scientific disciplines (see, for example, Grabski [60], Bulla and Bulla [32], Barbu and Limnios [21], Xia et al. [128], Stefanov and Manca [117], Isguder and Uzunoglu-Kocer [69], D'Amico[42], Votsi et al. [121, 122], Barbu et al. [18]). We also cite the hidden form of these processes, known as hidden semi-Markov processes, which are of special relevance in a variety of applications such as DNA investigations, speech and writing recognition, finance, and reliability. As stated in Yu's pioneering book [132], this type of process has evolved into one of the most important models in the field of machine learning and artificial intelligence, which has entered a period of intensive development.

Semi-Markov models in continuous time are appropriate for describing the evolution of a system. However, it is well known that they are difficult to solve numerically. Nonetheless, several methods are proposed in the literature to this end. Csenki [41] investigated the instability of the numerical solution of the Laplace transform, Cocozza-Thivent and Eymard [38] investigated an algorithm for computing an SMP's marginal distribution, and

Corradi et al. [39] proposed a general quadrature method for numerically solving the process evolution equation of a homogeneous semi-Markov process (HSMP), Limnios [81] proposed an algebraic method, and Hou et al. [68] studied the existence and uniqueness of a solution for the Markov renewal equation (MRE) of a semi-Markov process with countable state space that can be discretized and handled in discrete time.

Only a few R packages have been developed to deal with semi-Markov or hidden semi-Markov models. The **hsmm** R package (Bulla et al. [33]) implements hidden semi-Markov model simulation and maximum likelihood estimation. The **mhsmm** package (O'Connell and Højsgaard [91]) estimates and predicts hidden semi-Markov models for multiple observation sequences. The **msSurv** package (Ferguson et al. [54]) offers nonparametric estimation in semi-Markov models, but covariates are not taken into account. The **semiMarkov** package (Król and Saint Pierre [85]), performs maximum likelihood estimation for parametric continuous time semi-Markov processes and associated hazard rates. The **smm** package (Barbu et al. [16]), is dedicated to the simulation and parametric and nonparametric estimation of discrete time multi-state semi-Markov and Markov processes. Finally, **smmR** (Barbu et al. [15]) package deals with estimation and simulation, as well as reliability indicators, for multi-state discrete-time semi-Markov processes, in both parametric and nonparametric frameworks.

Most of the existing estimation procedures for continuous-time semi-Markov processes consider empirical estimators. However, one shortcoming of the empirical distribution function is that it is discontinuous. In particular, if the true distribution is known to be continuous, the empirical distribution may yield poor approximations. Kernel smoothing solves this discontinuity problem.

Although it may appear surprising, nonparametric Parzen-Rosenblatt kernel estimators in a semi-Markov context are almost non-existent in the literature, with only a few works addressing them. Shamsuddinov [113] studied the asymptotic unbiasedness and consistency of a kernel estimator of the density of the sojourn time distribution. This work considered a specific case of semi-Markov kernel (dependence of sojourn time distributions only on the current state, not on the next state to be visited), the smoothing parameter does not depend on the states (neither on the current one, nor on the next state to be visited), and asymptotic normality is not investigated. Dumitrescu et al. [47] proposed kernel estimators for general semi-Markov processes and investigated the L^1 convergence associated with them.

We can cite the works of Laksaci and Yousfate [77], Roussas [107, 108] and of Athreya and Atuncar [8] for nonparametric Parzen-Rosenblatt kernel estimators for Markov processes. We also refer the reader to Atuncar et al. [11], who provide some important conditions for strong consistency of kernel density estimators for some reliability measures of ergodic Markov processes.

Note that there is a possible terminological confusion when using the term « kernel estimator »: indeed, one can either understand (a) the nonparametric Parzen-Rosenblatt kernel estimator (of some probability density function, distribution function, etc.); or (b) the/an estimator of the semi-Markov kernel. Although it will be (more or less) clear which of these two meanings we are referring to in this thesis, this confusion/ambiguity can occur in the scientific literature at times.

The main goal of this dissertation is to use the kernel estimation method to perform nonparametric estimation for continuous-time semi-Markov processes.

Kernel density estimation is a well-known nonparametric method for estimating the density of a continuous random variable, and it is also useful statistical analysis tool in data mining. Rosenblatt [105] and Parzen [99] were the first to introduce the kenel method for density estimation. Cacoullos [35] and Epanechnikov [51] led the multivariate extension, while Nadaraya [88] and Watson [125] led the regression estimation. Originally, this method was proposed in the classical independent and identically distributed (i.i.d.) case. Let $X_1, X_2, \ldots, X_n$ be a sample of n i.i.d. random variables with the probability density f. The Parzen-Rosenblatt kernel estimator of the density function is defined by

$$\widehat{f}_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h_n}\right), \tag{1.1}$$

where: K is a bounded kernel function that is generally smooth and symmetric function; h_n, the so-called bandwidth or smoothing parameter, is a sequence of positive numbers tending to 0, as n goes to ∞.

Explicit formula defined in (1.1) can be interpreted as a transformation of the empirical density into a continuous function, where the probability mass of $1/n$ corresponding to each data point is redistributed along a neighborhood of the point according to the function K. The properties of this estimator were first investigated by Parzen[99], that obtained asymptotic expressions for the bias and variance and proved that, under certain conditions on the kernel K and the smoothing parameter h_n, the kernel density estimate $\widehat{f}_n$ is pointwise consistent and asymptotically normal.

The choice of the smoothing parameter is critical for the quality of the smoothness because varying the bandwidth along the support of the sample data provides flexibility to reduce the variance of the estimates in the few observations while also reducing the bias of the estimates in the many observations. Various methods for selecting this parameter have been proposed in the literature in virtually all nonparametric estimation; see, e.g., Jones et al. [71], del Río [43], Silverman [115], Bowman and Azzalini [29], etc. We can cite the methods based on plug-in and cross-validation introduced by Sarda [111], Altman and Leger [4], Bowman et al. [31], and Polanski and Baker [100] as methods of selecting the smoothing parameter.

The asymptotic properties of the kernel estimators, such as consistency and normality, were well documented. For the first time, Parzen [99] and Silverman [115] proved the pointwise consistency, Nadaraya [89] established the uniform almost sure convergence, Woodroofe [126] investigated the context of studying the asymptotic normality.

The kernel density estimation can also be used for modeling and simulation in different domains. For applications data mining and econometrics, we can see Scott [112], Wand and Jones [123], Wolfgang et al. [66], Alexandre [3]; for applications in the health field one can see, Yang et al. [131] and Rushton et al. [109]; for applications in archeology, see, e.g., Santos et al. [110], Bonnier et al. [28], Mariani et al. [86], etc.

Many packages exist that can perform kernel density estimation in R like **spatstat** (Baddeley and Turner [14]), kdde in **ks** (Duong [48]), **np** (Tristen and Jeffrey [67]), **KernSmooth** (Wand and Ripley [124]), and **spatialkernel** (Zheng and Diggle [46]), **kerdiest** (Quintela-del Río and Estévez-Pérez [44]), **sm** (Bowman and Azzalini [30]), and **feature** (Duong and Matt [49]), **kedd** (Guidoum [63]).

In this thesis, we construct kernel estimators of the semi-Markov propriety, and we study their asymptotic properties, mainly consistency and asymptotic normality, with numerical examples and applications.

1.2 Contributions of the thesis

1.2.1 Objectives

The following points summarize the primary goals of this project.
(i) Introduce kernel estimators for the sojourn time distribution function (whether conditional or not), the semi-Markov kernel, and the accompanying densities; establish asymptotic properties of the estimators, namely uniform strong consistency and asymptotic normality. We would like to emphasize that this work is an important step forward in the theory of statistical methods for semi-Markov processes; by doing so, we close a gap in the use of an important and already classical class of estimators (nonparametric Parzen-Rosenblatt kernel estimators) for this type of stochastic process.
(ii) For the reliability analysis of semi-Markov systems, modeling and estimating the reliability indicators, we introduce nonparametric estimators for the reliability and for its associated measures, such as availability, failure rate

of the semi-Markov and we study their asymptotic properties. To accomplish this, we firstly show how estimators of the basic quantities of a semi-Markov system are obtained by the nonparametric kernel method.

Numerical examples are used to support all of the theoretical findings developed in this thesis.

1.2.2 Outline of the thesis

The first chapter is dedicated to bibliographical notes, in order to better position the works presented in this thesis in the literature. In the same chapter, we present the principal aims of our work.

In the second chapter, we introduce the basic notations and definitions of Markov renewal theory and semi-Markov continuous-time processes in a probabilistic and statistical context, which will be used throughout the thesis.

In the third chapter, we estimate the conditional sojourn time, the continuous-time semi-Markov kernel, and the corresponding densities. The following section of this chapter is devoted to the investigation of the asymptotic properties of the proposed estimators, particularly uniform strong consistency and asymptotic normality; a numerical example illustrates the theoretical results.

The fourth chapter is devoted to the reliability and related measurements, as availability, failure rate of a repairable finite state space system which is described by a semi-Markov continuous time process. We obtain explicit expressions for the reliability function of such systems and for its associated measures from probabilistic and statistical point of view. We show that there are many methods for studying the reliability and related measurements and we can consider that the approach based on Markov renewal theory is more attractive due to its generality: first we find the Markov renewal equation associated to the respective quantity; then we solve this equation and get the desired result (solution of a Markov renewal equation). In this context, we propose kernel estimators of the reliability and for related measures, then we investigate asymptotic properties of these estimators as the uniform strong consistency and the asymptotic normality. Next, we consider a three-state semi-Markov system and present numerical results for the reliability.

We conclude our research with a broad conclusion, remarks on the work presented. We anticipate that we will investigate in the future several areas to improve and expand our performance on specific directions. Finally, a general bibliography concludes this thesis.

Our work can have multiple applications in multiple areas, such as biostatistics, reliability, queue theory, operations research, maintenance, communication, etc.

Finally, the most important definitions and tools that clarify many notions and will be used in the proofs of the main results are gathered in Appendix: for example, introducing Slutsky's theorem, Anscombe's theorem, limit theorems for Markov renewal process, etc.

Chapter 2

General concepts on continuous-time semi-Markov processes

This introductory chapter presents the notations and the essential notions necessary for describing the continuous-time semi-Markov model. We give the basic probabilistic properties of this process. In addition, we present the continue-time Markov renewal theory, and associated quantities. Furthermore, we present the basics of the statistical inference associated with these processes.

2.1 Semi-Markov processes and their associated measures

In what follows we present some of the basic tools and concepts on semi-Markov processes and Markov renewal processes, which will be used in the remainder of this thesis.

2.1.1 Semi-Markov processes

Let us consider a stochastic process $Z = (Z_t)_{t\in\mathbb{R}_+}$ with finite state space $E = \{1, \dots, s\}$, continuous to the right and having left limits in any time point. Let

$$0 = S_0 < S_1 < \cdots < S_n < S_{n+1} < \cdots ,$$

be the jump times of Z and $J_0, J_1, J_2, \dots$ the successively visited states of Z, that form a stochastic process $J = (J_n)_{n\in\mathbb{N}}$ defined on the probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$, that takes values in the finite state space E, note also that the process $S = (S_n)_{n\in\mathbb{N}}$ takes values in $[0, \infty)$. Set also $X = (X_n)_{n\in\mathbb{N}}$ for the successive sojourn times in the visited states. Thus, $X_n = S_n - S_{n-1}$, $n \in \mathbb{N}^*$, and by convention, we set $X_0 = S_0 = 0$.

The semi-Markov process can be defined, by a two-dimensional Markov renewal process, where one variable represents the states and the other the times of state changes (jump times). Thus, all properties of a semi-Markov process can be deduced from the properties of this Markov renewal process. The definition of a semi-Markov process in its usual form, essentially requires to points: A restrictive Markov condition which causes invariance of the renewal process under time shift (stationarity), and the homogeneity, i.e., the independence of the transition probabilities of the number of renewals.

Definition 2.1.1. *The stochastic process $(J, S) = (J_n, S_n)_{n\in\mathbb{N}}$ is called a Markov Renewal Process (MRP) with state space E, if the following relation holds true:*

$$\begin{aligned}&\mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t \mid J_0, ..., J_n; S_0, ..., S_n)\\ &= \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t|J_n),\ j \in E;\ t \in \mathbb{R}_+,\ \ n \in \mathbb{N}.\end{aligned}$$

If this property is verified, then the process $Z = \{Z_t,\ t \geq 0\}$ is called the semi-Markov process associated to the MRP (J, S)

$$Z_t = J_n \text{ with } S_n \leq t < S_{n+1}.$$

The relationships between the SMP Z and the MC J can be written as

$$Z_t = J_{N(t)} \quad \Leftrightarrow \quad J_n = Z_{S_n},$$

with

$$N(t) := \sup\{n \geq 0 \mid S_n \leq t\},\ t \in \mathbb{R}_+, \tag{2.1}$$

the counting process of the number of jumps in the time interval $(0, t]$. Thus, Z_t gives the state of the system at time t.
Figure 2.1 presents the evolution of a MRP.

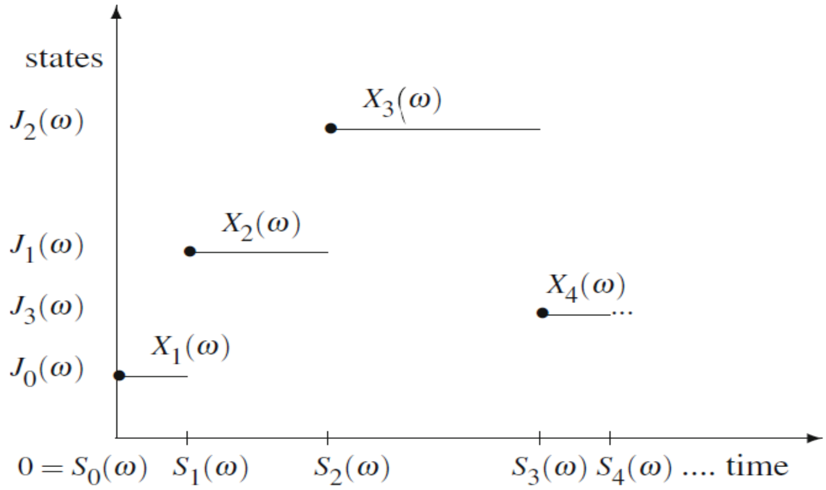


Figure 2.1: A typical semi-Markov sample.

The semi-Markov kernel Q is the essential quantity for the probabilistic and statistical study of the semi-Markov process, because most of the relative measures can be expressed directly as a function of it.

Definition 2.1.2. *Let us denote by* $Q(t) = \{Q_{ij}(t), i, j \in E\}$, $t \geq 0$, *the semi-Markov kernel of* Z, *defined by*

$$\begin{aligned} Q_{ij}(t) &:= \mathbb{P}(J_{n+1} = j, X_{n+1} \leq t \mid J_0, \ldots, J_n = i, X_1, \ldots, X_n) \\ &:= \mathbb{P}(J_{n+1} = j, X_{n+1} \leq t \mid J_n = i). \end{aligned} \tag{2.2}$$

$Q_{ij}(t)$ are absolutely continuous with respect to the Lebesgue measure, and let $q_{ij}(t)$ be the corresponding Radon-Nikodym derivative.

If $(J_n, S_n)_{n\in\mathbb{N}}$ is a MRP, it can be immediately checked that $J = (J_n)_{n\in\mathbb{N}}$ is a Markov chain, called the embedded Markov chain (EMC) with state space E and initial distribution

$$\alpha_i := \mathbb{P}(J_0 = i) = \mathbb{P}(Z_0 = i),\ i \in E.$$

Let us denote by $\nu = (\nu_i,\ i \in E)$ the stationary distribution of the EMC, and let the transition probabilities be denoted by

$$p_{ij} := \mathbb{P}(J_{n+1} = j | J_n = i) = Q_{ij}(\infty),\ i, j \in E.$$

It should be mentioned that the chain J does not act on the calendar time, but on the time indexed by the number of jumps.
The MRP and the SMP are considered homogeneous with respect to the time, in the sense that Equation (2.2) is independent of n. All along this work we consider homogeneous MRPs/SMPs only. Also, we do not allow transitions to the same state, i.e., $p_{ii} = 0$ for all $i \in E$, or equivalently $Q_{ii}(t) = 0$, for all $i \in E,\ t \in \mathbb{R}_+$. We also assume that there are not instantaneous transitions, that is $Q_{ij}(0) = 0$; note that this implies that S is a strictly increasing sequence.
In the next definition we introduce the sojourn time distributions and associated measures.

Definition 2.1.3. *For all $i,\ j \in E$ such that $p_{ij} \neq 0$, let us denote by:*

(i) F_i, the distribution function of the sojourn time in state i,

$$F_i(t) := \mathbb{P}(X_{n+1} \leq t \mid J_n = i) = \sum_{j=1}^{s} Q_{ij}(t), \quad t \in \mathbb{R}_+;$$

(ii) $\bar{F}(t) = (\bar{F}_i(t); i \in E) = (1 - F_i(t); i \in E)$, $t \in \mathbb{R}^+$ the survival function of $F(t)$.

(iii) F_{ij}, the conditional distribution of the sojourn time in state i before going to state j,

$$F_{ij}(t) := \mathbb{P}(X_{n+1} \leq t \mid J_n = i, J_{n+1} = j), \quad t \in \mathbb{R}_+.$$

Obviously, for all $i, j \in E$ and $t \in \mathbb{R}_+$, we have

$$Q_{ij}(t) = p_{ij} F_{ij}(t). \tag{2.3}$$

We assume that $F_{ij}(t)$ has a probability density $f_{ij}(t)$ with respect to the Lebesgue measure, such that

$$F_{ij}(t) = \int_0^t f_{ij}(x)dx = \frac{1}{p_{ij}} \int_0^t q_{ij}(x)dx.$$

That is, $f_{ij}(x) = \dfrac{q_{ij}(x)}{p_{ij}}$, $x \geq 0$.

The following definition gives the mean sojourn times of a semi-Markov process Z.

Definition 2.1.4. *Let m_{ij} be the mean sojourn times of SMP Z in state i when the next state is j, defined by*

$$m_{ij} = \frac{1}{p_{ij}} \int_0^\infty t q_{ij}(t)dt = \int_0^\infty t f_{ij}(t)dt. \tag{2.4}$$

The mean sojourn time in state i, m_i is defined by

$$m_i = \mathbb{E}[S_1 | J_0 = i] = \sum_{j \in E} p_{ij} m_{ij} = \int_0^\infty \overline{F}_i(t)dt. \tag{2.5}$$

The mean sojourn time m is defined by

$$m = \sum_{i \in E} \nu_i m_i = \sum_{i \in E} \sum_{j \in E} \nu_i p_{ij} m_{ij}.$$

In general, Q_{ij} is a sub-distribution, i.e., $Q_{ij}(\infty) \leq 1$, hence, F_i is a distribution function, $F_i(\infty) = 1$, and $Q_{ij}(0-) = F_i(0-) = 0$.
Another type of semi-Markov process can be obtained if $F_{ij}(\cdot)$ does not depend on j, i.e., $F_{ij}(t) \equiv F_i(t)$ and

$$Q_{ij}(t) = p_{ij} F_i(t).$$

Let $(S_n^i)_{n \geq 0}$ be the renewal process (eventually delayed) of the times of successive passages in state i. Then $N_i(t)$ is the counting process of renewals.

In the case where $Z_0 = i$, we have $S_0^i = 0$ and the renewal process S_n^i is an ordinary one; otherwise, if $Z_0 \neq i$, it is a delayed renewal process.
Denote by μ_{ii} the mean recurrence time of the state i of Z. This is the mean interarrival times of the (possibly delayed) renewal process (S_n^i), $n \geq 0$:

$$\mu_{ii} := \mathbb{E}[S_2^i - S_1^i].$$

Let us consider an irreducible positive recurrent Markov renewal process. For any $i \in E$ (see Limnios and Oprişan [83]) we have the following equality

$$\mu_{ii} = \frac{m}{\nu_i}. \tag{2.6}$$

Let $\mu^* := (\mu_{ii}^*)$ be the vector of mean recurrence times in the state i for the EMC J, defined by

$$\mu_{ii}^* := \mathbb{E}[T_i^* | J_0 = i],$$

where T_i^* is the first entry time in i defined by $T_i^* := \min\{n \geq 1 : J_n = i\}$, i.e. it is the minimal number of jumps to arrive in state i.
The relationship between the mean recurrence time and the stationary distribution (see Kemeny and Snell [73]) is given by

$$\mu_{ii}^* = \frac{1}{\nu_i}, \tag{2.7}$$

and, from relation (2.6), we have

$$\frac{\mu_{ii}}{\mu_{ii}^*} = m.$$

2.1.2 Renewal processes

Renewal processes (RPs) provide a theoretical framework for investigating the occurrence of patterns in repeated independent trials. The term « renewal » comes from the mean hypothesis that when the pattern of interest occurs for the first time, the process starts anew, in the sense that the initial situation is reestablished. More precisely, starting from this « renewal instant », the waiting time for the second occurrence of the pattern has the same distribution as the time needed for the first occurrence, and so on.
The Markov renewal equation is an essential tool in the theory of semi-Markov processes like the renewal equation in the case of the renewal process

theory on the half-real line.
The purpose of this subsection is to provide an introduction to the theory of continuous-time renewal processes.

Renewal function

The following definition of the convolution product is needed for the renewal function.

Definition 2.1.5. *Let $\phi(i,t)$, $i \in E$, $t \geq 0$, be a real-valued measurable function and define the convolution of ϕ by Q as follows*

$$Q * \phi(i,t) = \sum_{k\in E} \int_0^t Q_{ik}(ds)\phi(k,t-s).$$

The n-fold Stieltjes convolution of $Q_{ij}(t)$ by itself, for any $i,j \in E$ is defined by,

$$Q_{ij}^{(n)}(t) = \begin{cases} \delta_{ij}\mathbb{1}_{\{t\geq 0\}} & \text{if } n = 0, \\ Q_{ij}(t) & \text{if } n = 1, \\ \sum_{k\in E} \int_0^t Q_{ik}(ds) Q_{kj}^{(n-1)}(t-s) & \text{if } n \geq 2, \end{cases}$$

where δ_{ij} is the Kronocker symbol defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, we have

$$Q_{ij}^{(n)}(t) = \mathbb{P}_i(J_n = j, S_n \leq t). \tag{2.8}$$

Here, it is clear that, $\mathbb{P}_i(J_n = j, S_n \leq t)$ means $\mathbb{P}(J_n = j, S_n \leq t | J_0 = i)$, and $\mathbb{E}_i$ is the corresponding expectation.
Let us define the renewal function $\Psi_{ij}(t) := \mathbb{E}_i[N_j(t)]$ of the renewal process $(S_n^j,\ n \geq 1)$ with counting function $N_j(t)$. This is called the Markov renewal

function of the semi-Markov process. We have

$$\begin{aligned}
\Psi_{ij}(t) &= \mathbb{E}_i[N_j(t)] \\
&= \mathbb{E}_i \sum_{n=0}^{\infty} \mathbf{1}_{\{J_n=j, S_n \le t\}} \\
&= \sum_{n=0}^{\infty} P_i(J_n = j, S_n \le t) \\
&= \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t). \qquad (2.9)
\end{aligned}$$

Definition 2.1.6. *[83] The semi-Markov process Z is said to be regular if*

$$\mathbb{P}_i(N(t) < \infty) = 1,$$

for any $t \ge 0$ and any $i \in E$.
For regular semi-Markov processes, we have $S_n \le S_{n+1}$, for any $n \in \mathbb{N}$, and $S_n \to \infty$.

Therefore, a MRP is regular if and only if $\sum_j Q_{ij}^{(n)}(t) \to 0$, as $n \to \infty$, for all i.

The following theorem gives a criterion for regularity.

Theorem 2.1.1. *[106] If some real numbers, say $\alpha > 0$ and $\beta > 0$, exist, such that $F_i(\alpha) < 1-\beta$, for all $i \in E$, then the semi-Markov process is regular.*

Let us write the Markov renewal function (2.9) in matrix form

$$\Psi(t) = (\mathrm{I} - Q(t))^{(-1)} = \sum_{n=0}^{\infty} Q^{(n)}(t). \qquad (2.10)$$

This can also be written as

$$\Psi(t) = \mathrm{I}(t) + Q * \Psi(t), \qquad (2.11)$$

where $\mathrm{I} = \mathrm{I}(t)$ (the identity matrix), if $t \ge 0$ and $\mathrm{I}(t) = 0$, if $t < 0$. The upper index (-1) in the matrix $(\mathrm{I}-Q(t))$ means its inverse in the convolution sense.

Equation (2.11) is a special case of what is called a Markov Renewal Equation (MRE). A general MRE is as follows

$$\Theta(t) = \mathbf{L}(t) + Q * \Theta(t), \tag{2.12}$$

where $\Theta(t) = (\Theta_{i,j}(t))_{i,j\in E}$, $\mathbf{L}(t) = (L_{i,j}(t))_{i,j\in E}$ are matrix-valued measurable functions, with $\Theta_{i,j}(t) = L_{i,j}(t) = 0$ for $t < 0$. The function $\mathbf{L}(t)$ is a given matrix valued function whereas $\Theta(t)$ is an unknown matrix-valued function. We may also consider a vector version of Equation (2.12), i.e., consider corresponding columns of the matrices Θ and $\mathbf{L}$.
Let $\mathbf{A}$ be the space of all bounded on compact sets of $\mathbb{R}_+$ matrix-valued functions $\Theta(t)$, i.e., $\|\Theta(t)\| = \sup_{i,j} |\Theta_{i,j}(t)|$ is bounded on sets $[0, \eta]$ for all $\eta \in \mathbb{R}_+$.

We say that a matrix function $\Theta(t) = \Theta_{i,j}(t)$ belongs to $\mathbf{A}$, if for any fixed $j \in E$ the column vector function $\Theta_{\cdot,j}(\cdot)$ belongs to $\mathbf{A}$.

Equation (2.12) has a unique solution $\Theta = \Psi * \mathbf{L}(t)$ belonging to $\mathbf{A}$ when $\mathbf{L}(t)$ belongs to $\mathbf{A}$ (see Gámiz et al. [57]).

Let us give an example of a particular type of renewal process.

Example 1. *[57] Alternating Renewal Process:*
Let us consider an alternating renewal process with lifetime and repair time distributions F and G.

- *Up times:* $X_1^{'}, X_2^{'}, \ldots$
- *Down times:* $X_1^{''}, X_2^{''}, \ldots$

Denote by S_n the starting (arrival) time of the $(n+1)^{th}$ cycle, that is

$$S_n = \sum_{i=1}^{n} (X_i^{'} + X_i^{''}), \quad n \geq 1.$$

The process

$$Z_t = \sum_{n\geq 0} \mathbf{1}_{\{S_n \leq t < S_n + X_{n+1}^{'}\}}, \quad t \geq 0,$$

is a semi-Markov process, with states:

- 1 *for functioning,*
- 0 *for failure. The semi-Markov kernel is defined by*

$$Q(t) = \begin{pmatrix} 0 & F(t) \\ G(t) & 0 \end{pmatrix}.$$

The embedded Markov chain (J_n) *is a deterministic chain with transition matrix*

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and then we have the corresponding MRP (J_n, S_n)*.*

Transition matrix

Another important function is $P(\cdot) = (P_{ij}(\cdot),\ i, j \in E)$ called the transition matrix function of the semi-Markov process Z defined by

$$P_{ij}(t) = \mathbb{P}(Z_t = j | Z_0 = i), \quad i, j \in E,\ t \geq 0.$$

It is known, cf. Pyke [102], that

$$P_{ij}(t) = \mathbf{1}_{\{i=j\}} \left(1 - \sum_{k=1}^{s} Q_{ik}(t)\right) + \sum_{k \in E} \int_0^t P_{kj}(t-s) Q_{ik}(ds).$$

By solving the above Markov renewal equation, cf. Limnios [82], in a matrix notation, we obtain the solution,

$$P(t) = (I - Q(t))^{(-1)} * (I - diag(Q(t)\mathbf{1})), \tag{2.13}$$

where $diag(\cdot)$ is a diagonal matrix with i^{th} element $\sum_{j=1}^{s} Q_{ij}(t)$ and $\mathbf{1} = (1, 1, \ldots, 1)'$.

Definition 2.1.7. *For a semi-Markov process* Z, *the stationary distribution* $\pi = (\pi_i,\ i \in E)$ *is defined, when it exists, by*

$$\pi_i = \lim_{t \to \infty} P_{ij}(t), \quad \textit{for every } i,\ j \in E.$$

Proposition 2.1.1. *[84] If the MRP is positive recurrent, the limit distribution is given by*

$$\pi_i = \frac{\nu_i m_i}{\sum_{k=1}^{s} \nu_k m_k}. \tag{2.14}$$

The transition function is illustrated by the following example.

Example 2. *[57] Alternating Renewal Process:*
Let us continue the previous example. The transition function of the SMP Z is

$$P(t) = M * \begin{pmatrix} 1-F & F*(1-G) \\ G*(1-F) & 1-G \end{pmatrix}(t),$$

where M is the renewal function of the distribution function $F*G$, i.e.,

$$M(t) = \sum_{n=0}^{\infty} (F*G)^{(n)}(t).$$

Finally, by the Markov renewal theorem, we get also

$$\lim_{t\to\infty} P(t) = \begin{pmatrix} m_1 & m_0 \\ m_1 & m_0 \end{pmatrix} \Big/ (m_1 + m_0),$$

where m_0 and m_1 are the mean values of F and G, respectively. So, the limiting probability of the semi-Markov process is

$$\pi_1 = \frac{m_1}{m_1+m_0}, \quad \pi_0 = \frac{m_0}{m_1+m_0}.$$

2.2 Statistical inference of semi-Markov processes

The general theory of statistical inference in semi-Markov processes began with Moore and Pyke [87] who studied empirical estimators for finite semi-Markov kernels; Lagakos, Sommer, and Zelen [76] gave maximum likelihood estimators for nonergodic finite semi-Markov kernels; Akritas and Roussas [2] gave parametric local asymptotic normality results for semi-Markov processes; Ouhbi and Limnios [96] studied nonparametric estimators of semi-Markov kernels, etc.

We give the following observational procedure for statistical inference of stochastic processes: a single trajectory of this process is observed on the interval $[0, T]$, where $T \in \mathbb{R}_+$ is an arbitrary fixed censoring instant.

Definition 2.2.1. *Let us consider a sample path of the Markov renewal processes* $(J_l, S_l)_{l\in\mathbb{N}}$*,*

$$\mathcal{H}(T) := \left(J_0, X_1, \ldots, J_{N(T)-1}, X_{N(T)}, J_{N(T)}, u_T\right),\ T \in \mathbb{R}_+,$$

where $u_T := T - S_{N(T)}$*.*
For all $i,\ j \in E$ *and* $t \leq T$*, let us define the following counting processes:*

(i) $N_i(T) = \sum_{l=1}^{N(T)} \mathbb{1}_{\{J_{l-1}=i\}} = \sum_{l=1}^{\infty} \mathbb{1}_{\{J_{l-1}=i, S_l \leq T\}}$*: the number of visits to state* i*, up to time* T*;*

(ii) $N_{ij}(T) = \sum_{l=1}^{N(T)} \mathbb{1}_{\{J_{l-1}=i, J_l=j\}} = \sum_{l=1}^{\infty} \mathbb{1}_{\{J_{l-1}=i, J_l=j, S_l \leq T\}}$*: the number of transitions from* i *to* j*, up to time* T*;*

(iii) $N_{ij}(t,\ T) = \sum_{l=1}^{N(T)} \mathbb{1}_{\{J_{l-1}=i, J_l=j, X_l \leq t\}}$*: the number of transitions from* i *to* j*, up to time* T*, with sojourn time in state* i *less than or equal to* t*.*

2.2.1 Useful technical results

We introduce the following technical results which will be needed for the proofs.

Lemma 2.2.1. *[83] Under the previous notations, if the EMC* $(J_n)_n$ *is positive recurrent, then, for any* $i, j \in E$ *we have:*

1. $\frac{N_i(T)}{N(T)} \xrightarrow[T\to\infty]{a.s.} \nu_i$,

2. $\frac{N_{ij}(T)}{N(T)} \xrightarrow[T\to\infty]{a.s.} \nu_i p_{ij}$,

3. $\frac{N_i(T)}{T} \xrightarrow[T\to\infty]{a.s.} \frac{1}{\mu_{ii}}$,

4. $\frac{N_{ij}(T)}{T} \xrightarrow[T\to\infty]{a.s.} \frac{p_{ij}}{\mu_{ii}}$,

5. $\frac{N(T)}{T} \xrightarrow[T\to\infty]{a.s.} \frac{1}{\nu_i \mu_{ii}}$.

Theorem 2.2.1. *[63] Suppose that $Y_1, Y_2, \ldots$ are random variables such that*

$$Y_n \xrightarrow[n\to\infty]{a.s.} Y,$$

and that $\{N(t), t \geq 0\}$ is a family of positive, integer valued random variables, such that

$$N(t) \xrightarrow[t\to\infty]{a.s.} \infty.$$

Then

$$Y_{N(t)} \xrightarrow[t\to\infty]{a.s.} Y.$$

Lemma 2.2.2. *For a positive recurrent MRP we have:*

1. $S_n \xrightarrow{a.s.} \infty$ *as* $n \to \infty$,

2. $N(T) \xrightarrow{a.s.} \infty$ *as* $T \to \infty$.

2.2.2 Empirical estimators

The empirical estimator of the semi-Markov kernel (see Moore and Pyke [87]; Ouhbi and Limnios [96]) is defined by

$$\widehat{Q}_{ij}(t,T) = \frac{1}{N_i(T)} \sum_{l=1}^{N(T)} \mathbf{1}_{\{J_{l-1}=i, J_l=j, X_l \leq t\}}.$$

The empirical estimator of the semi-Markov kernel is strongly consistent and asymptotically normal.
From the definition of $\widehat{Q}_{ij}(t,T)$ we obtain $\widehat{Q}_{ij}(t,T) = \widehat{F}_{ij}(t,T)\widehat{p}_{ij}(T)$

$$\widehat{p}_{ij} = \frac{N_{ij}(T)}{N_i(T)},$$

and then

$$\widehat{F}_{ij}(t,T) = \frac{1}{N_{ij}(T)} \sum_{l=1}^{N(T)} \mathbf{1}_{\{J_{l-1}=i, J_l=j, X_l \leq t\}}.$$

The quantities $\widehat{F}_{ij}(t,T)$ and $\widehat{p}_{ij}(T)$ are respectively the empirical estimators for the conditional distribution functions and the transition probabilities. The conditional transition mechanism describes the probability function of the process moving into each possible new state, given the old state and the new one.

2.3 Conclusion

In this second chapter, we have exposed the most important works that have been carried out on SMP and MRP. The first part was devoted to the semi-Markov model, were their basic probabilistic properties and associated notions are established and basic elements of Markov renewal theory are presented. Having presented the semi-Markov framework, our objective is to study the statistical properties of various measures based on an observed trajectory.

Chapter 3

On the asymptotic properties of some kernel estimators for continuous-time semi-Markov processes

Our objectives in the present chapter are: (1) to introduce kernel estimators for the sojourn time distribution function (conditional or not), for the semi-Markov kernel and for the corresponding densities; (2) to establish asymptotic properties of the estimators, namely the uniform strong consistency and the asymptotic normality. We would like to stress that this work is an important step in the theory of statistical methods for semi-Markov processes; in this way we fill a gap in the use of the important and already classical class of estimators (nonparametric Parzen-Rosenblatt kernel estimators) for this type of stochastic processes.

This chapter is organized as follows. In Section 3.1, we impose some assumptions and we construct kernel estimators of the conditional sojourn time, continuous-time semi-Markov kernel, and unconditional sojourn time distribution. Section 3.2 is devoted to the study of the asymptotic properties of the proposed estimators, namely the uniform strong consistency and the asymptotic normality. Moreover, a numerical example has been conducted in Section 3.3 in order to highlight, on three state space, the superiority of our method to the empirical one. in Section 3.4 we prove our main results. We end the chapter by some concluding remarks.

3.1 Nonparametric kernel estimators

For fixed states i and j and $l \in \mathbb{N}^*$, let us denote by $X_{i;l}$ the corresponding sojourn time in state i during l^{th} visit of this state and by $X_{ij;l}$ the corresponding sojourn time in state i before going to state j, during l^{th} visit.

Taking a sample path $\mathcal{H}(T)$ of a Markov renewal process, for all $i, j \in E$ and $t \in \mathbb{R}_+$, $t \leq T$, we define the kernel estimators of $F_i(t)$, $F_{ij}(t)$, $Q_{ij}(t)$ and of the derivatives $f_i(t)$, $f_{ij}(t)$, $q_{ij}(t)$ as follows:

$$\begin{aligned} \widehat{F}_i(t,T) &= \frac{1}{N_i(T)} \sum_{l=1}^{N_i(T)} H\left(\frac{t - X_{i;l}}{h_{i,T}}\right) \\ &= \frac{1}{N_i(T)} \sum_{l=1}^{N(T)} H\left(\frac{t - X_l}{h_{i,T}}\right) \mathbb{1}_{\{J_{l-1}=i\}}; \qquad (3.1) \\ \widehat{F}_{ij}(t,T) &= \frac{1}{N_{ij}(T)} \sum_{l=1}^{N_{ij}(T)} H\left(\frac{t - X_{ij;l}}{h_{ij,T}}\right) \\ &= \frac{1}{N_{ij}(T)} \sum_{l=1}^{N(T)} H\left(\frac{t - X_l}{h_{ij,T}}\right) \mathbb{1}_{\{J_{l-1}=i, J_l=j\}}; \qquad (3.2) \end{aligned}$$

$$\begin{aligned} \widehat{f}_{ij}(t,T) &= \frac{1}{N_{ij}(T)} \sum_{l=1}^{N_{ij}(T)} \frac{1}{h_{ij,T}} K\left(\frac{t - X_{ij;l}}{h_{ij,T}}\right); \qquad (3.3) \\ \widehat{f}_i(t,T) &= \frac{1}{N_i(T)} \sum_{l=1}^{N_i(T)} \frac{1}{h_{i,T}} K\left(\frac{t - X_{i;l}}{h_{i,T}}\right), \qquad (3.4) \end{aligned}$$

where $H(t) = \displaystyle\int_{-\infty}^{t} K(t)dt$. It should be noted that the smoothing parameter of the previous estimators depends on the sample size, so we should write $h_{i,T} = h_{i,N_i(T)}$ (resp. $h_{ij,T} = h_{ij,N_{ij}(T)}$); however we prefer to use a simpler

notation.
Second, we can introduce an estimator of $Q_{ij}(T)$ defined by:

$$\widehat{Q}_{ij}(t,T) = \widehat{p}_{ij}(T)\widehat{F}_{ij}(t,T) = \frac{N_{ij}(T)}{N_i(T)}\widehat{F}_{ij}(t,T),$$

where $\widehat{p}_{ij}(T) := \frac{N_{ij}(T)}{N_i(T)}$ is the empirical estimator of p_{ij}. So, we get the corresponding kernel estimators of $Q_{ij}(t)$ and $q_{ij}(t)$ given by

$$\widehat{Q}_{ij}(t,T) = \frac{1}{N_i(T)} \sum_{l=1}^{N_{ij}(T)} H\left(\frac{t - X_{ij;l}}{h_{ij,T}}\right), \tag{3.5}$$

$$\widehat{q}_{ij}(t,T) = \frac{1}{N_i(T)} \sum_{l=1}^{N_{ij}(T)} \frac{1}{h_{ij,T}} K\left(\frac{t - X_{ij;l}}{h_{ij,T}}\right). \tag{3.6}$$

3.1.1 Asymptotic properties of the estimators

Let us first focus on the assumptions we need to derive the asymptotic behavior of our estimators.

Assumptions

All along this chapter we are working under the following three assumptions.

(H.1) The EMC $(J_n)_{n\in\mathbb{N}}$ is an ergodic irreducible Markov chain, with stationary distribution ν.

(H.2) The SMP is irreducible, aperiodic, with finite mean sojourn times.

(H.3) The SMP (or equivalently, the MRP) is regular, that is $\mathbb{P}_i(N(t) < \infty) = 1$ for all $t > 0$, $i \in E$, where $\mathbb{P}_i(\cdot)$ means $\mathbb{P}(\cdot|J_0 = i)$.

In addition, we need to introduce the following conditions.

(H.4) i) $Q_{ij}(t), F_i(t)$ and $F_{ij}(t)$ are absolutely continuous with respect to the Lebesgue measure, and let $q_{ij}(t), f_i(t)$ and $f_{ij}(t)$ be respectively the corresponding Radon-Nikodym derivatives.

ii) The first derivatives f_{ij} and f_i are bounded.

(H.5) i) The function H is a distribution function.

ii) The kernel K is density function of bounded variation such that

$$\lim_{x\to\infty} |xK(x)| = 0 \text{ and } |\int t^j K^k(t)dt| < \infty \text{ for } j = 0,1, \text{ and } k = 1,2,$$

where K is the derivative of H.

(H.6) The smoothing parameters $h_{i,n}$, $h_{ij,n}$ satisfy

$$\lim_{n\longrightarrow\infty} h_{i,n} = 0 \text{ and } \lim_{n\longrightarrow\infty} h_{ij,n} = 0.$$

(H.7) The series $\sum_{n=1}^{\infty} e^{-\gamma n h_{i,n}^2}$ and $\sum_{n=1}^{\infty} e^{-\gamma n h_{ij,n}^2}$ converge for every positive value of γ.

Comments on the assumptions

Notice that these conditions are usually assumed in this context. Indeed, Conditions (H.1) and (H.2) are classical assumptions for semi-Markov processes, commonly used in the literature (see, for instance Dumitrescu et al. [47]; Barbu and Limnios [19]). Note also that under condition (H.3), $S_n < S_{n+1}, n \in \mathbb{N}$, $S_n \xrightarrow[n\to\infty]{a.s.} \infty$, $N(t) \xrightarrow[t\to\infty]{a.s.} \infty$ (see Limnios and Oprişan [83]). Assumption (H.4) imposed on $Q_{ij}(t), F_{ij}(t)$ and $F_i(t)$ is a regularity type hypothesis. Precisely, Hypothesis (H.4)(i) is a continuity-type constraint which will allow us to get strong consistency. Moreover, as soon as one wishes to state the asymptotic normality of our estimators, one has to introduce more restrictive constraints, which is the role played by the second derivative hypothesis (H.4)(ii). The technical conditions on the kernels are imposed for a sake of brevity of proofs. (H.6) and (H.7) are other technical constraints. Furthermore, (H.6) is also satisfied for $h_{ij,T}$ and $h_{i,T}$.

3.2 Asymptotic properties

3.2.1 Uniform strong consistency

Our first results concern the uniform strong consistency of the proposed estimators.

Theorem 3.2.1. *For any fixed arbitrary states $i, j \in E$ and any fixed arbitrary positive $t \in \mathbb{R}_+$, $t \leq T$, under Assumptions (H.5)-(H.6), the kernel estimator $\widehat{F}_i(t,T)$ introduced in (3.1) is uniformly strongly consistent, i.e.,*

$$\max_i \sup_{t\in[0,T]} |\widehat{F}_i(t,T) - F_i(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

According to the definition of the kernel estimators (3.2), (3.3) and (3.5), we can establish their uniform strong consistency, whose proofs are straightforward adaptations of the proof of Theorem 3.2.1.

Corollary 3.2.1. *For any fixed arbitrary states $i, j \in E$ and any fixed arbitrary positive $t \in \mathbb{R}_+$, $t \leq T$, under Assumptions (H.5)-(H.6) and additional hypothesis (H.7) for (ii), (iii) and (iv), the following statements stand true.*

(i) *The kernel estimator $\widehat{F}_{ij}(t,T)$ introduced in (3.2) is uniformly strong consistent, i.e.,*

$$\max_{i,j} \sup_{t\in[0,T]} |\widehat{F}_{ij}(t,T) - F_{ij}(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

(ii) *The kernel estimator $\widehat{f}_{ij}(t,T)$ proposed in (3.3) is uniformly strong consistent, i.e.,*

$$\max_{i,j} \sup_{t\in[0,T]} |\widehat{f}_{ij}(t,T) - f_{ij}(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

(iii) *The kernel estimator $\widehat{f}_i(t,T)$ introduced in (3.4) is uniformly strong consistent, i.e.,*

$$\max_i \sup_{t\in[0,T]} |\widehat{f}_i(t,T) - f_i(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

(iv) *The kernel estimator of the semi-Markov kernel density proposed in (3.6) is uniformly strongly consistent, i.e.,*

$$\max_{i,j} \sup_{t\in[0,T]} |\widehat{q}_{ij}(t,T) - q_{ij}(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

(v) *Since $\widehat{Q}_{ij}(t,T) = \widehat{p}_{ij}(T)\,\widehat{F}_{ij}(t,T)$, the uniform strong consistency of the estimators $\widehat{p}_{ij}(T)$ and $\widehat{F}_{ij}(t,T)$ allow us to deduce that:*

$$\max_{i,j} \sup_{t\in[0,T]} |\widehat{Q}_{ij}(t,T) - Q_{ij}(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

3.2.2 Asymptotic normality

The following results concern the asymptotic normality of the proposed estimators.

Theorem 3.2.2. *For any fixed arbitrary states $i, j \in E$ and any fixed arbitrary positive $t \in \mathbb{R}_+$, $t \leq T$, under Assumptions (H.4), (H.5) and (H.6), the following statements stand true.*

(i)

$$\sqrt{T}[\widehat{F}_i(t,T) - F_i(t)] \xrightarrow[T\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_F^2(i,t)),$$

with the asymptotic variance

$$\sigma_F^2(i,t) = \mu_{ii} F_i(t)\left[1 - F_i(t)\right].$$

(ii) *Under the condition $\lim_{T\longrightarrow\infty} Th_{ij,T} = \infty$, we have*

$$\sqrt{Th_{ij,T}}[\widehat{q}_{ij}(t,T) - q_{ij}(t)] \xrightarrow[T\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_q^2(i,j,t)),$$

with the asymptotic variance

$$\sigma_q^2(i,j,t) = \mu_{ii} q_{ij}(t) \int_{-\infty}^{\infty} K^2(z)\,dz.$$

Similarly to Theorem 3.2.2, we establish the following asymptotic results.

Corollary 3.2.2. *Under the same conditions than those of Theorem 3.2.2, the following statements stand true.*

(i)

$$\sqrt{T}[\widehat{F}_{ij}(t,T) - F_{ij}(t)] \xrightarrow[T\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_F^2(i,j,t)),$$

with the asymptotic variance

$$\sigma_F^2(i,j,t) = \frac{\mu_{ii}}{p_{ij}} F_{ij}(t) \left[1 - F_{ij}(t)\right].$$

(ii)

$$\sqrt{T}[\widehat{Q}_{ij}(t,T) - Q_{ij}(t)] \xrightarrow[T\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_Q^2(i,j,t)),$$

with the asymptotic variance

$$\sigma_Q^2(i,j,t) = \mu_{ii} Q_{ij}(t) \left[1 - Q_{ij}(t)\right].$$

(iii) If $\lim\limits_{T\longrightarrow\infty} Th_{ij,T} = \infty$ *holds, we have*

$$\sqrt{Th_{ij,T}}[\widehat{f}_{ij}(t,T) - f_{ij}(t)] \xrightarrow[T\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_f^2(i,j,t)),$$

with the asymptotic variance

$$\sigma_f^2(i,j,t) = \frac{\mu_{ii}}{p_{ij}} f_{ij}(t) \int_{-\infty}^{\infty} K^2(z)\, dz.$$

(iv) If $\lim\limits_{T\longrightarrow\infty} Th_{i,T} = \infty$ *holds, we have*

$$\sqrt{Th_{i,T}}[\widehat{f}_{i}(t,T) - f_{i}(t)] \xrightarrow[T\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_f^2(i,t)),$$

with the asymptotic variance

$$\sigma_f^2(i,t) = \mu_{ii} f_{i}(t) \int_{-\infty}^{\infty} K^2(z)\, dz.$$

3.3 Numerical example

In this section we carry out a simulation study to evaluate the finite sample performance of the estimation procedure described in the previous sections. We will apply our results to a three-state semi-Markov process. The state space of the system is given by $E = \{1, 2, 3\}$.

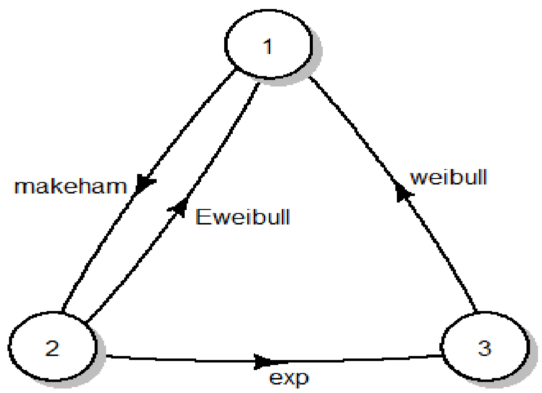


Figure 3.1: A three-state semi-Markov system.

The possible transitions between states are given in Figure 3.1. The system is defined by:

- The initial distribution $\alpha = (1/3, 1/3, 1/3)$.

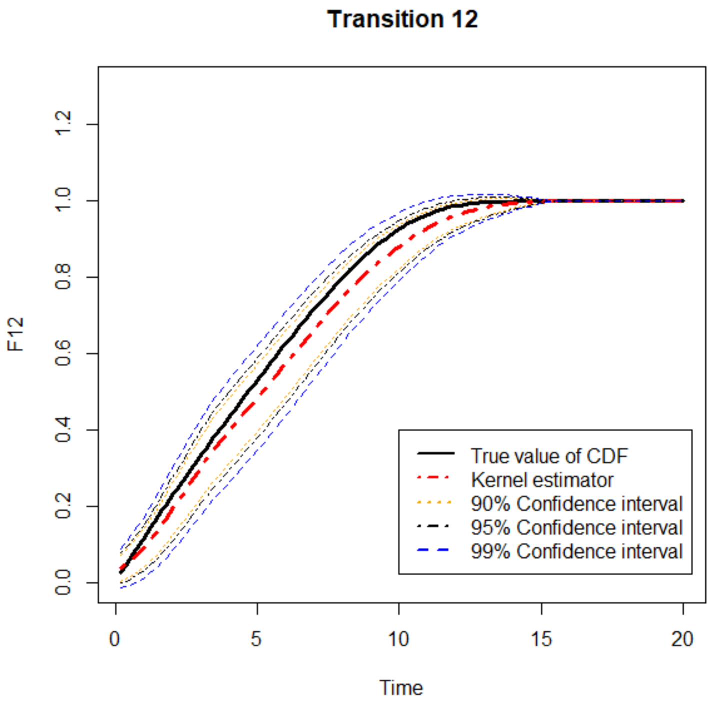
- The transition matrix $\mathbf{p}$ of the embedded Markov chain $(J_n)_{n\in\mathbb{N}}$

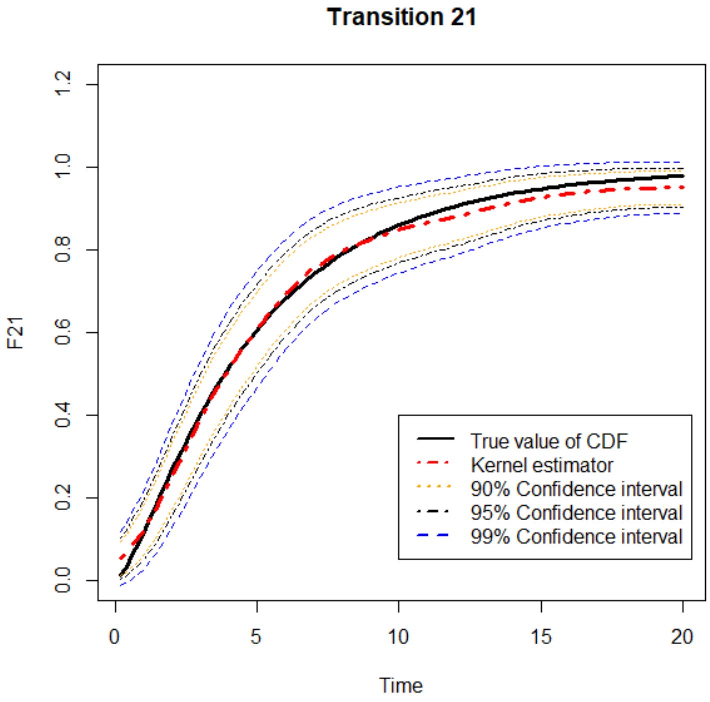
$$\mathbf{p} = \begin{pmatrix} 0 & 1 & 0 \\ 0.95 & 0 & 0.05 \\ 1 & 0 & 0 \end{pmatrix}.$$

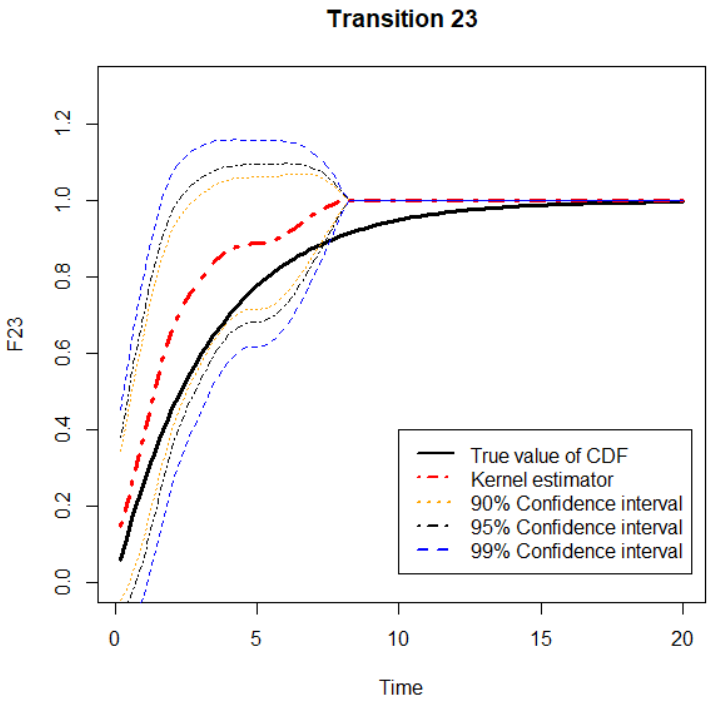
- The sojourn time in state 1 before going to state 2 is Gompertz-Makeham distributed with parameters $\mu = 0.02$, $\nu = 0.1$ and $\gamma = 3$. When state 2 is entered, the next state will be 1 or 3, if the next state to be visited is 1, then the sojourn time is exponentially Weibull distributed, $W(\theta; k; \gamma)$ with $\theta = 0.8$, $k = 3$ and $\gamma = 2$; otherwise the next state to be entered is 3 and the sojourn time in this state is exponentially distributed, with parameter $\lambda = 0.3$. When state 3 is entered, the next state to be visited is state 1 and the sojourn time in state 3 is Weibull distributed $W(\alpha; \beta)$ with parameters $\alpha = 2$ and $\beta = 0.5$.

3.3.1 Confidence intervals

To construct the smoothed estimators, the kernel $K(\cdot)$ is chosen to be the quadratic function defined as $K(u) = \frac{3}{4}(1-u^2)$ for $|u| \leq 1$ and the cumulative distribution function $H(u)$ is defined by $H(u) = \displaystyle\int_{-\infty}^{u} \frac{3}{4}(1 - z^2)\mathbb{1}_{[-1,1]}(z)dz$. The bandwidth h_T has been obtained by the « PBbw » method, that computes the plug-in bandwidth of Polansky and Baker method, cf. [100]. Figure 3.2 presents the true values of the conditional distribution of sojourn time $F_{12}(t)$, $F_{21}(t)$, $F_{23}(t)$ and $F_{31}(t)$, their estimators and the corresponding confidence intervals at levels 99%, 95% and 90%. Notice that we have considered that the observation period is the interval $[0, T]$ with $T = 1000$.


Transition 12
F12
Time
True value of CDF
Kernel estimator
90% Confidence interval
95% Confidence interval
99% Confidence interval


Transition 21
F21
Time
True value of CDF
Kernel estimator
90% Confidence interval
95% Confidence interval
99% Confidence interval



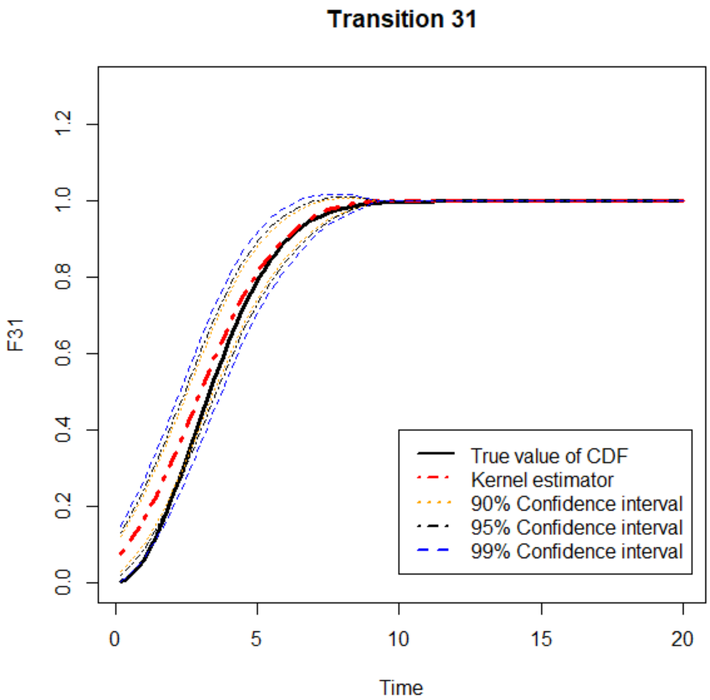


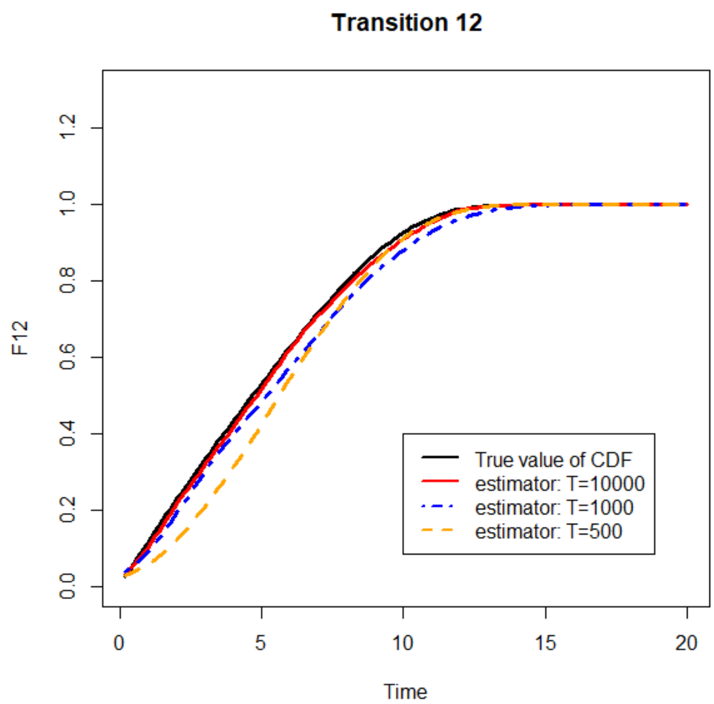
Figure 3.2: Confidence interval of the conditional distribution estimators of sojourn time of the system described in Figure 3.1.

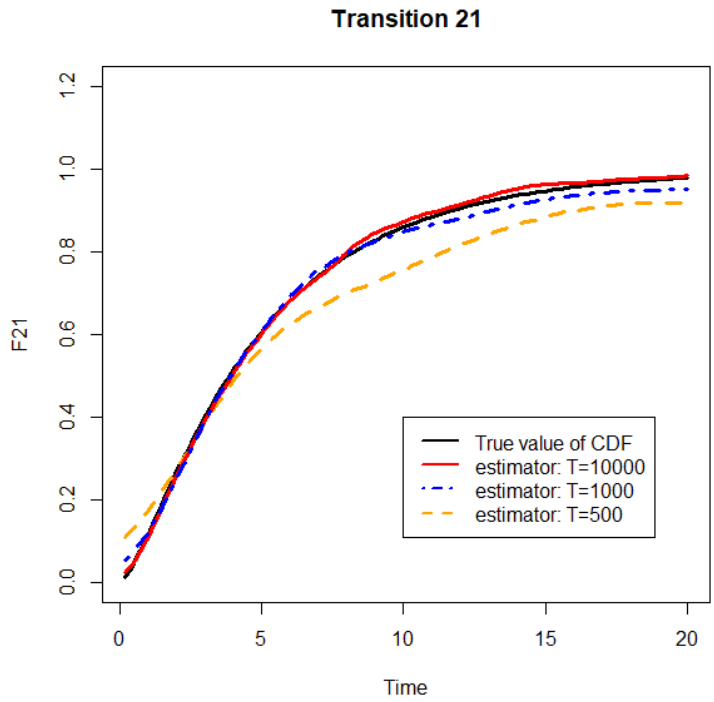
3.3.2 Mean integrate square error

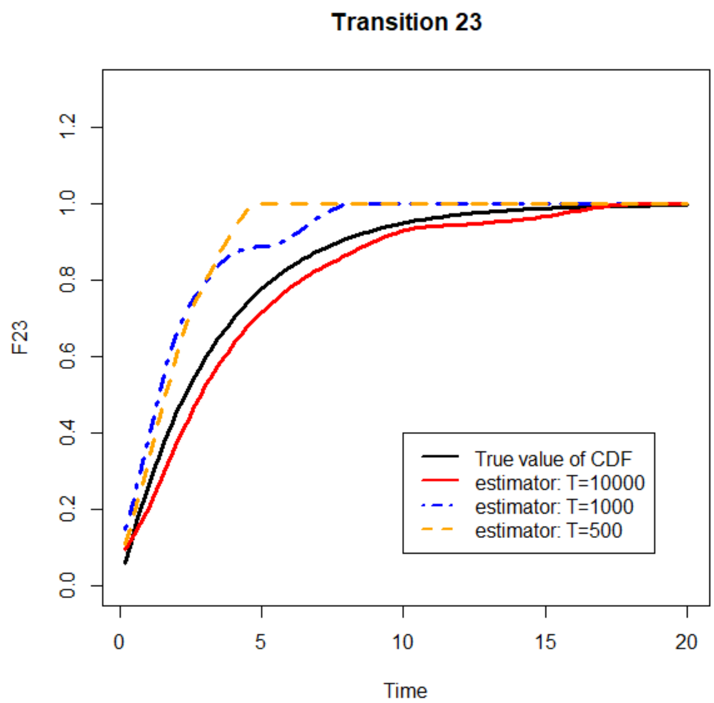
One way of illustrating the accuracy of the estimators is by providing the Mean integrate square error (MISE)

$$MISE(\widehat{F}_{ij}) = \frac{1}{R}\sum_{r=1}^{R}\int_0^{\infty}(\widehat{F}_{ij}(t) - F_{ij}(t))^2 dt.$$

We have carried out $R = 100$ repetitions of the experiment and we have taken $m = 100$ points of discretization.
Figure 3.3 and Table 3.1 give a comparison between the conditional distribution of sojourn time estimators obtained for different sample sizes ($T = 500$, $T = 1000$ and $T = 10000$). We observe that the estimators approach the true value as T increases.




Transition 21
F21
Time
True value of CDF
estimator: T=10000
estimator: T=1000
estimator: T=500


Transition 23
F23
Time
True value of CDF
estimator: T=10000
estimator: T=1000
estimator: T=500

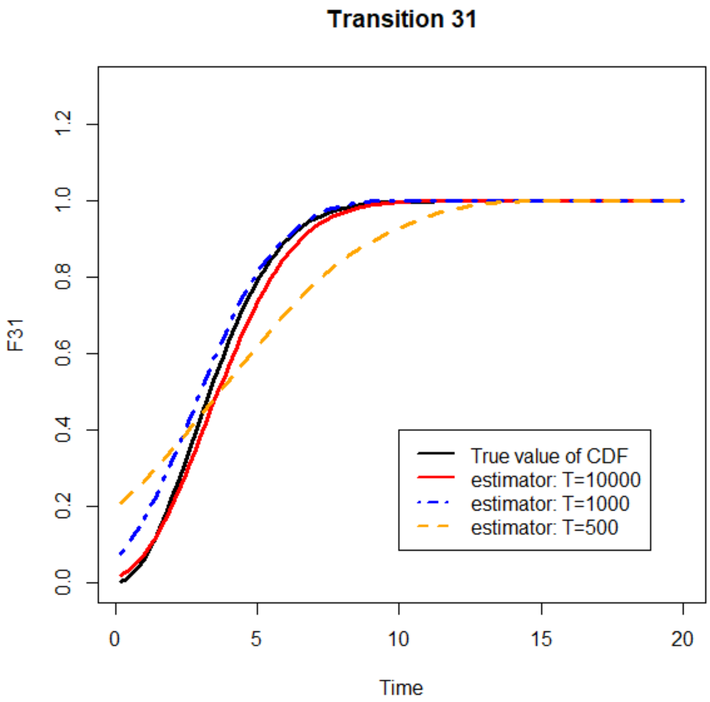


Figure 3.3: Comparison between the conditional distribution estimators of the sojourn time for different sample sizes and the true value.

	$T = 1000$	$T = 5000$	$T = 10000$
F_{12}	0.1249947	0.01793063	0.003740013
F_{21}	0.0502854	0.04285931	0.01664149
F_{23}	0.5886443	0.2435079	0.1527249
F_{31}	0.145625	0.01719063	0.004212039

Table 3.1: Estimators of the conditional distribution of sojourn times obtained for different sample sizes.

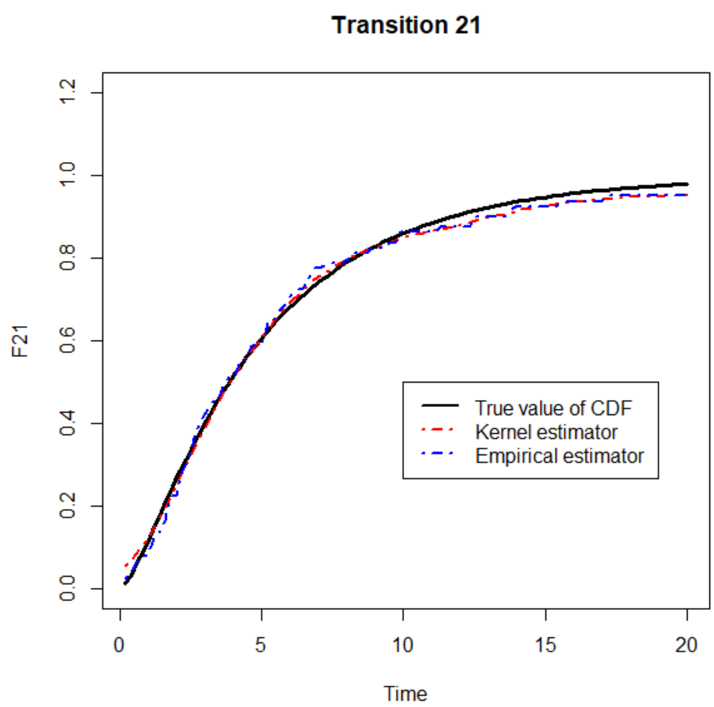
3.3.3 Comparison between the empirical and the kernel estimation

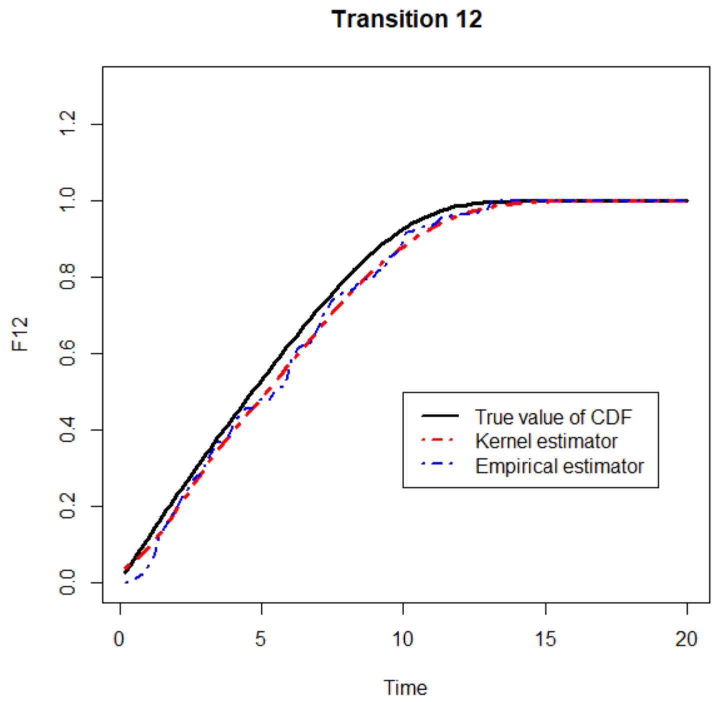
Table 3.2 presents the MSIE's values for both methods. We remark easily that the kernel method gives better results than the empirical one. This superiority is important in F_{31}.

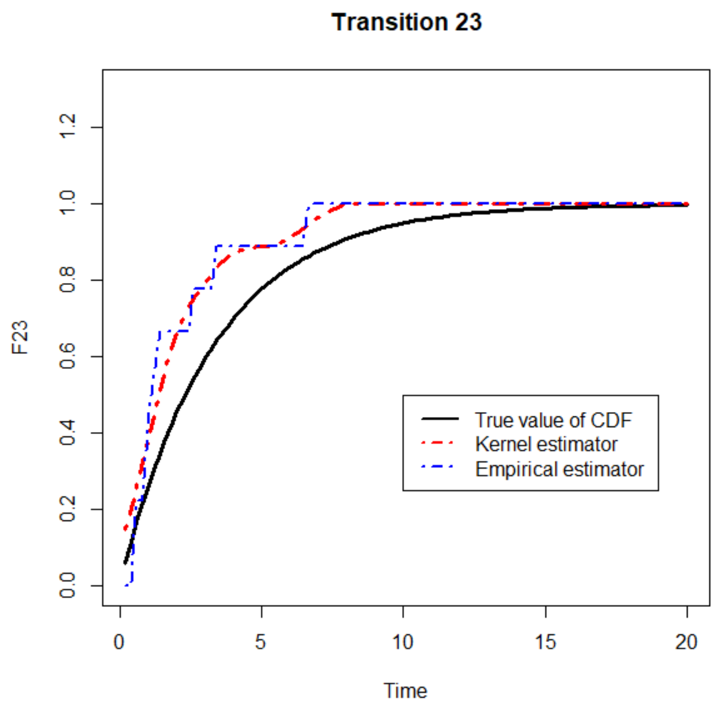
We present the results for the Weibull, exponential, exponential Weibull and Makeham distributions. The mean squared error of the kernel method is smaller than that of the empirical method. Furthermore, the mean squared is the smallest when we use the exponential Weibull and Gomperty-Makeham distributions.

	Kernel estimation	Empirical estimation
F_{12}	0.003740013	0.004806194
F_{21}	0.01664149	0.01906831
F_{23}	0.1527249	0.1964222
F_{31}	0.004212039	0.02892303

Table 3.2: MISEs for both methods, kernel estimation and empirical estimation.




Transition 12
F12
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0
5
10
15
20
Time
True value of CDF
Kernel estimator
Empirical estimator


Transition 23
F23
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0
5
10
15
20
Time
True value of CDF
Kernel estimator
Empirical estimator

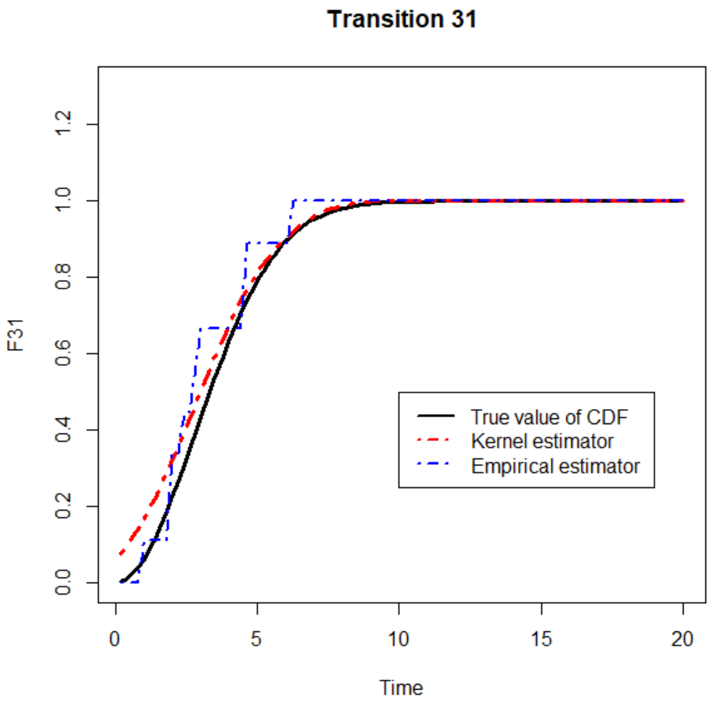


Figure 3.4: Comparison between the conditional distribution of the sojourn time estimators of the empirical and the kernel method.

3.4 Proofs of main results

Proof of Theorem 3.2.1

Let

$$\widehat{F}_i^*(t) = \frac{1}{n}\sum_{l=1}^{n} H\left(\frac{t - X_{i;l}}{h_{i,n}}\right).$$

Under assumptions (H.5) and (H.6), applying Theorem 1 of [88] we have

$$\max_i \sup_{t\in[0,T]} |\widehat{F}_i^*(t) - F_i(t)| \xrightarrow[n\to\infty]{a.s.} 0.$$

Taking into account Theorem 2.2.1, we obtain the desired result. □

Proof of Corollary 3.2.1

We give here only the proof of (iv). For the other points, according to the definition of the kernel estimators (3.2), (3.3) and (3.5), we can establish their uniform strong consistency whose proofs are a straightforward adaptation of the proof of Theorem 3.2.1.

(iv) We have for all i, $j \in E$, $\widehat{q}_{ij}(t,T) = \widehat{f}_{ij}(t,T)\widehat{p}_{ij}(T)$, $|\widehat{p}_{ij}(T)| \le 1$, and $|f_{ij}(t,T)| \le 1$. After some computation we obtain

$$\begin{aligned} & \mathbb{P}\left(\max_{i,j} \sup_{t\in[0,T]} \mid \widehat{q}_{ij}(t,T) - q_{ij}(t) \mid > \epsilon\right) \\ \le \; & \mathbb{P}\left(\max_{i,j} \mid \widehat{p}_{ij}(T) - p_{ij} \mid > \frac{\epsilon}{2}\right) + \mathbb{P}\left(\max_{i,j} \sup_{t\in[0,T]} \mid \widehat{f}_{ij}(t,T) - f_{ij}(t) \mid > \frac{\epsilon}{2}\right). \end{aligned}$$

Hence the result is a consequence of (ii) in Corollary 3.2.1 under assumptions (H.6), (H.7) and of the strong consistency of $\widehat{p}_{ij}(T)$. □

Proof of Theorem 3.2.2

(i) Remark that we can write

$$\sqrt{T}[\widehat{F}_i(t,T) - F_i(t)] = \tfrac{T}{N_i(T)} \tfrac{1}{\sqrt{T}} \sum_{l=1}^{N(T)} \left[H\left(\frac{t - X_l}{h_{i,T}}\right) \mathbb{1}_{\{J_{l-1}=i\}} - F_i(t) \mathbb{1}_{\{J_{l-1}=i\}} \right].$$

Taking into account Theorem 2.2.1, we introduce

$$U_l = \left(H\left(\frac{t - X_l}{h_{i,n}}\right) - F_i(t) \right) \mathbb{1}_{\{J_{l-1}=i\}}. \tag{3.7}$$

If we denote by $\mathcal{F}_l$ the σ-algebra $\mathcal{F}_l := \sigma(J_n, X_n; n \le l)$, $l \ge 0$, thus U_l is $\mathcal{F}_l$-measurable and $\mathcal{F}_l \subseteq \mathcal{F}_{l+1}$, for all $l \in \mathbb{N}$. Moreover, we have

$$\begin{aligned} \mathbb{E}(U_l|\mathcal{F}_{l-1}) &= \mathbb{E}\left(\left[H\left(\tfrac{t-X_l}{h_{i,n}}\right)\mathbb{1}_{\{J_{l-1}=i\}} - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}\right]|\mathcal{F}_{l-1}\right) \\ &= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} H\left(\frac{t-x}{h_{i,n}}\right) f_i(x)dx - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}, \end{aligned}$$

where the last equation is obtained by the fact that $\mathbb{1}_{\{J_{l-1}=i\}}$ is $\mathcal{F}_{l-1}$-measurable. On the other hand, using a change of variable, an integration by parts followed by Taylor's expansion of $F_i(t - h_{i,n}z)$ in a neighborhood of t, combined with assumptions (H.4), (H.5) and (H.6), we get

$$\begin{aligned}
\mathbb{E}(U_l|\mathcal{F}_{l-1}) &= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} H(z)\, dF_i(t - zh_{i,n}) - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}} \\
&= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} K(z)\, F_i(t - zh_{i,n})dz - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}} \\
&= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} K(z)\, (F_i(t - zh_{i,n}) - F_i(t))dz \\
&= \mathbb{1}_{\{J_{l-1}=i\}} \int_{-\infty}^{\infty} K(z)\, (-h_{i,n}zF_i'(t^*))dz,
\end{aligned}$$

where t^* is between t and $t - h_{i,n}z$. It follows

$$|\int_{-\infty}^{\infty} K(z)\, F_i'(t^*)h_{i,n}zdz| \leq Ch_{i,n} \int_{-\infty}^{\infty} |z|K(z)\, dz = O(h_{i,n}).$$

This implies that

$$\mathbb{E}(U_l|\mathcal{F}_{l-1}) \;\; \to 0,\ as\ n \to \infty.$$

By using the CLT for martingales, we have

$$\frac{1}{\sqrt{n}} \sum_{l=1}^{n} U_l \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2(i,t)).$$

To obtain the asymptotic variance $\sigma^2(i,t)$, we need to compute

$$\sigma^2(i,t) = \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}(U_l^2|\mathcal{F}_{l-1}) > 0.$$

Firstly,

$$\begin{aligned}\mathbb{E}(U_l^2|\mathcal{F}_{l-1}) &= \mathbb{E}\left(H^2\left(\tfrac{t-X_l}{h_{i,n}}\right)\mathbb{1}_{\{J_{l-1}=i\}}|\mathcal{F}_{l-1}\right)+\mathbb{E}\left(F_i^2(t)\mathbb{1}_{\{J_{l-1}=i\}}|\mathcal{F}_{l-1}\right)\\ &\quad -\mathbb{E}\left(2F_i(t)H\left(\tfrac{t-X_l}{h_{i,n}}\right)\mathbb{1}_{\{J_{l-1}=i\}}|\mathcal{F}_{l-1}\right)\\ &= \mathbb{1}_{\{J_{l-1}=i\}}\left(\int_{-\infty}^{\infty}H^2\left(\frac{t-x}{h_{i,n}}\right)f_i(x)dx\right)+F_i^2(t)\mathbb{1}_{\{J_{l-1}=i\}}\\ &\quad -2F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}\left(\int_{-\infty}^{\infty}H\left(\frac{t-x}{h_{i,n}}\right)f_i(x)dx\right).\end{aligned}$$

Secondly, under Assumptions (H.4), (H.5), (H.6) and a change of variable, an integration by parts followed by Taylor's expansion, we obtain

$$\begin{aligned}\mathbb{E}(U_l^2|\mathcal{F}_{l-1}) &= \mathbb{1}_{\{J_{l-1}=i\}}\int_{-\infty}^{\infty}H^2(z)\,dF_i(t-zh_{i,n})+F_i^2(t)\mathbb{1}_{\{J_{l-1}=i\}}\\ &\quad -2F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}\int_{-\infty}^{\infty}H(z)\,dF_i(t-zh_{i,n})\\ &= \mathbb{1}_{\{J_{l-1}=i\}}\int_{-\infty}^{\infty}2K(z)\,H(z)\,F_i(t-zh_{i,n})dz+F_i^2(t)\mathbb{1}_{\{J_{l-1}=i\}}\\ &\quad -2F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}[F_i(t)+O(h_{i,n})]\\ &= \mathbb{1}_{\{J_{l-1}=i\}}F_i(t)\int_{-\infty}^{\infty}2K(z)\,H(z)\,dz+O(h_{i,n})-F_i^2(t)\mathbb{1}_{\{J_{l-1}=i\}}\\ &= \mathbb{1}_{\{J_{l-1}=i\}}F_i(t)\left[\int_{-\infty}^{\infty}\left(H^2\right)'(z)\,dz-F_i(t)\right]+O(h_{i,n}).\end{aligned}$$

Thus,

$$\sigma^2(i,t)=\nu_iF_i(t)\left[1-F_i(t)\right]. \tag{3.8}$$

Furthermore,

$$\sqrt{T}[\widehat{F}_i(t,T)-F_i(t)] = \tfrac{T}{N_i(T)}\sqrt{\tfrac{N(T)}{T}}\tfrac{1}{\sqrt{N(T)}}\sum_{l=1}^{N(T)}\left[H\left(\tfrac{t-X_l}{h_{i,T}}\right)\mathbb{1}_{\{J_{l-1}=i\}}-F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}\right].$$

Combining the statements (3) and (5) of Lemma 2.2.1 with Equation (3.8) and applying Anscombe's Theorem (see [25]), we get

$$\begin{aligned}\sigma_F^2(i,t) &= \left(\mu_{ii}\sqrt{1/\mu_{ii}\nu_i}\right)^2 \nu_i F_i(t)\left[1-F_i(t)\right] \\ &= \mu_{ii}F_i(t)\left[1-F_i(t)\right].\end{aligned}$$

Now, it suffices to show that

$$\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(U_l^2 \mathbb{1}_{\{|U_l|>\epsilon\sqrt{n}\}}) \xrightarrow[n\to\infty]{} 0.$$

Indeed, using successively inequalities of *Holder, Markov, Jensen* and then *Minkowski*, we obtain for any $\epsilon > 0$ and any p and q such that $\frac{1}{p}+\frac{1}{q}=1$:

$$\begin{aligned}\mathbb{E}(U_l^2 \mathbb{1}_{\{|U_l|>\epsilon\sqrt{n}\}}) &\leq (\mathbb{E}(U_l^{2q}))^{1/q} P\{\mid U_l \mid > \epsilon\sqrt{n}\})^{1/p} \leq (\epsilon\sqrt{n})^{-2q/p}\mathbb{E}(\mid U_l \mid^{2q}) \\ &\leq (\epsilon\sqrt{n})^{-2q/p}\mathbb{E}\left(|H\left(\tfrac{t-X_l}{h_{i,n}}\right)\mathbb{1}_{\{J_{l-1}=i\}} - F_i(t)\mathbb{1}_{\{J_{l-1}=i\}}|^{2q}\right) \\ &\leq (\epsilon\sqrt{n})^{-2q/p}(\mathbb{E}\left(H^{2q}\left(\tfrac{t-X_l}{h_{i,n}}\right)\mathbb{1}_{\{J_{l-1}=i\}}\right) + F_i^{2q}(t)\mathbb{1}_{\{J_{l-1}=i\}}) \\ &\leq (\epsilon\sqrt{n})^{-2q/p}(\mathbb{1}_{\{J_{l-1}=i\}}\int_{-\infty}^{\infty} H^{2q}\left(\frac{t-x}{h_{i,n}}\right) f_i(x)dx + F_i^{2q}(t)\mathbb{1}_{\{J_{l-1}=i\}}) \\ &\leq (\epsilon\sqrt{n})^{-2q/p}\left(h_{i,n}\int_{-\infty}^{\infty} H^{2q}(z)\, f_i(t-zh_{i,n})dz + F_i^{2q}(t)\right)\mathbb{1}_{\{J_{l-1}=i\}}.\end{aligned}$$

Consequently, since f is a density function and using Lemma 2.2.1, it follows that

$$\tfrac{1}{n}\sum_{l=1}^{n}\mathbb{E}(U_l^2 \mathbb{1}_{\{|U_l|>\epsilon\sqrt{n}\}}) \leq \; (\epsilon\sqrt{n})^{-2q/p}\nu_i(h_{i,n}\|H\|_\infty^{2q} + F_i^{2q}(t)) \xrightarrow[n\to\infty]{} 0.$$

From the results above and the functional central limit Theorem for martingale differences (see Billingsley [25]; Hall and Heyde [65]) we get the desired result. □

(*ii*)
We start by writing

$$\begin{aligned}\sqrt{Th_{ij,T}}[\widehat{q}_{ij}(t,T)-q_{ij}(t)] &= \tfrac{T}{N_i(T)}\tfrac{1}{\sqrt{T}}\sum_{l=1}^{N_{ij}(T)}\frac{1}{\sqrt{h_{ij,T}}}K\left(\frac{t-X_n}{h_{ij,T}}\right)-q_{ij}(t)\sqrt{h_{ij,T}}\\ &= \tfrac{T}{N_i(T)}\tfrac{1}{\sqrt{T}}\sum_{l=1}^{N(T)}\left[\frac{1}{\sqrt{h_{ij,T}}}K\left(\frac{t-X_l}{h_{ij,T}}\right)\mathbb{1}_{\{J_{l-1}=i,J_l=j\}}\right.\\ &- \left. q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,T}}\right].\end{aligned}$$

According to Theorem 2.2.1, we denote

$$V_l=\frac{1}{\sqrt{h_{ij,n}}}K\left(\frac{t-X_l}{h_{ij,n}}\right)\mathbb{1}_{\{J_{l-1}=i,J_l=j\}}-q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,n}}. \tag{3.9}$$

Then, we use the same steps as in (i), so we give only the main details. Let us denote by $\mathcal{F}_l$ the σ-algebra $\mathcal{F}_l:=\sigma(J_n,X_n;n\leq l)$, $l\geq 0$, for all $l\in\mathbb{N}$. Thus, we have

$$\begin{aligned}\mathbb{E}(V_l|\mathcal{F}_{l-1}) &= \tfrac{1}{\sqrt{h_{ij,n}}}\mathbb{1}_{\{J_{l-1}=i\}}p_{ij}\int_{-\infty}^{\infty}K\left(\frac{t-x}{h_{ij,n}}\right)f_{ij}(x)dx-q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,n}}\\ &= \tfrac{1}{\sqrt{h_{ij,n}}}\mathbb{1}_{\{J_{l-1}=i\}}p_{ij}\int_{-\infty}^{\infty}K(z)\,dF_{ij}(t-zh_{ij,n})-q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,n}}\\ &= \sqrt{h_{ij,n}}\mathbb{1}_{\{J_{l-1}=i\}}q_{ij}(t)\int_{-\infty}^{\infty}K(z)\,dz+O(h_{ij,n})-q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,n}}\\ &\to 0,\ as\ n\to\infty,\end{aligned}$$

the previous result is obtained by using the $\mathcal{F}_{l-1}$-measurability of $\mathbb{1}_{\{J_{l-1}=i\}}$, a change of variable, Taylor's expansions of order one and Assumptions (H.4), (H.5) and (H.6).
To get the asymptotic variance, we need first to compute $\mathbb{E}(V_l^2|\mathcal{F}_{l-1})$ and then to obtain

$$\sigma^2(i,j,t) = \lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(V_l^2|\mathcal{F}_{l-1}) > 0.$$

Firstly,

$$\begin{aligned}\mathbb{E}(V_l^2|\mathcal{F}_{l-1}) &= \tfrac{1}{h_{ij,n}}\mathbb{1}_{\{J_{l-1}=i\}}p_{ij}\left(\int_{-\infty}^{\infty}K^2\left(\frac{t-x}{h_{ij,n}}\right)f_{ij}(x)dx\right)+h_{ij,n}q_{ij}^2(t)\mathbb{1}_{\{J_{l-1}=i\}}\\ &\quad -2q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}p_{ij}\left(\int_{-\infty}^{\infty}K\left(\frac{t-x}{h_{ij,n}}\right)f_{ij}(x)dx\right).\end{aligned}$$

Using the same steps as before, we obtain

$$\begin{aligned}\mathbb{E}(V_l^2|\mathcal{F}_{l-1}) &= \mathbb{1}_{\{J_{l-1}=i\}}p_{ij}\int_{-\infty}^{\infty}K^2(z)\,f_{ij}(t-zh_{ij,n})dz+h_{ij,n}q_{ij}^2(t)\mathbb{1}_{\{J_{l-1}=i\}}\\ &\quad -2q_{ij}(t)h_{ij,n}\mathbb{1}_{\{J_{l-1}=i\}}p_{ij}\int_{-\infty}^{\infty}K(z)\,f_{ij}(t-zh_{ij,n})dz\\ &= \mathbb{1}_{\{J_{l-1}=i\}}q_{ij}(t)\left[\int_{-\infty}^{\infty}K^2(z)\,dz-h_{ij,n}q_{ij}(t)\right]+O(h_{ij,n}).\end{aligned}$$

Secondly,

$$\sigma^2(i,j,t) = \nu_i q_{ij}(t)\int_{-\infty}^{\infty}K^2(z)\,dz. \tag{3.10}$$

Using (3.9) with Theorem 2.2.1, we can write

$$\begin{aligned}\sqrt{Th_{ij,T}}[\widehat{q}_{ij}(t,T)-q_{ij}(t)] &= \tfrac{T}{N_i(T)}\sqrt{\tfrac{N(T)}{T}}\tfrac{1}{\sqrt{N(T)}}\textstyle\sum_{l=1}^{N(T)}\left[\tfrac{1}{\sqrt{h_{ij,T}}}K\left(\tfrac{t-X_l}{h_{ij,T}}\right)\mathbb{1}_{\{J_{l-1}=i,J_l=j\}}\right.\\ &\quad \left.-q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\sqrt{h_{ij,T}}\right].\end{aligned}$$

Using statements (3) and (5) of Lemma 2.2.1, with the application of Anscombe's Theorem, we deduce that

$$\begin{aligned}\sigma_q^2(i,j,t) &= \left(\mu_{ii}\sqrt{1/\mu_{ii}\nu_i}\right)^2\nu_i q_{ij}(t)\int_{-\infty}^{\infty}K^2(z)\,dz\\ &= \mu_{ii}q_{ij}(t)\int_{-\infty}^{\infty}K^2(z)\,dz.\end{aligned}$$

Now, it suffices to show that

$$\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(V_l^2\mathbb{1}_{\{|V_l|>\epsilon\sqrt{n}\}}) \xrightarrow[n\to\infty]{} 0$$

and we use the same steps as in (i) to prove this result.

From the results above and the functional central limit theorem for martingale differences (see Billingsley [25]; Hall and Heyde [65]), we get the desired result. □

3.5 Concluding remarks

The main contribution of the work presented in this chapter is the use of the classical technique of nonparametric Parzen-Rosenblatt kernel estimation for estimating the main characteristics of a continuous-time semi-Markov process, namely the sojourn time distribution functions (conditional or not), the semi-Markov kernel and the corresponding densities. We have proposed kernel estimators for these quantities and investigated the uniform strong consistency and the asymptotic normality of these estimators. There are several advantages of using this kernel method approach compared to the empirical estimator. In particular, the kernel smoothing avoids discontinuities in the empirical. Thus, if it is known that the true distribution is continuous, the empirical distribution may be viewed as a poor approximation.

Note that in this chapter we have neglected the last censored sojourn time. Since we are concerned here by one long trajectory in the time interval $[0, T]$ and the asymptotic properties are obtained as T goes to infinity, this is not a real limitation. Nonetheless, this is a very important question that will represent the topic of a future research.

A symmetric problem is to consider the case when the process is not observed from time 0; this generates a first observed sojourn time that is also right censored, since we have the information that the real sojourn time is in fact greater than the length that we observe. So, in this case also, this censored time can be neglected asymptotically, as T goes to infinity (our framework). As previously mentioned for the last censored time, taking also into account this situation of first censored sojourn time will be developed during a future research.

As already mentioned in the introduction, semi-Markov processes are extremely important in many fields of applied sciences. For this reason, the work that we develop in the present chapter has an important potential impact for those studies to come that will use semi-Markov processes as a modelling or forecasting tool.

Chapter 4

Nonparametric estimators of the reliability and related functions for semi-Markov systems

In this chapter, we introduce a kernel estimator of the reliability, availability and failure rate of a semi-Markov system when the semi-Markov process is homogeneous and time continuous. We establish the strong consistency and the asymptotic normality of the proposed estimators.

4.1 Introduction

In the last years, several works have been carried out in the field of the estimation of the semi-Markov processes (SMP) for the reliability and related measurements, as availability, failure rate etc., by using empirical and maximum likelihood estimators for discrete or continue time. Key references on this topic is Greenwood and Wefelmeyer [62] that studied efficiency of empirical estimators for linear functionals of semi-Markov kernels for general state spaces; Ouhbi and Limnios [96] studied empirical estimators of non-linear functionals of semi-Markov kernels including the Markov renewal matrix and reliability functions [98], the failure rate of a semi-Markov system [95]; Limnios [79] gave the invariance principle for the empirical estimator of semi-Markov kernels. Barbu and Limnios [19] studied empirical estimation for discrete time semi-Markov processes with applications in reliability, [20], [17].

More precisely, we are interested in a continue-time finite state space semi-Markov model estimated by the kernel method, because one problem with the empirical estimator is that it is discontinuous. In particular, if it is known that the true distribution is continuous, the empirical distribution may yield bad approximations. Kernel smoothing avoids this problem of discontinuity.
From mathematical point of view, reliability theory is essentially an application of the theory of stochastic processes. It serves with its related measurements, as failure rate, availability etc., to assist in the modeling of technical systems as far as their dependability aspects are concerned. From a theoretical point of view, the problems related to reliability are mostly concerned by the hitting time of a so-called failure or down subset of states of the system. In practice, there are cases where the Markov approach is inadequate because it is necessary to allow sojourn times in a state that are more general than those which are exponentially distributed.
To summarize, in this chapter we introduce the kernel estimators for the reliability, availability and failure rate of a semi-Markov system and we study their asymptotic properties, when T tends to infinity.

This chapter is organized as follows. Section 4.2 we impose some assumptions and we construct kernel estimators of the Markov renewal function. Section 4.3 is devoted to the construction of estimators of reliability and availability of semi-Markov systems, as well as to the asymptotic properties of the proposed estimators. Section 4.4 is devoted to the failure rate estimation and presents the strong consistency and the asymptotic normality. Finally, in Section 4.6 we give the proof of all the above results and in Section 4.7 we give a numerical application.

4.2 Estimation of the Markov renewal function

4.2.1 Assumptions

All along this Chapter we are working under the following three assumptions.

(H.1) The EMC $(J_n)_{n\in\mathbb{N}}$ is an ergodic irreducible Markov chain, with stationary distribution ν.

(H.2) The SMP is irreducible, aperiodic, with finite mean sojourn times.

$$m_i = \int_0^{\infty} \overline{F}_i(t)dt \leq C < +\infty,$$

$$m = \sum_{i\in E} \nu_i m_i > 0.$$

(H.3) The SMP (or equivalently, the MRP) is regular, that is $\mathbb{P}_i(N(t) < \infty) = 1$ for all $t > 0$, $i \in E$.

Furthermore, we need to introduce the following conditions.

(H.4)
- i) $Q_{ij}(t), F_i(t)$ and $F_{ij}(t)$ are absolutely continuous with respect to the Lebesgue measure, and let $q_{ij}(t), f_i(t)$ and $f_{ij}(t)$ be respectively the corresponding Radon-Nikodym derivatives.
- ii) The first derivatives $f_{ij}(t)$, $f_i(t)$ and $q_{ij}(t)$ are bounded.

(H.5)
- i) The function H is a distribution function.
- ii) The kernel K is density function of bounded from below by some constant $\beta > 0$ on its support such that

$$\lim_{x\to\infty} |xK(x)| = 0 \text{ and } |\int t^j K^k(t)dt| < \infty \text{ for } j = 0, 1, \text{ and } k = 1, 2.$$

(H.6) The smoothing parameters h_n satisfy

$$\lim_{n\longrightarrow\infty} h_n = 0 \text{ and } \lim_{n\longrightarrow\infty} nh_n = \infty.$$

(H.7) The series $\sum_{n=1}^{\infty} e^{-\gamma n h_n^2}$ converge for every positive value of γ.

4.2.2 Comments on the assumptions

The comments on these hypotheses are the same comments those in the above Chapter.

4.2.3 Nonparametric estimation

The Markov renewal matrix is of considerable importance when studying the behavior of MRP. The aim of this section is to give an estimator of this matrix and estimator of transition matrix function.
Let us denote by $\widehat{\Psi}(t)$ the estimators of $\Psi(t)$, defined by

$$\widehat{\Psi}(t,T) = \sum_{n=0}^{\infty} \widehat{Q}^{(n)}(t,T). \tag{4.1}$$

Let $\widehat{P}$ be the estimator of the transition function of the semi-Markov process, given by

$$\widehat{P}(t,T) = \widehat{\Psi} * (I - \widehat{F}(t,T)). \tag{4.2}$$

The estimator for the stationary distribution of the SMP is determined by:

$$\widehat{\pi}_i(T) = \frac{\widehat{\nu}_i(T)\widehat{m}_i(T)}{\sum_{k=1}^{s} \widehat{\nu}_k(T)\widehat{m}_k(T)}. \tag{4.3}$$

Proposition 4.2.1. *For any fixed $t \in \mathbb{R}_+$ and $i,\ j \in E$ and under Assumptions (H.5)-(H.7), for all $i,\ j \in E$, we have*

(a) *For $n \in \mathbb{N}$, we have*

$$\max_{i,j} \sup_{t\in[0,\infty]} |\hat{Q}_{ij}^{(n)}(t,T) - Q_{ij}^{(n)}(t)| \xrightarrow{a.s.} 0, \ as\ T \to \infty.$$

(b) *(Strong consistency) For any fixed $L > 0$ we have*

$$\max_{i,j} \sup_{t\in[0,L]} |\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t)| \xrightarrow{a.s.} 0, \ as\ T \to \infty.$$

(c) *(Strong consistency) For any fixed $L > 0$ we have*

$$\max_{i,j} \sup_{t\in[0,L]} |\hat{P}_{ij}(t,T) - P_{ij}(t)| \xrightarrow{a.s.} 0, \ as\ T \to \infty.$$

4.3 Reliability of semi-Markov systems

4.3.1 Reliability modeling

For a stochastic system with state space E described by a semi-Markov process Z, semi-Markov kernel $Q(t)$ and initial distribution α, describing the stochastic behavior of a repairable semi-Markov system, let us consider a partition U, D of E, i.e., $E = U \cup D$, with $U \cap D = \emptyset$, $U \neq \emptyset$, and $D \neq \emptyset$. The set U contains the up states and D contains the down states of the system. The transition from one state to another state means, physically speaking, the failure or the repair of the system. The system is operational in U. No service is delivered if the system is in D. But, one repair will return the system from D to U. For more details, see Limnios and Oprişan [83].
In the sequel, we will consider the kernels and the functions defined in the previous sections $Q(t)$, $\Psi(t)$, $P(t)$, under their matrix form and we will denote their restrictions on the sets U, $U \times U$, $U \times D$. For example, $Q^U(t)$ is the restriction of the matrix $Q(t)$ on $U \times U$, and α_U is the restriction of the probability distribution α_i of the r.v. Z^U on U, and for $\Psi^U(t)$ (resp. $P^U(t)$) we consider the restrictions to $U \times U$ (resp. $U \times D$) induced by the corresponding restrictions of the semi-Markov kernel $Q^U(t)$. Define the hitting time T of D, that is,

$$T = \inf\{t \geq 0 : Z_t \in D\}, \quad (\inf \emptyset = +\infty).$$

For the finite state space case, without loss of generality, let us enumerate first the up states and next the down states, i.e., for $E = \{1, 2, \ldots, s\}$, we have $U = \{1, \ldots, r\}$ and $D = \{r + 1, \ldots, s\}$.
The conditional and unconditional reliability and availability, $R_i(t)$, $R(t)$, and $A_i(t)$, $A(t)$, of a semi-Markov system are defined as follows:

$$R_i(t) = \mathbb{P}_i(T > t) = \mathbb{P}_i(Z_s \in U, \forall s \in [0, t]), \quad R(t) = \mathbb{P}(Z_s \in U, \forall s \in [0, t]),$$

with

$$R(t) = \sum_{i \in U} \alpha_i R_i(t),$$

and

$$A_i(t) = \mathbb{P}_i(Z_t \in U), \quad \forall i \in E, \quad A(t) = \mathbb{P}(Z_t \in U), \quad \textit{and} \quad A(t) = \sum_{i \in E} \alpha_i A_i(t).$$

The reliability and availability functions of a semi-Markov system verify Markov renewal equations., for instance, for the reliability

$$R_i(t) = 1 - F_i(t) + \sum_{j\in U} \int_0^t Q_{ij}(ds) R_j(t-s), \quad i \in U. \tag{4.4}$$

The solution of the above MRE is given by the following formula:

$$R(t) = \alpha_U \Psi^U * (I - F^U)(t)\mathbf{1},$$

where $\mathbf{1} = (1, \ldots, 1)'$, and $\Psi^U = (I - Q^U(t))^{(-1)}$ is the Markov renewal function, and $t \geq 0$, $F^U(t) = diag(F_i(t)), i \in U$ a diagonal matrix.
Given that the process started from state $i \in U$. So

$$R(t) = \alpha_U \cdot P^U(t)\mathbf{1},$$

where

$$P^U(t) = \Psi^U * (I - F^U(t)).$$

In the same way, we get for the availability

$$A_i(t) = \mathbb{1}_U(i)(1 - F_i(t)) + \sum_{j\in E} \int_0^t Q_{ij}(ds) A_j(t-s). \tag{4.5}$$

The solution of the above MRE is given by the following formula:

$$A(t) = \alpha\Psi * (I - F)(t)\mathbf{e},$$

where $\mathbf{e} = (e_1, \ldots, e_s)'$ is an s-dimensional column-vector, with $e_i = 1$, if $i \in U$, and $e_i = 0$, if $i \in D$.
In matrix form

$$A(t) = \alpha \cdot P(t)\mathbf{e}.$$

4.3.2 Reliability estimation

Let $\hat{Q}$ be the kernel estimation (3.5) of the semi-Markov kernel Q. Then we propose the following estimator for $P^U(t)$,

$$\hat{P}^U(t,T) = \hat{\Psi}^U * (I - \hat{F}^U(t,T)).$$

The reliability estimator of the system is:

$$\hat{R}(t,T) = \hat{\alpha}_U \cdot \hat{P}^U(t,T)\mathbf{1}.$$

Then we propose the following estimator for the availability of the system:

$$\hat{A}(t,T) = \hat{\alpha} \cdot \hat{P}(t,T)\mathbf{e}. \tag{4.6}$$

We obtain plug-in estimators for the availability and reliability by substitution into the above formulas of the estimators of Ψ and Q. For these estimators, we have the following properties.

Theorem 4.3.1. *For any fixed $t > 0$ and for any $L \in \mathbb{R}_+$, we have:*

(a) *(Strong consistency)Under Assumptions (H.5)-(H.7), for all i, $j \in E$ we have.*

$$\sup_{0\le t\le L} |\hat{A}_{ij}(t,T) - A_{ij}(t)| \xrightarrow{a.s.} 0, \text{ as } T \to \infty,$$

(b) *(Asymptotic normality) Under (H.4), (H.5) and (H.6), setting $\min\limits_{ij} h_{ij,T} = h_T$, we have*

$$\sqrt{Th_T}(\hat{A}(t,T) - A(t)) \xrightarrow[T\to\infty]{D} \mathcal{N}(0, \sigma_A^2(i,j,t)),$$

where

$$\sigma_A^2(i,j,t) \quad \le \sum_{i=1}^{s} \mu_{ii} \sum_{j=1}^{s} \left[D_{ij} - \mathbb{1}_{\{i\in U\}} \sum_{l=1}^{s} \alpha_l \Psi_{li} \right]^2 * Q_{ij}(t) \int K^2(z)\, dz, \tag{4.7}$$

and

$$D_{ij} = \sum_{n=1}^{s} \sum_{r\in U} \alpha_n \Psi_{ni} * \Psi_{jr} * (I - diag(Q\cdot I))_{rr}\,. \tag{4.8}$$

Theorem 4.3.2. *For any fixed $t > 0$ and for any $L \in \mathbb{R}_+$, we have:*

(a) *(Strong consistency) Under Assumptions (H.5)-(H.7), for all i, $j \in E$ we have.*

$$\sup_{0\le t\le L} |\hat{R}_{ij}(t,T) - R_{ij}(t)| \xrightarrow{a.s.} 0, \text{ as } T \to \infty,$$

(b) *(Asymptotic normality) Under (H.4), (H.5) and (H.6), setting* $\min_{ij} h_{ij,T} = h_T$*, we have*

$$\sqrt{Th_T}(\hat{R}(t,T) - R(t)) \xrightarrow[T\to\infty]{D} \mathcal{N}(0, \sigma_R^2(i,j,t)),$$

where

$$\sigma_R^2(i,j,t) \quad \leq \sum_{i=1}^{s} \mu_{ii} \sum_{j=1}^{s} \left[D_{ij}^U - \mathbb{1}_{\{i\in U\}} \sum_{l\in U} \alpha_l \Psi_{li} \right]^2 * Q_{ij}(t) \int K^2(z)\,dz. \tag{4.9}$$

4.4 Failure rate estimation

An interesting introduction to the stochastic process approach of the failure rate is given by Aalen and Gjessing [1]. The failure rate of a semi-Markov system is defined as follows

$$\lambda(t) := \lim_{h\downarrow 0} \frac{1}{h} P(Z_{t+h} \in D | Z_u \in U, \forall u \leq t). \tag{4.10}$$

From this definition we get:

$$\lambda(t) = \frac{\alpha_U \Psi^U * F'^U(t)\mathbf{1}}{\alpha_U \Psi^U * (I - F^U(t))\mathbf{1}}, \tag{4.11}$$

where $F'^U(t)$ is the diagonal matrix of derivatives of $F_i(t)$, i.e., $F'^U(t) = diag(F'_i(t), i \in U)$. By replacing Q, Ψ, F by their estimators in Equation (4.11), we get the kernel estimator for failure rate, i.e.,

$$\hat{\lambda}(t,T) = \frac{\hat{\alpha}_U \hat{\Psi}^U * \hat{F}'^U(t,T)\mathbf{1}}{\hat{\alpha}_U \hat{\Psi}^U * (I - \hat{F}^U(t,T))\mathbf{1}}, \tag{4.12}$$

where the derivative of $\hat{F}(t,T)$ is $\hat{f}(t,T)$.

Theorem 4.4.1. (a) *(Strong consistency) If the semi-Markov kernel is continuously differentiable, and under (H.5), (H.6), (H.7) then the estimator is uniformly strongly consistent, i.e., for any fixed* $L \in \mathbb{R}_+$

$$\sup_{0\leq t\leq L} |\hat{\lambda}(t,T) - \lambda(t)| \xrightarrow{a.s.} 0, \text{ as } T \to \infty.$$

(b) *(Asymptotic normality) If f_{ij} is twice continuously differentiable for all $i, j \in E$, and under (H.4), (H.5) and (H.6), setting $\min_{ij} h_{ij,T} = h_T$, we have*

$$\sqrt{Th_T}(\hat{\lambda}(t,T) - \lambda(t)) \xrightarrow[T\to\infty]{D} \mathcal{N}(0, \sigma^2_\lambda(i,j,t)),$$

$$\sigma^2_\lambda(i,j,t) \quad = \frac{\sigma_1^2(t)}{(R(t))^2}, \tag{4.13}$$

where

$$\sigma_1^2(t) \quad = \sum_{i=1}^{s} \mu_{ii} (\sum_{j=1}^{s} \delta_j \cdot \Psi_{ij}^{U'})^2 * F_i(t) \int K^2(z)\, dz. \tag{4.14}$$

4.5 The evolution equation numerical solution

An approximate solution of (4.4) can be solved numerically using discretization to numerically evaluate the integrals. Let $v > 0$ be the step size of discretization, then we have the countable linear system given by

$$R_i^v(kv) = d_i^v(kv) + \sum_{l\in U} \sum_{\tau=1}^{k} R_l^v(kv - \tau v) q_{il}^v(\tau v), \tag{4.15}$$

where $k \leq N$, $k, N \in \mathbb{N}$, such that $Nv = T$ and $[0,T]$ is the integration interval.

$$q_{ij}^v(kv) = \begin{cases} Q_{ij}^v(kv) - Q_{ij}^v((k-1)v) & if \quad k > 0; \\ 0 & if \quad k = 0. \end{cases}$$

$$d_i^v(kv) = 1 - F_i^v(kv)$$

Now the Equation (4.15) is rewritten in matrix form:

$$\mathbf{R}^v(kv) - \sum_{\tau=1}^{k} \mathbf{R}^v(kv - \tau v)\mathbf{q}^v(\tau v) = \mathbf{D}^v(kv). \tag{4.16}$$

4.6 Proofs

Proof of Proposition 4.2.1

(a)

For $t \in [0, T]$ and under Assumptions (H.5)-(H.6), we will give the proof by induction.

For $n = 1$, it is the result of Corollary 3.2.1 (v). Suppose that this result is verified by order $n-1$, for i and j two states, we have

$$\begin{aligned}\max_{i,j} \sup_{t\in[0,\infty]} |\hat{Q}_{ij}^{(n)}(t,T) - Q_{ij}^{(n)}(t)| &\leq \max_{i,j} \sup_{t\in[0,\infty]} |\hat{Q}_{ij}^{(n-1)}(t,T) - Q_{ij}^{(n-1)}(t)| \\ &\quad + s \cdot \max_{i,j} \sup_{t\in[0,\infty]} |\hat{Q}_{ij}(t,T) - Q_{ij}(t)|.\end{aligned}$$

This result converges to zero a.s., as $T \to \infty$ by the fact that $\sum_{j=1}^{s} |\hat{Q}_{ij}^{m}(t,T)| \leq s$ for every $m \geq 0$ and the induction hypothesis. $\square$

(b)

Since $S_n \to \infty$ (Lemma 2.2.2), there exists a constant $n_0 > 0$ such that $\max_i \sum_{j=1}^{s} Q_{ij}^{(n_0)}(t) < 1$.

Using (a), for all $n \geq 1$, if we denote $\epsilon = 1 - \max_i \sum_{j=1}^{s} Q_{ij}^{(n_0)}(t)$, we get

$$\begin{aligned}\max_i \sum_{j=1}^{s} \hat{Q}_{ij}^{(n_0)}(t,T) &\leq \max_i \left| \sum_{j=1}^{s} [\hat{Q}_{ij}^{(n_0)}(t,T) - Q_{ij}^{(n_0)}(t)] \right| \\ &\quad + \max_i \sum_{j=1}^{s} Q_{ij}^{(n_0)}(t) \leq 1 - \frac{\epsilon}{2}.\end{aligned}$$

Moreover, for all $d > n_0$, there exists $(z, u) \in \mathbb{N}^* \times \mathbb{N}$ such that $d = zn_0 + u$ where $0 \leq u < n_0$ and observe that,

$$\begin{aligned}\max_{i,j} \hat{Q}_{ij}^{(d)}(t,T) &= \max_{i,j} \sum_{m=1}^{s} \hat{Q}_{im}^{(u)} * \hat{Q}_{mj}^{(zn_0)}(t,T) \\ &\leq \max_{i,j} \sum_{m=1}^{s} \hat{Q}_{im}^{(u)}(t,T) \cdot \hat{Q}_{mj}^{(zn_0)}(t,T) \\ &\leq \max_{i,j} \hat{Q}_{ij}^{(zn_0)}(t,T).\end{aligned}$$

Let us now prove that, for all $z \in \mathbb{N}^*$,

$$\max_i \sum_{j=1}^{s} \hat{Q}_{ij}^{(zn_0)}(t,T) \quad \leq \quad \left(1-\tfrac{\epsilon}{2}\right)^z . \tag{4.17}$$

In fact, for $z = 1$, (4.17) is true. Suppose now, that this result is valid until order z and we prove it to order $z+1$.

$$\begin{aligned}
\max_i \sum_{j=1}^{s} \hat{Q}_{ij}^{((z+1)n_0)}(t,T) &= \max_i \sum_{j=1}^{s}\sum_{m=1}^{s} \hat{Q}_{im}^{(zn_0)} * Q_{mj}^{(n_0)}(t,T) \\
&\leq \max_i \sum_{m=1}^{s} \hat{Q}_{im}^{(zn_0)}(t,T) \cdot \max_i \sum_{j=1}^{s} Q_{ij}^{(n_0)}(t,T) \\
&\leq \left(1-\tfrac{\epsilon}{2}\right)^z \cdot \left(1-\tfrac{\epsilon}{2}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\hat{\Psi}_{ij}(t,T) &= \sum_{n=0}^{\infty} \hat{Q}_{ij}^{(n)}(t,T) \\
&= \sum_{n=0}^{n_0} \hat{Q}_{ij}^{(n)}(t,T) + \sum_{n=n_0+1}^{2n_0} \hat{Q}_{ij}^{(n)}(t,T) + \sum_{n=2n_0+1}^{3n_0} \hat{Q}_{ij}^{(n)}(t,T) + \ldots .
\end{aligned}$$

Let $\gamma_{ij}^d(t)$ be a sequence of functions defined by

$$\gamma_{ij}^d(t) = \begin{cases} \hat{Q}_{ij}^{(d)}(t,T) & \text{if } d \leq n_0 \\ n_0(1-\frac{\epsilon}{2})^{[d/n_0]} & \text{otherwise.} \end{cases}$$

Where $[x]$ is the integer part of x. We have, $\hat{\Psi}_{ij}(t,T) \leq \sum_{d=0}^{\infty} \gamma_{ij}^d(t) < \infty$. Thus by (a) and the Lebesgue's dominated convergence theorem, we get

$$\hat{\Psi}_{ij}(t,T) \xrightarrow{a.s.} \Psi_{ij}(t), \quad \text{as } T \to \infty.$$

The uniform consistency of $\hat{\Psi}_{ij}(t,T)$ to $\Psi_{ij}(t,T)$ is deduced by the monotony and the continuity of $\Psi_{ij}(t)$. □

(c)

Let us consider the matrices $B(t) = I - diag(Q(t)\mathbf{1})$ and then

$\hat{B}(t,T) = I - diag(\hat{Q}(t,T)\mathbf{1})$, for $(i,j) \in E \times E$ be fixed,

$$\begin{aligned}
\sup_{t\in[0,L]} |\hat{P}_{ij}(t,T) - P_{ij}(t)| &= \sup_{t\in[0,L]} |(\hat{\Psi} * \hat{B})_{ij}(t,T) - (\Psi * B)_{ij}(t)| \\
&\leq \sup_{t\in[0,L]} |(\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t))| \\
&+ \sup_{t\in[0,L]} |(\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t))| \cdot diag(\hat{Q}(t,T)\mathbf{1}) \\
&+ \sup_{t\in[0,L]} |diag((\hat{Q} - Q)\mathbf{1})_{jj}(t,T)|\Psi_{ij}(L).
\end{aligned}$$

Combining (B) with Corollary 3.2.1(v) and by the fact that $\Psi_{ij}(t)$ is finite (cf. [103]), we deduce that $diag((\hat{Q}-Q)\mathbf{1})_{jj}(t,T) \xrightarrow{a.s} 0$ and $\Delta\Psi_{ij}(t,T) \xrightarrow{a.s} 0$ on $[0,L]$ as T tends to infinity. □

Proof of Theorem 4.3.1

(a)

The strong consistency of the availability estimator is obtained from the strong consistency of the semi-Markov transition function estimator $P_{ij}(t,T)$ Proposition 4.2.1(c) and from the following inequality:

$$\begin{aligned}
\sqrt{T}[\hat{A}(t,T) - A(t)] &= \Big| \sum_{i\in E}\sum_{j\in U} \alpha_i \hat{P}_{ij}(t,T) - \sum_{i\in E}\sum_{j\in U} \alpha_i P_{ij}(t,T) \Big| \\
&\leq \sum_{i\in E}\sum_{j\in U} | \alpha_i \hat{P}_{ij}(t,T) - \alpha_i P_{ij}(t,T) | .
\end{aligned}$$

□

(b)

From (2.13) and (4.6), we get that

$$\begin{aligned}
\sqrt{Th_T}[\hat{A}(t,T) - A(t)] &= \sqrt{Th_T} \sum_{i=1}^{s}\sum_{j\in U} \Big[\alpha_i \hat{\Psi}_{ij} * \Big(I - diag(\hat{Q}\cdot\mathbf{1})\Big)_{jj} \\
&\quad -\alpha_i \Psi_{ij} * (I - diag(Q\cdot\mathbf{1}))_{jj}\Big](t),
\end{aligned}$$

which has the same limit in law as

$$\begin{aligned}
&\sqrt{Th_T} \sum_{i=1}^{s}\sum_{j\in U} \alpha_i \Big[(\hat{\Psi}_{ij} - \Psi_{ij}) * (I - diag(Q\cdot\mathbf{1}))_{jj} \\
&-\Psi_{ij} * (diag(\Delta Q\cdot\mathbf{1}))_{jj} - (\hat{\Psi}_{ij} - \Psi_{ij}) * (diag(\Delta Q\cdot\mathbf{1}))_{jj}\Big](t).
\end{aligned}$$

We have

$$\begin{aligned}
\sqrt{Th_T}[\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t)] &= \sqrt{Th_T}[\hat{\Psi}_{ij} - (\hat{\Psi} * \Psi)_{ij} + (\hat{\Psi} * \Psi)_{ij} - \Psi_{ij}](t) \\
&= \sqrt{Th_T}[(\hat{\Psi} - \Psi) * \Delta Q * \Psi]_{ij}(t) \\
&+ \sqrt{Th_T}[\Psi * \Delta Q * \Psi]_{ij}(t). \qquad (4.18)
\end{aligned}$$

For every $t \geq 0$, $t \leq T$, and for every $r, m, u, v \in E$, $\sqrt{T}(\hat{Q}_{rm}(t,T) - Q_{rm}(t))$ converges in distribution to a normal random variable (Corollary 3.2.2 (ii)), as $T \to \infty$, and $\hat{\Psi}_{uv}(t,T) - \Psi_{uv}(t) \xrightarrow{P} 0$ as $T \to \infty$ is deduced from Proposition 4.2.1 (b).
Thus, using Slutsky's Theorem we obtain that $\sqrt{Th_T}[(\hat{\Psi} - \Psi) * (\hat{Q} - Q) * \Psi]_{ij}(t) \xrightarrow{P} 0$ as $T \to \infty$. Consequently, applying again Slutsky's Theorem we get that $\sqrt{Th_T}[\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t)]$ has the same limit in distribution as $\sqrt{Th_T}[\Psi * (\hat{Q} - Q) * \Psi]_{ij}(t)$.
The last term can be written as follows:

$$\begin{aligned}
\sqrt{Th_T}(\Psi * (\hat{Q} - Q) * \Psi)_{ij}(t) &= \sqrt{Th_T}\sum_{m=1}^{s}\sum_{r=1}^{s}(\Psi_{im} * (\hat{Q}(\cdot,T) - Q)_{mr} * \Psi_{rj})(t) \\
&= \sqrt{Th_T}\sum_{m=1}^{s}\sum_{r=1}^{s}(\Psi_{im} * \hat{Q}(\cdot,T)_{mr} * \Psi_{rj})(t) \\
&- \sqrt{Th_T}\sum_{m=1}^{s}\sum_{r=1}^{s}(\Psi_{im} * Q_{mr} * \Psi_{rj})(t) \\
&= \sqrt{\frac{h_T}{T}}\sum_{l=1}^{N(T)}\sum_{m=1}^{s}\frac{T}{N_m(T)}\sum_{r=1}^{s}[(\Psi_{im} * H\left(\frac{\cdot - X_l}{h_{mr,T}}\right) \\
&\mathbb{1}_{\{J_{l-1}=m, J_l=r\}} * \Psi_{rj})(t) - (\Psi_{im} * Q_{mr}\mathbb{1}_{\{J_{l-1}=m\}} * \Psi_{rj})(t)].
\end{aligned}$$

Since $\dfrac{N_m(T)}{T} \xrightarrow{a.s.} \dfrac{1}{\mu_{mm}}$ (see Theorem 1.7 in Limnios and Oprişan [83]), using Slutsky's Theorem we obtain that $\sqrt{Th_T}[\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t)]$ has the same limit in distribution as

$$\sqrt{\tfrac{h_T}{T}}\sum_{l=1}^{N(T)}\sum_{m=1}^{s}\mu_{mm}\sum_{r=1}^{s}[(\Psi_{im}*H\left(\frac{\cdot-X_l}{h_{mr,T}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=r\}}*\Psi_{rj})(t)$$

$$-\quad(\Psi_{im}*Q_{mr}\mathbb{1}_{\{J_{l-1}=m\}}*\Psi_{rj})(t)]$$

$$=\quad\sqrt{\tfrac{N(T)}{T}}\frac{1}{\sqrt{N(T)}}\textstyle\sum_{l=1}^{N(T)}Y_l.$$

Taking into account Theorem 8.1, pp.302 of [64], denote

$$Y_l\;=\sum_{m=1}^{s}\mu_{mm}\sqrt{h_n}\sum_{r=1}^{s}[(\Psi_{im}*H\left(\frac{\cdot-X_l}{h_{mr,n}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=r\}}*\Psi_{rj})(t)$$

$$-(\Psi_{im}*Q_{mr}\mathbb{1}_{\{J_{l-1}=m\}}*\Psi_{rj})(t)].$$

Let $\mathcal{F}_l$ be the σ-algebra defined by $\mathcal{F}_l:=\sigma(J_m,X_m;m\leq l)$. Note that Y_l is $\mathcal{F}_l$ -measurable, for all $l\geq 1$. Moreover, we have

$$\mathbb{E}(Y_l|\mathcal{F}_{l-1}))\;=\mathbb{E}(\sum_{m=1}^{s}\mu_{mm}\sqrt{h_n}\sum_{r=1}^{s}[(\Psi_{im}*H\left(\frac{\cdot-X_l}{h_{mr,n}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=r\}}*\Psi_{rj})(t)$$

$$-(\Psi_{im}*Q_{mr}\mathbb{1}_{\{J_{l-1}=m\}}*\Psi_{rj})(t)]|\mathcal{F}_{l-1})$$

$$=\sum_{m=1}^{s}\mu_{mm}\sum_{r=1}^{s}[\int_0^t(\Psi_{im}*\Psi_{rj})(t-t_1)\mathbb{1}_{\{J_{l-1}=m\}}p_{mr}\frac{\sqrt{h_n}}{h_{mr,n}}\int K\left(\frac{t_1-x}{h_{mr,n}}\right)$$

$$f_{mr}(x)dxdt_1-(\Psi_{im}*\Psi_{rj})(t)*Q_{mr}(t)\mathbb{1}_{\{J_{l-1}=m\}}\sqrt{h_n}].$$

The last equation is obtained by the fact that $\mathbb{1}_{\{J_{l-1}=m\}}$ is $\mathcal{F}_{n-1}$-measurable. Be the change of variable, Taylor's expansion of $f_{mr}(t-h_{mr,n}z)$ in a neighborhood of t with assumptions (H.4), (H.5) and (H.6) we get

$$\begin{aligned}
\mathbb{E}(Y_l|\mathcal{F}_{l-1}) &= \sum_{m=1}^{s}\mu_{mm}\sum_{r=1}^{s}[\int_0^t(\Psi_{im}*\Psi_{rj})(t-t_1)\mathbb{1}_{\{J_{l-1}=m\}}p_{mr}\sqrt{h_n}\int K(z) \\
& f_{mr}(t_1-zh_{mr,n})dzdt_1-(\Psi_{im}*\Psi_{rj})(t)*Q_{mr}(t)\mathbb{1}_{\{J_{l-1}=m\}}\sqrt{h_n}] \\
&= \sum_{m=1}^{s}\mu_{mm}\sum_{r=1}^{s}[\int_0^t(\Psi_{im}*\Psi_{rj})(t-t_1)\mathbb{1}_{\{J_{l-1}=m\}}\sqrt{h_n}(q_{mr}(t)\int K(z)\,dz \\
& +O(h_{mr,n}))dt_1-(\Psi_{im}*\Psi_{rj})(t)*Q_{mr}(t_1)\mathbb{1}_{\{J_{l-1}=m\}}\sqrt{h_n}] \\
&= \sum_{m=1}^{s}\mu_{mm}\sum_{r=1}^{s}[(\Psi_{im}*\Psi_{rj})(t)*\mathbb{1}_{\{J_{l-1}=m\}}\sqrt{h_n}(Q_{mr}(t)\int K(z)\,dz+O(h_{mr,n})) \\
& -(\Psi_{im}*\Psi_{rj})(t)*Q_{mr}(t)\mathbb{1}_{\{J_{l-1}=m\}}\sqrt{h_n}].
\end{aligned}$$

This implies that

$$\mathbb{E}(Y_l|\mathcal{F}_{l-1}) \quad = 0,\ as\ n\to\infty.$$

To get the asymptotic variance, we need to compute $\mathbb{E}(Y_l^2|\mathcal{F}_{l-1})$ and then we get,

$$\sigma^2=\lim_{n\to\infty}\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(Y_l^2|\mathcal{F}_{l-1})>0. \tag{4.19}$$

Firstly, we have

$$\begin{aligned}
Y_l^2 &= \sum_{m=1}^{s}\mu_{mm}^2[\sum_{r=1}^{s}(\int_0^t(\Psi_{im}*\Psi_{rj})(t-t_1)\frac{\sqrt{h_n}}{h_{mr,n}}K\left(\frac{t_1-X_l}{h_{mr,n}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=r\}}dt_1)^2 \\
& +\mathbb{1}_{\{J_{l-1}=m\}}\sum_{r_1,r_2=1}^{s}[\int_0^t(\Psi_{im}*\Psi_{r_1j})(t-t_1)q_{mr_1}(t_1)\sqrt{h_n}dt_1\cdot\int_0^t(\Psi_{im}*\Psi_{r_2j})(t-t_2) \\
& q_{mr_2}(t_2)\sqrt{h_n}dt_2]-2\sum_{r_1,r_2=1}^{s}[\int_0^t(\Psi_{im}*\Psi_{r_1j})(t-t_1)\frac{\sqrt{h_n}}{h_{mr_1,n}}K\left(\frac{t_1-X_l}{h_{mr_1,n}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=r_1\}}dt_1 \\
& \cdot\int_0^t(\Psi_{im}*\Psi_{r_2j})(t-t_2)q_{mr_2}(t_2)\sqrt{h_n}dt_2]].
\end{aligned}$$

Secondly, using Jensen's inequality we obtain:

$$\mathbb{E}(Y_l^2|\mathcal{F}_{l-1}) \leq \sum_{m=1}^{s} \mu_{mm}^2 [\sum_{r=1}^{s} \int_0^t (\Psi_{im} * \Psi_{rj})^2 (t-t_1) \frac{h_n}{h_{mr,n}^2} \mathbb{E}(K^2 \left(\frac{t_1 - X_l}{h_{mr,n}} \right)$$

$$\mathbb{1}_{\{J_{l-1}=m, J_l=r\}} |\mathcal{F}_{l-1}) dt_1$$

$$+ \sum_{r_1, r_2 = 1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1) q_{mr_1}(t_1) \sqrt{h_n} dt_1$$

$$\cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_2) q_{mr_2}(t_2) \sqrt{h_n} dt_2 \mathbb{E}(\mathbb{1}_{\{J_{l-1}=m\}} |\mathcal{F}_{l-1}) - 2 \sum_{r_1, r_2=1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})$$

$$(t-t_1) \tfrac{\sqrt{h_n}}{h_{mr_1,n}} \mathbb{E}(K \left(\tfrac{t_1 - X_l}{h_{mr_1,n}} \right) \mathbb{1}_{\{J_{l-1}=m, J_l=r_1\}} |\mathcal{F}_{l-1}) dt_1 \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_2)$$

$$q_{mr_2}(t_2) \sqrt{h_n} dt_2].$$

$$= \sum_{m=1}^{s} \mu_{mm}^2 [\sum_{r=1}^{s} \int_0^t (\Psi_{im} * \Psi_{rj})^2 (t-t_1) \mathbb{1}_{\{J_{l-1}=m\}} p_{mr} \frac{h_n}{h_{mr,n}^2} \int K^2 \left(\frac{t_1 - x}{h_{mr,n}} \right)$$

$$f_{mr}(x) dx dt_1 + \sum_{r_1, r_2=1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1) q_{mr_1}(t_1) \sqrt{h_n} dt_1 \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_2)$$

$$q_{mr_2}(t_2) \sqrt{h_n} dt_2 \mathbb{1}_{\{J_{l-1}=m\}} - 2 \sum_{r_1, r_2=1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1) \mathbb{1}_{\{J_{l-1}=m\}} p_{mr_1} \frac{\sqrt{h_n}}{h_{mr_1,n}}$$

$$\int K \left(\frac{t_1 - x}{h_{mr_1,n}} \right) f_{mr_1}(x) dx dt_1 \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_2) q_{mr_2}(t_2) \sqrt{h_n} dt_2].$$

Under Assumptions (H.4), (H.5), (H.6) and a change of variable, followed by Taylor's expansion, we obtain

$$
\begin{aligned}
\mathbb{E}(Y_l^2|\mathcal{F}_{l-1}) \quad & \leq \sum_{m=1}^{s} \mu_{mm}^2 [\sum_{r=1}^{s} \int_0^t (\Psi_{im} * \Psi_{rj})^2 (t-t_1) \mathbb{1}_{\{J_{l-1}=m\}} p_{mr} \int K^2\,(z)\, f_{mr}(t_1 - z h_{mr,n}) \\
& dz dt_1 + \sum_{r_1,r_2=1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1) q_{mr_1}(t_1) \sqrt{h_n} dt_1 \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_2) \\
& q_{mr_2}(t_2) \sqrt{h_n} dt_2 \mathbb{1}_{\{J_{l-1}=m\}} - 2 \sum_{r_1,r_2=1}^{s} \mathbb{1}_{\{J_{l-1}=m\}} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1) p_{mr_1} \sqrt{h_n} dt_1 \\
& \int K\,(z)\, f_{mr_1}(t_1 - z h_{mr,n}) dz \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_2) q_{mr_2}(t_2) \sqrt{h_n} dt_2] \\
& = \sum_{m=1}^{s} \mu_{mm}^2 \mathbb{1}_{\{J_{l-1}=m\}} [\sum_{r=1}^{s} \int_0^t (\Psi_{im} * \Psi_{rj})^2 (t-t_1) q_{mr}(t_1) (\int K^2\,(z)\, dz + O(h_{mr,n})) dt_1 \\
& + \left(\sum_{r=1}^{s} \sqrt{h_n} \Psi_{im} * \Psi_{rj} * Q_{mr}(t) \right)^2 - 2 \sum_{r_1,r_2=1}^{s} (\int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1) \\
& \sqrt{h_n} q_{mr_1}(t_1) (\int K\,(z)\, dz + O(h_{mr,n})) dt_1) \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_2) q_{mr_2}(t_2) \sqrt{h_n} dt_2] \\
& = \sum_{m=1}^{s} \mu_{mm}^2 \mathbb{1}_{\{J_{l-1}=m\}} [\sum_{r=1}^{s} (\Psi_{im} * \Psi_{rj})^2 * Q_{mr}(t) \int K^2\,(z)\, dz \\
& - \left(\sum_{r=1}^{s} \sqrt{h_n} \Psi_{im} * \Psi_{rj} * Q_{mr}(t) \right)^2 + O(h_{mr,n})].
\end{aligned}
$$

Using Equation (4.19) and the fact that $\dfrac{N_m(T)}{N(T)} \xrightarrow[T\to\infty]{a.s.} \nu(m)$ (see Limnios [79]), and applying Anscombe's Theorem (see Billingsley [25]), we obtain:

$$
\sigma^2 \;= \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}(Y_l^2|\mathcal{F}_{l-1})
$$

$$\leq \sum_{m=1}^{s} \mu_{mm}^2 \nu(m) \sum_{r=1}^{s} (\Psi_{im} * \Psi_{rj})^2 * Q_{mr}(t) \int K^2(z)\, dz.$$

As a final step, we have to show that the variance σ^2 does not vanish. Assuming the condition (H.4), (H.5) and (H.6), observe that:

$$\mathbb{E}(Y_l^2|\mathcal{F}_{l-1}) = \sum_{m=1}^{s} \mu_{mm}^2 \mathbb{E}[\sum_{r=1}^{s}(\int_0^t (\Psi_{im} * \Psi_{rj})(t-t_1)\frac{h_n}{h_{mr,n}^2} K\left(\frac{t_1 - X_l}{h_{mr,n}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=r\}} dt_1)$$

$$\cdot(\int_0^t (\Psi_{im} * \Psi_{rj})(t-t_1) K\left(\frac{t_1 - X_l}{h_{mr,n}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=r\}} dt_1) + \mathbb{1}_{\{J_{l-1}=m\}}$$

$$\sum_{r_1, r_2=1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1) q_{mr_1}(t_1)\sqrt{h_n} dt_1 \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_1)$$

$$q_{mr_2}(t_1)\sqrt{h_n} dt_1$$

$$-2 \sum_{r_1, r_2=1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1)\frac{\sqrt{h_n}}{h_{mr_1,n}} K\left(\frac{t_1 - X_l}{h_{mr_1,n}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=r_1\}} dt_1$$

$$\cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_1) q_{mr_2}(t_1)\sqrt{h_n} dt_1 |\mathcal{F}_{l-1}]$$

$$\geq \sum_{m=1}^{s} \mu_{mm}^2 [\beta \sum_{r=1}^{s} \int_0^t (\Psi_{im} * \Psi_{rj})(t-t_1)\frac{h_n}{h_{mr,n}^2} \mathbb{E}(K\left(\frac{t_1 - X_l}{h_{mr,n}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=r\}} |\mathcal{F}_{l-1})$$

$$dt_1 \int_0^t (\Psi_{im} * \Psi_{rj})(t-t_1) dt_1$$

$$+ \sum_{r_1, r_2=1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1) q_{mr_1}(t_1)\sqrt{h_n} dt_1 \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_1) q_{mr_2}(t_1)$$

$$\sqrt{h_n} dt_1 \mathbb{E}(\mathbb{1}_{\{J_{l-1}=m\}} |\mathcal{F}_{l-1}) - 2 \sum_{r_1, r_2=1}^{s} \int_0^t (\Psi_{im} * \Psi_{r_1 j})(t-t_1)\frac{\sqrt{h_n}}{h_{mr_1,n}} \mathbb{E}(K\left(\frac{t_1 - X_l}{h_{mr_1,n}}\right)$$

$$\mathbb{1}_{\{J_{l-1}=m, J_l=r_1\}} |\mathcal{F}_{l-1}) dt_1 \cdot \int_0^t (\Psi_{im} * \Psi_{r_2 j})(t-t_1) q_{mr_2}(t_1) dt_1 \sqrt{h_n}].$$

Therefore, we have:

$$
\begin{aligned}
\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(Y_l^2|\mathcal{F}_{l-1}) \quad & \geq \frac{1}{n}\sum_{l=1}^{n}\sum_{m=1}^{s}\mu_{mm}^2[\beta\sum_{r=1}^{s}\int_0^t(\Psi_{im}*\Psi_{rj})(t-t_1)\frac{h_n}{h_{mr,n}^2}\mathbb{E}(K\left(\frac{t_1-X_l}{h_{mr,n}}\right)\\
& \mathbb{1}_{\{J_{l-1}=m,J_l=r\}}|\mathcal{F}_{l-1})dt_1\int_0^t(\Psi_{im}*\Psi_{rj})(t-t_1)dt_1\\
& +\sum_{r_1,r_2=1}^{s}\int_0^t(\Psi_{im}*\Psi_{r_1j})(t-t_1)q_{mr_1}(t_1)\sqrt{h_n}dt_1\cdot\int_0^t(\Psi_{im}*\Psi_{r_2j})(t-t_1)\\
& q_{mr_2}(t_1)\sqrt{h_n}\mathbb{E}(\mathbb{1}_{\{J_{l-1}=m\}}|\mathcal{F}_{l-1})dt_1\\
& -2\sum_{r_1,r_2=1}^{s}\int_0^t(\Psi_{im}*\Psi_{r_1j})(t-t_1)\frac{\sqrt{h_n}}{h_{mr_1,n}}\mathbb{E}(K\left(\frac{t_1-X_l}{h_{mr_1,n}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=r_1\}}|\mathcal{F}_{l-1})\\
& dt_1\cdot\int_0^t(\Psi_{im}*\Psi_{r_2j})(t-t_1)q_{mr_2}(t_1)\sqrt{h_n}dt_1]\\
& =\frac{1}{n}\sum_{l=1}^{n}\sum_{m=1}^{s}\mu_{mm}^2\mathbb{1}_{\{J_{l-1}=m\}}[\beta\sum_{r=1}^{s}\int_0^t(\Psi_{im}*\Psi_{rj})(t-t_1)p_{mr}\frac{h_n}{h_{mr,n}^2}\\
& \int K\left(\frac{t_1-x}{h_{mr,n}}\right)f_{mr}(x)dxdt_1\int_0^t(\Psi_{im}*\Psi_{rj})(t-t_1)dt_1\\
& +\left(\sum_{r=1}^{s}\sqrt{h_n}\Psi_{im}*\Psi_{rj}*Q_{mr}(t)\right)^2-2\sum_{r_1,r_2=1}^{s}\int_0^t(\Psi_{im}*\Psi_{r_1j})(t-t_1)\\
& p_{mr_1}\frac{\sqrt{h_n}}{h_{mr_1,n}}\int K\left(\frac{t_1-x}{h_{mr_1,n}}\right)f_{mr_1}(x)dxdt_1\cdot\int_0^t(\Psi_{im}*\Psi_{r_2j})(t-t_1)\\
& q_{mr_2}(t_1)dt_1\sqrt{h_n}]
\end{aligned}
$$

$$= \frac{1}{n}\sum_{l=1}^{n}\sum_{m=1}^{s}\mu_{mm}^{2}\mathbb{1}_{\{J_{l-1}=m\}}[\beta\sum_{r=1}^{s}\int_{0}^{t}(\Psi_{im}*\Psi_{rj})(t-t_1)p_{mr}\frac{h_n}{h_{mr,n}}\int K(z)$$

$$f_{mr}(t_1 - zh_{mr,n})dzdt_1\int_{0}^{t}(\Psi_{im}*\Psi_{rj})(t-t_1)dt_1$$

$$+\left(\sum_{r=1}^{s}\sqrt{h_n}\Psi_{im}*\Psi_{rj}*Q_{mr}(t)\right)^2 - 2\sum_{r_1,r_2=1}^{s}\int_{0}^{t}(\Psi_{im}*\Psi_{r_1j})(t-t_1)p_{mr_1}$$

$$\sqrt{h_n}\int K(z)\, f_{mr_1}(t_1 - zh_{mr,n})dzdt_1\cdot\int_{0}^{t}(\Psi_{im}*\Psi_{r_2j})(t-t_1)q_{mr_2}(t_1)\sqrt{h_n}dt_1]$$

$$\geq \frac{1}{n}\sum_{l=1}^{n}\sum_{m=1}^{s}\mu_{mm}^{2}\mathbb{1}_{\{J_{l-1}=m\}}[c\beta\sum_{r=1}^{s}\int_{0}^{t}(\Psi_{im}*\Psi_{rj})(t-t_1)q_{mr}(t_1)(\int_{0}^{\infty}K(z)\,dz)dt_1$$

$$-\left(\sum_{r=1}^{s}\sqrt{h_n}\Psi_{im}*\Psi_{rj}*Q_{mr}(t)\right)^2 + O(h_{mr,n})]$$

$$= c\beta\sum_{m=1}^{s}\mu_{mm}^{2}\nu(m)\sum_{r=1}^{s}(\Psi_{im}*\Psi_{rj})*Q_{mr}(t)\int_{0}^{t}(\Psi_{im}*\Psi_{rj})(t-t_1)dt_1 \quad n\to\infty.$$

$$\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(Y_l^2|\mathcal{F}_{l-1})\xrightarrow{P}\sigma^2.$$

To complete the proof we need to satisfy Lindeberg condition.
Now, it suffices to show that

$$\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(Y_l^2\mathbb{1}_{\{|Y_l|>\epsilon\sqrt{n}\}})\xrightarrow[n\to\infty]{} 0. \tag{4.20}$$

Indeed, using successively inequalities of *Holder, Markov, Jensen* and then *Minkowski*, we obtain for any $\epsilon > 0$ and any p and q such that $\frac{1}{p}+\frac{1}{q}=1:$

$$\begin{aligned}\mathbb{E}(Y_l^2\mathbb{1}_{\{|Y_l|>\epsilon\sqrt{n}\}}) &\leq (\mathbb{E}(Y_l^{2q}))^{1/q}\mathbb{P}\{|\ Y_l\ |>\epsilon\sqrt{n}\})^{1/p}\leq(\epsilon\sqrt{n})^{-2q/p}\mathbb{E}(|\ Y_l\ |^{2q})\\ &\leq(\epsilon\sqrt{n})^{-2q/p}\mathbb{E}(|\sum_{m=1}^{s}\mu_{mm}\sqrt{h_n}\sum_{r=1}^{s}[(\Psi_{im}*H\left(\frac{\cdot - X_l}{h_{mr,n}}\right)\mathbb{1}_{\{J_{l-1}=m,J_l=r\}}*\Psi_{rj})(t)\end{aligned}$$

$$-(\Psi_{im} * Q_{mr}\mathbb{1}_{\{J_{l-1}=m\}} * \Psi_{rj})(t)] \,|^{2q})$$

$$\leq (\epsilon\sqrt{n})^{-2q/p}\sum_{m=1}^{s}\mu_{mm}^{2q}\mathbb{1}_{\{J_{l-1}=m\}}[\sum_{r=1}^{s}\frac{h_n^q}{h_{mr,n}^{2q-1}}\int_0^t(\Psi_{im} * \Psi_{rj})^{2q}(t-t_1)p_{mr}$$

$$\frac{1}{h_{mr,n}}\int K^{2q}\left(\frac{t_1-x}{h_{mr,n}}\right)f_{mr}(x)dxdt_1 + (\sum_{r=1}^{s}\sqrt{h_n}\Psi_{im} * Q_{mr} * \Psi_{rj})^{2q}(t)]$$

$$\leq (\epsilon\sqrt{n})^{-2q/p}\sum_{m=1}^{s}\mu_{mm}^{2q}\mathbb{1}_{\{J_{l-1}=m\}}[\sum_{r=1}^{s}\frac{1}{h_{mr,n}^{q-1}}\int_0^t(\Psi_{im} * \Psi_{rj})^{2q}(t-t_1)p_{mr}$$

$$\int K^{2q}(z)\,f_{mr}(t_1 - zh_{mr,n})dzdt_1 + (\sum_{r=1}^{s}\sqrt{h_n}\Psi_{im} * Q_{mr} * \Psi_{rj})^{2q}(t)].$$

By a first order Taylor's expansion, we have

$$\frac{1}{n}\sum_{l=1}^{n}\mathbb{E}(Y_l^2\mathbb{1}_{\{|Y_l|>\epsilon\sqrt{n}\}}) \leq \quad (\epsilon\sqrt{n})^{-2q/p}\sum_{m=1}^{s}\mu_{mm}^{2q}\nu(m)\frac{1}{h_{mr,n}^{q-1}}[\sum_{r=1}^{s}\int_0^t(\Psi_{im} * \Psi_{rj})^{2q}(t-t_1)$$

$$\| K \|_\infty^{2q} dt_1 + (\sum_{r=1}^{s}\sqrt{h_n}\Psi_{im} * Q_{mr} * \Psi_{rj})^{2q}(t)] \xrightarrow[n\to\infty]{} 0. \tag{4.21}$$

From the results above and the functional central limit Theorem for martingale differences [see Billingsley [25], Hall and Heyde [65]] we get the desired result. We have

$$\sigma_\Psi^2(i,j,t) \leq \sum_{l=1}^{s}\mu_{ll}\sum_{r=1}^{s}(\Psi_{il} * \Psi_{rj})^2 * Q_{lr}(t)\int K^2(z)\,dz. \tag{4.22}$$

So the last term, i.e., $\sqrt{Th_T}(\hat{\Psi}_{ij} - \Psi_{ij}) * (diag(\Delta Q \cdot \mathbf{1}))_{jj}(t)$, converges in probability to zero, as T tends to infinity. By using Slutsky's Theorem, $\sqrt{Th_T}(\hat{\Psi}_{ij} - \Psi_{ij})(t)$ converges in distribution to a normal random variable (4.22), and $\Delta Q_{jr}(t)$ converges in probability to zero (Corollary 3.2.1 (v)), we can show that $\sqrt{Th_T}[\hat{A}(t,T) - A(t)]$ has the same limit in law as

$$\sqrt{Th_T}\sum_{i=1}^{s}\sum_{j\in U}[\sum_{l=1}^{s}\sum_{r=1}^{s}\alpha_i\Psi_{ir}*\Psi_{lj}*(I - diag(Q\cdot I))_{jj}*\Delta Q_{rl}(t) - \alpha_i\Psi_{ij}*(diag(\Delta Q)\mathbf{1})_{jj}](t),$$

which can be written as

$$\sqrt{Th_T}\sum_{l=1}^{s}\sum_{r=1}^{s} D_{rl} * \Delta Q_{rl}(t) - \sqrt{Th_T}\sum_{l=1}^{s}\sum_{r\in U}\left(\sum_{n=1}^{s}\alpha_n \Psi_{nr}\right) * \Delta Q_{rl}(t). \quad (4.23)$$

We can use two different proofs of this theorem. The first one is based on the CLT for Markov renewal process such as the before result. The second one relies on the Lindeberg-Lévy CLT for martingales. We use here the second proofs

Since $\dfrac{N_l(T)}{T} \xrightarrow{a.s.} \dfrac{1}{\mu_{ll}}$ (Limnios and Oprişan [83]), let us consider the function

$$\begin{aligned} f(i,j,x) &= \sum_{l=1}^{s}\sum_{r=1}^{s}\mu_{rr}\sqrt{h_T}\Big[D_{rl} * \left(H\left(\frac{\cdot - x}{h_{rl,T}}\right)\mathbb{1}_{\{i=r,j=l\}} - Q_{rl}(\cdot)\mathbb{1}_{\{i=r\}}\right)(t) \\ &\quad - \mathbb{1}_{\{r\in U\}}\left(\sum_{k=1}^{s}\alpha_k\Psi_{kr}\right) * \left(H\left(\tfrac{\cdot - x}{h_{rl,T}}\right)\mathbb{1}_{\{i=r,j=l\}} - Q_{rl}(\cdot)\mathbb{1}_{\{i=r\}}\right)(t)\Big] \\ &= \mu_{ii}\sqrt{h_T}\Big[D_{ij} * H\left(\frac{\cdot - x}{h_{ij,T}}\right)(t) - \mathbb{1}_{\{i\in U\}}\left(\sum_{k=1}^{s}\alpha_k\Psi_{ki}\right) * H\left(\frac{\cdot - x}{h_{ij,T}}\right)(t) \\ &\quad - \sum_{l=1}^{s}[D_{il} * Q_{il}(t) - \mathbb{1}_{\{i\in U\}}\left(\sum_{k=1}^{s}\alpha_k\Psi_{ki}\right) * Q_{il}(t)]\Big]. \end{aligned}$$

We apply the central limit Theorem related to semi-Markov processes (see for instance [103]) to the function

$$\begin{aligned} W_f(T) &= \sum_{m=1}^{N(T)} f(J_{m-1}, J_m, X_m) \\ &= \sum_{m=1}^{N(T)}\sum_{r,l=1}^{s}\mu_{rr}\sqrt{h_T}\Big[D_{rl} * \left(H\left(\frac{\cdot - X_m}{h_{rl,T}}\right)\mathbb{1}_{\{J_{m-1}=r,J_m=l\}} - Q_{rl}(\cdot)\mathbb{1}_{\{J_{m-1}=r\}}\right)(t) \\ &\quad - \mathbb{1}_{\{r\in U\}}\left(\sum_{k=1}^{s}\alpha_k\Psi_{kr}\right) * \left(H\left(\frac{\cdot - X_m}{h_{rl,T}}\right)\mathbb{1}_{\{J_{m-1}=r,J_m=l\}} - Q_{rl}(\cdot)\mathbb{1}_{\{J_{m-1}=r\}}\right)(t)\Big]. \end{aligned}$$

For this function

$$A_{ij} = \int_0^\infty f(i,j,x)dQ_{ij}(x), \qquad A_i = \sum_{j=1}^{s} A_{ij},$$

$$B_{ij} = \int_0^\infty [f(i,j,x)]^2 dQ_{ij}(x), \qquad B_i = \sum_{j=1}^{s} B_{ij}.$$

So, using a change of variable, an integration by parts followed by Taylor's expansion with assumptions (H.4), (H.5) and (H.6), we get

$$\begin{aligned}
A_i &= \mu_{ii}\Big[\Big[\sum_{j=1}^{s} D_{ij} - \sum_{j=1}^{s} \mathbb{1}_{\{i\in U\}}\left(\sum_{k=1}^{s}\alpha_k\Psi_{ki}\right)\Big] * \sqrt{h_T}(Q_{ij}(t)\int_0^\infty K\,(z)\,dz + O(h_{ij,T}))\\
&- \sum_{l=1}^{s}\Big[D_{il} - \mathbb{1}_{\{i\in U\}}\left(\sum_{k=1}^{s}\alpha_k\Psi_{ki}\right)\Big] * Q_{il}(t)\sqrt{h_T}\sum_{j=1}^{s} p_{ij}\Big]\\
&= 0 \quad as \quad T\to\infty.
\end{aligned}$$

By using Jensen's inequality followed by the same steps as before, we get:

$$B_i \le \mu_{ii}^2 \sum_{j=1}^{s}\left[D_{ij} - \mathbb{1}_{\{i\in U\}}\left(\sum_{k=1}^{s}\alpha_k\Psi_{ki}\right)\right]^2 * Q_{ij}(t)\int_0^\infty K^2\,(z)\,dz; \quad as \quad T\to\infty.$$

Finally, write

$$r_j = \sum_{i=1}^{s} A_i \frac{\mu_{jj}^*}{\mu_{ii}^*} = 0, \quad as \quad T\to\infty.$$

Then

$$\sigma_j^2 = \sum_{i=1}^{s} B_i\frac{\mu_{jj}^*}{\mu_{ii}^*} \le \sum_{i=1}^{s}\frac{\mu_{jj}^*}{\mu_{ii}^*}\mu_{ii}^2\sum_{j=1}^{s}\left[D_{ij} - \mathbb{1}_{\{i\in U\}}\left(\sum_{k=1}^{s}\alpha_k\Psi_{ki}\right)\right]^2 * Q_{ij}(t)\int_0^\infty K^2\,(z)\,dz.$$

As a final step, we have to show that the variance σ_j^2 does not vanish. Assuming the condition (H.4), (H.5) and (H.6), we observe that:

$$\sum_{i=1}^{s} B_i\frac{\mu_{jj}^*}{\mu_{ii}^*} = \sigma_j^2 > 0.$$

□

Proof of Theorem 4.3.2

(a)
The proof of the uniform strong consistency of the reliability estimator is the same as the proof of the uniform strong consistency of the estimator of the availability. □

(b)
The proof of this theorem is the same as for the central limit Theorem for the estimator of the availability except that we make restrictions of different matrices to U instead of the whole state space. □

Proof of Theorem 4.4.1

(a)

We have in Theorem 4.3.2 (a) the uniformly strongly consistent in the sense that,

$$\sup_{0\le t\le L} |\hat{R}_{ij}(t,T) - R_{ij}(t)| \xrightarrow{a.s.} 0, \text{ as } T \to \infty.$$

Thus to prove the uniform strong consistency of the failure rate estimator, it is sufficient to prove the same property for the numerator in (4.12). The Markov renewal matrix estimator $\hat{\Psi}$ is uniformly strongly consistent on $[0, L]$, for all $L \in \mathbb{R}_+$ (see Proposition 4.2.1 (b)) in the sense that

$$\max_{i,j} \sup_{t\in[0,L]} |\hat{\Psi}_{ij}(t,T) - \Psi_{ij}(t)| \xrightarrow{a.s.} 0, \text{ as } T \to \infty.$$

To derive the almost sure convergence of the estimator of $f_j(t)$, it can be written as

$$\hat{f}_j(t,T) = \frac{1}{N_j}\sum_{l=1}^{N_j}\frac{1}{h_{j,T}}K\left(\frac{t-X_{j;l}}{h_{j,T}}\right).$$

In Chapter 1 Corollary (iii) 3.2.1, it was proved that the estimator of $f_j(t,T)$, is uniformly strongly consistent in the sense that,

$$\max_{j} \sup_{t\in[0,T]} |\widehat{f}_j(t,T) - f_j(t)| \xrightarrow[T\to\infty]{a.s.} 0.$$

Thus the estimator of $[\hat{F}^U]'$ buy is uniformly strongly consistent on $[0, L]$, for all $L \in \mathbb{R}_+$. Now, by a generalization of a Helly-Bray Theorem, cf. Baxter and Li [22], it is easy to see that $\hat{\Psi}^U * (I - diag(\hat{Q}'(t,T) \cdot \mathbf{1})^U$ is uniformly strongly consistent.
Finally, since the number of states is finite, the desired result is obtained.

□

(b)

From above result, we have the convergence in probability of $\hat{R}(t)$ to $R(t)$. So, we have

$$\sqrt{Th_T}\left[\hat{\alpha}_U \cdot \hat{\Psi}^U * diag[\hat{F}^U]'(t) \cdot \mathbf{1} - \alpha_U \cdot \Psi^U * diag[F^U]'(t) \cdot \mathbf{1}\right].$$

Firstly, remark that

$$\sqrt{Th_T}\left[\hat{\alpha}_U \cdot \hat{\Psi}^U * diag[\hat{F}^U]' \cdot \mathbf{1}(t) - \alpha_U \cdot \Psi^U * diag[F^U]' \cdot \mathbf{1}(t)\right]$$

$$= \sqrt{Th_T}\left[\hat{\alpha}_U \cdot \Delta\Psi^U * diag[\hat{F}^U]' \cdot \mathbf{1}(t) + \alpha_U \cdot \Psi^U * diag[\Delta F^U]' \cdot \mathbf{1}(t)\right]$$

$$= \sqrt{Th_T}\Big[\hat{\alpha}_U \cdot \Delta\Psi^U * diag[\Delta F^U]' \cdot \mathbf{1}(t) + \alpha_U \cdot \Psi^U * diag[\Delta F^U]' \cdot \mathbf{1}(t)$$
$$+\alpha_U \cdot \Delta\Psi^U * diagF' \cdot \mathbf{1}(t)\Big].$$

It is clear that, if for all $(i,j) \in E \times E$, $f_{ij}(\cdot)$ is twice continuously differentiable, then $\sqrt{Th_T}[\alpha_U \cdot \Delta\Psi^U * diagF'(t) \cdot \mathbf{1}](t)$ converges to zero, in probability as $T \to \infty$.
On the other hand, the term $\sqrt{Th_T}\hat{\alpha}_U \cdot \Delta\Psi^U * diag[\Delta F^U]'(t) \cdot \mathbf{1}$ converges in probability to zero, as T tends to infinity by applying Slutsky's Theorem.
Thus, $\sqrt{Th_T}[\hat{\lambda}_{ij}(t,T) - \lambda_{ij}(t)]$ has the same limit in distribution as

$$\sqrt{Th_T}\frac{\alpha_U \cdot \Psi^U * diag[\Delta F^U]' \cdot \mathbf{1}(t)}{R(t)},$$

and we obtain that

$$\frac{\sqrt{Th_T}}{R(t)}[\alpha_U \cdot \Psi^U * diag[\hat{F}^U]' \cdot \mathbf{1} - \alpha_U \cdot \Psi^U * diag[F^U]' \cdot \mathbf{1}](t)$$

$$= \frac{1}{R(t)} \frac{1}{\sqrt{T}} \sum_{l=1}^{N(T)} \sum_{i \in U} \sum_{j \in U} \frac{T}{N_i(T)} \sqrt{h_T}$$
$$\times \left[\alpha_j \cdot \Psi_{ij} * \left(\frac{1}{h_{i,T}} K \left(\frac{\cdot - X_l}{h_{i,T}} \right) \mathbb{1}_{\{J_{l-1}=i\}} - F_i'(\cdot) \mathbb{1}_{\{J_{l-1}=i\}} \right)(t) \right].$$

Since $\frac{N_i(T)}{T} \xrightarrow{a.s.} \frac{1}{\mu_{ii}}$ (see Limnios and Oprişan [83]), let us consider the function

$$f(d,r,x) = \frac{1}{R(t)} \sum_{i \in U} \sum_{j \in U} \mu_{ii} \sqrt{h_T} \times \left[\alpha_j \cdot \Psi_{ij} * \left(\frac{1}{h_{i,T}} K \left(\frac{\cdot - x}{h_{i,T}} \right) \mathbb{1}_{\{d=i\}} - F_i'(\cdot) \mathbb{1}_{\{d=i\}} \right)(t) \right].$$

We apply the central limit Theorem related to semi-Markov processes (see for instance [103]) to the function

$$W_f(T) = \sum_{l=1}^{N(T)} f(J_{l-1}, J_l, X_l)$$

$$= \sum_{l=1}^{N(T)} \frac{1}{R(t)} \sum_{i \in U} \sum_{j \in U} \mu_{ii} \sqrt{h_T} \times \left[\alpha_j \cdot \Psi_{ij} * \left(\frac{1}{h_{i,T}} K \left(\frac{\cdot - X_l}{h_{i,T}} \right) \mathbb{1}_{\{J_{l-1}=i\}} \right.\right.$$

$$\left.\left. - F_i'(\cdot) \mathbb{1}_{\{J_{l-1}=i\}} \right)(t) \right].$$

For this function

$$A_{dr} = \int_0^\infty f(d,r,x) dQ_{dr}(x), \qquad A_d = \sum_{r=1}^{s} A_{dr},$$

$$B_{dr} = \int_0^\infty [f(d,r,x)]^2 dQ_{dr}(x), \qquad B_d = \sum_{r=1}^{s} B_{dr}.$$

So using a change of variable, an integration by parts followed by Taylor's expansion and assumptions (H.4)-(H.6), we get

$$\begin{aligned}A_d &= \sum_{i\in U}\sum_{j\in U}\frac{\mu_{ii}}{R(t)}[\alpha_j\cdot\Psi_{ij}(t)*\mathbb{1}_{\{d=i\}}\sqrt{h_T}(f_i(t)\int_0^\infty K\left(z\right)dz+O(h_{i,T}))\\ &- \alpha_j\cdot\Psi_{ij}*F_i'(t)\mathbb{1}_{\{d=i\}}\sqrt{h_T}].\end{aligned}$$

By using Jensen's inequality followed by the same steps as before, we obtain:

$$B_d \leq \sum_{i\in U}\frac{\mu_{ii}^2}{R(t)^2}\mathbb{1}_{\{d=i\}}(\sum_{j\in U}\alpha_j\cdot\Psi_{ij})^2*f_i(t)\int_0^\infty K^2\left(z\right)dz;\quad as\quad T\to\infty.$$

Let

$$r_i = \sum_{d=1}^{s}A_d\frac{\mu_{ii}^*}{\mu_{dd}^*}=0,\quad as\quad T\to\infty.$$

Then

$$\sigma_i^2 = \sum_{d=1}^{s}B_d\frac{\mu_{ii}^*}{\mu_{dd}^*}\leq\sum_{i\in U}\frac{\mu_{ii}^2}{R(t)^2}(\sum_{j\in U}\alpha_j\cdot\Psi_{ij})^2*f_i(t)\int_0^\infty K^2\left(z\right)dz;\quad as\quad T\to\infty.$$

As a final step, we have to show that the variance σ_i^2 does not vanish. Assuming the condition (H.4), (H.5) and (H.6), observe that:

$$\sum_{d=1}^{s}B_d\frac{\mu_{ii}^*}{\mu_{dd}^*}=\sigma_i^2\geq c\beta\sum_{i\in U}\frac{\mu_{ii}^2}{R(t)^2}\sum_{j_1,j_2\in U}^{s}\alpha_j\cdot\Psi_{ij_1}*f_i(t)\int_0^t d(\alpha_j\cdot\Psi_{ij_2}(t_1)).$$

□

4.7 Simulation study

In this section we apply the previous results to the three-state continuous time semi-Markov process described in figure 4.1. Note that we study here a semi-Markov system in a strict sense that cannot be reduced to a Markov one.

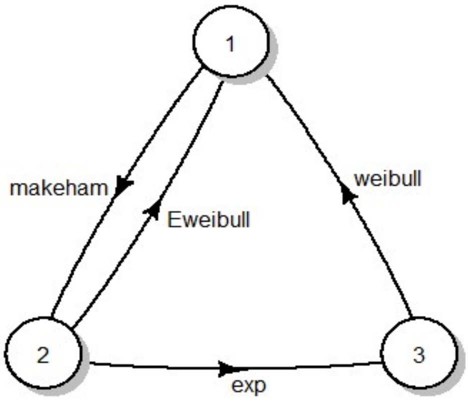


Figure 4.1: A three-states semi-Markov system.

Let us consider that the state space $E = \{1, 2, 3\}$ is partitioned into the upstate set $U = \{1, 2\}$ and the downstate set $D = \{3\}$.
The system is defined by the initial distribution $\alpha = (1, 0, 0)$.
The transition probability matrix $\mathbf{p}$ of the embedded Markov chain $(J_n)_{n\in\mathbb{N}}$

$$\mathbf{p} = \begin{pmatrix} 0 & 1 & 0 \\ 0.95 & 0 & 0.05 \\ 1 & 0 & 0 \end{pmatrix}$$

Afterward, kernel estimators for all the characteristics of the semi-Markov system (Q, F, R, A) are obtained.
Figure (4.2) and (4.3) gives a comparison between the reliability estimators and availability estimators obtained for different sample sizes (T = 1000 and T = 20000). We observe that the estimators approach to the true value as T increases.

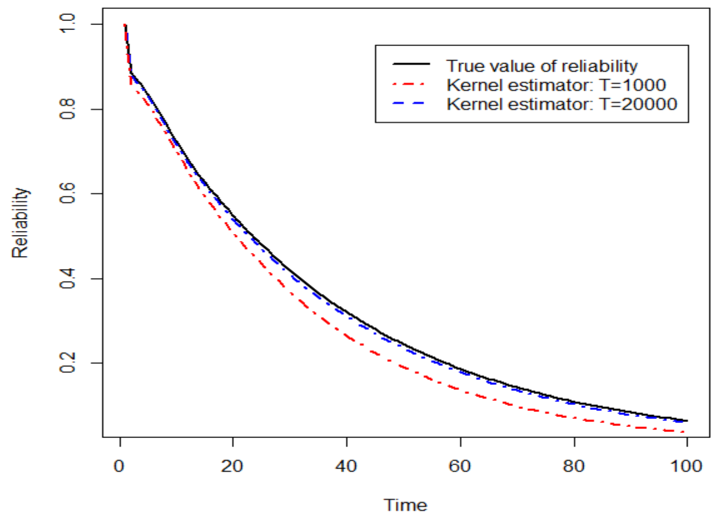


Figure 4.2: Comparison between the true values of the reliability and its estimator.

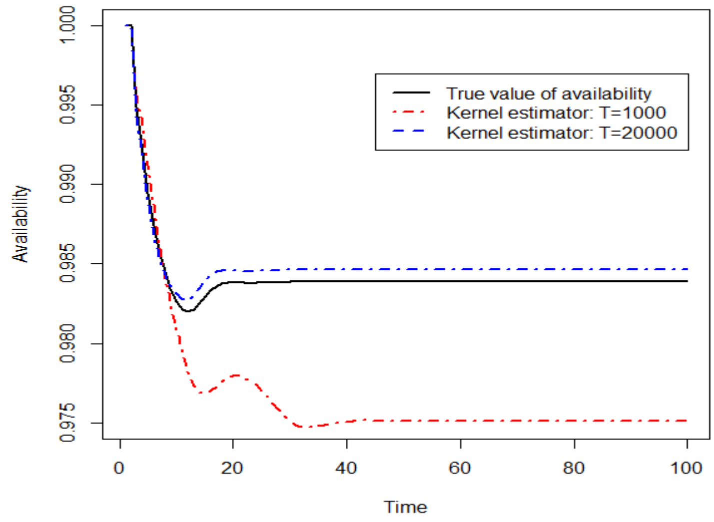


Figure 4.3: Comparison between the true values of the availability and its estimator.

General conclusion and perspectives

4.8 General conclusion

Throughout this work we were interested in presenting a study based on kernel estimation of semi-Markov systems for the main parameters, as well as in offering numerical illustration of our findings. The following points can be used to summarize the findings of this research:

i) In the first contribution of this thesis we used the classical technique of nonparametric Parzen-Rosenblatt kernel estimation to construct the conditional sojourn time estimators for a continuous-time semi-Markov process: the semi-Markov kernel, the sojourn time distribution functions (conditional or not) and the corresponding densities. We have proved their asymptotic properties of convergence and asymptotic normality by using the central limit theorem (CLT) for Markov renewal processes or by using the Lindeberg-Lévy CLT for martingales. Semi-Markov processes are crucial in many sectors of applied sciences; for this reason, we illustrated our results by a numerical example with three states with different distributions.

ii) In the second contribution of this work, we considered a theoretical study for the reliability and related measurements like availability and failure rate. First, we define the probabilistic expressions of the reliability indicators of a semi-Markov system. Second, we construct plug-in estimators of these indicators and we prove their asymptotic properties. We also illustrate our findings by a numerical example of a three-state semi-Markov system.

4.9 Perspectives

We can identify possible extensions of our thesis that would allow us to complete the study of semi-Markov processes as a direct continuation of our thesis work.

Empirical estimators were used in previous works to estimate the properties of semi-Markov systems and many works are based on these estimators. However, one problem with the empirical distribution function is that it is always a discontinuous function. For continuous distribution it is more suitable to use a continuous estimator, as the kernel estimator, instead of the empirical one. That is what has been established in the first Chapter, where we have established asymptotic properties for the different important indicators and measures of semi-Markov systems. This is merely the first stage in the estimating process. We can apply our method in order to estimate the corresponding indicators in survival analysis, rocof, maintainability, and hitting times etc. Obtaining these types of results is of particular interest in applied sciences, for instance for engineering and biomedical studies as suggested in [90].

This thesis presents the estimation of continuous-time finite state space semi-Markov processus under an observable semi-Markov process Z. The second concept that needs our investigation is the case of an unobservable semi-Markov process Z with a companion observable process Y depending on Z. This last setting, described by a coupled process (Z, Y), is called a hidden semi-Markov model (HSMM).

Another future question is the generalization of our approach to the local linear, recursive, k-nearest neighbors, etc. (see for instance [45], [6], [34]).

We can study the same type of estimator, as well as the corresponding asymptotic properties, in the case of continuous state space.

Another matter that needs our attention concerns the applications and simulation. Indeed, for the implementation of the empirical estimators and their asymptotic proprieties, the R package **SMM** [16], **smmR** [15], were devoted to the simulation and estimation of discrete-time multi-state semi-Markov and Markov models, that can be adjusted by the nonparametric kernel method. Furthermore, we can extend the existing **Semi-Markov** Package [85], that is specially designed for fitting multi-state semi-Markov models to longitudinal data, by applying the kernel estimation methodology that we have developed.

Appendix

In this appendix, we present some results used throughout this thesis, for proving the asymptotic properties of all estimators obtained for semi-Markov characteristics and for the associated reliability indicators.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(\mathrm{E}, \varepsilon)$ be a measurable space and let I be a set called a parameter set. Generally, I is a subset of $\mathbb{R}$, usually $\mathbb{N}$ or $\mathbb{R}_+$.

4.10 Stochastic processes state space

Definition 4.10.1. *A stochastic process is a family of random variables* $\{X(t), t \in I\}$ *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* E*. For every* $t \in I$*,* $X(t)$ *is a random variable* $X(t) : \Omega \to \mathrm{E}$*, whose value for the outcome* $\omega \in \Omega$ *is noted* $X(t, \omega)$*. If instead of* t *we fix an* $\omega \in \Omega$*, we obtain the function* $X(., \omega) : I \to \mathrm{E}$ *which is called a trajectory or a path-function or a sample function of the process.*

The set E *is called the state space of the stochastic process* $X = (X(t), t \in I)$*. The stochastic process may be denoted by* X_t *instead of* $X(t)$ *(respectively,* X_n *if* $I = \mathbb{N}$*).*

4.11 Theorem of strong law of large numbers

The following result concern the SLLN.

Theorem 4.11.1. *Let* $(X_1, X_2, \ldots)$ *be an infinite sequence of i.i.d. Lebesgue integrable random variables with expected value* $\mathbb{E}[X_1] = \mathbb{E}[X_2] = \ldots$*, then* *we have*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{a.s} \mathbb{E}[X_1].$$

4.12 Slutsky's theorem

Theorem 4.12.1. *[120]*
Let X, X_n, Y_n, $n \in \mathbb{N}$, be random variables or vectors. If

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} X,$$

and

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} c,$$

with c a constant, then

1. $X_n + Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X + c,$

2. $Y_n X_n \xrightarrow[n\to\infty]{\mathcal{D}} cX,$

3. $Y_n^{-1} X_n \xrightarrow[n\to\infty]{\mathcal{D}} c^{-1}X$, *for* $c \neq 0.$

4.13 Theorem of strong consistency

The following result concerns the the strong consistency given by Nadaraya [89].

Theorem 4.13.1. *Suppose that $K(x)$ is a function of bounded variation, $f(x)$ is a uniformly continuous density function, and the series $\sum_{n=1}^{\infty} e^{-\gamma n h^2}$ converges for every positive value of γ. Then*

$$\sup_{-\infty<x<\infty} |f_n(x) - f(x)| \longrightarrow 0,$$

with probability one as $n \to \infty$.

4.14 Central limit theorems

4.14.1 CLT for martingales

The following results is the Lindeberg-Lévy Central limit theorem for martingales by Billingsley [23].

Definition 4.14.1. *(**Martingale**) Let* $\mathbf{F} = (\mathcal{F}_n, n \geq 0)$ *be a family of sub-*σ*-algebras of* $\mathcal{F}$ *such that* $\mathcal{F}_n \subset \mathcal{F}_m$*, when* $n < m$*. We say that* $\mathbf{F}$ *is a filtration of* $\mathcal{F}$*. A real-valued* $\mathbf{F}$*-adapted stochastic process* X_n *is called a martingale with respect to a filtration* $\mathbf{F}$ *if, for every* $n = 0, 1, \ldots,$

1. $\mathbb{E}|X_n| < \infty$,
2. $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ *a.s.*

Theorem 4.14.1. *(**CLT for martingales**)*
Let $(X_n)_{n\in\mathbb{N}^\star}$ *be a martingale with respect to the filtration* $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ *and define the process* $Y_n = X_n - X_{n-1}$, $n \in \mathbb{N}^\star$ *(with* $Y_1 := X_1$*), called a difference martingale. If*

1. $\displaystyle \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[Y_k^2|\mathcal{F}_{k-1}] \xrightarrow[n\to\infty]{P} \sigma^2 > 0;$
2. $\displaystyle \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[Y_k^2 \mathbf{1}_{\{|Y_k|>\epsilon\sqrt{n}\}}] \xrightarrow[n\to\infty]{} 0,$ *for all* $\epsilon > 0$,

then

$$\frac{X_n}{n} \xrightarrow[n\to\infty]{a.s} 0,$$

and

$$\frac{1}{\sqrt{n}} X_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2).$$

4.14.2 Anscombe's theorem

The following Theorem is from Billingsley [25].

Theorem 4.14.2. *Let* $(Y_n)_{n\in\mathbb{N}}$ *be a sequence of random variables and* $(N_n)_{n\in\mathbb{N}}$ *a positive integer-valued stochastic process. Suppose that*

$$\frac{1}{\sqrt{n}}\sum_{m=1}^{n} Y_m \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma^2) \quad \text{and} \quad N_n/n \xrightarrow[n\to\infty]{P} \theta,$$

where θ *is a constant,* $0<\theta<\infty$. *Then,*

$$\frac{1}{\sqrt{N_n}}\sum_{m=1}^{N_n} Y_m \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma^2).$$

4.15 Limit Theorems for Markov renewal process

We present the strong law of large numbers and the central limit theorem for additive functional of MRPs. Pyke and Schaufele [103] gave these results. The notation used here comes from Moore and Pyke [87].

For a real measurable function f, defined on $\mathrm{E}\times\mathrm{E}\times\mathbb{R}$, define, for each $T>0$, the functional $W_f(T)$ as

$$W_f(T) := \sum_{n=1}^{N(T)} f(J_{n-1}, J_n, X_n). \tag{4.24}$$

Set

$$\begin{aligned} A_{ij} &:= \int_0^\infty f(i,j,x)dQ_{ij}(x), \quad A_i := \sum_{j=1}^{s} A_{ij}, \\ B_{ij} &:= \int_0^\infty (f(i,j,x))^2 dQ_{ij}(x), \quad B_i := \sum_{j=1}^{s} B_{ij}. \end{aligned}$$

Let μ_{ij} and μ^*_{ij} denote the mean first passage times from state i to j in the MRP (J_n, S_n) and in the corresponding Markov chain $(J_n)_{n\in\mathbb{N}}$, respectively.

Write

$$
\begin{aligned}
r_i &:= \sum_{u=1}^{s} A_u \frac{\mu_{ii}^*}{\mu_{uu}^*}, \\
\sigma_i^2 &:= -r_i^2 + \sum_{u=1}^{s} B_u \frac{\mu_{ii}^*}{\mu_{uu}^*} + 2\sum_{u=1}^{s}\sum_{l\neq i}\sum_{j\neq i} A_{ul}A_j\mu_{ii}^* \frac{\mu_{li}^* + \mu_{ij}^* - \mu_{lj}^*}{\mu_{uu}^*\mu_{jj}^*}.
\end{aligned}
$$

Finally, put

$$
\begin{aligned}
m_f &:= \frac{r_i}{\mu_{ii}}, \\
B_f &:= \frac{\sigma_i^2}{\mu_{ii}}.
\end{aligned}
$$

Theorem 4.15.1. ***(strong law of large numbers).***
For an aperiodic MRPs that satisfies Assumptions H.1 and H.2 (see Chapter 4*) we have*

$$\frac{W_f(T)}{T} \xrightarrow[T\to\infty]{a.s.} m_f.$$

Theorem 4.15.2. ***(Central Limit Theorem)***

For an irreducible recurrent MRPs that satisfies Assumptions H.1 and H.2 (see Chapter 4*) we have*

$$T^{-1/2}\left[W_f(T) - T\cdot m_f\right] \xrightarrow[T\to\infty]{\mathcal{D}} \mathcal{N}(0, B_f).$$

4.16 Classification of states

Let $S_i^j, i, \ldots, n \ldots$ be the recurrence times for a fixed state j with commun distribution G_{ij} on $\{J_0 = i\}$; these random variables represent a renewal process. Furthermore, the random variables $S_{n+1}^j - S_n^j$, $n \geq 1$, are i.i.d. with a common distribution denoted by G_{jj}. If j is the initial state, i.e., $\{J_0 = i\}$ (a.s.), then S_1^j, $S_{n+1}^j - S_n^j$, $n \geq 1$, are i.i.d.. Let μ_{ij} be the first moment of G_{ij}.
Then

$$G_{ij}(t) = \mathbb{P}_i(S_1^j \leq t) = \mathbb{P}_i(N_j(t) > 0).$$

Let $(\nu_i; i \in E)$ be an invariant measure for $p = (p_{ij}; i, j \in E)$, i.e., $\nu p = \nu$.

Definition 4.16.1. *1. States i and j are said to communicate if $i = j$ or $G_{ij}(\infty)G_{ji}(\infty) > 0$. Communication is an equivalence relation.*

2. A state i is said to be recurrent if $G_{ii}(\infty) = 1$, otherwise it is called transient.

3. A recurrent state i is said to be positive-recurrent if $\mu_{ii} < \infty$ and null-recurrent if $\mu_{ii} = \infty$.

4. A state i is said to be periodic with period $c > 0$ if $G_{ii}(\cdot)$ is arithmetic, i.e., concentrated on $\{nc : n \in \mathbb{N}\}$. In the opposite case it is called aperiodic.

Definition 4.16.2. *An MRP whose all states are:*

1. *communicating is called irreducible;*
2. *positive (respectively null) recurrent is called positive (respectively null) recurrent.*

Proposition 4.16.1. *Let us consider an irreducible positive recurrent MRP. For j fixed, $(\mu_{ij}, i \in E)$ is the unique bounded solution of*

$$\mu_{ij} = m_i + \sum_{k \neq j} p_{ik}\mu_{kj}.$$

Proposition 4.16.2. *1. An MRP is irreducible if and only if its EMC is irreducible.*

2. A state i is recurrent (transient) in the MRP, if and only if it is recurrent (transient) in the EMC.

3. For an irreducible finite MRP, a state i is positive recurrent in the MRP, if and only if it is recurrent in the EMC and if for all $j \in E$, $m_j < \infty$.

4. If the EMC of an MRP is irreducible and recurrent, then all states are:

- *positive recurrent if and only if $\sum_i \nu_i m_i < \infty$,*

- *null recurrent if and only if* $\sum_i \nu_i m_i = \infty$.

Theorem 4.16.1. *[103] A regular MRP is positive recurrent if and only if $m_j < \infty$ for all $j \in E$ and if there exists a convergent sequence $(y_i, i \in E)$ of positive numbers such that $\sum_{i \in E} y_i[p_{ij} - \delta_{ij}]/\mu_i = 0$. The sequence is unique up to a multiplicative constant.*

4.17 Basic definitions and properties

For $B \in \theta$, $B \neq \emptyset$, we consider the random variable

$$\tau_B = \inf\{t > S_1, Z(t) \in B\},$$

we suppose that the family of σ-algebras $\mathcal{F}_t = \sigma(Z(s), s \leq t)$ satisfied $\mathcal{F}_{t+} = \mathcal{F}_t$ for all $t \in \mathbb{R}_+$. Obviously τ_B is a stoping time with respect to the σ-algebras $(\mathcal{F}_t, t \in \mathbb{R}_+)$.
Set

$$L(x, B, t) = \mathbb{P}_x(\tau_B \leq t), \ x \in E, \ t \in \mathbb{R}_+.$$

Definition 4.17.1. *The set B is said to be accessible from the state $x \in E$ if $L(x, B, \infty) > 0$.*

Let φ be a σ-finite measure on (E, θ) such that $\varphi(E) > 0$.

Definition 4.17.2.

1. *The semi-Markov process $(Z(t), t \in \mathbb{R}_+)$ is called φ-irreducible if, whenever $\varphi(B) > 0$, the set B is accessible from any $x \in E$.*

2. *The semi-Markov process is called φ-recurrent if whenever $\varphi(B) > 0$, we have $L(x, B, \infty) = 1$ for any $x \in E$.*

Obviously, a φ-recurrent process is also φ-irreducible.

Theorem 4.17.1. *[83] The semi-Markov process $(Z(t), t \in \mathbb{R}_+)$ is φ-irreducible (φ-recurrent) if and only if the embedded Markov chain $(J_n)_{n \in \mathbb{N}}$ is φ-irreducible (φ-recurrent).*

Definition 4.17.3. *A set $B \in \theta$ is called recurrent (respectively, transient) for the semi-Markov process $(Z(t), t \in \mathbb{R}_+)$ if it is recurrent (respectively, transient) for the Markov chain $(J_n)_{n\in\mathbb{N}}$, i.e., if $\mathbb{P}_x(J_n \in B, i.o.) = 1$ (respectively, $= 0$) for any $x \in E$.*

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« التقدير اللامعلمي لعمليات شبه ماركوف مع التطبيقات »

ملخص:

يتعلق العمل الحالي بتقدير نظام شبه ماركوف (SMS) بطريقة غير معلميه مع حالات محدودة. نقدم بناء مقدرات النواة لمؤشرات ومقاييس مهمة مختلفة لعملية شبه ماركوف ثم نظهر التقارب القوي والحالة الطبيعية المقاربة.
أولاً، نوفر مقدرات النواة للخصائص الرئيسية لعملية شبه ماركوف ذات الوقت المستمر، مثل أوقات الإقامة المشروطة وغير المشروطة في حالة، ونواة شبه ماركوف، بالإضافة إلى مشتقات الرادون-نيكوديم المرتبطة بها. الهدف الرئيسي هو إنشاء خصائص مقاربة مثل الاتساق القوي الموحد والحالة الطبيعية المقاربة.
ثانيًا، ندرس موثوقية أنظمة شبه ماركوف. نقدم مقدر النواة الموثوقية والقياسات ذات الصلة: معدل الفشل، والتوافر، وندرس الخصائص المقاربة للمقدرات المقترحة.
من أجل إثبات فعالية نتائجنا النظرية، يتم تحقيق كل جزء من خلال مثال رقمي.

الكلمات الرئيسية:

عمليات شبه ماركوف، مقدر النواة، أوقات الإقامة، نواة شبه ماركوف، مصفوفة تجديد ماركوف، مصفوفة انتقال شبه ماركوف، التوافر، الموثوقية، معدل الفشل، الاتساق، الوضع الطبيعي المقارب.

« Estimation non paramétrique pour les processus semi-markoviens avec applications »

Résumé :

Le présent travail porte sur l'estimation d'un système semi-markovien (SMS) à états finis par une méthode non paramétrique. Nous présentons la construction des estimateurs à noyau pour différents indicateurs et mesures qui sont importants pour un processus semi-markovien, puis nous établissons la convergence forte ainsi que la normalité asymptotique de ces estimateurs.
Premièrement, par la méthode d'estimation à noyau, nous construisons des estimateurs des principales caractéristiques d'un processus semi-markovien en temps continu, telles que les temps de séjour conditionnel et inconditionnel, le noyau semi-markovien, ainsi que les premières dérivées des mesures précédentes. L'objectif principal est donc d'établir certaines propriétés asymptotiques des estimateurs construits.
Dans un second temps, nous étudions la fiabilité des systèmes semi-markoviens. Nous introduisons un estimateur à noyau de la fiabilité ainsi que du taux de défaillance et de la disponibilité. Ensuite, nous étudions les propriétés asymptotique des estimateurs proposés.
Afin de prouver l'efficacité de nos résultats théoriques, chaque partie est illustré à travers un exemple numérique.

Mots clés processus semi-markoviens, estimateur à noyau, temps de séjour, noyau semi-markovien, matrice de renouvellement de markovienne, matrice de transition semi-markovienne, disponibilité, fiabilité, taux de défaillance, consistance, normalité asymptotique.

« Nonparametric Estimation for semi-Markov Processes with Applications»

Abstract :

The present work concerns the estimation of a finite state semi-Markov system (SMS) by a nonparametric method. We present the construction of kernel estimators for different important indicators and measures of the semi-Markov process, then we prove their strong convergence and asymptotic normality.
Firstly, we provide kernel estimators of the main characteristics of a continuous-time semi-Markov process, like conditional and unconditional sojourn times in a state, semi-Markov kernel, as well as their associated derivatives. The main goal is to establish asymptotic properties as the uniform strong consistency and asymptotic normality.
Secondly, we study the reliability of semi-Markov systems. We introduce a kernel estimator of the reliability and its related measurements, as failure rate and availability. We also study the asymptotic properties of the proposed estimators.
In order to illustrate the quality of our theoretical results, each part is achieved by a numerical example.

Key words: semi-Markov processes, kernel estimator, sojourn times, semi-Markov kernel, Markov renewal matrix, semi-Markov transition matrix, availability, reliability, failure rate, consistency, asymptotic normality.