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Dedication

To the sym- bol of sweet-
ness, tenderness, love and affection,
to my mother for her continued sup-
port as well as for the sacrifices she
made for my education. I hope
that this work be the testi-
mony of the love that
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♡

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List Of Works

Publications

1. O. Kebiri, H. Bouanani, A. Kandouci. On the existence and uniqueness of solutions to forward backward stochastic differential equations driven by G-brownian motion. Bulletin of the Institute of Mathematics Academia Sinica(New Series),15(3):217-236, 2020.

Submitted Papers

1. Hafida Bouanani, Carsten Hartmann, and Omar Kebiri. Model reduction and uncertainty quantification of multiscale diffusions with parameter uncertainties using nonlinear expectations [16].

2. O. Kebiri, H. Bouanani, A. Redjil, C. Hartmann. Existence of optimal relaxed control of decoupled forward-backward stochastic differential equations in the G-framework.

Communications

- Hafida Bouanani. Communication in Febrary, 2019. Forward-backward stochastic differential equations, in the 19^{th} edition of TAMTAM conference in Tlemcen.

- Hafida Bouanani. Presentation in April, 2018. Stochastic optimal control, in the presence of Dr. O. Kebiri. Within weekly seminar of LMSSA.

- Hafida Bouanani. Presentation in September, $28^{th}, 2019$. Stochastic optimal control & G-Brownian motion, within the day of the progress state.

- Hafida Bouanani. Presentation in September, $28^{th}, 2020$. Forward-backward stochastic differential equations & G-Brownian motion, within the day of the progress state.

- Hafida Bouanani. Communication in June, $26^{th}, 2021$. Notes On G-SDE's in national conference on applied mathematics and didactics. Constantine-Algeria.

- Hafida Bouanani. Talk in June, $30^{th}, 2021$, On the existence and uniqueness of solution to forward backward stochastic differential equations driven by G-Brownian motion. Stochastic seminar, BTU Cottbus-Seftenberg- Germany.

Other training

- Participation in the CIMPA'ASA 2019 school on "Stochastic Analysis and Applications" organized by de Dr. Moulay Tahar-Saida University.

- Participation in Spring school on "Backward Stochastic Differential Equation & Stochastic Control", organized by LMSSA (April, $22^{th} - 26^{th}, 2018$).

- Participation in autumn school on "l'Inférence Statistique et Théorie de l'information pour des processus *Markoviens* et semi-*Markoviens*" organized by LMSSA (October, $23^{rd} - 28^{th}, 2018$).

Contents

Introduction

The birth of the control theory has been marked by the *Bernoulli*'s result (1697) on the solution of the brachistochrone problem [91] except that its real rises was in $1950 - 1960s$, by the work of *Pontryagin* and his students on the maximum principle. Since then, this theory draws attention of many researchers of the pure mathematic as well as the applied fields. One of the branches of the control theory that have found its way into many areas of control engineering and modeling is the stochastic optimal control which is very useful tool in many applied sciences fields such as medicine(e.g. optimal therapies), biology, economics (e. g. strategies), finance, mechanic, ...etc.

Originally, *Pontryagin* and his team introduce the well-known maximum principle while studying the optimal control issues. Additionally, the maximum principle application leads to the so called *Hamiltonian* system with the state equation and its adjoint equation, consequently, forward backward differential equations have been raised. Similarly to the deterministic area, forward backward stochastic differential equations (FBSDEs in short) have been introduced via the definition of backward stochastic differential equations (BSDEs, in short). Formerly, the earliest version of the BSDE's (the linear form) has been introduced in 1973 by *Bismut* [13] as the adjoint equation in the stochastic version of the *Pontryagin* maximum principle.

In 1983, *Bensoussan* [11] used the martingale representation theorem to prove the wellposedness result of general linear BSDE's. In addition, the non-linear case has been introduced by *Bismut* [14] in 1978. Next, the study of the general *Pontryagin* maximum principle for stochastic optimal controls done by *Peng* and *Pardoux* [72] (1990) demonstrates the first wellposedness result for non-linear BSDE's. A non-exhaustive list of the work carried out on this field of research is cited. However, note the works of *Peng* and *Pardoux* [72, 73], the article of *El Karoui*, *Peng* and *Quenez* [30] and the book [27].

Furthermore, the first result for a coupled FBSDE (the solution of the forward appear in the backward one and also the solution of the backward appear in the forward one) was obtained by *Antonelli* [2] in his Ph.D. thesis in 1993, since then, many authors studied this system using different methods such as four step scheme [54] and the method of continuation [51] and many other several authors [84, 23, 70]. Equally important, the applications of FBSDEs attract many authors, we set among them [84, 70].

Furthermore, in the FBSDE's theory framework and particularly, in a *Markovian* framework, the solution of a BSDE describes the viscosity solution of the associated semi-linear partial differential equation (PDE in short). For more details, we refer to *El Karoui*, *Peng* and *Quenez*'s paper [30] and the references therein. *Peng* [76] obtained the *Hamilton-Jacobi-Bellman* equation and proved that the value function is its viscosity solution. Many authors studied the theory of BSDEs and its applications in stochastic control (see *Peng* [74]), finance (see *El Karoui*, *Peng* and *Quenez* [30]), and for partial differential equations theory (see *Peng* [75], etc.). Since then, BSDE's have been widely used in stochastic control and especially in mathematical finance.

For instance, any pricing problem by replication can be written in terms of linear BSDEs or non-linear BSDEs when portfolios constraints are taken into account as in *Peng* [75]. More importantly, a deep study has been done on the controlled forward backward stochastic differential equations, indeed the existence of an optimal controls theorems was the central topic developed in this framework. For example the existence of an optimal control for the decoupled FBSDE has been established by *Buckdahn, Labed, Rainer* and *Tamer* [17]; by using the associated *Hamilton-Jacoi-Belman* equation to construct a sequence of optimal feedback controls, then, passing to the limit and using the result of [28], they get the existence of a relaxed optimal control. Then again, they use the *Filippov* convexity condition to get the existence of a strict optimal control. This last step was also used by [5] in which they use the *Jakubowsky* S-topology and compactness method. The authors shown directly the existence of a relaxed control and established the existence of an optimal control. In the coupled case, exploiting the result of [17] and [5] to general case (coupled), *Bahlali* and *Kebiri*[55] proved the existence of an optimal control for degenerate FBSDE where they consider the coefficients satisfying the monotony condition given in[84]. Because of the degeneracy of the diffusion they transform the coefficient of the hessian uniformly elliptic by adding a strictly positive number on the contrary to the non-degenerate case, *Bahlali, Kebiri* and *Mezerdi* [6] was not obligated to add it. However, it does not affect any change on the system where the proof differ from of the degenerate case, it is in some sense like [17, 5] for more details see [6].

All the results discussed above are introduced on a filtered probabilistic space where the (additive) probability measure is very important tool and the mathematical expectation is linear; the uncertain phenomena are expressed by using the *Brownian* motion. It is natural to wonder if it is common to find a perfect situation in which an exact probability can be precisely determined.

In our world, it is not; moreover, even the probability uncertainty itself becomes difficult to deal with it. Furthermore, the basic tools those are widely used in the probability theory, and in financial, commercial and many other industrial domains have been criticized. For instance, there exists many situations where a precise probability for each feature outcome cannot be attributed. As it is in the real world because of the randomness and the ambiguity, due to an imprecise probability, incomplete information and vague data . . . etc. They emphasized with different kinds of uncertainty, a kind that affected the humankind's efforts. This kind of uncertainty has been introduced first by *Frank Knight* in his book "Risk, Uncertainty and profit" [60], in which a distinction about uncertainty and risk has been established i.e. there is a crucial difference between the feature for taking a known risk which we can convert into an efficacious certainty easily and for supposing a "true uncertainty" -a risk whose its value is not known- as in the real business world, all events are complex that is not susceptible to measurement. The "true uncertainty" which is known as *Knightian* uncertainty in economy and it is applicable in the situations where the level of uncertainty is higher.

On the other hand, *Von Neumann-Morgenstern* (1947) [92] set necessary and sufficient conditions under which the expected utility hypothesis holds. Since then, the modern economic theory is based on the *Von Neumann-Morgenstern* utility which deals with the mathematical expectation.

However, in 1953 the *Allais*'s paradox [1] rises in which the *Von Neumann-Morgenstern* axioms (completeness, transitivity, continuity and independence) have been criticized. Actually, the *Allais*'s paradox was presented by a counterexample to the independence axiom which implies that the linear mathematical expectation cannot be used to explain the uncertain phenomena. So, it was necessary to find a way to measure such phenomena. The non-linear expectation was the solution. In fact, an attractive definition was given to the non-linear expectation by *Gustave Choquet* (1953) [20] and it is known by "*Choquet* non-linear expectation" which attract with great interest many scientists for its potential applications in uncertainty issues (Risk measurement and super hedging in finance). But unfortunately, it was difficult to define the conditional *Choquet* non-linear expectation.

In 1997, based on backward stochastic differential equation theory, the concept of g-expectation [77] as well as the conditional g-expectation was subsequently introduced by *Peng*. Unfortunately, *Peng*'s g-expectation can be defined only on a BSDE framework. Thus, it was necessary to think at a concept that makes a sense in more general setting. For that, independently of the BSDE theory and inspired by uncertainty problems in economics, *Peng*[80] constructs a type of fully non-linear expectation dynamically coherent named G-expectation using a non-linear partial differential equations approach. More precisely, following *Kolmogorov*'s method, in 2006, *Peng* [80] introduced the one-dimensional *G-Brownian* motion (G-BM, in short) as well as the G-expectation under which the canonical process is the *G-Brownian* motion. Then, by the method used in *Peng*'s paper [78], a multidimensional *G-Brownian* motion [81] has been defined.

Unlike the classical situation, the quadratic variation of this process $\langle B\rangle$ is as the *G-Brownian* motion B itself, that is an important and a helpful tool in financial markets especially for capturing the volatility fluctuations. *Denis* and *Martini* [25] define the G-expectation based on a quasi-sure analysis from the abstract potential theory to construct a similar structure using a tight family of possibly mutually singular probability.

The main difference between the small g-framework and the big G-framework is that in the g-framework we can't let the uncertainty in the diffusion, whereas in the G-framework, both the drift and the diffusion can have uncertainty.

A related stochastic calculus to the *G-Brownian* motion has been developed in *S. Peng*'s papers (see [80, 79, 81] and [82]).

The study of a forward stochastic differential equation driven by a *G-Brownian* motion under different conditions on the coefficients attracts the attention of many authors we set among them *Peng* [79, 80] and [81], *Gao* [40], *Bai* and *Lin* [85] ...etc. On the other hand, a varied result has been attended on the backward stochastic differential equation driven by the G-BM such as [68, 95, 36, 50]...etc.

In recent years, G-expectation framework has found increasing applications in the domain of finance and economics; for example, *Epstein* and *Ji* [32, 31] study the asset pricing with ambiguity preferences, and *Beissner* [8] has studied the equilibrium theory with ambiguous volatility, in addition to many others see e.g. [93, 12, 48].

However, many applications of stochastic optimal control required to solve the corresponding *Hamilton-Jacobi-Belmann* equation with probabilistic methods, especially in high dimensional problems see e.g. [41]. Furthermore, the case where we do not know the exact value of the volatility, but only a range of it, like the case of finance, the corresponding HJB equation in a fully non-linear G-partial differential equation, and in case of high dimension, we cannot solve this end by the usual methods like the finite difference, so a probabilistic representation is required, and when the control enter the diffusion see e.g. [61, 15], this will produce a fully coupled G-FBSDE, also the application of the stochastic maximum principle gives a G-FBSDE system. So, it becomes challenging to study the G-FBSDE.

Thus, motivated by the above results, the existence of solution to a G-FBSDE is considered as the first issue solved in this thesis. However, we argued the existence and uniqueness of the solution of a coupled forward-backward stochastic differential equation in the G-framework by constructing a mapping for which the fixed point is the solution of our G-FBSDE, where we prove that this mapping is a contraction. In this thesis, we do not require the monotonicity condition that *Wang* and *Yuan.* [94] supposed. In addition, we allow that the solution of the forward equation X can be multidimensional process, not necessarily one-dimensional like the case of [94], where they needed it for the comparison reasons.

Therefore, the developments reached on the G-framework allowed to wonder about many other fields; how to deal with it in the G-framework. Hence, to characterize the real world problems with ambiguity and randomness in a more precise way, it is important to deal with stochastic controlled equations that are perturbed by a *G-Brownian* motion. One of the relevant studies that contributed in the development of the stochastic optimal control under G-expectation is the *Peng*'s [76] one. Moreover, the stochastic optimal control problems under G-framework have been defined. Also, the dynamic programming has been obtained by *Zhang* [96]. Since then, the theory of optimal control in this framework received a strong attention. Furthermore, many of results are drawn from the varied spectrum of the classical framework into the G-framework. It becomes challengeable to generalize the methods and techniques of the classical situation into the G-framework, we set among them ([49, 44, 37, 48, 45]). One of the remarkable results is in particular for stochastic differential equations driven by a G-Brownian motion(G-SDEs) which was studied by *Redjil* and *Choutri*[86], where they introduced a space of G-relaxed controls and proved the so called G-Chattering lemma, the existence of an optimal relaxed control was established. Therefore, one of the main results in this thesis is the study of a system driven by decoupled forward-backward stochastic differential equations in the G-framework.

Our objective is to investigate the problem of the existence of optimal relaxed stochastic control given by a G-FBSDE and a cost function as the first component of the solution of the backward stochastic differential equation. The idea is to use an approximated control problem of our original issue, to prove the existence of relaxed-optimal control for the approximated system then pass to the limit.

As a result of the deep study done on the G-framework, we used the link between it and the result obtained by *Denis* and *Martini* [26] as well as the result shown in [25] to study model reduction of linear and bilinear quadratic stochastic control problems with parameter uncertainties. More precisely, we consider slow-fast systems with unknown diffusion coefficient and study the convergence of the slow process in the limit of infinite scale separation. Until now, there is no result in the G-framework that deal with the averaging and homogenization of multiscale systems.

In this thesis, we consider a slow-fast system in which the drift and the diffusion coefficients are uncertain parameters, then by reformulating the slow-fast system as an optimal control problem in which the unknown parameter plays the role of a control variable that can take values in a closed bounded set. For systems with unknown diffusion coefficient, the underlying stochastic control problem admits an interpretation in terms of a stochastic differential equation driven by a G-*Brownian* motion. Then again, we quantify the uncertainty in the reduced system by deriving a limit equation that represents a worst-case scenario for any given (possibly path-dependent) quantity of interest. The idea here is to formulate the nonlinear dynamic programming equation of the underlying control problem as a forward-backward stochastic differential equation in the G-*Brownian* motion framework. Next, we prove convergence of the slow process with respect to the nonlinear expectation on the probability space induced by the G-*Brownian* motion. At the end, two numerical examples are given in order to illustrate the theoretical results.

Outline of the thesis

The results presented in this thesis are very interesting for the theoretical and applied fields. The results can help to deal with many problems in finance, economics, games theory and optimal control theory as well as model reduction with uncertainty problem. The thesis statements are organized as follow:

Chapter one "Generalities": We recall some important notations, basic results and theorems obtained in the G-expectation framework, which concern the formulation of the G-expectation, the *G-Brownian* motion, and the G-stochastic differential equations. . . etc.

Chapter two "G-Forward Backward Stochastic Differential Equations" in which we present the existed result on this kind of systems, then establish the existence and uniqueness of solution of such coupled FBSDE in the G-framework that is the subject of our published research paper [56].

Chapter three Optimal Control of a decoupled FBSDEs in the G-framework" in which we establish the existence of relaxed optimal control for decoupled forward-backward system defined on a sub-linear expectation (decoupled forward-backward stochastic differential equations that is driven by the so called *G-Brownian* motion).

Chapter four "Model Reduction And Uncertainty Quantification Of Multiscale Under G-Expectation: In this last chapter we propose a general framework for the averaging and homogenization of multiscale with uncertain parameter then we aim to use this results to quantify the uncertainty in the reduce system and we close the chapter by two simple numerical examples. This chapter is a subject of the submitted paper [16].

Chapter 1

Generalities

Within this chapter, we introduce the basic notions and recall all the results that will be used throughout this thesis. In the first section, a brief summary on the forward backward stochastic differential equations (in the classical situation). Then, we introduce the g-expectation in the second section. Finally, the third section is devoted for the G-expectation framework.

1.1 Forward-backward stochastic differential equations

Once the backward stochastic differential equation (BSDE's in short) appeared, the theory of forward-backward stochastic differential equations (FBSDE's) has been risen in it decoupled form. So, it is necessary to recall some basics tools in the BSDE's theory which was the phenomenon of the last century that has been received a great attention because of its interest in various applied fields such as mathematical finance and especially for its connection with non-linear partial differential equation.

Backward stochastic differential equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with the natural filtration of the *Brownian* motion $(\mathcal{F}_t)_{t\geq 0}$, and ξ is a $\mathcal{F}_T$-measurable random variable. We aim to solve the following equation:

$$-\frac{dY_t}{dt} = g(Y_t), \qquad t \in [0,T], \quad \text{with} \quad Y_T = \xi.$$

A stochastic differential equation with a given terminal condition must have an adapted solution and a non-anticipating solution is unlikely. In general, by the martingale-representation of Y, the backward stochastic differential equations have the following form:

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dB_t, \quad Y_T = \xi, \tag{1.1}$$

equivalently

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + \int_t^T Z dB_s, \tag{1.2}$$

where the terminal condition ξ is $\mathcal{F}_T$-measurable, square integrable random variable and $(B_t)_{t\geq 0}$ are d-dimensional *Brownian* motion processes.

Let consider the following notations:

- $\mathcal{S}^2(\mathbb{R}^k)$ is the space of progressively measurable processes Y such that

$$\| Y \|_{\mathcal{S}^2}^2 := \mathbb{E}(\sup_{0\leq t\leq T} |Y|^2) < \infty,$$

  and $\mathcal{S}_c^2(\mathbb{R}^k)$ denotes the subspace of the continuous process.

- $\mathcal{M}^2(\mathbb{R}^{k\times d})$ is the space of progressively measurable processes Z such that

$$\| Z \|_{\mathcal{M}^2} := \mathbb{E}\left(\int_0^T \| Z_t \|^2 \, dt\right) < \infty,$$

  where, for $z \in \mathbb{R}^{k\times d}$, $\| z \|^2 = trace(zz^*)$, and $M^2(\mathbb{R}^{k\times d})$ denotes the $\mathcal{M}^2(\mathbb{R}^{k\times d})$ equivalent classes.

Definition 1.1.1. *A solution of the equation* (1.2) *is the couple process* $\{(Y_t, Z_t)\}_{0\leq t\leq T}$ *that verify*

1. *Y and Z are progressively measurable processes with values in $\mathbb{R}^k$ and $\mathbb{R}^{k\times d}$ respectively.*
2. $\mathbb{P}$*-p.s.* $\int_0^T \{|g(r, Y_r, Z_r)| + \| Z_r \|^2\} dr < \infty$.
3. $\mathbb{P}$*-p.s. we have*

$$Y_t = \xi + \int_t^T g(r, Y_r, Z_r) dr - \int_t^T Z_r dB_r, \qquad 0 \leq t \leq T.$$

We consider a random variable $\xi, \mathcal{F}_T-$ measurable with value in $\mathbb{R}^k$.
Here are the hypothesis under which we work

1. *Lipschitz* condition: There exist $L > 0$ such that, for any $y_1, y_2 \in \mathbb{R}^k, z_1, z_2 \in \mathbb{R}^{k\times d}$ we have :

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq L(|y_1 - y_2| + \| z_1 - z_2 \|).$$

2. Integrability condition:

$$\mathbb{E}\left[|\xi|^2 + \int_0 T|g(r, 0, 0)|^2 dr\right] < \infty.$$

Theorem 1.1. *[72] Under hypothesis* (1) *and* (2)*, the BSDE (1.2) has a unique solution* (Y, Z) *where* $Z \in M^2$.

Remark 1.1.1. *The role of the process Z is to make the process Y adapted and when it is not necessary, the process Z will be equal to zero.*

Now, let distinct between decoupled FBSDE system and the coupled one.

Decoupled forward-backward stochastic differential equations

A decoupled FBSDE is a system consisting of backward stochastic differential equations that depend on the solution of the associated forward stochastic differential equations i.e. on the standard filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of a d-dimensional *Brownian* motion. A decoupled FBSDE system is given by:

$$\begin{cases} Y_r^{t,x} = \phi(X_T^{t,x}) + \int_r^T g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})du + \int_r^T Z_u^{t,x} dB_u, \quad 0 \le r \le T, \\ \\ Y_T^{t,x} = \phi(X_T^{t,x}), \end{cases} \tag{1.3}$$

where, the parameter $(X_r)_{0\le r\le T}$ of the BSDE (1.3) is the solution of the forward stochastic differential equation:

$$\begin{cases} X_r^{t,x} = x + \int_t^r b(u, X_u^{t,x})du + \int_t^r \sigma(u, X_u^{t,x})dB_u, \\ \\ X_t = x. \end{cases} \tag{1.4}$$

In order to solve the BSDE (1.3), we start by solving the SDE (1.4) and substitute the result in (1.3). We use $P^{t,x}$ to denote the process that starts in x at the initial time t.

Coupled forward-backward stochastic differential equations

It is the case where the solution of the BSDE appears in the forward equation and conversely. In other words, on a filtered probability space $(\Omega,\ \mathcal{F},\ \mathbb{P},\ (\mathcal{F}_t))$ of an m-dimensional *Brownian* motion and which satisfies the usual conditions, we define the FBSDE system as follow:

$$\begin{cases} X_t^{t,x} = x + \int_t^T b(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + \int_t^T \sigma(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})dB_s, \\ \\ Y_t^{t,x} = \Phi(X_T^{t,x}) - \int_t^T g(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + \int_t^T Z_s^{t,x} dB_s, \end{cases} \tag{1.5}$$

where, the deterministic functions b, σ, g and Φ are defined by:

$$b : \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \longmapsto \mathbb{R}^d, \qquad \sigma : \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \longmapsto \mathbb{R}^{d\times m},$$

$$g : \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times m} \longmapsto \mathbb{R}^p, \qquad \Phi : \mathbb{R}^d \longmapsto \mathbb{R}^p.$$

The processes $X^{t,x}$, $Y^{t,x}$, $Z^{t,x}$ are $(\mathcal{F}_t)$-adapted square integrable processes.

- $\mathcal{S}^2(t, T; \mathbb{R}^m)$ denote the set of $\mathbb{R}^m$-valued, $\mathbb{F}$-adapted, continuous processes $(X_s, s \in [t, T])$ which satisfy $\mathbb{E}[\sup_{t\le s\le T} |X_s|^2] < \infty$.

- $\mathcal{H}^2(t,T;\mathbb{R}^m)$ is the set of $\mathbb{R}^m$-valued, $\mathbb{F}$-predictable processes $(Z_s, s \in [t,T])$ which satisfy $\mathbb{E}[\int_t^T |Z_s|^2 ds] < \infty$.
- $\mathcal{M}^2(t,T;\mathbb{R}^m)$ denotes the set of all $\mathbb{R}^m$-valued, square integrable càdlàg martingales $M = (M_s)_{s\in[t,T]}$ with respect to $\mathbb{F}$, with $M_t = 0$.
- $\mathcal{K}_t^{d,k,p\times m} = \mathcal{S}^2(t,T;\mathbb{R}^d) \times \mathcal{S}^2(t,T;\mathbb{R}^k) \times \mathcal{H}^2(t,T;\mathbb{R}^{p\times m})$

We have to mention that the solution of FBSDEs is not necessary defined on the *Brownian* filtration. In fact we have :

Definition 1.1.2. *A solution of FBSDE (1.5) is a process $(X^{t,x}, Y^{t,x}, Z^{t,x}) \in \mathcal{K}_t^{m,p,p\times m}$ which satisfies equation (1.5).*

Existence and uniqueness of the solution of a FBSDE's :
In the following, we suppose that the diffusion σ is independent of Z. The existence and uniqueness of the solution has been proven in two cases :

Non-degenerate case: it means that the forward diffusion is non-degenerate.
Let consider the following assumptions which are a special case of *Delarue*'s result [23]:
There exist two constants K and $\lambda > 0$, such that the functions b, σ, g and Φ satisfy the following assumptions:

(A1) The functions b, σ, g and Φ are bounded and satisfy for any (x,y,z) and $(x',y',z') \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$:

$$|b(x,y,z) - b(x',y',z')| \leq K(|x-x'| + |y-y'| + |z-z'|);$$

$$|\sigma(x,y) - \sigma(x',y')|^2 \leq K^2(|x-x'|^2 + |y-y'|^2);$$

$$|g(x,y,z) - g(x',y',z')| \leq K(|x-x'| + |y-y'| + |z-z'|);$$

$$|\Phi(x) - \Phi(x')| \leq K|x-x'|.$$

(A2) For every $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$;

$$\forall \zeta \in \mathbb{R}^d, \quad \langle \zeta, \sigma(t,x,y)\zeta \rangle \geq \lambda |\zeta|^2.$$

Therefore, the equation (1.5) has a unique solution $(X^{t,x}, Y^{t,x}, Z^{t,x})$ in the space

$$\mathcal{S}^2(t,T;\mathbb{R}^d) \times \mathcal{S}^2(t,T;\mathbb{R}) \times \mathcal{H}^2(t,T;\mathbb{R}^d).$$

Degenerate case: For a given $1\times d$ matrix G (with G^T be the transpose of G) and $\lambda := (x, y, z)$, we put

$$A(t,\lambda) := \begin{pmatrix} -G^T g \\ Gb \\ G\sigma \end{pmatrix}(t,\lambda),$$

we assume that there exists a $1 \times d$ full rank matrix G such that the following assumptions are satisfied.

(**H1**)

1. $A(t, \lambda)$ is uniformly *Lipschitz* in λ uniformly on t, and for any λ,

$$A(\cdot,\lambda) \in \mathcal{H}^2(0,T;\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d).$$

2. $\Phi(x)$ is uniformly *Lipschitz* and for any $x \in \mathbb{R}^d$, $\Phi(x) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$.

   We denote by K the *Lipschitz* constant of A and Φ.

(**H2**)

1. $\langle A(t,\lambda) - A(t,\widehat{\lambda}), \lambda - \widehat{\lambda}\rangle \leq -\beta_1|G\overline{x}|^2 - \beta_2(|G^T\overline{y}|^2 + |G^T\overline{z}|^2)$.
2. $\langle \Phi(x) - \Phi(\widehat{x}), G(x - \widehat{x})\rangle \geq \mu_1|G\overline{x}|^2$, $\overline{x} = x - \widehat{x}$, $\overline{y} = y - \widehat{y}$, $\overline{z} = z - \widehat{z}$,
   where β_1, β_2, μ_1 are strictly positive constants.

Theorem 1.2. *[84] We suppose that the assumptions (**H1-H2**) hold. Then, there exists a unique adapted solution (X, Y, Z) to the FBSDE (1.5).*

The systems considered in this thesis are driven by the so called *G-Brownian* motion (see section (1.3.4)), which is the canonical process of the "G-expectation" (capital G) in which it plays a crucial role. In contrast, the *G-Brownian* motion do not have any relation with the so called g-expectation (g lowercase) [77]. Therefore, for the clearness of the study and to make difference between our framework "G-expectation" and g-expectation, we set the following section.

1.2 g-Expectation

When the *Allais* paradox was brought to light in 1952, many economists wondered about the possibility to find a new notion that generalize the linear mathematical expectation concept and that conserve as much as possible it properties because the theory of "expected utility" based on the linear expectation is questionable. As response, in 1997, *Peng* [77] suggests a non-linear expectation named g-expectation which is introduced via a non-linear BSDE (1.3) as follow: Let consider the generator $g : \mathbb{R}^k \times \mathbb{R}^{k\times d} \times [0,T] \to \mathbb{R}^k$ as a function satisfying :

H.1 For all $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$, $g(y,z,t)$ is continuous in t and $\int_0^T g^2 dt < \infty$;

H.2 There exists a constant $L > 0$, for all $y_1, y_2 \in \mathbb{R}^k$, $z_1, z_2 \in \mathbb{R}^{k\times d}$

$$|g(y_1,z_1,t) - g(y_2,z_2,t)| \leq L(|y_1 - y_2| + \| z_1 - z_2 \|),$$

i.e. g is uniformly *Lipschitz* continuous in (y,z).

H.3 $g(y,0,t) = 0$, $\forall (y,t) \in \mathbb{R}^k \times [0,T]$.

Definition 1.2.1. *Assume that g satisfies $H.1, H.2$ and $H.3$. Let the couple (Y,Z) be the solution of the BSDE (1.2) and $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, we call the function $\mathbb{E}_g$*

$$\mathbb{E}_g(\xi) = Y_0, \qquad Y_0 \text{ is the initial solution of the BSDE (1.2)}$$

*the "g-**expectation**" of ξ related to g.*

Definition 1.2.2. *For a random variable $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, we call the random variable η satisfying*

1. *η is $\mathcal{F}_t$-measurable;*
2. *$\mathbb{E}_g(\mathbf{1}_A X) = \mathbb{E}_g(\mathbf{1}_A \eta)$, for all $A \in \mathcal{F}_t$*

the conditional g-expectation of a process X under $\mathcal{F}_t$ which coincide with y_t, the value of the solution of BSDE (1.2) at the time t. We denote it by

$$\mathbb{E}_g(X/\mathcal{F}_t) = Y_t.$$

Remark 1.2.1. *The new function $\mathbb{E}_g(.)$ have many properties as the classical mathematical expectation and as the g-expectation and the conditional g-expectation depend on the choice of the function g (if g is nonlinear, then the g-expectation is usually also nonlinear), the two functions $\mathbb{E}_g(.)$ and $\mathbb{E}_g(./\mathcal{F}_t)$ do not preserve the linearity.*

Proposition 1.2.1. *[77] The conditional g-expectation have the following properties :*

- *preserving of constants: If c is a constant then $\mathbb{E}_g(c) = c$;*
- *monotonicity : If $\xi_1 \geq \xi_2$, then $\mathbb{E}_g(\xi_1) \geq \mathbb{E}_g(\xi_2)$;*
- $\mathbb{E}_g\left(\mathbb{E}_g(\xi/\mathcal{F}_t)\right) = \mathbb{E}_g(\xi)$;
- *If ξ is $\mathcal{F}_t-$mesurable then $\mathbb{E}_g(\xi/\mathcal{F}_t) = \xi$;*

- *For the real function defined on* $\mathbb{R}^k \times \mathbb{R}^{k\times d} \times [0,T]$*, if* ξ *is independent of* $\mathcal{F}_t$*, then,* $\mathbb{E}_g(\xi/\mathcal{F}_t) = \mathbb{E}_g(\xi)$

- *If* g *is convex (resp. concave) in* (y,z)*, then for any* $\xi, \eta \in L^2(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathbb{E}_g(\xi+\eta) \le \mathbb{E}_g(\xi) + \mathbb{E}_g(\eta)\,(resp.\mathbb{E}_g(\xi+\eta) \ge \mathbb{E}_g(\xi) + \mathbb{E}_g(\eta)).$$

Since *Peng*'s g-expectation can be defined only on BSDE framework, *Peng* obtains a nonlinear generalization of the *Kolmogorov*'s consistent theorem, then use it to define the filtration-consistent nonlinear expectations via nonlinear *Markov* chains [78]. Since 2006, the **G-expectation** been the preferable notion to deal with probability models under uncertainty problems.

1.3 G-Expectation

This theory is a non-axiomatically generalization of the concept of g-expectation and it is not based on the classical probability space given a priori as it is mentioned in [80] i.e. our space will be equipped with nonlinear(sub-linear) expectation which is, in fact, at an equivalent place as the probability measure itself.

Let start this section by presenting the following spaces notations that I use in whole of my thesis:

1.3.1 Notations

$C_{b.lip}(\mathbb{R}^d)$ is the space of bounded and *Lipschitz* continuous functions on $\mathbb{R}^d$;

$C_{l.lip}(\mathbb{R}^d)$ is the Linear space of functions satisfying locally *Lipschitz* condition on $\mathbb{R}^d$

$L_{ip}(\Omega_T) := \left\{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \ge 1, t_1, \ldots, t_n \in [0,T], \varphi \in C_{b.lip}(\mathbb{R}^{d\times n})\right\}$;

$L^p_G(\Omega_T)$ is the completion of $L_{ip}(\Omega_T)$ under the norm $\parallel \eta \parallel_{p,G} = \left\{\hat{\mathbb{E}}\left(|\eta|^p\right)\right\}^{\frac{1}{p}}$;

$$M^0_G(0,T) := \left\{\eta_t = \sum_{i=0}^{N-1} \xi_i \mathbb{1}_{\{t_i, t_{i+1}\}} : 0 = t_0 < \ldots < t_N = T, \xi \in L_{ip}(\Omega_{t_i})\right\};$$

$$M^{p,0}_G(0,T) := \left\{\eta_t = \sum_{i=0}^{N-1} \xi_i \mathbb{1}_{\{t_i, t_{i+1}\}} : 0 = t_0 < \ldots < t_N = T, \xi_i \in L^p_G(\Omega_{t_i})\right\};$$

$\bar{M}^p_G(0,T)$ is the completion of $M^{p,0}_G(0,T)$ under the norm $\parallel \eta \parallel_{\bar{M}^p_G} = \left\{\int_0^T \hat{\mathbb{E}}\left(|\eta_s|^p ds\right)\right\}^{\frac{1}{p}}$;

$M^p_G(0,T)$ is the completion of $M^0_G(0,T)$ under the norm $\parallel \eta \parallel_{M^p_G} = \left\{\hat{\mathbb{E}}\left(\int_0^T |\eta_s|^p ds\right)\right\}^{\frac{1}{p}}$;

$H_G^p(0,T) :=$ the completion of $M_G^0(0,T)$ under the norm $\| \eta \|_{\mathbb{H}^p} = \left\{ \hat{\mathbb{E}}\left[\left(\int_0^T |\eta_s|^2 ds \right)^{\frac{p}{2}} \right] \right\}^{\frac{1}{p}}$;

$S_G^0(0,T) := \left\{ h(B_{t_1 \wedge t}, \ldots, B_{t_n \wedge t}) : t_1, \ldots, t_n \in [0,T], h \in C_{b.lip}(\mathbb{R}^{n+1}) \right\}$;

$S_G^p(0,T)$ is the completion of $S_G^0(0,T)$ under the norm $\| \eta \|_{S_G^p} = \left\{ \hat{\mathbb{E}}\left(\sup_{s\in[0,T]} |\eta_s|^p \right) \right\}^{\frac{1}{p}}$;

$\mathbb{L}_G^p(\Omega_T)$ is the space of decreasing G-martingales with $K_0 = 0$ and $K_T \in L_G^p(\Omega_T)$;

$\mathfrak{S}_G^p(0,T)$ is the collection of processes (Y, Z, K) such that $Y \in S_G^p(0,T)$, $Z \in H_G^p(0,T)$, K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in \mathbb{L}_G^p(\Omega_T)$;

$H_{G,T}^{q,\alpha} := \bar{M}_G^q([0,T]) \times \mathfrak{S}_G^\alpha(0,T)$.

1.3.2 Sub-linear expectation

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω. We suppose that $\mathcal{H}$ satisfies the following two conditions

1. $c \in \mathcal{H}$ for each constant c;
2. $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$.

In the following, we will use the space $\mathcal{H}$ as the space of random variables.

Definition 1.3.1. *A nonlinear expectation is a functional $\bar{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying:*

i) *Monotonicity: For $X, Y \in \mathcal{H}$ $\quad \bar{\mathbb{E}}(X) \le \bar{\mathbb{E}}(Y) \quad$ if $X \le Y$.*

ii) *Preservation of constants: $\quad \bar{\mathbb{E}}(c) = c \quad$ for $c \in \mathbb{R}$.*

The triplet $(\Omega, \mathcal{H}, \bar{\mathbb{E}})$ is called a ***nonlinear expectation space****.*
Furthermore, if the following properties are satisfied we call $\bar{\mathbb{E}}$ a ***Sub-linear expectation*** *and the triplet $(\Omega, \mathcal{H}, \bar{\mathbb{E}})$ is called a* ***sub-linear expectation space****.*

iii) *Subadditivity (or the self-dominated property): $\forall X, Y \in \mathcal{H}$,*

$$\bar{\mathbb{E}}(X+Y) \le \bar{\mathbb{E}}(X) + \bar{\mathbb{E}}(Y).$$

iv) *Positive homogeneity: $\forall \lambda \ge 0, X \in \mathcal{H}$, $\quad \bar{\mathbb{E}}(\lambda X) = \lambda \bar{\mathbb{E}}(X)$.*

It is clear that the preservation of constants with the Subadditivity implies:

v) Translation by constants:

$$\bar{\mathbb{E}}(X+c) = \bar{\mathbb{E}}(X) + c.$$

Remark 1.3.1.

- *An equivalent form to the condition (iv) is given by*

$$\bar{\mathbb{E}}(\lambda X) = \lambda^{+}\bar{\mathbb{E}}(X) + \lambda^{-}\bar{\mathbb{E}}(-X),$$

where

$$\lambda^{+} = \max\{\lambda, 0\}, \text{ and } \quad \lambda^{-} = \max\{-\lambda, 0\}.$$

- *The sub-linear expectation is called also the upper expectation or the upper prevision.*
- *Systematically, the above notion of sub-linear expectation was introduced by* Artzner, Delbaen, Eber *and* Heath *[3], [4], in the case where* Ω *is finite set. For the general situation with the notation of risk measure, the sub-linear expectation has been introduced by* Delbaen *[24]. For earlier study of this notion, see [52].*
- *In* $2005-2006$*, following Daniell's integration [22], the notion of nonlinear expectation was introduced in [78]* and *[80].*

Now, let move on to define the G-Normal distribution introduced on the sub-linear expectation.

1.3.3 G-Normal distribution

It is well known that the fundamental important distribution in the classical probability space is the Normal distribution. Since *Bachelier* 1900 and *Einstein* 1950, the standard normal distribution of a random variable $X \sim N(0,1)$ under expectation coincides with the solution of a heat equation in $(0,1)$. With the same way, we define the fundamental crucial notion "the G-Normal distribution" in the sub-linear expectation space, but let introduce some important notions related with the distribution of random variables which is less probabilistic but rather functional on sub-linear expectation space.

Definition 1.3.2. *Let* $X = (X_1, \ldots, X_n)$ *be a given n-dimensional random vector on a nonlinear expectation* $(\Omega, \mathcal{H}, \bar{\mathbb{E}})$*. On* $C_{l.lip}(\mathbb{R}^n)$*, we define the functional* $\mathbb{F}_X$ *which is* ***the distribution*** *of* X *under* $\bar{\mathbb{E}}$ *by*

$$\mathbb{F}_X(\phi) := \bar{\mathbb{E}}(\phi(X)) : \phi \in C_{l.lip}(\mathbb{R}^n) \mapsto \mathbb{R}.$$

The triplet $(\mathbb{R}^n, C_{l.lip}(\mathbb{R}^n, \mathbb{F}_X)$ *forms a nonlinear expectation space.*

For the case of the sub-linear expectation $\bar{E}$, the functional $\mathbb{F}_X$ is also a sub-linear expectation and it is very useful notion in this case.

Remark 1.3.2. *(Theorem*1.2.2 *[82])* $\mathbb{F}_X$ *characterizes the uncertainty of the distributions of* X*. In fact, it can be proved that there exists a family of probability measures* $\{F_X(\theta,.)\}_{\theta\in\Theta}$ *defined on* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ *such that*

$$\mathbb{F}_X(\phi) = \sup_{\theta\in\Theta} \int_{\mathbb{R}^n} \phi(x) F_X(\theta, dx), \text{ for each } \phi \in C_{l.lip}.$$

Definition 1.3.3. *Let X_1 and X_2 be two n-dimensional random vectors defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \bar{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \bar{\mathbb{E}}_2)$ respectively. They are called* ***identically distributed*** *if*

$$\bar{\mathbb{E}}_1(\phi(X_1)) = \bar{\mathbb{E}}_2(\phi(X_2)), \quad \text{for all}\, \phi \in C_{b.lip}(\mathbb{R}^n),$$

and we denote $X_1 \overset{d}{=} X_2$. We say that the distribution of X_1 is stronger than that of X_2 if

$$\bar{\mathbb{E}}_1(\phi(X_1)) \geq \bar{\mathbb{E}}_2(\phi(X_2)), \forall \phi \in C_{b.lip}(\mathbb{R}^n).$$

The distribution of $X \in \mathcal{H}$ has the following four typical parameters:

$$\bar{\mu} := \bar{\mathbb{E}}(X); \qquad \bar{l}^2 := \bar{\mathbb{E}}(X^2);$$

$$\underline{\mu} := -\bar{\mathbb{E}}(-X); \qquad \underline{l}^2 := -\bar{\mathbb{E}}(-X^2).$$

The identically distributed property can also be characterized by the following:

Proposition 1.3.1. *[82] Suppose that $X_1 \overset{d}{=} X_2$. Then*

$$\bar{\mathbb{E}}_1(\phi(X_1)) = \bar{\mathbb{E}}_2(\phi(X_2)) \quad \text{for all}\, \phi \in C_{l.lip}(\mathbb{R}^n).$$

Moreover, $X_1 \overset{d}{=} X_2$ if their distributions coincide.
It is well known that the following notion of independence is one of the most important notions which plays a crucial role in the theory of non-linear expectation.

Definition 1.3.4. *In a sub-linear expectation space $(\Omega, \mathcal{H}, \bar{\mathbb{E}})$, a random vector $Y = (Y_1, \dots, Y_n), Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \dots, X_n), X_i \in \mathcal{H}$ under $\bar{\mathbb{E}}(.)$, if for each test function $\phi \in C_{l.lip}(\mathbb{R}^m \times \mathbb{R}^n)$, we have*

$$\bar{\mathbb{E}}[\phi(X, Y)] = \bar{\mathbb{E}}(\bar{\mathbb{E}}(\phi(x, Y))_{x=X}).$$

It means that the uncertainty of distributions of Y does not change after the realization of $X = x$ i.e. $\bar{\mathbb{E}}[\phi(x, X)]_{X=x}$ is the conditional sub-linear expectation of Y with respect to X.

Remark 1.3.3. *Under sub-linear expectations if "X is independent from Y" it does not imply automatically that "Y is independent from X".*

The G-normal distribution is introduced as follow :

Definition 1.3.5. *A random variable $X \in L_{ip}(\Omega_T)$ is G-normal distributed with parameters $(0, [\underline{l}^2, \bar{l}^2])$, i.e. $X \sim N(0, [\underline{l}^2, \bar{l}^2])$ if for each $\phi \in C_{b.Lip}(\mathbb{R}), u(t, x) := \bar{\mathbb{E}}(\phi(x + \sqrt{t}X))$ is a viscosity solution (Section (1.3.7)) to the following nonlinear partial differential equation on $\mathbb{R}^+ \times \mathbb{R}$:*

$$\begin{cases} \partial_t u - G(\partial^2_{xx} u) = 0, \\ u(t_0, x) = \phi(x), \end{cases} \tag{1.6}$$

where G is called the ***generating function*** *of (1.6) is given by the following sub-linear real function $G(a) := \frac{1}{2}(a^+ \bar{l}^2 - a^- \underline{l}^2), a \in \mathbb{R}$. The PDE (1.6) is called* ***G-heat equation***.

Proposition 1.3.2. *[79] Let X be an $N(0,[\underline{l}^2,\bar{l}^2])$-distributed random variable. For each $\phi \in C_{l.Lip}(\mathbb{R})$, we define a function*

$$u(t,x) := \bar{\mathbb{E}}(\phi(x+\sqrt{t}X)), \qquad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Then we have

$$u(t+s,x) = \bar{\mathbb{E}}(u(t,x+\sqrt{s}X)), \qquad s \geq 0,$$

we also have the estimates: For each $T > 0$ there exist constants $c, k > 0$ such that, for all $t, s \in [0,T]$ and $x, y \in \mathbb{R}$,

$$|u(t,x) - u(t,y)| \leq C(1+|x|^k+|y|^k)|x-y|, \tag{1.7}$$

and

$$|u(t,x) - u(t+s,x)| \leq C(1+|x|^k)|s|^{\frac{1}{2}}. \tag{1.8}$$

Moreover, u is the unique viscosity solution (see Section (1.3.7)) continuous in the sense of (1.7) and (1.8) of the generating PDE (1.6).

It is shown that the unique sub-linear distribution defined on $(\mathbb{R}, C_{l.Lip}(\mathbb{R}))$ is the G-normal distribution $N(0,[\underline{l}^2,\bar{l}^2])$(see [79]).
Noting that a random variable with $N(0,[\underline{l}^2,\bar{l}^2])$ has no mean uncertainty. The following corollary shows that the classical normal distribution is a special case of the G-normal distribution.

Corollary 1.3.3.1. *[79] In the case where $\underline{l}^2 = \bar{l}^2 > 0$, $N(0,[\underline{l}^2,\bar{l}^2])$ is just the classical normal distribution $N(0,\bar{l}^2)$.*

The calculation of $\bar{\mathbb{E}}(\phi(X))$ is very easy in two interesting situations: When ϕ is a concave function then $\bar{\mathbb{E}}(\phi(X))$ coincide with

$$P_t^G = \frac{1}{\sqrt{2\pi\underline{l}^2}} \int_{-\infty}^{\infty} \phi(x) \exp(-\frac{x^2}{2\underline{l}^2})dx.$$

In case where ϕ is convex function, we replace $\underline{l}^2$ by $\bar{l}^2$ i.e. the value of $\bar{\mathbb{E}}(\phi(X))$ becomes

$$P_t^G = \frac{1}{\sqrt{2\pi\bar{l}^2}} \int_{-\infty}^{\infty} \phi(x) \exp(-\frac{x^2}{2\bar{l}^2})dx.$$

Multidimensional G-normal distribution

For a given positive integer n, (x,y) denotes the scalar product of $x, y \in \mathbb{R}^n$, and $|x| = (x,x)^2$ denotes the *Euclidian* norm of x. The space of all bounded and *Lipschitz* real functions on $\mathbb{R}^n$ is denoted $lip(\mathbb{R}^n)$.
Let $\mathbb{R}^d$ be considered as Ω and $lip(\mathbb{R}^d)$ as $\mathcal{H}$. The G-normal distribution is a sub-linear expectation (moreover, a nonlinear expectation) defined on $lip(\mathbb{R}^d)$ as follow:

$$P_1^G(\Phi) = u(1,0) : \Phi \in lip(\mathbb{R}^d) \mapsto \mathbb{R},$$

where the $[0,\infty)\times\mathbb{R}^d$-bounded continuous function u is the viscosity solution (see Section (1.3.7)) of the following sub-linear(moreover, nonlinear) partial differential equation:

$$\begin{cases} \partial_t u - G(D^2u) = 0, \\ u(0,x) = \Phi(x), \qquad (t,x)\in[0,\infty)\times\mathbb{R}^d, \end{cases} \tag{1.9}$$

where $D^2u = (\partial^2_{x^ix^j}u)^d_{i,j=1}$ i.e. it is the Hessian matrix of u and

$$G(A) = G_\Gamma(A) = \frac{1}{2}\sup_{\gamma\in\Gamma} tr[\gamma\gamma^T A], \qquad A = (A_{ij})^d_{i,j=1}\in\mathbb{S}_d.$$

$\mathbb{S}_d$ is the space of $d\times d$ symmetric matrices. Γ denotes the given non-empty, bounded and closed subset of $\mathbb{R}^{d\times d}$, the space of all $d\times d$ matrices.

Remark 1.3.4. *If Γ is a singleton $\{\gamma_0\}$, the above G-PDE becomes a standard linear PDE. Thus the corresponding G-distribution is the d-dimensional classical normal distribution with parameters $(0,\gamma_0\gamma_0^T)$.*

The sub-linear PDE (1.9) is a special kind of Hamilton-Jacobi-Bellman *equation.*

the following section, crucial notions in the G-expectation theory will be introduced that are the (one-dimensional, multidimensional) *G-Brownian* motion and its properties as well as the G-expectation $\hat{\mathbb{E}}$.

1.3.4 *G-Brownian* motion, *G*-expectation

Let start with the definition of the **stochastic process** on sub-linear expectation space :

Definition 1.3.6. *Let $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ be a sub-linear expectation space. $(X_t)_{t\geq 0}$ is called a d-dimensional stochastic process if for each $t\geq 0$, X_t is a d-dimensional random vector in $\mathcal{H}$.*

In the following, we are interested to the case where the *G-Brownian* motion is **symmetric** that its properties are very important for the stochastic analysis of the *G-Brownian* motion.

Definition 1.3.7. *A d-dimensional stochastic process $(B_t)_{t\geq 0}$ on a sub-linear expectation space $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ is called G-*Brownian *motion if the following properties are satisfied:*

1. *$B_0(\omega) = 0$;*
2. *For each $t,s\geq 0$, $B_{t+s}-B_t$ and B_s are identically distributed and $B_{t+s}-B_t$ is independent from $(B_{t_1},\dots,B_{t_n})$, for each $n\in\mathbb{N}$ and $0\leq t_t\leq\dots\leq t_n\leq t$;*
3. *$\lim\limits_{t\downarrow 0}\hat{\mathbb{E}}(|B_t|^3)t^{-1} = 0$.*

Moreover, if $\hat{\mathbb{E}}(B_t) = \hat{\mathbb{E}}(-B_t) = 0$*, then* $(B_t)_{t\geq 0}$ *is called a* ***symmetric*** *G*-Brownian *motion.*

A characterization of the symmetric *G-Brownian* motion is given in the following theorem:

Theorem 1.3. *[82] Let* $(B_t)_{t\geq 0}$ *be a given* $\mathbb{R}^d$*-valued symmetric G*-Brownian *motion on a sub-linear expectation space* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$*. Then, for each fixed* $\phi \in C_{b.Lip}(\mathbb{R}^d)$*, the function*

$$u(t,x) := \hat{\mathbb{E}}(\phi(x+B_t)), \qquad (t,x) \in [0,\infty)\times\mathbb{R}^d, \tag{1.10}$$

is the viscosity solution of the following parabolic PDE:

$$\partial_t u - G(D^2u) = 0, \qquad u|_{t=0} = \phi, \tag{1.11}$$

where

$$G(A) = \frac{1}{2}\hat{\mathbb{E}}(\langle AB_1, B_1\rangle), \qquad A \in \mathbb{S}_d. \tag{1.12}$$

In particular, B_1 *is G-normally distributed and* $B_t \overset{d}{=} \sqrt{t}B_1$*.*

It is important to note that if the mean uncertainty and variance uncertainty of the symmetric *G-Brownian* motion vanish i.e. if

$$\hat{\mathbb{E}}(B_1) = -\hat{\mathbb{E}}(-B_1), \qquad \text{and} \qquad \hat{\mathbb{E}}(B_1^2) = -\hat{\mathbb{E}}(-B_1^2),$$

then it becomes the classical *Brownian* motion.
The following proposition is very important in stochastic calculus. B_t^a denotes $\langle a, B_t\rangle$, for each $a = (a_1,\ldots,a_d)^T \in \mathbb{R}^d$.

Proposition 1.3.3. *[82] Let* $(B_t)_{t\geq 0}$ *be a d-dimensional G*-Brownian *motion on a sub-linear expectation space* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$*. Then* $(B_t^a)_{t\geq 0}$ *is a one-dimensional* G_a-Brownian *motion for each* $a \in \mathbb{R}^d$*, where* $G_a(\alpha) = \frac{1}{2}(l^2_{aa^T}\alpha^+ - l^2_{-aa^T}\alpha^-)$*,*

$$l^2_{aa^T} = 2G(aa^T) = \hat{\mathbb{E}}(\langle a, B_1\rangle^2), \qquad l^2_{-aa^T} = 2G(-aa^T) = -\hat{\mathbb{E}}(-\langle a, B_1\rangle^2).$$

In particular, for each $t, s \geq 0$*,* $B^a_{t+s} - B^a_t \overset{d}{=} N(\{0\}\times[sl^2_{-aa^T}, sl^2_{aa^T}])$*.*

Proposition 1.3.4. *[82] For each convex function* $\phi \in C_{l.Lip}(\mathbb{R})$*, we have:*

$$\hat{\mathbb{E}}(\phi(B^a_{t+s} - B^a_t)) = \frac{1}{\sqrt{2\pi s l^2_{aa^T}}}\int_{-\infty}^{\infty}\phi(x)\exp(-\frac{x^2}{2sl^2_{aa^T}})dx.$$

For each concave function $\phi \in C_{l.Lip}(\mathbb{R})$ *and* $l_{-aa^T} > 0$*, we have:*

$$\hat{\mathbb{E}}(\phi(B^a_{t+s} - B^a_t)) = \frac{1}{\sqrt{2\pi s l^2_{-aa^T}}}\int_{-\infty}^{\infty}\phi(x)\exp(-\frac{x^2}{2sl^2_{-aa^T}})dx.$$

In particular, we have the following relations:

$$\hat{\mathbb{E}}((B^a_t - B^a_s)^2) = l^2_{aa^T}(t-s), \qquad \hat{\mathbb{E}}((B^a_t - B^a_s)^4) = 3l^4_{aa^T}(t-s)^2,$$

$$\hat{\mathbb{E}}(-(B^a_t - B^a_s)^2) = -l^2_{-aa^T}(t-s), \hat{\mathbb{E}}(-(B^a_t - B^a_s)^4) = -3l^4_{-aa^T}(t-s)^2.$$

G-Expectation

Let $\Omega_T := \{\omega_{.\wedge T:\omega\in\Omega}\}$ and let $B_t(\omega) = \omega_t, t \in [0,\infty)$ for each $\omega \in \Omega$, where $\Omega = C_0^d(\mathbb{R}^+)$ denotes the space of all $\mathbb{R}^d$-valued continuous path $(\omega_t)_{t\mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^i [(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1].$$

For each $T \in [0,\infty)$, we put

$$L_{ip}(\Omega_T) = \{\phi(B_{t_{1\wedge T}}, \ldots, B_{t_{n\wedge T}}) : n \in \mathbb{N}, t_1, \ldots, t_n \in [0,\infty), \phi \in C_{l.Lip}(\mathbb{R}^{d\times n})\}.$$

It is clear that for $t \le T, L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$. We also have

$$L_{ip}(\Omega) = \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

Let $G(.) : \mathbb{S}_d \mapsto \mathbb{R}$ be a given monotone and sub-linear function.

Definition 1.3.8. *The sub-linear expectation $\hat{\mathbb{E}}(.) : L_{ip}(\Omega) \mapsto \mathbb{R}$ defined through the following procedure is called a G-**expectation**. And the corresponding canonical process $(B_t)_{t\ge 0}$ on the sub-linear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$ is called a G-*Brownian *motion.*

Let $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ be a sub-linear expectation space. On this space, we construct a sequence of d-dimensional random vectors $(\xi_i)_{i=1}^{\infty}$ such that ξ_i is G-normal distributed and ξ_{i+1} is independent from $(\xi_1, \ldots, \xi_i)$ for each $i = 1, 2, \ldots$.
Now, for each $X \in L_{ip}$ such that

$$X = \phi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_t}, \ldots, B_{t_n} - B_{t_{n-1}}),$$

and for some $\phi \in C_{l.Lip}(\mathbb{R}^{d\times n})$ and $0 = t_0 < t_1 < \ldots < t_n < \infty$, we set

$$\hat{\mathbb{E}}(X) := \tilde{\mathbb{E}}(\phi(\sqrt{t_1 - t_0}\xi_1, \ldots, \sqrt{t_n - t_{n-1}}\xi_n)).$$

The related conditional expectation of $X = \phi(B_{t_1}, B_{t_2} - B_{t_t}, \ldots, B_{t_n} - B_{t_{n-1}})$ under Ω_{t_j} is defined by:

$$\begin{aligned} \hat{\mathbb{E}}(X/\Omega_{t_j}) &= \hat{\mathbb{E}}(\phi(B_{t_1}, B_{t_2} - B_{t_t}, \ldots, B_{t_n} - B_{t_{n-1}})/\Omega_{t_j}) \\ &:= \psi(B_{t_1}, B_{t_2} - B_{t_t}, \ldots, B_{t_j} - B_{t_{j-1}}), \end{aligned} \tag{1.13}$$

where

$$\psi(x_1, \ldots, x_j) = \tilde{\mathbb{E}}(\phi(x_1, \ldots, x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, \ldots, \sqrt{t_n - t_{n-1}}\xi_n)).$$

It is easy to check that $\hat{\mathbb{E}}$ consistently defines a sub-linear expectation on $L_{ip}(\Omega)$ and the corresponding canonical process $(B_t)_{t\ge 0}$ is a *G-Brownian* motion. Since $L_{ip}(\Omega_T) \subseteq L_{ip}(\Omega)$, then $\hat{\mathbb{E}}$ is also sub-linear expectation on $L_{ip}(\Omega_T)$.
For each $X, Y \in L_{ip}(\Omega)$, $\hat{\mathbb{E}}(./\Omega_t)$ has the following properties:

Proposition 1.3.5. *[80]*

1. *If $X \geq Y$, then $\hat{\mathbb{E}}(X/\Omega_t) \geq \hat{\mathbb{E}}(Y/\Omega_t)$;*
2. *$\hat{\mathbb{E}}(\eta/\Omega_t) = \eta$, for each $t \in [0, \infty)$ and $\eta \in L_{ip}(\Omega_t)$;*
3. *$\hat{\mathbb{E}}(X/\Omega_t) - \hat{\mathbb{E}}(Y/\Omega_t) \leq \hat{\mathbb{E}}(X - Y/\Omega_t)$;*
4. *$\hat{\mathbb{E}}(\eta X/\Omega_t) = \eta^+\hat{\mathbb{E}}(X/\Omega_t) + \eta^-\hat{\mathbb{E}}(-X/\Omega_t)$ for each $\eta \in L_{ip}(\Omega_t)$.*
5. *$\hat{\mathbb{E}}(\hat{\mathbb{E}}(Y/\Omega_t)/\Omega_s) = \hat{\mathbb{E}}(Y/\Omega_{t\wedge s})$, In particular $\hat{\mathbb{E}}(\hat{\mathbb{E}}(Y/\Omega_t)) = \hat{\mathbb{E}}(Y)$,*

for each $X \in Lip(\Omega^t), \hat{\mathbb{E}}(X/\Omega_t) = \hat{\mathbb{E}}(X)$, where $Lip(\Omega^t)$ is the linear space of random variables with the form

$$\phi(B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_{n+1}} - B_{t_n}), n = 1, 2, \ldots, \phi \in C_{l.Lip}(\mathbb{R}^{d\times d}), t_1, \ldots, t_{n+1} \in [t, \infty).$$

Remark 1.3.5. *Properties* (2) *and* (3) *imply*

$$\hat{\mathbb{E}}(X + \eta/\Omega_t) = \hat{\mathbb{E}}(X/\Omega_t) = \eta + \eta, \qquad \textit{for each} \qquad \eta \in L_{ip}(\Omega_t).$$

Now, let consider the completion of $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.
For $p \geq 0$, we denote by

$$L^p_G(\Omega) := \{\text{the com pletion of the space } L_{ip}(\Omega) \text{ under the norm } \| X \|_p := (\hat{\mathbb{E}}(|X|^p))^{\frac{1}{p}}\}.$$

Similarly, It is clear that for each $0 \leq t \leq T < \infty$,

$$L^p_G(\Omega_t) \subseteq L^p_G(\Omega_T) \subseteq L^p_G(\Omega).$$

Due to Section 1.4 in Chapter 1 in [82], $\hat{\mathbb{E}}$ can be continuously extended to a sub-linear expectation on $(\Omega, L^1_G(\Omega))$ and still denoted by $\hat{\mathbb{E}}$. For each fixed $t \leq T$, the conditional G-expectation $\hat{\mathbb{E}}(./\Omega_t) : L_{ip}(\Omega_T) \mapsto L_{ip}(\Omega_t)$ is a continuous mapping under $\| \,.\, \|$.

Similar to the classical situation, the independence of a random vectors is defined as follow:

Definition 1.3.9. *An n-dimensional random vector $Y \in (L^1_G(\Omega))^n$ is said to be independent from Ω_t for some given t if for each $\phi \in C_{b.Lip}(\mathbb{R}^n)$, we have*

$$\hat{\mathbb{E}}(\phi(Y)/\Omega_t) = \hat{\mathbb{E}}(\phi(Y)).$$

Remark 1.3.6. *Just as in the classical situation, the increments of G-*Brownian *motion $(B_{t+s} - B_t)_{s\geq 0}$ are independent from Ω_t, for each $t \geq 0$. For example: For each $0 \leq s \leq t$ and for each fixed $a \in \mathbb{R}^d$, we have*

$$\hat{\mathbb{E}}(B^a_s - B^a_t/\Omega_t) = 0, \qquad \hat{\mathbb{E}}(-(B^a_s - B^a_t)/\Omega_t) = 0,$$

$$\hat{\mathbb{E}}((B^a_s - B^a_t)^2/\Omega_t) = l^2_{aa^T}(t - s), \ \hat{\mathbb{E}}((B^a_s - B^a_t)^2/\Omega_t) = -l^2_{-aa^T}(t - s),$$

where $l^2_{aa^T} = 2G(aa^T)$ and $l^2_{-aa^T} = -2G(-aa^T)$.

The following proposition is a very useful property:

Proposition 1.3.6. *[82] Let $X, Y \in L^1_G(\Omega)$ be such that $\hat{\mathbb{E}}(Y/\Omega_t) = -\hat{\mathbb{E}}(-Y/\Omega_t)$, for some $t \in [0, T]$. Then we have*

$$\hat{\mathbb{E}}(X + Y/\Omega_t) = \hat{\mathbb{E}}(X/\Omega_t) + \hat{\mathbb{E}}(Y/\Omega_t).$$

In particular, if $\hat{\mathbb{E}}(Y/\Omega_t) = \hat{\mathbb{E}}(-Y/\Omega_t) = 0$, then $\hat{\mathbb{E}}(X + Y/\Omega_t) = \hat{\mathbb{E}}(X/\Omega_t)$.

Recall that the almost surely property plays a crucial tool in stochastic calculus theory. In the G-stochastic analysis and because of the particularity of the space, we use the quasi-surely property in its place.

Theorem 1.4. *[25] Let $\mathcal{M}(\Omega_T)$ be a set of probability measures on $(\Omega_T, \mathcal{F}_T)$ Then there exists a weakly compact subset $\mathcal{P} \subset \mathcal{M}(\Omega_T)$, such that*

$$\hat{\mathbb{E}}(\xi) = \sup_{P \in \mathcal{P}} \mathbb{E}_P(\xi)\,, \forall \xi \in L^1_G(\Omega_T) \tag{1.14}$$

$\mathcal{P}$ is the set that represents $\hat{\mathbb{E}}$

The G-associated capacity is defined as follow:

$$c(A) = \sup_{P \in \mathcal{P}} \mathbb{P}(A), \quad A \in \mathcal{F}_T.$$

Definition 1.3.10. *If $c(A) = 0$, the set A is named polar set i.e. $\mathbb{P}(A) = 0$ for any $P \in \mathcal{P}$. And a property holds "quasi-surely" (q.s.) if it holds outside a polar set.*

Let take an overview on the quadratic variation that was established by *Peng* [78].

Quadratic variation

The *G-Brownian* motion is characterized by its **quadratic variation** process $\langle B \rangle_t$ which is not deterministic except if $\underline{l} = \bar{l}$, i.e. when $(B_t)_{t \geq 0}$ coincides with the classical *Brownian* motion. This interesting process is defined as follow:

$$\langle B \rangle_t = \lim_{\mu(\pi_t^N) \to 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2 = (B_t)^2 - 2 \int_0^t B_s dB_s,$$

where $\pi_t^N = \{t_0, \ldots, t_N\}$ such that $0 = t_0 \leq \ldots, T_N$ is a partition of $[0, T]$ and $\mu(\pi_t^N) := \max_{0 \leq i \leq N} |t_i^N - t_{i-1}^N|$, i.e., $\mu(\pi_t^N)$ is a partition of $[0, T]$ with $\lim_{N \to \infty} \mu(\pi_t^N) = 0$. By construction, $\langle B \rangle_t$ is an increasing process with $\langle B \rangle_0 = 0$.

Lemma 1.3.1. *[82] For each fixed $s, t \geq 0, \langle B \rangle_{s+t} - \langle B \rangle_s$ is identically distributed with $\langle B \rangle_t$ and is independent from Ω_s for any $s \geq 0$.*

Let $a=(a_1,\dots,a_d)^T$ and $\bar{a}=(\bar{a}_1,\dots,\bar{a}_d)^T$ be a given vectors in $\mathbb{R}^d$. We have the quadratic variation processes for the *G-Brownian* motion $\langle B^a\rangle$ and $\langle B^{\bar{a}}\rangle$. Then we can define the mutual variation processes as follow

$$\langle B^a, B^{\bar{a}}\rangle_t := \frac{1}{4}\left[\langle B^a + B^{\bar{a}}\rangle_t - \langle B^a - B^{\bar{a}}\rangle_t\right] = \frac{1}{4}\left[\langle B^{a+\bar{a}}\rangle_t - \langle B^{a-\bar{a}}\rangle_t\right]$$

In particular we have

$$\langle B^a, B^a\rangle_t = \langle B^a\rangle_t$$

The *Itô* integrals with respect to the *G-Brownian*, the quadratic variation of the *G-Brownian* and the mutual variation for the *G-Brownian* are well defined see [82].

G-Martingale

Definition 1.3.11. *For each $t\in[0,\infty)$ and for each $s\in[0,t]$, we called a process $M_t\in Lip^0(\Omega), t\geq 0$: G-martingale if*

$$\hat{\mathbb{E}}(M_t/\mathcal{H}_s) = M_s;$$

G-super-martingale(respectively, G-sub-martingale) if

$$\hat{\mathbb{E}}(M_t/\mathcal{H}_s) \leq M_s (respectively, \hat{\mathbb{E}}(M_t/\mathcal{H}_s) \geq M_s);$$

The *G-Brownian* motions $(B_t)_{t\geq 0}$ and $(-B_t)_{t\geq 0}$ are *G*-martingales.

Remark 1.3.7. *If M is a G-martingale, then $-M$ is not G-martingale in general.*

BDG's inequalities

The following BDG's type inequalities (Theorems 2.1 and 2.2 in [40]) are very useful tool in the structure of the proofs in this thesis:

Lemma 1.3.2. *[40] Let $p\geq 1, \eta\in M_G^p([0,T])$ and $0\leq s\leq t\leq T$. Then*

$$\hat{\mathbb{E}}\left(\sup_{s\leq u\leq t}\left|\int_s^u \eta_r d\langle B\rangle_r\right|^p\right) \leq \left(\frac{\underline{l}+\bar{l}}{4}\right)^p (t-s)^{p-1}\hat{\mathbb{E}}\left(\int_s^t |\eta_u|^p du\right). \tag{1.15}$$

Lemma 1.3.3. *[40] Let $p\geq 2, \eta\in M_G^p([0,T])$ and $0\leq s\leq t\leq T$. Then,*

$$\hat{\mathbb{E}}\left(\sup_{s\leq u\leq t}\left|\int_s^u \eta_r dB\right|^p\right) \leq C_p\bar{l}^{\frac{p}{2}}|t-s|^{\frac{p}{2}-1}\hat{\mathbb{E}}\left(\int_s^t |\eta_u|^p du\right), \tag{1.16}$$

where $0<C_p<\infty$ is a constant.

1.3.5 G-Stochastic differential equations

A G-Stochastic differential equation has the following form:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t h_{ij}(s, X_s)d\langle B\rangle_s^{ij} + \int_0^t \sigma_j(s, X_s)dB_s^j, \quad t \in [0, T], \tag{1.17}$$

where the initial condition $X_0 \in \mathbb{R}^n$ is a given constant, B and $\langle B\rangle$ are the *G-Brownian* motion and its quadratic variation respectively and the functions $b, h_{ij}, \sigma : \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfying the following assumption:

(H) There exist some constant k such that $|\phi(t, x) - \phi(t, y)| \leq k|x - y|$, for each $t \in [0, T]$, $x, y \in \mathbb{R}^n$, with $\phi = b, h_{ij}, \sigma_j$.

Theorem 1.5. *[81] There exists a unique solution $(X_t)_{0\leq t\leq T} \in \bar{M}_G^2([0, T], \mathbb{R}^n)$ to the stochastic differential equation (1.17).*

The existence and uniqueness of the solution of the G-SDE (1.17) under *Lipschitz* condition was first established in 2007 by *Peng* [79, 80, 81] by using the contracting mapping theorem. Then, in 2009 by *Gao* [40] also by supposing the *Lipschitz* condition. Then again, under integral-*Lipschitz* condition, *Bai* and *Lin* [85] studied its solvability. Since then, many developments generate this kind of G-SDE, for more details we refer to [87, 69, 64, 65, 88]. Also *Faizullah* has studied this kind of G-SDE under different conditions on coefficients [35, 34] and [33] in which the G-SDE (1.17) is studied under discontinuous coefficients. Furthermore, under measurable coefficients, it results:

Theorem 1.6. *[33] Suppose that*

1. *The functions $b(t, x)$ and h are measurable with $\int_0^t \bar{\mathbb{E}}(|\Phi(s, .)|^2)ds < \infty$ for $\Phi = b$ and h respectively, where σ is* Lipschitz *continuous in x.*

2. *The respective upper and lower solutions U_t and L_t of the G-SDE (1.17)*

$$\bar{\mathbb{E}}(|U_t|^2) < \infty, \bar{\mathbb{E}}(|L_t|^2) < \infty$$

   satisfy $L_t \leq U_t$ for all $t \in [0, T]$.

3. *Also, $X_0 \in \mathbb{R}^n$ is a given initial value with $\bar{\mathbb{E}}(|X_0|^2) < \infty$ and $L_0 \leq X_0 \leq U_0$.*

Then, there exists a unique solution X_t for the G-SDE (1.17) such that $L_t \leq X_t \leq U_t$, for all $t \in [0, T]$ q.s.

1.3.6 G-Backward stochastic differential equation

Thanks to the complete representation theorem of G-martingale obtained by *Peng*, *Song* and *Zhang* [83], a naturel formulation of a Backward stochastic differential equation driven by *G-Brownian* motion has been founded by *Hu* et al. [46]. A BSDE driven by *G-Brownian*(G-BSDE) is defined as follow:

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s, M_s)ds + \int_t^T g(s, X_s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - \int_t^T dM_s, \quad (1.18)$$

where the solution of the above equation consists of a triplet (X, Y, Z). And X, Y, Z are a square integrable adapted processes and M is a decreasing G-martingale.
The following theorem shows the existence and uniqueness of the solution for (1.18):

Theorem 1.7. *[46] Assume that $\xi \in L_G^\beta(\Omega_T)$ and f, g satisfy:*

(C1) *There exists some $\beta > 1$ such that$f(., \omega, Y, Z), g$ for any $Y \in S_G^p(0,T), Z \in H_G^p(0,T)$, $f, g \in M_G^\beta(0,T)$*

(C2) $|\phi(., \omega, Y, Z) - \phi(., \omega, Y', Z')| \leq L(|Y - Y'| + |Z - Z'|)$ *for some* $L > 0$,

then, the equation (1.18) has a unique solution (X, Y, Z).

Remark 1.3.8. *The result obtained in Theorem (1.7) still hold for the case $d > 1$.*

Note that, many development and research have been done in this frame work among them the study of the wellposedness of multi-dimensional backward stochastic differential equations driven by *G-Brownian* motion [68] and the existence and uniqueness of solutions to a class of non-*Lipschitz* scalar-valued G-BSDE [95] and also G-BSDE with discontinuous drift coefficients [36] as well as under quadratic assumptions on coefficients (see [50]).

For a complete theory of G-BSDE, comparison theorem, *Feynman-Kac* formula and *Girsanov* transformation for the G-BSDE see [47]. In the next section, we are interested by *Feynman-Kac* formula because of its important role in linking the BSDE's theory and the PDE's theory. First, let define the viscosity solution.

1.3.7 Viscosity solution

Let consider the following parabolic PDE:

$$\begin{cases} \partial_t u + G(t, x, u, Du, D^2 u) = 0, & \text{on } (0,T) \times \mathbb{R}^d, \\ V(0,x) = \Phi(x), \ x \in \mathbb{R}^d, \end{cases} \quad (1.19)$$

where $G : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \mapsto \mathbb{R}, \quad \Phi \in C(\mathbb{R}^d)$. Assuming that the function G is continuous and satisfies the degenerate elliptic condition i.e.,

$$G(t, x, r, p, X) \geq G(t, x, r, p, Y) \qquad \text{when} \qquad X \geq Y.$$

Now, let $T > 0$ be fixed and let $\mathcal{I} \subset [0,T] \times \mathbb{R}^d$. We set

$$\text{LSC}(\mathcal{I})=\{\text{lower semi-continuous functions } u:\mathcal{I}\mapsto\mathbb{R}\}$$

$$\text{USC}(\mathcal{I})=\{\text{upper semi-continuous functions } u:\mathcal{I}\mapsto\mathbb{R}\}.$$

The viscosity solution is one of the important results in the theory of partial differential equation. It was introduced for the first-order *Hamilton-Jacobi-Belman* equation by *Crandall* and *Lions* [66] while the argument for the second-order HJB was founded by *Lions* [67]. Now, we give the definition of the viscosity solution

Definition 1.3.12. **(i)** *A viscosity sub-solution of (1.19), or G-sub-solution, on* $(0,T)\times\mathbb{R}^d$ *is a function* $u\in USC((0,T)\times\mathbb{R}^d)$ *such that for all* $(t,x)\in(0,T)\times\mathbb{R}^d, \Phi\in C^2((0,T)\times\mathbb{R}^d)$ *such that* $u(t,x)=\Phi(t,x)$ *and* $u<\Phi$ *on* $(0,T)\in\mathbb{R}^d\setminus(t,x)$, *we have*

$$\partial_t\Phi(t,x)-G(t,x,\Phi(t,x),D\Phi(t,x),D^2\Phi(t,x))\leq 0.$$

(ii) *A viscosity super-solution of (1.19), or G-super-solution, on* $(0,T)\times\mathbb{R}^d$ *is a function* $u\in LSC((0,T)\times\mathbb{R}^d)$ *such that for all* $(t,x)\in(0,T)\times\mathbb{R}^d, \Phi\in C^2((0,T)\times\mathbb{R}^d)$ *such that* $u(t,x)=\Phi(t,x)$ *and* $u>\Phi$ *on* $(0,T)\in\mathbb{R}^d\setminus(t,x)$, *we have*

$$\partial_t\Phi(t,x)-G(t,x,\Phi(t,x),D\Phi(t,x),D^2\Phi(t,x))\geq 0.$$

A viscosity solution of (1.19) on $(0,T)\times\mathbb{R}^d$ *is a function which is simultaneously a viscosity sub-solution and viscosity super-solution of (1.19) on* $(0,T)\times\mathbb{R}^d$.

1.3.8 *Feynman-Kac* formula

Let define the *Feynman-Kac* formula

Theorem 1.8. *[47] Let* $u(t,x):=Y_t^{t,x}$ *for* $(t,x)\in[0,T]\times\mathbb{R}^n$. *Then* $u(t,x)$ *is the unique viscosity solution of the following PDE:*

$$\begin{cases}\partial_t u(t,x)+H(D^2_{xx}u,D_xu,u,x,t)=0\\ u(T,x)=\Phi(x),\end{cases}\tag{1.20}$$

where,

$$\begin{aligned}H(D^2_{xx}u,D_xu,u,x,t)&=G(F(D^2_{xx}u,D_xu,u,x,t))+\langle b(t,x,u),D_xu\rangle;\\ &\quad+f(t,x,u,\langle\sigma_1(t,x),D_xu\rangle,\dots\langle\sigma_d(t,x),D_xu\rangle),\end{aligned}\tag{1.21}$$

with

$$\begin{aligned}F_{ij}(D^2_{xx}u,D_xu,u,x,t)&=\langle D^2_{xx}u\sigma_i(t,x),\sigma_j(t,x)\rangle+2\langle D_xu,h_{ij}(t,x)\rangle\\ &\quad+2g_{ij}(t,x,\langle\sigma_1(t,x),D_xV\rangle\dots\langle\sigma_d(t,x),D_xV\rangle).\end{aligned}\tag{1.22}$$

Chapter 2

G-Forward-Backward Stochastic Differential Equations

This chapter is devised into two sections; first, we briefly recall the result of *Wang* et al. [94] in which only the existence of the solution has been proved under monotone coefficients condition. Then in the second section, we prove that there exists a unique solution to the coupled G-FBSDE system by constructing a mapping for which the fixed point is the solution of the G-FBSDE, without requiring the monotonicity condition to prove the existence. It is important to mention that there are no results in this context except the following results.

2.1 Existence of solution

Let consider the following system

$$\begin{cases} dX_s = b(s, X_s, Y_s)ds + \sigma(s, X_s)dB_s + h(s, X_s, Y_s)d\langle B\rangle_s, \\ dY_s = -f(s, X_s, Y_s, Z_s)ds - g(s, X_s, Y_s, Z_s)d\langle B\rangle_s + Z_s dB_s + dM_s, \\ X_t = x \quad Y_T = \xi = \Phi(X_T)\,, \quad M_t = 0, \, , s \in [t, T], \end{cases} \tag{2.1}$$

where $\langle B\rangle$ denotes the quadratic variation of the 1-dimensional G-*Brownian* motion $B = (B_s)_{s\geq 0}$, (X, Y, Z, M) denotes the solution of our G-FBSDE where M is a decreasing G-martingale. $X \in M^2_G([0,T])$and $(Y, Z, M) \in \mathfrak{S}^2_G(0, T)$.

Let consider the following assumption: For all $t \in [0, T], \beta > 2, x, x', y, y', z, z' \in \mathbb{R}$

A.1 $\xi \in L^\beta_G(\Omega_T), f(., x, y, z, u), g(., x, y, z) \in \bar{M}^\beta_G(0, T),\ b(., x, y, u), h(., x, y) \in \bar{M}^2_G(0, T)$ and $\sigma(., x) \in \bar{M}^\beta_G(0, T)$.

A.2 $b(.,x,y,u), h(.,x,y,u)$ are increasing in y and $f(.,x,y,z,u), g(.,x,y,z,u)$ are increasing in x.

A.3 There exists a constant $k > 0$ such that

$$|\sigma(s,x) - \sigma(s,x')| \leq k|x - x'|,$$

$$|b(s,x,y,u) - b(s,x',y',u)| \vee |h(s,x,y,u) - h(s,x',y',u)| \leq k(|x - x'| + |y - y'|),$$

$$|f(s,x,y,z,u) - f(s,x',y',z',u)| \vee |g(s,x,y,z) - g(s,x',y',z')| \leq k(|x-x'| + |y-y'| + |z-z'|).$$

A.4 There exists a constant $L > 0$ such that

$$|\sigma(s,x)| \leq L(1 + |x|),$$

$$|b(s,x,y,u)| \vee |h(s,x,y,u)| \leq L(1 + |x| + |y|),$$

$$|f(s,x,y,z,u)| \vee |g(s,x,y,z)| \leq L(1 + |x| + |y| + |z|).$$

The following theorem shows the existence of solution for G-FBSDE (2.1)

Theorem 2.1. *[94] If (A1)-(A4) hold, then there exists a solution (X,Y,Z,M) of equation (2.1) Besides, this solution is the minimal one, in the sense that if $(\tilde{X},\tilde{Y},\tilde{Z},\tilde{M})$ is another solution of (2.1), then for any $t \in [0,T]$ we have*

$$X_t \leq \tilde{X}_t, \qquad Y_t \leq \tilde{Y}_t, \qquad q.s.$$

Now, we sketch the proof of this theorem just to have an idea about the technique used to argued Theorem (2.1).

Sketch of the proof:

The aim is to show that the solution of the equation (2.1) coincide with the limit of the sequence $\{(X^n, Y^n, Z^n, M^n)\}_{n\in\mathbb{N}}$ which is the solution of the following constructed iteration:

$$\begin{cases} X_t^n = x + \int_0^t b(s, X_s^n, Y_s^n)ds + \int_0^t \sigma(s, X_s^n)dB_s + \int_0^t h(s, X_s^n, Y_s^n)d\langle B\rangle_s, \\ Y_t^n = \xi + \int_t^T f(s, X_s^{n-1}, Y_s^n, Z_s^n)ds + \int_t^T g(s, X_s^{n-1}, Y_s^n, Z_s^n)d\langle B\rangle_s - \int_t^T Z_s^n dB_s - (M_T^n - M_t^n). \end{cases} \tag{2.2}$$

In fact, via theorem 31 in [25] and by Taking the *Lebesgue* dominated convergence on integral with respect to time variable t, then by taking the limit on the forward equation in (2.2), thus $X \in M_G^2(0,T), Y \in M_G^2(0,T)$ is a solution of the following G-FSDE:

$$X_s = x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s)dB_s + \int_0^t h(s, X_s, Y_s)d\langle B\rangle_s, \qquad q.s.. \tag{2.3}$$

On the other hand, for any $n \geq 1$, we have the BSDE part

$$Y_t^n = \xi + \int_t^T f(s, X_s^n, Y_s^n, Z_s^n)ds + \int_t^T g(s, X_s^n, Y_s^n, Z_s^n)d\langle B\rangle_s - \int_t^T Z_s^n dB_s - (M_T^n - M_t^n) \quad (2.4)$$

We apply *Itô*'s formula on $|Y^n|^2$, we got

$$\begin{aligned}|Y^n|^2 + \int_t^T |Z_s^n|^2 d\langle B\rangle_s = |\xi|^2 - \int_t^T 2y_t^n z_s^n dB_s + \int_t^T 2y_t^n f(s, X_s^n, Y_s^n, Z_s^n)ds \\ + \int_t^T 2y_t^n g(s, X_s^n, Y_s^n, Z_s^n)d\langle B\rangle_s \\ - \int_t^T 2y_t^n dM_t^n\end{aligned} \quad (2.5)$$

For $t = 0$, via Lemma 1.3.2, proposition 2.6 in [46], assumption (A4) and some calculation, it results

$$\hat{\mathbb{E}}\left[\left(\int_0^T |Z_t^n|^2 ds\right)^{\frac{\alpha}{2}}\right] \leq C\left\{\hat{\mathbb{E}}\left(\sup_{0\leq t\leq T}|Y_t^n|^\alpha\right) + \left(\hat{\mathbb{E}}\left(\sup_{0\leq t\leq T}|Y_t^n|^\alpha\right)\right)^{\frac{1}{2}} \hat{\mathbb{E}}\left(|M_t^n|^\alpha\right)^{\frac{1}{2}}\right\}, \quad (2.6)$$

where C is a positive constant which may change from line to line. Furthermore, we have

$$M_t^n = -Y_0^n + \xi + \int_t^T f(s, X_s^n, Y_s^n, Z_s^n)ds + \int_t^T g(s, X_s^n, Y_s^n, Z_s^n)d\langle B\rangle_s - \int_t^T Z_s^n dB_s, \quad (2.7)$$

following simple calculation, it results

$$\hat{\mathbb{E}}\left(|M_t^n|^\alpha\right)^{\frac{1}{2}} \leq C\left\{\hat{\mathbb{E}}\left(\sup_{0\leq t\leq T}|Y_t^n|^\alpha\right) + \hat{\mathbb{E}}\left[\left(\int_0^T |Z_t^n|^2 ds\right)^{\frac{\alpha}{2}}\right]\right\}. \quad (2.8)$$

By (2.6), (2.8) and since

$$\left\{\hat{\mathbb{E}}\left(\sup_{0\leq t\leq T}|Y_t^n|^\alpha\right)\right\} \leq C, \quad (2.9)$$

it result that there exists a positive real constant C independent of n, so that

$$\hat{\mathbb{E}}\left(|M_t^n|^\alpha\right)^{\frac{1}{2}} + \hat{\mathbb{E}}\left[\left(\int_0^T |Z_t^n|^2 ds\right)^{\frac{\alpha}{2}}\right] \leq C. \quad (2.10)$$

Moreover, by applying *Itô*'s formula to $|Y^m - Y^n|^2$ and via Lemma 3.4 in [46], lemma 2.5 in [94], Lemma (1.3.2), *Hölder* inequality and assumption $(A4)$, the sequence $(Y_t^n)_{n\in\mathbb{N}}$ is a *Cauchy* sequence in $\mathfrak{S}_G^2(0,T)$. Consequently, the sequence $(Y_t^n)_{n\in\mathbb{N}}$ is a *Cauchy* sequence in $M_G^2([0,T])$. As the sequences $(X_t^n)_{n\in\mathbb{N}}, (Y_t^n)_{n\in\mathbb{N}}, (Z_t^n)_{n\in\mathbb{N}}$ are a *Cauchy* sequence in $M_G^2([0,T]), \mathfrak{S}_G^2(0,T), M_G^2([0,T])$ respectively, then via simple calculation we get that the sequence $(M_t^n)_{n\in\mathbb{N}}$ is also a *Cauchy* sequence in $\mathbb{L}_G^2(\Omega_T)$. Then, we have

$$X = \lim_{n\to\infty} X_t^n, \qquad Y = \lim_{n\to\infty} Y_t^n, \qquad Z = \lim_{n\to\infty} Z_t^n, \qquad \textit{and } M = \lim_{n\to\infty} M_t^n$$

To finish the proof of the existence taking the limit on both side of (2.4), thus (X, Y, Z, M) satisfies

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds + \int_t^T g(s, X_s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - (M_T - M_t). \quad (2.11)$$

It is clear that (X, Y) satisfies (2.3). So, (X, Y, Z, M) is a solution of (2.1).

At the end, by using backward comparison theorem, forward comparison theorem with simple calculation the constructed solution is the minimal one. Thus, there exists $(X_t^n)_t \in [0, T]$ and $(Y_t^n)_{t\in[0,T]}$ such that $X_t := \lim_{n\to\infty} X_t^n$ and $Y_t = \lim_{n\to\infty} Y_t^n$ for each $t \in [0, T]$ $q.s.$.

For a complete proof; see [94].

2.2 Existence and uniqueness of solution to G-forward backward stochastic differential equations

In this section we present our result on the solution of forward backward stochastic differential equations in the G-framework [56] in which we prove the existence and the uniqueness of the solution based on the fixed point method. Let at first present the problem.

2.2.1 Problem and hypothesis

Consider the following coupled G-forward backward stochastic differential equation:

$$\begin{cases} dX_s = b(s, X_s, Y_s, Z_s)ds + \sigma(s, X_s, Y_s)dB_s + h_{ij}(s, X_s, Y_s)d\langle B^i, B^j\rangle_s, \\ dY_s = -f(s, X_s, Y_s, Z_s, M_s)ds - g_{ij}(s, X_s, Y_s)d\langle B^i, B^j\rangle_s + Z_s dB_s + dM_s, \ s \in [t, T], \\ X_t = x, \ Y_T = \Phi(X_T), M_t = 0, \end{cases} \quad (2.12)$$

where X, Y, Z are square integrable adapted processes and M is a decreasing G-martingale, and the initial value $x \in \mathbb{R}^d$ is a given vector, B is a l-dimensional G-*Brownian* motion, $\langle B\rangle$ is the quadratic variation of the process $(B_s)_{s\geq 0}$. We set:

$$f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^n \to \mathbb{R}^n; \sigma : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^{d\times l};$$

$$b : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \to \mathbb{R}^d, \qquad h_{ij} : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d,$$

$$g_{ij} : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n, \qquad \Phi : \mathbb{R}^d \to \mathbb{R}^n.$$

We aim to prove the existence and uniqueness of the solution (X, Y, Z, M) to our G-FBSDE in one dimension. To reach out our goal, let suppose the following :

Hypothesis(H)

(H1) For fixed $x \in \mathbb{R}^d, y \in \mathbb{R}^n, z \in \mathbb{R}^{n\times l}$ suppose that

$$b(., x, y, z), \sigma(., x, y), h_{ij}(., x, y) \in M_G^2([0,T]),$$

also for fixed x, y, z, m, we have

$$f(., x, y, z, m), g_{ij}(., x, y) \in M_G^2([0,T]), \Phi \in L_G^p(\Omega_T)$$

(H2) We suppose also that:

$$|f(s,x,y,z,m) - f(s,x',y',z',m')|^2 \leq k(|x-x'|^2 + |y-y'|^2 + |z-z'|^2 + |m-m'|^2),$$

$$|b(s,x,y,z) - b(s,x',y',z')|^2 \leq k(|x-x'|^2 + |y-y'|^2 + |z-z'|^2),$$

$$|g_{i,j}(s,x,y) - g_{i,j}(s,x',y')|^2 \leq k(|x-x'|^2 + |y-y'|^2),$$

$$|h_{i,j}(s,x,y) - h_{i,j}(s,x',y')|^2 \leq k(|x-x'|^2 + |y-y'|^2),$$

$$|\sigma(s,x,y) - \sigma(s,x',y')|^2 \leq k_1|x-x'|^2 + k_2|y-y'|^2,$$

$$|\Phi(x) - \Phi(x')|^2 \leq k|x-x'|^2.$$

Remark 2.2.1. *For a comparison reasons,* Wang *and* Yuan *[94] used the monotonicity condition to ensure the existence of the solution, unlike them, we are not in need to this condition to establish the existence and uniqueness of the solution. In addition to that here we can establish a solution for the forward equation X even in higher dimension, not necessarily one-dimension like the case of [94].*

2.2.2 Important inequalities

In addition to the BDG's inequalities shown in the first chapter we are also in need to the following BDG's inequalities type:

Proposition 2.2.1. *[82]Let $\beta \in M_G^p(0,T)$ with $p \geq 2$. Then we have $\int_0^T \beta_t dB_t \in L_G^p(\Omega_T)$ and*

$$\hat{\mathbb{E}}\left(\left|\int_0^T \beta_t dB_t\right|^p\right) \leq C_p \hat{\mathbb{E}}\left(\left|\int_0^T \beta_t^2 d\langle B\rangle_t\right|^{\frac{p}{2}}\right). \tag{2.13}$$

Proposition 2.2.2. *[46]For each $\eta \in H_G^{\alpha}(0,T)$ with $\alpha \geq 1$ and $p \in (0,\alpha]$, we have*

$$\underline{l}^p c_p \hat{\mathbb{E}}\left([\int_0^T \eta_s^2 ds]^{\frac{p}{2}} \right) \leq \hat{\mathbb{E}}\left(\sup_{t\in[0,T]} |\int_0^t \eta_s dB_s|^p \right) \leq \bar{l}^p C_p \hat{\mathbb{E}}\left([\int_0^T \eta_s^2 ds]^{\frac{p}{2}} \right), \tag{2.14}$$

where, $0 < c_p < C_p < \infty$ are constants.

Lemma 2.2.1. *[82] For $\theta \in S_G^2$, we have:*

$$\hat{\mathbb{E}}(\int_0^T |\theta_s|^2 d\langle B\rangle_s) \leq T\bar{l}^2 \hat{\mathbb{E}}(\sup_{s\in[0,T]} |\theta_s|^2).$$

Furthermore, for $\eta \in H_G^2(0,T)$, we have that $(\int_0^t \eta_s\theta_s dB_s)_{t\in[0,T]}$ is an uniformly integrable martingale, equal to 0 at time $t = 0$, so,

$$\hat{\mathbb{E}}(\int_t^T \eta_s\theta_s dB) = 0.$$

The following inequalities are very useful tools:

Lemma 2.2.2. *[82] For $r > 0$ and $1 < q, p < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$|a+b|^r \leq \max\{1, 2^{r-1}\}(|a|^r + |b|^r) \quad \textit{for} \quad a, b \in \mathbb{R} \tag{2.15}$$

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}. \tag{2.16}$$

Proposition 2.2.3. *[80] For each $X, Y \in \mathcal{H}$, we have*

$$\mathbb{E}\left(|X+Y|^r\right) \leq 2^{r-1}\left(\mathbb{E}(|X|^r) + \mathbb{E}(|Y|^r)\right) \tag{2.17}$$

$$\mathbb{E}(XY) \leq \left(\mathbb{E}(|X|^p)^{\frac{1}{p}} + \mathbb{E}(|Y|^q)^{\frac{1}{q}}\right) \tag{2.18}$$

$$\left(\mathbb{E}(|X+Y|^p)\right)^{\frac{1}{p}} \leq \left(\mathbb{E}(|X|^p)\right)^{\frac{1}{p}} + \left(\mathbb{E}(|Y|^p)\right)^{\frac{1}{p}}, \tag{2.19}$$

where $r \geq 1$ and $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, for $1 \leq p < p'$, we have

$$\left(\mathbb{E}(|X|^p)\right)^{\frac{1}{p}} \leq \left(\mathbb{E}(|X|^{p'})\right)^{\frac{1}{p'}}.$$

2.2.3 Main result

The following theorem shows that a G-FBSDE system has a unique solution.

Theorem 2.2. *Under assumption (H), there exists a constant $C_k > 0$ dependant on the* Lipschitz *coefficients k, k_1, k_2, such that for all $0 < T \leq C_k$, the G-FBSDE (2.12) has a unique solution $(X, Y, Z, K) \in H_{G,T}^{2,2}$.*

To go forward in the proof of the Theorem (2.2) we are in need to the following lemmas.

2.2.3.1 Important lemmas

The following lemmas hold for β large enough.

Lemma 2.2.3. *For a given $\beta > 0$, there exist positive constants C_1, C_2 depending only on $k, k_1, k_2, \bar{l}, \underline{l}, T, \beta$ s.t.:*

$$\int_0^T e^{-2\beta t}\hat{\mathbb{E}}(|\bar{x}_t|^2)dt \leq C_1\hat{\mathbb{E}}(\sup_{t\in[0,T]}|y_t|^2) + C_2\hat{\mathbb{E}}(\int_0^T |z_s|^2 ds). \tag{2.20}$$

Proof. We have from (2.29),

$$\bar{x}_t = \int_0^t (b(s,\tilde{X}_s,Y_s,Z_s) - b(s,\tilde{U}_s,V_s,W_s))ds + \int_0^t (\sigma(s,\tilde{X}_s,Y_s) - \sigma(s,\tilde{U}_s,V_s))dB_s$$
$$+ \int_0^t (h(s,\tilde{X}_s,Y_s) - h(s,\tilde{U}_s,V_s))d\langle B\rangle_s$$

By *Young*'s inequality and simple calculations, we have for $\varepsilon_1, \varepsilon_2, \varepsilon_3$:

$$\begin{aligned}\hat{\mathbb{E}}(|\bar{x}_t|^2) \leq & \left(\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2} + \frac{\varepsilon_3}{2}\right)\hat{\mathbb{E}}(|\bar{x}_t|^2) \\ & + \tfrac{t}{2\varepsilon_1}\hat{\mathbb{E}}(\int_0^t |(b(s,\tilde{X}_s,Y_s,Z_s) - b(s,\tilde{U}_s,V_s,W_s))|^2 ds) \\ & + \tfrac{1}{2\varepsilon_2}\hat{\mathbb{E}}(|\int_0^t (\sigma(s,\tilde{X}_s,Y_s) - \sigma(s,\tilde{U}_s,V_s))dB_s|^2) \\ & + \tfrac{1}{2\varepsilon_3}\hat{\mathbb{E}}(|\int_0^t (h(s,\tilde{X}_s,Y_s) - h(s,\tilde{U}_s,V_s))d\langle B\rangle_s|^2).\end{aligned}$$

Using Lemma (1.3.2) and *Lipschitz* conditions, we get:

$$\begin{aligned}\hat{\mathbb{E}}(|\bar{x}_t|^2) \leq & \left(\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2} + \frac{\varepsilon_3}{2}\right)\hat{\mathbb{E}}(|\bar{x}_t|^2) \\ & + \tfrac{kt}{2\varepsilon_1}\hat{\mathbb{E}}(\int_0^t (|\bar{x}_s|^2 + |y_s|^2 + |z_s|^2)ds) \\ & + \tfrac{C_2\bar{l}^2}{2\varepsilon_2}\hat{\mathbb{E}}(\int_0^t (k_1|\bar{x}_s|^2 + k_2|y_s|^2)ds) \\ & + \tfrac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\hat{\mathbb{E}}(\int_0^t (|\bar{x}_s|^2 + |y_s|^2)ds),\end{aligned}$$

$$\begin{aligned}\left(1 - (\frac{\varepsilon_1}{2} + \frac{k_1\varepsilon_2}{2} + \frac{\varepsilon_3}{2})\right)\hat{\mathbb{E}}(|\bar{x}_t|^2) \leq & \left(\frac{k_1 T}{2\varepsilon_1} + \frac{kC_2\bar{l}^2}{2\varepsilon_2} + \frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\right)\int_0^t \hat{\mathbb{E}}(|\bar{x}_s|^2)ds \\ & + T\left(\frac{kT}{2\varepsilon_1} + \frac{k\bar{l}^2}{2\varepsilon_2} + \frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\right)\hat{\mathbb{E}}(\sup_{s\in[0,T]}|y_s|^2) \\ & + \tfrac{kT}{2\varepsilon_1}\hat{\mathbb{E}}(\int_0^T |z_s|^2 ds).\end{aligned}$$

We multiply both sides of the inequality by $e^{-2\beta t}$ and integrate them on $[0,T]$, then, simple calculations gives:

$$\left(1-(\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{2}+\frac{\varepsilon_3}{2})\right)\int_0^T e^{-2\beta t}\hat{\mathbb{E}}(|\bar{x}_t|^2)dt \leq \frac{1}{2\beta}\left(\frac{kT}{2\varepsilon_1}+\frac{k_1C_2\bar{l}^2}{2\varepsilon_2}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\right)\int_0^T e^{-2\beta s}\hat{\mathbb{E}}(|\bar{x}_s|^2)ds$$
$$+\frac{T}{2\beta}\left(\frac{kT}{2\varepsilon_1}+\frac{k_2C_2\bar{l}^2}{2\varepsilon_2}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\right)(1-e^{-2\beta T})\hat{\mathbb{E}}(\sup_{s\in[0,T]}|y_s|^2)$$
$$+\tfrac{kT}{4\beta\varepsilon_1}(1-e^{-2\beta T})\hat{\mathbb{E}}(\int_0^T |z_s|^2 ds).$$

Now, let

$$\varepsilon=\left(1-\left((\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{2}+\frac{\varepsilon_3}{2}+\frac{1}{2\beta}\left(\frac{kT}{2\varepsilon_1}+\frac{k_1C_2\bar{l}^2}{2\varepsilon_2}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\right)\right)\right),$$

for big β and small $\varepsilon_1,\varepsilon_2,\varepsilon_3$, we have that $\varepsilon>0$, then

$$\int_0^T e^{-2\beta t}\hat{\mathbb{E}}(|\bar{x}_t|^2)dt \leq \frac{T}{2\varepsilon\beta}\left(\frac{kT}{2\varepsilon_1}+\frac{k_2C_2\bar{l}^2}{2\varepsilon_2}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\right)(1-e^{-2\beta T})\hat{\mathbb{E}}(\sup_{s\in[0,T]}|y_s|^2)$$
$$+\tfrac{k}{4\varepsilon\beta\varepsilon_1}(1-e^{-2\beta T})\hat{\mathbb{E}}(\int_0^T |z_s|^2 ds).$$

□

Similarly, to the previous technique used in the Lemma 2.2.3's proof and for

$$\bar{x}_T=\int_0^T(b(s,\tilde{X}_s,Y_s,Z_s)-b(s,\tilde{U}_s,V_s,W_s))ds+\int_0^T(\sigma(s,\tilde{X}_s,Y_s)-\sigma(s,\tilde{U}_s,V_s))dB_s$$
$$+\int_0^T(h(s,\tilde{X}_s,Y_s)-h(s,\tilde{U}_s,V_s))d\langle B\rangle_s,$$

we reach the following inequality:

$$\hat{\mathbb{E}}(|\bar{x}_T|^2)\leq\frac{C}{\bar{\varepsilon}'}\left(\frac{kT}{2\varepsilon_1'}+\frac{k_1\bar{l}^2}{2\varepsilon_2'}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3'}\right)\int_0^T\hat{\mathbb{E}}(|\bar{x}_t|^2)e^{-2\beta s}ds$$
$$+\frac{T}{\bar{\varepsilon}'}\left(\frac{kT}{2\varepsilon_1'}+\frac{k_2\bar{l}^2}{2\varepsilon_2'}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3'}\right)\hat{\mathbb{E}}(\sup_{t\in[0,T]}|y_t|^2)$$
$$+\tfrac{kT}{2\bar{\varepsilon}'\varepsilon_1'}\hat{\mathbb{E}}(\int_0^T|z_s|^2ds).$$

Then, for $\bar{\varepsilon}'=\left(1-(\frac{\varepsilon_1'}{2}+\frac{\varepsilon_2'}{2}+\frac{\varepsilon_3'}{2})\right)$, for strictly positive, and small enough $\varepsilon_1',\varepsilon_2',\varepsilon_3'$, the following lemma is proved.

Lemma 2.2.4. *For a given $\beta>0$, there exist positive constants $\tilde{C}_1,\tilde{C}_2,\tilde{C}_3$ depending only on $k,k_1,k_2,\bar{l},\underline{l},T,\beta$ s.t.:*

$$\hat{\mathbb{E}}(|\bar{x}_T|^2)\leq\tilde{C}_1\hat{\mathbb{E}}(\sup_{t\in[0,T]}|y_t|^2)+\tilde{C}_2\hat{\mathbb{E}}(\int_0^T|z_s|^2ds)+\tilde{C}_3\int_0^T e^{-2\beta t}\hat{\mathbb{E}}(|\bar{x}_t|^2)dt. \tag{2.21}$$

Let move to the backward stochastic differential equation part.

Lemma 2.2.5. *There exist positive constants C_3, C_4, C_5, C_6, depending only on $T, k, \bar{l}, \underline{l}, \beta$ s.t.:*

$$\begin{aligned} \hat{\mathbb{E}}(\sup_{s\in[0,T]} |\bar{y}_s|^2) \leq C_3 \hat{\mathbb{E}}\left(|m_T|^2\right) + C_4 \int_0^T e^{-2\beta s} \hat{\mathbb{E}}(|x_s|^2) ds \\ + C_5 \hat{\mathbb{E}}(\int_0^T |\bar{z}_s|^2 ds) + C_6 \hat{\mathbb{E}}(|\bar{x}_T|^2). \end{aligned} \tag{2.22}$$

Proof. We have

$$\begin{aligned} \bar{y}_t - \bar{y}_T = & \int_t^T (f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s)) ds \\ & + \int_t^T (g(s, X_s, \tilde{Y}_s) - g(s, U_s, \tilde{V}_s)) d\langle B \rangle_s \\ & - \int_t^T (\tilde{Z}_s - \tilde{W}_s) dB_s - \int_t^T d(\tilde{M}_t - \tilde{R}_t). \end{aligned}$$

We apply *Itô*'s formula on $|\bar{y}_t|^2$

$$\begin{aligned} |\bar{y}_t|^2 = & - \int_t^T 2\bar{y}_t \bar{z}_s dB_s + |\varphi(\tilde{X_T}) - \varphi(\tilde{U_T})|^2 \\ & + \int_t^T 2\bar{y}_t (f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s)) ds \\ & + \int_t^T 2\bar{y}_t (g(s, X_s, \tilde{Y}_s) - g(s, U_s, \tilde{V}_s)) d\langle B \rangle_s \\ & - \int_t^T |\bar{Z}_s|^2 d\langle B \rangle_s - \int_t^T 2\bar{y}_t d(\tilde{M}_t - \tilde{R}_t), \end{aligned}$$

$$\begin{aligned} |\bar{y}_t|^2 + \int_t^T |\bar{z}_s|^2 d\langle B \rangle_s \leq & - \int_t^T 2\bar{y}_s \bar{z}_s dB_s + k|\bar{x_T}|^2 \\ & + k \int_t^T 2(x_s \bar{y}_s + |\bar{y}_s|^2 + \bar{y}_s \bar{z}_s + \bar{y}_s m_s) ds \\ & + k \int_t^T 2\bar{y}_s (x_s + \bar{y}_s) d\langle B \rangle_s \\ & - \int_t^T 2\bar{y}_s d\bar{m}_s, \end{aligned} \tag{2.23}$$

let $J_t = \int_0^T 2|\bar{y}_s| d\bar{m}_s + \int_0^T 2\bar{y}_s \bar{z}_s dB_s$. Then,

$$\sup_{s\in[0,T]}|\bar{y}_s|^2 \leq 2k\sup_{s\in[0,T]}|\bar{y}_s|\int_0^T x_s ds + 2k\sup_{s\in[0,T]}|\bar{y}_s|^2\int_0^T ds$$
$$+2k\sup_{s\in[0,T]}|\bar{y}_s|\int_0^T|\bar{z}_s|ds + 2k\sup_{s\in[0,T]}|\bar{y}_s|\int_0^T|m_s|ds$$
$$+k\int_0^T(\frac{1}{\varsigma_1}|x_s|^2+\varsigma_1|\bar{y}_s|^2)d\langle B\rangle_s + k\int_t^T|\bar{y}_s|^2 d\langle B\rangle_s + k|\bar{x}_T|^2 + J_T - J_t.$$

Lemma 3.4 in [46] shows that J_t is a G-martingale. Using *Young*'s and the BDG inequalities, with simple calculations, we get

$$\hat{\mathbb{E}}(\sup_{s\in[0,T]}|\bar{y}_s|^2) \leq \left(\varsigma_3 k + 2kT + \varsigma_4 k + \varsigma_5 k + C_2\bar{l}T\,(k+k\varsigma_1)\right)\hat{\mathbb{E}}(\sup_{s\in[0,T]}|\bar{y}_s|^2)$$
$$+(\tfrac{kT}{\varsigma_3}+\tfrac{k(\bar{l}+l)^2T}{16\varsigma_1})\int_0^T\hat{\mathbb{E}}(|x_s|^2)ds + \frac{kT}{\varsigma_4}\hat{\mathbb{E}}(\int_0^T|\bar{z}_s|^2 ds)$$
$$+k\hat{\mathbb{E}}(|\bar{x}_T|^2) + \tfrac{kT^2}{\varsigma_5}\hat{\mathbb{E}}(|m_T|^2).$$

Let

$$\varsigma = 1 - \left(\varsigma_3 k + 2kT + \varsigma_4 k + \varsigma_5 k + C_2\bar{l}T\,(k+k\varsigma_1)\right).$$

Then,

$$\varsigma\hat{\mathbb{E}}(\sup_{s\in[0,T]}|\bar{y}_s|^2) \leq (\tfrac{kT}{\varsigma_3}+\tfrac{k(\bar{l}+l)^2T}{16\varsigma_1})\int_0^T\hat{\mathbb{E}}(|x_s|^2)ds + \frac{kT}{\varsigma_4}\hat{\mathbb{E}}(\int_0^T|\bar{z}_s|^2 ds)$$
$$+k\hat{\mathbb{E}}(|\bar{x}_T|^2) + \tfrac{kT^2}{\varsigma_5}\hat{\mathbb{E}}(|m_T|^2),$$

$$\hat{\mathbb{E}}(\sup_{s\in[0,T]}|\bar{y}_s|^2) \leq \tfrac{1}{\varsigma}(\tfrac{kT}{\varsigma_3}+\tfrac{k(\bar{l}+l)^2T}{16\varsigma_1})\int_0^T\hat{\mathbb{E}}(|x_s|^2)ds + \frac{kT}{\varsigma\varsigma_4}\hat{\mathbb{E}}(\int_0^T|\bar{z}_s|^2 ds)$$
$$+\tfrac{k}{\varsigma}\hat{\mathbb{E}}(|\bar{x}_T|^2) + \tfrac{kT^2}{\varsigma\varsigma_5}\hat{\mathbb{E}}(|m_T|^2).$$

□

Subsequently, from the equation (2.23), we get

$$\int_t^T|\bar{z}_s|^2 d\langle B\rangle_s \leq -\int_t^T 2\bar{y}_s\bar{z}_s dB_s + k|\bar{x}_T|^2$$
$$+k\int_t^T 2(x_s\bar{y}_s + |\bar{y}_s|^2 + \bar{y}_s\bar{z}_s + \bar{y}_s m_s)ds$$
$$+k\int_t^T 2\bar{y}_s(x_s+\bar{y}_s)d\langle B\rangle_s$$
$$-\int_t^T 2\bar{y}_s d\bar{m}_s$$

$$\hat{\mathbb{E}}(\int_t^T |\bar{z}_s|^2 d\langle B\rangle_s) \leq \hat{\mathbb{E}}(k|\bar{x_T}|^2 \\ +k\int_t^T 2(x_s\bar{y}_s + |\bar{y}_s|^2 + \bar{y}_s\bar{z}_s + \bar{y}_s m_s)ds \\ +k\int_t^T 2\bar{y}_s(x_s+\bar{y}_s)d\langle B\rangle_s \\ -\left(\int_t^T 2\bar{y}_s d\bar{m}_s + \int_t^T 2\bar{y}_s\bar{z}_s dB_s\right)),$$

$$\hat{\mathbb{E}}(\int_t^T |\bar{z}_s|^2 d\langle B\rangle_s) \leq \hat{\mathbb{E}}(k|\bar{x_T}|^2 \\ +k\int_t^T 2(x_s\bar{y}_s + |\bar{y}_s|^2 + \bar{y}_s\bar{z}_s + \bar{y}_s m_s)ds \\ +k\int_t^T 2\bar{y}_s(x_s+\bar{y}_s)d\langle B\rangle_s) \\ +\hat{\mathbb{E}}\left(-\left(\int_t^T 2\bar{y}_s d\bar{m}_s + \int_t^T 2\bar{y}_s\bar{z}_s dB_s\right)\right).$$

With some simple calculations, we have for some strictly positive $\varsigma_1', \varsigma_3', \varsigma_4', \varsigma_5'$

$$\hat{\mathbb{E}}(\int_0^T |\bar{z}_s|^2 d\langle B\rangle_s) \leq \hat{\mathbb{E}}((\varsigma_3' k + 2kT + \varsigma_4' k + \varsigma_5' k+) \sup_{s\in[0,T]} |\bar{y}_s|^2 \\ +\tfrac{kT}{\varsigma_3}' \int_0^T |x_s|^2 ds + \frac{kT}{\varsigma_4'}\int_0^T |\bar{z}_s|^2 ds \\ +k|\bar{x}_T|^2 + \tfrac{kT^2}{\varsigma_5'}|m_T|^2 + (k+k\varsigma_1')\int_0^T |\bar{y}_s|^2 d\langle B\rangle_s + \frac{k}{\varsigma_1'}\int_0^T |x_s|^2 d\langle B\rangle_s).$$

From Proposition (2.2.2)

$$\hat{\mathbb{E}}(\int_0^T |\bar{z}_s|^2 ds) \leq \tfrac{1}{\underline{l}^2 c_2}(\varsigma_3' k + 2kT + \varsigma_4' k + \varsigma_5' k)\,\hat{\mathbb{E}}(\sup_{s\in[0,T]} |\bar{y}_s|^2) + \frac{kT}{\underline{l}^2 c_2 \varsigma_3'}\int_0^T \hat{\mathbb{E}}(|x_s|^2)ds \\ +\tfrac{kT}{\underline{l}^2 c_2 \varsigma_4'}\hat{\mathbb{E}}(\int_0^T |\bar{z}_s|^2 ds) + \frac{k}{\underline{l}^2 c_2}\hat{\mathbb{E}}(|\bar{x}_T|^2) + \frac{kT^2}{\underline{l}^2 c_2 \varsigma_5'}\hat{\mathbb{E}}(|m_T|^2) \\ +\tfrac{C_2 \bar{l} T}{\underline{l}^2 c_2}(k+k\varsigma_1')\,\hat{\mathbb{E}}(\sup_{s\in[0,T]} |\bar{y}_s|^2) + \frac{k(\bar{l}+\underline{l})^2 T}{16\varsigma_1'}\int_0^T \hat{\mathbb{E}}(|x_s|^2)ds,$$

$$\left(1 - \tfrac{kT}{\underline{l}^2 c_2 \varsigma_4'}\right)\hat{\mathbb{E}}(\int_0^T |\bar{z}_s|^2 ds) \leq \tfrac{1}{\underline{l}^2 c_2}\left(\varsigma_3' k + 2kT + \varsigma_4' k + \varsigma_5' k + \tfrac{C_2 \bar{l} T}{\underline{l}^2 c_2}(k+k\varsigma_1')\right)\hat{\mathbb{E}}(\sup_{s\in[0,T]} |\bar{y}_s|^2) \\ +C\left(\tfrac{k(\bar{l}+\underline{l})^2 T}{16\varsigma_1'} + \tfrac{kT}{\underline{l}^2 c_2 \varsigma_3'}\right)\int_0^T \hat{\mathbb{E}}(|x_s|^2)e^{-2\beta s}ds \\ +\tfrac{k}{\underline{l}^2 c_2}\hat{\mathbb{E}}(|\bar{x}_T|^2) + \tfrac{kT^2}{\underline{l}^2 c_2 \varsigma_5'}\hat{\mathbb{E}}(|m_T|^2).$$

Thus, the following lemma is argued.

Lemma 2.2.6. *There exist positive constants C_7, C_8, C_9, C_{10}, depending only on $T, k, \bar{l}, \underline{l}, \beta$ s.t.:*

$$\begin{aligned}\hat{\mathbb{E}}\left(\int_0^T |\bar{z}_s|^2 ds\right) \leq C_7 \hat{\mathbb{E}}(\sup_{s\in[0,T]} |\bar{y}_s|^2) + C_8 \hat{\mathbb{E}}\left(|m_T|^2 ds\right) \\ + C_9 \int_0^T e^{-2\beta s} \hat{\mathbb{E}}(|x_s|^2 ds) + C_{10} \hat{\mathbb{E}}(|\bar{x}_T|^2).\end{aligned} \tag{2.24}$$

Lemma 2.2.7. *There exist positive constants $C_{11}, C_{12}, C_{13}, C_{14}, C_{15}$, depending only on $T, k, \bar{l}, \underline{l}, \beta$ s.t.:*

$$\begin{aligned}\hat{\mathbb{E}}(|\bar{m}_T|^2) \leq C_{11} \hat{\mathbb{E}}(\sup_{s\in[0,T]} |m_s|^2) + C_{12} \hat{\mathbb{E}}(\sup_{s\in[0,T]} |\bar{y}_s|^2) + C_{13} \hat{\mathbb{E}}(\int_0^T |\bar{z}_s|^2 ds) \\ + C_{14} \int_0^T e^{-2\beta s} \hat{\mathbb{E}}(|x_s|^2) ds + C_{15} \hat{\mathbb{E}}(|\bar{x}_T|^2).\end{aligned} \tag{2.25}$$

Proof.

$$\begin{aligned}\bar{m}_T - \bar{m}_t = -\bar{y}_t + \bar{y}_T + \int_t^T (f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s)) ds \\ + \int_t^T (g(s, X_s, \tilde{Y}_s) - g(s, U_s, \tilde{V}_s)) d\langle B\rangle_s \\ - \int_t^T (\tilde{Z}_s - \tilde{W}_s) dB_s.\end{aligned}$$

$$\begin{aligned}\bar{m}_T = \bar{m}_t - \bar{Y}_t + \varphi(\bar{x}_T) + \int_t^T (f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s)) ds \\ + \int_t^T (g(s, X_s, \tilde{Y}_s) - g(s, U_s, \tilde{V}_s)) d\langle B\rangle_s \\ - \int_t^T \bar{z}_s dB_s.\end{aligned}$$

By taking $t = 0$, and using that $\bar{m}_t = 0$, we have for some $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$

$$\begin{aligned}|\bar{m}_T|^2 \leq \tfrac{\delta_1}{2} |\bar{m}_T|^2 + \tfrac{1}{2\delta_1} \sup_{s\in[0,T]} |\bar{Y}_s|^2 + \frac{\delta_2}{2} |\bar{m}_T|^2 + \frac{1}{2\delta_2} |\int_0^T (f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s)) ds|^2 \\ + \tfrac{\delta_3}{2} |\bar{m}_T|^2 + \tfrac{1}{2\delta_3} |\int_0^T (g(s, X_s, \tilde{Y}_s) - g(s, U_s, \tilde{V}_s)) d\langle B\rangle_s|^2 \\ + \tfrac{\delta_4}{2} |\bar{m}_T|^2 + \tfrac{1}{2\delta_4} |\int_0^T \bar{z}_s dB_s|^2 + \frac{k\delta_5}{2} |\bar{m}_T|^2 + \frac{k}{2\delta_5} |\bar{x}_T|^2.\end{aligned}$$

$$\hat{\mathbb{E}}(|\bar{m}_T|^2) \leq \tfrac{1}{2}(\delta_1+\delta_2+\delta_3+\delta_4+k\delta_5)\hat{\mathbb{E}}(|\bar{m}_T|^2) + \tfrac{1}{2\delta_1}\hat{\mathbb{E}}(\sup_{s\in[0,T]}|\bar{y}_s|^2)$$
$$+\tfrac{T}{2\delta_2}\hat{\mathbb{E}}(\int_0^T |(f(s,X_s,\tilde{Y}_s,\tilde{Z}_s,M_s)-f(s,U_s,\tilde{V}_s,\tilde{W}_s,R_s))|^2 ds)$$
$$+\tfrac{1}{2\delta_3}\hat{\mathbb{E}}(\sup_{t\in[0,T]}|\int_0^T (g(s,X_s,\tilde{Y}_s)-g(s,U_s,\tilde{V}_s))d\langle B\rangle_s|^2)$$
$$+\tfrac{1}{2\delta_4}\hat{\mathbb{E}}(\sup_{t\in[0,T]}|\int_0^T \bar{z}_s dB_s|^2) + \frac{k}{2\delta_5}\hat{\mathbb{E}}(|\bar{x}_T|^2).$$

From Proposition (2.2.2), we have for $\delta = 1-\frac{1}{2}(\delta_1+\delta_2+\delta_3+\delta_4+k\delta_5)$, and we chose δ_i for $i=1,2,3,4,5$ small enough such that $\delta>0$ and

$$\hat{\mathbb{E}}(|\bar{m}_T|^2) \leq \tfrac{C}{\delta}\left(\tfrac{Tk(l+\bar{l})^2}{32\delta_3}+\tfrac{Tk}{2\delta_2}\right)\hat{\mathbb{E}}(\int_0^T |x_s|^2 e^{-2\beta s}ds)$$
$$+\tfrac{1}{\delta}\left(\tfrac{T^2k(l+\bar{l})^2}{32\delta_3}+\tfrac{T^2k}{2\delta_2}+\tfrac{1}{2\delta_1}\right)\hat{\mathbb{E}}(\sup_{s\in[0,T]}|\bar{y}_s|^2)$$
$$+\tfrac{1}{\delta}\left(\tfrac{Tk}{2\delta_2}+\tfrac{\bar{l}^2C_2}{2\delta_4}\right)\hat{\mathbb{E}}(\int_0^T |\bar{z}_s|^2 ds) + \frac{Tk}{2\delta\delta_2}\hat{\mathbb{E}}(|m_T|^2) + \frac{k}{2\delta\delta_5}\hat{\mathbb{E}}(|\bar{x}_T|^2).$$

□

2.2.3.2 Proof of the main Theorem

We start by defining the map $\digamma$ as follow:
For $(X,Y,Z,M),(U,V,W,R) \in M_G^2(0,T)\times\mathfrak{S}_G^2(0,T) := H_{G,T}^{2,2}$ we define $(\tilde{X},\tilde{Y},\tilde{Z},\tilde{M})$(resp. $(\tilde{U},\tilde{V},\tilde{W},\tilde{R})$) as the image of (X,Y,Z,M) (resp.(U,V,W,R)) by the map $\digamma$ where:
$\mathfrak{S}_G^2(0,T) := S_G^2(0,T)\times H_G^2(0,T)\times\mathbb{L}_G^2(\Omega_T)$, and

$$\digamma : H_{G,T}^{2,2} \to H_{G,T}^{2,2},\ (X,Y,Z,M) \to \digamma(X,Y,Z,M) := (\tilde{X},\tilde{Y},\tilde{Z},\tilde{M}), \tag{2.26}$$

where $(\tilde{X},\tilde{Y},\tilde{Z},\tilde{M})$ are defined by:
for $t\in[0,T]$,

$$\tilde{X}_t = x + \int_0^t b(s,\tilde{X}_s,Y_s,Z_s)ds + \int_0^t \sigma(s,\tilde{X}_s,Y_s)dB_s + \int_0^t h(s,\tilde{X}_s,Y_s)d\langle B\rangle_s, \tag{2.27}$$

and,

$$\tilde{Y}_t = \Phi(X_T)+\int_t^T f(s,X_s,\tilde{Y}_s,\tilde{Z}_s,M_s)ds+\int_t^T g(s,X_s,\tilde{Y}_s)d\langle B\rangle_s-\int_t^T Z_s dB_s-\int_t^T d\tilde{M}_s. \tag{2.28}$$

Remark 2.2.2. *1. The space $H_{G,T}^{2,2}$ is a Banach space as a product of Banach spaces $M_G^2(0,T)$, $S_G^2(0,T)$, $H_G^2(0,T)$, and $L_G^2(\Omega_T)$*

2. *The map $\digamma$ is well defined. Indeed, because Y and Z are given respectively in $S_G^2(0,T)$, $H_G^2(0,T)$, and the coefficients b, σ and h hold the conditions (H), then $\tilde{X}$ in equation (2.27) exists (see e.g. [82]) as the solution of this equation and belongs to $M_G^2([0,T])$, and so we plug-in $\tilde{X}$ in the G-BSDE equation (2.28); then we have also $\tilde{Y}_s, \tilde{Z}_s, \tilde{M}_s$ which exist (see e.g. [46]) as the solution of the BSDE (2.28) and belong to $\mathfrak{S}_G^2(0,T) = S_G^2(0,T) \times H_G^2(0,T) \times \mathbb{L}_G^2(\Omega_T)$ for fixed $(\tilde{X}, M) \in M_G^2(0,T) \times \mathbb{L}_G^2(\Omega_T)$.*

Now, we aim to proving that the map $\digamma$ is a contraction, and for this, let consider the following notations:
$\bar{x}_s = \tilde{X}_s - \tilde{U}_s, \bar{z} = \tilde{Z}_s - \tilde{W}_s$ and $\bar{y}_s = \tilde{Y}_s - \tilde{V}_s, x_s = X_s - U_s, y_s = Y_s - V_s, w_s = Z_s - W_s, \bar{m}_t = \tilde{M}_t - \tilde{R}_t, m_t = M_t - R_t$.
So,

$$\begin{aligned}\bar{x}_t = &\int_0^t (b(s, \tilde{X}_s, Y_s, Z_s) - b(s, \tilde{U}_s, V_s, W_s))ds + \int_0^t (\sigma(s, \tilde{X}_s, Y_s) - \sigma(s, \tilde{U}_s, V_s))dB_s \\ &+ \int_0^t (h(s, \tilde{X}_s, Y_s) - h(s, \tilde{U}_s, V_s))d\langle B\rangle_s\end{aligned} \tag{2.29}$$

and,

$$\begin{aligned}\bar{y}_t = \tilde{Y}_t - \tilde{V}_t = \bar{y}_T &+ \int_t^T (f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s))ds \\ &+ \int_t^T (g(s, X_s, \tilde{Y}_s) - g(s, U_s, \tilde{V}_s))d\langle B\rangle_s \\ &- \int_t^T (\tilde{Z}_s - \tilde{W}_s)dB_s - \int_t^T d(\tilde{M}_t - \tilde{R}_t).\end{aligned} \tag{2.30}$$

Proof. From Lemmas (2.2.3)-(2.2.7), for a given $\beta > 0$, there exists a constant $C = C_{k,k_1,k_2,\bar{l},\underline{l},\beta} > 0$, such that $\forall T; 0 < T \leq C$, and there exist constants $\omega_1, \omega_2, \omega_3, \omega_4 \in (0,1)$ depending only on $k, T, \beta, \bar{l}, \underline{l}$ s.t.:

$$\begin{aligned}\int_0^T e^{-2\beta s}\hat{\mathbb{E}}(|\bar{x}_s|^2)ds + \hat{\mathbb{E}}(\sup_{s\in[0,T]} |\bar{y}_s|^2) + \hat{\mathbb{E}}\left(\int_0^T |\bar{z}_s|^2 ds\right) + \hat{\mathbb{E}}(\sup_{s\in[0,T]} |\bar{m}_s|^2) \leq \omega_1 \int_0^T e^{-2\beta s}\hat{\mathbb{E}}(|x_s|^2)ds \\ +\omega_2\hat{\mathbb{E}}(\sup_{s\in[0,T]} |y_s|^2) + \omega_3\hat{\mathbb{E}}(\int_0^T |z_s|^2 ds) + \omega_4\hat{\mathbb{E}}(\sup_{s\in[0,T]} |m_s|^2).\end{aligned} \tag{2.31}$$

It is noted that the following two norms are equivalent on $\bar{M}_G^p(0,T)$,

$$\int_0^T e^{-2\beta t}\hat{\mathbb{E}}(|\bar{x}_t|^2)dt \sim \int_0^T \hat{\mathbb{E}}(|\bar{x}_t|^2)dt.$$

So, the map $\digamma$ is a contracting mapping from the Banach space $H_{G,T}^{2,2}$ to itself, which ensures the existence of a unique fixed point $(X, Y, Z, M) \in H_{G,T}^{2,2}$ which is (from the definition of $\digamma$) the solution of the FBSDE (2.12). □

Remark 2.2.3. *This result can't be extended to the case where σ depend to Z with*

$$|\sigma(s,x,y,z)-\sigma(s,x',y',z')|^2 \leq k_1|x-x'|^2+k_2|y-y'|^2+k_3|z-z'|^2,$$

indeed, the system:

$$\begin{cases} dX_s = Z_s dB_s, \\ dY_s = Z_s dB_s + dM_s,\ s\in[t,T], \\ X_t = x,\ Y_T = X_T, M_t = 0, \end{cases} \tag{2.32}$$

has an infinity number of solutions, because for any $Z \in H_G^p(0,T)$ and any decreasing G-martingales M such that $M_t = 0$ and $M_T \in \mathbb{L}_G^p(\Omega_T)$; the tuple (X,Y,Z,M) with $X_u := x+\int_t^u Z_s dB_s$ and $Y_u := x - M_T$ is a solution of the G-FBSDE (2.32).
The result is still valid in a multi-dimension case.

Chapter 3

Optimal Control For Decoupled Forward-Backward Stochastic Differential Equations in the G-framework

In this chapter, we study a controlled system for decoupled forward-backward stochastic differential equations driven by G-*Brownian* motion. Our aim is to investigate the problem of the existence of optimal relaxed stochastic control given by a G-FBSDE and a cost function as the first component of the solution of the backward stochastic differential equation.
In first section, we recall some generalities on classical optimal control for FBSDE in classical space. Then, we define the stochastic optimal control (SOC, for short) under G-frame work in the second section. The third section is devoted for the G-relaxed optimal control in brief. In the last section, we present the main issue of this chapter as well as the assumptions needed to solve it, then, we introduce the approximated system of our original one and we show the existence of an optimal control for it. Next, we study the convergence of the approximating control problem; we show that this problem converges to the value function of the original problem. Finally, to conclude the section as well as the chapter we establish the existence of an optimal control for our system of G-decoupled BSDE as limit of the approximated control problem.

3.1 General definitions

A control problem is to optimize a functional which depend on a solution of a dynamical system. This dynamical system can be deterministic (Ordinary differential equation, partial differential equation) [38] as it can be stochastic dynamical system (SDE, BSDE, FBSDE). In our case we are interested by the study of the stochastic case. In general, it is formulated according to the following characteristics:

System state: Considering a dynamical system characterized by its state at all times; time can be discrete or continuous. The horizon (time variation interval) can be finite or infinite. The state of the system is the set of quantitative variables constituting an "exhaustive" description of the system. The state variables are assumed to be finite numbers with real values. We denote by $X_t(\omega)$ the state of the system (state process) at the instant t in a scenario $\omega \in \Omega$ a measurable space endowed with a probability $\mathbb{P}$.
Once the state is defined, it is a question of defining down the laws of evolution of that state in a function of time. The application $t \to X_t$ describes the evolution of the system. This evolution is provided by a probabilistic model.

Control: The dynamics X_t of the state of the system is influenced by a control that it is modeled as a process $(u_t)_t$ and in order to the stochastic integral be defined, an adaptation constraint with respect to certain filtration is required on the control u that takes its values in a control space $\mathcal{U}$.

Cost criterion/ performance: The objective is to maximize (or minimize) the functional $J(X;u)$. In general, the functional is considered of the form:

$$J(X;u) = \mathbb{E}[\int_t^T f(X_s,u_s)ds + g(X_T)], \qquad \text{on finite horizon } T < \infty$$

and,

$$J(X;u) = \mathbb{E}[\int_t^\infty e^{-\beta s} f(X_s,u_s)ds], \qquad \text{on infinte horizon.}$$

The function f is the integral cost, g is the final cost and $\beta > 0$ is the actualization coefficient. The value function is defined by

$$v = \inf_u J(x;u),$$

The objective will be to determine the value function, as well as the extremum for these criteria and the optimal controls, if there exist whom realize them.

Definition 3.1.1. *(Admissible control)*
Let U be a given compact set. For each $s \in [t,T]$, we say u is an admissible control on $[t,T]$, if it satisfies the following conditions:

1. $u : [t,T] \times \Omega \to U$.
2. $u \in M_G^p$.

$\mathcal{U}[t,T]$ denotes the set of admissible controls on $[t,T]$.

Definition 3.1.2. ***(Strict control)****: a strict control is an $\mathcal{F}_t-$adapted process with values in some subset U of $\mathbb{R}^n$.*

In some situations, to study the value function of stochastic optimal control, it is important to define the essential infimum (essential supremum, resp) of the cost functional $Y_t^{t,x;u}$ that is a solution of a G-backward SDE ($Y_t^{t,x;u}$ is deterministic).

Definition 3.1.3. • *The essential infimum of* $\{Y_t^{t,x;u}, u \in \mathcal{U}[t,T]\}$*, denoted by* $\underset{u(.)\in\mathcal{U}[t,T]}{ess\inf} Y_t^{t,x;u}$ *is a random variable* $\varrho \in L_G^2(\Omega_t)$ *satisfying:*

1. $\forall u \in \mathcal{U}[t,T], \varrho \leq Y_t^{t,x;u}$;
2. *If* η *is a random variable satisfying* $\eta \leq Y_t^{t,x;u}$ *q.s., for any* $u \in \mathcal{U}[t,T]$*, then* $\varrho \geq \eta$ *q.s..*

- *The essential supremum of* $\{Y_t^{t,x;u}, u \in \mathcal{U}[t,T]\}$*, denoted by* $\underset{u(.)\in\mathcal{U}[t,T]}{ess\sup} Y_t^{t,x;u}$ *is a random variable* $\varrho \in L_G^2(\Omega_t)$ *satisfying:*

1. $\forall u \in \mathcal{U}[t,T], \varrho \geq Y_t^{t,x;u}$;
2. *If* η *is a random variable satisfying* $\eta \geq Y_t^{t,x;u}$ *q.s., for any* $u \in \mathcal{U}[t,T]$*, then* $\varrho \leq \eta$ *q.s..*

3.1.1 Relaxed control

Occasionally, the limit which should be the natural candidate to optimality of the sequence u_n is lacking in the space of controls. So, it will necessary to look for space in which this limit exists. On $\mathcal{U}$, characterize $u_n(t)$ with the *Dirac* measure: $\delta_{u_n(t)}(\mathrm{du})$. Set q_n a measure over the space $[0,1] \times \mathcal{U}$ defined by:

$$\delta_{u_n(t)}(dt, du) = \delta_{u_n(t)}(du)dt,$$

and q_n converges weakly to $\tilde{q}_n(du, dt) = \frac{1}{2}[\delta_{-1} + \delta_1](du)dt$. All the measures $q(du, dt)$ should be taken as controls. However, to reach the proof of the existence of an optimal control, we have to restrict to a compact space which contain the "classical" controls. For this, we set the following definition

Definition 3.1.4. *Let* $\mathcal{U} \in \mathbb{R}^d$*. A relaxed control with values in* $\mathcal{U}$ *is a random measure* q *over* $[0,T] \times \mathcal{U}$ *such that the projection on* $[0,T]$ *is the* Lebesgue *measure.*
If there exist $u : [0,T] \to \mathcal{U}$*, such that* $q_n(du, dt) = \delta_{u_n(t)}(du)dt$*,* q *is identified with* u_t *and said to be control process.*

In the relaxed control problems, we use probability measure μ_t on a set A of controls values instead of using $u_t \in A$. Then the problem is modeling with the following decoupled forward backward stochastic differential equation

$$
\begin{aligned}
dX_s &= \int_A b(s, X_s, a) d\mu_s(a) ds + \int_A \sigma(s, X_s, a) d\mu_s(a) dB_s, \qquad s \in [0, T] \\
dY_s &= -\int_A g(s, X_s, Y_s, a) d\mu_s(a) ds + Z_s dB_s, \qquad s \in [t, T] \\
Y_T &= \phi(X_T) \qquad X_0 = x, \qquad x \in \mathbb{R}^m,
\end{aligned} \tag{3.1}
$$

Remark 3.1.1. ***(Relation between Relaxed and Strict control)****: Any strict $U-$valued control process u_t can be represented as a relaxed control by setting $\nu_t(du) = \delta_{u_t}(du)$. Moreover, the so-called Chattering Lemma, show that any relaxed control is a weak limit of a sequence of strict controls.*

Remark 3.1.2. *In the classical stochastic optimal control, there exists two important and famous principals; stochastic maximum principal (SMP, for short) and dynamic programming principal. By the first one, for a stochastic optimal control problem, one can derives necessary conditions for optimality. Any optimal control with its corresponding optimal state have to solve a system consists of a forward stochastic differential equation and the adjoint equation*[1] *associated with a condition of optimization of a function called the* Hamiltonian, *such system is called the* Hamiltonian *system. Many research have been done on this principle e.g. [42, 43].*

Equivalently important, the dynamic programming principle [9, 10] has been initiated by Richard Bellman *and co-workers in* 1950. *The basic idea of this principle is to consider a family of control problems at different initial states and times. The infinitesimal version of the dynamic programming principle is the well-known* **Hamilton-Jacobi-Bellman***(HJB) equations which are second-order, possibly degenerate elliptic, fully nonlinear partial differential equations in the value function(i.e. under a suitable condition, its solution is the value function itself). In general, the HJB equations are of the following form:*

$$\frac{\partial v}{\partial t}(t, x) + H(t, x, D_x v(t, x), D_x^2 v(t, x)) = 0,$$

where the function H is called the Hamiltonian *which should taking its maximum(minimum) to obtain the optimal control. Its study has been the interest of many authors especially the existence of its solution which has been studied well by many authors among them [21, 19].*

3.1.2 Controlled forward-backward stochastic differential equation

In stochastic optimal control, for a given dynamical system, the system states can be modeled by a controlled SDE, a controlled BSDE as well as a controlled FBSDE in which the control problem is described as follow:
Let $\mathbb{U}$ be a compact metric space. Let $t \in [0,\ T]$ where $T > 0$ is a finite horizon. Let $(\Omega,\ \mathcal{F},\ \mathbb{P},\ (\mathcal{F}_t))$ be a filtered probability space which satisfies the usual conditions. Let B be

[1] adjoint equation: is a linear backward stochastic differential equation.

a d-dimensional *Brownian* motion with respect to the filtration $(\mathcal{F}_t)$(not necessary *Brownian* filtration).

Decoupled controlled FBSDE:
Let the deterministic functions b, σ, f and ϕ be defined as follow:

$$b : \mathbb{R}^d \times \mathbb{U} \longmapsto \mathbb{R}^d,$$

$$\sigma : \mathbb{R}^d \times \mathbb{U} \longmapsto \mathbb{R}^{d\times d},$$

$$f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U} \longmapsto \mathbb{R},$$

$$\phi : \mathbb{R}^d \longmapsto \mathbb{R}.$$

So, the control problem is presented as follow:

$$\begin{cases} dX_s^u = b(X_s^u, u_s)ds + \sigma(X_s^u, u_s)dB_s, \\ dY_s^u = -f(X_s^{;u}, Y_s^u, Z_s^u, u_s)ds + Z_s^u dB_s + dM_s^u, \\ \langle M^u,\ B\rangle_s = 0, \\ X_t^u = x, \quad Y_T^u = \phi(X_T^u), \quad M_t^u = 0. \end{cases} \tag{3.2}$$

The existence of an optimal control for the decoupled FBSDE has been established by *Buckdahn* et al. [17] where, in order to get the existence of a relaxed optimal control they used the associated *Hamilton-Jacoi-Belman* equation(see next section) to construct a sequence of optimal feedback controls then, they analyze to the limit and use the result of [28] and they use the *Filippov* convexity condition to get the existence of a strict optimal control this last step was also used by [5] in which using the *Jakubowsky* S-topology and compactness method the authors shown directly the existence of a relaxed control and thus they established the existence of an optimal control by different methods.

Coupled controlled FBSDE: Let the deterministic functions b, σ, f and ϕ be defined as follow:

$$b : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U} \longmapsto \mathbb{R}^d,$$

$$\sigma : \mathbb{R}^d \times \mathbb{R} \times \mathbb{U} \longmapsto \mathbb{R}^{d\times d},$$

$$f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U} \longmapsto \mathbb{R},$$

$$\phi : \mathbb{R}^d \longmapsto \mathbb{R}.$$

The control problem is presented as follow:

$$\begin{cases} dX_s^u = b(X_s^u, Y_s^u, Z_s^u, u_s)ds + \sigma(X_s^u, Y_s^u, u_s)dB_s, \\ dY_s^u = -f(X_s^u, Y_s^u, Z_s^u, u_s)ds + Z_s^{t,x;u}dB_s + dM_s^u, \\ \langle M^u,\ B\rangle_s = 0, \\ X_t^u = x, \quad Y_T^u = \phi(X_T^{t,x;u}), \quad M_t^u = 0, \end{cases} \tag{3.3}$$

where, in both cases, X^u, Y^u, Z^u are $(\mathcal{F}_t)$-adapted square integrable processes and M^u is an $(\mathcal{F}_t)$-adapted square integrable martingale which is orthogonal to B, and the control variable u is an $\mathcal{F}_t$ adapted process with values in a given compact metric space $\mathbb{U}$.
The cost functional is defined by:

$$J(u) := Y_t^u, \text{ for } u \in \mathcal{U} \tag{3.4}$$

which will be optimize by a supremum, essential sup, infrumum or essential inf. If an $\mathcal{F}_t$-adapted control $\widehat{u}$ minimize (3.4) i.e.:

$$Y_t^{\widehat{u}} = \text{essinf}\left\{Y_t^u,\ \ u \in \mathcal{U}(t)\right\},$$

then, $\widehat{u}$ is called an optimal control.

The coupled case issue, exploiting the result of [17] and [5] to the more general case(coupled) *Bahlali et al.*[55] proved the existence of an optimal control for degenerate FBSDE. In fact, they consider that the coefficients satisfying the G-monotony condition given in[84] to guarantee the existence of a unique solution to the coupled system states, also they was obligated to transform coefficient of the hessian uniformly elliptic by adding a strictly positive number because of the degeneracy of the diffusion unlike the non-degenerate case where *Kebiri et al.* [6] was not obligated to add it which does not affect any change on the system (reverse of the generate case) where the prove differ from of the degenerate case, it is in some sense like [17, 5] for more details see [6].

3.2 Stochastic optimal control under G-framework

It is worthy to mention that the issue of randomness and ambiguity of the real word as well as the inability of the classical stochastic optimal control to consider a model uncertainty necessitate the study of the stochastic optimal control on nonlinear (i. e. the systems states are perturbed by a G-*Brownian* motion) and developed it as the classical one. Thus, for economic perspectives *Fei* & *Fei* [37] set up an optimality principle of stochastic control issue and investigate an optimal consumption and portfolio decision with a volatility ambiguity. *Sun et al.*[90] argued the stochastic maximum principle for processes driven by G-*Brownian* motion

either *Hu & Ji*[45] generalized the dynamic programming obtained by *Peng* [76] to make it suitable for the G-framework.
Let consider the following FBSDE system

$$
\begin{cases}
dX_s^{t,x;u} = b(s, X_s^{t,x;u}; u_s)ds + \sigma(s, X_s^{t,x;u}; u_s)dW_s + h(s, X_s^{t,x;u}; u_s)d\langle W\rangle_s, \\
Y_s^{t,x;u} = \Phi(X_T^u) + \int_t^T f(s, X_s^{t,x;u}, Y_s^{t,x;u}, Z_s^{t,x;u}; u_s)ds + \int_t^T g(s, X_s^{t,x;u}, Y_s^{t,x;u}, Z_s^{t,x;u}; u_s)d\langle W\rangle_s \\
\qquad - \int_t^T Z_s^{t,x;u}dW_s - (M_T^{t,x;u} - M_t^{t,x;u}), \; s \in [t,T], \\
X_t^{t,x;u} = x, \qquad M_t^u = 0.
\end{cases}
\tag{3.5}
$$

The above G-SDE and G-BSDE in (3.5) have a unique solution $X_t^{t,x;u}$ and $(Y_t^{t,x;u}, Z_t^{t,x;u})$ respectively as it is shown in (1.5) and (1.7) respectively. Then the forward G-SDE in (3.5) govern the state equation of the stochastic optimal control where its value function is :
For a given $x \in \mathbb{R}^n$, the problem is to minimize the cost function

$$
\begin{aligned}
J(t,x;u) &:= \max_{\mathbb{P}\in\mathcal{P}} \mathbb{E}\left[\Phi_1(s, x(s); u(s))ds + \Phi(x(T))\right] \\
&= \hat{\mathbb{E}}\left[\Phi_1(s, x(s); u(s))ds + \Phi(x(T))\right],
\end{aligned}
$$

where u denote the control process.
The value function for a given $x \in \mathbb{R}^n$ is defined by:

$$
v(t,x) := \inf_{u\in\mathcal{U}[t,T]} J(t,x;u) = \inf_{u\in\mathcal{U}[t,T]} \hat{\mathbb{E}}\left[\Phi_1(s, x(s); u(s))ds + \Phi(x(T))\right], \text{ for } x \in \mathbb{R}^n. \tag{3.6}
$$

Now, we take an overview on the dynamic programming and the related HJB. For more details on this subject see [48] and [49].

Dynamic programming principle

For a given initial data (t,x), and on the time horizon $[0, t+\delta]$, defining:

$$
\mathbb{G}_{s,t+s}^{t,x;u}(\eta) := \hat{Y}_s^{t,x;u},
$$

where, δ is positive number in $[0, T-t]$, $\eta \in L_G^1(R)$ and $\hat{Y}_s^{t,x;u}$ is the solution of the following G-BSDE

$$
\begin{aligned}
\hat{Y}_s^{t,x;u} &= \eta + \int_s^T f(r, X_r^{t,x;u}, \hat{Y}_r^{t,x;u}, \hat{Z}_r^{t,x;u}, u_r)dr \\
&\quad + \int_s^T g(r, X_r^{t,x;u}, \hat{Y}_r^{t,x;u}, \hat{Z}_r^{t,x;u}, u_r)d\langle W\rangle_r - \int_s^T Z_r^{t,x;u}dW_r - (\hat{M}_T^{t,x;u} - \hat{M}_s^{t,x;u}),
\end{aligned}
$$

and $X_s^{t,x;u}$ is the solution of the forward G-SDE of (3.5). In the case where $Y_s^{t,x;u}$ is the solution of the backward G-SDE in (3.5), we have

$$\mathbb{G}_{s,T}^{t,x,u}(\Phi(X_T^u)) = \mathbb{G}_{s,t+\delta}^{t,x;u}(Yt + \delta^{t,x;u}).$$

A generalization of the well-known dynamic programming principle is given via the following theorem.

Theorem 3.1. *[48] For any $\delta \in [0, T-t]$, we have*

$$V(t,x) = \sup_{u(.)\in\mathcal{U}} \mathbb{G}_{t,t+\delta}^{t,x;u}(u(t,\delta, X_{t+\delta}^{t,x;u})).$$

Now, we define $G : \mathbb{S}_d \to \mathbb{R}$ by

$$G(A) := \frac{1}{2}\hat{\mathbb{E}}(\langle AB_1, B_1\rangle) \tag{3.7}$$

and for $(t,x,v,p,A,u) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n \times U$, let set:

$$F_{ij}(t,x,v,p,A,u) = \langle A\sigma(t,x), \sigma(t,x)\rangle + 2\langle p, h_{ij}^{\delta}(t,x;u)\rangle$$

$$+2g_{ij}(t,x,v,\langle\sigma(t,x;u),p\rangle,u);$$

the function $H(t,x,v,p,A,u)$ by

$$H(t,x,v,p,A,u) = G(F((t,x,v,p,A,u)) + \langle b(t,x,v),p\rangle + f_{\delta}(t,x,V,\partial_x V, \partial_{xx}^2 V, v),$$

and the *Hamilton-Jacobi-Belman* equation is the following second order partial differential equation:

$$\begin{cases} \partial V(t,x) + \inf\limits_{v\in U} H(t,x,v,p,A,u) = 0 \\ V(T,x) = \Phi(x),\ x \in \mathbb{R}^n. \end{cases} \tag{3.8}$$

The relationship between the above PDE and the value function (3.6) is given by the following theorem

Theorem 3.2. *[48] The value function V defined by (3.6) is the unique viscosity solution of the second-order partial differential equation (3.8).*

Maximum principle in the G-framework

Let the control process $u(t) : [0,T] \times \Omega \to U$, where $U \in \mathbb{R}$ is a given compact convex set. On a G-expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, let $B = (B_t^1, B_t^2, \ldots, B_t^d)_{t\geq 0}^T$ be d-dimensional *G-Brownian* motion. Let consider the following controlled system:

$$\begin{cases} dX_t = b(t,X_t,u_t)dt + \sum_{j=1}^{d}\sigma^j(t,X_t,u_t)dB_t^j + \sum_{i,j=1}^{d} h^{ij}(t,X_t,u_t)d\langle B^i,B^j\rangle_t, \\ Y_t = f(t,X_t,Y_t,Z_t,u_t)dt + \sum_{i,j=1}^{d} g^{ij}(t,X_t,Y_t,Z_t,u_t)d\langle B^i,B^j\rangle_t \\ \qquad -Z_t dB_t - dM_t,\ t\in[0,T], \\ X_0 = x, \qquad Y_T = \Phi(X_T^u). \end{cases} \tag{3.9}$$

The problem is to find an optimal control $\hat{u}(.) \in \mathcal{U}[0,T]$, such that

$$J(\hat{u}(.)) = \inf_{u\in\mathcal{U}[0,T]} J(u(.)),$$

where the cost functional J is defined as follow

$$J(u(.)) := \hat{\mathbb{E}}[\int_0^T \eta(t,X_t,Y_t,Z_t,u_t)dt + \gamma(X_T) + l(Y_0)].$$

Under suitable conditions on the deterministic functions $b, h^{ij}, \sigma^j, f, g^{ij}, \eta, \gamma$ and l, we have

$$\langle H_u(t,\hat{X}_t,\hat{Y}_t,\hat{Z}_t,\hat{u}_t,p_t,q_t), u-\hat{u}\rangle + G(F(t,\hat{X}_t,\hat{Y}_t,\hat{Z}_t,\hat{u}_t,p_t,q_t,\xi_t)) \geq 0, \quad \forall u\in U \quad t\in[0,T],$$

where

$$H(t,\hat{X}_t,\hat{Y}_t,\hat{Z}_t,\hat{u}_t,p_t,q_t) = \langle p, b(t,x;u)\rangle + \langle q, f(t,x,y,z,u)\rangle + \eta(t,x,y,z,u),$$

and, $F = (F_{ij})_{1\leq i,j\leq d}$;

$$\begin{aligned} F_{ij}(t,x,y,z,u,p,q,\xi) = &\langle \xi^i, \sigma_u^j(t,x,\hat{u})(u-\hat{u})\rangle \\ &+\langle p, h_u^{ij}(t,x,\hat{u})(u-\hat{u})\rangle + \langle p, h_u^{ji}(t,x,\hat{u})(u-\hat{u})\rangle \\ &+\langle q, g_u^{ij}(t,x,\hat{u})(u-\hat{u})\rangle + \langle q, g_u^{ji}(t,x,\hat{u})(u-\hat{u})\rangle \end{aligned}$$

$\xi = (\xi^1,\xi^2,\ldots,\xi^d)$, and $(\hat{X}_t,\hat{Y}_t,\hat{Z}_t)$ represents the corresponding optimal trajectory and p_t, q_t are the solutions of the adjoint equations. For more details, we refer to [89].

3.2.1 G-Relaxed stochastic optimal control

It is worthy to mention that the famous Chattering Lemma has been extended to the following lemma which implies that each control in the class of relaxed controls $\mathcal{R}$ can be approximated with a sequence of strict controls from the set of strict controls constituted of $\mathbb{F}^{\mathcal{P}}$-adapted processes u taking values in the set U that is denoted $\mathcal{U}[0,T]$. The set $\mathcal{U}[0,T]$ embeds to $\mathcal{R}$ through the mapping

$$\varphi : u \in \mathcal{U}[0,T] \mapsto \varphi(u)(dt, d\alpha) = \delta_{u_t} dt \in \mathcal{R}. \tag{3.10}$$

Lemma 3.2.1. *Let (U,d) be a compact separable metric space. Let $(\mu_t)_t$ be an $\mathbb{F}^{\mathcal{P}}$-progressively measurable process with values in $\mathcal{P}(U)$. Then there exists a sequence $(u_n)_{n\geq 0}$ of $\mathbb{F}^{\mathcal{P}}$-progressively measurable processes with values in U such that the sequence of random measures $\delta_{u_n}(d\alpha)dt$ converges in the sense of stable convergence (thus, weakly) to $\mu_t(d\alpha)dt)$ quasi-surely.*

Proof. Given the $\mathbb{F}^{\mathcal{P}}$-progressively measurable relaxed control μ, the detailed pathwise construction of the approximating sequence $(\delta_{u_n}(d\alpha)dt)_{n\geq 0}$ of $\mu_t(d\alpha)dt$ in [29] (Theorem 2.2) extends easily to make the strict controls $(u_n)_n$ $\mathbb{F}^{\mathcal{P}}$-progressively measurable. □

Remark 3.2.1. *Note that (U,d) is a separable metric space and we denote the space of probability measures on the set U endowed with its* Borel *σ-algebra $\mathcal{B}(U)$ by $\mathcal{P}(U)$. The class $\mathcal{M}([0,T]\times U)$ of relaxed controls is considered as a subset of the set $\mathbb{M}([0,T]\times U)$ of* Radon *measures $\nu(dt,d\alpha)$ on $[0,T]\times U$ equipped with the topology of stable convergence[2]. of measures, whose projections on $[0,T]$ coincide with the* Lebesgue *measure dt, moreover, whose projection on U coincide with some probability measure $\mu_t(d\alpha)\in\mathcal{P}(U)$, it mean that $\nu(d\alpha,dt) := \mu_t(d\alpha)dt$. for fixed continuous $t,\phi(t,\Delta)$, the coarsest topology which makes the function*

$$q \mapsto \int_0^T \int_U \phi(t,\alpha) q(dt,d\alpha) \textit{ for all bounded measurable functions } \phi(t,\alpha)$$

continuous is the topology of stable convergence of measures.

Now, on $(\Omega, Lip(\Omega_T), \hat{\mathbb{E}})$, let

$$\begin{cases} dX_s^\mu = \int_U b(s,X_s^\mu,\alpha)\mu_s(d\alpha)ds + \sigma(s,X_s^\mu)dB_s + \int_U h(s,X_s^\mu,\alpha)\mu_s(d\alpha)d\langle B\rangle_s, \\ X_0^\mu = x, \end{cases} \tag{3.11}$$

where

$$b : [0,T]\times\mathbb{R}^d\times U \to \mathbb{R}^d, \sigma, h : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$$

are deterministic functions, the problem is to minimize the following cost function

$$J(\mu) = \hat{\mathbb{E}}(\int_0^T \int_U f(s,X_s^\mu,\alpha)\mu_s(d\alpha)ds + l(X_t^\mu)), \tag{3.12}$$

where

$$f : [0,T]\times\mathbb{R}^d\times U \to \mathbb{R}, l : \mathbb{R}^d \to \mathbb{R}$$

[2]topology of stable convergence (weak topology) is is the coarsest topology for which all mappings: $\mathbb{Q} \mapsto \mathbb{E}^{\mathbb{P}}(.)$ are continuous. For more details on this topology see *Jacod* & *Mémin* [53].

are also deterministic functions. In view of (2.17) $J(\mu) = J(\delta_u)$ when $\mu = \delta_u, u \in \mathcal{U}([0,T])$ and the process $X^{\delta_u} := X^\mu$, solves

$$\begin{cases} dX_s^u = b(s, X_s^u, u_s)ds + \sigma(s, X_s^u)dB_s + h(s, X_s^u, u_t)d\langle B\rangle_s, \\ X_0^u = x. \end{cases} \tag{3.13}$$

Furthermore, the following assumptions hold

(A.1) The functions b, h and σ are continuous and bounded. Moreover, they are *Lipschitz* continuous with respect to the space variable uniformly in (t, u).

(A.2) The functions f and l are continuous and bounded.

The following theorem establish the existence of an optimal control i.e. there exists a minimizer for the problem (3.12)

Theorem 3.3. *[86] We have*

$$\inf_{u\in\mathcal{U}[0,T]} J(u) = \inf_{\mu\in\mathcal{R}} J(\mu). \tag{3.14}$$

Moreover, there exists a relaxed control $\hat{\mu} \in \mathcal{R}$ such that

$$J(\hat{\mu}) = \inf_{\mu\in\mathcal{R}} J(\mu). \tag{3.15}$$

Recall that

$$J(\mu) = \sup_{\mathbb{P}\in\mathcal{P}} J^{\mathbb{P}}(\mu), \tag{3.16}$$

where the relaxed performance functional associated to each $\mathbb{P} \in \mathcal{P}$ is given by

$$J^{\mathbb{P}}(\mu) = \mathbb{E}^{\mathbb{P}}\Big(\int_0^T \int_U f(s, X_s^\mu, \alpha)\mu_s(d\alpha)ds + l(X_t^\mu)\Big). \tag{3.17}$$

3.3 On the existence of an optimal control to decoupled forward-backward stochastic differential equations in the G-framework

3.3.1 Presentation of the problem and hypothesis

Let $\Omega_T = C_0([t,T], \mathbb{R})$ be the space of real-valued continuous functions on $[t,T]$, where $T > 0$. $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sub-linear expectation space and $\mathcal{U}$ be a compact metric space. For any initial condition $(t,x) \in [0,T] \times \mathbb{R}^n$ and any admissible control $u := (u_s, s \in [t,T]) \in \mathcal{U}(t)$, we set the deterministic functions b, σ, h, f, g and Φ as follow:

$$b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n; \qquad h : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \qquad \sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d};$$

$$f:[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times U\to\mathbb{R};\qquad g:[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times U\to\mathbb{R},$$
$$\Phi:\mathbb{R}^n\to\mathbb{R}.$$

Then, consider the following controlled decoupled G-FBSDE system:

$$\begin{cases} dX_s^u = b(s,X_s^u,u_s)ds+\sigma(s,X_s^u)dW_s+h(s,X_s^u,u_s)d\langle W\rangle_s,\\ dY_s^u=-f(s,X_s^u,Y_s^u,Z_s^u,u_s)ds-g(s,X_s^u,Y_s^u,Z_s^u,u_s)d\langle W\rangle_s+Z_s^udW_s+dM_s^u,\ s\in[t,T],\\ X_t^u=x,\ Y_T^u=\Phi(X_T^u), M_t^u=0,\end{cases} \tag{3.18}$$

where $s\in[t,T]$ and $\langle W\rangle$ denotes the quadratic variation of the 1-dimensional G-*Brownian* motion $W=(W_s)_{s\geq 0}$, and (X^u,Y^u,Z^u,M^u) denotes the solution of the G-FBSDE, with M^u is a decreasing G-martingale, and $X^u\in M_G^2([0,T])$ and $(Y^u,Z^u,M^u)\in\mathfrak{S}_G^2(0,T)$.
Let consider the following 0:

Hypothesis(H)

(H1)
The functions b,h,σ,f and $g\in M_G^2([0,T];\mathbb{R}^n)$ for each $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d$, and $\Phi(x)\in L_G^\beta(\Omega_T)$, with $\beta>1$.

(H2)

a) For every fixed $(x,\upsilon)\in\mathbb{R}^n\times\mathcal{U}(t)$, $b(\cdot,x,\upsilon)$, $h_{ij}(\cdot,x,\upsilon)$, and $\sigma_j(\cdot,x)$ are continuous in t;

b) b,h_{ij},σ_j are given functions satisfying $b(.,x,\upsilon),h_{ij}(.,x,\upsilon),\sigma_j(.,x)\in M_G^2([0,T],\mathbb{R}^n)$;

c) There exists a constant $L>0$ For each $s\in[t,T]$, for each fixed control $u\in\mathcal{U}(s)$, x, $x'\in\mathbb{R}^n$, for $\phi=b,\ h_{ij}$:
$$\left|\phi(s,x;u)-\phi(s,x',u)\right|\leq L\left|x-x'\right|.$$
We suppose also that σ_j is *Lipschitz* in x.

(H3)

a) There exists some $\beta>2$ such that for any y,z,u, we have $f(\cdot,\cdot,y,z,u),g_{ij}(\cdot,\cdot,y,z,u)\in M_G^\beta([0,T],\mathbb{R}^n)$.

b) There exists some $L>0$, for $s\in[t,T]$, for each fixed control $u\in\mathcal{U}(s)$, y, $y'\in\mathbb{R}$, z, $z'\in\mathbb{R}^d$, such that
$$|f(t,x,y,z,u)-f(t,x,y',z',u)|+\sum_{i,j=1}^d|g_{ij}(t,x,y,z,u)-g_{ij}(t,x,y',z',u)|\leq L(|y-y'|+|z-z'|).$$

c) $f(\cdot,x,y,z,u), g(\cdot,x,y,z,u)$ are continuous in $s \in [t,T]$, for every fixed (x,y,z,u)

d) There exist a constant $L>0$, for $s\in[t,T]$, $u\in U(s)$, for $x\in\mathbb{R}^n$, $y,y'\in\mathbb{R}$, $z,z'\in\mathbb{R}^d$, we have

$$|\Phi(x)-\Phi(x')|\leq L(|x-x'|),$$

$$|\ f(t,x,y,z,u)-f(t,x,y',z',u)\ |+|g(t,x,y,z,u)-g(t,x,y',z',u)|\ \ \leq L(|y-y'|+|z-z'|).$$

(H4) The functions b,σ,h,f,g,Φ are bounded.

For every fixed initial time $s\in[t,T]$, and initial state $x\in\mathbb{R}^n$, and for every fixed control $u\in\mathcal{U}(s)$. Under hypothesis $(H1)$, the forward stochastic differential equation in (3.18) has a unique solution $X^{t,x;u}$, see [82]. Moreover, under hypothesis $(H4)$ the backward stochastic differential equation of (3.18) has a unique solution noted by: $Y^{t,x;u}, Z^{t,x;u}, M^{t,x;u}$, see [46]. Now, for all $(s,x)\in[t,T]\times\mathbb{R}^n$ and $u\in\mathcal{U}(s)$, consider the first component of the solution of the backward stochastic differential equation in (3.18). Since $Y_s^{t,x;u}$ is deterministic function of (s,x), see [47], let defined the value function as follow:

$$V(s,x)=\operatorname*{essinf}_{u\in\mathcal{U}(s)} J(s,x;u).$$

We have also that $V(s,x)$ is the unique viscosity solution of the following G-PDE:

$$\begin{cases}\partial_t V(s,x)+\inf\limits_{u\in\mathcal{U}(t)} H(D_x^2V,D_xV,V,x,s,u)=0,\\ \\ v(T,x)=\Phi(x),\end{cases}\tag{3.19}$$

where

$$\begin{aligned}H(D_x^2V,D_xV,V,x,s,u)&=G(H(D_x^2V,D_xV,V,x,s,u))+\langle b(s,x;u),D_xV\rangle\\&\quad+f(s,x,V,\langle\sigma_1(s,x),D_xV,u\rangle,\ldots,\langle\sigma_d(s,x),D_xV\rangle,u),\end{aligned}$$

and

$$\begin{aligned}H_{ij}(D_x^2V,D_xV,V,x,s,u)=&\langle D_x^2V\sigma_i(s,x),\sigma_j(s,x)\rangle+2\langle D_xV,h_{ij}(s,x;u)\rangle\\&+2g_{ij}(s,x,V,\langle\sigma_1(s,x),D_xV\rangle,\ldots,\langle\sigma_d(s,x),D_xV\rangle,u).\end{aligned}$$

3.3.2 The approximative *Hamilton-Jacobi-Bellman* equation

In this section, we aim to determine explicitly an optimal feedback control process from a sequence of stochastic control problems which his value functions converge to that of our problem.

The coefficients of our original control problem are not smooth enough to get a smooth solution; we replace the coefficients by their mollification.
Let define the mollification of a given function as follow:

Definition 3.3.1. *For any integer $m \geq 1$, we consider $\varphi : \mathbb{R}^m \to \mathbb{R}$ verified some properties*
(i) φ *be a non-negative smooth function.*
(ii) $supp(\varphi) \subset B_{\mathbb{R}^m}(0,1)$ *(the support of φ included in the unit ball of $\mathbb{R}^m$).*
(iii) $\int_{\mathbb{R}^m} \varphi(\xi)\, d\xi = 1.$
We can define for any Lipschitz function $l : \mathbb{R}^m \to \mathbb{R}$:

$$l_\delta(\xi) = \delta^{-m} \int_{\mathbb{R}^m} l\left(\xi - \xi'\right) \varphi\left(\delta^{-1}\xi'\right) d\xi', \quad \xi \in \mathbb{R}^m,\, \delta > 0.$$

It is called the mollification of l.

Properties 3.4. *The mollification function verifies the following properties*
(i) $|l_\delta(\xi) - l(\xi)| \leq C_l \delta$
(ii) $|l_\delta(\xi) - l_{\delta'}(\xi)| \leq C_l |\delta - \delta'|,$
(iii) $|l_\delta(\xi) - l_\delta(\xi')| \leq C_l |\xi - \xi'|,$
for all $\xi, \xi' \in \mathbb{R}^m$, $\delta, \delta' > 0$,
where C_l denotes the Lipschitz constant of l independently of δ.

Proof.

$$\begin{aligned} |l_\delta(\xi) - l_\delta(\xi')| &= \delta^{-m} \int_{\mathbb{R}^m} |l(\xi - x)\varphi(\delta^{-1}x) - l(\xi' - x)\varphi(\delta^{-1}x)| dx \\ &\leq \delta^{-m} \int_{\mathbb{R}^m} \varphi(\delta^{-1}x) C_l |\xi - \xi'| dx \\ &\leq \delta^{-m} C_l |\xi - \xi'| \int_{\mathbb{R}^m} \varphi(\delta^{-1}x) dx \end{aligned}$$

we pose $M = \delta^{-1}x$, then

$$\begin{aligned} dM &= \delta^{-m} dx \\ dx &= \delta^m dM \end{aligned}$$

Finally,

$$\begin{aligned} |l_\delta(\xi) - l_\delta(\xi')| &\leq C_l |\xi - \xi'| \int_{\mathbb{R}^m} \varphi(M) dM \\ &\leq C_l |\xi - \xi'| \end{aligned}$$

Then $$|l_\delta(\xi) - l_\delta(\xi')| \leq C_l |\xi - \xi'|, .$$ □

Definition 3.3.2. *For each $\delta \in (0,1]$, we denote by $b_\delta, \sigma_\delta, f_\delta$ and Φ_δ the mollification of the functions b, σ, f and Φ, respectively, introduced in Section (3.3.1), with $l = b(.,v)$, $\sigma(.,v)$, $f(.,v)$ or $\Phi(.)$.*

Now, let (**H**) hold, and $\delta \in (0,1]$ be an arbitrarily fixed number. We define the function F^δ by:

$$F^\delta_{ij}(t,x,V,\partial_x V,\partial^2_{xx}V,v) = \langle \partial^2_{xx}V\sigma^\delta_i(t,x),\sigma^\delta_j(t,x)\rangle + 2\langle \partial_x V, h^\delta_{ij}(t,x,V,v)\rangle$$

$$+2g^\delta_{ij}(t,x,V,\langle\sigma^\delta_1(t,x),D_xV\rangle\ldots\langle\sigma^\delta_d(t,x),D_xV\rangle);$$

and the function $H^\delta(t,x,V,\partial_x V,\partial^2_{xx}V,u)$ by

$$H^\delta(t,x,V,\partial_x V,\partial^2_{xx}V,v) = G^\delta(F^\delta(t,x,V,\partial_x V,\partial^2_{xx}V,v))+\langle b_\delta(t,x,v),\partial_x V\rangle+f_\delta(t,x,V,\partial_x V,\partial^2_{xx}V,v),$$

and consider the Hamilton-Jacobi-Bellman equation

$$\begin{cases}\partial V^\delta(t,x) + \inf\limits_{v\in U} H^\delta(t,x,V^\delta,\partial_x V^\delta,\partial^2_{xx}V^\delta,v) = 0\\ V^\delta(T,x) = \Phi_\delta(x),\ x\in\mathbb{R}^n.\end{cases} \tag{3.20}$$

Since H^δ is smooth function and G^δ is uniformly elliptic, then, the unique bounded continuous viscosity solution V^δ of the equation (3.20) according to the regularity result of *Krylov* [62] becomes a classical $C^{1+\frac{l}{2},2+l}([0,T]\times\mathbb{R}^n)$ solution. The regularity of V^δ and the compactness of the control state space $\mathbb{U}$ allow to find a measurable function $v^\delta : [t,T]\times\mathbb{R}^n \mapsto \mathbb{U}$ such that for all $(s,x)\in(t,T]\times\mathbb{R}^n$,

$$H^\delta(x,(V,\partial_x V,\partial^2_{xx}V)(s,x),v^\delta) = \inf_{v\in\mathbb{U}} H^\delta(x,(V^\delta,\partial_x V^\delta,\partial^2_{xx}V^\delta)(s,x),v) \tag{3.21}$$

Lemma 3.3.1. *Assume that the assumptions in **(H)** are satisfied, then :*

$$J^\delta(u^\delta) = V^\delta(t,x) = \underset{u\in\mathbb{U}(t)}{essinf}\, J^\delta(u).$$

Moreover, $u^\delta_s := v^\delta_s(s,X^\delta_s), s\in[t,T]$ is an admissible control.

Proof. Let $(s,x)\in[t,T]\times\mathbb{R}^n$ a fixed arbitrary initial datum and for $\delta\in(0,1]$, let V^δ be the solution of (3.20) and the function v^δ defined by (3.21), let consider the following G-SDE

$$\begin{cases} dX^\delta_s &= b_\delta(s,X^\delta_s,v^\delta(s,X^\delta_s))ds + \sigma_\delta(s,X^u_s)dW_s + h_\delta(s,X^\delta_s,v^\delta((s,X^\delta_s)))d\langle W\rangle_s,\ s\in[0,t]\\ X^\delta_t &= x.\end{cases} \tag{3.22}$$

Since b_δ, h_δ are bounded measurable functions in $(t;x)$ and σ_δ is *Lipschitz* in x, and since $\int_t^T \hat{\mathbb{E}}(|\phi(t,.,.)|^2)dt < \infty$, for $\phi = b$ and h respectively. Then, according to Faizoellah (Theorem

1 page 695) [33] there exists a unique solution $X_t^\delta \in M_G^2([0,T];\mathbb{R}^n)$.
We define Y^δ and Z^δ by

$$Y_s^\delta = V^\delta(s, X_s^\delta), \qquad \textit{and} \qquad Z_s^\delta = \nabla_x V^\delta(s, X_s^\delta)\sigma_\delta(s, X_s^\delta). \tag{3.23}$$

Applying *G-Itô*'s formula to $V^\delta(s, X_s^\delta)$, we have

$$\begin{aligned} V^\delta(s, X_T^\delta) - V^\delta(s, X_t^\delta) = & \int_t^T \partial_x V^\delta(s, X_s^\delta)\sigma_\delta(s, X_s^\delta) dW_s \\ & + \int_t^T \partial_x V^\delta(s, X_s^\delta b_\delta(s, X_s^\delta, v^\delta(s, X_s^\delta)) ds \\ & + \int_t^T [\partial_x V^\delta(s, X_s^\delta) h_\delta(s, X_s^\delta v^\delta(s, X_s^\delta)), v^\delta((s, X_s^\delta)) \\ & + \frac{1}{2}\partial_{xx}^2 V^\delta(s, X_s^\delta)\sigma_\delta(s, X_s^\delta)] d\langle W\rangle_s, \end{aligned}$$

combined this formula with the HJB equation (3.20), we obtain that $(X_s^\delta, Y_s^\delta, Z_s^\delta, v_s^\delta)$ satisfies the following equation:

$$\begin{cases} dX_s^{\delta,u} = b_\delta(s, X_s^{\delta,u}, u_s^\delta) ds + \sigma_\delta(s, X_s^{\delta,u}) dW_s + h_\delta(s, X_s^{\delta,u}, u_s^\delta) d\langle W\rangle_s, \\ dY_s^{\delta,u} = -f_\delta(s, X_s^{\delta,u}, Y_s^{\delta,u}, Z_s^{\delta,u}, v^\delta(s, X_s^\delta)) ds - g_\delta(s, X_s^{\delta,u}, Y_s^{\delta,u}, Z_s^{\delta,u}) d\langle W\rangle_s + Z_s^{\delta,u} dW_s + dM_s^u \\ \qquad X_t^{\delta,u} = x, \qquad Y_T^{\delta,u} = \Phi_\delta(X_T^{\delta,u}), M_t^u = 0. \end{cases} \tag{3.24}$$

By [46] the backward equation of (3.24) has a unique solution $(Y^\delta, Z^\delta) \in S_G^2(0,T) \times H_G^2(0,T)$, therefore $(X^\delta, Y^\delta, Z^\delta) = (X^{\delta,t,x;u^\delta}, Y^{\delta,t,x;u^\delta}, Z^{\delta,t,x;u^\delta})$.
In particular $Y^{\delta,t,x;u^\delta} = Y^\delta = V^\delta(t,x)$. □

According to *Faizallah* [33], $X^{\delta',t',x';u^\delta}$ is the solution of the following forward stochastic differential equation

$$\begin{cases} dX_s^{\delta'} & = b_{\delta'}(s, X^{\delta',t',x';u^\delta}, v^{\delta'}(s, X^{\delta',t',x';u^\delta})) ds + \sigma_{\delta'}(s, X^{\delta',t',x';u^\delta}) dW_s \\ & + h_{\delta'}(s, X^{\delta',t',x';u^\delta}, v^{\delta'}(s, X^{\delta',t',x';u^\delta})) d\langle W\rangle_s,\ s \in [0,t] \\ X_{t'}^{\delta'} & = x'. \end{cases}$$

We extend this solution to the whole interval $[t,T]$, by putting $X_s^{\delta',x';u^\delta} = x'$ for $s < t$. We apply *G-Itô*'s formula to $V^{\delta'}$

$$\begin{cases} \partial V^{\delta'}(t', x) + \inf_{u\in U} H^{\delta'}(t', x, V^{\delta'}, \partial_x V^{\delta'}, \partial_{xx}^2 V^{\delta'}, u) = 0 \\ V^{\delta'}(T, x) = \Phi^{\delta'}(x), \qquad x \in \mathbb{R}^n. \end{cases} \tag{3.25}$$

$$\begin{cases} H^{\delta'}(t,x,V^{\delta'},\partial_x V^{\delta'},\partial^2_{xx}V^{\delta'},v^{\delta}) = \inf_{v\in U} H^{\delta'}(t,x,V^{\delta'},\partial_x V^{\delta'},\partial^2_{xx}V^{\delta'},v). \\ V^{\delta'}(T,x) = \Phi^{\delta'}(x), \qquad x\in\mathbb{R}^n. \end{cases} \tag{3.26}$$

we get

$$\begin{aligned} V^{\delta'}(s,X_T^{\delta',x';u^{\delta}}) - V^{\delta'}(s,X^{\delta',x';u^{\delta}}) &= \int_t^T \partial_x V^{\delta'}(s,X_s^{\delta',x';u^{\delta}})\sigma_{\delta'}(s,X_s^{\delta',x';u^{\delta}})dW_s \\ &+ \int_t^T \partial_x V^{\delta'}(s,X_s^{\delta,x';u^{\delta}})b_{\delta'}(s,X_s^{\delta',x';u^{\delta}},v^{\delta'}(s,X_s^{\delta',x';u^{\delta}}))ds \\ &+ \int_t^T [\partial_x V^{\delta'}(s,X_s^{\delta',x';u^{\delta}})h_{\delta'}(s,X_s^{\delta',x';u^{\delta}},v^{\delta'}((s,X_s^{\delta',x';u^{\delta}}))) \\ &+ \tfrac{1}{2}\partial^2_{xx}V^{\delta'}(s,X_s^{\delta',x';u^{\delta}})\sigma_{\delta'}(s,X_s^{\delta',x';u^{\delta}})]d\langle W\rangle_s \end{aligned}$$

Since,

$$Y^{\delta',t',x'} := V^{\delta'}(s,X_s^{\delta',x';u^{\delta}})$$

$$Z^{\delta',t',x'} := \sigma_{\delta'}(s,X_s^{\delta',x';u^{\delta}})\nabla_x V^{\delta'}(s,X_s^{\delta',x';u^{\delta}}).$$

and,

$$\begin{cases} dY^{\delta',t',x'} &= -\tilde{f}_{\delta',t',x'}ds - g_{\delta}(s,X^{\delta',t',x'},Y^{\delta',t',x'},Z^{\delta',t',x'})d\langle W\rangle_s + Z_s^{\delta',t',x'}dW_s + dM_s \\ Y_T^{\delta',t',x'} &= \Phi_{\delta'}(X_T^{\delta',t',u}). \end{cases} \tag{3.27}$$

where

$$\tilde{f}_{\delta',t',x'} = G^{\delta'}(F^{\delta'}(t,x,V^{\delta'},\partial_x V^{\delta'},\partial^2_{xx}V^{\delta'},u)) + \langle b_{\delta'}(t,x,V^{\delta'},u),\partial_x V^{\delta'}\rangle$$

and from the HJB equation (3.25) with the classical solution $V^{\delta'}$ we observe that

$$\tilde{f}_{\delta',t',x'} \le f_{\delta',t',x'}(s,X_s^{\prime u},Y_s^{\prime u},Z_s^{\prime u},u_s^{\delta}),$$

so, by the comparison theorem(Theorem 3.6 page 1183) [47], we have

$$Y^{\delta',x'} \le Y^{\delta',x';u^{\delta}}$$

Lemma 3.3.2. *Assume that assumptions in (**H**) hold, then there exists a non-negative constant $\bar{C}$, only depending on the* Lipshitz *constants of the coefficients and the time T, such that:*

$$|V_s^{\delta'} - V_s^{\delta}|^2 \le \bar{C}|\delta' - \delta|^2. \tag{3.28}$$

Proof. We start this proof by some notations. Let

$$X^{\delta,t,x;u^{\delta}} = X, \qquad Y^{\delta,t,x;u^{\delta}} = Y, \qquad Z^{\delta,t,x;u^{\delta}} = Z,$$

and

$$X^{\delta',t,x;u^{\delta'}} = X', \qquad Y^{\delta',t,x;u^{\delta'}} = Y', \qquad Z^{\delta',t,x;u^{\delta'}} = Z'.$$

Applying *Itô*'s formula to $|Y'_s - Y_s|^2$, then

$$\begin{aligned}
|Y'_s - Y_s|^2 + \int_t^T |Z'_s - Z_s|^2 d\langle W\rangle_s = -(&\int_t^T 2|Y'_s - Y_s||Z'_s - Z_s|dW_s + \int_t^T |Y'_s - Y_s|^2 dM_s)\\
&+\int_t^T 2|Y'_s - Y_s|(f_\delta(s, X_s, Y_s, Z_s, u^\delta) - f_{\delta'}(s, X'_s, Y'_s, Z'_s, u^{\delta'}))ds\\
&+\int_t^T 2|Y'_s - Y_s|(g_\delta(s, X_s, Y_s) - g_{\delta'}(s, X'_s, Y'_s))d\langle W\rangle_s\\
&+|\varphi_\delta(X'_T) - \varphi_{\delta'}(X_T)|^2.
\end{aligned}$$

Let $J_s = \int_t^T 2|Y'_s - Y_s||Z'_s - Z_s|dW_s + \int_t^T |Y'_s - Y_s|^2 dM_s$; then,

$$\begin{aligned}
|Y'_s - Y_s|^2 + J_s = |\varphi_\delta(X'_T) &- \varphi_{\delta'}(X_T)|^2\\
&+\int_t^T 2|Y'_s - Y_s|(f_\delta(s, X_s, Y_s, Z_s, u^\delta) - f_{\delta'}(s, X'_s, Y'_s, Z'_s, u^{\delta'}))ds\\
&+\int_t^T 2|Y'_s - Y_s|(g_\delta(s, X_s, Y_s) - g_{\delta'}(s, X'_s, Y'_s))d\langle W\rangle_s,
\end{aligned}$$

according to [46] the process J_s is a G-martingale, Then

$$\begin{aligned}
\hat{\mathbb{E}}(|Y'_s - Y_s|^2) \leq \hat{\mathbb{E}}(&|\varphi_{\delta'}(X'_T) - \varphi_\delta(X_T)|^2)\\
&+\hat{\mathbb{E}}(\int_t^T 2|Y'_s - Y_s|(f_\delta(s, X_s, Y_s, Z_s, u^\delta) - f_{\delta'}(s, X'_s, Y'_s, Z'_s, u^{\delta'}))ds)\\
&+\hat{\mathbb{E}}(\int_t^T 2|Y'_s - Y_s|(g_\delta(s, X_s, Y_s) - g_{\delta'}(s, X'_s, Y'_s))d\langle W\rangle_s).
\end{aligned}$$

By *young*'s inequality, we have

$$\begin{aligned}
\hat{\mathbb{E}}(|Y'_s - Y_s|^2) \leq \hat{\mathbb{E}}(&|\varphi_{\delta'}(X'_T) - \varphi_\delta(X'_T) + \varphi_\delta(X'_T) - \varphi_\delta(X_T)|^2)\\
&+\hat{\mathbb{E}}(\int_t^T \frac{1}{\epsilon}|Y'_s - Y_s|^2 + \epsilon|f_\delta(s, X_s, Y_s, Z_s, u^\delta) - f_{\delta'}(s, X'_s, Y'_s, Z'_s, u^{\delta'})|^2 ds)\\
&+\hat{\mathbb{E}}(\int_t^T \frac{1}{\epsilon_1}|Y'_s - Y_s|^2 + \epsilon_1|g_\delta(s, X_s, Y_s) - g_{\delta'}(s, X'_s, Y'_s)|^2 d\langle W\rangle_s).
\end{aligned}$$

Using the *BDG* inequality under G-expectation [85](Lemma 2.18) with $p = 1$, it yelds

$$\hat{\mathbb{E}}(|Y_s' - Y_s|^2) \leq 2\hat{\mathbb{E}}(|\varphi_{\delta'}(X_T') - \varphi_\delta(X_T')|^2) + 2\hat{\mathbb{E}}(|\varphi_\delta(X_T') - \varphi_\delta(X_T)|^2)$$
$$+\tfrac{1}{\epsilon}\int_t^T \hat{\mathbb{E}}(|Y_s' - Y_s|^2) + \epsilon\hat{\mathbb{E}}(\int_t^T |f_\delta(s, X_s, Y_s, Z_s, u^\delta) - f_{\delta'}(s, X_s', Y_s', Z_s', u^{\delta'})|^2 ds)$$
$$+\tfrac{\underline{l}+\bar{l}}{4\epsilon_1}\int_t^T \hat{\mathbb{E}}(|Y_s' - Y_s|^2)ds + \frac{\epsilon_1(\underline{l}+\bar{l})}{4}\hat{\mathbb{E}}(\int_t^T |g_\delta(s, X_s, Y_s) - g_{\delta'}(s, X_s', Y_s')|^2 ds);$$

$$\hat{\mathbb{E}}(|Y_s' - Y_s|^2) \leq 2\hat{\mathbb{E}}(|\varphi_{\delta'}(X_T') - \varphi_\delta(X_T')|^2) + 2\hat{\mathbb{E}}(|\varphi_\delta(X_T') - \varphi_\delta(X_T)|^2)$$
$$+\tfrac{1}{\epsilon}\int_t^T \hat{\mathbb{E}}(|Y_s' - Y_s|^2)ds + \epsilon\hat{\mathbb{E}}(\int_t^T |f_\delta(s, X_s, Y_s, Z_s, u^\delta) - f_\delta(s, X_s', Y_s', Z_s', u^\delta)$$
$$+f_\delta(s, X_s', Y_s', Z_s', u^\delta) - f_{\delta'}(s, X_s', Y_s', Z_s', u^\delta)$$
$$+f_{\delta'}(s, X_s', Y_s', Z_s', u^\delta) - f_{\delta'}(s, X_s', Y_s', Z_s', u^{\delta'})|^2 ds)$$
$$+\tfrac{\underline{l}+\bar{l}}{4\epsilon_1}\int_t^T \hat{\mathbb{E}}(|Y_s' - Y_s|^2)ds + \frac{\epsilon_1(\underline{l}+\bar{l})}{4}\hat{\mathbb{E}}(\int_0^T |g_\delta(s, X_s, Y_s) - g_{\delta'}(s, X_s', Y_s')|^2 ds).$$

Using the fact the function f is K-*Lipshitz*, bounded by b_f and by the properties of the mollifier function, we have:

$$\hat{\mathbb{E}}(|Y_s' - Y_s|^2) \leq 2k^2|\delta' - \delta|^2 + 2k^2\hat{\mathbb{E}}(|X_T' - X_T|^2)$$
$$+\tfrac{1}{\epsilon}\int_t^T \hat{\mathbb{E}}(|Y_s' - Y_s|^2)ds + 6k^2\epsilon\hat{\mathbb{E}}(\int_t^T |(|X_s - X_s'|^2 + |Y_s - Y_s'|^2 + |Z_s - Z_s'|^2)ds)$$
$$+2k^2(T-t)\epsilon|\delta' - \delta|^2 + 2k^2(T-t)\epsilon b_f$$
$$+\tfrac{\underline{l}+\bar{l}}{4\epsilon_1}\int_t^T \hat{\mathbb{E}}(|Y_s' - Y_s|^2)ds + \frac{\epsilon_1(\underline{l}+\bar{l})}{4}\hat{\mathbb{E}}(\int_t^T |g_\delta(s, X_s, Y_s) - g_{\delta'}(s, X_s, Y_s)|^2 ds)$$
$$+\tfrac{\epsilon_1(\underline{l}+\bar{l})}{4}\hat{\mathbb{E}}(\int_t^T |g_{\delta'}(s, X_s, Y_s) - g_{\delta'}(s, X_s', Y_s')|^2 ds);$$

so, we obtain

$$\begin{aligned}
\hat{\mathbb{E}}(|Y'_s - Y_s|^2) \leq{}& 2k^2|\delta' - \delta|^2 + 2k^2\hat{\mathbb{E}}(|X'_T - X_T|^2) \\
&+\tfrac{1}{\epsilon}\int_t^T \hat{\mathbb{E}}(|Y'_s - Y_s|^2)ds + 6k^2\epsilon\hat{\mathbb{E}}(\int_t^T |X_s - X'_s|^2 + |Y_s - Y'_s|^2 + |Z_s - Z'_s|^2 ds) \\
&+2k^2(T-t)\epsilon|\delta' - \delta|^2 + 2k^2(T-t)\epsilon b_f \\
&+\tfrac{\underline{l}+\bar{l}}{4\epsilon_1}\int_t^T \hat{\mathbb{E}}(|Y'_s - Y_s|^2)ds + (T-t)\frac{\epsilon_1(\underline{l}+\bar{l})k^2}{4}|\delta - \delta'|^2 \\
&+\tfrac{2k^2\epsilon_1(\underline{l}+\bar{l})}{4}\hat{\mathbb{E}}(\int_t^T |X_s - X'_s|^2 + |Y_s - Y'_s|^2 ds);
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbb{E}}(|Y'_s - Y_s|^2) \leq{}& \left(2k^2 + 6k^2(T-t)\epsilon + (T-t)\tfrac{2\epsilon_1(\underline{l}+\bar{l})k^2}{4}\right)|\delta' - \delta|^2 + 2k^2\hat{\mathbb{E}}(|X'_T - X_T|^2) \\
&+\left(\tfrac{1}{\epsilon} + 2k^2\epsilon + \tfrac{\underline{l}+\bar{l}}{4\epsilon_1} + \tfrac{2k^2\epsilon_1(\underline{l}+\bar{l})}{4}\right)\int_t^T \hat{\mathbb{E}}(|Y'_s - Y_s|^2)ds + 2k^2(T-t)\epsilon b_f \\
&+6k^2\epsilon\hat{\mathbb{E}}(\int_t^T |Z_s - Z'_s|^2 ds) + (6k^2\epsilon + \frac{2k^2\epsilon_1(\underline{l}+\bar{l})}{4})\hat{\mathbb{E}}(\int_0^T |X_s - X'_s|^2 ds).
\end{aligned} \tag{3.29}$$

On the other hand we have:

$$\begin{aligned}
X_t - X'_t ={}& \int_0^t (b_\delta(s, X_s, u^\delta) - b_{\delta'}(s, X'_s, u^{\delta'}))ds + \int_0^t (\sigma_\delta(s, X_s) - \sigma_{\delta'}(s, X'_s))dW_s \\
&+\int_0^t (h_\delta(s, X_s, u^\delta) - h_{\delta'}(s, X'_s, u^{\delta'}))d\langle W\rangle_s.
\end{aligned}$$

We apply *Itô*'s formula to $|X_s - X'_s|^2$, Here X_t and X'_t have the same initial conditions, so $|X_0 - X'_0| = 0$, then

$$\begin{aligned}
|X_t - X'_t|^2 ={}& \int_0^t 2|X_s - X'_s|(b_\delta(s, X_s, u^\delta) - b_{\delta'}(s, X'_s, u^{\delta'}))ds \\
&+\int_0^t 2|X_s - X'_s|(\sigma_\delta(s, X_s) - \sigma_{\delta'}(s, X'_s))dW_s \\
&+\int_0^t 2|X_s - X'_s|(h_\delta(s, X_s, u^\delta) - h_{\delta'}(s, X'_s, u^{\delta'})) + (\sigma_\delta(s, X_s) - \sigma_{\delta'}(s, X'_s))^2 d\langle W\rangle_s.
\end{aligned}$$

By *Young*'s inequality and simple calculations , we have for some positive constant $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$$\begin{aligned}
|X_t - X'_t|^2 \leq{}& \int_0^t \left(\frac{1}{\varepsilon_1}|X_s - X'_s|^2 + \varepsilon_1|b_\delta(s, X_s, u^\delta) - b_{\delta'}(s, X'_s, u^{\delta'})|^2\right) ds \\
&+\int_0^t 2|X_s - X'_s|(\sigma_\delta(s, X_s) - \sigma_{\delta'}(s, X'_s))dW_s \\
&+\int_0^t \varepsilon_2|X_s - X'_s|^2 + |h_\delta(s, X_s, u^\delta) - h_{\delta'}(s, X'_s, u^{\delta'})|^2 + (\sigma_\delta(s, X_s) - \sigma_{\delta'}(s, X'_s))^2 d\langle W\rangle_s.
\end{aligned}$$

Since,

$$\hat{\mathbb{E}}(\int_0^t \varepsilon_3 |X_s - X_s'|^2 + \frac{1}{\varepsilon_3}|\sigma_\delta(s, X_s) - \sigma_{\delta'}(s, X_s')|^2 dW_s) = 0,$$

we get

$$\begin{aligned}\hat{\mathbb{E}}(|X_t - X_t'|^2) \leq \hat{\mathbb{E}}\left(\int_0^t \left(\frac{1}{\varepsilon_1}|X_s - X_s'|^2 + \varepsilon_1 |b_\delta(s, X_s, u^\delta) - b_{\delta'}(s, X_s', u^{\delta'})|^2\right) ds\right) \\ +\hat{\mathbb{E}}(\int_0^t \frac{1}{\varepsilon_2}|X_s - X_s'|^2 + \varepsilon_2(|h_\delta(s, X_s, u^\delta) - h_{\delta'}(s, X_s', u^{\delta'})|^2 \\ +(\sigma_\delta(s, X_s) - \sigma_{\delta'}(s, X_s'))^2) d\langle W\rangle_s).\end{aligned}$$

Using the BDG inequality under G-framework [85], for $p = 1$, we obtain

$$\begin{aligned}\hat{\mathbb{E}}(|X_t - X_t'|^2) \leq (\tfrac{1}{\varepsilon_1} + \tfrac{(\bar{l}+l)}{4\varepsilon_2}) \int_0^t \hat{\mathbb{E}}(|X_s - X_s'|^2) + \varepsilon_1 \hat{\mathbb{E}}(\int_t^T |b_\delta(s, X_s, u^\delta) - b_{\delta'}(s, X_s', u^{\delta'})|^2 ds) \\ +\tfrac{(\bar{l}+l)\varepsilon_2}{4}\hat{\mathbb{E}}(\int_0^t |h_\delta(s, X_s, u^\delta) - h_{\delta'}(s, X_s', u^{\delta'})|^2 ds) \\ +\tfrac{(\bar{l}+l)\varepsilon_2}{4}\hat{\mathbb{E}}(\int_0^t |\sigma_\delta(s, X_s) - \sigma_{\delta'}(s, X_s')|^2 ds);\end{aligned}$$

therefore,

$$\begin{aligned}\hat{\mathbb{E}}(|X_t - X_t'|^2) \leq (\tfrac{1}{\varepsilon_1} + \tfrac{(\bar{l}+l)}{4\varepsilon_2}) \int_0^t \hat{\mathbb{E}}(|X_s - X_s'|^2) ds \\ +\varepsilon_1 \hat{\mathbb{E}}(\int_0^t |b_\delta(s, X_s, u^\delta) - b_{\delta'}(s, X_s, u^\delta)|^2 ds \\ + \int_0^t |b_{\delta'}(s, X_s, u^\delta) - b_{\delta'}(s, X_s', u^{\delta'})|^2 ds) \\ +\tfrac{(\bar{l}+l)\varepsilon_2}{4}\hat{\mathbb{E}}(\int_0^t |h_\delta(s, X_s, u^\delta) - h_\delta(s, X_s',^{\delta'})|^2 ds) \\ +\tfrac{(\bar{l}+l)\varepsilon_2}{4}\hat{\mathbb{E}}(\int_0^t |h_\delta(s, X_s', u^{\delta'}) - h_{\delta'}(s, X_s', u^{\delta'})|^2 ds) \\ +\tfrac{(\bar{l}+l)\varepsilon_2}{4}\hat{\mathbb{E}}(\int_0^t |\sigma_\delta(s, X_s) - \sigma_\delta(s, X_s')|^2 ds) \\ +\tfrac{(\bar{l}+l)\varepsilon_2}{4}\hat{\mathbb{E}}(\int_0^t |\sigma_\delta(s, X_s') - \sigma_{\delta'}(s, X_s')|^2 ds);\end{aligned}$$

$$
\begin{aligned}
\hat{\mathbb{E}}(|X_t - X'_t|^2) \leq (\tfrac{1}{\varepsilon_1} + \tfrac{(\bar{l}+\underline{l})}{4\varepsilon_2}) \int_0^t \hat{\mathbb{E}}(|X_s - X'_s|^2)ds \\
+\varepsilon_1(tk^2|\delta - \delta'|^2 + 2k^2 \int_0^t |X_s - X'_s|^2 ds) \\
+\tfrac{4(\bar{l}+\underline{l})k^2\varepsilon_2}{4}(\int_0^t \hat{\mathbb{E}}(|X_s - X'_s|^2)ds + \frac{2(\bar{l}+\underline{l})\varepsilon_2 t}{4} b_h^2 \\
+\tfrac{(\bar{l}+\underline{l})k^2\varepsilon_2}{4} \int_0^t \hat{\mathbb{E}}(|X_s - X'_s|^2)ds + \frac{(\bar{l}+\underline{l})\varepsilon_2 t}{4} b_\sigma^2;
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbb{E}}(|X_t - X'_t|^2) \leq C_1 \int_0^t \hat{\mathbb{E}}(|X_s - X'_s|^2)ds \\
+\varepsilon_1 tk^2|\delta - \delta'|^2 + b_{h\sigma}\varepsilon_2 t.
\end{aligned}
$$

Let

$$b_{h\sigma} = (\frac{2(\bar{l}+\underline{l})b_h^2}{4} + \frac{(\bar{l}+\underline{l})}{4} b_\sigma^2);$$

$$C_1 = (\frac{1}{\varepsilon_1} + \frac{(\bar{l}+\underline{l})}{\varepsilon_2} + \frac{(\bar{l}+\underline{l})k^2}{\varepsilon_2 4} + \varepsilon_1 k^2 + \frac{(\bar{l}+\underline{l})k^2}{\varepsilon_3 4}).$$

We choose ε_1 and ε_2 small as follow

$$\varepsilon_1 \leq \frac{k^2}{b_{h\sigma}},$$

so,

$$b_{h\sigma}\varepsilon_2 t \leq \varepsilon_1 tk^2|\delta - \delta'|^2.$$

Then,

$$\hat{\mathbb{E}}(|X_t - X'_t|^2) \leq C_1 \int_0^t \hat{\mathbb{E}}(|X_s - X'_s|^2)ds + 3\varepsilon_1 tk^2|\delta - \delta'|^2.$$

We applied the *Gronwall* inequality we obtain:

$$\hat{\mathbb{E}}(|X_t - X'_t|^2) \leq e^{C_1 t}(3\varepsilon_1 tk^2|\delta - \delta'|^2). \tag{3.30}$$

By the same steps done above, we get

$$\hat{\mathbb{E}}(|X_T - X'_T|^2) \leq e^{C'_1 T}(3\varepsilon'_1 Tk^2|\delta - \delta'|^2). \tag{3.31}$$

By applying *Itô*'s formula to $|Y'_s - Y_s|^2$, we extract the following inequality

$$\hat{\mathbb{E}}(\int_t^T |Z'_s - Z_s|^2 d\langle W\rangle_s) \leq \left(2k^2 + 6k^2(T-t)\rho + (T-t)\tfrac{2\rho_1(\underline{l}+\bar{l})k^2}{4}\right)|\delta' - \delta|^2 + 2k^2\hat{\mathbb{E}}(|X'_T - X_T|^2)$$
$$+(\tfrac{1}{\rho} + 2k^2\rho + \tfrac{\underline{l}+\bar{l}}{4\rho_1} + \tfrac{k^2\rho_1(\underline{l}+\bar{l})}{4})\int_t^T \hat{\mathbb{E}}(|Y'_s - Y_s|^2)ds + 2k^2(T-t)\rho b_f$$
$$+2k^2\rho\hat{\mathbb{E}}(\int_0^T |Z_s - Z'_s|^2 ds) + (2k^2\rho + \frac{k^2\rho_1(\underline{l}+\bar{l})}{4})\hat{\mathbb{E}}(\int_0^T |X_s - X'_s|^2 ds).$$

Using the fact that

$$\hat{\mathbb{E}}(|\int_t^T \xi dW_s|^2) = \hat{\mathbb{E}}(\int_t^T |\xi|^2 d\langle W\rangle_s)$$

and the *BDG* inequality under *G*-expectation [85](Lemma2. 19) for $p = 2$, we get

$$\hat{\mathbb{E}}(\int_0^T |Z'_s - Z_s|^2 ds) \leq \tfrac{1}{\underline{l}c_2}\left(2k^2 + 2k^2(T-t)\rho + (T-t)\tfrac{\rho_1(\underline{l}+\bar{l})k^2}{4}\right)|\delta' - \delta|^2 + \tfrac{2k^2}{\underline{l}c_2}\hat{\mathbb{E}}(|X'_T - X_T|^2)$$
$$+\tfrac{2k^2}{\underline{l}c_2}(\tfrac{1}{\rho} + 2k^2\rho + \tfrac{\underline{l}+\bar{l}}{4\rho_1} + \tfrac{k^2\rho_1(\underline{l}+\bar{l})}{4})\int_0^T \hat{\mathbb{E}}(|Y'_s - Y_s|^2)ds + \frac{2k^2}{\underline{l}c_2}(T-t)\rho b_f$$
$$+\tfrac{2k^2}{\underline{l}c_2}\rho\hat{\mathbb{E}}(\int_0^T |Z_s - Z'_s|^2 ds) + \frac{1}{\underline{l}c_2}(2k^2\rho + \frac{k^2\rho_1(\underline{l}+\bar{l})}{4})\hat{\mathbb{E}}(\int_0^T |X_s - X'_s|^2 ds).$$

We choose

$$\rho \leq \inf(\frac{\underline{l}c_2}{8k^2}, \frac{2k^2 + 2k^2}{2k^2 b_f});$$

which implies

$$(1 - \tfrac{2k^2}{\underline{l}c_2})\hat{\mathbb{E}}(\int_0^T |Z'_s - Z_s|^2 ds) \leq C_5|\delta' - \delta|^2 + \tfrac{2k^2}{\underline{l}c_2}\hat{\mathbb{E}}(|X'_T - X_T|^2)$$
$$+C_y\int_0^T \hat{\mathbb{E}}(|Y'_s - Y_s|^2)ds + \frac{2k^2}{\underline{l}c_2}(T-t)\rho b_f$$
$$+C_x\hat{\mathbb{E}}(\int_0^T |X_s - X'_s|^2 ds).$$

Let

$$C_5 = \frac{1}{\underline{l}c_2}(2k^2 + 2k^2(T-t)\rho + (T-t)\frac{\rho_1(\underline{l}+\bar{l})k^2}{4}),$$

$$C_y = \frac{2k^2}{\underline{l}c_2}(\frac{1}{\rho} + 2k^2\rho + \frac{\underline{l}+\bar{l}}{4\rho_1} + \frac{k^2\rho_1(\underline{l}+\bar{l})}{4}),$$

$$C_z = (1 - \frac{2k^2}{\underline{l}c_2}) = \frac{3}{4},$$

$$C_x = \frac{1}{\underline{l}c_2}(2k^2\rho + \frac{k^2\rho_1(\underline{l}+\bar{l})}{4}),$$

$$\frac{2k^2}{\underline{l}c_2}(T-t)\rho b_f \le C_5|\delta'-\delta|^2;$$

so,

$$\begin{aligned}\hat{\mathbb{E}}(\int_0^T |Z'_s - Z_s|^2 ds) \le \tfrac{2C_5}{C_z}|\delta'-\delta|^2 + \tfrac{2C_z k^2}{\underline{l}c_2}\hat{\mathbb{E}}(|X'_T - X_T|^2) \\ +\tfrac{C_y}{C_z}\int_0^T \hat{\mathbb{E}}(|Y'_s - Y_s|^2)ds + \frac{C_x}{C_z}\hat{\mathbb{E}}(\int_0^T |X_s - X'_s|^2 ds).\end{aligned} \tag{3.32}$$

By replacing (3.32) in (3.29) and choosing

$$\rho \le \inf(\frac{\underline{l}c_2}{8k^2}, \frac{2k^2+2k^2}{2k^2 b_f}),$$

and let:

$$\bar{C}_1 = (2(2k^2 + 2k^2(T-t)\epsilon + (T-t)\frac{3\epsilon_1(\underline{l}+\bar{l})k^2}{4}) + 6k^2\epsilon + \frac{2C_5}{C_z},$$

$$\bar{C}_2 = (\frac{1}{\epsilon} + 6k^2\epsilon + \frac{\underline{l}+\bar{l}}{4\epsilon_1} + \frac{3k^2\epsilon_1(\underline{l}+\bar{l})}{4}) + \frac{C_y 2k^2\epsilon}{C_z},$$

$$\bar{C}_3 = (2k^2\epsilon + \frac{3k^2\epsilon_1(\underline{l}+\bar{l})}{4}) + \frac{C_x 2k^2\epsilon}{C_z}.$$

So,

$$\begin{aligned}\hat{\mathbb{E}}(|Y'_s - Y_s|^2) \le \bar{C}_1|\delta'-\delta|^2 + (2k^2 + \tfrac{24C_z k^2 k^2\epsilon}{\underline{l}c_2})\hat{\mathbb{E}}(|X'_T - X_T|^2) \\ +\bar{C}_2\int_0^T \hat{\mathbb{E}}(|Y'_s - Y_s|^2)ds + \bar{C}_3\hat{\mathbb{E}}(\int_0^T |X_s - X'_s|^2 ds),\end{aligned} \tag{3.33}$$

we replace (3.30) and (3.31) in the inequality (3.33), we get

$$\begin{aligned}\hat{\mathbb{E}}(|Y'_s - Y_s|^2) \le \bar{C}_1|\delta'-\delta|^2 + (2k^2 + \tfrac{24C_z k^2 k^2\epsilon}{\underline{l}c_2})e^{C'_1 T}(3\varepsilon'_1 Tk^2|\delta-\delta'|^2) \\ +\bar{C}_2\int_0^T \hat{\mathbb{E}}(|Y'_s - Y_s|^2)ds + \bar{C}_3 e^{C_1 t}(3\varepsilon_1 tk^2|\delta-\delta'|^2),\end{aligned} \tag{3.34}$$

$$\begin{aligned}\hat{\mathbb{E}}(|Y'_s - Y_s|^2) \le \left(\bar{C}_1 + (2k^2 + \tfrac{24C_z k^2 k^2\epsilon}{\underline{l}c_2})e^{C'_1 T}(3\varepsilon'_1 Tk^2) + \bar{C}_3 e^{C_1 t}(3\varepsilon_1 tk^2)\right)|\delta-\delta'|^2 \\ +\bar{C}_2\int_0^T \hat{\mathbb{E}}(|Y'_s - Y_s|^2)ds,\end{aligned} \tag{3.35}$$

Let

$$\bar{C} = \left(\bar{C}_1 + (2k^2 + \frac{24C_z k^2 k^2\epsilon}{\underline{l}c_2})e^{C'_1 T}(3\varepsilon'_1 Tk^2) + \bar{C}_3 e^{C_1 t}(3\varepsilon_1 tk^2)\right).$$

By the *Gronwall*'s inequality we get

$$\hat{\mathbb{E}}(|Y'_s - Y_s|^2) \le \bar{C}e^{\bar{C}_2 T}|\delta'-\delta|^2. \tag{3.36}$$

□

Lemma 3.3.3. *If $f_\delta, b_\delta, \Phi_\delta, \sigma_\delta, h_\delta$ and g_δ are bounded $\mathbb{C}^\infty$ functions for every order the derivatives are bounded, then*

$$\begin{cases} \partial V^\delta(s,x) + \inf_{u\in U} \bar{H}^\delta(x, (V^\delta, \partial_x V^\delta, \partial^2_{xx} V^\delta)(s,x), v^\delta(s,x)) = 0 \quad (s,x) \in [t,T] \times \mathbb{R}^n \\ V^\delta(T,x) = \Phi(x), \ x \in \mathbb{R}^n. \end{cases} \tag{3.37}$$

admits a unique solution $V^\delta \in C_b^{1,2}([t,T] \times \mathbb{R}^n)$, and

$$\nabla_x V^\delta \quad \text{and} \quad \nabla^2_{xx} V^\delta \quad \text{are bounded on} \quad [t,T] \times \mathbb{R}^n. \tag{3.38}$$

Moreover, there exists a constant $\bar{C}$ only depending on T and constants $\bar{\Gamma}$ and $\bar{\kappa}$ only depending on K and T, such that

$$\sup_{(s,x)\in[t,T]\times\mathbb{R}^n} |V^\delta(s,x)| \le \bar{C} \tag{3.39}$$

$$\sup_{(s,x)\in[t,T]\times\mathbb{R}^n} |\nabla_x V^\delta(t,x)| \le \bar{\kappa} \tag{3.40}$$

$$\forall (s,s') \in [t,T]^2 \ |V^\delta(s',x) - V^\delta(s,x)| \le \bar{\kappa}|s'-s|^{\frac{1}{2}} \tag{3.41}$$

Proof. Since G satisfying the uniformly elliptic condition, then the unique bounded continuous viscosity solution V^δ of the equation (3.37) is smooth with regularity $C^{1,2}([t,T] \times \mathbb{R}^n)$, in this case we can apply the regularity results by *Krylov* [62](Theorems 6.4.3 and 6.4.4 in [62]). So V^δ satisfies (3.38).
Let define for every $(t;x) \in ([t,T] \times \mathbb{R}^n)$

$$B(t,x) = b_\delta(s, X^{\delta,t,x;u^\delta}, v^{\delta'}(s, X^{\delta',t',x';u^\delta})),$$

$$\Xi(t,x) = \sigma_\delta(s, X^{\delta,t,x;u^\delta})),$$

$$\Theta(t,x) = h_{\delta'}(s, X^{\delta',t',x';u^\delta}, v^{\delta'}(s, X^{\delta',t',x';u^\delta})).$$

For every $t \in [0;T]$, the SDE

$$X_s^{t,x,\delta} = x + \int_0^t B(r, X_r^{t,x,\delta})dr + \int_0^t \Xi(r, X_r^{t,x,\delta})dW_r + \int_0^t \Theta((r, X_r^{t,x,\delta})d\langle W\rangle_r,$$

has a unique solution. We define $Y^{t,x,\delta}$ and $Z^{t,x,\delta}$, $\forall t \le s \le T$

$$Y_s^\delta = V^\delta(s, X_s^\delta), \qquad \text{and} \qquad Z_s^\delta = \nabla_x V^\delta(s, X_s^\delta)\sigma_\delta(s, X_s^\delta).$$

Thus, by applying *Itô*'s formula to the function $(t,x) \to V^\delta(t,x)$ which satisfies the following system

$$\begin{cases} \partial V^\delta(t,x) + \bar{H}^\delta(t,x,V^\delta,\partial_x V^\delta,\partial^2_{xx}V^\delta,u) = 0 \\ V^\delta(T,x) = \Phi^\delta(x), \quad x \in \mathbb{R}^n, \end{cases}$$

We see that $\forall t \le s \le T$

$$\begin{aligned} Y_t^{t,x,\delta} &= \Phi^\delta(X_T^{t,x,\delta}) - \int_t^T f_\delta(s, X_s^{t,x,\delta}, Y_s^{t,x,\delta}, Z_s^{t,x,\delta}, v_s^\delta) ds \\ &- \int_t^T g_\delta(s, X_s^{t,x,\delta}, Y_s^{t,x,\delta}, Z_s^{t,x,\delta}) d\langle W^\delta \rangle_s + \int_t^T Z_s^{t,x,\delta} dW_s^\delta - (M_T - M_t). \end{aligned} \tag{3.42}$$

Subsequently, the process $(X_s^{t,x,\delta}, Y_s^{t,x,\delta}, Z_s^{t,x,\delta})$ is the solution of the G-FBSDE associated to the coefficients $\Phi_\delta, b_\delta, \sigma_\delta, h_\delta, f_\delta, g_\delta$ and to the initial condition (t,x). Now, we apply *Itô*'s formula to the function $(t,x) \to |y|^2$, where $y = Y_s^{t,x,\delta}$, for $t \le s \le T$ and $x \in \mathbb{R}^n$.

$$J'_s = + \int_t^T 2|Y_s||Z_s| dW_s + \int_t^T |y_s|^2 dM_s.$$

$$\begin{aligned} |Y_s|^2 + \int_t^T |Z_s|^2 d\langle W\rangle_s + J'_s = |\varphi_\delta(X_T)|^2 &+ \int_t^T 2|Y_s| f_\delta(s, X_s, Y_s, Z_s, u^\delta) ds \\ &+ \int_t^T 2|Y_s|(g_\delta(s, X_s, Y_s)) d\langle W\rangle_s, \end{aligned}$$

$$\begin{aligned} |Y_t|^2 + J'_s \le |\varphi_\delta(X_T)|^2 &+ \int_t^T 2|Y_s| f_\delta(s, X_s, Y_s, Z_s, u^\delta) ds \\ &+ \int_t^T 2|Y_s|(g_\delta(s, X_s, Y_s)) d\langle W\rangle_s, \end{aligned}$$

if we take the G-expectation of the both sides, we get

$$\begin{aligned} \hat{\mathbb{E}}(\sup_{s\in[t,T]} |Y_s|^2) \le \hat{\mathbb{E}}(\sup_{s\in[t,T]} \{|\varphi_\delta(X_T)|^2 &+ \int_t^T 2|Y_s| f_\delta(s, X_s, Y_s, Z_s, u^\delta) ds \\ &+ \int_t^T 2|Y_s|(g_\delta(s, X_s, Y_s)) d\langle W\rangle_s\}). \end{aligned}$$

We apply *young* and BDG (Lemma 2.19 in [85]) inequalities, we get

$$\begin{aligned} \hat{\mathbb{E}}(\sup_{s\in[t,T]} |Y_s|^2) \le \hat{\mathbb{E}}(C_\varphi^2 &+ \sup_{s\in[t,T]} \{\int_t^T \epsilon |Y_s|^2 + \frac{1}{\epsilon} |f_\delta(s, X_s, Y_s, Z_s, u^\delta)|^2 ds\}) \\ &+ C_2 \bar{l} \hat{\mathbb{E}}(\sup_{s\in[t,T]} \{\int_t^T \epsilon_1 |Y_s|^2 + \frac{1}{\epsilon_1} |g_\delta(s, X_s, Y_s)|^2 ds\}); \end{aligned}$$

$$\hat{\mathbb{E}}(\sup_{s\in[t,T]}|Y_s|^2) \leq C_\varphi^2 + (T-t)\epsilon\hat{\mathbb{E}}(\sup_{s\in[t,T]}|Y_s|^2) + \frac{(T-t)C_f^2}{\epsilon}$$
$$+C_2\bar{l}(T-t)\epsilon_1\hat{\mathbb{E}}(\sup_{s\in[t,T]}|Y_s|^2) + \frac{C_2\bar{l}C_g^2(T-t)}{\epsilon_1}.$$

We choose $\epsilon = \frac{1}{8(T-t)}$ and $\epsilon_1 = \frac{1}{8C_2\bar{l}(T-t)}$, then

$$\hat{\mathbb{E}}(\sup_{s\in[t,T]}|Y_s|^2) \leq \frac{4C_\varphi^2}{3} + \frac{32(T-t)^2C_f^2}{3} + \frac{32C_2^2\bar{l}^2C_g^2(T-t)^2}{3}, \tag{3.43}$$

we conclude that there exist a constant $\bar{C}$, depending only on C_φ, C_f, C_g and T, such that (3.39) is satisfied.

Based on the priori estimate of the supremum norm of $\left(|\nabla_x V(t,x)|^2\right)$ result of *Ladyzhenskaya* and al. (see Theorem 6.1 chapter VII in [63]), we can estimate this quantity on every compact of $[0,T]\times\mathbb{R}^n$, moreover, we can extend this result to the cylinder $[0,T]\times\{x\in\mathbb{R}^n, |x|\leq n\}$ and $[0,T]\times\{x\in\mathbb{R}^n, |x|\leq n+1\}$. The quantity $\sup_{[0,T]\times\{x\in\mathbb{R}^n,|x|\leq n\}}|\nabla_x V(t,x)|^2$ is estimated in terms of $\bar{C}, k$ and d. The distance between $\{x\in\mathbb{R}^n, |x|\leq n\}$ and $\{x\in\mathbb{R}^n, |x|\leq n+1\}$ being equal to 1. There exist a constant $\bar{\Gamma}$ depending only on k and T such that

$$\forall (t,x)\in[0,T]\times\mathbb{R}^n, \quad |V_x^\delta(t,x)| \leq \bar{\Gamma}.$$

It is easy to check that there exist a constant κ such that (see (3.3.2))

$$\begin{aligned} &\hat{\mathbb{E}}(|X_s - X_r|^2) \leq \bar{\kappa}(s-r); \\ &\hat{\mathbb{E}}(\sup_{s\in[r,s]}|Y_s^{t,x,\delta} - Y_r^{t,x,\delta}|^2) \leq \bar{\kappa}(s-r)^2. \end{aligned} \tag{3.44}$$

Hence, using the fact that $Y_r^{\delta,s,x} = V(r, X_r^{\delta,s})$, we obtain

$$\begin{aligned} \hat{\mathbb{E}}(|V^\delta(s,x) - V^\delta(r,x)|^2) &\leq 2\hat{\mathbb{E}}(|V^\delta(s,x) - Y_r^{\delta,s,x}|^2) + 2\hat{\mathbb{E}}(|Y_r^{\delta,s,x} - V^\delta(s,x)|^2) \\ &\leq 2\bar{\kappa}(s-r) + 2\kappa\hat{\mathbb{E}}(|X_s - X_r|^2) \\ &\leq 2\bar{\kappa}(s-r) + 2\bar{\kappa}(s-r), \qquad \text{modifying} \quad \bar{\kappa} \\ &\leq 4\bar{\kappa}(s-r). \end{aligned}$$

Then, (3.41) is proved. □

To be more precise, let do some simple checking

Simple verification of (3.44)

Let $t \le r \le s \le T$

$$\begin{aligned}Y_s^{t,x,\delta} - Y_r^{t,x,\delta} &= -(\int_s^T f_\delta(u, X_u^{t,x,\delta}, Y_u^{t,x,\delta}, Z_u^{t,x,\delta}, v_u^\delta)du - \int_t^T f_\delta(u, X_u^{t,x,\delta}, Y_s^{t,x,\delta}, Z_u^{t,x,\delta}, v_u^\delta)du) \\ &-(\int_s^T g_\delta(u, X_u^{t,x,\delta}, Y_u^{t,x,\delta}, Z_u^{t,x,\delta})d\langle W\rangle_u - \int_t^T g_\delta(u, X_u^{t,x,\delta}, Y_u^{t,x,\delta}, Z_u^{t,x,\delta})d\langle W\rangle_u) \\ &+(\int_s^T Z_u^{t,x,\delta}dW_u - \int_t^T Z_s^{t,x,\delta}dW_u) - ((M_T - M_s) - (M_T - M_r)),\end{aligned}$$

$$\begin{aligned}Y_r^{t,x,\delta} - Y_s^{t,x,\delta} = &- \int_r^s f_\delta(u, X_u^{t,x,\delta}, Y_u^{t,x,\delta}, Z_u^{t,x,\delta}, v_u^\delta)du \\ &- \int_r^s g_\delta(u, X_u^{t,x,\delta}, Y_u^{t,x,\delta}, Z_u^{t,x,\delta})d\langle W\rangle_u \\ &+ \int_r^s Z_u^{t,x,\delta}dW_u + (M_s - M_r),\end{aligned}$$

this looks like (3.42), then, we use the result obtain in (3.43), so

$$\hat{\mathbb{E}}(\sup_{s\in[t,T]} |Y_s^{t,x,\delta} - Y_r^{t,x,\delta}|^2) \le \tfrac{32(s-r)^2C_f^2}{3} + \tfrac{32C_2^2\bar{l}^2C_g^2(s-r)^2}{3}.$$

Then

$$\hat{\mathbb{E}}(\sup_{s\in[r,s]} |Y_s^{t,x,\delta} - Y_r^{t,x,\delta}|^2) \le \bar{\kappa}(s-r)^2. \tag{3.45}$$

At the other hand, we have

$$\begin{aligned}\hat{\mathbb{E}}(|Y_r^{t,x,\delta} - V(r,x)|^2) &= \hat{\mathbb{E}}(V(r, X_r^{t,x,\delta}) - V(r,x)) \\ &\le \bar{\kappa}\hat{\mathbb{E}}(|X_r^{t,x,\delta} - x|^2) \\ &\le \bar{\kappa}\hat{\mathbb{E}}(|X_r^{t,x,\delta} - X_s^{t,x,\delta}|^2).\end{aligned}$$

For $X_s - X_r$, we have

$$\begin{aligned}X_s - X_r = &\int_r^s (b_\delta(u, X_u)du + \int_r^s \sigma_\delta(u, X_u)dW_u \\ &+ \int_r^s h_\delta(u, X_u)d\langle W\rangle_u.\end{aligned}$$

Using the fact that b, σ, h are bounded, the BDG inequalities gives

$$\begin{aligned}\hat{\mathbb{E}}(|\ X_s - X_r\ |^2) &\le 2(s-r)^2C_b^2 + 2C_2\bar{l}(s-r)C_\sigma^2 \\ &+ \tfrac{(\underline{l}+\bar{l})^2}{8}(s-r)^2C_h^2.\end{aligned}$$

Then,

$$\hat{\mathbb{E}}(|\ X_s - X_r\ |^2) \leq \bar{\kappa}(s-r).$$

Since,

$$|V^{\delta'}(t',x') - V^{\delta}(t,x)| \leq |V^{\delta'}(t',x') - V^{\delta}(t',x')| + |V^{\delta}(t',x') - V^{\delta}(s,x)|, \tag{3.46}$$

then, (3.40) and (3.41) implies that

$$|V^{\delta'}(t',x') - V^{\delta}(t,x)| \leq \bar{\kappa}|t-t'|^{\frac{1}{2}} + \Gamma|x-x'|, \tag{3.47}$$

Using (3.28), and we modify the constants if necessary we get

$$|V^{\delta'}(t',x') - V^{\delta}(t,x)| \leq C\left(|t-t'|^{\frac{1}{2}} + |x-x'| + |\delta-\delta'|\right), \tag{3.48}$$

As V^{δ} is bounded in (t,x), we conclude that it converges (as $\delta \to 0$) to a function $\bar{V}$, moreover the *Hamiltonian* H^{δ} converges to H because of the stability of the viscosity solution, in fact $\bar{V}$ is also solution of (3.20). The uniqueness of the solution of equation (3.20) shows that $\bar{V} = V$. This prove that

$$V^{\delta'} \to V \ \text{ as } \ \delta' \to 0.$$

Then,

$$|V^{\delta}(t,x) - V| \leq C\delta, \ \text{ for all } \ \delta \in [0,1) \ \text{ and } \ (t,x) \in [0,T] \times \mathbb{R}^n.$$

3.3.3 Convergence of the approximating control problems

In this section, we show that the approximated problem (3.20) converge to the value function of our original problem, and show that the optimal control of our original stochastic optimal control is the limit of the sequence of the optimal control of the approximated systems. the result is given in the next theorem

Theorem 3.5. *Assume that the assumptions (H) are satisfied. Let $(t,x) \in [0,T] \times \mathbb{R}^n$ and $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers which tends to 0. Then, there exists a process $(\bar{X},\bar{Y},\bar{Z},\bar{M}) \in M_G^2([0,T]) \times \mathfrak{S}_G^2(0,T)$, with $\bar{M}$ is a decreasing martingale and an admissible control $\bar{u} \in \mathcal{U}(t)$, such that:*

1. *There is a subsequence of $(X^{\delta_n}, Y^{\delta_n})_{n\in\mathbb{N}}$ which converges in distribution to $(\bar{X},\bar{Y})$,*
2. *$(\bar{X},\bar{Y},\bar{Z},\bar{M})$ is a solution of the following system*

$$\begin{cases} d\bar{X}_s = b(s,\bar{X}_s,\bar{u}_s)ds + \sigma(s,\bar{X}_s)d\bar{W}_s + h(s,\bar{X}_s,u_s)d\langle W\rangle_s, \\ d\bar{Y}_s = -f(s,\bar{X}_s,\bar{Y}_s,\bar{Z}_s,\bar{u}_s)ds - g(s,\bar{X}_s,\bar{Y}_s,\bar{Z}_s)d\langle W\rangle_s + \bar{Z}_s dW_s + d\bar{M}_s, \\ \bar{X}_t = x \quad \bar{Y}_T = \xi = \Phi(\bar{X}_T), \quad \bar{M}_t = 0 \end{cases} \tag{3.49}$$

3. For every $(t,x) \in [0,T] \times \mathbb{R}^n$, it holds that

$$\bar{Y}_t = V(t,x) = \underset{u \in \mathcal{R}(t)}{essinf} J(t,x;u) \ ,$$

i.e. the relaxed control $\bar{u} \in \mathcal{R}$ is optimal for our original SOC problem (3.49).

In order to prove this theorem, we need the following lemma:

Lemma 3.3.4. *For all $n \in \mathbb{N}$. There exists a constant L such that*

$$\hat{\mathbb{E}}\left(\sup_{s\in[t,T]} |X_s^{\delta_n} - X_s^n|^2\right) \leq L\delta_n^2 \tag{3.50}$$

$$\hat{\mathbb{E}}\left(\sup_{s\in[t,T]} |Y_s^{\delta_n} - Y_s^n|^2\right) \leq L\delta_n^2 \tag{3.51}$$

Proof. Let the sequence of processes (X_s^n, Y_s^n) satisfy the following controlled G-FBSDE

$$\begin{cases} dX_s^n = b(s, X_s^n, Y_s^n, u_s^{\delta_n})ds + \sigma(s, X_s^n)dW_s + h(s, X_s^n, Y_s^n, u_s^{\delta_n})d\langle W\rangle_s, \\ dY_s^n = -f(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n})ds - g(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n))d\langle W\rangle_s \\ \qquad +w_s^{\delta_n}\sigma(X_s^{\delta_n})dW_s + dM_s^n, \\ X_t^n = x \qquad Y_t^n = V^{\delta_n}(t,x)\,, \quad M_t^n = 0. \end{cases} \tag{3.52}$$

Where, $w^{\delta_n} = \nabla_x V^{\delta_n}(t, X_t^{\delta_n})$. Consider now the subsequence $(X_s^{\delta_n}, Y_s^{\delta_n})$ satisfied the following controlled system

$$\begin{cases} dX_s^{\delta_n} = b_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n})ds + \sigma_{\delta_n}(s, X_s^{\delta_n})dW_s + h_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n})d\langle W\rangle_s, \\ dY_s^{\delta_n} = -f_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}), u_s^{\delta_n})ds - g_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}))d\langle W\rangle_s \\ \qquad +w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})dW_s + dM_s^{\delta_n}, \\ X_t^{\delta_n} = x \quad Y_t^{\delta_n} = V^{\delta_n}(t,x)\,, \quad M_t^{\delta_n} = 0 \end{cases} \tag{3.53}$$

We apply the *Itô*'s formula to $|X_s^{\delta_n} - X_s^n|^2$, we get

$$\begin{aligned} |X_t^n - X_t^{\delta_n}|^2 = &\int_0^t 2|X_t^n - X_s^{\delta_n}|(b(s, X_s^n, Y_s^n, u_s^{\delta_n}) - b_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}))ds \\ &+ \int_0^t 2|X_t^n - X_s^{\delta_n}|(\sigma(s, X_s^n) - \sigma_{\delta_n}(s, X_s^{\delta_n}))dW_s \\ &+ \int_0^t 2|X_t^n + X_s^{\delta_n}|(h(s, X_s^n, Y_s^n, u_s^{\delta_n}) - h_{\delta_n}(s, X_s^{\delta_n}, Y_s, u_s^{\delta_n})) \\ &+(\sigma(s, X_s^n) - \sigma_{\delta_n}(s, X_s^{\delta_n}))^2 d\langle W\rangle_s \end{aligned}$$

$$\hat{\mathbb{E}}(|X_t^n - X_t^{\delta_n}|^2) = \hat{\mathbb{E}}(\int_0^t 2|X_t^n - X_s^{\delta_n}|(b(s, X_s^n, Y_s^n, u_s^{\delta_n}) - b_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}))ds)$$
$$+\hat{\mathbb{E}}(\int_0^t 2|X_t^n - X_s^{\delta_n}(|\sigma(s, X_s^n) - (\sigma_{\delta_n}(s, X_s^{\delta_n}))dW_s))$$
$$+\hat{\mathbb{E}}(\int_0^t 2|X_t^n - X_s^{\delta_n}|(h(s, X_s^n, u_s^{\delta_n}) - h_{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n}))$$
$$+(\sigma(s, X_s^n) - \sigma_{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n}))^2 d\langle W\rangle_s).$$

By the *BDG* inequalities (Lemma 2.18 [85]) and Proposition 2.6 [46], for $p = 1$, we get

$$\hat{\mathbb{E}}(|X_t^n - X_t^{\delta_n}|^2) \leq \hat{\mathbb{E}}(\int_0^t 2|X_t^n - X_t^{\delta_n}|(b(s, X_s^n, u_s^{\delta_n}) - b^{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n}))ds)$$
$$+\tfrac{(\underline{l}+\bar{l})}{4}\hat{\mathbb{E}}(\int_0^t 2|X_t^n - X_t^{\delta_n}|h(s, X_s^n, u_s^{\delta_n}) - (h_{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n}))$$
$$+(\sigma(s, X_s^n) - \sigma_{\delta_n}(s, X_s^{\delta_n}))ds).$$

By *Young*'s inequality, we get

$$\hat{\mathbb{E}}(|X_t^n - X_t^{\delta_n}|^2) \leq \hat{\mathbb{E}}(\int_0^t \frac{1}{\epsilon}|X_t^n - X_t^{\delta_n}|^2 + \epsilon|b(s, X_s^n, u_s^{\delta_n}) - b^{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n})|2ds)$$
$$+\tfrac{(\underline{l}+\bar{l})}{4}\hat{\mathbb{E}}(\int_0^t \frac{1}{\epsilon_1}|X_t^n - X_t^{\delta_n}|^2 + 2\epsilon_1|h(s, X_s^n, u_s^{\delta_n}) - h_{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n})|^2$$
$$+2\epsilon_1|\sigma(s, X_s^n) - \sigma_{\delta_n}(s, X_s^{\delta_n})|^2 ds),$$

which also gives

$$\hat{\mathbb{E}}(|X_t^n - X_t^{\delta_n}|^2) \leq \hat{\mathbb{E}}(\int_0^t \frac{1}{\epsilon}|X_t^n - X_t^{\delta_n}|^2 + \epsilon|b(s, X_s^n, u_s^{\delta_n}) - b_{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n})|2ds)$$
$$+\tfrac{(\underline{l}+\bar{l})}{4}\hat{\mathbb{E}}(\int_0^t \frac{1}{\epsilon_1}|X_t^n - X_t^{\delta_n}|^2 + 2\epsilon_1|h(s, X_s^n, u_s^{\delta_n}) - h_{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n})|^2$$
$$+2\epsilon_1|\sigma(s, X_s^n) - \sigma_{\delta_n}(s, X_s^{\delta_n})|^2 ds);$$

Consequently,

$$\hat{\mathbb{E}}(|X_t^n - X_t^{\delta_n}|^2) \leq (\tfrac{1}{\epsilon} + \tfrac{(\underline{l}+\bar{l})}{4\epsilon_1})\int_0^t \hat{\mathbb{E}}(|X_t^n - X_t^{\delta_n}|^2)ds$$
$$+\epsilon\hat{\mathbb{E}}(\int_0^t |b(s, X_s^n, u_s^{\delta_n}) - b_{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n})|2ds)$$
$$+\tfrac{\epsilon_1(\underline{l}+\bar{l})}{2}\hat{\mathbb{E}}(\int_0^t |h(s, X_s^n, u_s^{\delta_n}) - h_{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n})|^2 ds)$$
$$\tfrac{(\underline{l}+\bar{l})\epsilon_1}{2}\hat{\mathbb{E}}(\int_0^t (\sigma(s, X_s^n) - \sigma_{\delta_n}(s, X_s^{\delta_n}))^2 ds).$$

We have

$$\hat{\mathbb{E}}(\int_0^t |b(s,X_s^n,u_s^{\delta_n}) - b_{\delta_n}(s,X_s^{\delta_n},u_s^{\delta_n})|^2 ds) \le 2\hat{\mathbb{E}}(\int_0^t |b(s,X_s^n,u_s^{\delta_n}) - b^{\delta_n}(s,X_s^n,u_s^{\delta_n})|^2 ds)$$
$$+2\hat{\mathbb{E}}(\int_0^t |b^{\delta_n}(s,X_s^n,u_s^{\delta_n}) - b^{\delta_n}(s,X_s^{\delta_n},u_s^{\delta_n})|^2 ds);$$

by the property of the mollification, we have

$$\hat{\mathbb{E}}(\int_0^t |b(s,X_s^n,u_s^{\delta_n}) - b_{\delta_n}(s,X_s^{\delta_n},u_s^{\delta_n})|^2 ds) \le 2Tk_\delta^2\delta_n^2 + 4k^2\hat{\mathbb{E}}(\int_0^T |X_s^n - X_s^{\delta_n}|^2 ds),$$

and

$$\hat{\mathbb{E}}(\int_0^T |h(s,X_s^n,u_s^{\delta_n}) - h_{\delta_n}(s,X_s^{\delta_n},u_s^{\delta_n})|^2 ds) \le 2\hat{\mathbb{E}}(\int_0^T |h(s,X_s^n,u_s^{\delta_n}) - h_{\delta_n}(s,X_s^n u_s^{\delta_n})|^2 ds)$$
$$+2\hat{\mathbb{E}}(\int_0^T |h_{\delta_n}(s,X_s^n,u_s^{\delta_n}) - h_{\delta_n}(s,X_s^{\delta_n},u_s^{\delta_n})|^2 ds),$$

also by the same technique

$$\hat{\mathbb{E}}(\int_0^t |h(s,X_s^n,u_s^{\delta_n}) - h_{\delta_n}(s,X_s^{\delta_n},u_s^{\delta_n})|2ds) \le 2Tk_\delta^2\delta_n^2 + 2k^2\hat{\mathbb{E}}(\int_0^T |X_s^n - X_s^{\delta_n}|^2 ds).$$

By the same way,

$$\hat{\mathbb{E}}(\int_0^T |\sigma(s,X_s^n) - \sigma_{\delta_n}(s,X_s^{\delta_n})|^2 ds) \le 2\hat{\mathbb{E}}(\int_0^T |\sigma(s,X_s^n) - \sigma_{\delta_n}(s,X_s^n)|^2 ds)$$
$$+2\hat{\mathbb{E}}(\int_0^T |\sigma_{\delta_n}(s,X_s^n) - \sigma_{\delta_n}(s,X_s^{\delta_n}) ds).$$

$$\hat{\mathbb{E}}(\int_0^t |\sigma(s,X_s^n) - \sigma_{\delta_n}(s,X_s^{\delta_n})|2ds) \le 2Tk_\delta^2\delta_n^2 + 2k^2\hat{\mathbb{E}}(\int_0^T |X_s^n - X_s^{\delta_n}|^2 ds).$$

Therefore,

$$\hat{\mathbb{E}}(|X_t^n - X_t^{\delta_n}|^2) \le (\tfrac{1}{\epsilon} + \tfrac{(\underline{l}+\bar{l})}{4\epsilon_1})\int_0^t \hat{\mathbb{E}}(|X_t^n - X_t^{\delta_n}|^2)ds$$
$$+\epsilon 2Tk_\delta^2\delta_n^2 + 2k^2\epsilon\hat{\mathbb{E}}(\int_0^T |X_s^n - X_s^{\delta_n}|^2 ds)$$
$$+\epsilon_1(\underline{l}+\bar{l})Tk_\delta^2\delta_n^2$$
$$+\epsilon_1(\underline{l}+\bar{l})k^2\hat{\mathbb{E}}(\int_0^T |X_s^n - X_s^{\delta_n}|^2 ds)$$
$$+\epsilon_1(\underline{l}+\bar{l})Tk_\delta^2\delta_n^2$$
$$+\epsilon_1(\underline{l}+\bar{l})k^2\hat{\mathbb{E}}(\int_0^T |X_s^n - X_s^{\delta_n}|^2 ds).$$

By *Gronwall*'s lemma, there exists a constant K independent of δ_n such that

$$\hat{\mathbb{E}}(|X_t^n - X_t^{\delta_n}|^2) \leq K\delta_n^2. \tag{3.54}$$

Now, for the estimation of the solution of the G-BSDE, we apply *Itô*'s formula to

$$|Y_s^n - Y_s^{\delta_n}|^2$$

let

$$J_s = \int_t^T 2|Y_s^n - Y_s||w^{\delta_n}\sigma(X_s^n) - w^{\delta_n}\sigma(X_s^{\delta^n})|dW_s + \int_t^T |Y_s^n - Y_s|^2 dM_s$$

so,

$$\begin{aligned}|Y_s^n - Y_s^{\delta_n}|^2 + J_s \leq\ & |\varphi(X_T^n) - \varphi^{\delta^n}(X_T^{\delta_n})|^2 \\ & + \int_t^T 2|Y_s^n - Y_s^{\delta_n}|(f(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n}) - f_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}), u_s^{\delta_n}))ds \\ & + \int_t^T 2|Y_s^n - Y_s^{\delta_n}|(g(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n)) - g_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})))d\langle W\rangle_s,\end{aligned}$$

where $M = M^n - M^{\delta_n}$. By taking the G-expectation of the both sides of the above equation, we get

$$\begin{aligned}\hat{\mathbb{E}}(|Y_s^n - Y_s^{\delta_n}|^2 + J_s) \leq\ & \hat{\mathbb{E}}(|\varphi(X_T^n) - \varphi^{\delta^n}(X_T^{\delta_n})|^2) \\ & + \hat{\mathbb{E}}(\int_t^T 2|Y_s^n - Y_s^{\delta_n}|(f(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n}) \\ & - f_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}), u_s^{\delta_n}))ds) \\ & + \hat{\mathbb{E}}(\int_t^T 2|Y_s^n - Y_s^{\delta_n}|(g(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n)) \\ & - g_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})))d\langle W\rangle_s).\end{aligned}$$

By the *BDG* inequalities in Lemma 2.18 [85] and Proposition 2.6 [46], for $p = 1$, we obtain

$$\begin{aligned}\hat{\mathbb{E}}(|Y_s^n - Y_s^{\delta_n}|^2) \leq\ & \hat{\mathbb{E}}(|\varphi(X_T^n) - \varphi^{\delta^n}(X_T^{\delta_n})|^2) \\ & + \hat{\mathbb{E}}(\int_t^T 2|Y_s^n - Y_s^{\delta_n}|(f(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n}) - f_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}), u_s^{\delta_n}))ds) \\ & + \tfrac{(l+\bar{l})}{4}\hat{\mathbb{E}}(\int_t^T 2|Y_s^n - Y_s^{\delta_n}|(g(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n)) - g_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})))ds),\end{aligned}$$

By *Young*'s inequality

$$\hat{\mathbb{E}}(|Y_s^n - Y_s^{\delta_n}|^2) \leq \hat{\mathbb{E}}(|\varphi(X_T^n) - \varphi^{\delta^n}(X_T^{\delta_n})|^2) + (\tfrac{1}{\varepsilon} + \tfrac{(l+\bar{l})}{4\varepsilon_1})\hat{\mathbb{E}}(\int_t^T |Y_s^n - Y_s^{\delta_n}|^2 ds)$$
$$+\varepsilon\hat{\mathbb{E}}(\int_t^T |f(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n}) - f_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}), u_s^{\delta_n})|^2 ds)$$
$$+\tfrac{(l+\bar{l})}{4\varepsilon_1}\hat{\mathbb{E}}(\int_t^T |g(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n)) - g_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}))|^2 ds),$$

we omit the variables by the following notation:

$$f - f_{\delta_n} = f(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n}) - f_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}), u_s^{\delta_n}),$$

then

$$\hat{\mathbb{E}}(\int_t^T |f - f_{\delta_n}|^2 ds) \leq 2\hat{\mathbb{E}}(\int_t^T |f(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n}) - f_{\delta_n}(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n})|^2 ds)$$
$$+2\hat{\mathbb{E}}(\int_t^T |f_{\delta_n}(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n}) - f_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}), u_s^{\delta_n})|^2 ds).$$

$$\hat{\mathbb{E}}(\int_t^T |f - f_{\delta_n}|^2 ds) \leq 2(T-t)k_\delta^2\delta_n^2$$
$$+6k\hat{\mathbb{E}}(\int_t^T (|X_s^n - X_s^{\delta_n}|^2 + |Y_s^n - Y_s^{\delta_n}|^2 + |w^{\delta_n}\sigma(X_s^n) - w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})|^2) ds).$$

also we note,

$$g - g_{\delta_n} = g(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n)) - g_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}))$$

so,

$$\hat{\mathbb{E}}(\int_t^T |g - g_{\delta_n}|^2 ds) \leq 2k\hat{\mathbb{E}}(\int_t^T |g(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n)) - g_{\delta_n}(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n))|^2 ds)$$
$$+2k\hat{\mathbb{E}}(\int_t^T |g_{\delta_n}(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n)) - g_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}))|^2 ds)$$

then,

$$\hat{\mathbb{E}}(\int_t^T |g - g_{\delta_n}|^2 ds) \leq 2(T-t)k_\delta^2\delta_n^2$$
$$+6k\hat{\mathbb{E}}(\int_t^T (|X_s^n - X_s^{\delta_n}|^2 + |Y_s^n - Y_s^{\delta_n}|^2 + |w^{\delta_n}\sigma(X_s^n) - w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})|^2) ds).$$

Hence,

$$\hat{\mathbb{E}}(|Y_s^n - Y_s^{\delta_n}|^2) \leq (\tfrac{1}{\varepsilon} + \tfrac{(\underline{l}+\bar{l})}{4\varepsilon_1})\hat{\mathbb{E}}(\int_t^T |Y_s^n - Y_s^{\delta_n}|^2 ds) + \varepsilon 2(T-t)k_\delta^2\delta_n^2 + \frac{(T-t)k_\delta^2(\underline{l}+\bar{l})}{2\varepsilon_1}\delta_n^2$$
$$+6k\varepsilon\hat{\mathbb{E}}(\int_t^T (|X_s^n - X_s^{\delta_n}|^2 + |Y_s^n - Y_s^{\delta_n}|^2 + |w^{\delta_n}\sigma(X_s^n) - w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})|^2)ds)$$
$$+\tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1}\hat{\mathbb{E}}(\int_t^T (|X_s^n - X_s^{\delta_n}|^2 + |Y_s^n - Y_s^{\delta_n}|^2 + |w^{\delta_n}\sigma(X_s^n) - w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})|^2)ds),$$

$$\hat{\mathbb{E}}(|Y_s^n - Y_s^{\delta_n}|^2) \leq (\tfrac{1}{\varepsilon} + \tfrac{(\underline{l}+\bar{l})}{4\varepsilon_1} + \tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1} + 6k\varepsilon)\hat{\mathbb{E}}(\int_t^T |Y_s^n - Y_s^{\delta_n}|^2 ds)$$
$$+\varepsilon 2(T-t)k_\delta^2\delta_n^2 + \tfrac{(T-t)k_\delta^2(\underline{l}+\bar{l})}{2\varepsilon_1}\delta_n^2$$
$$+(6k\varepsilon + \tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1})\hat{\mathbb{E}}(\int_t^T |X_s^n - X_s^{\delta_n}|^2 ds)$$
$$+(6k\varepsilon + \tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1})\hat{\mathbb{E}}(\int_t^T |w^{\delta_n}\sigma(X_s^n) - w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})|^2 ds),$$

$$\hat{\mathbb{E}}(|Y_s^n - Y_s^{\delta_n}|^2) \leq (\tfrac{1}{\varepsilon} + \tfrac{(\underline{l}+\bar{l})}{4\varepsilon_1} + \tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1} + 6k\varepsilon)\hat{\mathbb{E}}(\int_t^T |Y_s^n - Y_s^{\delta_n}|^2 ds)$$
$$+\varepsilon 2(T-t)k_\delta^2\delta_n^2 + \tfrac{(T-t)k_\delta^2(\underline{l}+\bar{l})}{2\varepsilon_1}\delta_n^2$$
$$+(6k\varepsilon + \tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1})\hat{\mathbb{E}}(\int_t^T |X_s^n - X_s^{\delta_n}|^2 ds)$$
$$+(6k\varepsilon + \tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1})\hat{\mathbb{E}}(\int_t^T |w_s^{\delta_n}|^2|\sigma(X_s^n) - \sigma_{\delta_n}(X_s^{\delta_n})|^2 ds),$$

using the fact that

$$\hat{\mathbb{E}}(\int_0^t |\sigma(s, X_s^n) - \sigma_{\delta_n}(s, X_s^{\delta_n})|^2 ds) \leq 2Tk_\delta^2\delta_n^2 + 2k^2\hat{\mathbb{E}}(\int_0^T |X_s^n - X_s^{\delta_n}|^2 ds)$$

and w^{δ_n} is bounded, we have:

$$\hat{\mathbb{E}}(|Y_s^n - Y_s^{\delta_n}|^2) \leq (\tfrac{1}{\varepsilon} + \tfrac{(\underline{l}+\bar{l})}{4\varepsilon_1} + \tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1} + 6k\varepsilon)\hat{\mathbb{E}}(\int_t^T |Y_s^n - Y_s^{\delta_n}|^2 ds)$$
$$+\varepsilon 2(T-t)k_\delta^2\delta_n^2 + \tfrac{(T-t)k_\delta^2(\underline{l}+\bar{l})}{2\varepsilon_1}\delta_n^2 + +(6k\varepsilon + \tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1})C_w^2 2Tk_\delta^2\delta_n^2$$
$$+(6k\varepsilon + \tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1} + 2k^2(6k\varepsilon + \tfrac{3k(\underline{l}+\bar{l})}{2\varepsilon_1})C_w^2)\hat{\mathbb{E}}(\int_t^T |X_s^n - X_s^{\delta_n}|^2 ds).$$

Moreover, we already have proved (3.54). So,

$$\hat{\mathbb{E}}(|Y_s^n - Y_s^{\delta_n}|^2) \leq e^{C_1(T-t)}C\delta_n^2. \tag{3.55}$$

Let $C_1 = ((\frac{1}{\varepsilon} + \frac{(\underline{l}+\bar{l})}{4\varepsilon_1} + \frac{3k(\underline{l}+\bar{l})}{2\varepsilon_1} + 6k\varepsilon) + 6k\varepsilon + \frac{3k(\underline{l}+\bar{l})}{2\varepsilon_1} + 2k^2(6k\varepsilon + \frac{3k(\underline{l}+\bar{l})}{2\varepsilon_1})C_w^2)e^{at}b$

$$C = ke^{a(T-t)}(\varepsilon 2(T-t)k_\delta^2) + \frac{(T-t)k_\delta^2(\underline{l}+\bar{l})}{2\varepsilon_1} + (6k\varepsilon + \frac{3k(\underline{l}+\bar{l})}{2\varepsilon_1})C_w^2 2Tk_\delta^2.$$

Then,

$$\hat{\mathbb{E}}(|X_t^n - X_t^{\delta_n}|^2) \le e^{at}(K + be^{C_1(T-t)}C)\delta_n^2. \tag{3.56}$$

□

Now, let prove the main result, Theorem (3.5))

Proof. We aim to prove that the limit of the sequence $(X^{\delta_n}, Y^{\delta_n})$ coincide with that of the auxiliary sequence of forward SDE for which we can extract a subsequence whose solutions converge in law to $(\bar{X}, \bar{Y})$ associated to a control that is optimal for our control problem in order to prove the existence of subsequence.
For $n \in \mathbb{N}$, we define the sequence of auxiliary processes (X_s^n, Y_s^n) as the unique solution of the following controlled forward system:

$$\begin{cases} dX_s^n = b(s, X_s^n, u_s^{\delta_n})ds + \sigma(s, X_s^n)dW_s + h(s, X_s^n, u_s^{\delta_n})d\langle W\rangle_s, \\ dY_s^n = -f(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n})ds - g(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n))d\langle W\rangle_s \\ \qquad +w_s^{\delta_n}\sigma(X_s^n)dW_s + \theta_n d\langle B\rangle_s - 2G(\theta_n)ds, \\ X_t^n = x \qquad Y_t^n = V^{\delta_n}(t,x) \end{cases} \tag{3.57}$$

where θ_n is a stochastic process, $u_s^{\delta_n} := v^{\delta_n}(s, X_s^{\delta_n})$ and $w_s^n = \nabla_x V^{\delta_n}(s, X_s^{\delta_n})$.
With $\theta_n = 0$, the process $(X_s^{\delta_n}, Y_s^{\delta_n})$ is a solution of the following controlled forward system:

$$\begin{cases} dX_s^{\delta_n} = b_{\delta_n}(s, X_s^{\delta_n}, u_s^{\delta_n})ds + \sigma_{\delta_n}(s, X_s^{\delta_n})dW_s + h_{\delta_n}(s, X_s^{\delta_n}, w_s^{\delta_n})d\langle W\rangle_s, \\ dY_s^{\delta_n} = -f_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}), u_s^{\delta_n})ds - g(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}))d\langle W\rangle_s \\ \qquad +w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})dW_s, \\ X_t^{\delta_n} = x \quad Y_t^{\delta_n} = V^{\delta_n}(t,x) \end{cases} \tag{3.58}$$

From (3.23), we have, for $t \le s \le T$,

$$Y_s^{\delta_n} = V^{\delta_n}(s, X^{\delta_n}) \qquad \text{and} \qquad u_s^{\delta_n} = v^{\delta_n}(t, X_s^{\delta_n}).$$

Since $(s,x) \mapsto V^{\delta_n}(s,x)$ is a $C^{1,2}$ function and satisfies equation (3.37) with $\delta = \delta_n$, so, using *G-Itô*'s formula we get for $t \le s \le T$

$$\begin{aligned} Y_s^{\delta_n} = \phi_{\delta_n}(X_T^{\delta_n}) &+ \int_t^T f_{\delta_n}(s, X_s^{\delta_n}, Y_s^{\delta_n}, u_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}), u_s^{\delta_n})ds \\ &+ \int_t^T g(s, X_s^{\delta_n}, Y_s^{\delta_n}, w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n}))d\langle W\rangle_s - \int_t^T w_s^{\delta_n}\sigma_{\delta_n}(X_s^{\delta_n})dW_s \end{aligned} \tag{3.59}$$

If we put

$$\chi_s^n := \begin{pmatrix} X_s^n \\ \\ Y_s^n \end{pmatrix}, \quad r_s^n := (w_s^n \sigma_n(X_s^n), 0, u_s^{\delta_n}), \qquad \mathcal{W} := \begin{pmatrix} W \\ \\ W \end{pmatrix}, \quad \text{and} \quad d\langle \mathcal{W} \rangle := \begin{pmatrix} d\langle W \rangle \\ \\ d\langle W \rangle \end{pmatrix}$$

then the system (3.57) becomes:

$$\begin{cases} d\chi_s^n = \beta(\chi_s^n, r_s^n)ds + \Pi(\chi_s^n, r_s^n)d\langle \mathcal{W} \rangle_s + \Sigma(\chi_s^n, r_s^n)d\mathcal{W}_s, \quad s \in [t,T], \\ \chi_t^n = \begin{pmatrix} x \\ V^{\delta_n}(t,x) \end{pmatrix}, \end{cases} \tag{3.60}$$

where,

$$\beta(\chi_s^n, r_s^n) = \begin{pmatrix} b(s, X_s^n, u_s^{\delta_n}) \\ \\ -f(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n), u_s^{\delta_n}) - 2G(\theta_n) \end{pmatrix},$$

$$\Pi(\chi_s^n, r_s^n) = \begin{pmatrix} h(s, X_s^n, u_s^{\delta_n}) \\ \\ -g(s, X_s^n, Y_s^n, w^{\delta_n}\sigma(X_s^n)) + \theta_n \end{pmatrix}, \quad \text{and} \quad \Sigma(\chi_s^n, r_s^n) = \begin{pmatrix} \sigma(s, X_s^n) \\ \\ w_s^{\delta_n}\sigma(X_s^n) \end{pmatrix}$$

From (3.3.3), we have $w_s^n = \nabla_x V^{\delta_n}(s, X_s^{\delta_n})$ is uniformly bounded. Then, we can interpret $(r_s^n, s \in [t,T])$ as a control with values in the compact set

$$A := \mathbb{U} \times \bar{B}_C(0) \times [0,K].$$

The next step is to take $n \to +\infty$, for this, let's consider the random measure:

$$q^n(\omega, ds, da) = \delta_{r_s^n(\omega)}(da)ds, \ (s,a) \in [0,T] \times A, \omega \in \Omega.$$

We identify the control process r^n with the measure q^n, this end show us that the control r^n is in the set of relaxed controls, i.e. we consider r^n as random variable with values in the space $\mathcal{V}$ of all *Borel* measures q^n on $[0,T] \times \mathbb{U} \times \bar{B}_C(0) \times [0,K]$, whose projection $q^n(\cdot \times \mathbb{U} \times \bar{B}_C(0) \times [0,K])$ coincides with the *Lebesgue* measure.
From the boundedness of our coefficients and by the compactness of $\mathcal{V}$ with respect to the topology induced by the weak convergence of measures, we get the tightness of the laws of (χ^n, q^n) on this space, and then, from this and the use of the G-Chattering Lemma [86] we can extract a subsequence that converges in law on this space to $(\chi, \bar{r})$, where $\bar{r}$ with values in $\mathcal{R}$, which satisfies

$$\begin{cases} d\chi_s = \beta(\chi, \bar{r}_s)ds + \Pi(\chi_s, \bar{r}_s)d\langle \mathcal{W}\rangle_s + \Sigma(\chi_s, \bar{r}_s)d\mathcal{W}_s, \qquad s \in [t,T], \\ \chi_t = \begin{pmatrix} x \\ V^{\delta_n}(t,x) \end{pmatrix}. \end{cases} \tag{3.61}$$

Replacing $\Sigma, \prod$ and β by their definition and setting

$$\chi := \begin{pmatrix} \bar{X} \\ \\ \bar{Y} \end{pmatrix}, \mathcal{W} := \begin{pmatrix} W \\ \\ W \end{pmatrix}, \quad \text{and} \quad d\langle \mathcal{W}\rangle := \begin{pmatrix} d\langle W\rangle \\ \\ d\langle W\rangle \end{pmatrix}$$

and $\bar{r} := (\bar{w}, \bar{\theta}, \bar{u})$, the system (3.61) can be rewritten as follows:

$$\begin{cases} d\bar{X}_s = b(s, \bar{X}_s, \bar{u}_s)ds + \sigma(s, \bar{X}_s)dW_s + h(s, \bar{X}_s, \bar{u}_s)d\langle W\rangle_s, \\ d\bar{Y}_s = -f(s, \bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{u}_s)ds - g(s, \bar{X}_s, \bar{Y}_s, \bar{Z}_s)d\langle W\rangle_s + \bar{Z}_s dW_s + \bar{\theta}_s d\langle B\rangle_s - 2G(\bar{\theta}_s)ds, \\ \bar{X}_t = x \quad \bar{Y}_t = V(t,x) \end{cases} \tag{3.62}$$

We put

$$\bar{M}_s := \int_t^s \bar{\theta}_r d\langle B\rangle_r - 2\int_t^s G(\bar{\theta}_r)dr,$$

$(\bar{M}_s)_{s\in[t,T]}$ is a decreasing G-martingale, and this prove 1.
Lemma (3.3.4) shows that if the sequence $(X^n, Y^n)_{n\in\mathbb{N}}$ converges in law, the same holds true for $(X^{\delta_n}, Y^{\delta_n})_{n\in\mathbb{N}}$, and the limits have the same distribution. Further, we deduce from ((3.50), (3.51)) and Proposition (3.3.3), that $\bar{Y}_s = V(s, \bar{X}_s)$ for each $s \in [t,T]$ quasi-surely. In particular, $Y_T = \Phi(X_T)$ q.s.. So, assertion 2 of the theorem is proved.
We have already seen that $\bar{Y}_s = V(s, \bar{X}_s)$ for all $s \in [t,T]$ q.s.. On the other hand, it is well known that, for the unique bounded viscosity solution V of the *Hamilton-Jacobi-Bellman* equation (3.19), we have

$$V(t,x) = \underset{u\in\mathcal{R}}{\text{essinf}} J(t,x;u), \text{ q.s.}$$

This proves assertion 3 of the theorem. □

Chapter 4

Model Reduction And Uncertainty Quantification Of Multiscale Under G-Expectation

This chapter is organized as follow: in the first section, the idea of using the *G-Brownian* motion framework to the uncertainty quantification for multiscale systems is explained. Second section records basic definitions and identities related to the stochastic representations of fully nonlinear partial differential equations in terms of second-order BSDE that are used to carry out the numerical simulations in Section (4.3). The key theoretical result of this chapter, the convergence of the value function and its derivative, are formulated and proved in Section 3. To illustrate the theoretical findings, we discuss two numerical examples with uncertain diffusions in Section 4; a linear quadratic Gaussian regulator with uncertain diffusion and an uncontrolled bilinear benchmark system from turbulence modeling.

4.1 Slow-fast system

Let $x = (r, u) \in \mathbb{R}^n = \mathbb{R}^{n_s} \times \mathbb{R}^{n_f}$ and $\epsilon > 0$ be a small parameter. We consider slow-fast multiscale SDE models of the form

$$dR_t^\epsilon = \left(f_0(R_t^\epsilon, U_t^\epsilon) + \frac{1}{\sqrt{\epsilon}} f_1(R_t^\epsilon, U_t^\epsilon) \right) dt + \alpha(R_t^\epsilon, U_t^\epsilon) dV_t \tag{4.1a}$$

$$dU_t^\epsilon = \frac{1}{\epsilon} g(R_t^\epsilon, U_t^\epsilon; \theta) dt + \frac{1}{\sqrt{\epsilon}} \beta(R_t^\epsilon, U_t^\epsilon; \theta) dW_t \,, \tag{4.1b}$$

where all coefficients are assumed to be such that the SDE has a unique strong solution for all times. We call R_t^ϵ the resolved (slow) variable and U_t^ϵ the unresolved (fast) variable that is not fully accessible and depends on an unknown parameter $\theta \in \Theta \subset \mathbb{R}^p$, where for convenience we suppress the dependence on θ.

The aim is to derive a closed equation for R^ϵ for $\epsilon \to 0$ that best approximates the resolved process whenever ϵ is sufficiently small. Since the fast process depends on an unknown parameter, the answer to the question what the *best approximation* is remaining ambiguous.

4.1.1 Goal-oriented uncertainty quantification

To illustrate the ambiguity in the reduced dynamics, let us consider the degenerate diffusion

$$dR_t = (R_t - U_t^3)dt\,, \quad R_0 = r \tag{4.2a}$$

$$dU_t = \frac{1}{\epsilon}(R_t^\epsilon - U_t)dt + \sqrt{\frac{2\theta}{\epsilon}}dW_t\,, \quad U_0 = u\,. \tag{4.2b}$$

for $\theta \in [0,1]$ where, for simplicity, we use the shorthand $(R,U) = (R^\epsilon, U^\epsilon) \in \mathbb{R}\times\mathbb{R}$ and suppress the dependence on the small parameter ϵ.

When $\epsilon \ll 1$, the fast dynamics becomes "slaved" by the slow dynamics and randomly fluctuates around R_t. The unique limiting invariant measure of the fast variables conditionally on $R_t = r$ is given by $\mu_r = \mathcal{N}(r,\theta)$ when $\theta \in (0,1]$, and singular, $\mu_r = \delta_r$ for $\theta = 0$. As $\epsilon \to 0$ it follows from the averaging principle (e.g. [39, Ch. 7]), that the slow process $R = R^\epsilon$ converges pathwise to a limit process that is the solution of the (here: deterministic) initial value problem

$$\frac{dr}{dt} = F(r;\theta)\,, \quad r(0) = r\,, \tag{4.3}$$

where

$$F(r,\theta) = -r^3 + r(1-3\theta)\,, \quad \theta \in [0,1]\,. \tag{4.4}$$

Figure (4.1) shows the vector field $F(\cdot,\theta)$ for three different values of θ and illustrates that the limit dynamics undergoes a supercritical pitchfork bifurcation at $\theta = 1/3$ at which two asymptotically stable fixed point and an unstable one collapse into one asymptotically stable one. Note that $F(r,\cdot)$ is continuous at $\theta = 0$, nevertheless, depending on value of θ, the qualitative properties of the limit dynamics change drastically as θ varies. It therefore makes sense to modify the best approximation question slightly and instead ask for a worst-case scenario in terms of the unknown parameter for a given quantity of interest (QoI).

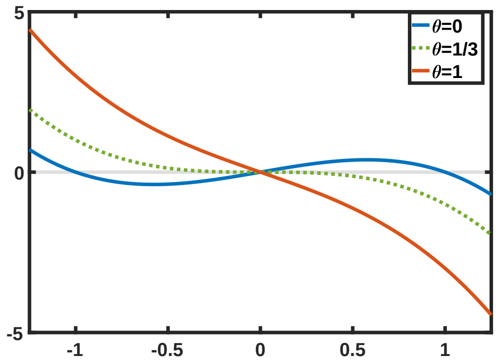

Figure 4.1: Limiting vector field $F(\cdot, \theta)$ for $\theta \in \{0, 1/3, 1\}$. The value $\theta = 1/3$ (green dotted curve) corresponds to a supercritical pitchfork bifurcation of the dynamics.

Let $\varphi\colon C([0,T]) \to \mathbb{R}$ be a suitable test function. The objects of interest are path functionals of the form $\phi^\epsilon = \varphi(R^\epsilon)$, with $R^\epsilon = (R_t^{\epsilon,\theta})_{t\in[0,T]}$. To this end let $r = (r_t^\theta)_{t\in[0,T]}$ denotes the candidate limit process as $\epsilon \to 0$ and denote $\phi = \varphi(r)$.
A worst-case scenario for the convergence of R^ϵ to the limiting process r can be expressed by the G-expectation using the representation formula (1.14):

$$\hat{\mathbb{E}}\left(|\phi^\epsilon - \phi|\right) = \sup_{\theta\in\Theta} \hat{\mathbb{E}}_\theta\left(|\phi^\epsilon - \phi|\right) . \tag{4.5}$$

For example, the worst-case approximation for the variance (or the second moment) may be different from the approximation of the slow process itself, in that they correspond to different values of the unknown parameter θ.

If the linear expectation on the right hand side of (4.5) converges for every fixed $\theta \in \Theta$, stability results (e.g. [97, Thm. 3.1]) for G-BSDE imply that

$$\lim_{\epsilon\to 0} \hat{\mathbb{E}}\left(|\phi^\epsilon - \phi|\right) = 0 . \tag{4.6}$$

If ϕ^ϵ is regarded as *data*, then the G-expectation defines some kind of tracking problem for the limit dynamics, with θ playing the role of the control variable (There may be an additional control variable in the equations though.). An equivalent statement is that the value function, i.e. the unique viscosity solution of the underlying dynamic programming equation converges as $\epsilon \to 0$.

One of the messages of the previous considerations is that robust approximations of a multiscale diffusion with parameter uncertainties may depend on the class of test functions φ via the optimal parameter θ^*. In general, by the dynamic programming principle, $\theta^* = \theta^*(t)$ will be time dependent or a feedback law, therefore the limit equations are not simply obtained by setting θ equal to some appropriate value. They are moreover goal-oriented, in that they depend on the QoI.

4.2 Convergence of the quantity of interest

In this section we study the convergence of the slow component of a slow-fast system driven by a *G-Brownian* motion. Specifically, we prove convergence of the corresponding value function that is associated with the QoI. For the sake of simplicity, the proof will be given for a linear controlled G-SDE only, but we stress that the proof carries over to the case of a nonlinear G-SDE with or without control and under standard *Lipschitz* conditions, using essentially the same techniques.

Controlled linear-quadratic slow-fast system and related QoI

We consider the following controlled stochastic differential equation

$$dX_s^\epsilon = (A^\epsilon X_s^\epsilon + B^\epsilon \alpha_s)ds + C^\epsilon dW_s; \quad X_t^\epsilon = x, \tag{4.7}$$

with $X_s^\epsilon = (U_s, R_s)$ taking values in $\mathbb{R}^{n_s} \times \mathbb{R}^{n_f}$ where $n_s + n_f = n$, and $x = (r, u)$ denotes the decomposition of the state vector x into slow (resolved) and fast (unresolved) components. We will suppress the dependence of R and U on ϵ, until further notice. Here $W = (W_t)_{t\geq 0}$ is a standard $\mathbb{R}^m$-valued *Brownian* motion on $(\Omega, \mathcal{F}, \mathbb{P})$ that is endowed with its own filtration $(\mathcal{F}_t)_{t\geq 0}$, and $\alpha = (\alpha_t)_{t\geq 0}$ denotes an adapted control variable with values in $\mathbb{R}^k$. Let

$$A^\epsilon = \begin{pmatrix} A_{11} & \epsilon^{-1/2} A_{12} \\ \epsilon^{-1/2} A_{21} & \epsilon^{-1} A_{22} \end{pmatrix} \in \mathbb{R}^{n\times n},$$

with the natural partitioning into $A_{11} \in \mathbb{R}^{n_s\times n_s}$, etc. where assume that the matrix $A_{22} \in \mathbb{R}^{n_f\times n_f}$ is Hurwitz, i.e. all of its eigenvalues are lying in the open left half-plane. The control and the noise coefficients are partitioned as follows:

$$B^\epsilon = \begin{pmatrix} B_1 \\ \epsilon^{-1/2} B_2 \end{pmatrix} \in \mathbb{R}^{n\times k}, \qquad C^\epsilon = \begin{pmatrix} C_1 \\ \epsilon^{-1/2} C_2 \end{pmatrix} \in \mathbb{R}^{n\times m}.$$

We assume that, for all $\epsilon > 0$, the columns of B^ϵ lie in the column space of the matrix C^ϵ, i.e. $\operatorname{ran}(B^\epsilon) \subset \operatorname{ran}(C^\epsilon)$ or, equivalently, the column space of B^ϵ is orthogonal to the kernel of $(C^\epsilon)^T$, so that the equation

$$C^\epsilon \xi = B^\epsilon c \tag{4.8}$$

has a (not necessarily unique) solution for every $c \in \mathbb{R}^k$. We seek a control α that minimizes the following quadratic cost functional

$$J(\alpha; t, x) = \mathbb{E}_{t,x}\left[\frac{1}{2}\int_t^\tau R_s^T Q_0 R_s + |\alpha_s|^2 ds + \frac{1}{2} R_\tau^T Q_1 R_\tau\right], \tag{4.9}$$

where τ is a bounded stopping time given by $\tau = \inf\{s \in [t, T] : X_t \notin S\}$ where S is a bounded subset of $\mathbb{R}^{n_s} \times \mathbb{R}^{n_f}$ which contain the initial state x, and where $Q_0, Q_1 \in \mathbb{R}^{n_s \times n_s}$ are any given symmetric positive semi-definite matrices. Note that even though the cost depends only on the slow process, the expected cost depends on the initial conditions of both r and u. We call

$$q_0 = r^T Q_0 r\,, \quad q_1 = r^T Q_1 r\,. \tag{4.10}$$

The corresponding value function is our QoI, it is given by

$$V^\epsilon(t, x) = \inf_{\alpha \in \mathcal{A}} J(\alpha; t, x). \tag{4.11}$$

where $\mathcal{A}$ is the space of all admissible controls α, such that (4.7) has a unique strong solution. (Likewise we may consider q_0, q_1 or J to be our quantities of interest.)
Assuming that all coefficients are known, the averaging principle for linear-quadratic control systems of the form (4.7)–(4.9) implies that, under mild conditions on the system matrices, the value function V^ϵ converges uniformly on any compact subset of $[0, T] \times \mathbb{R}^n$ to a value function $v = v(t, r)$; see e.g. [57]. The latter is the value function of the following linear-quadratic stochastic control problem: minimize the reduced cost functional

$$\bar{J}(\alpha; t, r) = \mathbb{E}_{t,r}\left[\frac{1}{2}\int_t^\tau q_0(\bar{R}_s) + |\alpha_s|^2\, ds + \frac{1}{2} q_1(\bar{R}_\tau)\right], \tag{4.12}$$

subject to

$$d\bar{R}_s = (\bar{A}\bar{R}_s + \bar{B}\alpha_s)ds + \bar{C}dW_s, \tag{4.13}$$

where the coefficients of the reduced system are given by

$$\bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}\,,\ \bar{B} = B_1 - A_{12}A_{22}^{-1}B_2\,,\ \bar{C} = C_1 - A_{12}A_{22}^{-1}C_2\,. \tag{4.14}$$

Multiscale system with unknown diffusion coefficient

We suppose that the noise coefficients C_1 and/or C_2 are unknown. This situation is common in many applications, since especially the diffusion coefficient of the unresolved variables is difficult to estimate. Very often, however, an educated guess can be made as to which set or interval the unknown coefficient lies in. Specifically, we suppose that $(C_1,\, C_2)^T \in \mathcal{A}^{\Theta}_{0,\infty}$ which is the collection of all Θ-valued adapted process on $[0, \infty)$ where Θ is a given bounded and closed subset in $\mathbb{R}^{(n_s+n_f)\times m}$.
Following the work by Denis and co-workers [25, 26] we exploit the link between the $G-$expectation framework and diffusion controlled processes and define

$$D^\epsilon \tilde{W}_t = \int_0^t C dW_s,$$

for each $C^\epsilon \in \mathcal{A}^{\Theta}_{0,\infty}$, such that

$$C^\epsilon = D^\epsilon \begin{pmatrix} C_1 \\ \\ C_2 \end{pmatrix}, \quad D^\epsilon = \begin{pmatrix} I_{n_s} & 0 \\ \\ 0 & \epsilon^{-1/2} I_{n_f} \end{pmatrix}$$

so that $(\tilde{W}_s)_{s\geq 0}$ is a $d-$dimensional G-Brownian motion. As the main result, we will show below that the value function converges uniformly on any compact subset of $[0,T]\times\mathbb{R}^n$. The result does not rely on any compactness or periodicity assumptions of the fast variables with unknown diffusion; the key idea is to recast the fully nonlinear dynamic programming (or: G-Hamilton-Jacobi-Bellman) equation of the full G-stochastic optimal control problem as a G-FBSDE and then study convergence to the limiting G-FBSDE, which implies convergence of the corresponding dynamic programming equation.

Nonlinear dynamic programming equation

By the dynamic programming principle for controlled G-SDE [37], the G-Hamilton-Jacobi-Bellman (G-HJB) equation associated with our uncertain stochastic control problem (4.7)–(4.9) reads

$$-\frac{\partial v^\epsilon}{\partial t}=\inf_c\{G(DD^T\colon\nabla^2 v^\epsilon)+\langle\nabla v^\epsilon,Ax+Bc\rangle+\frac{1}{2}q_0+\frac{1}{2}|c|^2)\},\tag{4.15}$$

with terminal condition

$$v^\epsilon(\tau,\cdot)=\frac{1}{2}q_1\,.\tag{4.16}$$

Note that we v^ϵ is different from the value function V^ϵ in (4.11), since the diffusion coefficient in (4.11) is assumed constant, whereas, here, it is part of the nonlinear generator that involves a maximisation over the coefficient. Further note that we have dropped the ϵ in $A=A^\epsilon$, $B=B^\epsilon$ and $D=D^\epsilon$. We can get rid of the outer infimum since the diffusion part is independent of the control variable, and

$$\inf_c\left\{\langle\nabla V^\epsilon,B^\epsilon c\rangle+\frac{1}{2}|c|^2\right\}=-\frac{1}{2}|c|^2_{BB^T}\,,$$

where $|c^2|_{BB^T}=\langle c,BB^Tc\rangle$. This implies that (4.15) is equivalent to

$$\frac{\partial v}{\partial t}+G(DD^T\colon\nabla^2 v)+\langle\nabla v,Ax\rangle-\frac{1}{2}|c|^2_{BB^T}+\frac{1}{2}q_0=0\,,\tag{4.17}$$

with the associated G-FBSDE system given by

$$\begin{aligned}dX_s^\epsilon=&AX_s^\epsilon ds+D\,d\tilde{W}_s\,,\;X_t^\epsilon=x\\ Y_t^\epsilon=&\frac{1}{2}q_1(R_\tau)-\frac{1}{2}\int_t^\tau q_0(R_s)\,ds+\frac{1}{2}\int_t^\tau|B^T(D^T)^\sharp Z_s^\epsilon|^2\,ds\\ &-\int_t^\tau Z_s^\epsilon d\tilde{W}_s-(K_\tau-K_t)\end{aligned}\tag{4.18}$$

Here

$$Y_s^\epsilon=v^\epsilon(s,X_s^\epsilon)\,,\quad Z_s^\epsilon=\nabla v^\epsilon(s,X_s^\epsilon)\,,\quad t\leq s\leq\tau\,,\tag{4.19}$$

and $\sharp$ denotes the Moore-Penrose pseudo inverse of a matrix. The process K is a decreasing G-martingale with $K_0=0$ that is a consequence of the G-martingale representation theorem [80].

Strong convergence of the quantity of interest

Since the G-FBSDE is decoupled and running and terminal cost q_0, q_1 depend only on the resolved variables, we can infer the candidate for the limiting process:

$$d\bar{R}_s = (\bar{A}\bar{R}_s + \bar{B}\alpha_s)\, ds + \bar{D}\, d\tilde{W}_s \tag{4.20}$$

with $\bar{D}\, d\tilde{W}$ given by

$$\begin{aligned} \bar{C} dW_s &= (C_1 - A_{12}A_{22}^{-1}C_2)dW_s \\ &= C_1 dW_s - A_{12}A_{22}^{-1}C_2 dW_s \\ &= d\tilde{W}_s - A_{12}A_{22}^{-1}d\tilde{W}_s \\ &=: \bar{D}\, d\tilde{W}\,. \end{aligned}$$

in other words, $\bar{D} = (I_{n_s}, -A_{12}A_{22}^{-1})$. The associated limiting G-FBSDE reads

$$\begin{aligned} d\bar{R}_s =& \bar{A}\bar{R}_s ds - \bar{D}\, d\tilde{W}_s\,, \ \ \bar{R}_t = r \\ \bar{Y}_s =& \frac{1}{2}q_1(\bar{R}_\tau) - \frac{1}{2}\int_t^\tau q_0(\bar{R}_s)\, ds + \frac{1}{2}\int_t^\tau |\bar{B}^T(\bar{D}^T)^\sharp \bar{Z}_s|^2 ds \\ &- \int_t^\tau \bar{Z}_s d\tilde{W}_s - (\bar{K}_\tau - \bar{K}_t) \end{aligned} \tag{4.21}$$

The corresponding limit G-HJB equation is then given by

$$\frac{\partial \bar{v}}{\partial t} + G(\bar{D}\bar{D}^T : \nabla^2 \bar{v}) + \langle \nabla \bar{v}, \bar{A}r\rangle - \frac{1}{2}|\nabla \bar{v}|^2_{\bar{B}\bar{B}^T} + \frac{1}{2}q_0 = 0. \tag{4.22}$$

with the natural terminal condition

$$\bar{v}(\tau, \cdot) = \frac{1}{2}q_1\,. \tag{4.23}$$

Theorem 4.1. *Let v^ϵ be the classical solution of the dynamic programming equation 4.17 and $\bar{v}$ be the solution of 4.22, then, as $\epsilon \to 0$*

$$v^\epsilon \to \bar{v}\,, \quad \nabla v^\epsilon \to \nabla \bar{v}$$

where the convergence of v^ϵ is uniform on any compact subset of $[0,T] \times \mathbb{R}^{n_s}$ and pointwise for ∇v^ϵ for all $(t,x) \in [0,T] \times \mathbb{R}^{n_s}$.

Proof. Subtracting the G-BSDE part of (4.21) from (4.18) yields

$$\begin{aligned} Y_t^\epsilon - \bar{Y}_t = &\frac{1}{2}q_1(R_\tau) - \frac{1}{2}q_1(\bar{R}_\tau) - \frac{1}{2}\int_t^\tau q_0(R_s)\, ds + \frac{1}{2}\int_t^\tau q_0(\bar{R}_s)\, ds \\ &+ \frac{1}{2}\int_t^\tau |B^T(D^T)^\sharp Z_s^\epsilon|^2 ds - \frac{1}{2}\int_t^\tau |\bar{B}^T(\bar{D}^T)^\sharp \bar{Z}_s|^2 ds \\ &- \int_t^\tau Z_s^\epsilon d\tilde{W}_s + \int_t^\tau \bar{Z}_s d\tilde{W}_s - (K_\tau - K_t) + (\bar{K}_\tau - \bar{K}_t) \end{aligned} \tag{4.24}$$

Let $\gamma > 0$ be arbitrary. Defining $y_t = Y_t^\epsilon - \bar{Y}_t, M_t = K_t - \bar{K}_t$, we can apply Itô's formula to $|y_t|^2 e^{\gamma t}$ for $0 \le t < \tau \le T$, which yields

$$\begin{aligned}|y_t|^2e^{\gamma t} &+ \int_t^\tau |Z_s^\epsilon - \bar{Z}_s|^2 d\langle \tilde{W}\rangle_s + \int_t^\tau \gamma |y_s|^2 e^{\gamma s} ds \\ &= \left|\frac{1}{2}q_1(R_\tau) - \frac{1}{2}q_1(\bar{R}_\tau)\right|^2 e^{\gamma\tau} - \int_t^\tau y_s e^{\gamma s}\left(q_0(R_s) - q_0(\bar{R}_s)\right) ds \\ &\quad + \int_t^\tau y_s e^{\gamma s}\left(|B^T(D^T)^\sharp Z_s^\epsilon|^2 - |\bar{B}^T(\bar{D}^T)^\sharp \bar{Z}_s|^2\right) ds - (\bar{M}_\tau - \bar{M}_t),\end{aligned} \tag{4.25}$$

where

$$\bar{M}_\tau - \bar{M}_t = 2\int_t^\tau y_s e^{\gamma s} dM_s + 2\int_t^\tau y_s e^{\frac{\gamma s}{2}}\left(Z_s^\epsilon - \bar{Z}_s\right) d\tilde{W}_s\,.$$

It is convenient to write $e^{\gamma s}$ on the right hand side as $e^{\gamma s/2}e^{\gamma s/2}$. Now dropping the quadratic variation term on the left and using Young's inequality (cf. Lemma 2.15) gives after rearranging terms

$$\begin{aligned}|y_t|^2e^{\gamma t} + \gamma\int_t^\tau |y_s|^2 e^{\gamma s} ds + (\bar{M}_\tau - \bar{M}_t) &\leq \left|\frac{1}{2}q_1(R_\tau) - \frac{1}{2}q_1(\bar{R}_\tau)\right|^2 e^{\gamma\tau} \\ &+ \int_t^\tau \left(\frac{\lambda_1}{2}|y_s|^2 e^{\gamma s} + \frac{e^{\gamma s}}{2\lambda_1}\left(q_0(\bar{R}_s) - q_0(R_s)\right)^2\right) ds \\ &+ \int_t^\tau \left(\frac{\lambda_2}{2}|y_s|^2 e^{\gamma s} + \frac{e^{\gamma s}}{2\lambda_2}\left(|B^T(D^T)^\sharp Z_s^\epsilon|^2 - |\bar{B}^T(\bar{D}^T)^\sharp \bar{Z}_s|^2\right)^2\right) ds,\end{aligned} \tag{4.26}$$

where we have defined λ_1, λ_2 by $\gamma = \lambda_1/2 + \lambda_2/2$. As a consequence,

$$\begin{aligned}|y_t|^2e^{\gamma t} + (\bar{M}_\tau - \bar{M}_t) \leq &\left|\frac{1}{2}q_1(R_\tau) - \frac{1}{2}q_1(\bar{R}_\tau)\right|^2 e^{\gamma\tau} \\ &+ \int_t^\tau \frac{e^{\gamma s}}{2\lambda_1}\left(q_0(\bar{R}_s) - q_0(R_s)\right)^2 ds \\ &+ \int_t^\tau \frac{e^{\gamma s}}{2\lambda_2}\left(|B^T(D^T)^\sharp Z_s^\epsilon|^2 - |\bar{B}^T(\bar{D}^T)^\sharp \bar{Z}_s|^2\right)^2 ds.\end{aligned} \tag{4.27}$$

Using the shorthands $N = (B_1, B_2)^T$ and $k_s = \left(NZ_s^\epsilon + N\bar{Z}_s\right)$, with

$$\left((B^T((D^\epsilon)^T)^\sharp Z_s^\epsilon) - (\bar{B}^T(D^T)^\sharp \bar{Z}_s)\right) = \left(NZ_s^\epsilon - N\bar{Z}_s\right),$$

the pathwise convergence

$$\mathbb{E}\left[\sup_{t\in[0,T]}|R_t - \bar{R}_t|^2\right] = \mathcal{O}(\epsilon)$$

as $\epsilon \to 0$ for any fixed diffusion coefficient (e.g. [58, 59]), together with the stability result of Zhang and Chen [97, Thm. 3.1], then implies that

$$\begin{aligned}|y_t|^2e^{\gamma t} + (\bar{M}_\tau - \bar{M}_t) \leq &\frac{l\epsilon^2 e^{\gamma\tau}}{4} + \int_t^\tau \frac{l\epsilon^2 e^{\gamma s}}{2\lambda_1} ds \\ &+ \int_t^\tau \frac{|k_s|^2\|NN^T\|_F}{2\lambda_2}|Z_s^\epsilon - \bar{Z}_s|^2 e^{\gamma s} ds\end{aligned} \tag{4.28}$$

for some generic constant $l \in (0, \infty)$ that may change from equation to equation. Taking the supremum and the using the fact that $\bar{M}$ is a symmetric G-martingale, it follows again by Young's inequality that

$$\hat{\mathbb{E}}\left(\sup_{s\in[t,\tau]}|y_s|^2e^{\gamma s}\right) \le \frac{l\epsilon^2e^{\gamma\tau}}{4}+\frac{l\epsilon^2e^{\gamma\tau}}{2\gamma\lambda_1}+\frac{l}{2\lambda_2}\hat{\mathbb{E}}\left(\int_t^\tau |Z_s^\epsilon-\bar{Z}_s|^2e^{\gamma s}ds\right). \tag{4.29}$$

Now using (4.25) again, together with the BDG-type inequalities (2.13)–(2.14) for the quadratic variation and Young's inequality for the integrals involving $y_se^{\gamma s}$ on the right hand side, we obtain after dropping the quadratic terms in y:

$$\begin{aligned}\underline{\sigma}^2\hat{\mathbb{E}}\left(\int_t^\tau |Z_s^\epsilon-\bar{Z}_s|^2e^{\gamma s}ds\right) \le & \frac{|l\epsilon|^2e^{\gamma\tau}}{4}+\int_t^\tau \frac{|l\epsilon|^2e^{\gamma s}}{2\alpha_1}ds\\ &+\frac{(k_1)^2NN^T}{2\alpha_2}\hat{\mathbb{E}}\left(\int_t^\tau |Z_s^\epsilon-\bar{Z}_s|^2e^{\gamma s}ds\right).\end{aligned} \tag{4.30}$$

where α_1,α_2 are defined by $\gamma=\alpha_1/2+\alpha_2/2$. Hence

$$\begin{aligned}\hat{\mathbb{E}}\left(\int_t^\tau |Z_s^\epsilon-\bar{Z}_s|^2e^{\gamma s}ds\right) \le & \frac{|l\epsilon|^2e^{\gamma\tau}}{4\underline{\sigma}^2}+\int_t^\tau \frac{|l\epsilon|^2e^{\gamma s}}{2\alpha_1\underline{\sigma}^2}ds\\ &+\frac{(k_1)^2NN^T}{2\underline{\sigma}^2\alpha_2}\hat{\mathbb{E}}\left(\int_t^\tau |Z_s^\epsilon-\bar{Z}_s|^2e^{\gamma s}ds\right),\end{aligned} \tag{4.31}$$

which can be rearranged to give

$$\left(1-\frac{(k_1)^2NN^T}{2l\underline{\sigma}^2\alpha_2}\right)\hat{\mathbb{E}}\left(\int_t^\tau |Z_s^\epsilon-\bar{Z}_s|^2e^{\gamma s}ds\right) \le \frac{|l\epsilon|^2e^{\gamma\tau}}{4l\underline{\sigma}^2}+\frac{|l_2\epsilon|^2(e^{\gamma\tau}-e^{\gamma t})}{2\gamma\alpha_1 l\underline{\sigma}^2}. \tag{4.32}$$

The last inequality can be combined with (4.29), so that we obtain

$$\begin{aligned}&\hat{\mathbb{E}}\left(\sup_{s\in[t,\tau]}|y_s|^2e^{\gamma s}\right)+\left(1-\frac{(k_1)^2NN^T}{2l\underline{\sigma}^2\alpha_2}-\frac{(k_1)^2NN^T}{2\lambda_2}\right)\hat{\mathbb{E}}\left(\int_t^\tau |Z_s^\epsilon-\bar{Z}_s|^2e^{\gamma s}ds\right)\\ &\le\frac{|l\epsilon|^2e^{\gamma\tau}}{4}+\int_t^\tau\frac{|l_2\epsilon|^2(e^{\gamma\tau}-e^{\gamma t})}{2\gamma\lambda_1}+\frac{|l\epsilon|^2e^{\gamma\tau}}{4l\underline{\sigma}^2}+\frac{|l_2\epsilon|^2(e^{\gamma\tau}-e^{\gamma t})}{2\gamma\alpha_1 l\underline{\sigma}^2}.\end{aligned} \tag{4.33}$$

As a consequence,

$$\|Y^\epsilon-\bar{Y}\|_\gamma := \hat{\mathbb{E}}\left(\sup_{s\in[t,\tau]}|y_s|^2e^{\gamma s}\right), \quad \|Z^\epsilon-\bar{Z}\|_\gamma := \hat{\mathbb{E}}\left(\int_t^\tau |Z_s^\epsilon-\bar{Z}_s|^2e^{\gamma s}ds\right)$$

go to zeros as $\epsilon\to 0$ at rate ϵ^2. Since $\|\cdot\|_\gamma$ and $\|\cdot\|_{\gamma=0}$ are equivalent, it follows that $Y_t^\epsilon\to\bar{Y}_t$ uniformly for $t\in[0,T]$, and therefore, as $\epsilon\to 0$,

$$v^\epsilon(\cdot,x)=Y^\epsilon\to\bar{Y}=\bar{v}(\cdot,x)$$

uniformly on any compact subset of $[0,T]\times\mathbb{R}^{n_s}$. Likewise,

$$\nabla v^\epsilon(t,x)=Z_t^\epsilon\to\bar{Z}_t=\nabla\bar{v}(t,x), \quad (t,x)\in[0,T]\times\mathbb{R}^{n_s}$$

as $\epsilon\to 0$, which implies the convergence of the optimal control in (4.7)–(4.9). □

Remark 4.2.1. *The theorem also holds if the underlying G-SDE is nonlinear, as long as the averaging principle applies (e.g. when the drift is uniformly Lipschitz).*

Remark 4.2.2. *When $B = D$ in (4.15) then the corresponding G-BSDE and the limit G-BSDE can be simplified to*

$$Y_t^\epsilon = \frac{1}{2}q_1(R_\tau) - \frac{1}{2}\int_t^\tau q_0(R_s)\,ds + \frac{1}{2}\int_t^\tau |Z_s^\epsilon|^2\,ds - \int_t^\tau Z_s^\epsilon d\tilde{W}_s - (K_\tau - K_t) \tag{4.34}$$

and

$$\bar{Y}_s = \frac{1}{2}q_1(\bar{R}_\tau) - \frac{1}{2}\int_t^\tau q_0(\bar{R}_s)\,ds + \frac{1}{2}\int_t^\tau |\bar{Z}_s|^2 ds - \int_t^\tau \bar{Z}_s d\tilde{W}_s - (\bar{K}_\tau - \bar{K}_t)\,. \tag{4.35}$$

4.3 Numerical illustration

In this section we present two numerical examples to verify that the value function of the original system (4.17) converges to the solution of the reduced system (4.22) as $\epsilon \to 0$. The corresponding fully nonlinear HJB equations (4.17) and (4.22) are numerically solved by exploiting the link between fully nonlinear PDE and second-order BSDE (2BSDE); see e.g. [18]. The numerical algorithm for solving 2BSDE is based on the deep 2BSDE solver introduced by Beck et al. [7].

4.3.1 Linear quadratic Gaussian regulator

The first example is a 2-dimensional linear quadratic regulator problem given by the SDE

$$dX_t^\epsilon = (A^\epsilon X_t^\epsilon + B^\epsilon u_t^\epsilon)dt + \sqrt{\sigma}B^\epsilon dW_t, X_0^\epsilon = x_0, \tag{4.36}$$

with unknown diffusion coefficient $\sigma \in [\underline{\sigma}, \overline{\sigma}]$ and the cost functional

$$J(u;t,x) = \frac{1}{2}\mathbb{E}\left[\int_t^T ((X_s^\epsilon)^T Q_0 X_s^\epsilon + |u_s^\epsilon|^2)ds + (X_T^\epsilon)^T Q_1 X_T^\epsilon\right]. \tag{4.37}$$

Here $x = (r,u) \in \mathbb{R}^2$ and the coefficients are given by

$$A^\epsilon = \begin{pmatrix} -2 & -1/\epsilon \\ 1/\epsilon & -2/\epsilon^2 \end{pmatrix}, \quad B^\epsilon = \begin{pmatrix} 0.1 \\ 2/\epsilon \end{pmatrix}, \quad Q_0 = 0\,, \quad Q_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and we define the value function as $v^\epsilon(t,x) = \inf_u J(u;t,x)$. The G-PDE corresponding to the Gaussian regulator problem (4.37)–(4.36) is then given by

$$\frac{\partial v^\epsilon}{\partial t} + G(a^\epsilon : \nabla^2 v^\epsilon) + \langle \nabla v^\epsilon, A^\epsilon x\rangle - \frac{1}{2}\langle B^\epsilon, z\rangle^2 = 0\,, \quad v^\epsilon(T,x) = x_1^2 \tag{4.38}$$

where we have used the shorthand $a^\epsilon = \sigma B^\epsilon (B^\epsilon)^T$. Calling $a = \sigma \bar{B}\bar{B}^T$, the G-PDE of the limiting value function $\bar{v} = \lim_{\epsilon\to 0} v^\epsilon$ then reads

$$\frac{\partial \bar{v}}{\partial t} + G(a : \nabla^2 \bar{v}) + \langle \nabla \bar{v}, \bar{A}\bar{x}\rangle - \frac{1}{2}\langle \nabla \bar{v}, \bar{B}\rangle^2 = 0\,, \quad \bar{v}(T,r) = r\,. \tag{4.39}$$

The function $G\colon \mathbb{R} \to \mathbb{R}$ is defined by:

$$G(x) = \frac{x}{2} \begin{cases} \bar{\sigma} \text{ if } x \geq 0 \\ \underline{\sigma} \text{ if } x < 0 \, . \end{cases}$$

Numerical results

We consider the two value functions in the time interval $[0, 0.1]$ with fixed initial condition $x = (r, u) = (1, 0.5)$. For the diffusion coefficient, we assume $\sigma \in [0.8, 1]$. The driver f of the 2BSDE corresponding to the G-PDE (4.38) of the original system is

$$f(t, x, y, z, S) = G(a^\epsilon : S) + \frac{1}{2}\langle B^\epsilon, z\rangle^2 + \frac{1}{2}\langle A^\epsilon x, z\rangle \, ,$$

whereas the 2BSDE corresponding to the limiting G-PDE (4.39) has the driver

$$\bar{f}(t, x, y, z, \bar{S}) = G(a : \bar{S}) + \frac{1}{2}\langle \bar{B}, z\rangle^2 + \frac{1}{2}\langle \bar{A}x, z\rangle \, .$$

We compare $v^\epsilon(0, x)$ and $\bar{v}(0, r)$ and call

$$\delta_v(\epsilon) = |v^\epsilon(0, x) - \bar{v}(0, r)| \, ,$$

Denoting by u^ϵ and u the corresponding optimal controls for any given noise coefficient σ (that can expressed in terms of the value function for fixed σ), we have

$$v^\epsilon(0, x) = \hat{\mathbb{E}}\left[\int_0^T |u_s^\epsilon|^2 ds\right] , \quad \bar{v}(0, r) = \hat{\mathbb{E}}\left[\int_0^T |u_s|^2 ds\right] .$$

The simulation results are shown in the following table:

ϵ	0.3	0.2	0.1
δ_v	0.15	0.06	0.01

4.3.2 Triad system for climate prediction

We consider a stochastic climate model which can be represented as a bilinear system with additive noise [71]

$$dX^\epsilon(t) = \frac{1}{\epsilon^2} L(X^\epsilon(t))dt + \frac{1}{\epsilon} B(X^\epsilon(t), X^\epsilon(t))dt + \frac{1}{\epsilon} \Sigma \, dW_t \, , \quad X^\epsilon(0) = x, \tag{4.40}$$

where $X^\epsilon(t) = (R_1^\epsilon(t), R_2^\epsilon(t), U^\epsilon(t)) \in \mathbb{R}^3$ and

$$L(x) = -\begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \quad B(x,x) = \begin{pmatrix} A_1 r_2 u \\ A_2 r_1 u \\ A_3 r_1 r_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix},$$

where $0 < \epsilon \ll 1$, and A_1, A_2, A_3 are real numbers such that

$$A_1 + A_2 + A_3 = 0\,,$$

and

$$\lambda \in [\underline{\sigma}, \overline{\sigma}]$$

is the unknown diffusion coefficient. Equation (4.40), which is a time rescaled version of (4.1a)–(4.1b), is a simplified stochastic turbulence model that comprises triad wave interactions between two climate variables r_1, r_2 and a single stochastic variable u.

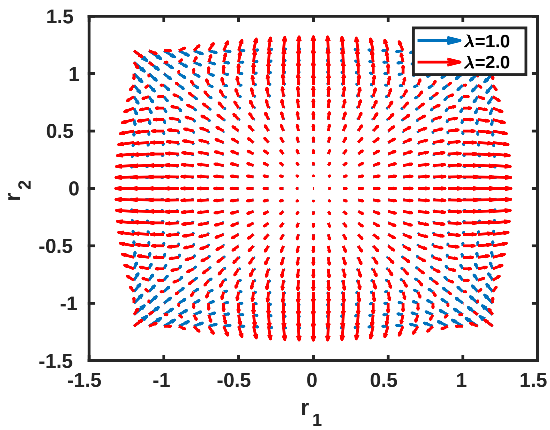

Figure 4.2: Vector field f of the limit triad system for $A_1 = A_2 = 1$ and $A_3 = -2$ and two different noise parameters λ.

The noise level λ cannot be accurately estimated, nevertheless it may have a huge impact on the dynamics, even though there are no bifurcations for $\lambda > 0$. Equation (4.40) can thus be considered an SDE driven by a *G-Brownian* motion. It is shown in [71] that, for any finite value $\lambda > 0$, the first two components $R^\epsilon = (R_1^\epsilon, R_2^\epsilon)$ converge strongly in L^p for $p = 1, 2$ and on any bounded time interval $[0, T]$ to the solution of the nonlinear SDE with multiplicative noise

$$dR(t) = f(R(t))dt + \sigma(R(t))dW_t\,, \quad R(0) = r, \tag{4.41}$$

where $R(t) = (R_1(t), R_2(t))$ and

$$f(r) = \begin{pmatrix} A_1 r_1 (A_3 r_2^2 + \frac{\lambda^2}{2} A_2) \\ \\ A_2 r_2 (A_3 r_1^2 + \frac{\lambda^2}{2} A_1) \end{pmatrix}, \quad \sigma(x) = \frac{\lambda}{\gamma} \begin{pmatrix} A_1 r_2 \\ \\ A_2 r_1 \end{pmatrix}.$$

The pathwise convergence $R^\epsilon \to R$ together with the stability result of *Zhang* and *Chen* [97, Thm. 3.1] implies that

$$\hat{\mathbb{E}}(\sup_{t\in[0,T]} |R^\epsilon(t) - R(t)|) \to 0 \quad \text{as} \quad \epsilon \to 0$$

We can study the qualitative features of the triad system (4.40) in terms of the reduced model (4.41). Using Itô's formula, it readily follows that

$$I(r_1, r_2) = A_1 r_2^2 - A_2 r_1^2$$

is a conserved quantity for both the reduced and the original system. We consider the case A_1, $A_2 > 0$ and $A_3 < 0$, in which case the level sets of I are hyperbola, and the origin is an unstable hyperbolic equilibrium. The rays that connect the origin with any of the four equilibria

$$r^*_{\pm,\pm} = \left(\pm\sigma\sqrt{\frac{A_1}{2|A_3|}}, \pm\sigma\sqrt{\frac{A_2}{2|A_3|}} \right), \quad A_1,\, A_2 > 0\,.$$

are (locally hyperbolically unstable) invariant sets. Figure 4.2 shows representative vector fields f of the limit system for different noise coefficients $\lambda = 1.0$ and $\lambda = 2.0$, when $A_1 = A_2 > 0$. It can be seen that the repulsive and attractive regions on the invariant diagonals change as the coefficient λ varies.

For illustration, Figure 4.3 shows three representative samples of $R(0.5)$ for $A_1 = 0.75$, $A_2 = 0.25$ and $A_3 = -1.0$, with $\lambda = 1.0$, $\lambda = 1.5$ and $\lambda = 1.0$, all starting from the same initial value $R(0) = (1, -2)$. Note that the sample means over 100 independent realizations each depend on λ in a non-trivial fashion.

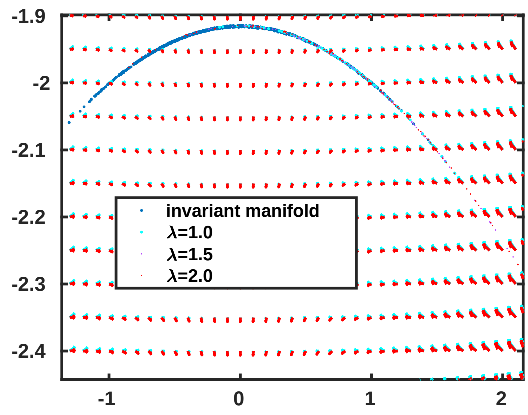

Figure 4.3: Independent realizations of the limit triad system for $A_1 = A_2 = 1$ and $A_3 = -2$ and different noise parameters $\lambda \in [1, 2]$ and fixed $T = 0.5$. For every parameter value, we have generated 100 independent realizations, all starting from the same initial value $r = (1, -2)$. Note that the invariant manifolds, to which the trajectories are confined, are independent of λ, nevertheless the dynamics on the invariant manifolds are different.

Goal-oriented uncertainty quantification

We now compare the full triad system (4.40) and the limit system (4.41) for a specific quantity of interest (QoI) using the G-BSDE framework. To this end, we consider the QoI *mean*

$$v^\epsilon(t,x) = \mathbb{E}_{t,x}(X_1^\epsilon(T))\,, \quad v(t,r) = \mathbb{E}_{t,r}(R_1(T)) \tag{4.42}$$

as a function of the initial data (t, x) and (t, r) where $x = (r, u) = (r_1, r_2, u)$ and $T > 0$ is fixed. By definition, the two value functions v^ϵ and v solve the following nonlinear dynamic programming (HJB-type) equations

$$\frac{\partial v^\epsilon}{\partial t} + G(a^\epsilon \colon \nabla^2 v^\epsilon) + \langle \nabla v^\epsilon, b^\epsilon \rangle = 0\,, \quad v^\epsilon(T,x) = x_1 \tag{4.43}$$

and

$$\frac{\partial v}{\partial t} + G(a \colon \nabla^2 v + \langle \nabla v, f_1 \rangle) + \langle \nabla v, f_2 \rangle = 0\,, \quad v(T,r) = r_1\,, \tag{4.44}$$

with the shorthands

$$b^\epsilon = \frac{1}{\epsilon^2} L + \frac{1}{\epsilon} B\,,\ a^\epsilon = \frac{1}{\epsilon^2}\Sigma\Sigma^T\,,\ a = \sigma\sigma^T\,,\ f_1 = \lambda^2 A_1 A_2 r\,,\ f_2 = f - \frac{f_1}{2}\,.$$

The nonlinearity G in (4.43) and (4.44) is defined by

$$G(x) = \frac{x}{2}\begin{cases} \overline{\sigma} \text{ if } x \geq 0 \\ \underline{\sigma} \text{ if } x < 0 \end{cases}$$

(We can think of G as the nonlinear generator of the parameter-dependent part of the corresponding G-SDE.) We solve the fully nonlinear HJB equations by exploiting the aforementioned relation to second-order BSDE (2BSDE) and using the deep learning approximation developed by Beck et al. [7].

Numerical results

As a first example, we consider the triad system and its homogenisation limit, with the parameters $A_1 = A_2 = 1$, $A_3 = -2$ and $\lambda \in [0.8, 1.2]$. Setting $T = 0.1$ and $x = (r, u) = (1, -2, -2)^T$ the 2BSDE solution for $\epsilon = 0.2$ yields the numerical approximations $v^\epsilon(0, x) = 0.9291$ and $v(0, r) = 0.9326$, i.e.

$$\frac{|v^\epsilon(0,x) - v(0,r)|}{v(0,r)} = 0.0038$$

in agreement with the theoretical prediction. We repeated the 2BSDE simulation for the same initial data and $\epsilon = 0.2$, but with the different set of parameters $A_1 = 1, A_2 = 2, A_3 = -3$, $\lambda \in [0.6, 1.2]$ and $T = 0.5$, and found $v^\epsilon(0, x) = 1.3202$ and the limiting PDE $v(0, r) = 1.3549$, i.e.

$$\frac{|v^\epsilon(0,x) - v(0,r)|}{v(0,r)} = 0.0256\,.$$

It is illustrative to consider the parameter for which the maximum in the nonlinear part G of the generator is attained. For example, for the original triad system,

$$G(a^\epsilon \colon \nabla^2 v^\epsilon) = \max_{\lambda \in [\underline{\sigma}, \overline{\sigma}]} a^\epsilon(\lambda) \colon \nabla^2 v^\epsilon = \frac{1}{\epsilon^2} \max_{\lambda \in [\underline{\sigma}, \overline{\sigma}]} \lambda \frac{\partial^2 v^\epsilon}{\partial u^2}\,, \tag{4.45}$$

which is identically equal to $\underline{\sigma}$ if v^ϵ is strictly concave in its third argument, u, and equal to $\overline{\sigma}$ if it is strictly convex in u. For a G-PDE of the form (4.43) that contains no running cost, one can show that the value function is strictly convex or concave if the terminal condition is strictly convex or concave (since the solution of the forward SDE is a strictly increasing function of the initial value). In general, however, it is not the convexity that determines, for which parameter value the maximum is attained, as the limit G-PDE (4.44) shows. In fact, the optimal parameter will be a feedback function that depends on (t, x) or (t, r).
Figure 4.4 shows the maximiser in (4.45) as function of t for a fixed value of x. It can be seen that the optimal parameter value is time-dependent, which underpins the fact that the optimal parameter depends on the QoI (here also through the initial data) in a nontrivial way; cf. Figure 4.3.

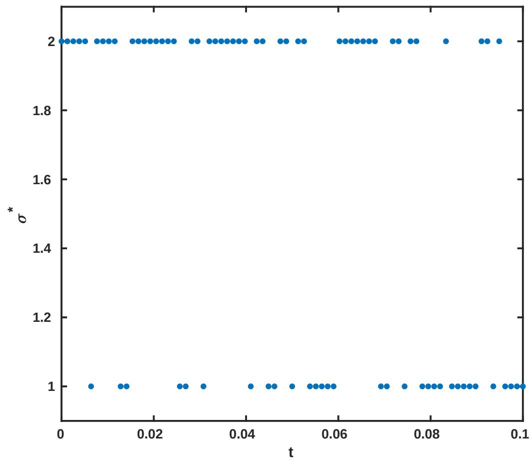

Figure 4.4: The plot shows the parameter σ^* that maximises the nonlinear part G of the generator in (4.45) for fixed initial condition over the noise coefficient $\lambda \in [1, 2]$.

As a final numerical test, we consider the triad system with $A_1 = 0.75, A_2 = 0.25, A_3 = -1$ and $\lambda \in [1, 2]$. For $T = 0.1$ and $\epsilon = 0.2$ we obtain $v^\epsilon(0, x) = 0.9752$ and the limiting PDE $v(0, r) = 0.9601$, i.e.

$$\frac{|v^\epsilon(0, x) - v(0, r)|}{v(0, r)} = 0.0157\,.$$

Conclusion

In this thesis, an attractive and challenging problems have been solved in which our results are obtained on a sub-linear expectation space where the systems are described by using the new developed process the so called G-*Brownian* motion. Investigating the existent results in the classical situations and exploiting the stochastic calculus related to the G-*Brownian* motion and its quadratic variation, we have shown that:

- A coupled forward backward stochastic differential equation driven by G-*Brownian* motion under suitable conditions has a unique solution where the non-linearity of the expectation and the nature of the G-BM did not forbid us to use the *Picard* iterations in the development of the proof as in the classical situation.

- An optimal control for decoupled forward backward stochastic differential equation driven by G-BM exists. In fact, we are content only with the proof of a relaxed control because of the modernistic of this framework which provide a lack of references and results that will developed in the near future.

- The slow component of a slow-fast system driven by a G-*Brownian* motion converges i.e. we prove the convergence of the corresponding value function that is associated with the QoI for a controlled linear-quadratic slow-fast system. The theoretical result has been illustrated by two numerical examples; linear quadratic Gaussian regulator and Goal-oriented uncertainty quantification.

The study done in this thesis can serve to develop, update and start new research on the G-framework in which the uncertain situation and the real-world phenomena can be modeled more precisely.
In future, there exists many issues that we aim to deal with it, such as:

- Prove the existence of an optimal control for coupled G-FBSDE.

- The nonlinear generator of the underlying G-*Brownian* motion may not be unambiguously defined then, this calls for a suitable regularization that is likely to have a Bayesian interpretation that may open up new algorithmic possibilities to quantify the uncertainty in the reduced system.

- In finance, we can investigate our results to deal with many uncertainty issues (e.g. risk measure, super-hedging, volatility uncertainty, ...).

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" التحكم الأمثل لنظام معادلات عشوائية مباشرة خلفية موجهة ب G- حركة براونية "

الملخص:

أهم أهداف هذه الأطروحة هو دراسة وجود تحكم أمثل لنظام معادلات عشوائية مباشرة-خلفية موجهة ب G- حركة براونية. من جهة أخرى تم في صفحات هذه الأطروحة البرهان على وجود حل وحيد لنظام المعادلات التفاضلية المقترنة المباشرة-الخلفية الموجهة ب G- حركة
براونية. وفي الأخير، تمت دراسة تقليص النظم للمشاكل الخطية و الثنائية الخطية التربيعية للتحكم الأمثل ذات معاملات غير مؤكدة.

كلمات مفتاحية: التحكم الأمثل، معادلات عشوائية مباشرة-خلفية، G- حركة براونية .

« Contrôle optimal pour les équations différentielles stochastiques progressives-rétrogrades dirigées par un G-mouvement Brownien »

Résumé :

Le but principal de cette thèse est la démonstration de l'existence d'un contrôle optimal pour un système d'équations progressives-rétrogrades découplées dirigées par un G-mouvement Brownien, D'autre part, dans cette thèse, on a établi l'existence d'une solution unique pour un système d'équations progressives-rétrogrades couplées dirigées par un G- mouvement Brownien. La dernière partie de cette thèse est consacrée pour l'étude de la réduction de modèle du problème de contrôle stochastique optimal linéaire et bilinéaire avec des paramètres incertains.

Mots clés : contrôle optimal, équations différentielle stochastique progressive rétrograde, G-mouvement Brownien

« Optimal control for forward backward stochastic differential equation driven by G-Brownian motion»

Abstract :

The main objective of this dissertation is to study the existence of an optimal control whose dynamical system is described by decoupled forward-backward stochastic differential equations driven by a G-Brownian motion. This dissertation established the existence of a unique solution to coupled forward-backward stochastic differential equations driven by G-Brownian motion. The present dissertation also offers a study on model reduction of linear and bilinear quadratic stochastic control problems with parameter uncertainties.

Key words : optimal control, FBSDEquations, G-Brownian motion