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# Nonparametric estimation for stochastic functional differential equations in infinite dimensional spaces

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*This modest work is dedicated to:*

*My dear mother.*

*The memory of my dear father.*

*My brothers and my sisters.*

*All my family.*

*All my friends.*

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# Résumé

La problématique abordée dans cette thèse est l'estimation non paramétrique pour la fonction de dérive dans certains modèles d'équations différentielles stochastiques dirigées par des processus auto-similaires Gaussienne généralisant le mouvement Brownien fractionnaire dans le cadre de petite diffusion. Dans un premier temps, nous considérons un modèle d'équation différentielle stochastique dirigé par un mouvement Brownien sous fractionnaire mixte et nous construisons un estimateur par la méthode du noyau pour la fonction du dérive. Nous étudions la normalité asymptotique ainsi que la convergence uniforme de cet estimateur en précisant la vitesse de convergence. Dans un second temps, nous généralisons nos résultats au cas d'un modèle d'équation différentielle stochastique dirigé par un mouvement Brownien bi-fractionnaire, et nous étudions les propriétés asymptotiques du l'estimateur à noyau pour la fonction de dérive dans le cadre de petite diffusion. Dans un troisième temps, un estimateur à noyau pour la fonction de dérive dans un modèle d'équation différentielle stochastique dirigé par un mouvement Brownien fractionnaire pondéré est construit, la convergence uniforme ainsi que la normalité asymptotique (avec vitesse de convergence) de cet estimateur sont établies.

Enfin, nous considérons le problème de l'estimation non paramétrique de la fonction de dérive et du multiplicateur linéaire pour un processus satisfaisant des équations différentielle stochastique dirigé par le drap Brownien fractionnaire. Nous proposons des estimateurs de type à noyau basés sur les trajectoires de l'équations différentielle stochastique avec un petite bruit, et nous étudions le comportement asymptotique de l'estimateur.



**Mots clés:** estimation non paramétrique, estimateur à noyau, fonction de dérive, équation différentielle stochastique, mouvement Brownien fractionnaire, mouvement Brownien sous fractionnaire mixte, mouvement Brownien bi-fractionnaire, mouvement Brownien fractionnaire pondéré, drap brownien fractionnaire.

# Abstract

In this thesis, we consider a drift estimation problem of nonparametric estimation for the drift function in some models of stochastic differential equations driven by Gaussian self-similar processes generalizing the fractional Brownian movement in the context of small diffusion. Firstly, we consider a model of stochastic differential equation driven by a mixed sub-fractional Brownian motion and we build an estimator by the kernel method for the drift function. We study the asymptotic normality, the uniform convergence (with rate) of convergence of this estimator. Secondly, we generalize our results to the case of a stochastic differential equation model driven by a bi-fractional Brownian motion, and we study the asymptotic properties of the kernel estimator for the drift function in the context of small diffusion. Thirdly, a kernel estimator for the drift function in a stochastic differential equation model driven by a weighted fractional Brownian motion is constructed, the uniform convergence and the rate of convergence of this estimator is established as well as its asymptotic normality. Finally, we considers the problem of estimating the drift function and linear multiplier for process satisfying stochastic differential equations driven by a fractional Brownian sheet. We propose kernel type estimators based on continuous observation of stochastic differential equations with small noise, and study the asymptotic behaviour of the estimator.



**Keywords:** nonparametric estimation, kernel estimator, drift function, stochastic differential equation, fractional Brownian motion, mixed fractional Brownian motion, bi-fractional Brownian motion, weighted fractional Brownian motion, fractional Brownian sheet.

# List of works

## Publications

- A. Keddi, F. Madani, A. A. Bouchentouf. **Asymptotic analysis of a kernel estimator for stochastic differential equations driven by a mixed sub-fractional Brownian motion**, has been published in the International Conference on Mathematics and Information Technology 2020 (ICMIT2020). IEEE, (2020) 91-97.

- A. Keddi, F. Madani, A. A. Bouchentouf. **Nonparametric estimation of trend function for stochastic differential equations driven by a bifractional Brownian motion**, has been published in Acta Univ. Sapientiae, Mathematica, 12, 1 (2020) 128–145.

- A. Keddi, F. Madani, A. A. Bouchentouf. **Nonparametric estimation of trend function for stochastic differential equations driven by a weighted fractional Brownian motion**, accepted for publication in "Applications and Applied Mathematics (AAM) journal".

- A. Keddi, F. Madani, A. A. Bouchentouf. **Asymptotic analysis of a kernel type estimator of trend function and linear multiplier for stochastic differential equation with additive fractional Brownian sheet**, submitted.

## Communications

- Oral presentation entitled **"Asymptotic analysis of a kernel estimator for stochastic differential equations driven by a mixed sub-fractional Brownian motion"** at the 2020 International Conference on Mathematics and Information Technology **(ICMIT 2020)** on February 18th - 19th, 2020, University of Ahmed Draya, Adrar, Algeria.

- Poster presentation entitled **"Asymptotic normality of Kernel type estimator of drift function for stochastic differential equations driven by a sub-fractional Brownian motion"** at the conference "Rencontre d'Analyse Mathématique et Applications" **(RAMA11)** on November 21th - 24th, 2019, Sidi Bel Abbes, Algeria.

## Participation in schools

- Participation in the spring school "Backward stochastic differential equations and stochastic control", Saida, Algeria, April 22th-26th, 2018.

- Participation in fall school "Statistical inference and information theory for Markov and semi-Markov processes", Saida, Algeria, October 23th-28th, 2018.

- Participation in the CIMPA research school "Stochastic Analysis and Applications", Saida, Algeria, March 01th-09th, 2019.

# Contents

# CHAPTER 1

# Introduction and Presentation

## Sommaire

Stochastic differential equations (SDEs) has been extensively applied in various fields, as diverse as biology, medicine, econometrics, physics, chemistry, finance, geophysics, and oceanography. (See, e.g. (Black & Scholes 1973), (Jennrich & Bright 1976), (Bergstrom & Wymer 1976), (Arató 1982), (Jones 1984), (Bergstrom 1988), (Adler, Müller & Rozovskii 2012) ). Moreover, over the past few years, self-similar stochastic processes with long-range dependence have been intensively used as models for various scientific areas, such as econometrics, hydrology, telecommunication, turbulence, image processing, finance and so on. The best known and most widely used process that exhibits the self-similar with long-range dependence property is fractional Brownian motion. The fractional Brownian motion is a suitable generalization of the standard Brownian motion but exhibits long-range dependence, self-similarity and it has stationary increments. This process was first introduced by (Kolmogorov 1940) and studied by (Mandelbrot & Van Ness 1968) and references therein.

Other extensions of the standard Brownian motion have been suggested mainly on the request of the application. The sub-fractional Brownian motion have properties analogous to those of fractional Brownian motion (self-similarity, long-range dependence and Hölder paths) but without stationary increments.

Recently, many authors have proposed to use other long-range dependence and self-similar Gaussian processes, such as mixed sub-fractional Brownian motion, bi-fractional Brownian motion, weighted fractional Brownian motion and fractional Brownian sheet.

In recent past, the theory of statistical inference for stochastic differential equations is a well-developed domain of mathematical statistics. Many papers and monographs have been devoted to this subject and the properties of the estimators for many models have been described (see e.g., (Banon 1978), (Banon & Nguyen 1981), (Kutoyants 1984a), (Kutoyants 1994), (Prakasa Rao 2011)). Let us consider the following stochastic differential equation:

$$dX_t = \mu(\theta, t, X_t)dt + \sigma(\vartheta, t, X_t)d\mathcal{G}_t, \quad t \geq 0, \; X_0 = x_0 \tag{1.1}$$

where $\{\mathcal{G}_t, \; t \geq 0\}$ is a long-range dependence and self-similar Gaussian process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $\mu : \Theta \times [0,T] \times \mathbb{R} \to \mathbb{R}$, called the drift (trend) coefficient depending parameters $\theta$, and $\sigma : \Xi \times [0,T] \times \mathbb{R} \to \mathbb{R}^+$, called the diffusion (volatility) coefficient depending parameters $\vartheta$, such that $\Theta \subset \mathbb{R}$, $\Xi \subset \mathbb{R}$ and $x_0 \in \mathbb{R}$. The existence and uniqueness of solutions to (1.1) under Lipschitz and the linear growth conditions on the coefficients $\mu(.)$ and $\sigma(.)$ are standard results in stochastic calculus (see e.g., (Rascanu et al. 2002), (Mishura & Shevchenko 2011), (El Barrimi & Ouknine 2019)).

In case the drift function (resp. diffusion function) is known with unknown parameter $\theta$ (resp. $\vartheta$), the estimation of $\theta$ (resp. $\vartheta$) based on discrete or continuous observations of $X_t$ was studied in the parametric framework. Parametric estimation of stochastic differential equations is largely based on parametric methods; least squares, maximum likelihood and methods of moment.

On the other hand, the nonparametric problems arise if we know only the degree of smoothness of drift function or diffusion function, and It is supposed that the parameters $\theta$ and $\vartheta$ are always known. Nonparametric method is now popular and in wide use with great success in applications. We can use nonparametric method to display the shape of a data set without relying on distribution assumptions. There are some popular smoothing techniques in nonparametric methods such as the kernel method, the local polynomial method, the Nadaraya-Watson method and so on.

In this thesis, we propose a nonparametric kernel method of trend function for stochastic differential equations driven by a long-range dependence and self-similar Gaussian process. We focus on stochastic differential equations driven by a mixed sub-fractional Brownian motion, bi-fractional Brownian motion, weighted fractional Brownian motion and sheet fractional Brownian. For each case, we establish the uniform convergence, rate of convergence and asymptotic normality of the kernel type estimator as $\sigma \longrightarrow 0$.

This chapter is organized as follows. Section 1.1 contains some preliminaries on long-range dependence and self-similar Gaussian processes such that, fBm, fBs, sfBm, msfBm, bifBm and wfBm. Then, we provide some basic proprieties that will be used in the forthcoming chapters. In Section 1.2 and Section 1.3 respectively, we present a literature review on parametric and nonparametric estimation for stochastic differential equations (SDEs). The description of the kernel type estimator is given in Section 1.4. Section 1.5 and Section 1.6 are dedicated to the contribution and the outline of the thesis, respectively.

## 1.1 Long-range dependence and self-similar processes

Long range dependence phenomena is said occurs in a time series $\{X_n, n \geq 0\}$ if the $Cov(X_0, X_n)$ of the time series tends to zero as $n \to \infty$ and satisfying the condition

$$\sum_{n=0}^{\infty} |Cov(X_0, X_n)| = \infty.$$

In other words, $Cov(X_0, X_n)$ tends to zero but so slowly that their sums diverge. This phenomenon was first observed by the hydrologist (Hurst 1951) on projects involving the design of reservoirs along the Nile river (cf. (Montanari 2003)) in hydrological time series. It was observed that a similar phenomenon occurs in problems concerned with modeling traffic patterns of packet flows in high-speed data net works such as Internet (cf. (Doukhan, Oppenheim & Taqqu 2002) and (Willinger, Paxson, Riedi & Taqqu 2003)). Long range dependence is related to the concept of self-similarity for a stochastic process.

**Definition 1.1.** A stochastic process $\{X(t);\ t \geq 0\}$ is said to be self-similar if for any $a > 0$, there exists $b > 0$ such that

$$\{X(at),\ t \geq 0\} \stackrel{\Delta}{=} \{bX(t),\ t \geq 0\}. \tag{1.2}$$

We say that $\{X(t),\ t > 0\}$ is stochastically continuous at $t$ if, for any $\varepsilon > 0$,

$$\lim_{h \to 0} \mathbb{P}(|X(t+h) - X(t)| > \varepsilon) = 0.$$

We also say that $\{X(t),\ t > 0\}$ is trivial if, $X(t)$ is a constant almost surely for every $t$.

**Definition 1.2.** If $\{X(t),\ t > 0\}$ is nontrivial, stochastically continuous at $t = 0$, and self-similar, then there exists a unique $H \geq 0$ such that $b$ in equation (1.2) can be expressed as $b = a^H$, so that

$$\{X(at),\ t \geq 0\} \stackrel{\Delta}{=} \left\{a^H X(t),\ t \geq 0\right\}. \tag{1.3}$$

**Definition 1.3.** A two parameter stochastic process $X = \{X(s,t),\ s,t \geq 0\}$ is called self-similar with self-similarity index $H = (H_1, H_2) \in (0, +\infty)^2$, if for any $h_1, h_2 > 0$,

$$\{X(h_1 s, h_2 t);\ s,t \geq 0\} \stackrel{\Delta}{=} \left\{h_1^{H_1} h_2^{H_2} X(s,t);\ s,t \geq 0\right\}. \tag{1.4}$$

### 1.1.1 Fractional Brownian motion

Fractional Brownian motion (fBm) is the only Gaussian self-similar process with stationary increments. It was introduced in (Kolmogorov 1940) and the first study dedicated to it (Mandelbrot & Van Ness 1968). The stochastic analysis of this process has been intensively developed, starting in the nineties, due to its various practical applications.

**Definition 1.4.** A centred Gaussian process $B^H = \left\{B_t^H,\ t \in \mathbb{R}^+\right\}$ is called fBm of parameter $H \in (0,1)$ if it has the covariance function

$$R_H(t,s) := \mathbb{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s,t \geq 0. \tag{1.5}$$

The index $H$ is called the Hurst parameter and it determines the main properties of the process $B^H$, such as self-similarity, regularity of the sample paths and long memory.

Note that, for $H = 1/2$, $B^{1/2}$ is the standard Brownian motion on $\mathbb{R}$ . Therefore, a fBm is an extension of a standard Brownian motion on $\mathbb{R}$. From (1.5) it follows that

$$\mathbb{E}\left|B_t^H - B_s^H\right|^2 = |t-s|^{2H}.$$

This implies that $B^H$ has stationary increments and, as a consequence, $B^H$ has $\alpha$-Hölder continuous paths for all $\alpha < H$.

A simple property of the fBm of parameter $H$ is its self-similarity: the process $\left\{B_{\alpha t}^H, t \geq 0\right\}$ has the same law as the process $\left\{\alpha^H B_t^H, t \geq 0\right\}$ for every $\alpha \geq 0$. This last property shows the interest of this process for the modelling of stock market fluctuations, traffic, telecommunications and networks (see, for example (Coelho & Decreusefond 1995)). In addition, several applications have been found in economics and natural sciences (see, for instance, (Mandelbrot 1983)).

Let $X_n = B_n^H - B_{n-1}^H$, $n \geq 1$. Then, $\{X_n, n \geq 1\}$ is a Gaussian stationary sequence with covariance function

$$\rho_H(n) = \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right).$$

This implies that two increments of the form $B_k^H - B_{k-1}^H$ and $B_{k+n}^H - B_{k+n-1}^H$ are positively correlated (i.e. $\rho_H(n) > 0$) if $H > 1/2$ and they are negatively correlated (i.e. $\rho_H(n) < 0$) if $H < 1/2$.

In the case $H > 1/2$ the stationary sequence $X_n$ exhibits long range dependence , that is,

$$\lim_{n\to\infty} \frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1,$$

and, as a consequence,

$$\sum_{n=1}^{\infty} \rho_H(n) = \infty.$$

In the case $H < 1/2$, we have

$$\sum_{n=1}^{\infty} \left|\rho_H(n)\right| < \infty.$$

If $H \neq 1/2$, the process $B^H$ is not Markov and it is not a semi-martingale and we cannot apply the stochastic calculus developed by Itô in order to define stochastic integrals with respect to $B^H$. The stochastic calculus of fBm began with the innovative work of (Mandelbrot & Van Ness 1968). They considered the moving average representation of $B^H$, via the Wiener process $\{W_t, t \geq 0\}$ over an infinite interval

$$B_t^H = C^{-1}(H) \int_{\mathbb{R}} \left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right] dW_s \tag{1.6}$$

where

$$C(H) = \left[\int_0^{+\infty} \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds + \frac{1}{2H}\right]^{\frac{1}{2}}.$$

#### 1.1.1.1 Wiener integral representation on a finite interval

Fix $H \in (1/2, 1)$. Let $B^H = B_t^H,\ t \in [0, T]$ be a fBm with parameter $H$. We assume that $B^H$ is defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ . We denote by $\mathcal{E} \subset \mathcal{H}$ the set of step functions on $[0, T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \longrightarrow B_t^H$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with $B^H$. We will denote this isometry by $\varphi \longrightarrow B^H(\varphi)$.
It is easy to see that

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2}\, du dr, \tag{1.7}$$

where $\alpha_H = H(2H-1)$. Formula (1.7) implies that

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2} \varphi(r)\psi(u) du dr,$$

for any pair of step functions $\varphi$ and $\psi$ in $\mathcal{E}$. On the other hand, by (Decreusefond et al. 1999) we know that the covariance kernel $R_H(t,s)$ can be written as

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,r) K_H(s,r) dr, \tag{1.8}$$

where $K_H$ is a square integrable kernel given by

$$K_H(t,s) = d_H (t-s)^{H-1/2} + s^{H-1/2} F\left(\frac{t}{s}\right), \tag{1.9}$$

where $d_H$ being a constant and

$$F(z) = d_H\, (1/2 - H) \int_0^{z-1} r^{H-3/2} \left(1 - (r+1)^{H-1/2}\right) dr.$$

Now, define the linear operator $K_H^* : \mathcal{E} \longrightarrow \mathbb{L}^2([0,T])$ by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K_H}{\partial r}(r,s) dr.$$

Due to (Mazet, Nualart et al. 2001), for all $\varphi,\ \psi \in \mathcal{E}$, there holds

$$\langle K_H^* \varphi, K_H^* \psi \rangle = \langle \varphi, \psi \rangle_{\mathcal{H}}$$

and then $K_H^*$ can be extended to an isometry between $\mathcal{H}$ and $\mathbb{L}^2([0,T])$. Hence, according to (Mazet et al. 2001) again, the process $W = \{W_t,\ t \in [0,T]\}$ defined by

$$W_t = B^H((K_H^*)^{-1} \mathbf{1}_{[0,t]}),\ \ t \in [0,T]$$

is a Wiener process and $B^H$ has the following Wiener integral representation:

$$B_t^H = \int_0^t K_H(t,s)dW_s.$$

We can find a linear space of functions contained in $\mathcal{H}$ in the following way. Let $|\mathcal{H}|$ be the linear space of measurable functions $\varphi$ on $[0,T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\varphi(r)||\varphi(u)||r-u|^{2H-2}\,drdu. \tag{1.10}$$

It is not difficult to show that $|\mathcal{H}|$ is a Banach space with the norm $\|.\|_{|\mathcal{H}|}$ and $\mathcal{E}$ is dense in $|\mathcal{H}|$ . On the other hand, it has been shown in (Pipiras & Taqqu 2000) that the space $|\mathcal{H}|$ equipped with the inner product $\langle\varphi,\psi\rangle_{|\mathcal{H}|}$ is not complete and it is isometric to a subspace of $\mathcal{H}$. We will identify $|\mathcal{H}|$ with this subspace. The following estimate has been proved in (Mémin, Mishura & Valkeila 2001). Let $H > 1/2$, we have the embeddings

$$\mathbb{L}^2([0,T]) \subset \mathbb{L}^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}. \tag{1.11}$$

### 1.1.2 Fractional Brownian sheet

**Definition 1.5.** The fractional Brownian sheet with Hurst index $(\alpha,\beta) \in (0,1)^2$ is a two-parameter Gaussian process $W^{\alpha,\beta} = \left\{W_{s,t}^{\alpha,\beta},\ 0 \le s,t \le T\right\}$ starting from zero with mean zero and covariance function

$$R_{\alpha,\beta}(s,t,u,v) := \frac{1}{2}\left(t^{2\alpha} + v^{2\alpha} - |t-v|^{2\alpha}\right)\frac{1}{2}\left(s^{2\beta} + u^{2\beta} - |s-u|^{2\beta}\right), \tag{1.12}$$

given for all $(s,t,u,v) \in [0,T]^4$.

**Definition 1.6.** We say that a two-parameter stochastic process (vanishing on the axis) $X = (X_{t,s})_{t,s\ge 0}$ has stationary rectangular increments if its rectangular increments are stationary, i.e. for every $h_1, h_2 > 0$, the process:

$$(X_{t+h_1,s+h_2} - X_{t+h_1,h_2} - X_{h_1,s+h_2} + X_{h_1,h_2})_{t,s\ge 0}$$

has the same finite dimensional distributions as $X$.

**Remark 1.1.** Note that the fractional Brownian sheet is a two-parameter self-similar process with stationary rectangular increments in the sense of Definitions 1.3 and 1.6. However, in contrast with the fractional Brownian motion, it is not the only Gaussian self-similar field with stationary rectangular increments.

The fractional Brownian sheet $W^{\alpha,\beta}$ can be represented as

$$W_{s,t}^{\alpha,\beta} = \int_0^t \int_0^s K_\alpha(s,u) K_\beta(t,v) dW_{u,v},$$

where $\{W_{u,v},\ 0 \leq u,v \leq T\}$ is a standard Brownian sheet and $K_\alpha(s,u)$ is given by (1.9). Denote by

$$K_{\alpha,\beta}(s,t) = K_\alpha(s,u) K_\beta(t,v), \tag{1.13}$$

and let $\mathcal{H}^{(2)}(\alpha,\beta) := \mathcal{H}^{(2)}$ be the canonical Hilbert space of the fractional Brownian sheet $W_{s,t}^{\alpha,\beta}$. That is, $\mathcal{H}^{(2)}$ is defined as the closure of the set of indicator functions $\left\{\mathbf{1}_{[0,t]\times[0,s]},\ t,s \in [0,T]\right\}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,s]\times[0,t]}; \mathbf{1}_{[0,u]\times[0,v]} \right\rangle_{\mathcal{H}^{(2)}} = R_{\alpha,\beta}(s,t,u,v); \tag{1.14}$$

with $(s,t,u,v) \in [0,T]^4$.

If $(\alpha,\beta) \in (1/2,1)^2$, the elements of $\mathcal{H}^{(2)}$ may be not functions but distributions. Thus, it is more convenient to work with subspaces of $\mathcal{H}^{(2)}$ that are sets of functions. We have actually the inclusions

$$\mathbb{L}^2([0,T]^2) \subset |\mathcal{H}|^{(2)} \subset \mathcal{H}^{(2)}, \tag{1.15}$$

where

$$|\mathcal{H}|^{(2)} = |\mathcal{H}(\alpha)| \otimes |\mathcal{H}(\beta)|$$

and $|\mathcal{H}(\alpha)|$ defined by (1.10).

A such space is the set $|\mathcal{H}^{(2)}|$ of measurable functions on $[0,T]^2$ such that

$$\|f\|^2_{|\mathcal{H}|^{(2)}} = \int_0^T \int_0^T \int_0^T \int_0^T |f(s,t)||f(u,v)| \phi_{\alpha,\beta}(s,t,u,v) ds dt du dv, \tag{1.16}$$

with

$$\phi_{\alpha,\beta}(s,t,u,v) = \alpha(2\alpha-1)\beta(2\beta-1)|s-u|^{2\alpha-2}|t-v|^{2\beta-2}. \tag{1.17}$$

Note that, if $f,\ g \in \mathcal{H}^{(2)}$, then their scalar product in $\mathcal{H}^{(2)}$ is given by

$$\langle f, g \rangle_{\mathcal{H}^{(2)}} = \int_0^T \int_0^T \int_0^T \int_0^T f(s,t) g(u,v) \phi_{\alpha,\beta}(s,t,u,v) ds dt du dv.$$

As in the one-parameter case (see (Norros, Valkeila, Virtamo et al. 1999)) one can associate to the fractional Brownian sheet a two-parameter martingale (the so-called fundamental martingale). We refer to (Tudor & Tudor 2005) for the two-parameter case. More precisely, let us define the deterministic function

$$c_\alpha = 2\alpha\Gamma(3/2-\alpha)\Gamma(\alpha+1/2), \quad k_\alpha(s,t) = c_\alpha^{-1} s^{1/2-\alpha}(t-s)^{1/2-\alpha}; \tag{1.18}$$

and

$$\lambda_\alpha = \frac{2\Gamma(3-2\alpha)\Gamma(\alpha+1/2)}{\Gamma(3/2-\alpha)}, \quad w_t^\alpha = \lambda_\alpha^{-1} t^{2-2\alpha}. \tag{1.19}$$

Then, the process

$$M_{s,t}^{\alpha,\beta} = \int_0^t \int_0^s k_\alpha(t,v) k_\beta(s,u) dW_{u,v}^{\alpha,\beta} \tag{1.20}$$

is a two-parameter Gaussian martingale with quadratic variation over $[0,s]\times[0,t]$ defined by

$$\left\langle M^{\alpha,\beta}\right\rangle_{s,t} = w_t^\alpha w_s^\beta = \lambda_\alpha^{-1} s^{2-2\alpha} \lambda_\beta^{-1} t^{2-2\beta}. \tag{1.21}$$

(the stochastic integral in (1.20) can be defined in a Wiener sense with respect to the fractional Brownian sheet). The filtration generated by $M^{\alpha,\beta}$ coincides to the one generated by $W^{\alpha,\beta}$.

### 1.1.3 Sub-fractional Brownian motion

The sub-fractional Brownian motion is an extension of a Brownian motion, which was investigated in many papers (e.g. (Bojdecki, Gorostiza & Talarczyk 2004) and (Tudor 2007)). It is a stochastic process $\xi^H = (\xi_t^H)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\forall t \in \mathbb{R}_+, \quad \xi^H(t) = \frac{B^H(t) + B^H(-t)}{\sqrt{2}}$$

where $B_t^H$ is fBm with covariance function (1.5) and $H \in (0,1)$. The sub-fractional Brownian motion (sfBm) is a centred Gaussian process $(\xi_t^H)_{t\geq 0}$ , starting from zero with covariance

$$C_H(t,s) = s^{2H} + t^{2H} - \frac{1}{2}\left((t+s)^{2H} + |t-s|^{2H}\right), \; s,t \geq 0, \tag{1.22}$$

with $H \in (0,1)$.

**Remark 1.2.** If $H = 1/2$, then $\xi^{1/2}$ is a standard Brownian motion on $\mathbb{R}$ because in this case,

$$C_{1/2}(t,s) = Cov(\xi_t^{1/2}, \xi_s^{1/2}) = s + t - \frac{1}{2}\left((t+s) + |t-s|\right) = s \wedge t.$$

Therefore, a sfBm is another extension of a standard Brownian motion.

#### 1.1.3.1 Main properties of the sub-fractional Brownian motion

**Proposition 1.1 ((Bojdecki et al. 2004)).** *Let* $\xi_t^H = \left\{\xi_t^H, \; t \geq 0\right\}$ *be a sfBm with Hurst parameter* $H$*. Then,*

1. *The process* $\xi^H$ *is self-similar, that is, for every* $\alpha > 0$,

$$\left\{\xi^H(\alpha t), \; t \geq 0\right\} \stackrel{\Delta}{=} \left\{\alpha^H \xi^H(t), \; t \geq 0\right\}.$$

2. *The sfBm does not have stationary increments unless* $H = 1/2$.

3. *For all* $s, t \geq 0$, *the covariance function* $C_H(s,t)$ *of the process* $\xi^H$ *is positive for all* $s, t \geq 0$. *Furthermore*

$$C_H(s,t) > R_H(s,t) \text{ if } H < 1/2,$$

$$C_H(s,t) < R_H(s,t) \text{ if } H > 1/2,$$

*with* $R_H(s,t)$ *the covariance function of the process* $B^H$.

4. *Let* $\beta_H = 2 - 2^{2H-1}$. *for all* $s, t > 0$,

$$(t-s)^{2H} \leq \mathbb{E}\left[\xi^H(t) - \xi^H(s)\right]^2 \leq \beta_H (t-s)^{2H}, \text{ if } H < 1/2,$$

*and*

$$\beta_H (t-s)^{2H} \leq \mathbb{E}\left[\xi^H(t) - \xi^H(s)\right]^2 \leq (t-s)^{2H}, \text{ if } H > 1/2,$$

*and the constants in the above inequalities are sharp.*

5. *The process* $\xi^H$ *is not Markov and it is not a semi-martingale.*

6. *The process* $\xi^H$ *has continuous sample paths almost surely and, for each* $0 < \epsilon < H$ *and* $T > 0$, *there exists a random variable* $K_{\epsilon,T}$ *such that*

$$\left|\xi^H(t) - \xi^H(s)\right| \leq K_{\epsilon,T} \left|t-s\right|^{H-\epsilon}, \; 0 \leq s, t \leq T \text{ a.s.}$$

#### 1.1.3.2 Moving average representation

**Proposition 1.2 ((Mishura & Zili 2018)).** *Let* $0 < H < 1$ *and* $\xi^H = \left\{\xi_t^H, \; t \in \mathbb{R}\right\}$ *be a sfBm with Hurst parameter* $H$. *Then,* $\xi^H$ *has the following integral representation: for any* $t \geq 0$

$$\xi^H(t) = \frac{1}{C_1(H)} \int_{\mathbb{R}} \left[(t-s)_+^{H-\frac{1}{2}} + (t+s)_-^{H-\frac{1}{2}} - 2(-s)_+^{H-\frac{1}{2}}\right] dW(s),$$

*where* $W$ *is the Brownian process on* $\mathbb{R}$ *and*

$$C_1(H) = \left[2 \int_0^{+\infty} \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds + \frac{1}{2H}\right]^{\frac{1}{2}}.$$

#### 1.1.3.3 Wiener integral representation on a finite interval

Fix a time interval $[0,T]$. We denote by $\mathcal{H}_{\xi^H}$ the canonical Hilbert space associated to the sfBm $\xi^H$ . That is, $\mathcal{H}_{\xi^H}$ is the closure of the linear span $\mathcal{E}$ generated by the indicator function $\left\{\mathbf{1}_{[0,t]},\ t\in[0,T]\right\}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}_{\xi^H}} = C_H(t,s).$$

We know that the covariance of sfBm can be written as

$$C_H(t,s) = \int_0^t \int_0^s \phi_H(u,v)dudr, \tag{1.23}$$

where

$$\phi_H(s,t) := \frac{\partial^2 C_H(s,t)}{\partial s \partial t} = H(2H-1)\left[|t-s|^{2H-2} - (s+t)^{2H-2}\right].$$

Formula (1.23) implies that

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{\xi^H}} = \int_0^t \int_0^s \varphi(u)\psi(v)\phi_H(u,v)dudv, \tag{1.24}$$

for any pair step functions $\varphi$ and $\psi$ on $[0,T]$. Consider the kernel

$$n_H(t,s) = \frac{2^{1-H}\sqrt{\pi}}{\Gamma\left(H-\frac{1}{2}\right)} s^{\frac{3}{2}-H} \left( \int_s^t (x^2-s^2)^{H-\frac{3}{2}} dx \right) \mathbf{1}_{[0,t]}(s). \tag{1.25}$$

By (Dzhaparidze & Van Zanten 2004), we have

$$C_H(t,s) = c_H^2 \int_0^{t\wedge s} n_H(t,u) n_H(s,u) du, \tag{1.26}$$

where

$$c_H^2 = \frac{\Gamma(1+2H)\sin(\pi H)}{\pi}.$$

The property (1.23) implies that $C_H(t,s)$ is non-negative definite. Consider the linear operator $n_H^*$ from $\mathcal{E}$ to $\mathbb{L}^2([0,T])$ defined by

$$(n_H^*\varphi)(s) := c_H \int_s^T \varphi(r) \frac{\partial n_H}{\partial r}(r,s) dr.$$

Using (1.24) and (1.26) we have

$$\langle n_H^*\varphi, n_H^*\psi \rangle_{\mathbb{L}^2([0,T])} = \langle \varphi, \psi \rangle_{\mathcal{H}_{\xi^H}}. \tag{1.27}$$

As a consequence, the operator $n_H^*$ provides an isometry between the Hilbert space $\mathcal{H}_{\xi^H}$ and $\mathbb{L}^2([0,T])$. Hence, the process $W_t$ defined by

$$W_t = \xi^H((n_H^*)^{-1}\mathbf{1}_{[0,t]}),\ \ t\in[0,T]$$

is a Wiener Process, and the process $\xi^H$ has an integral representation of the form

$$\xi_t^H = c_H \int_0^t n_H(t,s) dW_s.$$

By (Dzhaparidze & Van Zanten 2004), we have

$$W_t = \int_0^t \psi_H(t,s) d\xi_s^H,$$

where

$$\psi_H(t,s) = \frac{s^{H-\frac{1}{2}}}{\Gamma\left(\frac{3}{2}-H\right)} \left[ t^{H-\frac{3}{2}}(t^2-s^2)^{\frac{1}{2}-H} - (H-\frac{3}{2})^{\frac{1}{2}-H} x^{H-\frac{3}{2}} dx \right] \mathbf{1}_{[0,t]}(s).$$

We can find a linear space of functions contained in $\mathcal{H}_{\xi^H}$ in the following way. Let $\mathcal{H}_{\xi^H}$ be the linear space of measurable functions on $[0,T]$ such that

$$\|\varphi\|^2_{|\mathcal{H}_{\xi^H}|} = \int_0^t \int_0^s |\varphi(u)||\varphi(v)| \phi_H(u,v) du dv.$$

It is not difficult to show that $\left|\mathcal{H}_{\xi^H}\right|$ is a Banach space with the norm $\|.\|_{|\mathcal{H}_{\xi^H}|}$ and $\mathcal{E}$ is dense in $\left|\mathcal{H}_{\xi^H}\right|$. Moreover, for $H > 1/2$, (Mendy 2013) shows that

$$\mathbb{L}^2([0,T]) \subset \mathbb{L}^{\frac{1}{H}}([0,T]) \subset \left|\mathcal{H}_{\xi^H}\right| \subset \mathcal{H}_{\xi^H}$$

### 1.1.4 Mixed sub-fractional Brownian motion

The mixed sub-fractional Brownian motion is an extension of a sfBm, which was investigated in many papers (see (Charles & Mounir 2015) and (Zili 2006, Zili 2014)).

**Definition 1.7.** Let $N$ be a positive integer number, vector of indices $H = (H_1, H_2, ..., H_N) \in (0,1)^N$ and $a = (a_1, a_2, ..., a_N) \in \mathbb{R}^N$, with $a_i \neq 0$. A mixed sub-fractional Brownian motions (msfBm) that depend on parameters $N$, $a$ and $H$

$$S = \left\{ S_t^H(N,a),\, t \geq 0 \right\} = \left\{ S_t^H,\, t \geq 0 \right\} \tag{1.28}$$

defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$S_t^H(N,a) = \sum_{i=1}^N a_i \xi^{H_i}(t), \text{ for all } t \in \mathbb{R}_+, \tag{1.29}$$

where $(\xi^{H_i})_{i\in 1,...,N}$ is a family of independent sfBm of Hurst parameters $H_i \in (0,1)$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

**Remark 1.3.** If we assume that $N = 1$ and $a_1 = 1$, then $S^H = \xi^H$ is a sfBm. If $N = 1$, $H_1 = 1/2$ and $a_1 = 1$, then both $S^H$ are the standard Brownian motions. Therefore, the msfBm are obviously more general processes than standard Brownian motions and sfBm.

#### 1.1.4.1 Main properties of the msfBm

**Proposition 1.3.** *According to (Charles & Mounir 2015) and (Zili 2006, Zili 2014, Zili 2016). The msfBm $\left\{S_t^H(N,a) : t \geq 0\right\}$ satisfies the following properties:*

1. *The msfBm is a centred Gaussian process.*
2. *For all $(t,u) \in \mathbb{R}_+^2$,*

$$R_{a,H}(t,u) = Cov(S_t^H(a), S_u^H(a)) = \sum_{i=1}^{N} a_i^2 \left[ t^{2H_i} + u^{2H_i} - \frac{1}{2}\left[ (u+t)^{2H_i} + |u-t|^{2H_i} \right]\right]. \tag{1.30}$$

3. *For all $i \in \{1,...,N\}$, such that $H_i \in (1/2,1)$, we have*

$$\mathbb{E}((S_t^H(a))^2) = \sum_{i=1}^{N} a_i^2 \left[ (2 - 2^{2H_i-1}) t^{2H_i} \right], \ \forall t \in \mathbb{R}_+. \tag{1.31}$$

4. *$S_t^H(a)$ is said to be mixed self-similar*

$$\left\{S_{\alpha t}^H(a),\, t \geq 0\right\} \stackrel{\Delta}{=} \left\{S_t^H(a_1\alpha^{H_1}, a_2\alpha^{H_2}, ..., a_N\alpha^{H_N}),\, t \geq 0\right\}; \ \forall \alpha > 0. \tag{1.32}$$

5. *For every $a = (a_1,...,a_N) \in \mathbb{R}^N$ and $H = (H_1,...,H_N) \in (0,1)^N$ such that there exists $j \in \{1,...,N\}$ such that $a_j \neq 0$ and $H_j \neq 1/2$, $(S_t^H(a))_{t\in\mathbb{R}_+}$ is not a Markovian process.*
6. *If there exists $i \in \{1,...,N\}$; $H_i < 1/2$ and $a_i \neq 0$, or if $H_i > 1/2$, for every $i \in \{1,...,N\}$, then the msfBm $(S_t^H(a))_{t\in\mathbb{R}_+}$ is not a semimartingale.*

#### 1.1.4.2 Moving average representation

**Proposition 1.4 ((Mishura & Zili 2018)).** *Let $S = \left\{S_t^H(N,a),\, t \geq 0\right\} = \left\{S_t^H,\, t \geq 0\right\}$ be a msfBm defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $N \in \mathbb{N}\backslash\{0\}$, $H = (H_1, H_2, ..., H_N) \in (1/2,1)^N$, and vector $a = (a_1, a_2, ..., a_N) \in \mathbb{R}^N$, $a_i \neq 0$. Then, $S_t^H(N,a)$ has the following integral representation, for any $t \geq 0$*

$$S_t^H(N,a) = \sum_{i=1}^{N} a_i C_1^{-1}(H_i)\left(H_i - \frac{1}{2}\right) \int_{\mathbb{R}} \int_0^t \left[ (u-s)_+^{H_i - \frac{3}{2}} + (u+s)_-^{H_i-\frac{3}{2}} \right] du\, dW_i(s), \tag{1.33}$$

*where $\{W_i,\ 1 \leq i \leq N\}$ are independent Wiener processes on $\mathbb{R}$ and*

$$C_1(H_i) = \left[ 2\int_0^{+\infty} \left( (1+s)^{H_i - \frac{1}{2}} - s^{H_i-\frac{1}{2}} \right)^2 ds + \frac{1}{2H_i} \right]^{\frac{1}{2}}.$$

#### 1.1.4.3 The canonical Hilbert space

Let $\mathcal{H}_{a,H}$ be the completion of the linear space $\mathcal{E}$ generated by the indicator functions $\left\{\mathbf{1}_{[0,t]},\ t \in [0,T]\right\}$ with respect to the inner product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,u]} \right\rangle_{\mathcal{H}_{a,H}} = R_{a,H}(t,u), \tag{1.34}$$

where $R_{a,H}(t,u)$ is the covariance of $S_t^H(a)$ and $S_u^H(a)$. When $1/2 < H_i < 1$ and $a_i \neq 0$ for all $i \in \{1,...,N\}$, We will use the subspace $\left|\mathcal{H}_{a,H}\right|$ of $\mathcal{H}_{a,H}$ which is defined as the set of measurable function $\varphi$ on $[0,T]$ with

$$\left|\mathcal{H}_{a,H}\right| = \left\{\varphi : [0,T] \longrightarrow \mathbb{R};\ \|\varphi\|_{\mathcal{H}_{a,H}} < \infty\right\};$$

and

$$\|\varphi\|^2_{\mathcal{H}_{a,H}} = \int_0^T \int_0^T |\varphi(t)||\varphi(u)| \phi_{a,H}(t,u) dt du, \tag{1.35}$$

where

$$\phi_{a,H}(t,u) := \frac{\partial^2 R_{a,H}(t,u)}{\partial t \partial u} = \sum_{i=1}^{N} a_i^2 H_i (2H_i - 1)\left[|u-t|^{2H_i-2} - (u+t)^{2H_i-2}\right].$$

For every $\varphi,\ \psi \in \left|\mathcal{H}_{a,H}\right|$ the equation (1.34) implies that:

$$\begin{aligned} \langle \varphi, \psi \rangle_{\mathcal{H}_{a,H}} &= \mathbb{E}\left( \int_0^T \varphi(t) dS_t^H(a) \int_0^T \psi(u) dS_u^H(a) \right) \\ &= \int_0^T \int_0^T |\varphi(t)||\varphi(u)| \phi_{a,H}(t,u) dt du. \end{aligned} \tag{1.36}$$

### 1.1.5 Bi-fractional Brownian motion

As an extension of fBm, (Houdré & Villa 2003) introduced and studied a rather special class of self-similar Gaussian processes which preserves many properties of the fBm and involving fBm as a special case. This process is called the bi-fractional Brownian motion $B^{H,K}$ ( hereafter bfBm). Many of the powerful techniques from the Itô stochastic analysis are not available when dealing with $B^{H,K}$. However, as a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to $B^{H,K}$. The bfBm admits self-similarity and Hölder paths, and its increments are not stationary. Moreover, it happens that this process is a quasi-helix, as defined, for instance, in (Kahane 1981, Kahane 1993). We refer to (Bojdecki, Gorostiza & Talarczyk 2007), (El-Nouty 2009), (El-Nouty & Journé 2011), (Lei & Nualart 2009), (Russo & Tudor 2006), and (Tudor, Xiao et al. 2007) for further information on this process.

**Definition 1.8.** The bi-fractional Brownian motion $(B_t^{H,K})_{t\geq 0}$ is a centered Gaussian process, starting from zero, with covariance

$$R_{H,K}(t,s) = \frac{1}{2^K}\left((t^{2H} + s^{2H})^K - |t-s|^{2HK}\right)$$

with $H \in (0,1)$ and $K \in (0,1]$.

**Remark 1.4.** Note that, if $K = 1$, then $B^{H,1}$ is a fBm with Hurst parameter $H \in (0,1)$. In particular, if $K = 1$ and $H = 1/2$, then $B^{1/2,1}$ is a standard Brownian motion. Therefore, a bfBm is an extension of a fractional Brownian motion.

#### 1.1.5.1 Main properties of the bfBm

**Proposition 1.5 ((Russo & Tudor 2006)).** *The bfBm satisfies the following properties:*

1. *The process $B^{HK}$ is $HK$-self-similar.*
$$\left\{B_{\alpha t}^{H,K},\, t \geq 0\right\} \stackrel{\Delta}{=} \left\{\alpha^{HK} B_t^{H,K},\, t \geq 0\right\}, \text{ for each } \alpha > 0. \tag{1.37}$$
2. *The process is Hölder continuous of order $\delta$ for any $\delta < HK$.*
3. *For $H \neq 1/2$ and $K \in (0,1]$, $B^{H,K}$ is not stationary increments.*
4. *The process $B^{H,K}$ is a quasi-helix in the sense of Kahane. Let $T > 0$. For every $s,t \in [0,T]$, we have*
$$2^{-K}(t-s)^{2HK} \leq \mathbb{E}\left[B^{H,K}(t) - B^{H,K}(s)\right]^2 \leq 2^{1-K}(t-s)^{2HK}. \tag{1.38}$$
*(see (Kahane 1981, Kahane 1993) for various properties and applications of quasi-helices)*
5. *For every $H \in (0,1)$ and $K \in (0,1]$,*
$$\lim_{\epsilon \longrightarrow 0} \sup_{t\in[t_0-\epsilon,t_0+\epsilon]} \left|\frac{B_t^{H,K} - B_{t_0}^{H,K}}{t-t_0}\right| = +\infty$$
*with probability one for every $t_0$.*
6. *If $HK < 1/2$ (or $HK = 1/2$ with $K \neq 1$), $B^{H,K}$ is a short-memory process*
$$\sum_{j=m}^{\infty} \mathbb{E}\left(B_{j+1}^{H,K} - B_j^{H,K}\right)\left(B_{m+1}^{H,K} - B_m^{H,K}\right) < \infty, \text{ for all } m \geq 0.$$
7. *If $HK > 1/2$, the process $B^{H,K}$ exhibits long-range dependence*
$$\sum_{j=m}^{\infty} \mathbb{E}\left(B_{j+1}^{H,K} - B_j^{H,K}\right)\left(B_{m+1}^{H,K} - B_m^{H,K}\right) = \infty, \text{ for all } m \geq 0.$$
8. *The process $B^{H,K}$ is not Markov and it is not a semi-martingale if $HK \neq 1/2$.*

#### 1.1.5.2 The canonical Hilbert space

Fix a time interval $[0,T]$, we denote by $\mathcal{E}$ the set of step function on [0, T]. Let $\mathcal{H}_{B^{H,K}}$ be the canonical Hilbert space associated to the bfBm defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}_{B^{H,K}}} = R_{H,K}(t,s).$$

We know that the covariance of bifractional Brownian motion can be written as

$$\mathbb{E}\left[B_t^{H,K} B_s^{H,K}\right] = \int_0^T \int_0^T \mathbf{1}_{[0,t]}(u)\mathbf{1}_{[0,s]}(v)\phi_{H,K}(u,v)dudv = R_{H,K}(t,s),$$

such that

$$\phi_{H,K}(t,s) \quad := \quad \frac{\partial^2 R_{H,K}(t,s)}{\partial t \partial s} = \alpha_{H,K}\left(t^{2H}+s^{2H}\right)^{K-2}(ts)^{2H-1} + \beta_{H,K}|t-s|^{2HK-2},$$

where

$$\alpha_{H,K} = 2^{-K+2}H^2K(K-1) \quad \text{and} \quad \beta_{H,K} = 2^{-K+1}HK(2HK-1).$$

The application $\varphi \in \mathcal{E} \longrightarrow B^{H,K}(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^{H,K}$ and it can be extended to $\mathcal{H}_{B^{H,K}}$. Sometimes working with the space $\mathcal{H}_{B^{H,K}}$ is not convenient; once, because this space may contain also distributions (as, e.g. in the case K = 1, see (Pipiras & Taqqu 2000)) and twice, because the norm in this space is not always tractable.

When $1/2 < HK < 1$ We will use the subspace $|\mathcal{H}_{B^{H,K}}|$ of $\mathcal{H}_{B^{H,K}}$ which is defined as the set of measurable function $\varphi$ on $[0,T]$ with

$$\|\varphi\|_{|\mathcal{H}_{B^{H,K}}|} := \int_0^T \int_0^T |\varphi(u)||\varphi(v)|\,\phi_{H,K}(u,v)dudv < \infty,$$

Note that, if $\varphi,\ \psi \in |\mathcal{H}_{B^{H,K}}|$, then their scalar product in $\mathcal{H}_{B^{H,K}}$ is given by

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{B^{H,K}}} = \int_0^T \int_0^T \varphi(u)\psi(v)\phi_{H,K}(u,v)dudv,$$

For $\varphi,\ \psi \in |\mathcal{H}_{B^{H,K}}|$, we have

$$\mathbb{E}\left(\int_0^T \varphi(u)dB_u^{H,K}\right) = 0,\ \mathbb{E}\left(\int_0^T \varphi(u)dB_u^{H,K}\int_0^T \psi(v)dB_v^{H,K}\right) = \langle \varphi, \psi \rangle_{\mathcal{H}_{B^{H,K}}}.$$

It actually follows from (Kruk, Russo & Tudor 2007), (Pipiras & Taqqu 2000) that the space $\mathcal{H}_{B^{H,K}}$ is a Banach space for the norm $\|.\|_{|\mathcal{H}_{B^{H,K}}|}$ and it is included in $\mathcal{H}_{B^{H,K}}$. In fact,

$$\mathbb{L}^2([0,T]) \subset \mathbb{L}^{1/HK}([0,T]) \subset |\mathcal{H}_{B^{H,K}}| \subset \mathcal{H}_{B^{H,K}},$$

where $H \in (0,1)$, $K \in (0,1]$ and $HK \in (1/2,1)$.

### 1.1.6 Weighted fractional Brownian motion

As an extension of fBm, (Bojdecki et al. 2007) introduced and studied a special class of self-similar Gaussian processes which is called the weighted-fractional Brownian motion (wfBm in short) $B^{a,b}$. This process appeared in (Bojdecki, Gorostiza & Talarczyk 2008b) in a high density limit of occupation time fluctuations of the above mentioned particles system, where the initial Poisson configuration has finite intensity measure. Moreover, the wfBm has properties analogous to those of the fBm (self-similarity, long-range dependence, Hölder paths). However, in comparison with the fBm, the wfBm has non-stationary increments and satisfies the following estimates (see (Bojdecki et al. 2007)):

$$c_{a,b}(t\vee s)^a|t-s|^{b+1} \le \mathbb{E}\left[B_t^{a,b}-B_s^{a,b}\right]^2 \le C_{a,b}(t\vee s)^a|t-s|^{b+1}.$$

More studies on wfBm could be found in (Bojdecki, Gorostiza & Talarczyk 2008a), (Bojdecki et al. 2008b), (Garzón 2009), (Shen, Yan & Cui 2013), and (Yan, Wang & Jing 2014). Let the function $R^{a,b}$ defined by

$$R_{a,b}(t,s) = \int_0^{s\wedge t} u^a\left[(t-u)^b+(s-u)^b\right]du, \quad \forall s,t\in[0,T]. \tag{1.39}$$

Next, we present conditions under which this function (symmetric, continuous) is the covariance of a stochastic process.

**Theorem 1.6 ((Bojdecki et al. 2007)).** *The function $R_{a,b}$ defined by (1.39) is positive-definite if and only if $a$ and $b$ satisfy the conditions*

$$a>-1, \ \ -1<b\le 1, \ \ |b|\le a+1. \tag{1.40}$$

**Proof.** The proof of this theorem given in (Bojdecki et al. 2007).

**Definition 1.9.** The weighted fractional Brownian motion (wfBm for short) $B_t^{a,b}$ with parameters $a$ and $b$, such that $a>-1$, $\mid b\mid< 1$, and $\mid b\mid< a+1$ is a centered and self-similar Gaussian process with long/short-range dependence and the covariance function:

$$R_{a,b}(t,s) = \mathbb{E}\left(B_t^{a,b}B_s^{a,b}\right) = \int_0^{s\wedge t} u^a\left[(t-u)^b+(s-u)^b\right]du, \quad \forall s,t\in[0,T].$$

**Remark 1.5.** i) Clearly, for $a=b=0$, $B^{a,b}$ coincides with the standard Brownian motion $B$. When $a=0$, we have

$$\mathbb{E}\left(B_t^{0,b}B_s^{0,b}\right) = \frac{1}{b+1}\left[t^{b+1}+s^{b+1}-|s-t|^{b+1}\right]; \quad \forall s,t\in[0,T],$$

which is the covariance function of the fBm with Hurst index $\dfrac{b+1}{2}$ when $|b|<1$.

ii) Analogously to the case of fBm, in the definition of wfBm is excluded $b = 1$. It is easy to see that for $b = 1$ and $a \geq 0$, (1.39) is the covariance of the process

$$B_t^{a,1} = \int_0^t W_{r^a} dr,$$

where $W$ is a standard Brownian motion.

iii) For $b = 0$, $B^{a,0}$ is a time-inhomogeneous standard Brownian motion.

#### 1.1.6.1 Main properties of the wfBm

**Proposition 1.7 ((Bojdecki et al. 2007)).** *The weighted fractional Brownian motion $B_t^{a,b}$ with parameters $a$ and $b$ has the following properties:*

1. *The process $B^{a,b}$ is $(a+b+1)/2$-self-similar.*

$$\left\{B_{\alpha t}^{a,b}, t \geq 0\right\} \stackrel{\Delta}{=} \left\{\alpha^{(a+b+1)/2} B_t^{a,b}, t \geq 0\right\}, \text{ for each } \alpha > 0. \tag{1.41}$$

2. *Second moments of increments: for $0 \leq s < t$*

$$\mathbb{E}\left(B_t^{a,b} - B_s^{a,b}\right)^2 = 2\int_s^t u^a (t-u)^b \, du,$$

*and*

$$\mathbb{E}\left(B_t^{a,b} - B_s^{a,b}\right)^2 \leq C\,|t-u|^{b+1},$$

*if $a \geq 0$, $s,t \leq T$ for any $T > 0$ with $C = C(T)$, and also if $a < 0$, $s,t \geq \epsilon > 0$ for any $\epsilon > 0$, with $C = C(\epsilon)$;*

$$\mathbb{E}\left(B_t^{a,b} - B_s^{a,b}\right)^2 \leq C\,|t-u|^{a+b+1}, \ s,t \geq 0,$$

*if $a < 0$, $1 + a + b > 0$;*

$$\mathbb{E}\left(B_t^{a,b} - B_s^{a,b}\right)^2 \geq C\,|t-u|^{b+1}, \ s,t \geq 0,$$

*if $a > 0$, $s,t \geq \epsilon > 0$ for any $\epsilon > 0$ with $C = C(\epsilon)$ and also if $a \leq 0$, $s,t \leq T$ for any $T > 0$ with $C = C(T)$.*

3. *Path continuity: $B^{a,b}$ is a continuous process with the only exception of the case $a < 0$, $b < 0$, $a + b = -1$, where $B^{a,b}$ is discontinuous at 0.*

4. *If $b \neq 0$ the process $B^{a,b}$ is not Markov and it is not a semi-martingale.*

5. *Covariance of increments: For $0 \leq r < v \leq s < t$,*

$$R_{a,b}(r,v,s,t) = \mathbb{E}\left[\left(B_t^{a,b} - B_s^{a,b}\right)\left(B_v^{a,b} - B_r^{a,b}\right)\right] = \int_r^v u^a \left[(t-u)^b - (s-u)^b\right] du,$$

*hence*

$$R_{a,b}(r,v,s,t)\begin{cases} >0 & \textit{if } b>0,\\ =0 & \textit{if } b=0,\\ <0 & \textit{if } b<0. \end{cases} \tag{1.42}$$

6. *The process $B^{a,b}$ is not Markov and it is not a semi-martingale if $b \neq 0$.*

#### 1.1.6.2 The canonical Hilbert space

Let $B^{a,b}$ be a wfBm defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with parameters $a$ et $b$ $(a > -1,\ 0 < b < 1,\ b < a+1)$. It is possible for researchers to construct a variety of stochastic calculus with respect to the wfBm $B^{a,b}$ associated with the Malliavin calculus. More studies and references can be found in (Nualart 2006), (Mazet et al. 2001) and the references therein. Here, we need to review the basic concepts and some results of the Malliavin calculus. The crucial ingredient is the canonical Hilbert space $\mathcal{H}_{B^{a,b}}$ associated with the wfBm $B^{a,b}$ defined as the closure of the linear space $\mathcal{E}$ generated by the indicator functions $\{\mathbf{1}_{[0,t]},\ t \in [0,T]\}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}_{B^{a,b}}} = R_{a,b}(t,s)$$

The mapping $\varphi \in \mathcal{E} \longrightarrow B^{a,b}(\varphi) = \displaystyle\int_0^T \varphi(s)dB_s^{a,b}$ ( $B^{a,b}(\varphi)$ is a Gaussian process on $\mathcal{H}_{B^{a,b}}$ , and $\mathbb{E}\left(B^{a,b}(\varphi)B^{a,b}(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}_{B^{a,b}}}$ for all $\varphi, \in \psi \in \mathcal{H}$ ) is an isometry from the space $\mathcal{E}$ to the Gaussian space generated by the wfBm $B_t^{a,b}$, and it can be extended to the Hilbert space $\mathcal{H}_{B^{a,b}}$. We can find a linear space of functions contained in $\mathcal{H}_{B^{a,b}}$ in the following way. Let $|\mathcal{H}_{B^{a,b}}|$ be the linear space of measurable functions $\varphi$ on $[0,T]$ such that

$$\|\varphi\|^2_{|\mathcal{H}_{B^{a,b}}|} := \int_0^T \int_0^T |\varphi(s)||\varphi(r)|\phi_{a,b}(s,r)dsdr < \infty, \tag{1.43}$$

with

$$\phi_{a,b}(s,r) = b(s \wedge r)^a(s \vee r - s \wedge r)^{b-1}.$$

It is not difficult to show that $|\mathcal{H}_{B^{a,b}}|$ is a Banach space with the norm $\|\varphi\|_{|\mathcal{H}_{B^{a,b}}|}$ and $\mathcal{E}$ is dense in $|\mathcal{H}_{B^{a,b}}|$. (see (Guangjun, Xiuwei & Litan 2016) and (Pipiras & Taqqu 2000)). Moreover,

$$\mathbb{L}^2([0,T]) \subset \mathbb{L}^{\frac{2}{a+b+1}}([0,T]) \subset |\mathcal{H}_{B^{a,b}}| \subset \mathcal{H}_{B^{a,b}},$$

## 1.2 Parametric estimation for stochastic differential equations

### SDEs driven by the standard Brownian motion

Parameter estimation in SDEs driven by the standard Brownian motion was first studied by (Arató, Kolmogorov & Sinai 1962) who applied it to a geophysical problem. For long time asymptotic of parameter estimation in stochastic differential equations see the books by (Kabanov, Liptser & Shiryaev 1978), (Basawa & Prakasa Rao 1980), (Arató 1982), (Evstropov, Linkov, Morozenko & Pikus 1993), (Prakasa Rao 1999) and (Kutoyants & Spokoiny 1999). For small noise asymptotic of parameter estimation see the books by (Ibragimov & Has' Minskii 1981), (Kutoyants 1984b) (Kutoyants 1994).

### SDEs driven by the fractional Brownian motion

Drift parameter estimation in diffusion processes driven by fBm has been studied by many authors. In the case of small diffusion, (Prakasa Rao 2003) investigated the asymptotic properties of the maximum likelihood estimator and Bayes estimator of the drift parameter for stochastic processes satisfying linear SDEs driven by fBm. Then, (Prakasa Rao 2005) investigated the asymptotic properties of the minimum $\mathbb{L}_1$-norm estimator of the drift parameter for fractional Ornstein-Uhlenbeck type process satisfying a linear SDE driven by a fBm. Later, (Mishra & Prakasa Rao 2006) investigated the probabilities of large deviations of the maximum likelihood estimator and Bayes estimator of the drift parameter for a fractional Ornstein-Uhlenbeck type process. After that, (Prakasa Rao 2007) investigated the asymptotic properties of instrumental variable estimators of the drift parameter for stochastic processes satisfying linear SDEs driven by fBm. Then, (Kouame, N'Zi & Yode 2008) studied the properties of the minimum distance estimator of the drift parameter concerning the diffusion processes driven by a fBm as the diffusion coefficient tends to zero. (Shen & Xu 2014) studied the minimum $\mathbb{L}_1$-norm estimator of the drift parameter of a fractional Ornstein-Uhlenbeck type process and proved the asymptotic law of limit distribution for $T \to +\infty$, when $\varepsilon \to 0$.

In the discrete time instants, (Filatova, Grzywaczewski, Shybanova & Zili 2007) introduced a methodology for estimating the parameters of SDEs driven by fBm. The idea is connected with simulated maximum likelihood.Then, (Yaozhong, David, Weilin & Weiguo 2011) devoted the problems of consistency and strong consistency of the maximum likelihood estimators of the mean and variance of the drift fBm. Later, (Kozachenko, Melnikov & Mishura 2015) derived the standard maximum likelihood estimate and proposed non-standard estimates for the unknown drift parameter in a stochastic differential equation involving fractional Brownian motion.

In the continues observations, (Belfadli, Es-Sebaiy & Ouknine 2011) studied the consistency and the asymptotic distributions of the least-squares estimator for the non-ergodic fractional Ornstein-Uhlenbeck process if $H \in (1/2, 1)$. Then, (Neuenkirch & Tindel 2014) established the strong consistency of least square-type estimator for an unknown parameter in the drift coefficient of a stochastic differential equation with additive fractional noise of Hurst parameter $H > 1/2$. Later, (Kukush, Mishura, Ralchenko et al. 2017) was proposed a new method to test the hypothesis of the sign of the parameter and proved the consistency of the test, and also studied the estimators for drift parameter for continuous and discrete observations and proved their strong consistency for all $H \in (0, 1)$. Drift Parameter estimation based on a data recorded from continuous (discrete) trajectories for fractional Ornstein-Uhlenbeck process of the second kind was discussed in (Azmoodeh & Morlanes 2015), (Azmoodeh & Viitasaari 2015), and references therein.

### SDEs driven by the sub-fractional Brownian motion

In case of diffusion type processes driven by sfBm, many authors have investigated parametric estimation for the drift parametric. (Diedhiou, Manga & Mendy 2011) studied the maximum likelihood estimator for SDEs with additive sfBm. Then, (Nenghui Kuang2013) constructed a maximum likelihood estimator for the drift parameter by using a random walk approximation of the sfBm and studied the asymptotic behaviours of the estimator. Later, (Kuang & Xie 2015) investigated the $\mathbb{L}^2$-consistency and the strong consistency of the maximum likelihood estimators of the mean and variance of the sfBm with drift at discrete observation. After that, (Prakasa Rao 2017c) established the asymptotic properties of the maximum likelihood estimator and Bayes estimator of the drift parameter for stochastic processes satisfying linear SDEs driven by a sfBm and also obtained a Bernstein–von Mises type theorem for this class of processes. Recently, (Prakasa Rao 2018b) investigated the asymptotic properties of instrumental variable estimators of the drift parameter for stochastic processes satisfying linear SDEs driven by a sfBm. Then, (Li & Dong 2018) tackled the least squares estimators of the Vasicek-type model driven by sfBm. Very recently, (Prakasa Rao 2018a) obtained a Berry-Esseen type bound for the distribution of the maximum likelihood estimator of the drift parameter for fractional Ornstein-Uhlenbeck type process driven by sfBm.

## SDEs driven by the mixed fractional Brownian motion

Parametric inference for stochastic processes satisfying SDEs driven by a mixed fBm have been studied earlier and a comprehensive survey of various methods is given in (Mishura & Zili 2018) and (Prakasa Rao 2011). There has been a recent interest in the study of similar problems for stochastic processes driven by a mixed fractional Brownian motion (mfBm). (Rudomino-Dusyats'ka 2004) studied the properties of maximum likelihood estimates in diffusion and fractional-Brownian models. Then, the asymptotic properties of the minimum $\mathbb{L}_1$-norm estimator of the drift parameter for mixed fractional Ornstein-Uhlenbeck type process satisfying a linear stochastic differential equation driven by a mixed fractional Brownian motion were obtained in (Miao 2010). Later, under some regularity conditions and the diffusion coefficient tends to zero, (Song & Liu 2014) studied the asymptotic properties of minimum distance estimator of drift parameter for a class of nonlinear scalar stochastic differential equations driven by mixed fractional Brownian motion. After that, (Marushkevych 2016) investigated large deviation properties of the maximum likelihood drift parameter estimator for Ornstein–Uhlenbeck process driven by mixed fractional Brownian motion. Then, the method of instrumental variable estimation for such parametric models was presented in (Prakasa Rao 2017a). In addition, Maximum likelihood estimation for estimation of drift parameter in a linear stochastic differential equations driven by a mfBm was investigated in (Prakasa Rao 2018c). For related works on parametric inference for processes driven by mfBm, see (Prakasa Rao 2009), (Mishra & Prakasa Rao 2017) and (Chigansky & Kleptsyna 2019).

## SDEs driven by the weighted-fractional Brownian motion

Recently, there has been a large interest in the study of estimation of parameters for SDEs driven by weighted fractional Brownian motion. (Shen et al. 2013) derived the explicit bounds for the Kolmogorov distance in the central limit theorem and obtained the almost sure central limit theorem for the quadratic variation of the weighted fractional Brownian motion. (Guangjun et al. 2016) established the consistency and the asymptotic distribution of the least squares estimator for the Ornstein-Uhlenbeck process. Then, (Mei & Yin 2016) studied the asymptotic properties of a least squares estimator for the $\alpha$-weighted fractional bridge with discrete observations. Further, (Cheng, Shen & Chen 2017) studied the parameter estimation for nonergodic Ornstein-Uhlenbeck process driven by the weighted fractional Brownian motion. Recently, (Sun & Yan 2018) considered the problem of central limit theorems and parameter estimation associated with a weighted-fractional Brownian motion. Very recently, (Alsenafi, Al-Foraih & Es-Sebaiy 2020) provided the least squares estimation for non-ergodic weighted fractional Ornstein-Uhlenbeck process of general parameters.

## 1.3 Nonparametric estimation for stochastic differential equations

Over the last two decades, many authors studied nonparametric estimators from observations drawn from stochastic differential equations driven by Brownian motion. The first results of nonparametric estimation for diffusion processes appeared in (Banon 1978), (Banon & Nguyen 1978) and (Banon & Nguyen 1981).

### Continues observations

Nonparametric estimation for the coefficients of stochastic differential equations based on a data recorded from continuous trajectories is an interesting problem in the realm of mathematical statistics. For the small noise, the first reference in this context is (Kutoyants 1994) who proposed different nonparametric estimators for SDEs with small noise. After that, (Mishra & Prakasa Rao 2011b) studied the asymptotic behaviour of the kernel type estimator of trend function for stochastic differential equations driven by fractional Brownian motion. Later, (Mishra & Prakasa Rao 2011a) discussed the problem of nonparametric estimation of linear multiplier function for fractional diffusion processes with known Hurst index $H \in (1/2, 3/4)$ and studied the asymptotic behaviour of the estimator as $\varepsilon \to 0$. Then, (Prakasa Rao 2019c) considered nonparametric estimation of trend coefficient in models governed by a stochastic differential equation driven by a mixed fractional Brownian motion with small noise. Later, (Prakasa Rao 2019b) discussed the problem of nonparametric estimation of linear multiplier function for processes driven by sub-fractional Brownian motion. Nonparametric estimation of linear multiplier for processes driven by mixed fractional Brownian motion was investigated in (Prakasa Rao 2019a). Then, (Kutoyants 2019) dealt with the problem of drift function estimation of inhomogeneous stochastic differential equation with delay. It is shown that kernel-type estimator is consistent and asymptotically efficient. Very recently, (Marie 2020) proved the consistency, a rate of convergence and the asymptotic distribution of a nonparametric estimator of the trend in the Skorokhod reflection problem defined by a fractional SDE and a Moreau sweeping process.

For long time asymptotic of nonparameter estimation in stochastic differential equations, (Reiß 2002) established the minimax rates for nonparametric estimation of the drift functional in affine stochastic delay equations. Then, (Saussereau 2014) studied locally linear estimator and the Nadaraya–Watson kernel type estimator of the drift function for stochastic differential equations driven by a fractional Brownian motion. Later, (Comte & Marie 2019) established the consistency and a rate of convergence for a Nadaraya-Watson estimator of the drift function of a stochastic differential equation driven by an additive fractional noise. Recently, (Comte & Marie 2020) considered nonparametric projection estimators of the drift function computed from independent continuous observations, on a compact time interval, of the solution of a stochastic differential equation driven by the fractional Brownian motion and established the consistency and rate of convergence a for these estimators in the case of the compactly supported trigonometric basis or the $\mathbb{R}$−supported Hermite basis.

## Discrete observations

Nonparametric estimation based on discretely observed diffusion is very important from a practical point of view and it has also benefited from numerous studies. Many authors have investigated nonparametric techniques to estimate the SDEs. (Dinh Tuan 1981) proposed an estimator of the diffusion coefficient based on discrete observation of SDE throughout a given finite time interval, and proved the consistency and asymptotic normality of this estimator. In (Kutoyants 1984a), the trend coefficient in a diffusion process was estimated from the $N$ i.i.d. copies of the process. Then, (Nicolau 2007) proposed the nonparametric estimators of the infinitesimal coefficients associated with second-order stochastic differential equations, and stated appropriate conditions ensuring the asymptotic normality of these estimators. Later, (Wang & Lin 2011) presented the local linear estimations for the diffusion coefficient and drift coefficient in the second-order diffusion model, and showed that under mild conditions, the estimators are weak consistent. Then, (Dehay & El Waled 2013) constructed a kernel estimator of a periodic signal for SDE driven by a standard Brownian motion, and stated the consistency as well as the asymptotic normality of this estimator. In addition, (Gugushvili & Spreij 2014) established the posterior consistency for nonparametric Bayesian estimation of the drift coefficient of a multidimensional stochastic differential equation from discrete-time observations on the solution of this equation.

## Partial stochastic differential equations

The nonparametric estimation of the coefficients of partial stochastic differential equations has attracted a great deal of attention in the past. (Huebner & Lototsky 2000) studied the problem of estimating a coefficient of a strongly elliptic partial differential operator in stochastic parabolic equations. They computed the maximum likelihood estimate of the function on an approximating space (sieve) using a finite number of the spatial Fourier coefficients of the solution and established the consistency and the asymptotic normality of the resulting estimate as the number of the coefficients increases. (Prakasa Rao 2001b) used the Bernstein-von Mises theorems for parabolic SPDE and presented some new results on the problem of estimation of a linear multiplier for a class of SPDE using the methods of nonparametric inference following the approach of (Kutoyants 1994). Then, (Prakasa Rao 2001a) obtained a kernel-type estimator for the linear multiplier function of stochastic partial differential equation and established the consistency, rate of convergence and asymptotic normality of this estimator as $\varepsilon \longrightarrow 0$. After that, (Prakasa Rao 2002) discussed nonparametric estimation of a function $\theta(t)$ involved in the "forcing" term for a class of SPDE's and studied its asymptotic properties based on the Fourier coefficients of the random field observed at discrete times. Recently, (Wang & Jiang 2018) obtained a kernel estimator and the convergence rates of the time-varying coefficient a parabolic diagonalizable stochastic equation driven by fractional noises.

### SDEs with random effects

Nonparametric estimation in SDEs with random effects have been proposed in literature. The main purposes are the estimation of the density of the random effects in a nonparametric way. The first reference in this context is (Comte, Genon-Catalot & Samson 2013) who studied the nonparametric estimation of the density of the random effect in two kinds of mixed models. Then, (Dion 2016) proposed two adaptive nonparametric estimation methods for the density of the random effects in a mixed-effect Ornstein–Uhlenbeck model. Further, (Dion & Genon-Catalot 2016) dealt with the problem of nonparametric estimation for a mixed stochastic differential model with two random effects in the drift. Recently, (El Omari, El Maroufy & Fuchs 2019) build ordinary kernel estimators and histogram estimators for a class of fractional stochastic differential equations with random effects and studied their $L^p$ risk ( p = 1 or 2), when $H > 1/2$.

## 1.4 Kernel-type estimators

One of the most popular methods in estimating the functions is the kernel method. This method was first introduced in 1951 in a paper not published by Fix and Hodges (see (Silverman & Jones 1989) ). Later, the first form of the kernel estimator was introduced in the work of (Rosenblatt et al. 1956) and (Parzen 1962) which have defined a real application $G$ satisfying the condition

$$\int_{\mathbb{R}} G(u)du = 1 \tag{1.44}$$

and has since been known as the Parzen-Rosenblatt kernel. Let $(X_i)_{i\geq 1}$ be a stationary sequence of random variables. We denote by $\mathbb{P}$ the law of probability on $\mathbb{R}$ governing this process and $f$ its density with respect to the Lebesgue measure. The Parzen-Rosenblatt estimator is defined, for all $x \in \mathbb{R}$, by

$$\hat{f}_n(x) = \frac{1}{nh_n} \sum_{i=1}^{n} G\left(\frac{x - X_i}{h_n}\right),$$

where $h_n$, the bandwidth, is a parameter tending to 0 when $n$ tends to $+\infty$. The first version of this estimator was given by (Rosenblatt et al. 1956), by choosing the uniform kernel $G = \frac{1}{2}\mathbf{I}_{[-1,1]}$. (Banon 1978), was the first to be interested in the estimation of the density in continuous time from the observation of a stationary process $(X_t)_{0\leq t\leq T}$. Similarly, in the discrete case, the form of this estimator is given, for all $x \in \mathbb{R}$, by

$$\hat{f}_T(x) = \frac{1}{Th_T} \int_0^T G\left(\frac{x - X_t}{h_T}\right) dt.$$

The first results of nonparametric estimation for diffusion processes were appeared in (Banon 1978), (Banon & Nguyen 1978), (Banon & Nguyen 1981).

Thereafter, (Ibragimov & Has' Minskii 1981) studied simultaneously nonparametric results for two models: when an observed signal $X(t)$ on the interval $[0,1]$ is of the form

$$dX_t = S(t)dt + \varepsilon dW_t, \;\; 0 \leq t \leq 1, \tag{1.45}$$

with $\varepsilon \longrightarrow 0$ and when $S(t)$ is a periodic function of period 1 and the observation spans over $n$ ($n \longrightarrow \infty$) periods:

$$dX_t = S(t)dt + dW_t, \;\; 0 \leq t \leq n. \tag{1.46}$$

Then, (Ibragimov & Has' Minskii 1981) introduced the Kernel-type estimators

$$\hat{F}_1 := \frac{1}{\varphi_\varepsilon} \int_0^1 G\left(\frac{t - t_0}{\varphi_\varepsilon}\right) dX_t, \tag{1.47}$$

in the case of observations (1.45) and

$$\hat{F}_2 := \frac{1}{n\psi_n} \sum_{i=1}^{n} \int_i^{i+1} G\left(\frac{t - t_0 - i}{\psi_n}\right) dX_t, \tag{1.48}$$

in the case of observations (1.46). Where $\phi_\varepsilon$ and $\psi_n$ be functions which approach zero as $\varepsilon \longrightarrow 0$ and $n \longrightarrow \infty$ respectively and let $G(u)$, $u \in \mathbb{R}$, as a bounded function of finite support satisfying for simplicity the condition

$$\int_{-\infty}^{+\infty} G(u) = 1.$$

These estimators are uniformly asymptotically unbiased and uniformly converge to a quadratic mean. After that, (Kutoyants 1994) extended the work of (Ibragimov & Has' Minskii 1981) to continuous observation of $\{X_t\}$ on the interval $[0,T]$ defined by

$$dX_t = S(X_t)dt + \varepsilon dW_t, \ \ 0 \le t \le T. \tag{1.49}$$

(Kutoyants 1994) introduced the kernel type estimator of the unknown function $S(.)$ by

$$\hat{S}_t := \frac{1}{\varphi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon}\right) dX_\tau, \tag{1.50}$$

and under certain regularity conditions, (Kutoyants 1994) proved that this estimator is consistent and its rate of convergence is optimal in appropriate sense. This work has been expanded to nonparametric estimation of trend function involved in processes driven by fractional Brownian motion, mixed fractional Brownian, sub-fractional Brownian. (see, (Mishra & Prakasa Rao 2011a, Mishra & Prakasa Rao 2011b, Prakasa Rao 2019b, Prakasa Rao 2019d)).

## 1.5 Description of results obtained in this thesis

### a) Asymptotic analysis of a kernel estimator for stochastic differential equations driven by a mixed sub-fractional Brownian motion

In this work, we are interested in the problem of estimating the trend function $b_t = b(x_t)$ for process satisfying stochastic differential equations of the type

$$dX_t = b(X_t)dt + \varepsilon dS_t^H(a), \ X_0 = x_0, \ 0 \le t \le T, \tag{1.51}$$

where $\left\{S_t^H(a), t \ge 0\right\}$ is a mixed sub-fractional Brownian motion with known parameters $N$, $a$ and $H$, such that $N \in \mathbb{N}^*$, $H \in (1/2,1)^N$, and $a \in \mathbb{R}^N \setminus \left\{0_N\right\}$. By the observations $X = \{X_t, \ 0 \le t \le T\}$ of this process we will estimate the unknown function $b(x_t)$ by a kernel estimator and obtain the uniform convergence, rate of convergence and asymptotic normality of the kernel estimator (as $\varepsilon \longrightarrow 0$).

Let $S_t^H(a)$ be a msfBm with Hurst index $H_i \in (1/2,1)$ and $a_i \neq 0$ for all $i \in \{1,...,N\}$, we put

$$H_{i_*} = \min\{H_i : \ i \in \{1,...,N\}; \ 1/2 < H_i < 1 \text{ and } a_i \neq 0\}.$$

Using the method developed by (Kutoyants 1994). Then, the kernel estimator of $b_t$ is given by

$$\hat{b}_t = \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau, \tag{1.52}$$

where $G(u)$ is a bounded function with finite support $[A,B]$.

※ **Uniform convergence**

**Theorem 1.8.** *Under certain assumptions in Chapter 2, we have:*

$$\lim_{\varepsilon \longrightarrow 0} \sup_{b(x)\in\Sigma_0(L)} \sup_{c\le t\le d} \mathbb{E}_b(|\hat{b}_t - b(x_t)|^2) = 0.$$

※ **The rate of convergence**

**Theorem 1.9.** *Under certain assumptions in Chapter 2, we have:*

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{b(x)\in\Sigma_k(L)} \sup_{c\le t\le d} \mathbb{E}_b(|\hat{b}_t - b(x_t)|^2)\varepsilon^{\frac{-2(k+1)}{k-H_{i_*}+2}} < \infty.$$

※ **The asymptotic normality**

**Theorem 1.10.** *Under certain assumptions in Chapter 2, we have:*

$$\varepsilon^{\frac{-(k+1)}{k-H_{i_*}+2}}\left(\hat{b}_t - b(x_t)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(m, \sigma^2_{H,N,a}\right),$$

*where*

$$m = \frac{b^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G(u)u^{k+1}du,$$

*and*

$$\sigma^2_{H,N,a} = \left(\sum_{i=1}^N a_i^2\right) H_{i_*}(2H_{i_*}-1)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\left(|u-v|^{2H_{i_*}-2} - |u+v|^{2H_{i_*}-2}\right)dudv,$$

*as* $\varepsilon \longrightarrow 0$, *with* $\xrightarrow{\mathcal{D}}$ *denotes the convergence in distribution.*

## b) Nonparametric estimation of trend function for stochastic differential equations driven by a bifractional Brownian motion

The main objective of this work is to investigate the problem of estimating the trend function $S_t = S(x_t)$ for process satisfying stochastic differential equations of the type

$$dX_t = S(X_t)dt + \varepsilon dB_t^{H,K},\ X_0 = x_0,\ 0 \le t \le T, \tag{1.53}$$

where $\left\{B_t^{H,K}, t \ge 0\right\}$ is a bifractional Brownian motion with known parameters $H \in (0,1)$, $K \in (0,1]$ and $HK \in (1/2, 1)$. We estimate the unknown function $S(x_t)$ by a kernel estimator $\hat{S}_t$ and obtain the asymptotic properties as $\varepsilon \longrightarrow 0$.

The kernel estimator of $S_t$ is given by

$$\hat{S}_t = \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau, \tag{1.54}$$

where $G(u)$ is a bounded function with finite support $[A, B]$

※ **Uniform convergence**

**Theorem 1.11.** *Under certain assumptions in Chapter 3, we have:*

$$\lim_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_0(L)} \sup_{c\le t\le d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2) = 0.$$

※**The rate of convergence**

**Theorem 1.12.** *Under certain assumptions in Chapter 3, we have:*

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_k(L)} \sup_{c\le t\le d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2)\varepsilon^{\frac{-2(k+1)}{k-HK+2}} < \infty.$$

※ **The asymptotic normality**

**Theorem 1.13.** *Under certain assumptions in Chapter 3, we have:*

$$\varepsilon^{\frac{-(k+1)}{k-HK+2}}\left(\hat{S}_t - S(x_t)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(m, \sigma^2_{H,K}\right), \text{ as } \varepsilon \longrightarrow 0,$$

*where*

$$m = \frac{S^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G(u)u^{k+1}du,$$

*and*

$$\sigma^2_{H,K} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\left[\alpha_{H,K}\left(u^{2H}+v^{2H}\right)^{K-2}(uv)^{2H-1} + \beta_{H,K}|u-v|^{2HK-2}\right] du dv,$$

*where*

$$\alpha_{H,K} = 2^{-K+2}H^2K(K-1) \quad \text{and} \quad \beta_{H,K} = 2^{-K+1}HK(2HK-1).$$

## c) Nonparametric estimation of trend function for stochastic differential equations driven by a weighted fractional Brownian motion

In this work, we investigate the problem of estimating the trend function $S_t = S(x_t)$ for process satisfying stochastic differential equations of the type

$$dX_t = S(X_t)dt + \varepsilon dB^{a,b}_t, \; X_0 = x_0, \; 0 \le t \le T, \tag{1.55}$$

where $\left\{B^{a,b}_t, t \ge 0\right\}$ is a weighted fractional Brownian motion with known parameters $a$ and $b$, such that $a > -1$, $0 < b < 1$, $b < a+1$ and $a+b > 0$. We estimate the unknown function $S(x_t)$ by a kernel estimator $\hat{S}_t$ and obtain the asymptotic properties as $\varepsilon \longrightarrow 0$.

Using the method developed by (Kutoyants 1994). Then, the kernel estimator of $S_t$ is given by

$$\hat{S}_t = \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau, \tag{1.56}$$

where $G(u)$ is a bounded function with finite support $[A, B]$

⋇ **Uniform convergence**

**Theorem 1.14.** *Under certain assumptions in Chapter 4, we have:*

$$\lim_{\varepsilon\longrightarrow 0}\sup_{S(x)\in\Sigma_0(L)}\sup_{c\le t\le d}\mathbb{E}_S(|\hat{S}_t - S(x_t)|^2) = 0.$$

⋇**The rate of convergence**

**Theorem 1.15.** *Under certain assumptions in Chapter 4, we have:*

$$\limsup_{\varepsilon\longrightarrow 0}\sup_{S(x)\in\Sigma_k(L)}\sup_{c\le t\le d}\mathbb{E}_S(|\hat{S}_t - S(x_t)|^2)\varepsilon^{\frac{-4(k+1)}{2k-a-b+3}} < \infty.$$

⋇ **The asymptotic normality**

**Theorem 1.16.** *Under certain assumptions in Chapter 4, we have:*

$$\varepsilon^{\frac{-2(k+1)}{2k-a-b+3}}\left(\hat{S}_t - S(x_t)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(m, \sigma^2_{a,b}), \text{ as } \varepsilon \longrightarrow 0,$$

*where*

$$m = \frac{S^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G(u)u^{k+1}du,$$

*and*

$$\sigma^2_{a,b} = b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)(u\wedge v)^a(u\vee v - u\wedge v)^{b-1}dudv,$$

## d) Asymptotic analysis of a kernel type estimator of trend function and linear multiplier for stochastic differential equation with additive fractional Brownian sheet

**1.Trend function estimation**

Let $\{X_{s,t},\ 0\le s,t\le T\}$ be a process governed by the following equation:

$$dX_{s,t} = \rho(X_{s,t})dsdt + \varepsilon dW^{\alpha,\beta}_{s,t},\ X_{0,t} = X_{s,0} = x_0,\ 0\le s,t\le T, \tag{1.57}$$

where $x_0 \in \mathbb{R}$, $\varepsilon > 0$ and $W^{\alpha,\beta}_{s,t}$ a fractional Brownian sheet, and $\rho(.)$ is an unknown function. The main goal is to build an estimator of the trend function $\rho_{s,t} = \rho(x_{s,t})$, $0 \leq s,t \leq T$ by the two dimensional kernel estimator

$$\tilde{\rho}_{s,t} = \int_0^T \int_0^T G_{\varphi,\psi}(u,v)\, dX_{u,v}, \;\; 0 \leq s,t \leq T. \tag{1.58}$$

and study the asymptotic properties as $\varepsilon \longrightarrow 0$.

※ **Consistency of the estimator**

**Theorem 1.17.** *Under certain assumptions in Chapter 5, we have:*

$$\lim_{\varepsilon \longrightarrow 0} \sup_{c \leq s,t \leq d} \mathbb{E}_\rho(|\tilde{\rho}_{s,t} - \rho(x_{s,t})|^2) = 0.$$

※ **Rate of convergence**

**Theorem 1.18.** *Under certain assumptions in Chapter 5, we have:*

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{\rho \in \mathcal{N}(k,L)} \sup_{c \leq s,t \leq d} \mathbb{E}_\rho(|\tilde{\rho}_{s,t} - \rho(x_{s,t})|^2) = \mathcal{O}\left(\varepsilon^{\frac{2(k_1+1)(k_2+1)}{(k_1+1)(k_2+1)+(k_1+1)(1-\beta)+(k_2+1)(1-\alpha)}}\right).$$

※ **Asymptotic normality**

**Theorem 1.19.** *Under certain assumptions in Chapter 5, we have:*

$$\varepsilon^{\frac{-\bar{k}}{2(\bar{k}+1)}} \left(\tilde{\rho}_{s,t} - \rho(x_{s,t})\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(m_{k_1,k_2}, \sigma^2_{\alpha,\beta}\right), \;\; \text{as } \varepsilon \longrightarrow 0,$$

*where*

$$m_{k_1,k_2} = m_{k_1} + m_{k_2},$$

*and*

$$\sigma^2_{\alpha,\beta} = \alpha(2\alpha-1)\beta(2\beta-1)\sigma^2_\alpha \sigma^2_\beta,$$

*with*

$$m_{k_1} = \frac{\rho_s^{(k_1+1)}(x_{s,t})}{(k_1+1)!} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G_1(\mu)\, G_2(\nu)\, \mu^{k_1+1} d\mu d\nu;$$

$$m_{k_2} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_t^{(k_2+1)}(x_{s,t})}{(k_2+1)!} G_1(\mu)\, G_2(\nu)\, \nu^{k_2+1} d\mu d\nu;$$

$$\sigma^2_\alpha = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G_1(\mu_1) G_1(\mu_2) |\mu_1 - \mu_2|^{2\alpha-2} d\mu_1 d\mu_2;$$

*and*

$$\sigma^2_\beta = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G_2(\nu_1) G_2(\nu_2) |\nu_1 - \nu_2|^{2\beta-2} d\nu_1 d\nu_2.$$

**2. Linear multiplier estimation**
Let $X = \{X_{s,t},\ 0 \le s,t \le T\}$ be a stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{s,t\ge 0}, \mathbb{P})$ and assume the process $X$ satisfies

$$dX_{s,t} = \theta(s,t)X_{s,t}dsdt + \varepsilon dW_{s,t}^{\alpha,\beta},\ \ 0 \le s,t \le T; \tag{1.59}$$

where $\varepsilon > 0$, $\left\{W_{s,t}^{\alpha,\beta}\ 0 \le s,t \le T\right\}$ a fractional Brownian sheet, and $\theta(s,t)$ is an unknown function (linear multiplier). and study the asymptotic behavior of the kernel type estimator as $\varepsilon \longrightarrow 0$.

※ **Consistency of the estimator**

**Theorem 1.20.** *Under certain assumptions in Chapter 5, we have:*

$$\lim_{\varepsilon\longrightarrow 0}\sup_{0\le s,t\le T}\mathbb{E}\left(|\tilde{R}_{\alpha,\beta,\theta}(s,t) - R^{*}_{\alpha,\beta,\theta}(s,t)|^2\right) = 0.$$

※ **Rate of convergence**

**Theorem 1.21.** *Under certain assumptions in Chapter 5, we have:*

$$\lim_{\varepsilon\longrightarrow 0}\sup_{|\theta(.,.)|\le L}\sup_{0\le s,t\le T}\mathbb{E}(|\tilde{R}_{\alpha,\beta,\theta}(s,t) - R^{*}_{\alpha,\beta,\theta}(s,t)|^2) = \mathcal{O}\left(\varepsilon^{\frac{4\gamma_1\gamma_2}{3\gamma_1+3\gamma_2+4\gamma_1\gamma_2}}\right).$$

## 1.6 Outline of the thesis

This thesis is divided in 6 main chapters including the introduction and presentation chapter.

**Chapter** 2 is devoted to the problem of nonparametric estimation of the trend function for stochastic differential equation driven by a mixed fractional Brownian motion, under some hypotheses, we establish the consistent uniform, the rate of convergence as well as the asymptotic normality of the kernel estimator.

**This chapter has been published in the International Conference on Mathematics and Information Technology 2020 (ICMIT2020). IEEE, p. 91-97, 2020.**

In **chapter 3**, we consider the problem of drift function estimation of stochastic differential equations driven by a bi-fractional Brownian motion with small noise. We propose kernel-type estimator and study its asymptotic ($\varepsilon \longrightarrow 0$) behaviour. We show its consistency with rate of convergence and asymptotically normal. Finally, a numerical example is provided.

**This chapter has been published in Acta Univ. Sapientiae, Mathematica, 12, 1 (2020) 128–145.**

**Chapter** 4 deals with the problem of nonparametric estimation of the trend function for stochastic differential equations driven by small weighted fractional

Brownian motion noises based on continuous-time observation. Under some certain conditions, we derive the uniform consistency and the rate of convergence of the nonparametric estimator.

**This chapter has been accepted for publication in "Applications and Applied Mathematics (AAM) journal".**

**Chapter 5** is devoted to the problem of nonparametric estimation of the trend function and linear multiplier for stochastic differential equation driven by a fractional Brownian sheet, under some hypotheses, we establish the properties of the kernel type estimator of the trend function and linear multiplier. **(submitted).**

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# CHAPTER 2

# Asymptotic analysis of a kernel estimator for stochastic differential equations driven by a mixed sub-fractional Brownian motion

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## Sommaire

## 2.1 Introduction

Statistical inference for stochastic differential equations (SDEs) driven by a fractional Brownian Motion (fBm) has been extensively discussed because its wide applications in different scientific areas, network traffic, economy, biology, medicine, hydrology and so on (see, e.g. (Granger 1966), (Collins & De Luca 1995), (McLeod & Hipel 1978), (Willinger, Taqqu, Leland, Wilson et al. 1995), (Kuklinski, Chandra, Ruttirmann & Webber 1989)).

The study of nonparametric estimation of trend function for stochastic differential equations (SDEs) has attracted the attention of many authors, readers may refer to (Kutoyants 1994), (Mishra & Prakasa Rao 2011b, Mishra & Prakasa Rao 2011a), (Saussereau 2014), (Prakasa Rao 2019c, Prakasa Rao 2019b), (Comte & Marie 2019) and the references therein.

The contents of this investigation are as follows. In Section 2.2, the basic properties of mixed sub-fractional Brownian motion are stated. Section 2.3 is devoted to the preliminaries. Then, in Section 2.4, we give the main results; under some hypotheses, we establish the consistent uniform (Theorem 2.3), the rate of convergence (Theorem 2.4) as well as the asymptotic normality (Theorem 2.5) of the estimator.

## 2.2 Mixed Sub-fractional Brownian motion

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ denote a stochastic basis satisfying the habitual conditions, i.e., a filtered probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and $\mathcal{F}_0$ contains every $\mathbb{P}$-null set.
Let $\{\xi^H(t), t \geq 0\}$ be a normalized sub-fractional Brownian motion with Hurst parameter $H \in (0,1)$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, that is, a Gaussian process with continuous sample paths that: $\xi^H(0) = 0$; and the covariance:

$$Cov(\xi^H(t), \xi^H(u)) = t^{2H} + u^{2H} - \frac{1}{2}\Big[(u+t)^{2H} + |u-t|^{2H}\Big],\ t \geq 0,\ u \geq 0.$$

According to (Bojdecki et al. 2004), the sub-fractional Brownian motion (sfBm) is given by means of the fractional Brownian motion

$$\xi^H(t) = \left\{\frac{1}{\sqrt{2}}\Big(B^H(t) + B^H(-t)\Big),\ t \geq 0\right\}. \tag{2.1}$$

The mixed sub-fractional Brownian motion (msfBm) is an extension of a sfBm ((Charles & Mounir 2015) and (Zili 2006, Zili 2014)). Particularly, for $N \in \mathbb{N}^*$, $H = (H_1, H_2, ..., H_N) \in (0,1)^N$ and $a = (a_1, a_2, ..., a_N) \in \mathbb{R}^N \backslash \{0_{_N}\}$, the msfBm of parameters $N, H$ and $a$ is the process

$$S = \left\{S_t^H(N,a);\ t \geq 0\right\} = \left\{S_t^H;\ t \geq 0\right\}$$

defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ by

$$S_t^H(N,a) = \sum_{i=1}^{N} a_i \xi^{H_i}(t), \text{ for all } t \in \mathbb{R}_+, \tag{2.2}$$

where $(\xi^{H_i})_{i\in\{1,\dots,N\}}$ is a family of independent sfBm of Hurst parameters $H_i \in (0,1)$.

According to (Charles & Mounir 2015) and (Zili 2006, Zili 2014, Zili 2016), the msfBm $\left\{S_t^H(N,a);\, t \geq 0\right\}$ satisfies the following properties:

1. The msfBm is a centered Gaussian process.

2. For all $(t,u) \in \mathbb{R}_+^2$,

$$R_{a,H}(t,u) := Cov(S_t^H(a), S_u^H(a)) = \sum_{i=1}^{N} a_i^2 \left[t^{2H_i} + u^{2H_i} - \frac{1}{2}\left[(u+t)^{2H_i} + |u-t|^{2H_i}\right]\right].$$

3. For all $i \in \{1,\dots,N\}$, such that $H_i \in (1/2,1)$, we have

$$\mathbb{V}ar(S_t^H(a)) = \sum_{i=1}^{N} a_i^2 \left[(2 - 2^{2H_i-1})t^{2H_i}\right],\ \forall t \in \mathbb{R}_+.$$

4. $S_t^H(a)$ is said to be mixed-self-similar, for any $\alpha > 0$

$$\left\{S_{\alpha t}^H(a),\ t \geq 0\right\} \overset{\Delta}{=} \left\{S_t^H(a_1\alpha^{H_1}, a_2\alpha^{H_2}, \dots, a_N\alpha^{H_N}),\ t \geq 0\right\}.$$

5. For every $a = (a_1,\dots,a_N) \in \mathbb{R}^N$ and $H = (H_1,\dots,H_N) \in (0,1)^N$ such that there exists $j \in \{1,\dots,N\}$ such that $a_j \neq 0$ and $H_j \neq 1/2$, $(S_t^H(a))_{t\in\mathbb{R}_+}$ is not a Markovian process.

6. If there exists $i \in \{1,\dots,N\}$; $H_i < 1/2$ and $a_i \neq 0$, or if $H_i > 1/2$, for every $i \in 1,\dots,N$ , then the msfBm $(S_t^H(a))_{t\in\mathbb{R}_+}$ is not a semimartingale.

Further, by (Zili 2014, Zili 2016), for all $i \in \{1,\dots,N\}$ such that $a_i \neq 0$ and $H_i \in (1/2,1)$, we have the moving average representation for a msfBm $S_t^H(a)$ :

$$S_t^H(a) = \sum_{i=1}^{N} a_i C(H_i) \int_{\mathbb{R}} \left[(t-u)_+^{H_i-\frac{1}{2}} + (t+u)_-^{H_i-\frac{1}{2}} - 2(-u)_+^{H_i-\frac{1}{2}}\right] dW_i(u), \tag{2.3}$$

where $\{W_i;\, 1 \leq i \leq N\}$ are independent Wiener processes on $\mathbb{R}$, and

$$C(H_i) = \frac{\sqrt{2H_i \sin(\pi H_i)\Gamma(2H_i)}}{\Gamma(H_i + 1/2)}. \tag{2.4}$$

Let $\mathcal{H}_{a,H}$ be the completion of the linear space $\mathcal{E}$ generated by the indicator functions $\left\{\mathbf{1}_{[0,t]},\ t \in [0,T]\right\}$ with respect to the inner product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,u]} \right\rangle_{\mathcal{H}_{a,H}} = R_{a,H}(t,u), \tag{2.5}$$

where $R_{a,H}(t,u)$ is the covariance of $S_t^H(a)$ and $S_u^H(a)$.

When $1/2 < H_i < 1$ and $a_i \neq 0$ for all $i \in \{1,...,N\}$, we will use the subspace $\left|\mathcal{H}_{a,H}\right|$ of $\mathcal{H}_{a,H}$ which is defined as the set of measurable function $\varphi$ on $[0,T]$ with

$$\left|\mathcal{H}_{a,H}\right| = \left\{\varphi : [0,T] \longrightarrow \mathbb{R};\ \|\varphi\|_{|\mathcal{H}_{a,H}|} < \infty\right\};$$

and

$$\|\varphi\|^2_{|\mathcal{H}_{a,H}|} = \int_0^T \int_0^T |\varphi(t)||\varphi(u)|\phi_{a,H}(t,u)dtdu, \tag{2.6}$$

where

$$\phi_{a,H}(t,u) := \frac{\partial^2 R_{a,H}}{\partial t \partial u}(t,u) = \sum_{i=1}^N a_i^2 H_i(2H_i - 1)\left[|u-t|^{2H_i-2} - (u+t)^{2H_i-2}\right].$$

For every $\varphi,\ \psi \in \left|\mathcal{H}_{a,H}\right|$ the equation (2.5) implies that:

$$\begin{aligned}\langle \varphi, \psi \rangle_{\mathcal{H}_{a,H}} &= \mathbb{E}\left(\int_0^T \varphi(t)dS_t^H(a) \int_0^T \psi(u)dS_u^H(a)\right) \\ &= \int_0^T \int_0^T \varphi(t)\varphi(u)\phi_{a,H}(t,u)dtdu.\end{aligned} \tag{2.7}$$

**Lemma 2.1.** *Let $S_t^H(a)$ be a msfBm with Hurst index $H_i \in (1/2,1)$ and $a_i \neq 0$ for all $i \in \{1,...,N\}$, we have*

$$\mathbb{E}\left(\left[\int_0^T \varphi(u)dS_u^H\right]^2\right) = \sum_{i=1}^N a_i^2 H_i(2H_i-1)\int_0^T\int_0^T \varphi(u)\varphi(v)\left[|u-v|^{2H_i-2} - |u+v|^{2H_i-2}\right]dudv, \tag{2.8}$$

*and there exists a constant $C(H_i)$ such that*

$$\mathbb{E}\left(\left|\int_0^T \varphi(u)dS_u^H\right|^2\right) \leq \sum_{i=1}^N C(H_i)a_i^2\left(\int_0^T |\varphi(u)|^{1/H_i}du\right)^{2H_i}. \tag{2.9}$$

**Proof.** Let $(\xi^{H_i})_{i\in 1,...,N}$ denote a family of independent sfBm defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with Hurst parameters $H_i \in (1/2,1)$. Making use of Equation (4.4) and inequality (4.5) in (Prakasa Rao 2017b), the stated Lemma (2.1) is easily obtained.

## 2.3 Preliminaries

Let $\{X_t, 0 \leq t \leq T\}$ be a process governed by the following equation:

$$dX_t = b(X_t)dt + \varepsilon dS_t^H,\ X_0 = x_0,\ 0 \leq t \leq T, \tag{2.10}$$

where $\varepsilon > 0$, $S_t^H$ a msfBm, and $b(.)$ is an unknown function. We suppose that $x_t$ is a solution of the equation

$$\frac{dx_t}{dt} = b(x_t),\ x_0,\ 0 \leq t \leq T. \tag{2.11}$$

We suppose also that the function $b : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following assumptions:

**(A1):** There exists $L > 0$ such that

$$\left|b(x) - b(y)\right| \leq L\left|x - y\right|,\ 0 \leq t \leq T. \tag{2.12}$$

**(A2):** There exists $M > 0$ such that

$$|b(x)| \leq M(1 + |x|),\ x \in \mathbb{R},\ 0 \leq t \leq T.$$

Then, the stochastic differential equation (2.10) has a unique solution $\{X_t, 0 \leq t \leq T\}$.

**(A3):** Assume that the function $b(x)$ is bounded by a constant $C$. Since the function $x_t$ satisfies (2.11), it follows that

$$|b(x_t) - b(x_s)| \leq L|x_t - x_s| = L\left|\int_s^t b(x_r)dr\right| \leq LC|t - s|;\ t, s \in [0, T].$$

Let $\Sigma_0(L)$ be the class of functions $b(x)$ which satisfy the assumption **(A1)** and uniformly bounded by a constant $C$. Let $\Sigma_k(L)$ be the class of all function $b(x)$ which are uniformly bounded by $C$ and which are k-times differentiable with respect to $x$ verifying the condition

$$\left|b^k(x) - b^k(y)\right| \leq L\left|x - y\right|;\ \ x, y \in \mathbb{R}, \tag{2.13}$$

where $b^k(x)$ denote the k-th derivative of $b(x)$. Moreover, in all the sequel of this paper we put

$$H_{i_*} = \min\{H_i : i \in \{1, ..., N\};\ 1/2 < H_i < 1 \text{ and } a_i \neq 0\}.$$

**Lemma 2.2.** *Suppose that the assumption **(A1)** is verified and let $X_t$ and $x_t$ be the solutions of the equations (2.10) and (2.11) respectively. Then we have*

$$(i)\ |X_t - x_t| \leq e^{Lt}\varepsilon|S_t^H|, \tag{2.14}$$

*and*

$$(ii)\ \sup_{0\leq t\leq T} \mathbb{E}(X_t - x_t)^2 \leq e^{2LT}\varepsilon^2 \sum_{i=1}^{N} a_i^2\left[\left(2 - 2^{2H_i - 1}\right)T^{2H_i}\right]. \tag{2.15}$$

**Proof.**

**Proof of (i).**

By (2.10) and (2.11), we have

$$X_t = x_0 + \int_0^t b(X_r)dr + \varepsilon S_t^H,$$

and

$$x_t = x_0 + \int_0^t b(x_r)dr.$$

This implies

$$X_t - x_t = \int_0^t (b(X_r) - b(x_r))\,dr + \varepsilon S_t^H.$$

Thus

$$\begin{aligned} |X_t - x_t| &\leq \int_0^t |b(X_r) - b(x_r)|\,dr + \varepsilon |S_t^H| \\ &\leq L \int_0^t |X_r - x_r|\,dr + \varepsilon |S_t^H|. \end{aligned}$$

Putting $u_t = |X_t - x_t|$, we have

$$u_t \leq \int_0^t u_r dr + \varepsilon |S_t^H|.$$

Finally, by using Gronwalls inequality, we obtain

$$|X_t - x_t| \leq e^{Lt} \varepsilon \left|S_t^H\right|. \tag{2.16}$$

**Proof of (ii).**

From (2.16), we have

$$|X_t - x_t|^2 \leq e^{2Lt} \varepsilon^2 |S_t^H|^2.$$

Then, since

$$\mathbb{E}\left(S_t^H\right)^2 = \sum_{i=1}^N a_i^2 \left[\left(2 - 2^{2H_i - 1}\right) t^{2H_i}\right],$$

we have

$$\mathbb{E}|X_t - x_t|^2 \leq e^{2Lt} \varepsilon^2 \sum_{i=1}^N a_i^2 \left[\left(2 - 2^{2H_i - 1}\right) t^{2H_i}\right].$$

Finally, we obtain

$$\sup_{0 \leq t \leq T} \mathbb{E}\,(X_t - x_t)^2 < e^{2LT} \varepsilon^2 \sum_{i=1}^N a_i^2 \left[\left(2 - 2^{2H_i - 1}\right) T^{2H_i}\right].$$

□

## 2.4 Main results

The main goal of this work is to build an estimator of the trend function $b_t$ in the model described by stochastic differential equation (2.10) using the method developed by (Kutoyants 1994), then study the asymptotic properties of the estimator as $\varepsilon \longrightarrow 0$.

For all $t \in [0,T]$, the kernel estimator $\hat{b}_t$ of $b_t$ is given by

$$\hat{b}_t = \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau, \tag{2.17}$$

where $G(u)$ is a bounded function with finite support $[A,B]$ satisfying the following hypotheses

**(H1):** $G(u) = 0$ for $u < A$ and $u > B$, and $\displaystyle\int_A^B G(u)du = 1$,

**(H2):** $\displaystyle\int_{-\infty}^{+\infty} G^2(u)du < \infty$,

**(H3):** $\displaystyle\int_{-\infty}^{+\infty} u^{2(k+1)}G^2(u)du < \infty$,

**(H4):** $\displaystyle\int_{-\infty}^{+\infty} |G(u)|^{\frac{1}{H_i}} du < \infty, \quad \text{for any} \quad 1 \leq i \leq N.$

Further, we suppose that the normalizing function $\phi_\varepsilon$ satisfies:

**(H5):** $\phi_\varepsilon \longrightarrow 0$ and $\varepsilon^2\phi_\varepsilon^{-1} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

The following Theorem give uniform convergence of the estimator $\hat{b}_t$.

**Theorem 2.3.** *Suppose that the assumptions* ***(A1)-(A3)*** *and* ***(H1)-(H5)*** *hold true. Further, assume that the trend function $b(x)$ belongs to $\Sigma_0(L)$. Then, for any $0 < c \leq d < T$, the estimator $\hat{b}_t$ is uniformly consistent, that is,*

$$\lim_{\varepsilon \longrightarrow 0} \sup_{b(x)\in\Sigma_0(L)} \sup_{c\leq t\leq d} \mathbb{E}_b(|\hat{b}_t - b(x_t)|^2) = 0. \tag{2.18}$$

The following additional assumptions are useful for the rest of the theoretical study. Assume that

**(H6)** $\displaystyle\int_{-\infty}^{+\infty} u^j G(u)du = 0$, for $j = 1, 2, ..., k$,

**(H7)** $\displaystyle\int_{-\infty}^{+\infty} u^{k+1}G(u)du < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} u^{2(k+2)}G^2(u)du < \infty.$

The following Theorem establishes the rate of convergence of the estimator $\hat{b}_t$.

**Theorem 2.4.** *Assume that $b(x) \in \Sigma_k(L)$, $0 < \phi_\varepsilon < 1$ and $\phi_\varepsilon = \varepsilon^{\frac{1}{k-H_{i_*}+2}}$. Then, under hypotheses* ***(A1)-(A3)*** *and* ***(H1)-(H7)***, *we have*

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{b(x)\in\Sigma_k(L)} \sup_{c\leq t\leq d} \mathbb{E}_b(|\hat{b}_t - b(x_t)|^2)\varepsilon^{\frac{-2(k+1)}{k-H_{i_*}+2}} < \infty. \tag{2.19}$$

Finally, the following Theorem presents the asymptotic normality of the kernel type estimator $\hat{b}_t$ of $b(x_t)$.

**Theorem 2.5.** *Assume that $b(x) \in \Sigma_{k+1}(L)$, $\phi_\varepsilon = \varepsilon^{\frac{1}{k-H_{i_*}+2}}$ and $H_{i_*} = H_i$ for all $i \in \{1,2,...,N\}$. Then, under hypotheses **(A1)**-**(A3)** and **(H1)**-**(H7)**, we have*

$$\varepsilon^{\frac{-(k+1)}{k-H_{i_*}+2}}\left(\hat{b}_t - b(x_t)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(m, \sigma^2_{H,N,a}\right),$$

*where*

$$m = \frac{b^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G(u)u^{k+1}du,$$

*and*

$$\sigma^2_{H,N,a} = \left(\sum_{i=1}^{N} a_i^2\right) H_{i_*}(2H_{i_*}-1)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\left(|u-v|^{2H_{i_*}-2} - |u+v|^{2H_{i_*}-2}\right)dudv,$$

*as $\varepsilon \longrightarrow 0$, with $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.*

## 2.5 Proof of Theorems

### Proof of Theorem 2.3

From (2.10) and (2.17), we can see that

$$\begin{aligned}
\hat{b}_t - b(x_t) &= \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau - b(x_t) \\
&= \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right)(b(X_\tau) - b(x_\tau))\,d\tau \\
&\quad + \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) b(x_\tau)d\tau - b(x_t) \\
&\quad + \frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dS^H_\tau.
\end{aligned}$$

Using the inequality $(\alpha+\beta+\gamma)^2 \leq 3\alpha^2 + 3\beta^2 + 3\gamma^2$, it yields

$$\begin{aligned}
\mathbb{E}_b\left[\hat{b}_t - b(x_t)\right]^2 &\leq 3\mathbb{E}_b\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right)(b(X_\tau) - b(x_\tau))\,d\tau\right]^2 \\
&\quad + 3\mathbb{E}_b\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) b(x_\tau)d\tau - b(x_t)\right]^2 \\
&\quad + 3\mathbb{E}_b\left[\frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dS^H_\tau\right]^2. \\
&\leq I_1 + I_2 + I_3.
\end{aligned} \tag{2.20}$$

• **Concerning** $I_1$ : Via the inequalities (2.12), (2.15) and hypotheses **(H1)-(H2)**, we get

$$
\begin{aligned}
I_1 &= 3\mathbb{E}_b\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)(b(X_\tau)-b(x_\tau))\,d\tau\right]^2 \\
&= 3\mathbb{E}_b\left[\int_{-\infty}^{+\infty} G(u)\left(b(X_{t+\phi_\varepsilon u})-b(x_{t+\phi_\varepsilon u})\right)du\right]^2 \\
&\leq 3(B-A)\mathbb{E}_b\left[\int_{-\infty}^{+\infty} G^2(u)\left(b(X_{t+\phi_\varepsilon u})-b(x_{t+\phi_\varepsilon u})\right)^2 du\right] \\
&\leq 3L^2(B-A)\mathbb{E}_b\left[\int_{-\infty}^{+\infty} G^2(u)\left(X_{t+\phi_\varepsilon u}-x_{t+\phi_\varepsilon u}\right)^2 du\right] \\
&\leq 3L^2(B-A)\left[\int_{-\infty}^{+\infty} G^2(u)\sup_{0\leq t+\phi_\varepsilon u\leq T}\mathbb{E}_b\left(X_{t+\phi_\varepsilon u}-x_{t+\phi_\varepsilon u}\right)^2 du\right] \\
&\leq 3L^2(B-A)e^{2LT}\left(\int_{-\infty}^{+\infty} G^2(u)\,du\right)\sum_{i=1}^{N} a_i^2\left(2-2^{2H_i-1}\right)T^{2H_i}\varepsilon^2 \\
&\leq C_1\varepsilon^2,
\end{aligned}
\tag{2.21}
$$

where $C_1$ is a positive constant which depends on $T, L, H, a, N$ and $(B-A)$.

• **Concerning** $I_2$ : Let

$$
\begin{aligned}
I_2 &= 3\mathbb{E}_b\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)b(x_\tau)d\tau - b(x_t)\right]^2 \\
&= 3\mathbb{E}_b\left[\int_{-\infty}^{+\infty} G(u)\,b(x_{t+\phi_\varepsilon u})du - b(x_t)\right]^2 \\
&= 3\mathbb{E}_b\left[\int_{-\infty}^{+\infty} G(u)\left(b(x_{t+\phi_\varepsilon u})-b(x_t)\right)du\right]^2.
\end{aligned}
$$

Next, by using hypotheses **(A3)**, **(H1)** and **(H3)**, we have

$$
\begin{aligned}
I_2 &\leq 3L^2\mathbb{E}_b\left[\int_{-\infty}^{+\infty} G(u)(\phi_\varepsilon u)\,du\right]^2 \\
&\leq 3L^2(B-A)\left[\int_{-\infty}^{+\infty} G^2(u)\,u^2du\right]\phi_\varepsilon^2 \\
&\leq C_2\phi_\varepsilon^2,
\end{aligned}
\tag{2.22}
$$

where $C_2$ is a positive constant which depends on $L$ and $(B-A)$.

- **Concerning** $I_3$ : Since $H_i \in (1/2, 1)$ and $a_i \neq 0$ for all $i \in \{1, ..., N\}$, we have

$$\begin{aligned}
I_3 &= 3\mathbb{E}_b\left[\frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dS_\tau^H\right]^2 \\
&= 3\frac{\varepsilon^2}{\phi_\varepsilon^2}\mathbb{E}_b\left[\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dS_\tau^H\right]^2 \\
&\leq 3\frac{\varepsilon^2}{\phi_\varepsilon^2}\sum_{i=1}^N a_i^2\left[C(H_i)\left(\int_0^T \left|G\left(\frac{\tau-t}{\phi_\varepsilon}\right)\right|^{\frac{1}{H_i}}d\tau\right)^{2H_i}\right] \\
&\leq 3\frac{\varepsilon^2}{\phi_\varepsilon^2}\sum_{i=1}^N a_i^2\left[C(H_i)\phi_\varepsilon^{2H_i}\left(\int_{-\infty}^{+\infty}|G(u)|^{\frac{1}{H_i}}du\right)^{2H_i}\right] \\
&\leq C_3\frac{\varepsilon^2}{\phi_\varepsilon}\sum_{i=1}^N a_i^2\phi_\varepsilon^{2H_i-1} \text{ (using hypothesis (H4)),}
\end{aligned} \tag{2.23}$$

where $C_3$ is a positive constant which depends on $H_i$.

Combining (2.20), (2.21), (2.22) and (2.23), we have

$$\sup_{b(x)\in\Sigma_0(L)}\sup_{c\leq t\leq d}\mathbb{E}_b\left[\hat{b}_t - b(x_t)\right]^2 \leq C_4\left(\varepsilon^2 + \phi_\varepsilon^2 + \frac{\varepsilon^2}{\phi_\varepsilon}\sum_{i=1}^N a_i^2\phi_\varepsilon^{2H_i-1}\right).$$

Finally, under the assumption **(H6)**, we obtain

$$\lim_{\varepsilon\longrightarrow 0}\sup_{b(x)\in\Sigma_0(L)}\sup_{c\leq t\leq d}\mathbb{E}_b\left[\hat{b}_t - b(x_t)\right]^2 = 0$$

.

## Proof of Theorem 2.4

Using the Taylor formula, we easily get

$$b\left(x_{t+\phi_\varepsilon u}\right) = b(x_t) + \sum_{j=1}^k b^j(x_t)\frac{(\phi_\varepsilon u)^j}{j!} + \left(b^k(x_{t+\lambda(\phi_\varepsilon u)}) - b^k(x_t)\right)\frac{(\phi_\varepsilon u)^k}{k!},\ \lambda \in (0,1).$$

Then, by substituting this expression in $I_2$ and using (2.13) and assumptions **(H6)**-**(H7)**, we obtain

$$
\begin{aligned}
I_2 &= 3\mathbb{E}_b\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)b(x_\tau)d\tau - b(x_t)\right]^2 \\
&= 3\mathbb{E}_b\left[\int_{-\infty}^{+\infty} G(u)\Big(b(x_{t+\phi_\varepsilon u})-b(x_t)\Big)du\right]^2 \\
&= 3\mathbb{E}_b\left[\frac{\phi_\varepsilon^k}{k!}\int_{-\infty}^{+\infty} G(u)\,u^k\Big(b^k(x_{t+\lambda(\phi_\varepsilon u)})-b^k(x_t)\Big)du\right]^2 \\
&\le 3L^2\left[\frac{\phi_\varepsilon^{k+1}}{k!}\int_{-\infty}^{+\infty} G(u)\,u^{k+1}du\right]^2 \\
&\le 3L^2(B-A)\frac{\phi_\varepsilon^{2(k+1)}}{(k!)^2}\left[\int_{-\infty}^{+\infty} G^2(u)\,u^{2(k+1)}du\right] \\
&\le C_5\phi_\varepsilon^{2(k+1)},
\end{aligned}
\tag{2.24}
$$

where $C_5$ is a positive constant which depends on $L$, $k$ and $(B-A)$.
From (2.23), we have

$$I_3 \le C_3\frac{\varepsilon^2}{\phi_\varepsilon}\sum_{i=1}^N a_i^2\phi_\varepsilon^{2H_i-1}. \tag{2.25}$$

Consider

$$H_{i_*} = \min\{H_i : i\in\{1,\dots,N\};\ 1/2 < H_i < 1 \text{ and } a_i \neq 0\}, \tag{2.26}$$

and

$$0<\phi_\varepsilon<1,$$

then,

$$I_3 \le C_6\varepsilon^2\phi_\varepsilon^{2H_{i_*}-2}, \tag{2.27}$$

where $C_6$ is a positive constant which depends on $H_i$ and $a_i^2$.
Next, from (2.21), (2.24) and (2.27), we find

$$\sup_{b(x)\in\Sigma_k(L)}\sup_{c\le t\le d}\mathbb{E}_b\Big(\hat{b}_t-b(x_t)\Big)^2 \le C_7\Big(\varepsilon^2\phi_\varepsilon^{2H_{i_*}-2}+\phi_\varepsilon^{2(k+1)}+\varepsilon^2\Big).$$

Putting $\phi_\varepsilon = \varepsilon^{\frac{1}{k-H_{i_*}+2}}$, it yields

$$\limsup_{\varepsilon\longrightarrow 0}\sup_{b(x)\in\Sigma_k(L)}\sup_{c\le t\le d}\mathbb{E}_b(|\hat{b}_t-b(x_t)|^2)\varepsilon^{\frac{-2(k+1)}{k-H_{i_*}+2}}<\infty.$$

## Proof of Theorem 2.5

From (2.10) and (2.17), we have

$$\hat{b}_t - b(x_t) = \int_{-\infty}^{+\infty} G(u)\Big(b(X_{t+\phi_\varepsilon u}) - b(x_{t+\phi_\varepsilon u})\Big)du$$
$$+ \int_{-\infty}^{+\infty} G(u)\Big(b(x_{t+\phi_\varepsilon u}) - b(x_t)\Big)du$$
$$+ \frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right)dS_\tau^H.$$

Thus

$$\varepsilon^{\frac{-(k+1)}{k-H_{i_*}+2}}\Big(\hat{b}_t - b(x_t)\Big) = r_1(t) + r_2(t) + \eta_\varepsilon(t).$$

Hence, by the Slutsky's Theorem, it suffices to show the following three claims:

$$r_1(t) \xrightarrow{\mathbb{P}} 0, \text{ as } \varepsilon \to 0. \tag{2.28}$$

$$r_2(t) \xrightarrow{\mathbb{P}} m, \text{ as } \varepsilon \to 0. \tag{2.29}$$

and

$$\eta_\varepsilon(t) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2_{H,N,a}\right), \textit{ as } \varepsilon \to 0. \tag{2.30}$$

***Proof of (2.28).*** Let

$$r_1(t) = \varepsilon^{\frac{-(k+1)}{k-H_{i_*}+2}}\int_{-\infty}^{+\infty} G(u)\Big(b(X_{t+\phi_\varepsilon u}) - b(x_{t+\phi_\varepsilon u})\Big)du.$$

By applying the inequality (2.21), we have

$$0 \le \mathbb{E}\Big[r_1^2(t)\Big] \le C_8\,\varepsilon^{\frac{2(1-H_{i_*})}{k-H_{i_*}+2}}.$$

Therefore, using the Bienaymé-Tchebychev's inequality, as $\varepsilon \longrightarrow 0$, we obtain, for all $\lambda > 0$

$$\mathbb{P}\left(|r_1(t)| > \lambda\right) \longrightarrow 0.$$

Then

$$r_1(t) \xrightarrow{\mathbb{P}} 0.$$

***Proof of (2.29).***

Let

$$r_2(t) = \varepsilon^{\frac{-(k+1)}{k-H_{i_*}+2}}\int_{-\infty}^{+\infty} G(u)\Big(b(x_{t+\phi_\varepsilon u}) - b(x_t)\Big)du.$$

By taking any $t, u \in [0,T]$ and $b(x) \in \Sigma_{k+1}(L)$, via the Taylor expansion, we get

$$b\Big(x_{t+\phi_\varepsilon u}\Big) = b(x_t) + \sum_{j=1}^{k+1} b^j(x_t)\frac{(\phi_\varepsilon u)^j}{j!} + \Big(b^{k+1}(x_{t+\lambda(\phi_\varepsilon u)}) - b^{k+1}(x_t)\Big)\frac{(\phi_\varepsilon u)^{k+1}}{(k+1)!},\ \lambda \in (0,1).$$

Making use of the conditions **(H6)**, **(H7)**, and choosing $\phi_\varepsilon = \varepsilon^{\frac{1}{k-H_{i_*}+2}}$, we obtain

$$\begin{aligned}
\mathbb{E}[r_2(t)-m]^2 &= \mathbb{E}\left[\int_{-\infty}^{+\infty} G(u)\left(b^{k+1}(x_{t+\lambda(\phi_\varepsilon u)}) - b^{k+1}(x_t)\right)\frac{(\phi_\varepsilon u)^{k+1}}{(k+1)!}du\right]^2 \\
&\leq L^2\left(\int_{-\infty}^{+\infty} G(u)u^{k+2}\frac{\phi_\varepsilon^{k+2}}{(k+1)!}du\right)^2 \\
&\leq L^2(B-A)\left(\int_{-\infty}^{+\infty} G^2(u)u^{2(k+2)}du\right)\phi_\varepsilon^{2(k+2)} \\
&\leq C_9\phi_\varepsilon^{2(k+2)},
\end{aligned}$$

where $C_9$ is a positive constant which depends on $L$, $k$ and $B-A$,
and

$$m = \frac{b^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G(u)\,u^{k+1}du.$$

Therefore,

$$\mathbb{E}[r_2(t)-m]^2 \longrightarrow 0 \ \text{ as } \ \varepsilon \longrightarrow 0.$$

Then

$$r_2(t) \xrightarrow{\mathbb{P}} m, \text{ as } \varepsilon \to 0.$$

***Proof of (2.30).*** Let

$$\eta_\varepsilon(t) = \varepsilon^{\frac{-(k+1)}{k-H_{i_*}+2}}\varepsilon\phi_\varepsilon^{-1}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dS_\tau^H. \tag{2.31}$$

In fact, we have to evaluate the variance of (2.31). Observe that

$$\mathbb{E}(\eta_\varepsilon(t))^2 = \left(\varepsilon^{\frac{1-H_{i_*}}{k-H_{i_*}+2}}\phi_\varepsilon^{-1}\right)^2\mathbb{E}\left(\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dS_\tau^H\right)^2.$$

Moreover, using Lemma (2.1), we have

$$\begin{aligned}
\mathbb{E}(\eta_\varepsilon(t))^2 &= \left(\varepsilon^{\frac{1-H_{i_*}}{k-H_{i_*}+2}}\phi_\varepsilon^{-1}\right)^2\sum_{i=1}^N a_i^2\Big[\phi_\varepsilon^{2H_i}H_i(2H_i-1) \\
&\quad\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\left[|u-v|^{2H_i-2}-|u+v|^{2H_i-2}\right]dudv\Big].
\end{aligned}$$

Then, by taking $\phi_\varepsilon = \varepsilon^{\frac{1}{k-H_{i_*}+2}}$ and $H_{i_*} = H_i$, for all $i \in \{1,2,\ldots,N\}$, we get

$$\mathbb{E}(\eta_\varepsilon(t))^2 = \sigma_{H,N,a}^2.$$

Finally, making use of this last equation, we achieve the proof of Theorem (2.5).

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# CHAPTER 3

# Nonparametric estimation of trend function for stochastic differential equations driven by a bi-fractional Brownian motion

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## Sommaire

# 3.1 Introduction

Fractional Brownian motion (fBm) is the most well-known and employed process with a long dependency-property for many real world applications including telecommunication, turbulence, finance, and so on. This process was introduced by (Kolmogorov 1940), then studied by many researchers including (Mandelbrot & Van Ness 1968) and (Norros et al. 1999).

The bifractional Brownian motion (bfBm) was introduced in (Houdré & Villa 2003), further studied by (Russo & Tudor 2006), and by (Tudor et al. 2007).

Nonparametric estimation of trend function for stochastic differential equations (SDEs) has caught the attention of different researchers. It was first investigated by (Kutoyants 1994) for the stochastic differential equation driven by a standard Brownian motion. After that, the problem was generalized by (Mishra & Prakasa Rao 2011b) for the stochastic differential equation driven by a fractional Brownian motion. Then, (Mishra & Prakasa Rao 2011a) presented nonparametric estimation of linear multiplier for fractional diffusion processes. Later, nonparametric inference for fractional diffusion were dealt by (Saussereau 2014). Very recently, (Prakasa Rao 2019c) investigated nonparametric estimation of trend function for SDEs driven by mixed fractional Brownian motion.

In this chapter, we use the method developed by (Kutoyants 1994) to construct an estimate of the trend function $S_t$ in a model described by stochastic differential equations driven by a bifractional Brownian motion. For this, let $\{X_t, 0 \leq t \leq T\}$ be the process governed by the following equation:

$$dX_t = S(X_t)dt + \varepsilon dB_t^{H,K},\ X_0 = x_0,\ 0 \leq t \leq T,$$

where, $\varepsilon > 0$, $B_t^{H,K}$ is a bfBm of parameters $H \in (0,1)$, $K \in (0,1]$, and $S(.)$ is an unknown function. In (Kutoyants 1994), the trend coefficient in a diffusion process was estimated from the process $\{X_t, 0 \leq t \leq T\}$. In this investigation, we use a similar approach and consider the estimate $\hat{S}_t$ of $S_t$ as follows:

$$\hat{S}_t = \frac{1}{\phi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau,$$

where $G$ is a bounded kernel with finite support with $\phi_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Under some hypotheses, we firstly prove the mean square consistency of the estimator. Then, we give a bound on the rate of convergence and prove the asymptotic normality of the estimator $\hat{S}_t$.

To the best of our knowledge, the problem of nonparametric estimation of trend function for stochastic differential equations driven by a bfBm has not been considered in the literature.

The rest of the chapter is structured as follows. In Section 3.2, the basic properties of bifractional Brownian motion are stated. Section 3.3 is devoted to the preliminaries. Then, in Section 3.4, we give the main results; under some hypotheses, we establish the uniform consistency (Theorem 3.2), the rate of convergence (Theorem 3.3) as well as the asymptotic normality (Theorem 3.4) of the estimator. Further, in Section 3.5, a simulation example is carried out to illuminate our theoretical study. Section 3.6 is devoted to the technical proofs. Finally, we conclude the paper in Section 3.7.

## 3.2 Bi-fractional Brownian motion

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis satisfying the habitual hypotheses, i.e., a filtered probability space with a right continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\mathcal{F}_0$ contains every $\mathbb{P}$-null set. Let $\{B_t^{H,K}, t \geq 0\}$ be a normalized bifractional Brownian motion with parameters $H \in (0,1)$ and $K \in (0,1]$, that is, a Gaussian process with continuous sample paths with $B_0^{H,K} = 0$ and the covariance:

$$R_{H,K}(t,s) := \mathbb{E}\left(B_t^{H,K} B_s^{H,K}\right) = \frac{1}{2^K}\left[(t^{2H} + s^{2H})^K + |u-t|^{2HK}\right], \quad t \geq 0, \; s \geq 0.$$

When $K = 1$, we retrieve the fractional Brownian motion, while the case $K = 1$ and $H = 1/2$ corresponds to the standard Brownian motion.

The bfBm is an extension of the fBm which preserves many properties of the fBm, but not the stationarity of the increments. (Russo & Tudor 2006) showed that the bfBm $B^{H,K}$ behaves as a fBm of Hurst parameter $HK$. According to (Houdré & Villa 2003) and (Tudor et al. 2007), the bfBm has the following properties:

1. $\mathbb{E}\left(B_t^{H,H}\right) = 0$ and $Var\left(B_t^{H,K}\right) = t^{2HK}$.

2. $B_t^{H,K}$ is said to be self-similar with index $HK \in (0,1)$ , that is, for every constant $a > 0$,
$$\left\{B_{at}^{H,K}, t \geq 0\right\} \triangleq \left\{a^{HK} B_t^{H,K}, t \geq 0\right\}, \text{ for each } a > 0. \tag{3.1}$$
in the sense that the processes, on both the sides of the equality sign, have the same finite dimensional distributions.

3. The process $B_t^{H,K}$ is not Markov and it is not a semi-martingale if $HK \neq 1/2$.

4. The trajectories of the process $B^{H,K}$ are Hölder continuous of order $\delta$ for any $\delta < HK$ and they are nowhere differentiable.

5. The bfBm $B^{H,K}$ is a quasi-helix in the sense of (Kahane 1981), for any $t, s \geq 0$ we have
$$2^{-K}(t-s)^{2HK} \leq \mathbb{E}\left[B_t^{H,K} - B_s^{H,K}\right]^2 \leq 2^{1-K}(t-s)^{2HK}.$$

The bfBm $B^{H,K}$ can be extended for $K \in (1,2)$ with $H \in (0,1)$ and $HK \in (0,1)$ (see (Bardina & Es-Sebaiy 2011), and (Lifshits & Volkova 2015)).

The stochastic calculus with respect to the bifractional Brownian motion has been recently developed by (Kruk et al. 2007). More works on bifractional Brownian motion can be found in (Tudor et al. 2007), (Es-Sebaiy & Tudor 2007) and the references therein.

Fix a time interval $[0,T]$, we denote by $\mathcal{E}$ the set of step function on $[0,T]$. Let $\mathcal{H}_{B^{H,K}}$ be the canonical Hilbert space associated to the bfBm defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}_{B^{H,K}}} = R_{H,K}(t,s) = \int_0^T \int_0^T 1_{[0,t]}(u) 1_{[0,s]}(v) \frac{\partial^2 R_{H,K}(u,v)}{\partial u \partial v} du dv,$$

where $R_{H,K}(t,s)$ is the covariance of $B_t^{H,K}$ and $B_s^{H,K}$. The application $\varphi \in \mathcal{E} \longrightarrow B^{H,K}(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^{H,K}$ and it can be extended to $\mathcal{H}_{B^{H,K}}$. In this study, as $HK \in (1/2,1)$ we will employ the subspace $|\mathcal{H}_{B^{H,K}}|$ of $\mathcal{H}_{B^{H,K}}$ which is defined as the set of measurable function $\varphi$ on $[0,T]$ satisfying

$$\|\varphi\|_{|\mathcal{H}_{B^{H,K}}|} := \int_0^T \int_0^T |\varphi(u)| |\varphi(v)| \frac{\partial^2 R_{H,K}(u,v)}{\partial u \partial v} du dv < \infty, \tag{3.2}$$

such that

$$\frac{\partial^2 R_{H,K}(u,v)}{\partial u \partial v} = \alpha_{H,K} \left( t^{2H} + s^{2H} \right)^{K-2} (ts)^{2H-1} + \beta_{H,K} |t-s|^{2HK-2},$$

where

$$\alpha_{H,K} = 2^{-K+2} H^2 K(K-1) \quad \text{and} \quad \beta_{H,K} = 2^{-K+1} HK(2HK-1).$$

For $\varphi,\ \psi \in |\mathcal{H}_{B^{H,K}}|$, we have

$$\mathbb{E}\left( \int_0^T \varphi(u) dB_u^{H,K} \right) = 0,\ \mathbb{E}\left( \int_0^T \varphi(u) dB_u^{H,K} \int_0^T \psi(v) dB_v^{H,K} \right) = \langle \varphi, \psi \rangle_{\mathcal{H}_{B^{H,K}}}.$$

It is worth being pointed out that the canonical Hilbert space $\mathcal{H}_{B^{H,K}}$ associated with $B^{H,K}$ satisfies:

$$\mathbb{L}^2([0,T]) \subset \mathbb{L}^{1/HK}([0,T]) \subset |\mathcal{H}_{B^{H,K}}| \subset \mathcal{H}_{B^{H,K}}, \tag{3.3}$$

where $H \in (0,1)$, $K \in (0,1]$ and $HK \in (1/2,1)$.

## 3.3 Preliminaries

Let $\{X_t,\ 0 \le t \le T\}$ be a process governed by the following equation:

$$dX_t = S(X_t)dt + \varepsilon dB_t^{H,K},\ X_0 = x_0,\ 0 \le t \le T \tag{3.4}$$

where $\varepsilon > 0$, $B_t^{H,K}$ a bifractional Brownian motion, and $S(.)$ is an unknown function. We suppose that $x_t$ is a solution of the equation

$$\frac{dx_t}{dt} = S(x_t),\ x_0,\ 0 \le t \le T. \tag{3.5}$$

We also suppose that the function $S : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(**A1**) There exists $L > 0$ such that

$$\left|S(x) - S(y)\right| \le L\left|x - y\right|;\ \ x, y \in \mathbb{R}, \tag{3.6}$$

(**A2**) There exists $M > 0$ such that

$$|S(x)| \le M(1 + |x|),\ x \in \mathbb{R},$$

Then, the stochastic differential equation (3.4) has a unique solution $\{X_t,\ 0 \le t \le T\}$.

(**A3**) Assume that the function $S(x)$ is bounded by a constant $C$. Since the function $x_t$ satisfies (3.5), it follows that

$$|S(x_t) - S(x_s)| \le L|x_t - x_s| = L\left|\int_s^t S(x_r)dr\right| \le LC|t - s|,\ t, s \in \mathbb{R}.$$

Let us define $\Sigma_0(L)$ as the class of all functions $S(x)$ satisfying the assumption **(A1)** and uniformly bounded by the same constant $C$. Let us also denote by $\Sigma_k(L)$ the class of all function $S(x)$ which are uniformly bounded by same constant $C$ and which are k-times differentiable with respect to $x$ satisfying the condition

$$\left|S^k(x) - S^k(y)\right| \le L\left|x - y\right|,\ x, y \in \mathbb{R}, \tag{3.7}$$

where $S^k(x)$ denote the k-th derivative of $S(x)$.

**Lemma 3.1.** *Assume that hypothesis **(A1)** is verified. Let $X_t$ and $x_t$ be the solutions of the equations (3.4) and (3.5) respectively. Then, we have*

$$\sup_{0 \le t \le T} \mathbb{E}(X_t - x_t)^2 \le e^{2LT}\varepsilon^2 T^{2HK}. \tag{3.8}$$

**Proof.** By (3.4) and (3.5), we have

$$X_t - x_t = \int_0^t (S(X_r) - S(x_r))\,dr + \varepsilon B_t^{H,K}.$$

Thus

$$\begin{aligned} |X_t - x_t| &\le \int_0^t |S(X_r) - S(x_r)|\,dr + \varepsilon|B_t^{H,K}| \\ &\le L\int_0^t |X_r - x_r|\,dr + \varepsilon|B_t^{H,K}|. \end{aligned} \tag{3.9}$$

Putting $u_t = |X_t - x_t|$, we have

$$u_t \leq \int_0^t u_r dr + \varepsilon |B_t^{H,K}|.$$

By using Grönwall's inequality, we obtain

$$|X_t - x_t| \leq e^{Lt} \varepsilon \left| B_t^{H,K} \right|.$$

Then, since $\mathbb{E}\left(B_t^{H,K}\right)^2 = t^{2HK}$, we have

$$\mathbb{E}|X_t - x_t|^2 \leq e^{2Lt} \varepsilon^2 t^{2HK}.$$

Finally, we find

$$\sup_{0 \leq t \leq T} \mathbb{E}(X_t - x_t)^2 < e^{2LT} \varepsilon^2 T^{2HK}.$$

□

## 3.4 Main results

The main goal of this work is to build an estimator of the trend function $S_t$ in the model described by stochastic differential equation (3.4) using the method developed by (Kutoyants 1994). Then, we study of asymptotic properties of the estimator as $\varepsilon \longrightarrow 0$.

For all $t \in [0, T]$, the kernel estimator $\hat{S}_t$ of $S_t$ is given by

$$\hat{S}_t = \frac{1}{\phi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau, \tag{3.10}$$

where $G(u)$ is a bounded function with finite support $[A, B]$ satisfying the following hypotheses

**(H1)** $G(u) = 0$ for $u < A$ and $u > B$, and $\int_A^B G(u)du = 1$,

**(H2)** $\int_{-\infty}^{+\infty} G^2(u)du < \infty$,

**(H3)** $\int_{-\infty}^{+\infty} u^{2(k+1)} G^2(u)du < \infty$,

**(H4)** $\int_{-\infty}^{+\infty} |G(u)|^{\frac{1}{HK}} du < \infty$,

Further, we suppose that the normalizing function $\phi_\varepsilon$ satisfies:

**(H5)** $\phi_\varepsilon \longrightarrow 0$ and $\varepsilon^2 \phi_\varepsilon^{-1} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

The following theorem gives uniform convergence of the estimator $\hat{S}_t$.

**Theorem 3.2.** *Suppose that the assumptions **(A1)**-**(A3)** and **(H1)**-**(H5)** hold true. Further, suppose that the trend function $S(x)$ belongs to $\Sigma_0(L)$. Then, for any $0 < c \leq d < T$ and $HK \in (1/2, 1)$, the estimator $\hat{S}_t$ is uniformly consistent, that is,*

$$\lim_{\varepsilon \longrightarrow 0} \sup_{S(x) \in \Sigma_0(L)} \sup_{c \leq t \leq d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2) = 0. \tag{3.11}$$

The following additional assumptions are useful for the rest of the theoretical study. Assume that

**(H6)** $\displaystyle\int_{-\infty}^{+\infty} u^j G(u)du = 0$ for $j = 1, 2, ..., k$,

**(H7)** $\displaystyle\int_{-\infty}^{+\infty} u^{k+1} G(u)du < \infty$ and $\displaystyle\int_{-\infty}^{+\infty} u^{2(k+2)} G^2(u)du < \infty$.

The rate of convergence of the estimator $\hat{S}_t$ is established in the following theorem.

**Theorem 3.3.** *Suppose that the function $S(x) \in \Sigma_k(L)$, $HK \in (1/2, 1)$ and $\phi_\varepsilon = \varepsilon^{\frac{1}{k-HK+2}}$. Then, under the hypotheses **(A1)-(A3)** and **(H1)-(H7)**, we have*

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_k(L)} \sup_{c\le t\le d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2)\varepsilon^{\frac{-2(k+1)}{k-HK+2}} < \infty. \tag{3.12}$$

Finally, the following theorem presents the asymptotic normality of the kernel type estimator $\hat{S}_t$ of $S(x_t)$.

**Theorem 3.4.** *Suppose that the function $S(x) \in \Sigma_{k+1}(L)$, $HK \in (1/2, 1)$ and $\phi_\varepsilon = \varepsilon^{\frac{1}{k-HK+2}}$. Then, under the hypotheses **(A1)-(A3)** and **(H1)-(H7)**, we have*

$$\varepsilon^{\frac{-(k+1)}{k-HK+2}} \left(\hat{S}_t - S(x_t)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(m, \sigma^2_{H,K}\right), \text{ as } \varepsilon \longrightarrow 0,$$

*where*

$$m = \frac{S^{k+1}(x_t)}{(k+1)!} \int_{-\infty}^{+\infty} G(u)u^{k+1} du,$$

*and*

$$\sigma^2_{H,K} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\left[\alpha_{H,K}\left(u^{2H} + v^{2H}\right)^{K-2}(uv)^{2H-1} + \beta_{H,K}|u-v|^{2HK-2}\right] du dv,$$

*where*

$$\alpha_{H,K} = 2^{-K+2}H^2K(K-1) \quad \text{and} \quad \beta_{H,K} = 2^{-K+1}HK(2HK-1).$$

## 3.5 Numerical example

The main objective of this part is to conduct a numerical study to illustrate our theoretical result. We compare our kernel estimator for stochastic differential equations driven by a bifractional Brownian motion to the kernel estimator for stochastic differential equations driven by fractional Brownian motion given in (Mishra & Prakasa Rao 2011b). We compare numerically the variance $\sigma^2_{H,K}$ of our estimator to $\sigma^2_H$.

Consider a function $G$ which satisfies hypotheses **(H1)-(H7)**:

$$G(t) = \frac{15}{128}\left(63t^4 - 70t^2 + 15\right), \ |t| \le 1.$$

– The variance of the kernel estimator for stochastic differential equations driven by fractional Brownian motion given in (Mishra & Prakasa Rao 2011b) is given as: For all $H \in (1/2, 1)$,

$$\sigma_H^2 = H(2H-1)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\,|u-v|^{2H-2}\,dudv,$$

– Using the result given in Theorem 3.4, the variance of our estimator is obtained as:

For all $H \in (0,1)$, $K \in (0,1]$ and $HK \in (1/2,1)$, we have

$$\sigma_{H,K}^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\left[\alpha_{H,K}\left(u^{2H}+v^{2H}\right)^{K-2}(uv)^{2H-1} + \beta_{H,K}|u-v|^{2HK-2}\right]dudv,$$

where

$$\alpha_{H,K} = 2^{-K+2}H^2K(K-1) \quad \text{and} \quad \beta_{H,K} = 2^{-K+1}HK(2HK-1).$$

Next, we compute the variances, the results are presented in the following Tables

Table 3.1: The variance values $\sigma_H^2$.

| $H$ | 0.7 | 0.75 | 0.8 | 0.85 | 0.9 | 0.95 |
|---|---|---|---|---|---|---|
| $\sigma_H^2$ | 1.1567 | 1.1900 | 1.1830 | 1.1506 | 1.1025 | 1.0452 |

Table 3.2: The variance values $\sigma_{H,K}^2$.

| $K \setminus H$ | 0.7 | 0.75 | 0.8 | 0.85 | 0.9 | 0.95 |
|---|---|---|---|---|---|---|
| 0.75 | 0.6006 | 0.9458 | 1.1647 | 1.2965 | 1.3709 | 1.4091 |
| 0.8 | 0.8733 | 1.1230 | 1.2696 | 1.3462 | 1.3774 | 1.3801 |
| 0.85 | 1.0362 | 1.2107 | 1.3019 | 1.3376 | 1.3378 | 1.3159 |
| 0.9 | 1.1227 | 1.2382 | 1.2873 | 1.2930 | 1.2712 | 1.2326 |
| 0.95 | 1.1570 | 1.2264 | 1.2437 | 1.2274 | 1.1901 | 1.1402 |
| 1 | 1.1567 | 1.1900 | 1.1830 | 1.1506 | 1.1025 | 1.0452 |

From the obtained results in Tables 3.1 and 3.2, we clearly see that the variance of our estimator is less than that of the kernel estimator for stochastic differential equations driven by fractional Brownian motion. We can conclude that our kernel estimator for stochastic differential equations driven by a bifractional Brownian motion is better than that given in (Mishra & Prakasa Rao 2011b).

## 3.6 Proof of Theorems

### Proof of Theorem 3.2

From (3.4) and (3.10), we can see that

$$\begin{aligned}
\hat{S}_t - S(x_t) &= \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dX_\tau - S(x_t) \\
&= \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)\left(S(X_\tau)d\tau + \varepsilon dB_\tau^{H,K}\right) - S(x_t) \\
&= \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)(S(X_\tau) - S(x_\tau))\,d\tau + \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t) \\
&\quad + \frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{H,K}.
\end{aligned}$$

Using the inequality $(\alpha+\beta+\gamma)^2 \le 3\alpha^2 + 3\beta^2 + 3\gamma^2$, it yields

$$\begin{aligned}
\mathbb{E}_S\left[\hat{S}_t - S(x_t)\right]^2 &\le 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)(S(X_\tau)-S(x_\tau))\,d\tau\right]^2 \\
&\quad +3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\
&\quad +3\mathbb{E}_S\left[\frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{H,K}\right]^2. \\
&\le I_1 + I_2 + I_3.
\end{aligned} \tag{3.13}$$

• **Concerning** $I_1$. Via the inequalities (3.6) and (3.8), and hypotheses **(H1)**-**(H2)**, we get

$$\begin{aligned}
I_1 &= 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)(S(X_\tau)-S(x_\tau))\,d\tau\right]^2 \\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\left(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u})\right)du\right]^2 \\
&\le 3(B-A)\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G^2(u)\left(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u})\right)^2 du\right] \\
&\le 3(B-A)L^2\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G^2(u)\left(X_{t+\phi_\varepsilon u} - x_{t+\phi_\varepsilon u}\right)^2 du\right] \\
&\le 3(B-A)L^2\left[\int_{-\infty}^{+\infty} G^2(u) \sup_{0\le t+\phi_\varepsilon u\le T} \mathbb{E}_S\left(X_{t+\phi_\varepsilon u} - x_{t+\phi_\varepsilon u}\right)^2 du\right] \\
&\le 3(B-A)L^2 e^{2LT} T^{2HK} \varepsilon^2 \\
&\le C_1\varepsilon^2,
\end{aligned} \tag{3.14}$$

where $C_1$ is a positive constant depending on $T, L, H, K$, and $(B-A)$.

• **Concerning** $I_2$. Let

$$\begin{aligned} I_2 &= 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\ &= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)S(x_{t+\phi_\varepsilon u})du - S(x_t)\right]^2 \\ &= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\Big(S(x_{t+\phi_\varepsilon u}) - S(x_t)\Big)du\right]^2 \end{aligned} \tag{3.15}$$

Next, by using hypotheses **(A3)** and **(H3)**, we have

$$\begin{aligned} I_2 &\le 3L^2\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)(\phi_\varepsilon u)du\right]^2 \\ &\le 3(B-A)L^2\left[\int_{-\infty}^{+\infty} G^2(u)u^2du\right]\phi_\varepsilon^2 \\ &\le C_2\phi_\varepsilon^2, \end{aligned}$$

where $C_2$ is a positive constant depending on $L$ and $(B-A)$.

• **Concerning** $I_3$. Since $HK \in (1/2, 1)$, we have

$$I_3 = 3\frac{\varepsilon^2}{\phi_\varepsilon^2}\mathbb{E}_S\left[\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{H,K}\right]^2. \tag{3.16}$$

By using (3.3), we have

$$\begin{aligned} I_3 &\le 3\frac{\varepsilon^2}{\phi_\varepsilon^2}\left[C(2,HK)\left(\int_0^T \left|G\left(\frac{\tau-t}{\phi_\varepsilon}\right)\right|^{\frac{1}{HK}} d\tau\right)^{2HK}\right] \\ &= 3\frac{\varepsilon^2}{\phi_\varepsilon^2}\left[C(2,HK)\phi_\varepsilon^{2HK}\left(\int_{-\infty}^{+\infty} |G(u)|^{\frac{1}{HK}} du\right)^{2HK}\right] \\ &\le C_3\frac{\varepsilon^2}{\phi_\varepsilon}\phi_\varepsilon^{2HK-1} \text{ (using hypothesis } \mathbf{(H4)}\text{)}, \end{aligned} \tag{3.17}$$

where $C_3$ is a positive constant depending on $H$ and $K$.

Combining (3.13), (3.14), (3.15), and (3.17), we have

$$\sup_{S(x)\in\Sigma_0(L)}\sup_{c\le t\le d}\mathbb{E}_S\left[\hat{S}_t - S(x_t)\right]^2 \le C_4\left(\varepsilon^2 + \phi_\varepsilon^2 + \frac{\varepsilon^2}{\phi_\varepsilon}\phi_\varepsilon^{2HK-1}\right).$$

Finally, under the assumption **(H6)**, we obtain

$$\lim_{\varepsilon\longrightarrow 0}\sup_{S(x)\in\Sigma_0(L)}\sup_{c\le t\le d}\mathbb{E}_S\left[\hat{S}_t - S(x_t)\right]^2 = 0.$$

## Proof of Theorem 3.3

Using the Taylor formula, we easily get

$$S\left(x_{t+\phi_\varepsilon u}\right) = S\left(x_t\right) + \sum_{j=1}^{k} S^j\left(x_t\right)\frac{(\phi_\varepsilon u)^j}{j!} + \left(S^k(x_{t+\lambda(\phi_\varepsilon u)}) - S^k(x_t)\right)\frac{(\phi_\varepsilon u)^k}{k!},\ \lambda \in (0,1).$$

Then, by substituting this expression in $I_2$ and using inequality (3.7), and assumptions **(H6)**-**(H7)**, we obtain

$$\begin{aligned}
I_2 &= 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\,S(x_{t+\phi_\varepsilon u})du - S(x_t)\right]^2 \\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\left(S(x_{t+\phi_\varepsilon u}) - S(x_t)\right)du\right]^2 \\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\left(\sum_{j=1}^{k} S^j(x_t)\frac{(\phi_\varepsilon u)^j}{j!} + \left(S^k(x_{t+\lambda(\phi_\varepsilon u)}) - S^k(x_t)\right)\frac{(\phi_\varepsilon u)^k}{k!}\right)du\right]^2 \\
&\leq 3L^2(B-A)\frac{\phi_\varepsilon^{2(k+1)}}{(k!)^2}\left[\int_{-\infty}^{+\infty} G^2(u)\,u^{2(k+1)}du\right] \\
&\leq C_5\phi_\varepsilon^{2(k+1)},
\end{aligned} \tag{3.18}$$

where $C_5$ is a positive constant depending on $L$, $k$ and $(B-A)$.
Next, from (3.14), (3.17), and (3.18), we find

$$\sup_{S(x)\in\Sigma_k(L)}\ \sup_{c\leq t\leq d} \mathbb{E}_S\left|\hat{S}_t - S(x_t)\right|^2 \leq C_6\left(\varepsilon^2\phi_\varepsilon^{2HK-2} + \phi_\varepsilon^{2(k+1)} + \varepsilon^2\right).$$

Putting $\phi_\varepsilon = \varepsilon^{\frac{1}{k-HK+2}}$, it yields

$$\limsup_{\varepsilon\longrightarrow 0}\ \sup_{S(x)\in\Sigma_k(L)}\ \sup_{c\leq t\leq d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2)\varepsilon^{\frac{-2(k+1)}{k-HK+2}} < \infty.$$

This completes the proof of Theorem 3.3.

## Proof of Theorem 3.4

From (3.4) and (3.10), we can see that

$$\begin{aligned}
\varepsilon^{\frac{-(k+1)}{k-HK+2}}\left(\hat{S}_t - S(x_t)\right) = \varepsilon^{\frac{-(k+1)}{k-HK+2}}&\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)(S(X_\tau) - S(x_\tau))d\tau\right. \\
&\left.+\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\ + \frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{H,K}\right]
\end{aligned}$$

Therefore

$$\varepsilon^{\frac{-(k+1)}{k-HK+2}}\left(\hat{S}_t - S(x_t)\right) = \varepsilon^{\frac{-(k+1)}{k-HK+2}}\left[\int_{-\infty}^{+\infty} G(u)\left(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u})\right)du + \int_{-\infty}^{+\infty} G(u)\left(S(x_{t+\phi_\varepsilon u}) - S(x_t)\right)du + \frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right)dS_\tau^H.\right]$$

Thus

$$\varepsilon^{\frac{-(k+1)}{k-HK+2}}\left(\hat{S}_t - S(x_t)\right) = r_1(t) + r_2(t) + \eta_\varepsilon(t).$$

Hence, by the Slutsky's Theorem, it suffices to show the following three claims:

$$r_1(t) \to 0,\ as\ \varepsilon \to 0 \text{ in probability.} \tag{3.19}$$

$$r_2(t) \to m,\ as\ \varepsilon \to 0 \text{ in probability.} \tag{3.20}$$

and

$$\eta_\varepsilon(t) \to \mathcal{N}\left(0, \sigma_{H,K}^2\right),\ as\ \varepsilon \to 0 \text{ in distribution.} \tag{3.21}$$

***Proof of (3.19).***
Let

$$r_1(t) = \varepsilon^{\frac{-(k+1)}{k-HK+2}}\int_{-\infty}^{+\infty} G(u)\left(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u})\right)du.$$

By applying the inequality (3.14), we have

$$0 \le \mathbb{E}\left[r_1^2(t)\right] \le \varepsilon^{\frac{-2(k+1)}{k-HK+2}} I_1 \le C_7 \varepsilon^{\frac{2(1-HK)}{k-HK+2}}.$$

Therefore, using the Bienaymé-Tchebychev's inequality, as $\varepsilon \longrightarrow 0$, we obtain, for all $\alpha > 0$

$$\mathbb{P}\left(|r_1(t)| > \alpha\right) \le \frac{\mathbb{E}\left[r_1^2(t)\right]}{\alpha^2} \le \frac{C_7 \varepsilon^{\frac{2(1-HK)}{k-HK+2}}}{\alpha^2} \longrightarrow 0.$$

***Proof of (3.20).***

Let,

$$r_2(t) = \varepsilon^{\frac{-(k+1)}{k-HK+2}}\int_{-\infty}^{+\infty} G(u)\left(S(x_{t+\phi_\varepsilon u}) - S(x_t)\right)du.$$

By taking any $t, u \in [0, T]$ and $b(x) \in \Sigma_{k+1}(L)$, via the Taylor expansion, we get

$$S\left(x_{t+\phi_\varepsilon u}\right) = S(x_t) + \sum_{j=1}^{k} S^j(x_t)\frac{(\phi_\varepsilon u)^j}{j!} + \frac{S^{k+1}(x_t)}{(k+1)!}(\phi_\varepsilon u)^{k+1} + \left(S^{k+1}(x_{t+\lambda(\phi_\varepsilon u)}) - S^{k+1}(x_t)\right)\frac{(\phi_\varepsilon u)^{k+1}}{(k+1)!},\ \lambda \in (0,1),$$

Making use of the conditions **(H6)**, **(H7)**, and choosing $\phi_\varepsilon = \varepsilon^{\frac{1}{k-HK+2}}$, we obtain

$$\begin{aligned}
\mathbb{E}[r_2(t)-m]^2 &= \mathbb{E}\left[\int_{-\infty}^{+\infty} G(u)\Big(S^{k+1}(x_{t+\lambda(\phi_\varepsilon u)}) - S^{k+1}(x_t)\Big)\frac{(\phi_\varepsilon u)^{k+1}}{(k+1)!}du\right]^2 \\
&\leq L^2\left(\int_{-\infty}^{+\infty} G(u)u^{k+2}\frac{\phi_\varepsilon^{k+2}}{(k+1)!}du\right)^2 \\
&\leq L^2(B-A)\left(\int_{-\infty}^{+\infty} G^2(u)u^{2(k+2)}du\right)\phi_\varepsilon^{2(k+2)} \\
&\leq C_8\phi_\varepsilon^{2(k+2)},
\end{aligned}$$

where $C_8$ is a positive constant which depends on $L$, $(B-A)$ and $k$, and

$$m = \frac{S^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G(u)u^{k+1}du.$$

Therefore,

$$\mathbb{E}[r_2(t)-m]^2 \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.$$

Then

$$r_2(t) \xrightarrow{\mathbb{P}} m.$$

***Proof of (3.21).***
Let

$$\eta_\varepsilon(t) = \varepsilon^{\frac{-(k+1)}{k-HK+2}}\varepsilon\phi_\varepsilon^{-1}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{H,K}. \tag{3.22}$$

In fact, we have to evaluate the variance of (3.22). To this end, let

$$\mathbb{E}[\eta_\varepsilon(t)]^2 = \left(\varepsilon^{\frac{1-HK}{k-HK+2}}\phi_\varepsilon^{-1}\right)^2\mathbb{E}\left(\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{H,K}\right)^2.$$

Moreover, using equation (3.2), we have

$$\mathbb{E}[\eta_\varepsilon(t)]^2 = \left(\varepsilon^{\frac{1-HK}{k-HK+2}}\phi_\varepsilon^{-1}\right)^2\left[\phi_\varepsilon^{2HK}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\frac{\partial^2 R_{H,K}(u,v)}{\partial u\partial v}dudv\right].$$

Then, by taking $\phi_\varepsilon = \varepsilon^{\frac{1}{k-HK+2}}$, we get

$$\mathbb{E}[\eta_\varepsilon(t)]^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\frac{\partial^2 R_{H,K}(u,v)}{\partial u\partial v}dudv,$$

with

$$\frac{\partial^2 R_{H,K}(u,v)}{\partial u\partial v} = \alpha_{H,K}\left(u^{2H}+v^{2H}\right)^{K-2}(uv)^{2H-1} + \beta_{H,K}|u-v|^{2HK-2},$$

where

$$\alpha_{H,K} = 2^{-K+2}H^2K(K-1) \quad \text{and} \quad \beta_{H,K} = 2^{-K+1}HK(2HK-1).$$

Finally, this last equation allows us to achieve the proof of Theorem 3.4.

## 3.7 Conclusion

This chapter considered a nonparametric estimation of trend function for stochastic differential equations driven by a bifractional Brownian motion. We constructed an estimate of the trend function. Then, under some assumptions, we established the uniform consistency, the rate of convergence and the asymptotic normality of the proposed estimator. Further, an numerical example is provided. The present study has many applications in practical phenomena including telecommunications and economics.

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# CHAPTER 4

# Nonparametric estimation of trend function for stochastic differential equations driven by a weighted fractional Brownian motion

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## Sommaire

## 4.1 Introduction

A long/short-term memory stochastic processes with self-similarity have been extensively employed as models for various physical phenomena. They seemed to play an crucial role in analyzing network traffic, economy, and telecommunications. As a result, some effective mathematical models based on long-term/short term dependence processes with self-similarity have been suggested in these directions.

Fractional Brownian motion is a straightforward stochastic process with long/ short term dependence and self-similarity, which is an appropriate generalization of standard Brownian motion. Intersting comprehensive surveys and literatures on fractional Brownian motion can be found in (Norros et al. 1999), (Hu 2005), (Gradinaru, Nourdin, Russo & Vallois 2005), (Mishura 2008), (Biagini, Hu, Øksendal & Zhang 2008), and references therein.

On the other hand, in contrast to the extensive studies on fractional Brownian motion, only few systematic investigations have been conducted on other self-similar Gaussian processes. The principal cause is the complexity of the dependency structures for self-similar Gaussian processes that don't have stationary increments. It therefore seems interesting to study some extensions of the fractional Brownian motion, such as weighted fractional Brownian motion and the bi-fractional Brownian motion. Recently, weighted fractional Brownian motion has attracted considerable attention from many researchers, see for instance (Shen et al. 2013), (Guangjun et al. 2016), and (Sun, Yan & Zhang 2017).

Nonparametric estimation of trend function for stochastic differential equations (SDEs) have been studied by several authors for their mathematical interest and their applications. The first papers on the subject is (Kutoyants 1994) for the stochastic differential equation driven by a standard Brownian motion. Then, the problem was generalized; (Kutoyants 1994) for the stochastic differential equation driven by a fractional Brownian motion; (Mishra & Prakasa Rao 2011a) for nonparametric estimation of linear multiplier for fractional diffusion processes; (Saussereau 2014) for nonparametric inference for fractional diffusion; and (Prakasa Rao 2019c) for nonparametric estimation of trend function for SDEs driven by mixed fractional Brownian motion.

In this investigation, we consider nonparametric estimation of trend function for stochastic differential equations driven by a weighted fractional Brownian motion (weighted-fBm). To the best of our knowledge, the problem of nonparametric estimation of trend function for stochastic differential equations driven by a weighted fractional Brownian motion (weighted-fBm) has not been attempted in the literature.

The rest of the chapter is organized as follows. In Section 4.2, the basic properties of weighted fractional Brownian motion are stated. Section 4.3 is devoted to the preliminaries. Then, in Section 4.4, we give the main results; under some hypotheses, we establish the consistent uniform (Theorem 4.2), the rate of convergence (Theorem 4.3) as well as the asymptotic normality (Theorem 4.4) of the estimator. Further, in Section 4.5, a numerical examples. Section 4.6 is dedicated to the technical proofs. Finally, we conclude the paper in Section 4.7.

## 4.2 Weighted fractional Brownian motion

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis satisfying the habitual hypotheses, i.e., a filtered probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and $\mathcal{F}_0$ contains every $\mathbb{P}$-null set.
Let $\{B_t^{a,b}, t \geq 0\}$ be a weighted fractional Brownian motion with parameters $(a,b)$ such that $a > -1$, $\mid b \mid< 1$, and $\mid b \mid< a+1$ is a centered and self-similar Gaussian process with long/short-range dependence and the covariance function:

$$R_{a,b}(t,s) = \mathbb{E}\left(B_t^{a,b} B_s^{a,b}\right) = \int_0^{s\wedge t} u^a \left[(t-u)^b + (s-u)^b\right] du, \quad s,t \geq 0.$$

Clearly, for $a = b = 0$, $B^{a,b}$ coincides with the standard Brownian motion $B$. When $a = 0$, we have

$$\mathbb{E}\left(B_t^{a,b} B_s^{a,b}\right) = \frac{1}{b+1}\left[t^{b+1} + s^{b+1} - |s-t|^{b+1}\right] du;$$

which is the covariance function of the fBm with Hurst index $(b+1)/2$ when $|b| < 1$.

According to (Bojdecki et al. 2007, Bojdecki et al. 2008a), and (Yan et al. 2014), the weighted-fBm has the following properties:

1. $\mathbb{E}\left(B_t^{a,b}\right) = 0$ and $Var\left(B_t^{a,b}\right) = 2\int_0^t u^a (t-u)^b du \quad \forall t \in [0,T]$.

2. $B_t^{a,b}$ is said to be self-similar with index $\dfrac{a+b+1}{2}$, that is, for every constant $\alpha > 0$,
$$\left\{B_{\alpha t}^{a,b}, t \geq 0\right\} \triangleq \left\{\alpha^{\frac{a+b+1}{2}} B_t^{a,b}, t \geq 0\right\}, \text{ for each } a > 0 \tag{4.1}$$
in the sense that the processes, on both the sides of the equality sign, have the same finite dimensional distributions.

3. The process $B_t^{a,b}$ is not Markov and it is not a semi-martingale if $b \neq 0$.

4. The process $B_t^{a,b}$ has independent increments for $b = 0$.

5. The trajectories of the process $B^{a,b}$ are Hölder continuous of order $\delta$ for any $\delta < \frac{1}{2}(b+1)$.

6. The trajectories of the process $B^{a,b}$ are continuous with the only exception of the case $a<0$, $b<0$, $a+b=-1$, where $B^{a,b}$ is discontinuous at 0.

7. The weighted-fBm $B_t^{a,b}$ has not stationary increments in general, we have

$$c_{a,b}(t \vee s)^a |t-s|^{b+1} \leq \mathbb{E}\left[B_t^{a,b} - B_s^{a,b}\right]^2 \leq C_{a,b}(t \vee s)^a |t-s|^{b+1}.$$

8. $B_t^{a,b}$ is long-range dependent for $b>0$ and short-range dependent for $b<0$.

The stochastic calculus with respect to the weighted-fBm has been recently developed by (Kruk et al. 2007). More works on weighted-fBm can be found in (Nualart 2006) and (Mazet et al. 2001).
Fix a time interval $[0,T]$, we denote by $\mathcal{E}$ the set of step function on $[0,T]$. Let $\mathcal{H}_{a,b}$ be the Hilbert space defined as the completion of the linear space $\mathcal{E}$ generated by the indicator functions $\left\{\mathbf{1}_{[0,t]};\ 0 \leq t \leq T\right\}$, with respect to the scalar product

$$\left\langle 1_{[0,t]}, 1_{[0,s]}\right\rangle_{\mathcal{H}_{a,b}} = R^{a,b}(t,s),$$

where $R^{a,b}(t,s)$ is the covariance of $B_t^{a,b}$ and $B_s^{a,b}$. The application

$$\varphi \in \mathcal{E} \longrightarrow B^{a,b}(\varphi) = \int_0^T \varphi(u) dB_u^{a,b}$$

is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^{a,b}$ and it can be extended to $\mathcal{H}_{a,b}$. Then $B^{a,b}(\varphi)$ is a Gaussian process on $\mathcal{H}_{a,b}$ such that, for all $\varphi, \psi \in \mathcal{H}_{a,b}$ we have

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{a,b}} = \mathbb{E}\left(B^{a,b}(\varphi) B^{a,b}(\psi)\right) = \int_0^T \int_0^T \varphi(u)\psi(v)\phi_{a,b}(u,v)\,du\,dv, \tag{4.2}$$

with

$$\phi_{a,b}(u,v) := \frac{\partial^2 R^{a,b}(u,v)}{\partial u \partial v} = b(u \wedge v)^a (u \vee v - u \wedge v)^{b-1}.$$

When $0<b<1$, the Hilbert space $\mathcal{H}_{a,b}$ can be written as

$$\mathcal{H}_{a,b} = \left\{\varphi : [0,T] \longrightarrow \mathbb{R};\ \|\varphi\|_{\mathcal{H}_{a,b}} < \infty\right\};$$

where

$$\|\varphi\|^2_{\mathcal{H}_{a,b}} = \int_0^T \int_0^T \varphi(u)\varphi(v)\phi_{a,b}(u,v)\,du\,dv. \tag{4.3}$$

We can use the subspace $\left|\mathcal{H}_{a,b}\right|$ of $\mathcal{H}_{a,b}$ that is defined as the set of measurable function $\varphi$ on $[0,T]$ such that

$$\|\varphi\|^2_{|\mathcal{H}_{a,b}|} = \int_0^T \int_0^T |\varphi(u)||\varphi(v)|\phi_{a,b}(u,v)\,du\,dv < \infty. \tag{4.4}$$

It was shown that $\left|\mathcal{H}_{a,b}\right|$ is a Banach space with the norm $\|\varphi\|_{|\mathcal{H}_{a,b}|}$ and $\mathcal{E}$ is dense in $\left|\mathcal{H}_{a,b}\right|$.

Assume that $a > -1$, $0 < b < 1$, $b < a+1$ and $a+b > 0$. Then, the canonical Hilbert space $\mathcal{H}_{a,b}$ associated to $B^{a,b}$ satisfies:

$$\mathbb{L}^2([0,T]) \subset \mathbb{L}^{\frac{2}{a+b+1}}([0,T]) \subset \left|\mathcal{H}_{a,b}\right| \subset \mathcal{H}_{a,b}. \tag{4.5}$$

## 4.3 Preliminaries

Let $\{X_t, 0 \le t \le T\}$ be a process governed by the following equation:

$$dX_t = S(X_t)dt + \varepsilon dB_t^{a,b},\ X_0 = x_0,\ 0 \le t \le T \tag{4.6}$$

where $\varepsilon > 0$, $B_t^{a,b}$ a weighted-fBm, and $S(.)$ is an unknown function. We suppose that $x_t$ is a solution of the equation

$$\frac{dx_t}{dt} = S(x_t),\ x_0,\ 0 \le t \le T. \tag{4.7}$$

We suppose also that the function $S : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(**A1**) There exists $L > 0$ such that

$$\left|S(x) - S(y)\right| \le L\left|x - y\right|,\ 0 \le t \le T. \tag{4.8}$$

(**A2**) There exists $M > 0$ such that

$$|S(x)| \le M(1 + |x|),\ x \in \mathbb{R},\ 0 \le t \le T.$$

Then, the stochastic differential equation (4.6) has a unique solution $\{X_t, 0 \le t \le T\}$.

(**A3**) Assume that the function $S(x)$ is bounded by a constant $C$.

Since the function $x_t$ satisfies (4.7), it follows that

$$|S(x_t) - S(x_s)| \le L|x_t - x_s| = L\left|\int_s^t S(x_r)dr\right| \le LC|t-s|,\ t,s \in \mathbb{R}.$$

Let us define $\Sigma_0(L)$ as the class of all functions $S(x)$ satisfying the assumption **(A1)** and uniformly bounded by the same constant $C$. Let us also denote by $\Sigma_k(L)$ the class of all function $S(x)$ which are uniformly bounded by same constant $C$ and which are k-times differentiable with respect to $x$ satisfying the condition

$$\left|S^k(x) - S^k(y)\right| \le L\left|x - y\right|,\ x,y \in \mathbb{R}, \tag{4.9}$$

where $S^k(x)$ denote the k-th derivative of $S(x)$.

**Lemma 4.1.** *Assume that hypothesis **(A1)** is verified. Let $X_t$ and $x_t$ be the solutions of the equations (4.6) and (4.7) respectively. Then, we have*

$$(i)\ |X_t - x_t| \leq e^{Lt}\varepsilon|B_t^{a,b}|. \tag{4.10}$$

*and*

$$(ii)\ \sup_{0\leq t\leq T} \mathbb{E}(X_t - x_t)^2 \leq e^{2LT}\varepsilon^2 2\int_0^T u^a(T-u)^b\,du. \tag{4.11}$$

**Proof.**

**Proof of (i).**

By (4.6) and (4.7), we have

$$\begin{aligned}|X_t - x_t| &\leq \int_0^t |S(X_r) - S(x_r)|\,dr + \varepsilon|B_t^{a,b}|\\ &\leq L\int_0^t |X_r - x_r|\,dr + \varepsilon|B_t^{a,b}|.\end{aligned}$$

Putting $u_t = |X_t - x_t|$, we have

$$u_t \leq \int_0^t u_r dr + \varepsilon|B_t^{a,b}|.$$

Finally, by using Gronwalls inequality, we obtain

$$|X_t - x_t| \leq e^{Lt}\varepsilon\left|B_t^{a,b}\right|. \tag{4.12}$$

**Proof of (ii).**

From (4.12), we have

$$\mathbb{E}|X_t - x_t|^2 \leq e^{2Lt}\varepsilon^2\mathbb{E}\left|B_t^{a,b}\right|^2.$$

Then

$$\mathbb{E}|X_t - x_t|^2 \leq e^{2Lt}\varepsilon^2 2\int_0^t u^a(t-u)^b\,du.$$

Finally, we obtain

$$\sup_{0\leq t\leq T} \mathbb{E}(X_t - x_t)^2 < e^{2LT}\varepsilon^2 2\int_0^T u^a(T-u)^b\,du.$$

□

## 4.4 Main results

The main goal of this work is to build a estimator of the trend function $S_t$ in the model described by stochastic differential equation (4.6) using the method developed by (Kutoyants 1994). Then, we study of asymptotic properties of the estimator

as $\varepsilon \longrightarrow 0$.
For all $t \in [0,T]$, the kernel estimator $\hat{S}_t$ of $S_t$ is given by

$$\hat{S}_t = \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right) dX_\tau, \tag{4.13}$$

where $G(u)$ is a bounded function with finite support $[A,B]$ satisfying the following hypotheses

**(H1)** $G(u) = 0$ for $u < A$ and $u > B$, and $\int_A^B G(u)du = 1$,

**(H2)** $\int_{-\infty}^{+\infty} G^2(u)du < \infty$,

**(H3)** $\int_{-\infty}^{+\infty} u^{2(k+1)}G^2(u)du < \infty$,

**(H4)** $\int_{-\infty}^{+\infty} |G(u)|^{\frac{2}{a+b+1}} du < \infty$,

Further, we suppose that the normalizing function $\phi_\varepsilon$ satisfies:
**(H5)** $\phi_\varepsilon \longrightarrow 0$ and $\varepsilon^2\phi_\varepsilon^{-1} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

The following Theorem gives uniform convergence of the estimator $\hat{S}_t$.

**Theorem 4.2.** *Suppose that the assumptions **(A1)**-**(A3)** and **(H1)**-**(H5)** hold true. Further, suppose that the trend function $S(x)$ belongs to $\Sigma_0(L)$. Then, for any $0 < c \le d < T$, $a > -1, 0 < b < 1, b < a+1$, and $a+b > 0$, the estimator $\hat{S}_t$ is uniformly consistent, that is,*

$$\lim_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_0(L)} \sup_{c\le t\le d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2) = 0. \tag{4.14}$$

The following additional assumptions are useful for the rest of the theoretical study. Assume that

**(H6)** $\int_{-\infty}^{+\infty} u^j G(u)du = 0$ for $j = 1,2,\ldots,k$,

**(H7)** $\int_{-\infty}^{+\infty} u^{k+1}G(u)du < \infty$; and $\int_{-\infty}^{+\infty} u^{2(k+2)}G^2(u)du < \infty$.

The rate of convergence of the estimator $\hat{S}_t$ is established in the following Theorem.

**Theorem 4.3.** *Suppose that the function $S(x) \in \Sigma_k(L)$, $a > -1, 0 < b < 1, b < a+1, a+b > 0$, and $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$. Then, under the hypotheses **(A1)**-**(A3)** and **(H1)**-**(H7)**, we have*

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_k(L)} \sup_{c\le t\le d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2)\varepsilon^{\frac{-4(k+1)}{2k-a-b+3}} < \infty. \tag{4.15}$$

Finally, the following Theorem presents the asymptotic normality of the kernel type estimator $\hat{S}_t$ of $S(x_t)$.

**Theorem 4.4.** *Suppose that the function $S(x) \in \Sigma_{k+1}(L)$, $a > -1$, $0 < b < 1$, $b < a+1$, $a+b > 0$, and $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$. Then, under the hypotheses **(A1)-(A3)** and **(H1)-(H7)**, we have*

$$\varepsilon^{\frac{-2(k+1)}{2k-a-b+3}}\left(\hat{S}_t - S(x_t)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(m, \sigma^2_{a,b}\right), \text{ as } \varepsilon \longrightarrow 0,$$

*where*

$$m = \frac{S^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G(u)u^{k+1}du,$$

*and*

$$\sigma^2_{a,b} = b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)(u\wedge v)^a(u\vee v - u\wedge v)^{b-1}dudv,$$

## 4.5 Numerical example

In this section, we present a numerical analysis illustrating our theoretical result. We compare our kernel estimator for stochastic differential equations driven by a weighted fractional Brownian motion to the kernel estimator for stochastic differential equations driven by fractional Brownian motion given in (Mishra & Prakasa Rao 2011b). The variances of the two kernel estimators are compared. $\sigma^2_{a,b}$ of our estimator is compared to that of the estimator $\sigma^2_H$.

Consider a function $G$ verifying hypotheses **(H1)**-**(H7)**:

$$G(t) = \frac{3}{8}\left(3 - 5t^2\right), \ |t| \le 1.$$

**1.** The variance of the kernel estimator for stochastic differential equations driven by fractional Brownian motion given in (Mishra & Prakasa Rao 2011b) is given as:

For all $H \in (1/2, 1)$,

$$\sigma^2_H = H(2H-1)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\,|u-v|^{2H-2}\,dudv.$$

**2.** From Theorem 4.4, the variance of our estimator is as follows:

For all $a > -1$, $0 < b < 1$, $b < a+1$ we have

$$\sigma^2_{a,b} = b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)(u\wedge v)^a(u\vee v - u\wedge v)^{b-1}dudv.$$

By developing a program in **R** software, we compute the numerical values of the variances. The obtained results are arranged in the following tables.

Table 4.1: The variance values $\sigma^2_H$.

| $H$ | 0.51 | 0.54 | 0.57 | 0.6 | 0.63 | 0.66 | 0.69 | 0.72 | 0.75 | 0.78 | 0.81 | 0.84 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma^2_H$ | 0.0911 | 0.3240 | 0.5056 | 0.6461 | 0.7536 | 0.8351 | 0.8958 | 0.9403 | 0.9720 | 0.9935 | 1,0070 | 1.0143 |

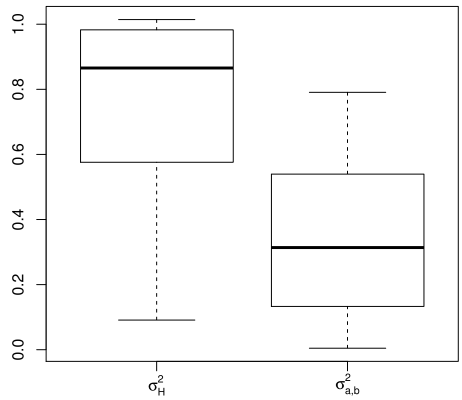
Table 4.2: The variance values $\sigma^2_{0,b}$.

| $b$ | 0.02 | 0.08 | 0.14 | 0.2 | 0.26 | 0.32 | 0.38 | 0.44 | 0.5 | 0.56 | 0.62 | 0.68 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma^2_{0,b}$ | 0.0911 | 0.3240 | 0.5056 | 0.6461 | 0.7536 | 0.8351 | 0.8958 | 0.9403 | 0.9720 | 0.9935 | 1.0070 | 1.0143 |

Table 4.3: The variance values $\sigma^2_{a,b}$.

| $a \setminus b$ | 0.51 | 0.54 | 0.57 | 0.6 | 0.63 | 0.66 | 0.69 | 0.72 | 0.75 | 0.78 | 0.81 | 0.84 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 0.7990 | 0.7907 | 0.7810 | 0.7702 | 0.7584 | 0.7459 | 0.7329 | 0.7195 | 0.7058 | 0.6919 | 0.6778 | 0.6637 |
| 0.3 | 0.5643 | 0.5569 | 0.5486 | 0.5395 | 0.5298 | 0.5196 | 0.5091 | 0.4983 | 0.4874 | 0.4764 | 0.4654 | 0.4543 |
| 0.4 | 0.3549 | 0.3484 | 0.3413 | 0.3338 | 0.3260 | 0.3179 | 0.3097 | 0.3014 | 0.2931 | 0.2848 | 0.2766 | 0.2684 |
| 0.5 | 0.1807 | 0.1751 | 0.1693 | 0.1633 | 0.1573 | 0.1512 | 0.1450 | 0.1390 | 0.1330 | 0.1271 | 0.1213 | 0.1157 |
| 0.6 | 0.0470 | 0.0424 | 0.0378 | 0.0332 | 0.0287 | 0.0243 | 0.0201 | 0.0160 | 0.0120 | 0.0082 | 0.0046 | 0.0012 |

Figure 4.1: $\sigma^2_H$ and $\sigma^2_{a,b}$ for $a \neq 0$.




According to Tables 4.1-4.3 and Figure 4.1, we have

- For $a = 0$, the variance values $\sigma^2_{a,b}$ are equal to the variance values $\sigma^2_H$. This is due to the property of weighted fractional Brownian motion (wfBm); when a = 0 this later coincides with the fractional Brownian motion (fBm) with Hurst index $H = (b+1)/2$.

- As expected, for a fixed $a$ (resp. $b$), the variance values of our estimator $\sigma^2_{a,b}$ decreases along the increasing of $b$ (resp. $a$). While, the numerical values of the variance $\sigma^2_H$ increases with the increases of the Hurst index $H$.

- For larger values of $a$, the values of $\sigma^2_{a,b}$ are smaller than $\sigma^2_H$, for instance, for $a =$ 0.5 (resp. $a = 0.4$) the values of $\sigma^2_{a,b}$ become smaller than that of $\sigma^2_H$ when $b \geq$ 0.54 (resp. $b \geq 0.57$). While for $a = 0.6$, and any values of $b$, the variance values of our estimator are smaller than that of (Mishra & Prakasa Rao 2011b). However, from Figure 4.1, we clearly observe that the variance values of our estimator is less than that of the kernel estimator for stochastic differential equations driven by fractional Brownian motion given in (Mishra & Prakasa Rao 2011b). Thus, we can say that our kernel estimator for trend function for stochastic differential equations driven by a weighted fractional Brownian motion is better than that given in (Mishra & Prakasa Rao 2011b).

## 4.6 Proof of Theorems

### Proof of Theorem 4.2

From (4.6) and (4.13), we can see that

$$\begin{aligned}
\mathbb{E}_S\left[\hat{S}_t - S(x_t)\right]^2 &\leq 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)(S(X_\tau)-S(x_\tau))\,d\tau\right]^2 \\
&\quad +3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\
&\quad +3\mathbb{E}_S\left[\frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{a,b}\right]^2. \\
&\leq I_1 + I_2 + I_3.
\end{aligned} \tag{4.16}$$

• **Concerning** $I_1$. Via the inequalities (4.8) and (4.11), and hypotheses **(H1)**-**(H2)**, we get

$$\begin{aligned}
I_1 &= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\left(S(X_{t+\phi_\varepsilon u})-S(x_{t+\phi_\varepsilon u})\right)du\right]^2 \\
&\leq 3(B-A)\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G^2(u)\left(S(X_{t+\phi_\varepsilon u})-S(x_{t+\phi_\varepsilon u})\right)^2 du\right] \\
&\leq 3(B-A)L^2\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G^2(u)\left(X_{t+\phi_\varepsilon u}-x_{t+\phi_\varepsilon u}\right)^2 du\right] \\
&\leq 3(B-A)L^2\left[\int_{-\infty}^{+\infty} G^2(u)\sup_{0\leq t\leq T}\mathbb{E}_S\left(X_{t+\phi_\varepsilon u}-x_{t+\phi_\varepsilon u}\right)^2 du\right] \\
&\leq 3(B-A)L^2e^{2LT}\varepsilon^2 2\int_0^T u^a(T-u)^b\,du \\
&\leq C_1\varepsilon^2,
\end{aligned} \tag{4.17}$$

where $C_1$ is a positive constant depending on $T, L, a, b$ and $(B-A)$.

• **Concerning** $I_2$. Let

$$\begin{aligned}
I_2 &= 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)S(x_{t+\phi_\varepsilon u})du - S(x_t)\right]^2 \\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\left(S(x_{t+\phi_\varepsilon u})-S(x_t)\right)du\right]^2
\end{aligned}$$

Next, by using hypotheses **(A3)**, **(H1)**, and **(H3)**, we have

$$
\begin{aligned}
I_2 &\leq 3L^2\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)(\phi_\varepsilon u)\,du\right]^2 \\
&\leq 3(B-A)L^2\left[\int_{-\infty}^{+\infty} G^2(u)u^2du\right]\phi_\varepsilon^2 \\
&\leq C_2\phi_\varepsilon^2,
\end{aligned} \tag{4.18}
$$

where $C_2$ is a positive constant depending on $L$ and $(B-A)$.

- **Concerning** $I_3$. Since $a>-1$, $0<b<1$, $b<a+1$, and $0<a+b<1$. we have

$$
\begin{aligned}
I_3 &= 3\mathbb{E}_S\left[\frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{a,b}\right]^2 \\
&\leq 3\frac{\varepsilon^2}{\phi_\varepsilon^2}\left[C(a,b)\left(\int_0^T\left|G\left(\frac{\tau-t}{\phi_\varepsilon}\right)\right|^{\frac{2}{a+b+1}}d\tau\right)^{a+b+1}\right] \\
&\leq C(a,b)\frac{\varepsilon^2}{\phi_\varepsilon^2}\left[\phi_\varepsilon^{a+b+1}\left(\int_{-\infty}^{+\infty}|G(u)|^{\frac{2}{a+b+1}}du\right)^{a+b+1}\right]
\end{aligned}
$$

Hence, by using the hypothesis **(H4)** we have

$$
I_3 \leq C_3\frac{\varepsilon^2}{\phi_\varepsilon}\phi_\varepsilon^{a+b} \tag{4.19}
$$

where $C_3$ is a positive constant depending on $a$ and $b$.

Combining (4.16), (4.17), (4.18), and (4.19), we have

$$
\sup_{S(x)\in\Sigma_0(L)}\sup_{c\leq t\leq d}\mathbb{E}_S\left[\hat{S}_t-S(x_t)\right]^2 \leq C_4\left(\varepsilon^2+\phi_\varepsilon^2+\frac{\varepsilon^2}{\phi_\varepsilon}\phi_\varepsilon^{a+b}\right).
$$

Finally, under the assumption **(H6)**, we obtain

$$
\lim_{\varepsilon\longrightarrow 0}\sup_{S(x)\in\Sigma_0(L)}\sup_{c\leq t\leq d}\mathbb{E}_S\left[\hat{S}_t-S(x_t)\right]^2=0.
$$

## Proof of Theorem 4.3

Using the Taylor formula, we easily get

$$
S\left(x_{t+\phi_\varepsilon u}\right)=S(x_t)+\sum_{j=1}^{k}S^j(x_t)\frac{(\phi_\varepsilon u)^j}{j!}+\left(S^k(x_{t+\lambda(\phi_\varepsilon u)})-S^k(x_t)\right)\frac{(\phi_\varepsilon u)^k}{k!},\ \lambda\in(0,1).
$$

Then, by substituting this expression in $I_2$, using (4.9), and assumptions (H6)-(H7), we obtain

$$\begin{aligned}
I_2 &= 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)S(x_{t+\phi_\varepsilon u})du - S(x_t)\right]^2 \\
&= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\Big(S(x_{t+\phi_\varepsilon u}) - S(x_t)\Big)du\right]^2 \\
&= 3\mathbb{E}_S\left[\frac{\phi_\varepsilon^k}{k!}\int_{-\infty}^{+\infty} G(u)u^k\Big(S^k(x_{t+\lambda(\phi_\varepsilon u)}) - S^k(x_t)\Big)du\right]^2 \\
&\leq 3C^2L^2\left[\frac{\phi_\varepsilon^{k+1}}{k!}\int_{-\infty}^{+\infty} G(u)u^{k+1}du\right]^2 \\
&\leq 3C^2L^2(B-A)\frac{\phi_\varepsilon^{2(k+1)}}{(k!)^2}\left[\int_{-\infty}^{+\infty} G^2(u)u^{2(k+1)}du\right] \\
&\leq C_5\phi_\varepsilon^{2(k+1)},
\end{aligned} \tag{4.20}$$

where $C_5$ is a positive constant depending on $L$ and $(B-A)$.
Next, from (4.17), (4.19), and (4.20), we find

$$\sup_{S(x)\in\Sigma_k(L)}\sup_{c\leq t\leq d}\mathbb{E}_S\left|\hat{S}_t - S(x_t)\right|^2 \leq C_6\left(\varepsilon^2\phi_\varepsilon^{a+b-1} + \phi_\varepsilon^{2(k+1)} + \varepsilon^2\right).$$

Putting $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$, it yields

$$\limsup_{\varepsilon\longrightarrow 0}\sup_{S(x)\in\Sigma_k(L)}\sup_{c\leq t\leq d}\mathbb{E}_S(|\hat{S}_t - S(x_t)|^2)\varepsilon^{\frac{-4(k+1)}{2k-a-b+3}} < \infty.$$

This completes the proof of Theorem (4.3).

## Proof of Theorem 4.4

From (4.6) and (4.13), we can see that

$$\begin{aligned}
\varepsilon^{\frac{-2(k+1)}{2k-a-b+3}}\left(\hat{S}_t - S(x_t)\right) &= \varepsilon^{\frac{-2(k+1)}{2k-a-b+3}}\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)(S(X_\tau) - S(x_\tau))d\tau\right. \\
&\quad \left. + \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t) + \frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{a,b}\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
\varepsilon^{\frac{-2(k+1)}{2k-a-b+3}}\left(\hat{S}_t - S(x_t)\right) &= \varepsilon^{\frac{-2(k+1)}{2k-a-b+3}}\left[\int_{-\infty}^{+\infty} G(u)\Big(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u})\Big)du\right. \\
&\quad \left. + \int_{-\infty}^{+\infty} G(u)\Big(S(x_{t+\phi_\varepsilon u}) - S(x_t)\Big)du + \frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{a,b}\right].
\end{aligned}$$

Thus
$$\varepsilon^{\frac{-2(k+1)}{2k-a-b+3}}\left(\hat{S}_t - S(x_t)\right) = r_1(t) + r_2(t) + \eta_\varepsilon(t).$$
Hence, by the Slutsky's Theorem, it suffices to show the following three claims:
$$r_1(t) \to 0,\ as\ \varepsilon \to 0 \text{ in probability,} \tag{4.21}$$

$$r_2(t) \to m,\ as\ \varepsilon \to 0 \text{ in probability.} \tag{4.22}$$
and
$$\eta_\varepsilon(t) \to \mathcal{N}\left(0, \sigma^2_{a,b}\right),\ as\ \varepsilon \to 0 \text{ in distribution.} \tag{4.23}$$

***Proof of (4.21).***

Let
$$r_1(t) = \varepsilon^{\frac{-2(k+1)}{2k-a-b+3}} \int_{-\infty}^{+\infty} G(u)\left(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u})\right) du.$$
By applying the inequality (4.17), we have
$$0 \le \mathbb{E}\left[r_1^2(t)\right] \le C_{10}\varepsilon^{\frac{-2(a+b-1)}{2k-a-b+3}}.$$

Then
$$r_1(t) \to 0,\ as\ \varepsilon \to 0 \text{ in probability.}$$

***Proof of (4.22).***

Let
$$r_2(t) = \varepsilon^{\frac{-2(k+1)}{2k-a-b+3}} \int_{-\infty}^{+\infty} G(u)\left(S(x_{t+\phi_\varepsilon u}) - S(x_t)\right) du.$$
Take any $t \in [0,T]$, $u \in [0,T]$, via the Taylor expansion
$$\begin{aligned} S\left(x_{t+\phi_\varepsilon u}\right) &= S(x_t) + \sum_{j=1}^{k} S^j(x_t)\frac{(\phi_\varepsilon u)^j}{j!} + \frac{S^{k+1}(x_t)}{(k+1)!}(\phi_\varepsilon u)^{k+1} \\ &\quad + \left(S^{k+1}(x_{t+\lambda(\phi_\varepsilon u)}) - S^{k+1}(x_t)\right)\frac{(\phi_\varepsilon u)^{k+1}}{(k+1)!},\ \lambda \in (0,1), \end{aligned}$$
using the conditions (H6), (H7), and choosing $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$, we get
$$\begin{aligned} \mathbb{E}[r_2(t) - m]^2 &= \mathbb{E}\left[\int_{-\infty}^{+\infty} G(u)\left(S^{k+1}(x_{t+\lambda(\phi_\varepsilon u)}) - S^{k+1}(x_t)\right)\frac{(\phi_\varepsilon u)^{k+1}}{(k+1)!}du\right]^2 \\ &\le L^2C^2\left(\int_{-\infty}^{+\infty} G(u)u^{k+2}\frac{\phi_\varepsilon^{k+2}}{(k+1)!}du\right)^2 \\ &\le C_8\phi_\varepsilon^{2(k+2)}, \end{aligned}$$

where $C_8$ is a positive constant depending on $T, L, a, b$ and $k$;
and

$$m = \frac{S^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G(u)u^{k+1}du.$$

Therefore

$$\mathbb{E}[r_2(t)-m]^2 \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.$$

Then

$$r_2(t) \longrightarrow m, \, as \; \varepsilon \to 0 \text{ in probability.}$$

***Proof of (4.23).***
Let

$$\eta_\varepsilon(t) = \varepsilon^{\frac{-2(k+1)}{2k-a-b+3}}\varepsilon\phi_\varepsilon^{-1}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{a,b}. \tag{4.24}$$

In fact, we have to evaluate the variance of (4.24). To this end, let

$$\mathbb{E}[\eta_\varepsilon(t)]^2 = \varepsilon^{\frac{-2(a+b-1)}{2k-a-b+3}}\phi_\varepsilon^{-2}\mathbb{E}\left(\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{a,b}\right)^2.$$

Moreover, using (4.2), $a > -1$, $0 < b < 1$, $b < a+1$, and $a+b > 0$, we have

$$\mathbb{E}[\eta_\varepsilon(t)]^2 = \varepsilon^{\frac{-2(a+b-1)}{2k-a-b+3}}\phi_\varepsilon^{-2}\left[\phi_\varepsilon^{a+b+1}b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)(u\wedge v)^a(u\vee v - u\wedge v)^{b-1}dudv\right].$$

Then, by taking $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$, we get

$$\mathbb{E}[\eta_\varepsilon(t)]^2 = b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)(u\wedge v)^a(u\vee v - u\wedge v)^{b-1}dudv,$$

Finally, this last equation allows us to achieve the proof of Theorem (4.4).

## 4.7 Conclusion

In this chapter, we studied the problem of nonparametric estimation of the trend function for stochastic differential equations driven by a weighted fractional Brownian motion. We presented the kernel estimator of the trend function based on the continuous observation and obtained the uniform convergence (with rate) and the asymptotic normality of the proposed estimator. For further work, it will be interesting to investigate the problem of estimating the trend function for SDE driven by a generalized fractional Brownian motion such as, sub-fractional Brownian motion, mixed sub-fractional Brownian motion, and Lévy fractional Brownian motion.

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# CHAPTER 5

# Asymptotic analysis of a kernel type estimator of trend function and linear multiplier for stochastic differential equation with additive fractional Brownian sheet

This chapter was submitted for publication

## Sommaire

## 5.1 Introduction

Over the two past decades, studies of statistical inference from observations drawn from stochastic differential equations driven by fractional Brownian motion (fBm) have experienced significant growth. This process was introduced by (Kolmogorov 1940), then studied by many researchers including (Mandelbrot & Van Ness 1968) and (Norros et al. 1999). The fractional Brownian sheet, an extension of fractional Brownian motion, is of great significance for diverse applied areas especially in satellite imaging (Pesquet-Popescu & Véhel 2002), radar image data classification (Chao & Lin 1997), the classification and segmentation of hydrological basins (Maître & Pinciroli 1999) and medical applications like early detection of osteoporosis from $X$-ray images (Léger 2000). The subject of this chapter concerns the nonparametric estimation problem of the drift coefficient of the stochastic differential equations driven by a fractional Brownian sheet

$$dX_{s,t} = \rho(X_{s,t})dsdt + \varepsilon dW^{\alpha,\beta}_{s,t},\ X_{0,t} = X_{s,0} = x_0,\ 0 \le s,t \le T, \tag{5.1}$$

where $x_0 \in \mathbb{R}$ and $W^{\alpha,\beta} = \left\{W^{\alpha,\beta}_{s,t};\ s,t \ge 0\right\}$ is a fractional Brownian sheet with parameter $(\alpha,\beta) \in (0,1)^2$.

Existence and uniqueness of the solution of the stochastic differential equation (5.1) for parameters $(\alpha,\beta) \in (1/2,1)^2$ has been considered in (El Barrimi & Ouknine 2019). It has been proved that, if $\rho(.)$ satisfy the linear growth condition, then the equation (5.1) admits a unique weak solution.

As a result, various researchers are interested in the study estimation problems for stochastic differential equations (5.1). The parametric estimation for stochastic differential equations with additive fractional Brownian sheet was given in (Sottinen & Tudor 2008). (Mendy & Yodé 2010) established a minimum distance estimation in $\mathbb{L}^2$-norm for a stochastic equation with additive fractional Brownian sheet. Later, (Prakasa Rao 2011) studied the asymptotic behaviour of the maximum likelihood estimator and the Bayes estimator for the parameter drift for SDEs driven by a fractional Brownian sheet. For more references, authors may refer to (De la Cerda & Tudor 2012), (Liu 2013) and references therein.

Nonparametric estimation for stochastic differential equations driven by a fractional Brownian motion has attracted the interest of many researchers. We refer, among others, to (Mishra & Prakasa Rao 2011b, Mishra & Prakasa Rao 2011a, Deriyeva & Shpyga 2016, Prakasa Rao 2019b, Prakasa Rao 2019b, Keddi, Madani & Bouchentouf 2020, Keddi, Madani & Bouchentouf 2020b)

In this chapter, we establish a nonparametric estimation of trend function and linear multiplier for stochastic differential equation driven by fractional Brownian sheet. To the best of authors's knowledge, no similar work has been attempted in the literature. The layout of the chapter is as follows. The layout of the chapter is as follows. In Section 5.2, we state the elementary properties of fractional Brownian sheet. Then, in Section 5.3, under some hypotheses, we establish the uniform consistency (Theorem 5.4), the rate of convergence (Theorem 5.5) as well as the asymptotic normality (Theorem 5.6) of the kernel estimator of trend function. Further, in Section 5.4, under some hypotheses, we establish the uniform consistency (Theorem 5.7) and the rate of convergence (Corollary 5.8) of the kernel estimator of multiplier function. Then, we conclude the chapter in Section 5.5. Finally, the proofs of the main results are given in Section 5.6.

## 5.2 Preliminaries

Consider $\{B_t^\alpha, t \geq 0\}$ a normalized fractional Brownian motion with parameter $\alpha \in (0,1)$, with $B_0^\alpha = 0$ let $R_\alpha$ its covariance function:

$$R_\alpha(t,s) = \mathbb{E}(B_t^\alpha B_s^\alpha) = \frac{1}{2}\left[t^{2\alpha} + s^{2\alpha} - |s-t|^{2\alpha}\right], \quad t \geq 0,\ s \geq 0, \tag{5.2}$$

for every $s,t \in [0,T]$. It is well-known that $B_\alpha$ admits the Wiener integral representation

$$B_t^\alpha = \int_0^t K_\alpha(t,s) dW_s,$$

where $W$ denotes a standard Wiener process and

$$K_\alpha(t,s) = d_\alpha (t-s)^{\alpha-1/2} + s^{\alpha-1/2} F\left(\frac{t}{s}\right), \tag{5.3}$$

$d_\alpha$ being a constant and

$$F(z) = d_\alpha (1/2 - \alpha) \int_0^{z-1} r^{\alpha-3/2}\left(1 - (r+1)^{\alpha-1/2}\right) dr.$$

Let $\mathcal{H}(\alpha)$ be the Hilbert space associated to $B^\alpha$ defined as the closure of the linear space generated by the indicator functions $\left\{1_{[0,t]},\ t \in [0,T]\right\}$ with respect to the scalar product

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}(\alpha)} = R_\alpha(t,s). \tag{5.4}$$

If $\alpha \in (1/2,1)$, the elements of $\mathcal{H}(\alpha)$ may be not functions but distributions of negative order (Pipiras & Taqqu 2000). It is therefore more practical to work with subspaces of $\mathcal{H}(\alpha)$. A such space is the set $|\mathcal{H}(\alpha)|$ of measurable functions on $[0,T]$ such that

$$\int_0^T \int_0^T |f(u)||f(v)||u-v|^{2\alpha-2} du dv < \infty \tag{5.5}$$

endowed with the scalar product

$$\langle f,g\rangle_{|\mathcal{H}(\alpha)|} = \alpha(2\alpha-1)\int_0^T\int_0^T |f(u)||g(v)||u-v|^{2\alpha-2}dudv. \tag{5.6}$$

Next, let us present he following inclusions

$$\mathbb{L}^2([0,T]) \subset \mathbb{L}^{\frac{1}{\alpha}}([0,T]) \subset |\mathcal{H}(\alpha)| \subset \mathcal{H}(\alpha). \tag{5.7}$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_{s,t}\}_{s,t\geq 0}, \mathbb{P})$ be a stochastic basis satisfying the habitual hypotheses, i.e., a filtered probability space with a right continuous filtration $\{\mathcal{F}_{s,t}\}_{s,t\geq 0}$ and $\mathcal{F}_0$ contains every $\mathbb{P}$-null set.

Let us consider now the two-parameter case. The fractional Brownian sheet with Hurst index $(\alpha,\beta)\in(0,1)^2$ is a Gaussian process $W^{\alpha,\beta} = \left\{W^{\alpha,\beta}_{s,t},\ 0\leq s,t\leq T\right\}$ starting from zero with mean zero and covariance function

$$R_{\alpha,\beta}(s,t,u,v) := \frac{1}{2}\left(t^{2\alpha}+v^{2\alpha}-|t-v|^{2\alpha}\right)\frac{1}{2}\left(s^{2\beta}+u^{2\beta}-|s-u|^{2\beta}\right), \tag{5.8}$$

for all $(s,t,u,v)\in[0,T]^4$. Note that in case $\alpha=\beta=1/2$, one can gets a standard Brownian sheet.

The fractional Brownian sheet $W^{\alpha,\beta}$ can be defined as

$$W^{\alpha,\beta}_{s,t} = \int_0^t\int_0^s K_\alpha(s,u)K_\beta(t,v)dW_{u,v},$$

where $\{W_{u,v},\ 0\leq u,v\leq T\}$ is a standard Brownian sheet and $K_\alpha(s,u)$ is given by (5.3). Let

$$K_{\alpha,\beta}(s,t) = K_\alpha(s,u)K_\beta(t,v), \tag{5.9}$$

and denote by $\mathcal{H}^{(2)}(\alpha,\beta) := \mathcal{H}^{(2)}$ the canonical Hilbert space of the fractional Brownian sheet $W^{\alpha,\beta}_{s,t}$. That is, $\mathcal{H}^{(2)}$ is defined as the closure of the set of indicator functions $\left\{1_{[0,t]\times[0,s]},\ t,s\in[0,T]\right\}$ with respect to the scalar product

$$\left\langle 1_{[0,s]\times[0,t]}; 1_{[0,u]\times[0,v]}\right\rangle_{\mathcal{H}^{(2)}} = R_{\alpha,\beta}(s,t,u,v), \tag{5.10}$$

with $(s,t,u,v)\in[0,T]^4$.

If $(\alpha,\beta)\in(1/2,1)^2$, the elements of $\mathcal{H}^{(2)}$ may be not functions but distributions. Therefore, it is more suitable to consider subspaces of $\mathcal{H}^{(2)}$ that are sets of functions. In fact, we suppose that we have the following inclusions

$$\mathbb{L}^2([0,T]^2) \subset |\mathcal{H}|^{(2)} \subset \mathcal{H}^{(2)}, \tag{5.11}$$

where

$$|\mathcal{H}|^{(2)} = |\mathcal{H}(\alpha)| \otimes |\mathcal{H}(\beta)|$$

and $|\mathcal{H}(\alpha)|$ defined by (5.6).

A such space is the set $|\mathcal{H}^{(2)}|$ of measurable functions on $[0,T]^2$, where

$$\|f\|^2_{|\mathcal{H}|^{(2)}} = \int_0^T \int_0^T \int_0^T \int_0^T |f(s,t)||f(u,v)|\phi_{\alpha,\beta}(s,t,u,v)dsdtdudv, \tag{5.12}$$

with

$$\phi_{\alpha,\beta}(s,t,u,v) = \alpha(2\alpha-1)\beta(2\beta-1)|s-u|^{2\alpha-2}|t-v|^{2\beta-2}. \tag{5.13}$$

Note that, if $f,\ g \in \mathcal{H}^{(2)}$, then their scalar product in $\mathcal{H}^{(2)}$ is given by

$$\langle f,g\rangle_{\mathcal{H}^{(2)}} = \int_0^T \int_0^T \int_0^T \int_0^T f(s,t)g(u,v)\phi_{\alpha,\beta}(s,t,u,v)dsdtdudv.$$

For the one-parameter case (see (Norros et al. 1999)) we can associate to the fractional Brownian sheet a two-parameter martingale (named fundamental martingale). We refer to (Tudor & Tudor 2005) for the two-parameter case. Specifically, let us define the deterministic function

$$c_\alpha = 2\alpha\Gamma(3/2-\alpha)\Gamma(\alpha+1/2), \quad k_\alpha(s,t) = c_\alpha^{-1}s^{1/2-\alpha}(t-s)^{1/2-\alpha}; \tag{5.14}$$

and

$$\lambda_\alpha = \frac{2\Gamma(3-2\alpha)\Gamma(\alpha+1/2)}{\Gamma(3/2-\alpha)}, \quad w_t^\alpha = \lambda_\alpha^{-1}t^{2-2\alpha}. \tag{5.15}$$

Then the process

$$M_{s,t}^{\alpha,\beta} = \int_0^t \int_0^s k_\alpha(t,v)k_\beta(s,u)dW_{u,v}^{\alpha,\beta} \tag{5.16}$$

is a two-parameter Gaussian martingale with quadratic variation over $[0,s]\times[0,t]$ given as

$$\left\langle M^{\alpha,\beta}\right\rangle_{s,t} = w_t^\alpha w_s^\beta = \lambda_\alpha^{-1}s^{2-2\alpha}\lambda_\beta^{-1}t^{2-2\beta}. \tag{5.17}$$

It should be pointed out that the stochastic integral in (5.16) can be defined in a Wiener sense with respect to the fractional Brownian sheet. The filtration generated by $M^{\alpha,\beta}$ coincides to the one generated by $W^{\alpha,\beta}$.

## 5.3 Trend function estimation

Let $\{X_{s,t},\ 0\le s,t\le T\}$ be a process governed by the following equation:

$$dX_{s,t} = \rho(X_{s,t})dsdt + \varepsilon dW_{s,t}^{\alpha,\beta},\ X_{0,t} = X_{s,0} = x_0,\ 0\le s,t\le T, \tag{5.18}$$

where $x_0\in\mathbb{R}$, $\varepsilon>0$ and $W_{s,t}^{\alpha,\beta}$ a fractional Brownian sheet, and $\rho(.)$ is an unknown function. Assume that the stochastic process $\{X_{s,t},\ 0\le s,t\le T\}$ is observed over $[0,T]^2$. The main goal of this section is to build an estimator of the trend function $\rho_{s,t} = \rho(x_{s,t})$, $0\le s,t\le T$ using the method developed by (Kutoyants 1994). Then, we study the uniform consistency, the rate of convergence and the asymptotic normality of this estimator as $\varepsilon \longrightarrow 0$. We suppose that $x_{s,t}$ is a solution of the following equation

$$dx_{s,t} = \rho(x_{s,t})dsdt,\ x_0,\ 0 \le s,t \le T. \tag{5.19}$$

We also suppose that the function $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(**A1**) There exists $L_1, M > 0$ such that

$$\begin{cases} (i) & \left|\rho(x) - \rho(y)\right| \le L_1 \left|x - y\right|,\ x, y \in \mathbb{R}. \\ (ii) & \left|\rho(x)\right| \le M(1 + |x|),\ x \in \mathbb{R}. \end{cases}$$

(**A2**) There exists $L_2 > 0$ such that

$$\left|\rho(x_{s_2,t_2}) - \rho(x_{s_1,t_1})\right| \le L_2\left(|s_2 - s_1| + |t_2 - t_1|\right);\ s_1, s_2, t_1, t_2 \ge 0. \tag{5.20}$$

Assumption **(A1)-(ii)** ensures existence and uniqueness of a weak solution $\{X_{s,t},\ 0 \le s,t \le T\}$ to (5.18) (see (El Barrimi & Ouknine 2019)).

Following (Kutoyants 1994) and (Mishra & Prakasa Rao 2011b), we define the two dimensional kernel estimator $\tilde{\rho}_{s,t}$ of $\rho(x_{s,t})$ by

$$\tilde{\rho}_{s,t} = \int_0^T \int_0^T G_{\varphi,\psi}(u,v)\, dX_{u,v},\ 0 \le s,t \le T, \tag{5.21}$$

with

$$G_{\varphi,\psi}(u,v) = \frac{1}{\varphi_\varepsilon \psi_\varepsilon} G\left(\frac{u-s}{\varphi_\varepsilon}, \frac{v-t}{\psi_\varepsilon}\right),\ 0 \le s,t \le T, \tag{5.22}$$

where $G(u,v)$ is a bounded function with finite support $[A,B]^2$. We assume that $G(u,v) = G_1(u)G_2(v)$ is satisfying the following hypotheses

$$\textbf{(H1)} \quad \begin{cases} (i) \quad G(u,v) = 0 \quad \text{for} \quad (u,v) \notin [A,B]^2, \text{ and } \displaystyle\int_A^B \int_A^B G(u,v)dudv = 1. \\ (ii) \quad \displaystyle\int_{-\infty}^{+\infty} G_1^2(u)du < \infty, \quad \text{and} \quad \int_{-\infty}^{+\infty} G_2^2(v)dv < \infty. \end{cases}$$

**(H2)** $\displaystyle\int_{-\infty}^{+\infty} |u|^2\, G_1^2(u)du < \infty,$ and $\displaystyle\int_{-\infty}^{+\infty} |v|^2\, G_2^2(v)dv < \infty.$

**(H3)** $\displaystyle\int_{-\infty}^{+\infty} |G_1(u)|^{\frac{1}{\alpha}}\, du < \infty,$ and $\displaystyle\int_{-\infty}^{+\infty} |G_2(v)|^{\frac{1}{\beta}}\, dv < \infty.$

Further, we suppose that the normalizing functions $\varphi_\varepsilon$ and $\psi_\varepsilon$ satisfies:

**(H4)** $\varphi_\varepsilon,\ \psi_\varepsilon \longrightarrow 0$ and $\varepsilon^2 \varphi_\varepsilon^{-1} \psi_\varepsilon^{-1} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

### 5.3.1 Consistency of the estimator

This subsection considers the consistency of the kernel estimator $\tilde{\rho}_{s,t}$. Before we state our main result, we have to present some preliminary lemmas needed for the sequel of the chapter.

**Lemma 5.1.** *(Pachpatte & Pachpatte 2002) Let $u(s,t)$, $a(s,t)$, and $b(s,t)$ be real-valued non negative continuous functions defined for $s,t \geq 0$ and suppose that $a(s,t)$ is non increasing in $t \geq 0$ and $s \geq 0$. If*

$$u(s,t) \leq a(s,t) + \int_t^\infty \int_s^\infty b(x,y)u(x,y)dxdy, \ \ s,t \geq 0.$$

*Then,*

$$u(s,t) \leq a(s,t)\exp\left\{\int_t^\infty \int_s^\infty b(x,y)dxdy\right\}, \ \ s,t \geq 0.$$

The property of self-similarity of the fractional Brownian sheet enables us to obtain the following lemma which is similar to the result of (Novikov & Valkeila 1999).

**Lemma 5.2 ((Mendy & Yodé 2010)).** *Let $T > 0$ and let $\left\{W_{s,t}^{\alpha,\beta},\ 0 \leq s,t \leq T\right\}$ be a fractional Brownian sheet with Hurst parameter $(\alpha,\beta) \in (0,1)^2$. Then,*

$$\mathbb{E}\left[\sup_{0\leq s,t\leq T} |W_{s,t}^{\alpha,\beta}|\right]^p = K(p,\alpha,\beta)T^{p(\alpha+\beta)}, \tag{5.23}$$

*for every $p > 0$, where $K(p,\alpha,\beta) = \mathbb{E}(|W_{1,1}^{\alpha,\beta}|)^p$.*

**Lemma 5.3.** *Assume that hypothesis $(A1)$ is satisfied and let $X_{s,t}$ and $x_{s,t}$ be the solutions of equations (5.18) and (5.19) respectively. Then, we have*

$$(i)\ \ |X_{s,t} - x_{s,t}| \leq e^{L_1 st}\varepsilon|W_{s,t}^{\alpha,\beta}|, \tag{5.24}$$

*and*

$$(ii)\ \ \sup_{0\leq s,t\leq T} \mathbb{E}(X_{s,t} - x_{s,t})^2 \leq e^{2L_1T^2}K(2,\alpha,\beta)T^{2(\alpha+\beta)}\varepsilon^2. \tag{5.25}$$

The following theorem gives the uniform convergence of the estimator $\tilde{\rho}_{s,t}$.

**Theorem 5.4.** *Suppose that assumptions **(A1)**-**(A2)** and **(H1)**-**(H5)** hold true. Then, for any $0 < c \leq d < T$ and $(\alpha,\beta) \in (1/2,1)^2$, the estimator $\tilde{\rho}_{s,t}$ is uniformly consistent, that is,*

$$\lim_{\varepsilon\longrightarrow 0} \sup_{c\leq s,t\leq d} \mathbb{E}_\rho(|\tilde{\rho}_{s,t} - \rho(x_{s,t})|^2) = 0. \tag{5.26}$$

### 5.3.2 Rate of convergence

A function $\rho$ belongs to the class $\mathcal{N}(k,L)$ if $\rho:\mathbb{R}^2 \longrightarrow \mathbb{R}$ admits partial derivatives of order $k_i,\ i\in\{1,2\}$ such that, with $z_1=(z,0),\ z_2=(0,z)$, for all $z\in\mathbb{R}$, for $i\in\{1,2\}$

$$\begin{cases} \textbf{(i)} & \left| \dfrac{\partial^{k_i}\rho}{(\partial t_i)^{k_i}}((t_1,t_2)+z_i) - \dfrac{\partial^{k_i}\rho}{(\partial t_i)^{k_i}}(t_1,t_2) \right| \le L|z|; \\ \textbf{(ii)} & \left\| \dfrac{\partial^{k_i}\rho}{(\partial t_i)^{k_i}} \right\| \le L; \end{cases}$$

with $\|.\|$ denotes the $\mathbb{L}^2(\mathbb{R}^2)$-norm.

Let $\rho_s^{k_1}(x_{s,t})$ be the $k_1$-th partial derivative of $\rho(x_{s,t})$ at $s$, and let $\rho_t^{k_2}(x_{s,t})$ be the $k_2$-th partial derivative of $\rho(x_{s,t})$ at $t$, i.e.

$$\rho_s^{k_1}(x_{s,t}) = \frac{\partial^{k_1}\rho(x_{s,t})}{\partial s^{k_1}}, \text{ and } \rho_t^{k_2}(x_{s,t}) = \frac{\partial^{k_2}\rho(x_{s,t})}{\partial t^{k_2}}.$$

Recall that kernel $G:\mathbb{R}\to\mathbb{R}$ is of order $k\in\mathbb{N}^*$ if for $j=1,\dots,k$,

$$\int_{-\infty}^{+\infty} |u|^j |G(u)| du \le \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} u^j G(u) du = 0$$

(see (Tsybakov 2008)). In this context we suppose

$$\textbf{(H5)} \quad \begin{cases} \textbf{(i)} & G_1 \text{ is a kernel of order } k_1, \text{ and } G_2 \text{ a kernel of order } k_2. \\ \textbf{(ii)} & \displaystyle\int_{-\infty}^{+\infty} u^{2(k_1+1)} G_1^2(u)\,du < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} v^{2(k_2+1)} G_2^2(v)\,dv < \infty. \end{cases}$$

Next, the rate of convergence of the estimator $\tilde{\rho}_{s,t}$ is established in the following theorem.

**Theorem 5.5.** *Suppose that the function $\rho(t,s)\in\mathcal{N}(k,L)$, with $k=(k_1,k_2)$ and $(\alpha,\beta)\in(1/2,1)^2$. We put*

$$\bar{k} = \frac{(k_1+1)(k_2+1)}{(k_1+1)(1-\beta)+(k_2+1)(1-\alpha)};$$

*by the optimal choice of $\varphi_\varepsilon$ and $\psi_\varepsilon$,*

$$\varphi_\varepsilon^* = \mathcal{O}\left(\varepsilon^{\frac{\bar{k}}{(k_2+1)(\bar{k}+1)}}\right), \quad \psi_\varepsilon^* = \mathcal{O}\left(\varepsilon^{\frac{\bar{k}}{(k_1+1)(\bar{k}+1)}}\right).$$

*Then, under hypotheses **(A1)**, **(H1)**-**(H3)** and **(H5)**, we have*

$$\limsup_{\varepsilon\longrightarrow 0} \sup_{\rho\in\mathcal{N}(k,L)} \sup_{c\le s,t\le d} \mathbb{E}_\rho(|\tilde{\rho}_{s,t} - \rho(x_{s,t})|^2) = \mathcal{O}\left(\varepsilon^{\frac{2(k_1+1)(k_2+1)}{(k_1+1)(k_2+1)+(k_1+1)(1-\beta)+(k_2+1)(1-\alpha)}}\right) = \mathcal{O}\left(\varepsilon^{\frac{2\bar{k}}{\bar{k}+1}}\right). \tag{5.27}$$

**Remark 5.1.** (i) Under conditions stated in Theorem 5.5, it follows that the mean square error of the estimator $\tilde{\rho}_{s,t}$ of the function $\rho(x_{s,t})$ is of the order

$$\mathcal{O}\left(\varepsilon^{\frac{2(k_1+1)(k_2+1)}{(k_1+1)(k_2+1)+(k_1+1)(1-\beta)+(k_2+1)(1-\alpha)}}\right)$$

as $\varepsilon \longrightarrow 0$ and the order of the mean square error decreases as $k_1$ and $k_2$ increase.

(ii) If $k_1 = k_2 = 0$, we have $\varphi^*_\varepsilon = \psi^*_\varepsilon = \mathcal{O}\left(\varepsilon^{\frac{2}{3-\alpha-\beta}}\right)$. Then, under hypotheses **(A1)**-**(A2)** and **(H1)**-**(H3)**, it can be shown that

$$\limsup_{\varepsilon\longrightarrow 0} \sup_{c\le s,t\le d} \mathbb{E}_\rho(|\tilde{\rho}_{s,t} - \rho(x_{s,t})|^2) = \mathcal{O}\left(\varepsilon^{\frac{2}{3-\alpha-\beta}}\right).$$

### 5.3.3 Asymptotic normality

The following additional assumption is useful for the asymptotic normality of the estimator. Assume that

**(H6)** $\displaystyle\int_{-\infty}^{+\infty} G_1^2(\mu)\,\mu^{2(k_1+2)} d\mu \le \infty,\quad$ and $\quad\displaystyle\int_{-\infty}^{+\infty} G_2^2(\nu)\,\nu^{2(k_2+2)} d\nu \le \infty.$

Our main result of this subsection is as follows.

**Theorem 5.6.** *Suppose that the function* $\rho(x_{s,t}) \in \mathcal{N}(k+1, L)$, $\varphi_\varepsilon = \varepsilon^{\frac{\bar{k}}{(k_2+1)(\bar{k}+1)}}$ *and* $\psi_\varepsilon = \varepsilon^{\frac{\bar{k}}{(k_1+1)(\bar{k}+1)}}$*; with*

$$\bar{k} = \frac{(k_1+1)(k_2+1)}{(k_1+1)(1-\beta) + (k_2+1)(1-\alpha)}.$$

*Then, under hypotheses* ***(A1)*****-*****(A2)*** *and* ***(H1)*****-*****(H5)****, we have*

$$\varepsilon^{\frac{-\bar{k}}{2(\bar{k}+1)}}\left(\tilde{\rho}_{s,t} - \rho(x_{s,t})\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(m_{k_1,k_2}, \sigma^2_{\alpha,\beta}\right), \text{ as } \varepsilon \longrightarrow 0, \tag{5.28}$$

*where*

$$m_{k_1,k_2} = m_{k_1} + m_{k_2},$$

*and*

$$\sigma^2_{\alpha,\beta} = \alpha(2\alpha-1)\beta(2\beta-1)\sigma^2_\alpha\sigma^2_\beta,$$

*with*

$$m_{k_1} = \frac{\rho_s^{(k_1+1)}(x_{s,t})}{(k_1+1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1(\mu)\,G_2(\nu)\,\mu^{k_1+1} d\mu d\nu;$$

$$m_{k_2} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\rho_t^{(k_2+1)}(x_{s,t})}{(k_2+1)!} G_1(\mu)\,G_2(\nu)\,\nu^{k_2+1} d\mu d\nu;$$

$$\sigma^2_\alpha = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1(\mu_1)G_1(\mu_2)|\mu_1-\mu_2|^{2\alpha-2} d\mu_1 d\mu_2;$$

*and*

$$\sigma^2_\beta = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_2(\nu_1)G_2(\nu_2)|\nu_1-\nu_2|^{2\beta-2} d\nu_1 d\nu_2.$$

## 5.4 Linear multiplier estimation

Let $X = \{X_{s,t},\ 0 \le s,t \le T\}$ be a stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{s,t\ge 0}, \mathbb{P})$. Assume that the process $X$ satisfies

$$dX_{s,t} = \theta(s,t)X_{s,t}dsdt + \varepsilon dW^{\alpha,\beta}_{s,t},\ \ 0 \le s,t \le T, \tag{5.29}$$

where $\varepsilon > 0$, $\left\{W^{\alpha,\beta}_{s,t}\ 0 \le s,t \le T\right\}$ is a fractional Brownian sheet, and $\theta(s,t)$ is an unknown function (linear multiplier). We suppose that $x_{s,t}$ is a solution of the equation

$$dx_{s,t} = \theta(s,t)x_{s,t}dsdt,\ x_0,\ 0 \le s,t \le T. \tag{5.30}$$

Existence and uniqueness for solutions of stochastic differential equations (5.29) are investigated in (El Barrimi & Ouknine 2019). Here, we also suppose that the function $\theta(.,.)$ satisfies the following assumption:
(**A1'**) Assume that the function $\theta(s,t)$ is bounded by a constant $L$.
Then, the stochastic differential equation (5.29) has a unique solution $\{X_{s,t},\ 0 \le s,t \le T\}$.

Let us consider the process $\{\theta(s,t),\ s,t \in [0,T]\}$ adapted to the filtration $(\mathcal{F}_{s,t})_{s,t\ge 0}$, such that the process

$$R_{\alpha,\beta,\theta}(s,t) = \frac{d}{dw^{\alpha}_t}\frac{d}{dw^{\beta}_s}\int_0^t\int_0^s k_\alpha(t,v)k_\beta(s,u)\theta(u,v)X_{u,v}dudv$$

is well-defined where the derivative is understood in the sense of absolute continuity with regard to the measure generated by the function $w^\alpha$. Differentiation with respect to $w^\alpha_t$ is understood in the sense:

$$dw^\alpha_t = \lambda_\alpha^{-1}(2-2H)t^{1-2H}dt, \tag{5.31}$$

and

$$\frac{df(t)}{dw^\alpha_t} = \frac{df(t)}{dt}\frac{dt}{dw^\alpha_t}. \tag{5.32}$$

Assume that the process $\left\{R_{\alpha,\beta,\theta}(s,t),\ 0 \le s,t \le T\right\}$, defined over the interval $[0,T]^2$ belongs to the space $\mathbb{L}^2\left([0,T]^2, dw^\alpha_t dw^\beta_s\right)$. Let

$$Z_{s,t} = \int_0^t\int_0^s k_\alpha(t,v)k_\beta(s,u)dX_{u,v}. \tag{5.33}$$

Then, the process $Z = \{Z_{s,t},\ 0 \le s,t \le T\}$ is an $(\mathcal{F}_{s,t})$-semimartingale with the decomposition

$$Z_{s,t} = \int_0^t\int_0^s k_\alpha(t,v)k_\beta(s,u)\theta(u,v)X_{u,v}dudv + \varepsilon M^{\alpha,\beta}_{s,t}, \tag{5.34}$$

where $M^{\alpha,\beta}_{s,t}$ is the fundamental martingale defined by

$$M^{\alpha,\beta}_{s,t} = \int_0^t\int_0^s k_\alpha(t,v)k_\beta(s,u)dW^{\alpha,\beta}_{u,v}. \tag{5.35}$$

Moreover, for $(\alpha,\beta)\in(1/2,1)^2$, it follows from (Kleptsyna, Le Breton & Roubaud 2000) and (Tudor & Tudor 2005) that if we take

$$K_\alpha(t,v)=\alpha(2\alpha-1)\int_s^t r^{\alpha-1/2}(r-v)^{\alpha-3/2}dr.$$

Then, the process $X$ admits the following representation

$$X_{s,t}=\int_0^t\int_0^s K_\alpha(t,v)K_\beta(s,u)dZ_{u,v},$$

and

$$dZ_{s,t}=R_{\alpha,\beta,\theta}(s,t)dw_t^\alpha dw_s^\beta+\varepsilon dM_{s,t}^{\alpha,\beta},\ 0\le s,t\le T, \tag{5.36}$$

According to (Deriyeva & Shpyga 2016), it is obvious that the filtration generated by $\{Z_{s,t},\ 0\le s,t\le T\}$ and $\{X_{s,t},\ 0\le s,t\le T\}$ coincides, i.e., it is possible to use one field instead of another without loss of information. Let $\mathbb{P}_\theta(s,t)$ be a measure generated by field $\{X_{s,t},\ 0\le s,t\le T\}$, when $\theta(s,t)$ is a real drift function. From Girsanov theorem (Knopov & Shtatland 1973), there exists the Radon-Nikodym derivative with respect to $\mathbb{P}_0(s,t)$, defined as follows:

$$\frac{d\mathbb{P}_\theta(T,T)}{d\mathbb{P}_0(T,T)}=exp\left\{\int_0^T\int_0^T R_{\alpha,\beta}(s,t)dZ_{s,t}-\frac{1}{2}\int_0^T\int_0^T R^2_{\alpha,\beta}(s,t)dw_s^\alpha dw_t^\beta\right\}.$$

From the definition of the process $Z_{s,t}$ we obtain that its distribution over the measure $\mathbb{P}_\theta(s,t)$ is the same as for the process $X_{s,t}$ over the measure $\mathbb{P}_0(s,t)$.

Similar to Lemma 5.3 in Section 5.3, under the hypothesis **(A1')**, if $X_{s,t}$ and $x_{s,t}$ are the solutions of the equations (5.29) and (5.30) respectively, then we obtain

$$|X_{s,t}-x_{s,t}|\le e^{Lst}\varepsilon|W_{s,t}^{\alpha,\beta}|, \tag{5.37}$$

and

$$\sup_{0\le s,t\le T}\mathbb{E}(X_{s,t}-x_{s,t})^2\le e^{2Lst}K(2,\alpha,\beta)s^{2\alpha}t^{2\beta}\varepsilon^2. \tag{5.38}$$

Let us consider the process $R^*_{\alpha,\beta,\theta}(s,t)$ of the form

$$R^*_{\alpha,\beta,\theta}(s,t)=\frac{d}{dw_t^\alpha}\frac{d}{dw_s^\beta}\int_0^t\int_0^s k_\alpha(t,v)k_\beta(s,u)\theta(u,v)x(u,v)dudv. \tag{5.39}$$

Next, we suppose that
**(A2')** The function $R^*_{\alpha,\beta,\theta}(s,t)$ satisfies the Hölder condition for any fixed $\theta(.,.)$, that is: $\forall\ (s,t,u,v)\in[0,T]^4$

$$|R^*_{\alpha,\beta,\theta}(s,t)-R^*_{\alpha,\beta,\theta}(u,v)|\le C\left(|s-u|^{\gamma_1}+|t-v|^{\gamma_2}\right),\ \ \gamma_1,\gamma_2>0. \tag{5.40}$$

Similar to (Mishra & Prakasa Rao 2011a), rather than estimating the function $\theta(s,t)$, we estimate the function $R^*_{\alpha,\beta,\theta}(s,t)$ defined in (5.39). This is due to the fact that the process $\{X_{s,t},\ 0 \le s,t \le T\}$ governed by the stochastic differential equations (5.29) and the corresponding related process $\{Z_{s,t},\ 0 \le s,t \le T\}$ as given in (5.36) have the same filtration (see (Deriyeva & Shpyga 2016)).

For all $(s,t) \in [0,T]^2$, we define the two dimensional kernel type estimator $\tilde{R}_{\alpha,\beta,\theta}(s,t)$ of the function $R^*_{\alpha,\beta,\theta}(s,t)$ by

$$\begin{aligned} \tilde{R}_{\alpha,\beta,\theta}(s,t) &= \int_0^T \int_0^T G_{\varphi,\psi}(u,v)\,dZ_{u,v} \\ &= \int_0^T \int_0^T G_{\varphi,\psi}(u,v)\Big(R_{\alpha,\beta,\theta}(u,v)\,du\,dv + \varepsilon\, dM^{\alpha,\beta}_{u,v}\Big), \end{aligned} \tag{5.41}$$

with

$$G_{\varphi,\psi}(u,v) = \frac{1}{\varphi_\varepsilon \psi_\varepsilon} G\left(\frac{u-s}{\varphi_\varepsilon}, \frac{v-t}{\psi_\varepsilon}\right), \tag{5.42}$$

(cf. (Mishra & Prakasa Rao 2011a) and (Prakasa Rao 2019c)), where $G(u,v)$ is a bounded function with finite support $[A,B]^2$ satisfying the following hypotheses

**(H2')** $\int_{-\infty}^{+\infty} |u|^{2\gamma_1} G_1^2(u)\,du < \infty$, and $\int_{-\infty}^{+\infty} |v|^{2\gamma_2} G_2^2(v)\,dv < \infty$.

Further, we suppose that the normalizing function $\phi_\varepsilon$ satisfies:

**(H3')** $\varphi_\varepsilon,\ \psi_\varepsilon \longrightarrow 0$ and $\varepsilon\varphi_\varepsilon^{-3/2},\ \varepsilon\psi_\varepsilon^{-3/2} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

The following Theorem gives uniform convergence of the estimator $\tilde{R}_{\alpha,\beta,\theta}(s,t)$.

**Theorem 5.7.** *Suppose that the assumptions* **(A1')**, **(A2')**, **(H1)**, **(H2')** *and* **(H3')** *hold true. Then, for any $0 \le s,t \le T$ and $(\alpha,\beta) \in (1/2,3/4)^2$, the estimator $\tilde{R}_{\alpha,\beta,\theta}(s,t)$ is uniformly consistent, that is,*

$$\lim_{\varepsilon \longrightarrow 0} \sup_{0 \le s,t \le T} \mathbb{E}\Big(|\tilde{R}_{\alpha,\beta,\theta}(s,t) - R^*_{\alpha,\beta,\theta}(s,t)|^2\Big) = 0.$$

The rate of convergence of the estimator $\tilde{R}_{\alpha,\beta,\theta}(s,t)$ is established by the following corollary.

**Corollary 5.8.** *Suppose that the assumptions* **(A1')**, **(A2')**, **(H1)**, **(H2')** *and* **(H3')** *are verified. Then, for any $0 \le s,t \le T$, $(\alpha,\beta) \in (1/2,3/4)^2$ , we have*

$$\lim_{\varepsilon \longrightarrow 0} \sup_{|\theta(.,.)| \le L} \sup_{0 \le s,t \le T} \mathbb{E}(|\tilde{R}_{\alpha,\beta,\theta}(s,t) - R^*_{\alpha,\beta,\theta}(s,t)|^2) = \mathcal{O}\left(\varepsilon^{\frac{4\gamma_1\gamma_2}{3\gamma_1+3\gamma_2+4\gamma_1\gamma_2}}\right).$$

## 5.5 Conclusion

This work considers a nonparametric estimation of trend function and linear multiplier for stochastic differential equation driven by fractional Brownian sheet. Under some hypotheses, the uniform consistency, the rate of convergence and the asymptotic normality of the kernel estimator of trend function are obtained. In addition, under further hypotheses, the uniform consistency and the rate of convergence of the kernel estimator of multiplier function are derived. The current investigation has a large number of applications leading to the powerful development of stochastic calculus. For further reteach work it will be interesting to consider the same study in functional setting.

## 5.6 Proofs

### Proof of Lemma 5.3

**(i)** By equations (5.18) and (5.19), we have

$$X_{s,t} - x_{s,t} = \int_0^t \int_0^s (\rho(X_{u,v}) - \rho(x_{u,v}))\, du dv + \varepsilon W_{s,t}^{\alpha,\beta}.$$

By Lipschitz condition on $\rho(x)$ in **(A1)**, we have

$$\left|X_{s,t} - x_{s,t}\right| \leq L_1 \int_0^t \int_0^s \left|X_{u,v} - x_{u,v}\right| du dv + \varepsilon \sup_{0 \leq s,t \leq T} |W_{s,t}^{\alpha,\beta}|.$$

Using Lemma 5.1, we obtain

$$|X_{s,t} - x_{s,t}| \leq e^{L_1 ts} |W_{s,t}^{\alpha,\beta}|. \tag{5.43}$$

**(ii)** From Equation (5.43), we have

$$|X_{s,t} - x_{s,t}|^2 \leq e^{2L_1 ts} \varepsilon^2 \left[ \sup_{0 \leq s,t \leq T} |W_{s,t}^{\alpha,\beta}| \right]^2.$$

Finally, by using Lemma 5.2, we get

$$\sup_{0 \leq s,t \leq T} \mathbb{E}(X_{s,t} - x_{s,t})^2 \leq e^{2L_1 T^2} K(2,\alpha,\beta) T^{2(\alpha+\beta)} \varepsilon^2.$$

### Proof of Theorem 5.4

From (5.18) and (5.21), the difference $\tilde{\rho}_{s,t} - \rho(x_{s,t})$ can be divided into three parts,

$$\begin{aligned}
\tilde{\rho}_{s,t} - \rho(x_{s,t}) &= \int_0^T \int_0^T G_{\varphi,\psi}(u,v)\, dX_{u,v} - \rho(x_{s,t}) \\
&= \int_0^T \int_0^T G_{\varphi,\psi}(u,v) (\rho(X_{u,v}) - \rho(x_{u,v}))\, du dv \\
&\quad + \int_0^T \int_0^T G_{\varphi,\psi}(u,v) \rho(x_{u,v}) du dv - \rho(x_{s,t}) \\
&\quad + \varepsilon \int_0^T \int_0^T G_{\varphi,\psi}(u,v)\, dW_{u,v}^{\alpha,\beta}.
\end{aligned}$$

Using the inequality $(a+b+c)^2 \leq 3a^2+3b^2+3c^2$, it yields

$$\begin{aligned}
\mathbb{E}_\rho\left[\tilde{\rho}_{s,t}-\rho(x_{s,t})\right]^2 &\leq 3\mathbb{E}_\rho\left[\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\left(\rho(X_{u,v})-\rho(x_{u,v})\right)dudv\right]^2 \\
&\quad +3\mathbb{E}_\rho\left[\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\rho(x_{u,v})duv-\rho(x_{s,t})\right]^2 \\
&\quad +3\varepsilon^2\mathbb{E}_\rho\left[\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\,dW^{\alpha,\beta}_{u,v}\right]^2 \\
&= 3I_1(\varepsilon)+3I_2(\varepsilon)+3I_3(\varepsilon).
\end{aligned} \tag{5.44}$$

We first consider $I_1(\varepsilon)$. Via hypotheses **(A1)(i)**, **(H1)** and inequality (5.25) in Lemma 5.3, we get

$$\begin{aligned}
I_1(\varepsilon) &= \mathbb{E}_\rho\left[\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\left(\rho(X_{u,v})-\rho(x_{u,v})\right)dudv\right]^2 \\
&\leq \mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G^2(\mu,\nu)\left(\rho(X_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon})-\rho(x_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon})\right)^2 d\mu d\nu\right] \\
&\leq L_1^2\mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1^2(\mu)G_2^2(\nu)\left(X_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon}-x_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon}\right)^2 d\mu d\nu\right] \\
&\leq L_1^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1^2(\mu)G_2^2(\nu)\sup_{0\leq u,v\leq T}\mathbb{E}_\rho\left(X_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon}-x_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon}\right)^2 d\mu d\nu \\
&\leq L_1^2 e^{2L_1T^2}T^{2(\alpha+\beta)}\varepsilon^2\int_{-\infty}^{+\infty} G_1^2(\mu)\,d\mu\int_{-\infty}^{+\infty} G_2^2(\nu)\,d\nu \\
&\leq C_1\varepsilon^2,
\end{aligned} \tag{5.45}$$

where $C_1$ is a positive constant depending on $T, L_1, \alpha, \beta$, and $(B-A)$.

Now, we treat the second term $I_2(\varepsilon)$,

$$\begin{aligned}
I_2(\varepsilon) &= \mathbb{E}_\rho\left[\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\rho(x_{u,v})dudv-\rho(x_{s,t})\right]^2 \\
&= \mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1(\mu)G_2(\nu)\left(\rho(x_{s+\mu\varphi_\varepsilon,t+\psi_\varepsilon\nu})-\rho(x_{s,t})\right)d\mu d\nu\right]^2.
\end{aligned}$$

By using hypotheses **(A2)**, **(H1)**, and **(H2)**, we have

$$\begin{aligned}
I_2(\varepsilon) &\leq L_2^2\mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1(\mu)G_2(\nu)\left(|\varphi_\varepsilon\mu|+|\psi_\varepsilon\nu|\right)d\mu d\nu\right]^2 \\
&\leq L_2^2\varphi_\varepsilon^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mu^2G_1^2(\mu)G_2^2(\nu)\,d\mu d\nu \\
&\quad +L_2^2\psi_\varepsilon^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\nu^2G_1^2(\mu)G_2^2(\nu)\,d\mu d\nu \\
&= L_2^2\varphi_\varepsilon^2\int_{-\infty}^{+\infty}\mu^2G_1^2(\mu)\,d\mu\int_{-\infty}^{+\infty}G_2^2(\nu)\,d\nu \\
&\quad +L_2^2\psi_\varepsilon^2\int_{-\infty}^{+\infty}\nu^2G_2^2(\nu)\,d\nu\int_{-\infty}^{+\infty}G_1^2(\mu)\,d\mu \\
&= C_2\varphi_\varepsilon^2+C_3\psi_\varepsilon^2,
\end{aligned} \tag{5.46}$$

where $C_2$ and $C_3$ are positive constants.

Finally, for $I_3(\varepsilon)$, we assume that, $G(u,v) = G_1(u)G_2(v)$, and $W^{\alpha,\beta}_{s,t} = W^\alpha_s W^\beta_t$, where $W^\alpha_s$ and $W^\beta_t$ are two independent one-parameter fractional Brownian motion with Hurst index $(\alpha,\beta) \in (1/2,1)^2$. By Fubini theorem, inclusion (5.7) and condition **(H4)**, we have

$$
\begin{aligned}
I_3(\varepsilon) &= \varepsilon^2 \mathbb{E}_\rho\left[\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\,dW^{\alpha,\beta}_{u,v}\right]^2 \\
&= \frac{\varepsilon^2}{\varphi_\varepsilon^2\psi_\varepsilon^2}\mathbb{E}_\rho\left[\int_0^T\int_0^T G_1\left(\frac{u-s}{\varphi_\varepsilon}\right)G_2\left(\frac{v-t}{\psi_\varepsilon}\right)dW^{\alpha,\beta}_{u,v}\right]^2 \\
&\leq \frac{\varepsilon^2}{\varphi_\varepsilon^2\psi_\varepsilon^2}\left(\int_0^T\left|G_1\left(\frac{u-s}{\varphi_\varepsilon}\right)\right|^{\frac{1}{\alpha}}du\right)^{2\alpha}\left(\int_0^T\left|G_2\left(\frac{v-t}{\psi_\varepsilon}\right)\right|^{\frac{1}{\beta}}dv\right)^{2\beta} \\
&= \frac{\varepsilon^2}{\varphi_\varepsilon^2\psi_\varepsilon^2}\left(\varphi_\varepsilon\int_{-\infty}^{+\infty}\left|G_1(\mu)\right|^{\frac{1}{\alpha}}d\mu\right)^{2\alpha}\left(\psi_\varepsilon\int_{-\infty}^{+\infty}\left|G_2(\nu)\right|^{\frac{1}{\beta}}d\nu\right)^{2\beta} \\
&= C_4\frac{\varepsilon^2}{\varphi_\varepsilon\psi_\varepsilon}\varphi_\varepsilon^{2\alpha-1}\psi_\varepsilon^{2\beta-1},
\end{aligned}
\tag{5.47}
$$

where $C_4$ is a positive constant depending on $\alpha$ and $\beta$.

Finally, combining (5.44), (5.45), (5.46) and (5.47), under the assumption **(H4)**, we obtain

$$\lim_{\varepsilon\longrightarrow 0}\sup_{c\leq s,t\leq d}\mathbb{E}_\rho\left[\tilde{\rho}_{s,t}-\rho(x_{s,t})\right]^2 = 0.$$

## Proof of Theorem 5.5

By (5.44), it follows that

$$
\begin{aligned}
I_2(\varepsilon) &= \mathbb{E}_\rho\left[\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\rho(x_{u,v})\,du\,dv - \rho(x_{s,t})\right]^2 \\
&= \mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(\mu,\nu)\Big(\rho(x_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon})-\rho(x_{s,t})\Big)d\mu\,d\nu\right]^2.
\end{aligned}
$$

Observe that,

$$\rho(x_{s+\mu\varphi_\varepsilon,t+\psi_\varepsilon\nu})-\rho(x_{s,t}) = \rho(x_{s+\mu\varphi_\varepsilon,t+\psi_\varepsilon\nu})-\rho(x_{s+\mu\varphi_\varepsilon,t})+\rho(x_{s+\mu\varphi_\varepsilon,t})-\rho(x_{s,t}).$$

Then,

$$
\begin{aligned}
I_2(\varepsilon) &\leq \mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(\mu,\nu)\Big(\rho(x_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon})-\rho(x_{s+\mu\varphi_\varepsilon,t})\Big)d\mu\,d\nu\right]^2 \\
&\quad+\mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(\mu,\nu)\Big(\rho(x_{s+\mu\phi_\varepsilon,t})-\rho(x_{s,t})\Big)d\mu\,d\nu\right]^2 \\
&= I_2(\psi_\varepsilon)+I_2(\varphi_\varepsilon).
\end{aligned}
\tag{5.48}
$$

For the first term $I_2(\psi_\varepsilon)$, we apply Taylor's formula to the partial function: $z \longrightarrow \rho(x_{s+\mu\varphi_\varepsilon,z})$, then we have

$$\begin{aligned}\rho(x_{s+\mu\varphi_\varepsilon,t+\psi_\varepsilon\nu}) - \rho(x_{s+\mu\varphi_\varepsilon,t}) &= \sum_{j=1}^{k_2} \rho_t^j(x_{s+\mu\varphi_\varepsilon,t})\frac{(\psi_\varepsilon\nu)^j}{j!} \\ &\quad + \left(\rho_t^{k_2}(x_{s+\mu\varphi_\varepsilon,t+\lambda(\nu\psi_\varepsilon)}) - \rho_t^{k_2}(x_{s+\mu\varphi_\varepsilon,t})\right)\frac{(\psi_\varepsilon\nu)^{k_2}}{k_2!},\end{aligned} \tag{5.49}$$

where $\lambda \in (0,1)$.

Next, by using hypotheses **(H1)**, **(H5)** and $\rho(x_{s,t}) \in \mathcal{N}(k,L)$, we have

$$\begin{aligned}I_2(\psi_\varepsilon) &= \mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1(\mu)G_2(\nu)\left(\rho(x_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon}) - \rho(x_{s+\mu\varphi_\varepsilon,t})\right)d\mu d\nu\right]^2 \\ &= \mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1(\mu)G_2(\nu)\left(\rho_t^{k_2}(x_{s+\mu\varphi_\varepsilon,t+\lambda(\nu\psi_\varepsilon)}) - \rho_t^{k_2}(x_{s+\mu\varphi_\varepsilon,t})\right)\frac{(\psi_\varepsilon\nu)^{k_2}}{k_2!}d\mu d\nu\right]^2 \\ &\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[G_1(\mu)G_2(\nu)L\left|\nu\psi_\varepsilon\right|\frac{(\psi_\varepsilon\nu)^{k_2}}{k_2!}\right]^2 d\mu d\nu \\ &= L^2(B-A)^2\frac{\psi_\varepsilon^{2(k_2+1)}}{k_2!}\int_{-\infty}^{+\infty} G_1^2(\mu)\,d\mu\int_{-\infty}^{+\infty} G_2^2(\nu)\,\nu^{2(k_2+1)}d\nu \\ &= C_5\psi_\varepsilon^{2(k_2+1)},\end{aligned} \tag{5.50}$$

where $C_5$ is a positive constant depending on $L$, $(B-A)$ and $k_2$.

Similarly as for $I_2(\varphi_\varepsilon)$, we apply Taylor's formula to the partial functions: $z \longrightarrow \rho(x_{z,t})$ :

$$\rho(x_{s+\mu\varphi_\varepsilon,t}) - \rho(x_{s,t}) = \sum_{j=1}^{k_1}\rho_s^j(x_{s,t})\frac{(\varphi_\varepsilon\mu)^j}{j!} + \left(\rho_s^{k_1}(x_{s+\lambda(\mu\varphi_\varepsilon),t}) - \rho_s^{k_1}(x_{s,t})\right)\frac{(\varphi_\varepsilon\mu)^{k_1}}{k_1!}, \tag{5.51}$$

where $\lambda \in (0,1)$. Under the condition **(H1)**, **(H5)** and $\rho(x_{s,t}) \in \mathcal{N}(k,L)$, we get

$$\begin{aligned}I_2(\varphi_\varepsilon) &= \mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1(\mu)G_2(\nu)\left(\rho(x_{s+\mu\varphi_\varepsilon,t}) - \rho(x_{s,t})\right)d\mu d\nu\right]^2 \\ &= \mathbb{E}_\rho\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1(\mu)G_2(\nu)\left(\rho_s^{k_1}(x_{s+\lambda(\mu\varphi_\varepsilon),t}) - \rho_s^{k_1}(x_{s,t})\right)\frac{(\varphi_\varepsilon\mu)^{k_1}}{k_1!}d\mu d\nu\right]^2 \\ &\leq \mathbb{E}_\rho\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[G_1(\mu)G_2(\nu)L\left|\mu\varphi_\varepsilon\right|\frac{(\mu\varphi_\varepsilon)^{k_1}}{k_1!}\right]^2 d\mu d\nu \\ &= L^2(B-A)^2\frac{\varphi_\varepsilon^{2(k_1+1)}}{k_1!}\int_{-\infty}^{+\infty} G_2^2(\nu)\,d\nu\int_{-\infty}^{+\infty} G_1^2(\mu)\,\mu^{2(k_1+1)}d\mu \\ &\leq C_6\varphi_\varepsilon^{2(k_1+1)},\end{aligned} \tag{5.52}$$

where $C_6$ is a positive constant depending on $L$, $(B-A)$ and $k_1$.

Next, from (5.45), (5.47), (5.50), and (5.52), we find

$$\sup_{\rho\in\mathcal{N}(k,L)}\sup_{c\le s,t\le d}\mathbb{E}_\rho\left|\tilde{\rho}_{s,t}-\rho(x_t)\right|^2\le C_7\left(\varepsilon^2\varphi_\varepsilon^{2\alpha-2}\psi_\varepsilon^{2\beta-2}+\varphi_\varepsilon^{2(k_1+1)}+\psi_\varepsilon^{(2k_2+1)}+\varepsilon^2\right).$$

In order to minimize the mean square error, we have to minimize the following function of two variables:

$$(\varphi_\varepsilon,\psi_\varepsilon)\longrightarrow\xi(\varphi_\varepsilon,\psi_\varepsilon)=\varepsilon^2\varphi_\varepsilon^{2\alpha-2}\psi_\varepsilon^{2\beta-2}+\varphi_\varepsilon^{2(k_1+1)}+\psi_\varepsilon^{2(k_2+1)}+\varepsilon^2,$$

with

$$\bar{k}=\frac{(k_1+1)(k_2+1)}{(k_1+1)(1-\beta)+(k_2+1)(1-\alpha)},$$

The minimizing values is obtained as

$$\varphi_\varepsilon^*=\mathcal{O}\left(\varepsilon^{\frac{\bar{k}}{(k_2+1)(\bar{k}+1)}}\right),\quad \psi_\varepsilon^*=\mathcal{O}\left(\varepsilon^{\frac{\bar{k}}{(k_1+1)(\bar{k}+1)}}\right).$$

This implies

$$\xi(\varphi_\varepsilon,\psi_\varepsilon)=\mathcal{O}\left(\varepsilon^{\frac{2(k_1+1)(k_2+1)}{(k_1+1)(k_2+1)+(k_1+1)(1-\beta)+(k_2+1)(1-\alpha)}}\right)=\mathcal{O}\left(\varepsilon^{\frac{2\bar{k}}{\bar{k}+1}}\right).$$

Therefore, we obtain

$$\limsup_{\varepsilon\longrightarrow 0}\sup_{\rho\in\mathcal{N}(k,L)}\sup_{c\le s,t\le d}\mathbb{E}_\rho(|\tilde{\rho}_{s,t}-\rho(x_t)|^2)=\mathcal{O}\left(\varepsilon^{\frac{2(k_1+1)(k_2+1)}{(k_1+1)(k_2+1)+(k_1+1)(1-\beta)+(k_2+1)(1-\alpha)}}\right).$$

This completes the proof of Theorem 5.5.

## Proof of Theorem 5.6

From (5.18) and (5.21), we can see that

$$\varepsilon^{\frac{-\bar{k}}{\bar{k}+1}}(\tilde{\rho}_{s,t}-\rho(x_{s,t}))=r_1(s,t)+r_2(s,t)+r_3(s,t)+r_4(s,t),$$

with

$$r_1(s,t)=\varepsilon^{\frac{-\bar{k}}{\bar{k}+1}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(\mu,\nu)\Big(\rho(X_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon})-\rho(x_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon})\Big)d\mu d\nu,$$
$$r_2(s,t)=\varepsilon^{\frac{-\bar{k}}{\bar{k}+1}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(\mu,\nu)\Big(\rho(x_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon})-\rho(x_{s+\mu\varphi_\varepsilon,t})\Big)d\mu d\nu,$$
$$r_3(s,t)=\varepsilon^{\frac{-\bar{k}}{\bar{k}+1}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(\mu,\nu)\Big(\rho(x_{s+\mu\phi_\varepsilon,t})-\rho(x_{s,t})\Big)d\mu d\nu,$$

and

$$r_4(s,t)=\varepsilon^{\frac{1}{\bar{k}+1}}\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\,dW_{u,v}^{\alpha,\beta}.$$

– **Concerning the term** $r_1(s,t)$**.**

Obviously, using the Bienaymé-Tchebychev's inequality and (5.45), as $\varepsilon \longrightarrow 0$, we obtain, for all $\delta > 0$

$$\begin{aligned}
\mathbb{P}\left(|r_1(s,t)| > \delta\right) &\leq \frac{\mathbb{E}(r_1^2(s,t))}{\delta} \\
&= \delta^{-1}\varepsilon^{\frac{-2\bar{k}}{\bar{k}+1}} I_1 \\
&\leq C_1\delta^{-1}\varepsilon^{\frac{-2\bar{k}}{\bar{k}+1}}\varepsilon^2 \\
&\leq C_1\delta^{-1}\varepsilon^{\frac{1}{\bar{k}+1}} \\
&= C_1\delta^{-1}\varepsilon^{\frac{(k_1+1)(1-\beta)+(k_2+1)(1-\alpha)}{(k_1+1)(k_2+1)+(k_1+1)(1-\beta)+(k_2+1)(1-\alpha)}} \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0,
\end{aligned}$$

which implies that

$$r_1(s,t) \xrightarrow{\mathbb{P}} 0 \text{ as } \varepsilon \to 0. \tag{5.53}$$

– **Concerning the term** $r_2(s,t)$**.** Let

$$r_2(s,t) = \varepsilon^{\frac{-\bar{k}}{\bar{k}+1}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(\mu,\nu)\Big(\rho(x_{s+\mu\varphi_\varepsilon,t+\nu\psi_\varepsilon}) - \rho(x_{s+\mu\varphi_\varepsilon,t})\Big)\,d\mu\,d\nu. \tag{5.54}$$

By Taylor's formula to the partial function: $z \longrightarrow \rho(x_{s+\mu\varphi_\varepsilon,z})$, we get for any $t + \psi_\varepsilon \nu \in [0,T]$,

$$\begin{aligned}
\rho(x_{s+\mu\varphi_\varepsilon,t+\psi_\varepsilon\nu}) - \rho(x_{s+\mu\varphi_\varepsilon,t}) &= \sum_{j=1}^{k_2+1}\rho_t^j(x_{s+\mu\varphi_\varepsilon,t})\frac{(\psi_\varepsilon\nu)^j}{j!} \\
&+\Big(\rho_t^{(k_2+1)}(x_{s+\mu\varphi_\varepsilon,t+\lambda(\nu\psi_\varepsilon)}) - \rho_t^{(k_2+1)}(x_{s+\mu\varphi_\varepsilon,t})\Big)\frac{(\psi_\varepsilon\nu)^{k_2+1}}{(k_2+1)!},
\end{aligned} \tag{5.55}$$

where $\lambda \in (0,1)$. Substituting (5.55) in (5.54) and under the condition **(H5)**, we get

$$\begin{aligned}
r_2(s,t) &= \varepsilon^{\frac{-\bar{k}}{\bar{k}+1}}\left[\sum_{j=1}^{k_2+1}\int_{-\infty}^{+\infty}\rho_t^j(x_{s+\mu\varphi_\varepsilon,t})G_1(\mu)\,d\mu\left(\int_{-\infty}^{+\infty}G_2(\nu)\,\nu^j\,d\nu\right)\psi_\varepsilon^j(j!)^{-1}\right. \\
&\left.+\frac{\psi_\varepsilon^{k_2+1}}{(k_2+1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Big(\rho_t^{(k_2+1)}(x_{s+\mu\varphi_\varepsilon,t+\lambda(\nu\psi_\varepsilon)}) - \rho_t^{(k_2+1)}(x_{s+\mu\varphi_\varepsilon,t})\Big)G_1(\mu)\,G_2(\nu)\,\nu^{k_2+1}\,d\mu\,d\nu\right] \\
&= \varepsilon^{\frac{-\bar{k}}{\bar{k}+1}}\left[\frac{\psi_\varepsilon^{k_2+1}}{(k_2+1)!}\int_{-\infty}^{+\infty}\rho_t^{(k_2+1)}(x_{s+\mu\varphi_\varepsilon,t})G_1(\mu)\,d\mu\left(\int_{-\infty}^{+\infty}G_2(\nu)\,\nu^{k_2+1}\,d\nu\right)\right. \\
&\left.+\frac{\psi_\varepsilon^{k_2+1}}{(k_2+1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Big(\rho_t^{(k_2+1)}(x_{s+\mu\varphi_\varepsilon,t+\lambda(\nu\psi_\varepsilon)}) - \rho_t^{(k_2+1)}(x_{s+\mu\varphi_\varepsilon,t})\Big)G_1(\mu)\,G_2(\nu)\,\nu^{k_2+1}\,d\mu\,d\nu\right].
\end{aligned}$$

Therefore, by $\rho(x_{s,t}) \in \mathcal{N}(k+1,L)$, **(H1)** and **(H6)**, we have

$$
\begin{aligned}
&\left|r_2(s,t)-m_{k_2}\right|^2\\
&\leq \frac{1}{(k_2+1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\left(\rho_t^{(k_2+1)}(x_{s+\mu\varphi_\varepsilon,t+\lambda(\nu\psi_\varepsilon)})-\rho_t^{(k_2+1)}(x_{s+\mu\varphi_\varepsilon,t})\right)G_1(\mu)\,G_2(\nu)\,\nu^{k_2+1}\right|^2 d\mu d\nu\\
&\leq \frac{L_2^2\psi_\varepsilon^2}{[(k_2+1)!]^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G_1^2(\mu)\,G_2^2(\nu)\,\nu^{2(k_2+2)}d\mu d\nu\\
&= \frac{L_2^2\psi_\varepsilon^2}{[(k_2+1)!]^2}\int_{-\infty}^{+\infty}G_1^2(\mu)\,d\mu\int_{-\infty}^{+\infty}G_2^2(\nu)\,\nu^{2(k_2+2)}d\nu,
\end{aligned}
$$

which tends to zero as $\varepsilon \longrightarrow 0$. It immediately follows that

$$
r_2(\psi_\varepsilon) \xrightarrow{\mathbb{P}} m_{k_2}. \tag{5.56}
$$

– **Concerning the term** $r_3(s,t)$**.** Let

$$
r_3(s,t) = \varepsilon^{\frac{-\bar{k}}{\bar{k}+1}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(\mu,\nu)\left(\rho(x_{s+\mu\phi_\varepsilon,t})-\rho(x_{s,t})\right)d\mu d\nu. \tag{5.57}
$$

As for $r_3(s,t)$, we apply Taylor's formula to the partial functions: $z \longrightarrow \rho(x_{z,t})$

$$
\rho(x_{s+\mu\varphi_\varepsilon,t})-\rho(x_{s,t}) = \sum_{j=1}^{k_1+1}\rho_s^j(x_{s,t})\frac{(\varphi_\varepsilon\mu)^j}{j!}+\left(\rho_s^{(k_1+1)}(x_{s+\lambda(\mu\varphi_\varepsilon),t})-\rho_s^{(k_1+1)}(x_{s,t})\right)\frac{(\varphi_\varepsilon\mu)^{k_1+1}}{(k_1+1)!}, \tag{5.58}
$$

where $\lambda \in (0,1)$. Combining equation (5.58) in (5.57), by $\rho(x_{s,t}) \in \mathcal{N}(k+1,L)$, **(H1)**, **(H5)** and **(H6)**, we obtain

$$
\begin{aligned}
&\left|r_3(s,t)-m_{k_1}\right|^2\\
&\leq \frac{1}{(k_1+1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\left(\rho_s^{(k_1+1)}(x_{s+\lambda(\mu\varphi_\varepsilon),t})-\rho_s^{(k_1+1)}(x_{s,t})\right)G_1(\mu)\,G_2(\nu)\,\mu^{k_1+1}\right|^2 d\mu d\nu\\
&\leq \frac{L_2^2\varphi_\varepsilon^2}{[(k_1+1)!]^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G_1^2(\mu)\,G_2^2(\nu)\,\mu^{2(k_1+2)}d\mu d\nu\\
&= \frac{L_2^2\varphi_\varepsilon^2}{[(k_1+1)!]^2}\int_{-\infty}^{+\infty}G_2^2(\nu)\,d\nu\int_{-\infty}^{+\infty}G_1^2(\mu)\,\mu^{2(k_1+2)}d\mu,
\end{aligned}
$$

which tends to zero as $\varepsilon \longrightarrow 0$. It immediately follows that

$$
r_3(s,t) \xrightarrow{\mathbb{P}} m_{k_1}. \tag{5.59}
$$

– **Concerning the term** $r_4(s,t)$**.** Let

$$
r_4(s,t) = \varepsilon^{\frac{1}{\bar{k}+1}}\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\,dW_{u,v}^{\alpha,\beta}
$$

is a centred Gaussian processes. In fact, the condition **(H3)** ensures that the variance of $r_4(s,t)$ is finite. Consequently, we have

$$
\mathbb{E}\left[r_4(s,t)\right]^2 = \varepsilon^{\frac{2}{\bar{k}+1}}\mathbb{E}\left[\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\,dW_{u,v}^{\alpha,\beta}\right]^2,
$$

Then, using (5.12) and (5.13), we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\,dW^{\alpha,\beta}_{u,v}\right]^2\\
&\qquad=\varphi_\varepsilon^{2\alpha-2}\psi_\varepsilon^{2\beta-2}\int_{\mathbb{R}^4} G_{\varphi,\psi}(u_1,v_1)\,G_{\varphi,\psi}(u_2,v_2)\,\phi_{\alpha,\beta}(u_1,v_1,u_2,v_2)du_1dv_1du_2dv_2\\
&\qquad=\varphi_\varepsilon^{2\alpha}\psi_\varepsilon^{2\beta}\int_{\mathbb{R}^4} G(\mu_1,\nu_1)G(\mu_2,\nu_2)\phi_{\alpha,\beta}(\mu_1,\nu_1,\mu_2,\nu_2)d\mu_1d\mu_2d\nu_1d\nu_2,
\end{aligned}$$

with

$$\phi_{\alpha,\beta}(u_1,v_1,u_2,v_2)=\alpha(2\alpha-1)\beta(2\beta-1)|u_1-u_2|^{2\alpha-2}|v_1-v_2|^{2\beta-2}.$$

Now, we put

$$\varphi_\varepsilon=\varepsilon^{\frac{\bar{k}}{(k_2+1)(\bar{k}+1)}}\quad\text{and}\quad\psi_\varepsilon=\varepsilon^{\frac{\bar{k}}{(k_1+1)(\bar{k}+1)}},$$

with

$$\bar{k}=\frac{(k_1+1)(k_2+1)}{(k_1+1)(1-\beta)+(k_2+1)(1-\alpha)}.$$

Therefore,

$$\mathbb{E}[r_4(t)]^2=\int_{\mathbb{R}^4} G(\mu_1,\nu_1)G(\mu_2,\nu_2)\phi_{\alpha,\beta}(\mu_1,\nu_1,\mu_2,\nu_2)d\mu_1d\mu_2d\nu_1d\nu_2.$$

Then, by Fubini theorem and the condition $G(u,v)=G_1(u)G_2(v)$, we obtain

$$\mathbb{E}[r_4(s,t)]^2=\sigma^2_{\alpha,\beta}=\alpha(2\alpha-1)\beta(2\beta-1)\sigma^2_\alpha\sigma^2_\beta, \tag{5.60}$$

where

$$\sigma^2_\alpha=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_1(\mu_1)G_1(\mu_2)|\mu_1-\mu_2|^{2\alpha-2}d\mu_1d\mu_2;$$

and

$$\sigma^2_\beta=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_2(\nu_1)G_2(\nu_2)|\nu_1-\nu_2|^{2\beta-2}d\nu_1d\nu_2.$$

Thus, combining (5.53), (5.56), (5.59), (5.60) and by Slutsky's theorem, we can conclude that (5.28) holds as $\varepsilon\longrightarrow 0$. This completes the proof.

## Proof of Theorem 5.7

From (5.29) and (5.41), we have

$$\begin{aligned}
\tilde{R}_{\alpha,\beta,\theta}(s,t)-R^*_{\alpha,\beta,\theta}(s,t) &= \int_0^T\int_0^T G_{\varphi,\psi}(u,v)\Big(R_{\alpha,\beta,\theta}(u,v)dudv+\varepsilon dM^{\alpha,\beta}_{u,v}\Big)-R^*_{\alpha,\beta,\theta}(s,t)\\
&= \int_0^T\int_0^T G_{\varphi,\psi}(u,v)\Big(R_{\alpha,\beta,\theta}(u,v)-R^*_{\alpha,\beta,\theta}(u,v)\Big)dudv\\
&\quad+\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\Big(R^*_{\alpha,\beta,\theta}(u,v)-R^*_{\alpha,\beta,\theta}(s,t)\Big)dudv\\
&\quad+\varepsilon\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\,dM^{\alpha,\beta}_{u,v}.
\end{aligned}$$

Using the inequality $(a+b+c)^2 \leq 3a^2+3b^2+3c^2$, it yields

$$\begin{aligned}\mathbb{E}\left[\tilde{R}^*_{H,\theta}(t)-R^*_{H,\theta}(t)\right]^2 &\leq 3\mathbb{E}\left|\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\left(R_{\alpha,\beta,\theta}(u,v)-R^*_{\alpha,\beta,\theta}(u,v)\right)dudv\right|^2\\ &\quad +3\mathbb{E}\left|\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\left(R^*_{\alpha,\beta,\theta}(u,v)-R^*_{\alpha,\beta,\theta}(s,t)\right)dudv\right|^2\\ &\quad +3\varepsilon\mathbb{E}\left|\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\,dM^{\alpha,\beta}_{u,v}\right|^2\\ &= 3I_1+3I_2+3I_3.\end{aligned} \tag{5.61}$$

–**Concerning** $I_1$. Via the Cauchy–Schwarz inequality, we get

$$\begin{aligned}I_1 &= \mathbb{E}\left|\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\left(R_{\alpha,\beta,\theta}(u,v)-R^*_{\alpha,\beta,\theta}(u,v)\right)dudv\right|^2\\ &\leq \left\{\int_0^T\int_0^T G^2_{\varphi,\psi}(u,v)\,dudv\right\}\left\{\mathbb{E}\int_0^T\int_0^T\left(R_{\alpha,\beta,\theta}(u,v)-R^*_{\alpha,\beta,\theta}(u,v)\right)^2dudv\right\}\\ &= \varphi^{-1}\psi^{-1}(B-A)^2\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G^2(\mu,\nu)\,d\mu d\nu\right\}\\ &\qquad\times\left\{\mathbb{E}\int_0^T\int_0^T\left(R_{\alpha,\beta,\theta}(u,v)-R^*_{\alpha,\beta,\theta}(u,v)\right)^2dudv\right\}.\end{aligned} \tag{5.62}$$

Now, by equations (5.31), (5.32), we obtain

$$\begin{aligned}&\mathbb{E}\left[\int_0^T\int_0^T\left(R_{\alpha,\beta,\theta}(u,v)-R^*_{\alpha,\beta,\theta}(u,v)\right)^2dudv\right]\\ &=\int_0^T\int_0^T\mathbb{E}\left\{\frac{d}{dw^\alpha_u}\frac{d}{dw^\beta_v}\int_0^u\int_0^v k_\alpha(u,\mu)k_\beta(v,\nu)\theta(\mu,\nu)(X(\mu,\nu)-x(\mu,\nu))\,d\mu d\nu\right\}^2dudv\\ &=H(\alpha,\beta,\lambda_\alpha,\lambda_\beta)\int_0^T\int_0^T\mathbb{E}\left\{\int_0^u\int_0^v\frac{\partial k_\alpha(u,\mu)}{\partial u}\frac{\partial k_\beta(v,\nu)}{\partial v}\theta(\mu,\nu)(X(\mu,\nu)-x(\mu,\nu))\,d\mu d\nu\right\}^2\\ &\qquad\times u^{4\alpha-2}v^{4\beta-2}dudv\\ &\leq H(\alpha,\beta,\lambda_\alpha,\lambda_\beta)\int_0^T\int_0^T\left\{\int_0^u\int_0^v\left(\frac{\partial k_\alpha(u,\mu)}{\partial u}\right)^2\left(\frac{\partial k_\beta(v,\nu)}{\partial v}\right)^2\theta^2(\mu,\nu)d\mu d\nu\right\}\\ &\qquad\times\left\{\int_0^u\int_0^v\mathbb{E}\left(X(\mu,\nu)-x(\mu,\nu)\right)^2d\mu d\nu\right\}u^{4\alpha-2}v^{4\beta-2}dudv,\end{aligned} \tag{5.63}$$

where the constant $H(\alpha,\beta,\lambda_\alpha,\lambda_\beta)$ depends on the quadratic variation of the martingale $M^{\alpha,\beta}_{s,t}$. Next, using inequality (5.38), we have

$$\mathbb{E}\left(X(\mu,\nu)-x(\mu,\nu)\right)^2 \leq e^{2L\mu\nu}\varepsilon^2\mu^{2\alpha}\nu^{2\beta}. \tag{5.64}$$

On the other hand by (5.14), note that

$$\left(\frac{\partial k_\alpha(u,\mu)}{\partial u}\right)^2 = C_\alpha^{-2}\left(\frac{1}{2}-\alpha\right)^2\frac{\mu^{1-2\alpha}}{(u-\mu)^{2\alpha+1}},$$

and

$$\left(\frac{\partial k_\beta(v,\nu)}{\partial v}\right)^2 = C_\beta^{-2}\left(\frac{1}{2}-\beta\right)^2 \frac{\nu^{1-2\beta}}{(v-\nu)^{2\beta+1}}.$$

Then,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\int_0^T \left(R_{\alpha,\beta,\theta}(u,v)-R^*_{\alpha,\beta,\theta}(u,v)\right)^2 dudv\right] \\
&\quad\le H'(\alpha,\beta,\lambda_\alpha,\lambda_\beta)\int_0^T\int_0^T\left\{\int_0^u\int_0^v \frac{\mu^{1-2\alpha}}{(u-\mu)^{2\alpha+1}}\frac{\nu^{1-2\beta}}{(v-\nu)^{2\beta+1}}d\mu d\nu\right\} \\
&\qquad\times\left\{\int_0^u\int_0^v e^{2L\mu\nu}\varepsilon^2\mu^{2\alpha}\nu^{2\beta}d\mu d\nu\right\}u^{4\alpha-2}v^{4\beta-2}dudv \\
&\quad\le H'(\alpha,\beta,\lambda_\alpha,\lambda_\beta)\int_0^T\int_0^T\left\{\int_0^u \frac{\mu^{1-2\alpha}}{(u-\mu)^{2\alpha+1}}d\mu\int_0^v\frac{\nu^{1-2\beta}}{(v-\nu)^{2\beta+1}}d\nu\right\} \\
&\qquad\times\left\{\int_0^u\int_0^v e^{2L\mu\nu}\varepsilon^2\mu^{2\alpha}\nu^{2\beta}d\mu d\nu\right\}u^{4\alpha-2}v^{4\beta-2}dudv \\
&\quad\le H''(\alpha,\beta,\lambda_\alpha,\lambda_\beta)\varepsilon^2,
\end{aligned} \tag{5.65}$$

where $H''(\alpha,\beta,\lambda_\alpha,\lambda_\beta)$ is a positive constant depending on $T, L, \alpha, \beta, \lambda_\alpha, \lambda_\beta, C_\alpha$ and $C_\beta$.

In addition, combining (5.62) and (5.65), and by the condition **(H1)**, we have

$$I_1 \le C_1\varepsilon^2\varphi_\varepsilon^{-1}\psi_\varepsilon^{-1}, \tag{5.66}$$

where $C_1$ is a positive constant depending on $H''(\alpha,\beta,\lambda_\alpha,\lambda_\beta)$ and $(B-A)$.

– **Concerning term** $I_2$.

$$\begin{aligned}
I_2 &= \mathbb{E}\left[\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\left(R^*_{\alpha,\beta,\theta}(u,v)-R^*_{\alpha,\beta,\theta}(s,t)\right)dudv\right]^2 \\
&= \mathbb{E}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(\mu,\nu)\left(R^*_{\alpha,\beta,\theta}(s+\phi_\varepsilon\mu, t+\phi_\varepsilon\nu)-R^*_{\alpha,\beta,\theta}(s,t)\right)d\mu d\nu\right]^2.
\end{aligned}$$

From assumption **(A2')**, **(H1)** and **(H2')**, we obtain

$$\begin{aligned}
I_2 &\le L^2\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(\mu,\nu)\left(\left|\varphi_\varepsilon\mu\right|^{\gamma_1}+\left|\psi_\varepsilon\nu\right|^{\gamma_2}\right)d\mu d\nu\right]^2 \\
&\le \varphi_\varepsilon^{2\gamma_1}L^2\int_{-\infty}^{+\infty}G_1^2(\mu)\left|\mu\right|^{2\gamma_1}d\mu\int_{-\infty}^{+\infty}G_2^2(\nu)d\nu \\
&\quad+\psi_\varepsilon^{2\gamma_2}L^2\int_{-\infty}^{+\infty}G_2^2(\nu)|\nu|^{2\gamma_2}d\nu\int_{-\infty}^{+\infty}G_1^2(\mu)d\mu \\
&\le C_2\varphi_\varepsilon^{2\gamma_1}+C_3\psi_\varepsilon^{2\gamma_2},
\end{aligned} \tag{5.67}$$

where $C_2$ and $C_3$ are positive constants.

– **Concerning** $I_3$. Since $(\alpha,\beta)\in(1/2,3/4)^2$, we have

$$\begin{aligned}I_3 &= \varepsilon^2\mathbb{E}\left|\int_0^T\int_0^T G_{\varphi,\psi}(u,v)\,dM^{\alpha,\beta}_{u,v}\right|^2\\ &= \frac{\varepsilon^2}{\varphi_\varepsilon^2\psi_\varepsilon^2}\int_0^T\int_0^T G^2\left(\frac{u-s}{\phi_\varepsilon},\frac{v-t}{\phi_\varepsilon}\right)dw^\beta_u dw^\alpha_v.\end{aligned}$$

In addition, under (5.31), (5.32) and **(H1)**, we deduce that

$$\begin{aligned}I_3 &= \frac{\varepsilon^2}{\varphi_\varepsilon^2\psi_\varepsilon^2}\frac{(1-\alpha)(1-\beta)}{\lambda_\alpha\lambda_\beta}\int_0^T\int_0^T G^2\left(\frac{u-s}{\varphi_\varepsilon},\frac{v-t}{\psi_\varepsilon}\right)v^{1-2\alpha}u^{1-2\beta}dudv\\ &\leq \frac{\varepsilon^2}{\varphi_\varepsilon^2\psi_\varepsilon^2}\frac{(1-\alpha)(1-\beta)}{\lambda_\alpha\lambda_\beta}\left\{\int_0^T\int_0^T G^2\left(\frac{u-s}{\varphi_\varepsilon},\frac{v-t}{\psi_\varepsilon}\right)dudv\int_0^T\int_0^T v^{2-4\alpha}u^{2-4\beta}dudv\right\}^{1/2}\\ &= \frac{\varepsilon^2}{\varphi_\varepsilon^2\psi_\varepsilon^2}\frac{(1-\alpha)(1-\beta)}{\lambda_\alpha\lambda_\beta}\left(\frac{T^{6-4(\alpha+\beta)}}{(3-4\alpha)(3-4\beta)}\right)^{1/2}\left\{\varphi_\varepsilon\psi_\varepsilon\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G^2(\mu,\nu)\,d\mu d\nu\right\}^{1/2}\\ &= C_4\varepsilon^2\varphi_\varepsilon^{-3/2}\psi_\varepsilon^{-3/2},\end{aligned}\tag{5.68}$$

where $C_4$ is a positive constant depending on $\alpha,\beta,T,\lambda_\alpha$ and $\lambda_\beta$.

Finally, under the assumption **(H3')**, combining (5.61), (5.66), (5.67) and (5.68), we obtain

$$\lim_{\varepsilon\longrightarrow0}\sup_{|\theta(.)|\leq L}\sup_{0\leq s,t\leq T}\mathbb{E}\left[\tilde{R}_{\alpha,\beta,\theta}(s,t)-R^*_{\alpha,\beta,\theta}(s,t)\right]^2=0.$$

## Proof of Corollary 5.5

Combining (5.61), (5.66), (5.67) and (5.68), we get

$$\sup_{|\theta(.)|\leq L}\sup_{0\leq s,t\leq T}\mathbb{E}\left[\tilde{R}^*_{\alpha,\beta,\theta}(s,t)-R^*_{\alpha,\beta,\theta}(s,t)\right]^2=\mathcal{O}\left(\varepsilon^2\varphi_\varepsilon^{-1}\psi_\varepsilon^{-1}+\varphi_\varepsilon^{2\gamma_1}+\psi_\varepsilon^{2\gamma_2}+\varepsilon^2\varphi_\varepsilon^{-3/2}\psi_\varepsilon^{-3/2}\right).\tag{5.69}$$

In order to minimize the mean square error, we have to minimize the following function of two variables:

$$(\varphi_\varepsilon,\psi_\varepsilon)\longrightarrow\xi(\varphi_\varepsilon,\psi_\varepsilon)=\varphi_\varepsilon^{2\gamma_1}+\psi_\varepsilon^{2\gamma_2}+\varepsilon^2\varphi_\varepsilon^{-3/2}\psi_\varepsilon^{-3/2}.$$

The minimizing values are as:

$$\varphi^*_\varepsilon=\mathcal{O}\left(\varepsilon^{\frac{4\gamma_2}{3\gamma_1+3\gamma_2+4\gamma_1\gamma_2}}\right),\quad\psi^*_\varepsilon=\mathcal{O}\left(\varepsilon^{\frac{4\gamma_1}{3\gamma_1+3\gamma_2+4\gamma_1\gamma_2}}\right).$$

Then,

$$\xi(\varphi_\varepsilon,\psi_\varepsilon)=\mathcal{O}\left(\varepsilon^{\frac{4\gamma_1\gamma_2}{3\gamma_1+3\gamma_2+4\gamma_1\gamma_2}}\right)$$

and

$$\varepsilon^2\varphi_\varepsilon^{-1}\psi_\varepsilon^{-1}=\mathcal{O}\left(\varepsilon^{\frac{2\gamma_1+2\gamma_2+8\gamma_1\gamma_2}{3\gamma_1+3\gamma_2+4\gamma_1\gamma_2}}\right).$$

Therefore

$$\lim_{\varepsilon \longrightarrow 0} \sup_{|\theta(.)| \leq L} \sup_{0 \leq t \leq T} \mathbb{E}\Big[\tilde{R}^*_{H,\theta}(t) - R^*_{H,\theta}(t)\Big]^2 = \mathcal{O}\Big(\varepsilon^{\frac{4\gamma_1\gamma_2}{3\gamma_1+3\gamma_2+4\gamma_1\gamma_2}}\Big).$$

This completes the proof of Corollary.

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# Conclusion and future work

## Conclusion

In this thesis, we investigated the problem of nonparametric estimation of the trend function for stochastic differential equations driven by a long-range dependence and self-similar Gaussian process with small noise. The main purpose of this study is to estimate the trend function by the kernel type estimator from continuous observation of $\{X_t,\ 0 \leq t \leq T\}$ and to discuss the asymptotic behaviour of the estimator as $\varepsilon \longrightarrow 0$, we focused on stochastic differential equations driven by a mixed sub-fractional Brownian motion, bi-fractional Brownian motion, weighted fractional Brownian motion and sheet fractional Brownian.

A literature review of existing approaches and mathematical tools necessary for the contributions proposed in this thesis is provided in the first Chapter.

In the second chapter, we considered the problem of nonparametric estimation of the trend function for stochastic differential equation driven by a mixed fractional Brownian motion, under some hypotheses, we established the consistent uniform, the rate of convergence as well as the asymptotic normality of the kernel estimator.

In the third chapter, we studied the problem of drift function estimation of stochastic differential equations driven by a bi-fractional Brownian motion. We have proposed a kernel-type estimator for the trend function. We showed this estimator is consistent with the rate of convergence and asymptotically normal.

In the fourth chapter, we dealt with the problem of nonparametric estimation for equation differential equation is driven by weighted fractional Brownian. In this case, we have proved under certain conditions the uniform consistency and the rate of convergence of the kernel type estimator.

In the fifth chapter, we presented the case where the stochastic differential equation is driven by sheet fractional Brownian motion. We proposed kernel type estimators based on continuous observation of SDE with small noise, and study the asymptotic behaviour of the estimator.

## Future work

The works developed in this thesis offers many perspectives:

- Nonparametric estimation for stochastic differential equations driven by a fractional Brownian motion with general Hurst parameter, i.e. $H \in (0, 1/2)$.
- Locally linear estimation and Nadaraya–Watson kernel type estimator of the drift function for diffusion model with an additive sub-fractional Brownian motion noise.
- Statistical inference for stochastic differential equations via Malliavin calculus.

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**ملخص:**

في هذه الأطروحة، نتطرق إلى مسألة التقدير اللامعلمي لدالة الاتجاه في بعض نماذج المعادلات التفاضلية العشوائية الموجهة بواسطة سيرورات متشابهة ذاتياً تعمم الحركة البراونية الكسرية وفي سياق الانتشار الصغير. أولاً، نعتبر نموذج معادلة تفاضلية عشوائية موجهة بحركة براونية مختلطة شبه كسرية, نعطي تقديراً باستعمال مقدر النواة لدالة الاتجاه و ندرس التقارب الطبيعي، التقارب الموحد و سرعة التقارب لهذا المقدّر. ثانيا، نعمّم نتائجنا في حالة نموذج معادلة تفاضلية عشوائية موجهة بحركة براونية ثنائية الكسر وندرس خصائص التقارب لمقدّر النواة لدالة الاتجاه في سياق الانتشار الصغير. ثالثا، نعالج مُقدر النواة للدالة الاتجاه في نموذج معادلة تفاضلية عشوائية موجهة بحركة براونية كسرية مرجحة، وسنتطرق للتقارب الموحد ومعدل تقارب هذا المقدّر بالإضافة إلى التقارب الطبيعي. اخيرًا، ندرس التقدير اللامعلمي لمعادلات تفاضلية عشوائية موجهة بورقة براونية كسرية. نقترح مُقدر النواة ونقوم بدراسة السلوك التقاربي للمقدر.

**كلمات مفتاحية:** تقدير لامعلمي، مقدّر النواة، دالة الإتجاه، معادلة تفاضلية عشوائية، حركة براونية كسرية، حركة براونية شبه كسرية مختلطة، حركة براونية ثنائية الكسر، حركة براونية كسرية مرجحة.

**Résumé** :

La problématique abordée dans cette thèse est l'estimation non paramétrique pour la fonction de dérive dans certains modèles d'équations différentielles stochastiques présentée par des processus stochastiques auto-similaires généralisant le mouvement Brownien fractionnaire dans le cadre de petite diffusion.

Dans un premier temps, nous considérons un modèle d'équation différentielle stochastique dirigé par un mouvement Brownien sous fractionnaire mixte et nous construisons un estimateur par la méthode du noyau pour la fonction du dérive. Nous étudions la normalité asymptotique ainsi que la convergence uniforme de cet estimateur en précisant la vitesse de convergence.

Dans un second temps, nous généralisons nos résultats au cas d'un modèle d'équation différentielle stochastique dirigé par un mouvement Brownien bi-fractionnaire, et nous étudions les propriétés asymptotiques du l'estimateur à noyau pour la fonction de dérive dans le cadre de petite diffusion.

Dans un troisième temps, un estimateur à noyau pour la fonction de dérive dans un modèle d'équation différentielle stochastique dirigé par un mouvement Brownien fractionnaire pondéré est construit, la convergence uniforme ainsi que la normalité asymptotique (avec vitesse de convergence) de cet estimateur sont établies.

Enfin, nous considérons le problème de l'estimation non paramétrique de la fonction de dérive et du multiplicateur linéaire pour un processus satisfaisant des équations différentielle stochastique dirigé par le drap Brownien fractionnaire. Nous proposons des estimateurs de type à noyau basés sur les trajectoires de l'équation différentielle stochastique avec un petite bruit et étudions le comportement asymptotique de l'estimateur.

**Mots clés** : estimation non paramétrique, estimateur à noyau, fonction de dérive, équation différentielle stochastique, mouvement Brownien fractionnaire, mouvement Brownien sous fractionnaire mixte, mouvement Brownien bi-fractionnaire, mouvement Brownien fractionnaire pondéré, drap brownien fractionnaire.

**Abstract**:

In this thesis, we consider a drift estimation problem of nonparametric estimation for the drift function in some models of stochastic differential equations written by self-similar stochastic processes generalizing the fractional Brownian movement in the context of small diffusion. Firstly, we consider a model of stochastic differential equation driven by a mixed sub-fractional Brownian motion and we build an estimator by the kernel method for the drift function. We study the asymptotic normality, the uniform convergence (with rate) of convergence of this estimator. Secondly, we generalize our results to the case of a stochastic differential equation model driven by a bi-fractional Brownian motion, and we study the asymptotic properties of the kernel estimator for the drift function in the context of small diffusion.

Thirdly, a kernel estimator for the drift function in a stochastic differential equation model driven by a weighted fractional Brownian motion is constructed, the uniform convergence and the rate of convergence of this estimator is established as well as its asymptotic normality. Finally, we considers the problem of estimating the drift function and linear multiplier for process satisfying stochastic differential equations driven by a fractional Brownian sheet. We propose kernel type estimators based on continuous observation of SDE with small noise, and study the asymptotic behavior of the estimator.

**Key words:** nonparametric estimation, kernel estimator, drift function, stochastic differential equation, fractional Brownian motion, mixed fractional Brownian motion, bi-fractional Brownian motion, weighted fractional Brownian motion , fractional Brownian sheet.