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Publications

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5. H. Gorine, S. Abbas and M. Benchohra, Existence and Attractivity results for Caputo-Fabrizio fractional differnetial equations, (Submitted)

Contents

Introduction

The concept of fractional differential calculus has a long history. One may wonder what meaning may be ascribed to the derivative of a fractional order, that is $\frac{d^n y}{dx^n}$, where n is a fraction. In fact L'Hopital himself considered this very possibility in a correspondence with Leibniz, In 1695, L'Hopital wrote to Leibniz to ask, "*What if n be $\frac{1}{2}$?*" From this question, the study of fractional calculus was born. Leibniz responded to the question, "*$d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx : x}$. This is an apparent paradox from which, one day, useful consequences will be drawn.*"

Many known mathematicians contributed to this theory over the years. Thus, 30 September 1695 is the exact date of birth of the "***fractional calculus***"! Therefore, the fractional calculus it its origin in the works by *Leibnitz*, *L'Hopital* (1695), *Bernoulli* (1697), *Euler* (1730), and *Lagrange* (1772). Some years later, *Laplace* (1812), *Fourier* (1822), *Abel* (1823), *Liouville* (1832), *Riemann* (1847), *Grünwald* (1867), *Letnikov* (1868), *Nekrasov* (1888), *Hadamard* (1892), *Heaviside* (1892), *Hardy* (1915), *Weyl* (1917), *Riesz* (1922), *P. Levy*(1923), *Davis* (1924), *Kober* (1940), *Zygmund* (1945), *Kuttner* (1953), *J. L. Lions* (1959), and *Liverman* (1964)... have developed the basic concept of fractional calculus.

In June 1974, Ross has organized the "*First Conference on Fractional Calculus and its Applications*" at the University of New Haven, and edited its proceedings [126]; Thereafter, Spanier published the first monograph devoted to "*Fractional Calculus*" in 1974 [117]. The integrals and derivatives of non-integer order, and the fractional integrodifferential equations have found many applications in recent studies in theoretical physics, mechanics and applied mathematics. There exists the remarkably comprehensive encyclopedic-type monograph by *Samko*, *Kilbas* and *Marichev* which was published in Russian in 1987 and in English in 1993 [130]. (for more details see [109]) The works devoted substantially to fractional differential equations are: the book of *Miller* and *Ross* (1993) [112], of *Podlubny* (1999) [121], by *Kilbas* et al. (2006) [96], by *Diethelm* (2010) [65], by *Ortigueira* (2011) [119], by *Abbas et al.* (2012) [1], and by *Baleanu et al.* (2012) [30].

In recent years, there has been a significant development in the theory of fractional differential equations. It is caused by its applications in the modeling of many phenomena in various fields of science and engineering such as acoustic, control theory, chaos and fractals, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, optics, economics, astrophysics, chaotic dynamics, statistical physics, thermodynamics, proteins, biosciences, bioengineering, etc. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. See for example [31, 32, 83, 85, 109, 120, 129, 133].

Fractional calculus is a generalization of differentiation and integration to arbitrary order (non-integer) fundamental operator D^{α}_{a+} where $\alpha, a, \in \mathbb{R}$. Several approaches to fractional derivatives exist: Riemann-Liouville (RL),Caputo, Hadamard, Grunwald-Letnikov (GL) and Weyl etc.

In recent times, a new fractional differential operator having a kernel with exponential decay has been introduced by Caputo and Fabrizio [58]. This approach of fractional derivative is known as the Caputo-Fabrizio operator which has attracted many research scholars due to the fact that it has a non-singular kernel. Several mathematicians were recently busy in development of Caputo-Fabrizio fractional differential equations, see; [36, 63, 75, 76, 77, 108, 131, 139], and the references therein. ***In this thesis, we use the Caputo-Hadamard and Caputo-Fabrizio derivatives***.

Fractional differential equations with nonlocal conditions have been discussed in [10, 15, 60, 68, 80, 115, 116] and references therein. Nonlocal conditions were initiated by Byszewski [56] when he proved the existence and uniqueness of mild and classical solutions of nonlocal **Cauchy problems** (**C.P.** for short). As remarked by Byszewski [54, 55], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

Implicit differential equations involving the regularized fractional derivative were analyzed by many authors, in the last year; see for instance [13] and the references therein.

There are two measures which are the most important ones. The Kuratowski measure of noncompactness $\alpha(B)$ of a bounded set B in a metric space is defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of sets of diameter smaller than r. The Hausdorf measure of noncompactness $\chi(B)$ defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of balls of radii smaller than r. Several authors have studied the measures of noncompactness in Banach spaces. See, for example, the books such as [19, 25, 33] and the articles [21, 34, 35, 42, 48, 51, 86, 113],and references therein.

Recently, considerable attention has been given to the existence of solutions of boundary value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo fractional derivative. See for example [12, 16, 17, 18, 28, 43, 44, 45, 48, 87, 94, 101, 102, 103, 105, 132, 145], and references therein.

In the theory of ordinary differential equations, of partial differential equations, and in the theory of ordinary differential equations in a Banach space there are several types of data dependence . On the other hand, in the theory of functional equations there are some special kind of data dependence: Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-

Bourgin, Aoki-Rassias [127].

The stability problem of functional equations originated from a question of Ulam [134, 135] concerning the stability of group homomorphisms: "*Under what conditions does there exist an additive mapping near an approximately additive mapping ?*" Hyers [88] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers Theorem was generalized by Aoki [24] for additive mappings and by T.M. Rassias [125] for linear mappings by considering an unbounded Cauchy difference. A generalization of the T.M. Rassias theorem was obtained by Gavruta [71].

After, many interesting results of the generalized Hyers-Ulam stability to a number of functional equations have been investigated by a number of mathematicians; see [4, 22, 40, 89, 90, 91, 92, 93, 98, 136, 137, 138] and the books [61, 123, 124] and references therein.

We have organized this thesis as follows:

Chapter 1.

This chapter consists of three Sections.

In Section one, we present "***A brief visit to the history of the Fractional Calculus***", and in Section two, we present some "***Applications of Fractional calculus***". Finally, in the last Section, we recall some preliminary : some basic concepts, and useful famous theorems and results (notations, definitions, lemmas and **fixed point theorems**) which are used throughout this thesis.

Chapter 2

In the first section; we discuss and establish the existence,the uniqueness of solutions for a class of boundary value problem of Caputo-Hadamard fractional derivative.

Next, we will give existence and uniqueness results for the following problem of fractional differential equations:

$$({}^{Hc}D_1^{\alpha}u)(t) = f(t, u(t)),\ t \in I := [1, T],$$
$$\begin{cases} a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1), \\ a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2), \end{cases}$$

where $\alpha \in (1, 2]$, $T > 1$, $a_1, b_1, d_1, a_2, b_2, d_2 \in \mathbb{R}$, $\xi_1, \xi_2 \in (1, T)$, $f : I \times \mathbb{R}^m \to \mathbb{R}^m$, $m \in \mathbb{N}^*$ is a given continuous function, and ${}^{Hc}D_1^{\alpha}$ is the Caputo–Hadamard fractional derivative of order α.

Finally, an example will be included to illustrate our main results.

In the second section; two results for a class of boundary value problem for implicit fractional differential equations in Banach spaces with Caputo-Hadamard fractional derivative are discussed. The argument are based on *Banach's fixed point theorem* and *Nonlinear alternative of Leray-Schauder type.*

We establish existence and uniqueness results of the following problem of implicit fractional differential equation :

$$({}^{Hc}D_1^{\alpha}u)(t) = f(t, u(t), ({}^{Hc}D_1^{\alpha}u)(t)), \ t \in I := [1, T],$$

$$\begin{cases} a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1), \\ a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2), \end{cases}$$

where $\alpha \in (1, 2], \ T > 1, \ a_1, b_1, d_1, a_2, b_2, d_2 \in \mathbb{R}, \ \xi_1, \xi_2 \in (1, T), \ f : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m, \ m \in \mathbb{N}^*$ is a given continuous function, and ${}^{Hc}D_1^{\alpha}$ is the Caputo-Hadamard fractional derivative of order α.

At last and as application, an example is included.

Chapter 3
We establish the existence and the stability results for a class of multipoint boundary conditions problem of fractional differential equations.
Here three results are discussed and based on the methode associated with the technique of the measure of non-compactness and the fixed point theorems of **Mönch** and **Darbo.**

In section 3.2; we discuss the existence, uniqueness and stability results to the following problem of fractional differential equations:

$$({}^{Hc}D_1^{\alpha}u)(t) = f(t, u(t)), \ t \in I := [1, T],$$

$$\begin{cases} a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1), \\ a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2), \end{cases}$$

where $T > 1, \ a_1, b_1, d_1, a_2, b_2, d_2 \in \mathbb{R}, \ \xi_1, \xi_2 \in (1, T), \ f : I \times E \to E$ is a given continuous function, $(E, \|\cdot\|)$ is a real or complex Banach space, and ${}^{Hc}D_1^{\alpha}$ is the Caputo–Hadamard fractional derivative of order $\alpha \in (1, 2]$.

In section 3.3, we present an example to show the applicability of our results.
Chapter 4
We establish some existence of solutions for a class of Caputo-Fabrizio fractional differential equations in fréchet spaces. Some applications are made of a generalization of the classical Darbo fixed point theorem for Fréchet spaces associate with the concept of measure of noncompactness.

In section 4.2, we are concerned with the existence results for the fractional differential equation

$$({}^{CF}D_0^{r}u)(t) = f(t, u(t)); \ t \in \mathbb{R}_+ := [0, \infty),$$

with the following initial condition

$$u(0) = u_0 \in E,$$

where $T > 0$, $(E, \|\cdot\|)$ is a (real or complex) Banach space, $r \in (0,1)$, $f : \mathbb{R}_+ \times E \to E$ is a given function, and ${}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0,1)$.

Next, we discuss the existence of solutions for the fractional differential equation

$$({}^{CF}D_0^r u)(t) = f(t, u(t));\ t \in \mathbb{R}_+ := [0, \infty),$$

with the following nonlocal condition

$$u(0) + Q(u) = u_0,$$

where $u_0 \in E$, $Q : C(\mathbb{R}_+, E) \to E$ is a given function.
The last section; illustrates our results with some examples.

Chapter 5

We discuss the existence and the attractivity of solutions for a class of Caputo-Fabrizio fractional differential equation.

In section 5.2, we investigate the existence and the attractivity of solutions for the following class of Caputo-Fabrizio fractional differential equation

$$({}^{CF}D_0^r u)(t) = f(t, u(t));\ t \in \mathbb{R}_+,$$

with the initial condition

$$u(0) = u_0 \in \mathbb{R},$$

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and ${}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0,1)$. Our results are based on Schauder's fixed point theorem. Next we prove that all solutions are uniformly locally attractive.
Finally, we present an example to show the applicability of our results.

Chapter 1

Basic Ingredients

1.1 A brief visit to the history of the Fractional Calculus

In 1695, in a letter to the French mathematician *L'Hospital*, *Leibniz* raised the following question: "*Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?*" *L'Hospital* was somewhat curious about that question and replied by another question to *Leibniz*: "*What if the order will be* $\frac{1}{2}$?" *Leibnitz* in a letter dated September 30, replied: "*It will lead to a paradox, from which one day useful consequences will be drawn.*" Many known mathematicians contributed to this theory over the years. Thus, September 30, 1695 is the exact date of birth of the "*fractional calculus*"! Therefore, the fractional calculus it its origin in the works by *Leibnitz*, *L'Hopital* (1695), *Bernoulli* (1697), *Euler* (1730), and *Lagrange* (1772). Some years later, *Laplace* (1812), *Fourier* (1822), *Abel* (1823), *Liouville* (1832), *Riemann* (1847), Grünwald (1867), *Letnikov* (1868), *Nekrasov* (1888), *Hadamard* (1892), *Heaviside* (1892), *Hardy* (1915), *Weyl* (1917), *Riesz* (1922), *P. Levy* (1923), *Davis* (1924), *Kober* (1940), *Zygmund* (1945), *Kuttner* (1953), *J. L. Lions* (1959), and *Liverman* (1964)... have developed the basic concept of fractional calculus.

In 1783, *Leonhard Euler* made his first comments on fractional order derivative. He worked on progressions of numbers and introduced first time the generalization of factorials to *Gamma* function. A little more than fifty year after the death of *Leibniz*, *Lagrange*, in 1772, indirectly contributed to the development of exponents law for differential operators of integer order, which can be transferred to arbitrary order under certain conditions. In 1812, *Laplace* has provided the first detailed definition for fractional derivative. *Laplace* states that fractional derivative can be defined for functions with representation by an integral, in modern notation it can be written as $\int y(t)t^{-x}dt$. Few years after, *Lacroix* worked on generalizing the integer order derivative of function $y(t) = t^m$ to fractional order, where m is some natural number. In modern notations, integer order n^{th} derivative derived by *Lacroix* can be given as

$$\frac{d^n y}{dt^n} = \frac{m!}{(m-n)!} t^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}, \ m > n$$

where, Γ is the *Euler's Gamma function.*

Thus, replacing n with $\frac{1}{2}$ and letting $m = 1$, one obtains the derivative or order $\frac{1}{2}$ of the function t

$$\frac{d^{\frac{1}{2}} y}{dt^{\frac{1}{2}}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}} \sqrt{t}$$

Euler's Gamma function (or ***Euler's integral*** of the second kind) has the same importance in the fractional-order calculus and it is basically given by integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

The exponential provides the convergence of this integral in ∞, the convergence at zero obviously occurs for all complex z from the right half of the complex plane ($Re(z) > 0$).

This function is generalization of a factorial in the following form:

$$\Gamma(n) = (n-1)!.$$

Other generalizations for values in the left half of the complex plane can be obtained in following way. If we substitute e^{-t} by the well-known limit

$$e^{-t} = \lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n$$

and then use n-times integration by parts, we obtain the following limit definition of the Gamma function

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{z(z+1)\ldots(z+n)}.$$

Therefore, historically the first discussion of a derivative of fractional order appeared in a calculus written by Lacroix in 1819.

It was Liouville who engaged in the first major study of fractional calculus. Liouville's first definition of a derivative of arbitrary order ν involved an infinite series. Here, the series must be convergent for some ν. Liouville's second definition succeeded in giving a fractional derivative of x^{-a} whenever both x and are positive. Based on the definite integral related to Euler's gamma integral, the integral formula can be calculated for x^{-a}. Note that in the integral

$$\int_0^\infty u^{a-1} e^{-xu} du,$$

if we change the variables $t = xu$, then

$$\int_0^\infty u^{a-1} e^{-xu} du = \int \left(\frac{t}{x}\right)^{a-1} e^{-t} \frac{1}{x} dt = \frac{1}{x^a} \int_0^\infty t^{a-1} e^{-t} dt.$$

Thus,

$$\int_0^\infty u^{a-1}e^{-xu}du = \frac{1}{x^a}\int_0^\infty t^{a-1}e^{-t}dt.$$

From the ***Gamma*** function, we obtain the integral formula

$$x^{-a} = \frac{1}{\Gamma(a)}\int_0^\infty u^{a-1}e^{-xu}du.$$

Consequently, by assuming that $\frac{d^\nu}{dx^\nu}e^{ax} = a^\nu e^{ax}$, for any $\nu > 0$, then

$$\frac{d^\nu}{dx^\nu}x^{-a} = \frac{\Gamma(a+\nu)}{\Gamma(a)}x^{-a-\nu} = (-1)^\nu\frac{\Gamma(a+\nu)}{\Gamma(a)}x^{-a-\nu}$$

In 1884 Laurent published what is now recognized as the definitive paper on the foundations of fractional calculus. Using Cauchy's integral formula for complex valued analytical functions and a simple change of notation to employ a positive ν rather than a negative ν will now yield Laurent's definition of integration of arbitrary order

$${}_{x_0}D_x^\alpha h(x) = \frac{1}{\Gamma(\nu)}\int_{x_0}^x (x-t)^{\nu-1}h(t)dts.$$

The **Riemann-Liouville** differential operator of fractional calculus of order α defined as

$$(D_{a+}^\alpha f)(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\left(\dfrac{d}{dt}\right)^n \displaystyle\int_a^t (t-s)^{n-\alpha-1}f(s)ds & \text{if } n-1<\alpha<n, \\ \left(\dfrac{d}{dt}\right)^n f(t), & \text{if } \alpha = n, \end{cases}$$

where $\alpha, a, t \in \mathbb{R}$, $t > a$, $n = [\alpha]+1$; $[\alpha]$ denotes the integer part of the real number α, and Γ is the *Gamma* function.

The **Grünwald-Letnikov** differential operator of fractional calculus of order α defined as

$$(D_{a+}^\alpha f)(t) := \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{[\frac{t-a}{h}]}(-1)^j\tbinom{\alpha}{j}f(t-jh).$$

Binomial coefficients with alternating signs for positive value of n are defined as

$$\tbinom{n}{j} = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!} = \frac{n!}{j!(n-j)!}.$$

For binomial coefficients calculation we can use the relation between Euler's *Gamma* function and factorial, defined as

$$\tbinom{\alpha}{j} = \frac{\alpha!}{j!(\alpha-j)!} = \frac{\Gamma(\alpha)}{\Gamma(j+1)\Gamma(\alpha-j+1)}.$$

The Grünwald-Letnikov definition of differ-integral starts from classical definitions of derivatives and integrals based on infinitesimal division and limit. The disadvantages of this approach are its technical difficulty of the computations and the proofs and large restrictions on functions. (see [146])

The **Caputo** (**1967**) differential operator of fractional calculus of order α defined as

$$({}^cD^\alpha_{a+}f)(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds & \text{if } n-1<\alpha<n, \\ \left(\dfrac{d}{dt}\right)^n f(t), & \text{if } \alpha = n, \end{cases}$$

where $\alpha, a, t \in \mathbb{R}$, $t > a$, $n = [\alpha]+1$. This operator is introduced in 1967 by the Italian Caputo.

This consideration is based on the fact that for a wide class of functions, the three best known definitions ((GL), (RL), and Caputo) are equivalent under some conditions. (see ([84]))

Unfortunately, fractional calculus still lacks a geometric interpretation of integration or differentiation of arbitrary order.

We refer readers, for example, to the books such as [1, 30, 85, 96, 104, 112, 117, 121, 130] and the articles [8, 11, 12, 26, 27, 38, 39, 43, 45, 46, 47, 48, 97, 132], and references therein.

1.2 Applications of Fractional calculus

The concept of fractional calculus has great potential to change the way we see, model and analyze the systems. It provides good opportunity to scientists and engineers for revisiting the origins. The theoretical and practical interests of using fractional order operators are increasing. The application domain of fractional calculus is ranging from accurate modeling of the microbiological processes to the analysis of astronomical images.
Next, we will present a brief survey of applications of fractional calculus in science and engineering.

The Tautochrone Problem (Historical Example):
This example was studied, for the first time, by Abel in the early 19^{th} century. It was one of the basic problems where the framework of the fractional calculus was used although it is not essentially necessary.

Signal and Image Processing:
In the last decade, the use of fractional calculus in *signal processing* has tremendously increased. In *signal processing*, the fractional operators are used in the design of differentiator and integrator of fractional order, fractional order differentiator **FIR** (finite impulse response), **IIR** type digital fractional order differentiator (infinite impulse response), a new **IIR** (infinite impulse response)-type digital fractional order differentiator (**DFOD**) and for modeling the speech signal. The fractional calculus allows the edge detection, enhances the quality of images, with interesting possibilities in various image enhancement applications such as image restoration image denoising and the texture enhancement. He is used, in particularly, in satellite image classification, and astronomical image processing.

Electromagnetic Theory:
The use of fractional calculus in electromagnetic theory has emerged in the last two decades. In 1998, Engheta [69] introduced the concept of fractional curl operators and this concept is extended by Naqvi and Abbas [114]. Engheta's work gave birth to the newfield of research in Electromagnetics, namely, "*Fractional Paradigms in Electromagnetic Theory*". Nowadays fractional calculus is widely used in Electromagnetics to explore new results; for example, Faryad and Naqvi [70] have used fractional calculus for the analysis of a Rectangular Waveguide.

Control Engineering:
In industrial environments, robots have to execute their tasks quickly and precisely, minimizing production time, and the robustness of control systems is becoming imperative these days. This requires flexible robots working in large workspaces, which means that they are influenced by nonlinear and fractional order dynamic effects.

Biological Population Model
The problems of the diffusion of biological populations occur nonlinearly and the fractional order differential equations appear more and more frequently in different research

areas.

Reaction-Diffusion Equations

Fractional equations can be used to describe some physical phenomenon more accurately than the classical integer order differential equation. The reaction-diffusion equations play an important role in dynamical systems of mathematics, physics, chemistry, bioinformatics, finance, and other research areas. There has been a wide variety of analytical and numerical methods proposed for fractional equations ([107, 144]), for example, finite difference method ([59]), finite element method, Adomian decomposition method ([122]), and spectral technique ([106]). Interest in fractional reaction-diffusion equations has increased.

1.3 Fractional Calculus Theory

In this section; definitions and some auxiliary results are given regarding the main objects of the monograph: some notations and definitions of fractional calculus theory, some definitions and properteis of the measure of non-compactness , some fixed point theorems.

Consider the complete metric space $C(I) := C(I, \mathbb{R}^m)$ of continuous functions from I into $\mathbb{R}^m$ equipped with the usual metric

$$d(u, v) := \max_{t \in I} \|u(t) - v(t)\|,$$

where $\|.\|$ is a suitable norm on $\mathbb{R}^m$. Notice that $C(I)$ is a Banach space with the supremum(uniform) norm

$$\|u\|_\infty := \sup_{t \in I} \|u(t)\|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}^m$, and $L^1(I)$ denotes the space measurable functions $v : I \to \mathbb{R}^m$ which are Lebesgue integrable with the norm

$$\|v\|_1 = \int_I \|v(t)\| dt.$$

For any $n \in \mathbb{N}^*$, we denote by $AC^n(I)$ the space defined by

$$AC^n(I) := \{w : I \to \mathbb{R}^m : \frac{d^n}{dt^n} w(t) \in AC(I)\}.$$

Let

$$\delta = t\frac{d}{dt},\ q > 0,\ n = [q] + 1,$$

where $[q]$ is the integer part of q. Define the space

$$AC^n_\delta := \{u : I \to \mathbb{R}^m : \delta^{n-1}[u(t)] \in AC(I)\}.$$

Let $X := C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into E, equipped with the family of seminorms

$$\|v\|_n = \sup_{t \in [0,n]} \|v(t)\|;\ n \in \mathbb{N},$$

and the distance

$$d(u, v) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|u - v\|_n}{1 + \|u - v\|_n};\ u, v \in X.$$

Definition 1. *A nonempty subset $B \subset X$ is said to be bounded if*

$$\sup_{v \in B} \|v\|_n < \infty;\ for\ n \in \mathbb{N}.$$

Let us recall some definitions and properties of fractional integration and differentiation.

Definition 2. *[96, 121]. The fractional (arbitrary) order integral of the function $f \in L^1([0,T],\mathbb{R})$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where Γ is the gamma function.

Theorem 3. *[96] For any $f \in C([a,b],\mathbb{R})$ the Riemann-Liouville fractional integral satisfies*

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t),$$

for $\alpha, \beta > 0$.

Definition 4. *[97]. For a function f given on the interval $[0,T]$, the Caputo fractional-order derivative of order α of h, is defined by*

$$({}^c D^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 5. *[112] Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Then*

$$I^\alpha({}^c D^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

Remark 6. *([112])The Caputo derivative of a constant is equal to zero.*

We need the following auxiliary lemmas.

Lemma 7. *([145]) Let $\alpha > 0$. Then the differential equation*

$${}^c D^\alpha f(t) = 0$$

has solutions $f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 8. *[145] Let $\alpha > 0$. Then*

$$I^{\alpha c} D^\alpha f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 9. *([65], Lemma 3.11) Let* $\alpha > 0$, $\alpha \notin \mathbb{N}$ *and* $m = [\alpha]$. *Moreover assume that* $f \in C^m[a,b]$. *Then*

$$^cD_a^\alpha f \in C[a,b],$$

and

$$^cD_a^\alpha f(a) = 0.$$

Theorem 10. *([96], Theorem 2.2) Let* $\alpha \geq 0$ *and let* n *such that* $n = [\mathfrak{Re}(\alpha)] + 1$ *for* $\alpha \notin \mathbb{N}$ *and* $n = \alpha$ *for* $\alpha \in \mathbb{N}$. *Also let* $y(x) \in C^n[a,b]$. *Then the Caputo fractional derivatives* $^cD_{a+}^\alpha y(x)$ *and* $^cD_{b-}^\alpha y(x)$ *are continuous on* $[a,b]$: $^cD_{a+}^\alpha y(x) \in C[a,b]$ *and* $^cD_{b-}^\alpha y(x) \in C[a,b]$.

Definition 11. *[96] (Hadamard fractional integral) The Hadamard fractional integral of order* $q > 0$ *for a function* $u \in L^1(I)$, *is defined as*

$$({}^HI_1^q u)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{u(s)}{s} ds,$$

provided the integral exists.

Example 12. *Let* $0 < q < 1$. *Then*

$${}^HI_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q}; \ \textit{for a.e. } t \in [1,e].$$

Definition 13. *[96] (Hadamard fractional derivative) The Hadamard fractional derivative of order* $q > 0$ *applied to the function* $u \in AC_\delta^n(I)$ *is defined as*

$$({}^HD_1^q u)(x) = \delta^n({}^HI_1^{n-q} u)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^HD_1^q u)(x) = \delta({}^HI_1^{1-q} u)(x).$$

Example 14. *Let* $0 < q < 1$. *Then*

$${}^HD_1^q \ln t = \frac{1}{\Gamma(2-q)} (\ln t)^{1-q}; \ \textit{for a.e. } t \in [1,e].$$

It has been proved (see e.g. Kilbas [[95], Theorem 4.8]) that in the space $L^1(I)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$({}^HD_1^q)({}^HI_1^q w)(x) = w(x).$$

From Theorem 2.3 of [96], we have

$$({}^HI_1^q)({}^HD_1^q w)(x) = w(x) - \frac{({}^HI_1^{1-q} w)(1)}{\Gamma(q)} (\ln x)^{q-1}.$$

Analogous to the Hadamard fractional operator, the Caputo–Hadamard fractional derivative is defined in the following way:

Definition 15. *(Caputo–Hadamard fractional derivative) The Caputo–Hadamard fractional derivative of order $q > 0$ of the function $u \in AC_\delta^n$ is defined as*

$$({}^{HC}D_1^q u)(x) = ({}^{H}I_1^{n-q}\delta^n u)(x).$$

In particular, if $q \in (0, 1]$, then

$$({}^{HC}D_1^q u)(x) = ({}^{H}I_1^{1-q}\delta u)(x).$$

Definition 16. *[58, 108] The Caputo-Fabrizio fractional integral of order $0 < r < 1$ for a function $h \in L^1(I)$ is defined by*

$${}^{CF}I^r h(\tau) = \frac{2(1-r)}{M(r)(2-r)}h(\tau) + \frac{2r}{M(r)(2-r)}\int_0^\tau h(x)dx, \quad \tau \geq 0$$

where $M(r)$ is normalization constant depending on r.

Definition 17. *[58, 108] The Caputo-Fabrizio fractional derivative for a function $h \in C^1(I)$ of order $0 < r < 1$, is defined by*

$${}^{CF}D^r h(\tau) = \frac{(2-r)M(r)}{2(1-r)}\int_0^\tau \exp\left(-\frac{r}{1-r}(\tau - x)\right) h'(x)dx; \ \tau \in I.$$

Note that $({}^{CF}D^r)(h) = 0$ if and only if h is a constant function.

We state the following generalization of Gronwall 's lemma for singular kernels.

Lemma 18. *[142] Let $v : [0, T] \to [0, +\infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, T]$. Assume that there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + a\int_0^t (t-s)^{-\alpha} v(s)ds,$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq w(t) + Ka\int_0^t (t-s)^{-\alpha} w(s)ds, \ \textit{for every } t \in [0, T].$$

Bainov and Hristova [29] introduced the following integral inequality of Gronwall type for piecewise continuous functions which can be used in the sequel.

Lemma 19. *Let for $t \geq t_0 \geq 0$ the following inequality hold*

$$x(t) \leq a(t) + \int_{t_0}^t g(t,s)x(s)ds + \sum_{t_0 < t_k < t} \beta_k(t)x(t_k),$$

where $\beta_k(t)(k \in \mathbb{N})$ are nondecreasing functions for $t \geq t_0$, $a \in PC([t_0, \infty), \mathbb{R}_+)$, a is nondecreasing and $g(t, s)$ is a continuous nonnegative function for $t, s \geq t_0$ and *nondecreasing with respect to t for any fixed $s \geq t_0$. Then, for $t \geq t_0$, the following inequality is valid:*

$$x(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \beta_k(t)) exp \left(\int_{t_0}^{t} g(t,s) ds \right).$$

Theorem 20. *[82](theorem of Ascoli-Arzela) Let $A \subset C(J, \mathbb{R})$, A is relatively compact (i.e $\overline{A}$ is compact) if:*

1. *A is uniformly bounded i.e, there exists $M > 0$ such that*

$$|f(x)| < M \text{ for every } f \in A \text{ and } x \in J.$$

2. *A is equicontinuous i.e, for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, \overline{x} \in J$, $|x - \overline{x}| \leq \delta$ implies $|f(x) - f(\overline{x})| \leq \epsilon$, for every $f \in A$.*

1.3.1 Some definitions and properties of the measure of non-compactness

In this section we define the Kuratowski (1896-1980) and Hausdorf measures of non-compactness (**MNC** for short) and give their basic properties.

Definition 21. *[99] Let (X, d) be a complete metric space and $\mathcal{B}$ the family of bounded subsets of X. For every $B \in \mathcal{B}$ we define the Kuratowski measure of non-compactness $\alpha(B)$ of the set B as the infimum of the numbers d such that B admits a finite covering by sets of diameter smaller than d.*

Remark 22. *The diameter of a set B is the number $\sup\{d(x, y) : x \in B, y \in B\}$ denoted by $diam(B)$, with $diam(\emptyset) = 0$.*
It is clear that $0 \leq \alpha(B) \leq diam(B) < +\infty$ for each nonempty bounded subset B of X and that $diam(B) = 0$ if and only if B is an empty set or consists of exactly one point.

Definition 23. *[33] Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty]$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E,$$

where

$$diam(B_i) = sup\{||x - y|| : x, y \in B_i\}.$$

The Kuratowski measure of noncompactness satisfies the following properties:

Lemma 24. *([19, 33, 34, 99]) Let A and B bounded sets.*

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is relatively compact), where $\overline{B}$ denotes the closure of B.*

(b) nonsingularity : α is equal to zero on every one element-set.

(c) If B is a finite set, then $\alpha(B) = 0$.

(d) $\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$, *where $convB$ is the convex hull of B.*

(e) monotonicity : $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(f) algebraic semi-additivity : $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, *where*

$$A + B = \{x + y : x \in A, \quad y \in B\}.$$

(g) semi-homogencity : $\alpha(\lambda B) = |\lambda|\alpha(B)$; $\lambda \in \mathbb{R}$, *where* $\lambda(B) = \{\lambda x : x \in B\}$.

(h) semi-additivity : $\alpha(A \bigcup B) = max\{\alpha(A), \alpha(B)\}$.

(i) $\alpha(A \bigcap B) = min\{\alpha(A), \alpha(B)\}$.

(j) invariance under translations : $\alpha(B + x_0) = \alpha(B)$ *for any* $x_0 \in E$.

Remark 25. *The a-measure of noncompactness was introduced by Kuratowski in order to generalize the Cantor intersection theorem*

Theorem 26. *[99] Let* (X, d) *be a complete metric space and* $\{B_n\}$ *be a decreasing sequence of nonempty, closed and bounded subsets of* X *and* $\lim_{n\to\infty} \alpha(B_n) = 0$*. Then the intersection* B_∞ *of all* B_n *is nonempty and compact.*

In [86], Horvath has proved the following generalization of the Kuratowski theorem:

Theorem 27. *[99] Let* (X, d) *be a complete metric space and* $\{B_i\}_{i\in I}$ *be a family of nonempty of closed and bounded subsets of* X *having the finite intersection property. If* $\inf_{i\in I} \alpha(B_i) = 0$ *then the intersection* B_∞ *of all* B_i *is nonempty and compact.*

Lemma 28. *[81] If* $V \subset C(J, E)$ *is a bounded and equicontinuous set, then*

(i) the function $t \to \alpha(V(t))$ *is continuous on* J*, and*

$$\alpha_c(V) = \sup_{0\le t\le T} \alpha(V(t)).$$

(ii) $\alpha\left(\int_0^T x(s)ds : x \in V\right) \le \int_0^T \alpha(V(s))ds,$

where

$$V(s) = \{x(s) : x \in V\}, \ s \in J.$$

In the definition of the Kuratowski measure we can consider balls instead of arbitrary sets. In this way we get the definition of the Hausdorff measure:

Definition 29. *([99]) The Hausdorff measure of non-compactness* $\chi(B)$ *of the set* B *is the infimum of the numbers* r *such that* B *admits a finite covering by balls of radius smaller than* r.

Theorem 30. *([99]) Let* $B(0, 1)$ *be the unit ball in a Banach space* X*. Then*

$$\alpha(B(0, 1)) = \chi(B(0, 1)) = 0$$

if X *is finite dimensional, and* $\alpha(B(0, 1)) = 2,\ \chi(B(0, 1)) = 1$ *otherwise.*

Theorem 31. *([99]) Let* $S(0, 1)$ *be the unit sphere in a Banach space* X*. Then* $\alpha(S(0, 1)) = \chi(S(0, 1)) = 0$ *if* X *is finite dimensional, and* $\alpha(S(0, 1)) = 2,\ \chi(S(0, 1)) = 1$ *otherwise.*

Theorem 32. *([99]) The Kuratowski and Hausdorff* **MNCs** *are related by the inequalities*

$$\chi(B) \leq \alpha(B) \leq 2\chi(B).$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

Example 33. *Let l^∞ be the space of all real bounded sequences with the supremum norm, and let A be a bounded set in l^∞. Then $\alpha(A) = 2\chi(A)$.*

For further facts concerning measures of non-compactness and their properties we refer to [19, 25, 33, 34, 99] and the references therein.

We recall the following definition of the notion of a sequence of measures of non-compactness [66, 67].

Definition 34. *Let $\mathcal{M}_F$ be the family of all nonempty and bounded subsets of a Fréchet space F. A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$ where $\mu_n : \mathcal{M}_F \to [0, \infty)$ is said to be a family of measures of non-compactness in the real Fréchet space F if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_F$:*

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is: $\mu_n(B) = 0$ for $n \in \mathbb{N}$ if and only if B is precompact,

(b) $\mu_n(B_1) \leq \mu_n(B_2)$ for $B_1 \subset B_2$ and $n \in \mathbb{N}$,

(c) $\mu_n(ConvB) = \mu_n(B)$ for $n \in \mathbb{N}$,

(d) If $\{B_i\}_{i=1,\cdots}$ is a sequence of closed sets from $\mathcal{M}_F$ such that $B_{i+1} \subset B_i$; $i = 1, \cdots$ and if $\lim\limits_{i\to\infty} \mu_n(B_i) = 0$, for each $n \in \mathbb{N}$, then the intersection set $B_\infty := \cap_{i=1}^{\infty} B_i$ is nonempty.

Some Properties:

(e) We call the family of measures of non-compactness $\{\mu_n\}_{n\in\mathbb{N}}$ to be homogeneous if $\mu_n(\lambda B) = |\lambda|\mu_n(B)$; for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.

(f) If the family $\{\mu_n\}_{n\in\mathbb{N}}$ satisfied the condition $\mu_n(B_1 \cup B_2) \leq \mu_n(B_1) + \mu_n(B_2)$, for $n \in \mathbb{N}$, it is called subadditive.

(g) It is sublinear if both conditions (e) and (f) hold.

(h) We say that the family of measures $\{\mu_n\}_{n\in\mathbb{N}}$ has the maximum property if

$$\mu_n(B_1 \cup B_2) = \max\{\mu_n(B_1), \mu_n(B_2)\},$$

(i) The family of measures of non-compactness $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be regular if if the conditions (a), (g) and (h) hold; (full sublinear and has maximum property).

Example 35. *[66, 118] For $B \in \mathcal{M}_X$, $x \in B$, $n \in \mathbb{N}$ and $\epsilon > 0$, let us denote by $\omega^n(x, \epsilon)$ the modulus of continuity of the function x on the interval $[0, n]$; that is,*

$$\omega^n(x, \epsilon) = \sup\{\|x(t) - x(s)\| : t, s \in [0, n], |t - s| \leq \epsilon\}.$$

Further, let us put

$$\omega^n(B, \epsilon) = \sup\{\omega^n(x, \epsilon) : x \in B\},$$

$$\omega_0^n(B) = \lim_{\epsilon\to 0^+} \omega^n(B, \epsilon),$$

$$\bar{\alpha}^n(B) = \sup_{t\in[0,n]} \alpha(B(t)) := \sup_{t\in[0,n]} \alpha(\{x(t) : x \in B\}),$$

and

$$\beta_n(B) = \omega_0^n(B) + \bar{\alpha}^n(B).$$

The family of mappings $\{\beta_n\}_{n\in\mathbb{N}}$ where $\beta_n : \mathcal{M}_X \to [0, \infty)$, satisfies the conditions (a)-(d) fom Definition 34.

Lemma 36. *[52] If Y is a bounded subset of a Fréchet space F, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\mu_n(Y) \leq 2\mu_n(\{y_k\}_{k=1}^{\infty}) + \epsilon; \ \textit{for } n \in \mathbb{N}.$$

Lemma 37. *[113] If $\{u_k\}_{k=1}^{\infty} \subset L^1([0, n])$ is uniformly integrable, then $\mu_n(\{u_k\}_{k=1}^{\infty})$ is measurable for $n \in \mathbb{N}^*$, and*

$$\mu_n\left(\left\{\int_1^t u_k(s)ds\right\}_{k=1}^{\infty}\right) \leq 2\int_1^t \mu_n(\{u_k(s)\}_{k=1}^{\infty})ds,$$

for each $t \in [0, n]$.

Definition 38. *Let Ω be a nonempty subset of a Fréchet space F, and let $A : \Omega \to F$ be a continuous operator which transforms bounded subsets of onto bounded ones. One says that A satisfies the Darbo condition with constants $(k_n)_{n\in\mathbb{N}}$ with respect to a family of measures of non-compactness $\{\mu_n\}_{n\in\mathbb{N}}$, if*

$$\mu_n(A(B)) \leq k_n \mu_n(B)$$

for each bounded set $B \subset \Omega$ and $n \in \mathbb{N}$.
If $k_n < 1$; $n \in \mathbb{N}$ then A is called a contraction with respect to $\{\mu_n\}_{n\in\mathbb{N}}$.

1.3.2 Some fixed point theorems

Theorem 39 (Banach's fixed point theorem (1922) [79])**.** *Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 40 (Schauder fixed point theorem [25])**.** *Let X be a Banach space, D be a bounded closed convex subset of X and $T : D \to D$ be a compact and continuous map. Then T has at least one fixed point in D.*

Theorem 41 (Nonlinear alternative of Leray-Schauder type [79])**.** *Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to C$ continuous and compact operator.*
Then,
(a) either T has fixed points,
(b) or there exist $u \in \partial U$ and $\lambda \in [0, 1]$ with $u = \lambda T(u)$.

Theorem 42 (Darbo's Fixed Point Theorem [33, 79])**.** *Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $N : C \to C$ is such that for all closed subsets D of C,*

$$\alpha(T(D)) \leq k\alpha(D), \tag{1.1}$$

where $0 \leq k < 1$, and α is the Kuratowski measure of noncompactness. Then T has a fixed point in C.

Remark 43. *Mappings satisfying the Darbo-condition (1.1) have subsequently been called k-set contractions.*

Theorem 44 (Mönch's Fixed Point Theorem [14, 113])**.** *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0 \tag{1.2}$$

holds for every subset V of D, then N has a fixed point.
Here α is the Kuratowski measure of noncompactness.

The following generalization of the classical Darbo fixed point theorem for Fréchet spaces.

Theorem 45. *[66, 67] Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space F and let $V : \Omega \to \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

For more details see [14, 23, 78, 79, 99, 143]

Chapter 2

Caputo-Hadamard Fractional Differential Equation

In this chapter, we will give existence and uniqueness results for a class of boundary value problem of fractional differential equations in banach spaces with Caputo-Hadamard fractional derivative.

2.1 Caputo-Hadamard Fractional Differential Equations

2.1.1 Introduction and Motivations

The purpose of this section, is to establish existence and uniqueness of solutions for the following class of Caputo-Hadamard fractional differential equation

$$({}^{Hc}D_1^{\alpha}u)(t) = f(t, u(t)),\ t \in I := [1, T], \tag{2.1}$$

$$\begin{cases} a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1), \\ a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2), \end{cases} \tag{2.2}$$

where $\alpha \in (1, 2],\ T > 1,\ a_1, b_1, d_1, a_2, b_2, d_2 \in \mathbb{R},\ \xi_1, \xi_2 \in (1, T),\ f : I \times \mathbb{R}^m \to \mathbb{R}^m,\ m \in \mathbb{N}^*$ is a given continuous function, and ${}^{Hc}D_1^{\alpha}$ is the Caputo–Hadamard fractional derivative of order α.

In [3], S. Abbas et al. studied the existence of solutions for the following problem of Caputo-Hadamard fractional differential equations of the form

$$\begin{cases} ({}^{Hc}D_1^{r}u)(t) = f(t, u(t)),\ t \in I := [1, T], \\ u(t)|_{t=1} = \phi, \end{cases} \tag{2.3}$$

where $r \in (0,1)$, $T > 1$, $\phi \in E$, $f : I \times E \to E$ is a given continuous function, E is a real(or complex) Banach space with a norm $\| \cdot \|$, ${}^{Hc}D_1^r$ is the Caputo-Hadamard fractional derivative of order r.

They next discussed the existence of solutions for the following problem of Caputo-Hadamard partial fractional differential equation of the form

$$\begin{cases} ({}^{Hc}D_\sigma^r u)(t,x) = f(t,x,u(t,x)), \ (t,x) \in J := [1,T] \times [1,b], \\ u(t,1) = \phi(t); \ t \in [1,T], \\ u(1,x) = \psi(x); \ x \in [1,b], \end{cases} \tag{2.4}$$

where $r = (r_1, r_2) \in (0,1] \times (0,1]$, $T, b > 1$, $\sigma = (1,1)$, $f : J \times E \to E$ is a given continuous function, $\phi : [1,T] \to E$ and $\psi : [1,b] \to E$ are given absolutely continuous functions with $\phi(1) = \psi(1)$, and ${}^{Hc}D_1^r$ is the Caputo-Hadamard partial fractional derivative of order r.

In [74]; the authors used the technique of measure of weak noncompactness combine with the fixed point theorem to discuss the existence theorem of weak solutions for a class of nonlinear fractional integrodifferential equations of the form

$$\begin{gathered} ({}^{c}D_{0^+}^{\alpha} u)(t) = f(t, x(t), T(x)(t), (Sx)(t)); \ t \in [0,1], \ \alpha \in (1,2], \\ \begin{cases} a_1 x(0) - b_1 x'(0) = d_1 x(\xi_1), \\ a_2 x(1) + b_2 x'(1) = d_2 x(\xi_2), \end{cases} \end{gathered} \tag{2.5}$$

where T and S are two operators defined by

$$\begin{cases} (Tu)(t) = \displaystyle\int_0^t k_1(t,s) g(s,u(s)) ds, \\ (Su)(t) = \displaystyle\int_0^a k_2(t,s) h(s,u(s)) ds, \end{cases}$$

E is a nonreflexive Banach space.

2.1.2 Existence of solutions

Consider the complete metric space $C(I) := C(I, \mathbb{R}^m)$ of continuous functions from I into $\mathbb{R}^m$ equipped with the usual metric

$$d(u, v) := \max_{t \in I} \|u(t) - v(t)\|,$$

where $\|\cdot\|$ is a suitable norm on $\mathbb{R}^m$.

Notice that $C(I)$ is a Banach space with the supremum (uniform) norm

$$\|u\|_\infty := \sup_{t \in I} \|u(t)\|.$$

By $BV(I, \mathbb{R})$, we denote the space of real bounded variation functions with its classical norm $\|\cdot\|_{BV}$.

Let us defining what we mean by a solution of problem (2.1)-(2.2).

Definition 46. *By a solution of the problem (2.1)-(2.2) we mean a continuous function u that satisfies the equation (2.1) on I and the conditions (2.2).*

For the existence of solutions for the problem (2.1)-(2.2); we need the following auxiliary lemma:

Lemma 47. *Let $h \in C$ and $\alpha \in (1, 2]$. Then the unique solution of the problem*

$$\begin{cases} ({}^{Hc}D_1^\alpha u)(t) = h(t), \quad t \in I, \\ a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1), \\ a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2), \end{cases}$$

is given by

$$u(t) = \int_1^T G(t, s) h(s) ds,$$

where G is the Green function with $G(t, s)$ given by

$$\begin{aligned} &\frac{\left(\ln \frac{t}{s}\right)^{\alpha-1}}{s\Gamma(\alpha)} \\ &+ \frac{b_1 + d_1 \ln \xi_1}{\Delta} \left[\frac{a_2 \left(\ln \frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b_2 \left(\ln \frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{d_2 \left(\ln \frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)} \right] \\ &- \frac{d_1 \left(\ln \frac{\xi_1}{s}\right)^{\alpha-1}}{s\Delta\Gamma(\alpha)} \left[a_2 \ln T + \frac{b_2}{T} - d_2 \ln \xi_2 \right] \end{aligned}$$

$$+\frac{d_1\ln t\left(\ln\frac{\xi_1}{s}\right)^{\alpha-1}}{s\Delta\Gamma(\alpha)}[a_2-d_2]$$

$$-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]$$

for $s\le\xi_1$ *and* $s\le t$,

$$\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]$$

$$-\frac{d_1\left(\ln\frac{\xi_1}{s}\right)^{\alpha-1}}{s\Delta\Gamma(\alpha)}\left[a_2\ln T+\frac{b_2}{T}-d_2\ln\xi_2\right]$$

$$+\frac{d_1\ln t\left(\ln\frac{\xi_1}{s}\right)^{\alpha-1}}{s\Delta\Gamma(\alpha)}[a_2-d_2]$$

$$-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]$$

for $s\le\xi_1$ *and* $t\le s$,

$$\frac{\left(\ln\frac{t}{s}\right)^{\alpha-1}}{s\Gamma(\alpha)}$$

$$+\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]$$

$$-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]$$

for $\xi_1\le s\le\xi_2$ *and* $s\le t$,

$$\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]$$

$$-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]$$

for $\xi_1\le s\le\xi_2$ *and* $t\le s$,

$$+\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]$$

$$-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]$$

for $\xi_2 \le s \le t$, *and*

$$\frac{\left(\ln\frac{t}{s}\right)^{\alpha-1}}{s\Gamma(\alpha)}$$
$$\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]$$
$$-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]$$

for $\xi_2 \le s$ *and* $t \le s$,

$$\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]$$
$$-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]$$

where

$$\Delta=(d_1-a_1)\{a_2\ln T+\frac{b_2}{T}-d_2\ln\xi_2\}-(a_2-d_2)(b_1+d_1\ln\xi_1)\neq 0.$$

Proof.

By Lemma 59, solving the linear fractional differential equation

$$({}^{Hc}D_1^{\alpha}u)(t)=h(t),$$

we obtain

$$u(t)=({}^{H}I_1^{\alpha}u)(t)+c_1+c_2\ln t. \tag{2.6}$$

On the other hand, by the relation

$$D_1^{\beta}I_1^{\alpha}u(t)=I_1^{\alpha-\beta}u(t),$$

we get

$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-2}h(s)\frac{\mathrm{S}}{s}+\frac{c_2}{t}.$$

From the boundary conditions, we have

$$(d_1-a_1)c_1+(b_1+d_1\ln\xi_1)c_2=a_1\alpha h(1)-b_1\alpha-1h(1)-d_1\alpha h(\xi_1)$$

and

$$(a_2 - d_2)c_1 + a_2 \ln T + \frac{b_2}{T} - d_2 \ln \xi_2 c_2 = d_2 \alpha h(\xi_2) - a_2 \alpha h(T) - b_2 \alpha - 1 h(T).$$

Thus, we get

$$\begin{aligned}
c_1 =& \frac{b_1 + d_1 \ln \xi_1}{\Delta} \left[a_2 \int_1^T \left(\ln \frac{T}{s} \right)^{\alpha-1} \frac{h(s)}{s\Gamma(\alpha)} \underset{.}{s} \right. \\
&\left. + b_2 \int_1^T \left(\ln \frac{T}{s} \right)^{\alpha-2} \frac{h(s)}{s\Gamma(\alpha-1)} \underset{.}{s} - d_2 \int_1^{\xi_2} \left(\ln \frac{\xi_2}{s} \right)^{\alpha-1} \frac{h(s)}{s\Gamma(\alpha)} \underset{.}{s} \right] \\
&- \frac{d_1}{\Delta\Gamma(\alpha)} \left[a_2 \ln T + \frac{b_2}{T} - d_2 \ln \xi_2 \right] \int_1^{\xi_1} \left(\ln \frac{\xi_1}{s} \right)^{\alpha-1} \frac{h(s)}{s} \underset{.}{s}
\end{aligned}$$

and

$$\begin{aligned}
c_2 =& \frac{d_1}{\Delta\Gamma(\alpha)} [a_2 - d_2] \int_1^{\xi_1} \left(\ln \frac{\xi_1}{s} \right)^{\alpha-1} \frac{h(s)}{s} \underset{.}{s} \\
&- \frac{d_1 - a_1}{\Delta} \left[a_2 \int_1^T \left(\ln \frac{T}{s} \right)^{\alpha-1} \frac{h(s)}{s\Gamma(\alpha)} \underset{.}{s} \right. \\
&\left. + b_2 \int_1^T \left(\ln \frac{T}{s} \right)^{\alpha-2} \frac{h(s)}{s\Gamma(\alpha-1)} \underset{.}{s} - d_2 \int_1^{\xi_2} \left(\ln \frac{\xi_2}{s} \right)^{\alpha-1} \frac{h(s)}{s\Gamma(\alpha)} \underset{.}{s} \right].
\end{aligned}$$

Substituting the values of c_1 and c_2 in 2.6, we get

$$\begin{aligned}
u(t) =& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{h(s)}{s} \underset{.}{s} \\
&+ \frac{b_1 + d_1 \ln \xi_1}{\Delta} \left[a_2 \int_1^T \left(\ln \frac{T}{s} \right)^{\alpha-1} \frac{h(s)}{s\Gamma(\alpha)} \underset{.}{s} \right. \\
&\left. + b_2 \int_1^T \left(\ln \frac{T}{s} \right)^{\alpha-2} \frac{h(s)}{s\Gamma(\alpha-1)} \underset{.}{s} - d_2 \int_1^{\xi_2} \left(\ln \frac{\xi_2}{s} \right)^{\alpha-1} \frac{h(s)}{s\Gamma(\alpha)} \underset{.}{s} \right] \\
&- \frac{d_1}{\Delta\Gamma(\alpha)} \left[a_2 \ln T + \frac{b_2}{T} - d_2 \ln \xi_2 \right] \int_1^{\xi_1} \left(\ln \frac{\xi_1}{s} \right)^{\alpha-1} \frac{h(s)}{s} \underset{.}{s} \\
&+ \frac{d_1 \ln t}{\Delta\Gamma(\alpha)} [a_2 - d_2] \int_1^{\xi_1} \left(\ln \frac{\xi_1}{s} \right)^{\alpha-1} \frac{h(s)}{s} \underset{.}{s} \\
&- \frac{(d_1 - a_1) \ln t}{\Delta} \left[a_2 \int_1^T \left(\ln \frac{T}{s} \right)^{\alpha-1} \frac{h(s)}{s\Gamma(\alpha)} \underset{.}{s} \right. \\
&\left. + b_2 \int_1^T \left(\ln \frac{T}{s} \right)^{\alpha-2} \frac{h(s)}{s\Gamma(\alpha-1)} \underset{.}{s} - d_2 \int_1^{\xi_2} \left(\ln \frac{\xi_2}{s} \right)^{\alpha-1} \frac{h(s)}{s\Gamma(\alpha)} \underset{.}{s} \right] \\
=& \int_1^T G(t,s) h(s) \underset{.}{s}.
\end{aligned}$$

Remark 48. *Notice that the function $G(\cdot,\cdot)$ is not continuous over $[1,T]\times[1,T]$, however the function $t \mapsto \int_1^t G(t,s)ds$ is continuous on $[1,T]$. Set*

$$G^* = \sup_{t\in[1,T]} \int_1^t |G(t,s)|ds.$$

Definition 49. *([128]) A nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it satisfies one of the following conditions:*

(1) *For any $t > 0$ we have*

$$\lim_{n\to\infty} \phi^{(n)}(t) = 0,$$

where $\phi^{(n)}$ denotes the n-th iteration of ϕ.

(2) *The function ϕ is right-continuous and satisfies*

$$\forall t > 0 : \phi(t) < t.$$

Remark 50. *The choice $\phi(t) = kt$ with $0 < k < 1$ gives the classical Banach contraction mapping principle.*

Definition 51. *[20] Let (M,d) be a metric space. A map $T : M \to M$ is said to be Lipschitzian if there exists a constant $k > 0$ (called Lipschitz constant) such that*

$$d(T(x),T(y)) \le kd(x,y);\ \ for\ all\ x,y \in M.$$

A Lipschitzian mapping with a Lipschitz constant $k < 1$ is called a contraction.

For our purpose we will need the following fixed point theorem:

Theorem 52. *[53, 110] Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(T(x),T(y)) \le \phi(d(x,y)),$$

where ϕ is a comparison function. Then T has a unique fixed point in X.

The following hypotheses will be used in the sequel.

(H_1) The function f satisfies the generalized Lipschitz condition:

$$\|f(t,u) - f(t,v)\| \le \frac{1}{G^*}\phi(\|u-v\|),$$

for $t \in I$ and $u,v \in \mathbb{R}^m$, where ϕ is a comparison function.

(H_2) There exist functions $p, q \in C(I, [0, \infty))$ such that

$$\|f(t,u)\| \le p(t) + q(t)\|u\|, \text{ for each } t \in I \text{ and } u \in \mathbb{R}^m.$$

Set

$$p^* = \sup_{t \in I} p(t), \;\; q^* = \sup_{t \in I} q(t).$$

First, we prove an existence and uniqueness result for the problem (2.1)- (2.2).

Theorem 53. *Assume that the hypothesis (H_1) holds. Then there exists a unique solution of problem (2.1)- (2.2) on I.*

Proof. By using Lemma 59, we transform the problem (2.1)- (2.2) into a fixed point problem.

Consider the operator $N : C(I) \to C(I)$ defined by

$$(Nu)(t) = \int_1^T G(t,s) f(s, u(s)) ds; \; t \in I. \tag{2.7}$$

For each $u, v \in C(I)$ and $t \in I$, we have

$$\begin{array}{rcl} \|(Nu)(t) - (Nv)(t)\| & = & \| \int_1^T G(t,s)[f(s,u(s)) - f(s,v(s))] ds \| \\ & \le & \int_1^T \|G(t,s)[f(s,u(s)) - f(s,v(s))]\| ds \\ & \le & \int_1^T |G(t,s)| \|[f(s,u(s)) - f(s,v(s))]\| ds \;\cdot \\ & \le & \phi(\|u(s) - v(s)\|) \\ & \le & \phi(d(u,v)). \end{array}$$

Thus, we get

$$d(N(u), N(v)) \le \phi(d(u,v)).$$

Consequently, from Theorem 52, the operator N has a unique fixed point, which is the unique solution of our problem (2.1)-(2.2) on I.

Now, we prove an existence result by using Schauder fixed point theorem.

Theorem 54. *Assume that the hypothesis (H_2) holds. If*

$$G^* q^* < 1,$$

then the problem (2.1)- (2.2) has at least one solution defined on I.

Proof. Let N be the operator defined in (2.7). Set

$$R \ge \frac{G^* p^*}{1 - G^* q^*},$$

and consider the closed and convex ball $B_R = \{u \in C(I) : \|u\|_\infty \leq R\}$.

Let $u \in B_R$. Then, for each $t \in I$, we have

$$\begin{array}{rcl} \|(Nu)(t)\| & \leq & \int_1^T \|G(t,s)\|\|f(s,u(s)\|ds \\ & \leq & \int_1^T |G(t,s)|\|f(s,u(s)\|ds \\ & \leq & \int_1^T |G(t,s)|(p(s)+q(s)\|u(s)\|)ds \\ & \leq & G^*(p^*+Rq^*). \end{array}.$$

Thus

$$\|N(u)\|_\infty \leq R.$$

Hence N maps the ball B_R into B_R. We shall show that the operator $N : B_R \to B_R$ satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Step1: N is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I$, we have

$$\begin{array}{rcl} \|(Nu_n)(t)-(Nu)(t)\| & \leq & \int_1^T \|G(t,s)[f(s,u_n(s))-f(s,u(s))]\|ds \\ & \leq & \int_1^T |G(t,s)|\|f(s,u_n(s))-f(s,u(s))\|ds. \end{array}$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, then by the Lebesgue dominated convergence theorem;

$$\|N(u_n)-N(u)\|_\infty \to 0 \;\; as \;\; n \to \infty.$$

Step2: $N(B_R)$ is bounded. This is clear since $N : B_R \to B_R$ and B_R is bounded.

Step3: N maps bounded sets into equicontinuous sets in B_R.
Let $t_1, t_2 \in I$, such that $t_1 < t_2$ and let $u \in B_R$. Then, we have

$$\begin{array}{rcl} \|(Nu)(t_1)-(Nu)(t_2)\| & \leq & \int_1^T |G(t_1,s)-G(t_2,s)|\|f(s,u(s))\|ds \\ & \leq & (p^*+Rq^*)\int_1^T |G(t_1,s)-G(t_2,s)|ds. \end{array}$$

As $t_1 \to t_2$, from Remark 2.1.3, the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3, together with the Arzelá–Ascoli theorem, we can conclude that $N : B_R \to B_R$ is continuous and compact. From an application of Theorem 40, we deduce that N has a fixed point u which is a solution of problem (2.1)-(2.2).

2.1.3 An Example

As application of our results, we consider the following class of Caputo–Hadamard fractional differential equation of the form

$$({}^{Hc}D_1^{\frac{3}{2}}u)(t) = \frac{ce^{-2t-1}}{1+e^{2t}|u(t)|};\ t \in [1,e], \tag{2.8}$$

with the four-point boundary conditions

$$\begin{cases} u(1) - cu'(1) = u(\frac{3}{2}), \\ u(e) + 2u'(e) = u(2), \end{cases} \tag{2.9}$$

where $c > 0$. Set $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ with

$$f(t,u) = \frac{ce^{-2t-1}}{1+e^{2t}|u|};\ t \in [1,e].$$

It is clear that f is continuous, and satisfies the hypothesis (H_1) with

$$\phi(x) = ce^{-1}G^*x;\ x > 0,$$

and a good choice of the constant c; like $c < \frac{e}{G^*}$. Hence by Theorem 53, the problem (2.8)-(2.13) has a unique solution defined on $[1,e]$.

2.2 Implicit Caputo-Hadamard Fractional Differential Equations

2.2.1 Introduction and motivations

In this section, we discuss the existence and uniqueness of solutions for the following class of Caputo-Hadamard fractional differential equation

$$({}^{Hc}D_1^{\alpha}u)(t) = f(t, u(t), ({}^{Hc}D_1^{\alpha}u)(t)),\ t \in I := [1, T], \tag{2.10}$$

$$\begin{cases} a_1u(1) - b_1u'(1) = d_1u(\xi_1), \\ a_2u(T) + b_2u'(T) = d_2u(\xi_2), \end{cases} \tag{2.11}$$

where $\alpha \in (1, 2],\ T > 1,\ a_1, b_1, d_1, a_2, b_2, d_2 \in \mathbb{R},\ \xi_1, \xi_2 \in (1, T),\ f : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,\ m \in \mathbb{N}^*$ is a given continuous function, and ${}^{Hc}D_1^{\alpha}$ is the Caputo–Hadamard fractional derivative of order α.

In [50]; the authors established the existence, uniqueness and stability results of solutions for the following initial value problem for imlicit fractional order differential equations

$$\begin{cases} {}^{H}D^{\alpha}y(t) = f(t, y(t), {}^{H}D^{\alpha}y(t)),\ t \in J,\ 0 < \alpha \leq 1, \\ y(1) = y_1, \end{cases}$$

where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function space, $y_1 \in \mathbb{R}$ and $J = [1, T]$, $T > 1$.

In [41]; the following classes of boundary value problems for the existence and stability of solutions for implicit fractional differential equations with Caputo fractional derivative:

$$\begin{cases} {}^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)),\ t \in J := [0, T],\ T > 0,\ 0 < \alpha \leq 1, \\ \alpha y(0) + by(T) = c, \end{cases}$$

where ${}^{c}D^{\alpha}$ is the fractional derivative of Caputo, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function, and a,b,c are real constants with $a + b \neq 0$, and

$$\begin{cases} {}^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)),\ t \in J := [0, T],\ T > 0,\ 0 < \alpha \leq 1, \\ y(0) + g(y) = y_0, \end{cases}$$

where $g : C([0, T], \mathbb{R}) \to \mathbb{R}$ a continuous function and y_0 a real constant; are studied. This type of non-local Cauchy problem was introduced by Byszewski[57]. The author observed that the non-local condition is more appropriate that the non-local condition(initial) to describe correctly some physics phenomenons[57] and proved the existence and the uniqueness of weak solutions and also classical solutions for this type of problems. We take an example of non-local conditions as follows:

$$g(y) = \sum_{i=1}^{p} c_i y(t_i)$$

where c_i, $i = 1...p$ are constants and $0 < t_1 < ... < t_p \leq T$.

2.2.2 Existence of solutions

Consider the complete metric space $C(I) := C(I, \mathbb{R}^m)$ of continuous functions from I into $\mathbb{R}^m$ equipped with the usual metric

$$d(u, v) := \max_{t \in I} \|u(t) - v(t)\|,$$

where $\| \cdot \|$ is a suitable norm on $\mathbb{R}^m$. Notice that $C(I)$ is a Banach space with the supremum (uniform) norm

$$\|u\|_\infty := \sup_{t \in I} \|u(t)\|.$$

Let us defining what we mean by a solution of problem (2.10)-(2.11).

Definition 55. *By a solution of the problem (2.10)-(2.11) we mean a continuous function u that satisfies the equation (2.10) on I and the conditions (2.11).*

For the existence of solutions for the problem (2.10)-(2.11); we need the Lemma 59.

The following hypotheses will be used in the sequel.

(H_1) The function f satisfies the generalized Lipschitz condition:

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \frac{1}{G^*}\phi_1(\|u_1 - u_2\|) + \phi_2(\|v_1 - v_2\|),$$

for $t \in I$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}^m$, where ϕ_1 and ϕ_2 are comparison functions.

(H_2) There exist functions $p, q, r \in C(I, [0, \infty))$with $r(t) < 1$ such that

$$(1 + \|u\|)\|f(t, u, v)\| \leq p(t) + q(t)\|u\| + r(t)\|v\|, \text{ for each } t \in I \text{ and } u, v \in \mathbb{R}^m.$$

Set

$$p^* = \sup_{t \in I} p(t), \; q^* = \sup_{t \in I} q(t), \; r^* = \sup_{t \in I} r(t).$$

First, we prove an existence and uniqueness result for the problem (2.10)- (2.11).

Theorem 56. *Assume that the hypothesis (H_1) holds. Then there exists a unique solution of problem (2.10)- (2.11) on I.*

Proof. By using of Lemma 59, we transform the problem (2.10)- (2.11) into a fixed point problem. Consider the operator $N : C(I) \to C(I)$ defined by

$$(Nu)(t) = \int_1^T G(t,s)g(s)ds; \; t \in I, \tag{2.12}$$

where $g \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)), \; or \; g(t) = f\left(t, \int_1^T G(t,s)g(s)ds, g(s)\right).$$

Let $u, v \in C(I)$. Then, for $t \in I$, we have

$$\|(Nu)(t) - (Nv)(t)\| \;\leq\; \textstyle\int_1^T \|G(t,s)(g(s) - h(s))\|ds,$$

where $g, h \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)), \; and \; h(t) = f(t, u(t), h(t)).$$

From (H_2), we get

$$\|g(t) - h(t)\| \leq \frac{1}{G^*}\phi_1(\|u(t) - v(t\|) + \phi_2(\|g(t) - h(t\|).$$

Thus

$$\|g(t) - h(t)\| \leq \frac{1}{G^*}(Id - \phi_2)^{-1}\phi_1(\|u(t) - v(t\|),$$

where Id is the identity function. Set $\phi := (Id - \phi_2)^{-1}\phi_1$. We obtain

$$\begin{array}{rcl} \|(Nu)(t) - (Nv)(t)\| & \leq & \phi(\|u(t) - v(t)\|) \\ & \leq & \phi(d(u,v)). \end{array}.$$

Hence, we get

$$d(N(u), N(v)) \leq \phi(d(u,v)).$$

Consequently, from Theorem 52, the operator N has a unique fixed point, which is the unique solution of our problem (2.10)-(2.11) on I.

Now, we prove an existence result by Nonlinear alternative of Leray-Schauder type.

Theorem 57. *Assume that the hypothesis* (H_2) *holds. Then the problem (2.10)- (2.11) has at least one solution defined on* I.

Proof. Let N be the operator defined in (2.12). Set

$$R \geq \frac{G^*(p^*+q^*)}{1-r^*},$$

and consider the closed and convex ball $B_R = \{u \in C(I) : \|u\|_\infty \leq R\}$.
Let $u \in B_R$. Then, for each $t \in I$, we have

$$\|(Nu)(t)\| \leq \int_1^T \|G(t,s)g(s)\|ds,$$

where $g \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)).$$

By (H_2), for each $t \in I$ we have

$$\begin{array}{rcl} \|g(t)\| & \leq & p(t) + q(t) + r(t)\|g(t)\| \\ & \leq & p^* + q^* + r^*\|g(t)\| \\ & \leq & p^* + q^* + r^*\|g(t)\|. \end{array}$$

This gives

$$\|g(t)\| \leq \frac{p^*+q^*}{1-r^*}.$$

Thus

$$\|N(u)\|_\infty \leq \frac{G^*(p^*+q^*)}{1-r^*}.$$

So,

$$\|N(u)\|_\infty \leq R.$$

Hence N maps the ball B_R into B_R.
We shall show that the operator $N : B_R \to B_R$ is continuous and compact.

The proof will be given in several steps.

Step1: N is continuous.
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I$, we have

$$\|(Nu_n)(t) - (Nu)(t)\| \quad \leq \quad \int_1^T \|G(t,s)(g_n(s) - g(s))\|ds,$$

where $g_n, g \in C(I)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous function, we get

$$g_n(t) \to g(t) \text{ as } n \to \infty, \text{ for each } t \in I.$$

Hence

$$\|N(u_n) - N(u)\|_\infty \leq G^* \|g_n - g\|_\infty \to 0 \ \ as \ \ n \to \infty.$$

Step2: $N(B_R)$ is bounded. This is clear since $N : B_R \subset B_R$ and B_R is bounded.

Step3: N maps bounded sets into equicontinuous sets in B_R.
Let $t_1, t_2 \in I$, such that $t_1 < t_2$ and let $u \in B_R$. Then, we have

$$\begin{array}{rcl} \|(Nu)(t_1) - (Nu)(t_2)\| & \leq & \int_1^T |G(t_1, s) - G(t_2, s)| \|g(s)\| ds \\ & \leq & \frac{p^*+q^*}{1-r^*} \int_1^T |G(t_1, s) - G(t_2, s)| ds. \end{array}.$$

As $t_1 \to t_2$, from Remark 2.1.3, the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that $N : B_R \to B_R$ is continuous and completely continuous.

Step4: A priori bounds.
We now show there exist an open set $U \subseteq C(I)$ with $u \neq \lambda N(u)$, for $\lambda \in (0, 1)$ and $u \in \partial U$. Let $u \in C(I)$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$.
Thus for each $t \in I$, we have

$$u(t) = \lambda \int_1^T G(t, s) g(s) ds.$$

This implies by (H_2), for each $t \in I$, we get $\|u\| \leq R$.
Set

$$U = \{u \in C(I) : \|u\|_\infty < R + 1\}.$$

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0, 1)$.
As a consequence of Theorem 41, we deduce that N has a fixed point u in $\overline{U}$ which is a solution of problem (2.10)- (2.11).

2.2.3 Example

Consider the following problem of implicit Caputo-Hadamard fractional differential equations

$$\begin{cases} ({}^{Hc}D_1^{\frac{3}{2}}u)(t) = f(t, u(t), ({}^{Hc}D_1^{\frac{3}{2}}u)(t)); \ t \in [1, e], \\ u(1) - u'(1) = d_1 u(u(\frac{3}{2})), \\ u(e) + 2u'(e) = d_2 u(2), \end{cases} \tag{2.13}$$

where

$$f(t, u(t), ({}^{Hc}D_1^{\frac{3}{2}}u)(t)) = \frac{t^2}{1 + \|u(t)\|_E + \|{}^{Hc}D_1^{\frac{3}{2}}u(t)\|_E} \left(e^{-7} + \frac{1}{e^{t+5}}\right) u(t); \quad t \in [0, 1].$$

The hypothesis (H_1) is satisfied with $\phi_1(t) = \phi_2(t) = t^2 \left(e^{-7} + \frac{1}{e^{t+5}}\right) t$. In addition, with good choice of the constants $d_i;\ i = 1, 2$, we can conclude that our problem (2.13) has a unique solution defined on $[1, e]$.

Chapter 3

Existence and Ulam Stabilities

In this chapter, we discuss the existence, uniqueness and the stability results for a class of multipoint boundary conditions problem of fractional differential equations in banach spaces with Caputo-Hadamard fractional derivative.

3.1 Introduction and motivations

In this section; we discuss the existence of solutions and the stability for the Caputo-Hadamard fractional differential equation

$$({}^{Hc}D_1^{\alpha}u)(t) = f(t, u(t)),\ t \in I := [1, T], \tag{3.1}$$

with the boundary conditions

$$\begin{cases} a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1), \\ a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2), \end{cases} \tag{3.2}$$

where $T > 1,\ a_1, b_1, d_1, a_2, b_2, d_2 \in \mathbb{R},\ \xi_1, \xi_2 \in (1, T),\ f : I \times E \to E$ is a given continuous function, $(E, \|\cdot\|)$ is a real or complex Banach space, and ${}^{Hc}D_1^{\alpha}$ is the Caputo–Hadamard fractional derivative of order $\alpha \in (1, 2]$.

In [7], S. Abbas et al. studied the existence and the Ulam stability of solutions for the following problem of Hilfer-Hadamard fractional differential equations of the form

$$\begin{cases} ({}^{H}D_1^{\alpha,\beta}u)(t) = f(t, u(t)),\ t \in I \\ ({}^{H}I_1^{1-\gamma}u)(t)|_{t=1} = \phi, \end{cases} \tag{3.3}$$

where $\alpha \in (0, 1), \beta \in [0, 1], \gamma = \alpha + \beta - \alpha\beta, T > 1, \phi \in \mathbb{R}, f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, ${}^{H}I_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1 - \gamma$, and ${}^{H}D_1^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order α and type β.

3.2 Existence and Ulam Stability Results

Consider the Banach space $C(I) := C(I, E)$ of continuous functions from I into E equipped with the usual supremum (uniform) norm

$$\|u\|_\infty := \sup_{t\in I} \|u(t)\|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into E, and $L^1(I)$ denotes the space measurable functions $v : I \to E$ which are Bochner integrable with the norm

$$\|v\|_1 = \int_I \|v(t)\|dt.$$

For any $n \in \mathbb{N}^*$, we denote by $AC^n(I)$ the space defined by

$$AC^n(I) := \{w : I \to E : \frac{d^n}{dt^n} w(t) \in AC(I)\}.$$

Let

$$\delta = t\frac{d}{dt},\ q > 0,\ n = [q] + 1,$$

where $[q]$ is the integer part of q. Define the space

$$AC^n_\delta := \{u : I \to E : \delta^{n-1}[u(t)] \in AC(I)\}.$$

Let us defining what we mean by a solution of problem (3.1)-(3.2).

Definition 58. *By a solution of the problem* (3.1)-(3.2) *we mean a continuous function u that satisfies the equation* (3.1) *on I and the conditions* (3.2).

For the existence of solutions for the problem (3.1)-(3.2), we need the auxiliary lemma:

Lemma 59. *Let $h \in C$ and $\alpha \in (1, 2]$. Then the unique solution of the problem*

$$\begin{cases} ({}^{Hc}D^\alpha_1 u)(t) = h(t), \quad t \in I, \\ a_1u(1) - b_1u'(1) = d_1u(\xi_1), \\ a_2u(T) + b_2u'(T) = d_2u(\xi_2), \end{cases}$$

is given by

$$u(t) = \int_1^T G(t, s)h(s)ds,$$

where G is the Green function with $G(t,s)$ given by

$$\begin{aligned}
&\frac{\left(\ln\frac{t}{s}\right)^{\alpha-1}}{s\Gamma(\alpha)}\\
&+\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]\\
&-\frac{d_1\left(\ln\frac{\xi_1}{s}\right)^{\alpha-1}}{s\Delta\Gamma(\alpha)}\left[a_2\ln T+\frac{b_2}{T}-d_2\ln\xi_2\right]\\
&+\frac{d_1\ln t\left(\ln\frac{\xi_1}{s}\right)^{\alpha-1}}{s\Delta\Gamma(\alpha)}\left[a_2-d_2\right]\\
&-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]
\end{aligned}$$

for $s\le\xi_1$ and $s\le t$,

$$\begin{aligned}
&\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]\\
&-\frac{d_1\left(\ln\frac{\xi_1}{s}\right)^{\alpha-1}}{s\Delta\Gamma(\alpha)}\left[a_2\ln T+\frac{b_2}{T}-d_2\ln\xi_2\right]\\
&+\frac{d_1\ln t\left(\ln\frac{\xi_1}{s}\right)^{\alpha-1}}{s\Delta\Gamma(\alpha)}\left[a_2-d_2\right]\\
&-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]
\end{aligned}$$

for $s\le\xi_1$ and $t\le s$,

$$\begin{aligned}
&\frac{\left(\ln\frac{t}{s}\right)^{\alpha-1}}{s\Gamma(\alpha)}\\
&+\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]\\
&-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]
\end{aligned}$$

for $\xi_1\le s\le\xi_2$ and $s\le t$,

$$\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]$$

$$-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{d_2\left(\ln\frac{\xi_2}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\right]$$

for $\xi_1 \le s \le \xi_2$ *and* $t \le s$,

$$\begin{aligned}&+\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\\&-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\end{aligned}$$

for $\xi_2 \le s \le t$, *and*

$$\begin{aligned}&\frac{\left(\ln\frac{t}{s}\right)^{\alpha-1}}{s\Gamma(\alpha)}\\&\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\\&-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\end{aligned}$$

for $\xi_2 \le s$ *and* $t \le s$,

$$\begin{aligned}&\frac{b_1+d_1\ln\xi_1}{\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\\&-\frac{(d_1-a_1)\ln t}{s\Delta}\left[\frac{a_2\left(\ln\frac{T}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{b_2\left(\ln\frac{T}{s}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\end{aligned}$$

where

$$\Delta=(d_1-a_1)\{a_2\ln T+\frac{b_2}{T}-d_2\ln\xi_2\}-(a_2-d_2)(b_1+d_1\ln\xi_1)\neq 0.$$

Now, we consider the Ulam stability for the problem (3.1)-(3.2). Let $\epsilon>0$ and $\Phi: I\to\mathbb{R}_+$ be a continuous function. We consider the following inequalities

$$\|({}^{Hc}D_1^{\alpha}u)(t)-f(t,u(t))\|\le\epsilon;\ t\in I. \tag{3.4}$$

$$\|({}^{Hc}D_1^{\alpha}u)(t)-f(t,u(t))\|\le\Phi(t);\ t\in I \tag{3.5}$$

$$\|({}^{Hc}D_1^{\alpha}u)(t)-f(t,u(t))\|\le\epsilon\Phi(t);\ t\in I \tag{3.6}$$

Definition 60. *[2] The problem(3.1)-(3.2) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C(I)$ of the inequality (3.4), there exists a solution $v \in C(I)$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq \epsilon c_f; \; t \in I.$$

Definition 61. *[2] The problem (3.1)-(3.2) is generalized Ulam-Hyers stable if there exists $c_f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $c_f(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in C(I)$ of the inequality (3.4), there exists a solution $v \in C(I)$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq c_f(\epsilon); \; t \in I.$$

Definition 62. *[2] The problem (3.1)-(3.2) is Ulam-Hyers-Rassias stable with respect to ϕ if there exists a real number $c_{f,\phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C(I)$ of the inequality (3.6), there exists a solution $v \in C(I)$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq \epsilon c_{f,\phi}\phi(t); \; t \in I.$$

Definition 63. *[2] The problem (3.1)-(3.2) is generalized Ulam-Hyers-Rassias stable with respect to ϕ if there exists a real number $c_{f,\phi} > 0$ such that for each solution $u \in C(I)$ of the inequality (3.5), there exists a solution $v \in C(I)$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq c_{f,\phi}\phi(t); \; t \in I.$$

Remark 64. *A function $u \in C$ is a solution of the inequality* (3.4) *if and only if there exist a function $g \in C$ (which depend on u) such that*

$$\|g(t)\|_E \leq \epsilon,$$

$$({}^{Hc}D_1^{\alpha}u)(t) = f(t, u(t)) + g(t); \; for \; t \in I.$$

Lemma 65. *If $u \in C$ is a solution of the inequality (3.4) then u is a solution of the following integral inequality*

$$\|u(t) - \int_1^T G(t,s)f(s,u(s))ds\| \leq \epsilon G^*; \; if \; t \in I. \tag{3.7}$$

Proof.

By Remark 64, for $t \in I$, we have

$$({}^{Hc}D_1^{\alpha}u)(t) = f(t, u(t)) + g(t).$$

Then, for $t \in I$, we get

$$u(t) = \int_1^T G(t,s)(f(s,u(s)) + g(s))ds.$$

Thus, for $t \in I$ we obtain

$$\begin{array}{rcl} \|u(t) - \int_1^T G(t,s)f(s,u(s))ds\| & = & \left\| \int_1^T G(t,s)g(s)ds \right\| \\ & \leq & \int_1^T |G(t,s)| \, \|g(s)\| \, ds \\ & \leq & \epsilon G^*. \end{array}$$

Hence, we obtain (3.7).

The following hypotheses will be used in the sequel.

(H_1) The function $t \mapsto f(t,u)$ is measurable on I for each $u \in E$, and the function $u \mapsto f(t,u)$ is continuous on E for a.e.$t \in I$,

(H_2) There exists a continuous function $p \in C(I,[0,\infty))$ such that

$$\|f(t,u)\| \leq p(t)(1 + \|u\|)$$

for a.e. $t \in I$ and $u \in E$,

(H_3) For each bounded set $B \subset E$ and for each $t \in I$; we have

$$\alpha(f(t,B)) \leq p(t)\alpha(B).$$

Set $p^* = \sup_{t \in I} p(t)$.

Now,we prove an existence result for the problem (3.1)-(3.2) based on concept of measures of non-compactness and Darbo's fixed point theorem.

Theorem 66. *Assume that the hypotheses* $(H_1) - (H_3)$ *hold. If*

$$L := p^* G^* < 1, \tag{3.8}$$

then the problem (3.1)-(3.2) *has at least one solution defined on* I.

Proof.

By using Lemma 59, we transform the problem (3.1)-(3.2) into fixed point problem. Consider the operator $N : C(I) \to C(I)$ defined by

$$(Nu)(t) = \int_1^T G(t,s)f(s,u(s))ds \; t \in I. \tag{3.9}$$

Set

$$R = \frac{L}{1-L},$$

and consider the closed and convex ball

$$B_R := \{w \in C : \|w\|_\infty \leq R\}.$$

Let $u \in B_R$. Then, for each $t \in I$, we have

$$\begin{array}{rcl} \|(Nu)(t)\| & \leq & \int_1^T |G(t,s)| \|f(s,u(s))\| ds \\ & \leq & \int_1^T |G(t,s)| (p(s)(1+\|u(s)\|)) ds \\ & \leq & p^* G^* (1+R) \\ & := & L(1+R). \end{array}$$

Thus

$$\|N(u)\|_\infty \leq R.$$

Hence $N(B_R) \subset B_R$. we shall show that the operator $N : B_R \to B_R$ satisfies the assumptions of Darbo's fixed point theorem.

The proof will be given in several steps.

Step 1. N is continuous and bounded.
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $B_R \subset C(I,E)$. Then, for each $t \in I$ we have

$$\begin{array}{rcl} \|(Nu_n)(t) - (Nu)(t)\| & \leq & \int_1^T \|G(t,s)[f(s,u_n(s)) - f(s,u(s))]\| \, ds \\ & \leq & \int_1^T |G(t,s)| \, \|f(s,u_n(s)) - f(s,u(s))\| \, ds. \end{array}$$

Since $u_n \to u$ as $n \to \infty$, and f is continuous then by the Lebesgue dominated convergence theorem;

$$\|N(u_n) - N(u)\|_\infty \to 0 \ \ as \ \ n \to \infty.$$

Hence, N is continuous. Since $N : B_R \to B_R$ and B_R is bounded, then $N(B_R)$ is bounded

Step 2. N maps bounded sets into equicontinuous sets in B_R.
Let $t_1, t_2 \in I$,such that $t_1 < t_2$ and let $u \in B_R$. Thus, we have

$$\begin{array}{rcl} \|(Nu)(t_2) - (Nu)(t_1)\| & \leq & \int_1^T |G(t_2,s) - G(t_1,s)| \|f(s,u(s))\| ds \\ & \leq & p^*(1+R) \int_1^T |G(t_2,s) - G(t_1,s)| ds. \end{array}$$

As $t_1 \to t_2$ and G is continuous function; the right-hand side of the above inequality tends to zero.

Step 3. The operator $N : B_R \to B_R$ is a strict set contraction.
Let $V \subset B_R$ and $t \in I$; then we have

$$\begin{array}{rcl} \alpha((NV)(t)) & = & \alpha(\{(Ny)(t), y \in V\}) \\ & \leq & \{\int_1^T |G(t,s)|\alpha(f(s,y(s)))ds : y \in V\}. \end{array}$$

By (H_3) and lemma (28), for each $s \in I$,

$$\begin{array}{rcl} \{\int_1^T |G(t,s)|\alpha(f(s,y(s)))ds : y \in V\} & \leq & \{\int_1^T |G(t,s)|p(s)\alpha(y(s))ds : y \in V\} \\ & \leq & p^*\{\int_1^T |G(t,s)|\alpha(y(s))ds : y \in V\} \\ & \leq & p^*\alpha_c(V)\int_1^T |G(t,s)|ds \\ & \leq & p^*G^*\alpha_c(V). \end{array}$$

Therefore

$$\alpha_c(NV) \leq p^*G^*\alpha_c(V).$$

So, by (3.8) the operator N is a set contraction. As a consequence of theorem 42, we deduce that N has a fixed point that is a solution of the problem (3.1)-(3.2).

Our next existence result for the problem (3.1)-(3.2) is based on Mönch's fixed point theorem.

Theorem 67. *Assume that $(H_1)-(H_3)$, and the condition (3.8) hold. Then the problem* (3.1)-(3.2) *has at least one solution.*

Proof. Consider the operator N defined in (3.9). We know from theorem 66 that $N : B_R \to B_R$ is continuous and bounded. We need to prove that the implication

$$V = \overline{convN(V)}, orV = N(V) \cup 0 \Rightarrow \alpha(V) = 0$$

holds for every subset V of B_R.

Let V be a subset of B_R such that $V \subset N(V) \cup \{0\}$. The set V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on I. By (H_3) and the properties of the measure α, for each $t \in I$ we have

$$\begin{array}{rcl} v(t) & \leq & \alpha((NV)(t) \cup 0) \\ & \leq & \alpha((NV)(t)) \\ & \leq & \alpha\{(Ny)(t) : y \in V\} \\ & \leq & p^* \int_1^T |G(t,s)|\alpha(\{y(s) : y \in V\})ds \\ & \leq & p^* \int_1^T |G(t,s)|v(s)ds. \end{array}$$

Thus,

$$\|v\|_\infty \leq p^*G^*\|v\|_\infty$$

From (3.8), we get $\|v\|_\infty = 0$, that is $v(t) = \alpha(V(t)) = 0$, for each $t \in I$, and then $V(t)$ is relatively compact in E. In view of the Arzelà-Ascoli theorem, V is relatively compact in B_R. Applying now Theorem 44 , we conclude that N has a fixed point which is a solution of the problem (3.1)-(3.2).

Now, we are concerned with the generalized Ulam-Hyers-Rassias stability of our problem (3.1)-(3.2).

The following hypothesis will be used in the sequel.

(H_4) There exists $l_f \in C(I, [0, \infty))$ such that

$$(1 + \|u - \overline{u}\|)\|f(t, u) - f(t, \overline{u})\| \leq l_f(t)\phi(t)\|u - \overline{u}\|,$$

for each $t \in I$ and each $u, \overline{u} \in E$.

(H_5) $\Phi \in L^1(I, [0, \infty))$ and there exists $\lambda_\Phi > 0$ such that, for each $t \in I$ we have

$$\int_1^T |G(t, s)|\Phi(s)ds \leq \lambda_\Phi \Phi(t).$$

Remark 68. *From (H_4), for each $u \in E$ and $t \in I$, we have that*

$$\|f(t, u)\| \leq \|f(t, 0)\| + l_f(t)\Phi(t)\|u\|.$$

So, (H_4) implies (H_2), with $p^* = \max\{l_f^*, f^*\Phi^*\}$, where $l_f^* = \sup_{t\in I} l_f(t)\phi(t)$, $\Phi^* = \sup_{t\in I} \Phi(t)$, and $f^* = \sup_{t\in I} |f(t, 0)|$.

Theorem 69. *Assume that the hypotheses (H_1), $(H_3) - (H_5)$ and the condition*

$$G^* \max\{l_f^* f, f^*\} < 1, \tag{3.10}$$

hold. Then the problem (3.1)-(3.2) has a solution on I and it is generalized Ulam-Hyers-Rassias stable.

Proof. Let u be a solution of the inequality (3.5), and let us assume that v is a solution of problem (3.1)-(3.2). Then; we have

$$v(t) = \int_1^T G(t, s)f(s, v(s))ds; \ t \in I.$$

By differential inequality (3.5), for each $t \in I$, we have

$$\|u(t) - \int_1^T G(t, s)f(s, u(s))ds\| \leq \int_1^T |G(t, s)|\Phi(s)ds$$

Thus, by (H_5) for each $t \in I$, we get

$$\|u(t) - \int_1^T G(t, s)f(s, u(s))ds\| \leq \lambda_\Phi \Phi(t).$$

Hence for each $t \in I$, it follows that

$$\begin{array}{rcl}\|u(t)-v(t)\| & \leq & \|u(t)-\int_1^T G(t,s)f(s,u(s))ds\| \\ & + & \int_1^T |G(t,s)|\|f(s,u(s))-f(s,v(s))\|ds \\ & \leq & \lambda_\Phi \Phi(t) + l_f^* \int_1^T |G(t,s)|\Phi(s)\frac{\|u(s)-v(s)\|}{1+\|u(s)-v(s)\|}ds \\ & \leq & \lambda_\Phi \Phi(t) + l_f^* \int_1^T |G(t,s)|\Phi(s)ds \\ & \leq & \lambda_\Phi \Phi(t) + l_f^* \lambda_\Phi \Phi(t) \\ & \leq & (1+l_f^*)\lambda_\Phi \Phi(t) \\ & := & c_{f,\phi}\phi(t).\end{array}$$

Thus

$$\|u(t)-v(t)\| \leq c_{f,\phi}\phi(t).$$

Hence, the problem (3.1)-(3.2) is generalized-Ulam-Hyers-Rassias stable.

3.3 Example

Let

$$l^1 = \left\{ u = (u_1, u_2, \dots, u_m, \dots), \sum_{m=1}^{\infty} |u_m| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{m=1}^{\infty} |u_m|.$$

We consider the following Caputo-Hadamard fractional differential equation

$$({}^{Hc}D_1^{\frac{3}{2}} u_n)(t) = f_n(t, u(t));\ if\ t \in I := [1, e], \tag{3.11}$$

with the boundary conditions

$$\begin{cases} u(1) - u'(1) = d_1 u(u(\frac{3}{2})), \\ u(e) + 2u'(e) = d_2 u(2), \end{cases} \tag{3.12}$$

where

$$f_n(t, u(t)) = \frac{t^{-2} e^{-t-5}(2^{-n} + u_n(t))}{1 + \|u(t)\|_{l^1}}; \quad t \in [1, e],$$

with $f = (f_1, f_2, \dots, f_n, \dots),\ and\ u = (u_1, u_2, \dots, u_n, \dots)$.
For each $t \in [1, e]$, we have

$$\begin{aligned} \|f(t, u(t))\|_{l^1} &= \sum_{n=1}^{\infty} |f_n(s, u_n(s))| \\ &\leq e^{-6}(1 + \|u\|_{l^1}). \end{aligned}$$

The hypothesis (H_2) is satisfied with $p^* \leq e^{-6}$. In addition, with good choice of the constants $d_i; i = 1, 2$, a simple computation show that all conditions of Theorem 66 are satisfied. Hence, the problem (3.11)-(3.12) has at least one solution defined on $[1, e]$.

Also; the hypotheses (H_5) and (H_4) are satisfied with $l_f = e^{-6},\ \Phi(t) = \dfrac{t}{|G(t,s)|}$, $\lambda_\phi = \frac{|G(t,s)|(e^2-1)}{t}$, and for each $t \in [1, e]$; we get

$$\int_1^e |G(t, s)| \phi(t) dt = \frac{1}{2}(e^2 - 1) \leq e^2 - 1 = \lambda_\phi \phi(t).$$

Consequently Theorem 69 implies that the problem (3.11)-(3.12) is generalized-Ulam-Hyers-Rassias stable.

Chapter 4

Caputo-Fabrizio fractional differential equations

In this chapter, we establish the existence and uniqueness results with initial and nonlocal conditions problem of fractional differential equations in fréchet spaces with Caputo-Fabrizio fractional derivative.

4.1 Introduction and Motivation

The purpose of this section; is to establish existence of solutions for the following Caputo-Fabrizio fractional differential equation

$$({}^{CF}D_0^r u)(t) = f(t, u(t));\ t \in \mathbb{R}_+ := [0, \infty), \tag{4.1}$$

with the following initial condition

$$u(0) = u_0 \in E, \tag{4.2}$$

where $(E, \|\cdot\|)$ is a (real or complex) Banach space, $r \in (0,1)$, $f : \mathbb{R}_+ \times E \to E$ is a given function, and ${}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0,1)$.

Next, we discuss the existence of solutions for the fractional differential equation (4.1), with the following nonlocal condition

$$u(0) + Q(u) = u_0, \tag{4.3}$$

where $u_0 \in E,\ Q : C(\mathbb{R}_+, E) \to E$ is a given function.

In [58]; the authors presented a new definition of fractional derivative with a smooth kernel which takes on two different representation for the temporal and spatial variable.

The first works on the time variables; thus it is suitable to use Laplace transform. The second definition is related to the spatial variables, by a non-local fractional derivative, for which it is more convenient to work with the Fourier transform. The interest for this new approach with a regular kernel was born from the prospect that there is a class of non-local systems, which have the ability to describe the material heterogeneities and the fluctuations of diffrent scales, which cannot be well described by classical local theories or by fractional models with singular kernel.

4.2 Existence of solution

Let C be the Banach space of all continuous functions v from $I := [0, T];\ T > 0$ into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t \in I} \|v(t)\|.$$

By $L^1(I)$, we denote the space of Bochner-integrable functions $v : I \to E$ with the norm

$$\|v\|_1 = \int_0^T \|v(t)\|dt.$$

Let $X := C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into E, equipped with the family of seminorms

$$\|v\|_n = \sup_{t \in [0,n]} \|v(t)\|;\ n \in \mathbb{N},$$

and the distance

$$d(u, v) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|u - v\|_n}{1 + \|u - v\|_n};\ u, v \in X.$$

4.2.1 The Initial Value Problem

Let us defining what we mean by a solution of problem (4.1)-(4.2)

Definition 70. *By a solution of the problem (4.1)-(4.2) we mean a continuous function $u \in X$ that satisfies the integral equation*

$$u(t) = c_r + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds,$$

where $c_r = u_0 - a_r f(0, u_0)$.

For the existence of solutions for the problem (4.1)-(4.2); we need the following auxiliary lemma:

Lemma 71. *Let $h \in L^1(I)$. A function $u \in C$ is a solution of problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) = h(t), \ t \in I, \\ u(0) = u_0, \end{cases} \tag{4.4}$$

if and only if u satisfies the following integral equation

$$u(t) = c_r + a_r h(t) + b_r \int_0^t h(s)ds, \tag{4.5}$$

where

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)}, \ c_r = u_0 - a_r h(0).$$

Proof. Suppose that u satisfies (4.4). From Proposition 1 in [108]; the equation

$$({}^{CF}D_0^r u)(t) = h(t),$$

implies that

$$u(t) - u(0) = a_r(h(t) - h(0)) + b_r \int_0^t h(s)ds.$$

Thus from the initial condition $u(0) = u_0$, we obtain

$$u(t) = u_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds.$$

Hence we get (4.5).
Conversely, if u satisfies (4.5), then $({}^{CF}D_0^r u)(t) = h(t)$; for $t \in I$, and $u(0) = u_0$.

Let us introduce the following hypotheses.

(H_1) The function $t \mapsto f(t,u)$ is measurable on $\mathbb{R}_+$ for each $u \in E$, and the function $u \mapsto f(t,u)$ is continuous on E for a.e. $t \in \mathbb{R}_+$.

(H_2) There exists a continuous function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,u)\| \le p(t)(1 + \|u\|), \ for \ a.e. \ t \in \mathbb{R}_+, \ and \ each \ u \in E.$$

(H_3) For each bounded set $B \subset E$ and for each $t \in \mathbb{R}_+$, we have

$$\mu(f(t,B)) \le p(t)\mu(B),$$

where μ is a measure of non-compactness on the Banach space E.

(H_4) The function $Q : C(\mathbb{R}_+, E) \to E$ is continuous, and there exists a constant $q^* > 0$, such that
$$\|Q(u)\| \leq q^*(1 + \|u\|);\ for\ each\ u \in C(\mathbb{R}_+, E).$$
Moreover, for each bounded set $B_1 \subset X$, we have
$$\mu(Q(B_1)) \leq q^* \sup_{t \in I_n} \mu(B_1(t)),$$
where $B_1(t) = \{u(t) : u \in B_1\};\ t \in I_n;\ n \in \mathbb{N}$.

For $n \in \mathbb{N}$, let
$$p_n^* = \sup_{t \in [0,n]} p(t),$$
and define on $X := C(\mathbb{R}_+, E)$ the family of measure of non-compactness by
$$\mu_n(D) = \omega_0^n(D) + \sup_{t \in [0,n]} \mu(D(t)),$$
where $D(t) = \{v(t) \in E : v \in D\};\ t \in [0, n]$.

Theorem 72. *Assume that the hypotheses $(H_1) - (H_3)$ hold. If*
$$l_n := p_n^*(2a_r + 4nb_r) < 1;$$
for each $n \in \mathbb{N}^*$, then the problem (4.1)-(4.2) has at least one solution.

Proof. Consider the operator $N : X \to X$ defined by:
$$(Nu)(t) = c_r + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds. \tag{4.6}$$
Clearly, the fixed points of the operator N are solution of the problem (4.1)-(4.2).

For any $n \in \mathbb{N}^*$, we set
$$R_n \geq \frac{\|c_r\| + p_n^*(a_r + nb_r)}{1 - p_n^*(a_r + nb_r)},$$
and we consider the ball
$$B_{R_n} := B(0, R_n) = \{w \in X : \|w\|_n \leq R_n\}.$$
For any $n \in \mathbb{N}^*$, and each $u \in B_{R_n}$ and $t \in [0, n]$ we have
$$|(Nu)(t)| \leq \|c_r\| + a_r\|f(t, u(t))\| + b_r \int_0^t \|f(s, u(s))\|ds$$

$$
\begin{aligned}
&\leq \|c_r\| + a_r p(t)(1 + \|u(t)\|) + b_r \int_0^t p(s)(1 + \|u(s)\|)ds \\
&\leq \|c_r\| + a_r p_n^*(1 + R_n) + b_r p_n^*(1 + R_n) \int_0^t ds \\
&\leq \|c_r\| + p_n^*(a_r + nb_r)(1 + R_n) \\
&\leq R_n.
\end{aligned}
$$

Thus

$$
\|N(u)\|_n \leq R_n. \tag{4.7}
$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 45.

The proof will be given in two steps.

Step 1. $N(B_{R_n})$ is bounded and $N : B_{R_n} \to B_{R_n}$ is continuous.
Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, then $N(B_{R_n})$ is bounded.
Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k \to u$ in B_{R_n}. Then, for each $t \in [0, n]$, we have

$$
\|(Nu_k)(t) - (Nu)(t)\| \leq a_r\|f(t, u_k(t)) - f(t, u(t))\|) + b_r \int_0^t \|f(s, u_k(s)) - f(s, u(s))\|ds.
$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$
\|N(u_k) - N(u)\|_n \to 0 \quad \text{as } k \to \infty.
$$

Step 2. For each bounded equicontinuous subset D of B_{R_n}, $\mu_n(N(D)) \leq \ell_n\mu_n(D)$.
From Lemmas 36 and 37, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0, n]$, we have

$$
\begin{aligned}
\mu((ND)(t)) &= \mu\left(\left\{c_r + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds;\ u \in D\right\}\right) \\
&\leq 2\mu\left(\{a_r f(t, u_k(t))\}_{k=1}^{\infty}\right) + 2\mu\left(\left\{b_r \int_0^t f(s, u_k(s))ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\leq 2a_r\mu\left(\{f(t, u_k(t))\}_{k=1}^{\infty}\right) + 4b_r \int_0^t \mu\left(\{f(s, u_k(s))\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 2a_r p(t)\mu\left(\{u_k(t)\}_{k=1}^{\infty}\right) + 4b_r \int_0^t p(s)\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 2a_r p_n^*\mu_n(D) + 4nb_r p_n^*\mu_n(D) + \epsilon \\
&= (2a_r + 4nb_r)p_n^*\ \mu_n(D) + \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, then

$$
\mu((ND)(t)) \leq p_n^*(2a_r + 4nb_r)\ \mu_n(D).
$$

Thus

$$\mu_n(N(D)) \leq p_n^*(2a_r + 4nb_r)\ \mu_n(D).$$

As a consequence of steps 1 and 2 together with Theorem 45, we can conclude that N has at least one fixed point in B_{R_n} which is a solution of problem (4.1)-(4.2).

4.2.2 The Problem with Nonlocal Condition

Now, we are concerned with the existence results of the problem (4.1)-(4.3).

Definition 73. *By a solution of the problem (4.1)-(4.3) we mean a continuous function $u \in X$ that satisfies the integral equation*

$$u(t) = c_r - Q(u) + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds,$$

where $c_r = u_0 - a_r f(0, u_0)$.

Now, we shall prove the following theorem concerning the existence of solutions of problem (4.1)-(4.3).

Theorem 74. *Assume that the hypotheses $(H_1) - (H_4)$ hold.*
If

$$\lambda_n := 2q^* + p_n^*(2a_r + 4nb_r) < 1,$$

for each $n \in \mathbb{N}^*$, then the problem (4.1)-(4.3) has at least one solution.

Proof. Consider the operator $G : X \to X$ defined by:

$$(Gu)(t) = c_r - Q(u) + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds. \tag{4.8}$$

Clearly, the fixed points of the operator G are solution of the problem (4.1)-(4.3).

For any $n \in \mathbb{N}^*$, we set

$$\rho_n \geq \frac{\|c_r\| + q^* + p_n^*(a_r + nb_r)}{1 - q^* - p_n^*(a_r + nb_r)},$$

and we consider the ball

$$B_{\rho_n} := B(0, \rho_n) = \{w \in X : \|w\|_n \leq \rho_n\}.$$

For any $n \in \mathbb{N}^*$, and each $u \in B_{\rho_n}$ and $t \in [0, n]$ we have

$$\|(Gu)(t)\| \leq \|c_r\| + \|Q(u)\| + a_r\|f(t, u(t))\| + b_r \int_0^t \|f(s, u(s))\|ds$$

$$
\begin{aligned}
&\leq \|c_r\| + q^*(1+\|u\|) + a_r p(t)(1+\|u(t)\|) + b_r \int_0^t p(s)(1+\|u(s)\|)ds \\
&\leq \|c_r\| + q^*(1+\rho_n) + a_r p_n^*(1+\rho_n) + b_r p_n^*(1+\rho_n) \int_0^t ds \\
&\leq \|c_r\| + q^*(1+\rho_n) + p_n^*(a_r + nb_r)(1+\rho_n) \\
&\leq \rho_n.
\end{aligned}
$$

Thus

$$\|G(u)\|_n \leq \rho_n. \tag{4.9}$$

This proves that G transforms the ball B_{R_n} into itself. As in the proof of Theorem 72, we can show that the operator $G : B_{\rho_n} \to B_{\rho_n}$ satisfies all the assumptions of Theorem 45. Indeed; $G(B_{\rho_n})$ is bounded, and we can easily prove that $G : B_{\rho_n} \to B_{\rho_n}$ is continuous. Next, from Lemmas 36 and 37, for any $D \subset B_{\rho_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0, n]$, we have

$$
\begin{aligned}
\mu((GD)(t)) &= \mu\left(\left\{c_r - Q(u) + a_r f(t,u(t)) + b_r \int_0^t f(s,u(s))ds;\ u \in D\right\}\right) \\
&\leq 2\mu\left(\{Q(u) + a_r f(t,u_k(t))\}_{k=1}^{\infty}\right) \\
&\quad +2\mu\left(\left\{b_r \int_0^t f(s,u_k(s))ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\leq 2\mu\left(\{Q(u_k)\}_{k=1}^{\infty}\right) + 2a_r\mu\left(\{f(t,u_k(t))\}_{k=1}^{\infty}\right) \\
&\quad +4b_r \int_0^t \mu\left(\{f(s,u_k(s))\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 2q^* mu\left(\{u_k(t)\}_{k=1}^{\infty}\right) + 2a_r p(t)\mu\left(\{u_k(t)\}_{k=1}^{\infty}\right) \\
&\quad +4b_r \int_0^t p(s)\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 2q^*\mu_n(D) + 2a_r p_n^*\mu_n(D) + 4nb_r p_n^*\mu_n(D) + \epsilon \\
&= [2q^* + p_n^*(2a_r + 4nb_r)]\mu_n(D) + \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((GD)(t)) \leq [2q^* + p_n^*(2a_r + 4nb_r)]\mu_n(D).$$

Thus

$$\mu_n(G(D)) \leq [2q^* + p_n^*(2a_r + 4nb_r)]\mu_n(D).$$

Hence, from Theorem 45, we can conclude that G has at least one fixed point in B_{ρ_n} which is a solution of problem(4.1)-(4.3).

4.3 Examples

Let

$$l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{k=1}^{\infty} |u_k| < \infty \right\}$$

be the Banach space with the norm

$$\|u\| = \sum_{k=1}^{\infty} |u_k|,$$

and $C(\mathbb{R}_+, l^1)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into l^1, equipped with the family of seminorms

$$\|v\|_n = \sup_{t \in [0,n]} \|v(t)\|; \; n \in \mathbb{N}.$$

Example 1. Consider the following problem of Caputo-Fabrizio fractional differential equations

$$\begin{cases} ({}^{CF}D_0^{\frac{1}{2}} u_k)(t) = f_k(t, u(t)); \; t \in \mathbb{R}_+, \\ u(0) = (1, 2^{-1}, 2^{-2}, \ldots, 2^{-n}, \cdots); \; t \in \mathbb{R}_+, \; k = 1, 2, \cdots, \end{cases} \tag{4.10}$$

where

$$\begin{cases} f_k(t,u) = \dfrac{(2^{-k} + u_k(t)) \sin t}{64(a_{\frac{1}{2}} + 2nb_{\frac{1}{2}})(1 + \sqrt{t})}; \; t \in (0, +\infty), \; u \in l^1, \\ \qquad f_k(0,u) = 0; \qquad\qquad u \in l^1, \end{cases}$$

for each $t \in [0,n]$; $n \in \mathbb{N}$, with

$$f = (f_1, f_2, \ldots, f_k, \ldots), \; \textit{and} \; u = (u_1, u_2, \ldots, u_k, \ldots).$$

The hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \dfrac{|\sin t|}{64(a_{\frac{1}{2}} + 2nb_{\frac{1}{2}})(1 + \sqrt{t})}; \; t \in (0, +\infty), \\ \qquad p(0) = 0. \end{cases}$$

So; for any $n \in \mathbb{N}$, we have $p_n^* = \frac{1}{64(a_{\frac{1}{2}} + 2nb_{\frac{1}{2}})}$, and

$$\ell_n := p_n^*(2a_r + 4nb_r) = \frac{1}{64(a_{\frac{1}{2}} + 2nb_{\frac{1}{2}})}(2a_{\frac{1}{2}} + 4nb_{\frac{1}{2}}) = \frac{1}{32} < 1.$$

Simple computations show that all conditions of Theorem 72 are satisfied. Consequently, the problem (4.10) has at least one solution defined on $\mathbb{R}_+$.

Example 2. Consider now the following problem of Caputo-Fabrizio fractional differential equations

$$\begin{cases} ({}^{CF}D_0^{\frac{1}{2}}u_k)(t) = f_k(t, u(t));\ t \in \mathbb{R}_+, \\ u(0) + Q(u) = (1, 2^{-1}, 2^{-2}, \ldots, 2^{-n}, \cdots);\ t \in \mathbb{R}_+,\ k = 1, 2, \cdots, \end{cases} \tag{4.11}$$

where $Q = (Q_1, Q_2, \ldots, Q_k, \ldots),\ Q : C(\mathbb{R}_+, l^1) \to l^1$, and

$$Q_k(u) = \frac{2^{-k} + u_k}{64};\ k = 1, 2, \cdots .$$

In addition to hypotheses $(H_1) - (H_3)$, the hypothesis (H_4) is satisfies with $q^* = \frac{1}{64}$. Also we have

$$\lambda_n := 2q_n^* + p_n^*(2a_r + 4nb_r) = \frac{1}{32} + \frac{1}{64(a_{\frac{1}{2}} + 2nb_{\frac{1}{2}})}(2a_{\frac{1}{2}} + 4nb_{\frac{1}{2}}) = \frac{1}{16} < 1.$$

Simple computations show that all conditions of Theorem 74 are satisfied. Consequently, the problem (4.11) has at least one solution defined on $\mathbb{R}_+$.

Chapter 5

Existence and Attractivity Results

In this chapter, we investigate the existence and attractivity results with initial condition problem of fractional differential equations with Caputo-Fabrizio fractional derivative.

5.1 Introduction and Motivation

The purpose of this section; is to investigate the existence and the attractivity of solutions for the following class of Caputo–Fabrizio fractional differential equation

$$({}^{CF}D_0^r u)(t) = f(t, u(t));\ t \in \mathbb{R}_+ := [0, +\infty), \tag{5.1}$$

with the initial condition

$$u(0) = u_0 \in \mathbb{R}, \tag{5.2}$$

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and ${}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0, 1)$.

In [9], S. Abbas et al. studied the existence and the attractivity of solutions to the following nonlinear fractional order Riemann-Liouville Volterra Stieltjes quadratic partial integral equations of the form,

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^{x} (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\ &\times\ h(t,x,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s);\ (t,x) \in J := \mathbb{R}_+ \times [a,b], \end{aligned}$$

where $b > 0$, $r_1, r_2 \in (0, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\alpha, \beta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,$g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $h : J' \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $\lim_{t\to\infty} \alpha(t) = \infty$, $J' = \{(t, x, s, y) \in J^2 : s \le t, y \le x\}$.

We use the Schauder fixed point theorem for the existence of solutions of the equation, and we prove that all solutions are uniformly globally attractive.

5.2 Existence and attractivity results

Let $BC := BC(\mathbb{R}_+)$ be the Banach space of all bounded and continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$, with the norm

$$\|v\|_{BC} := \sup_{t\in\mathbb{R}_+} |v(t)|.$$

Let us defining what we mean by a solution of problem (5.1)-(5.2).

Definition 75. *By a solution of the problem (5.1)-(5.2) we mean a function $u \in BC$ that satisfies the condition $u(0) = u_0$, and the equation $({}^{CF}D_0^r u)(t) = f(t, u(t))$ on $\mathbb{R}_+$.*

For the existence of solutions for the problem (5.1)-(5.2); we need the following auxiliary lemma:

Lemma 76. *[5, 37, 100] Let $h \in L^1(I)$. A function $u \in C$ is a solution of problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) = h(t), \quad t \in I, \\ u(0) = u_0, \end{cases} \tag{5.3}$$

if and only if u satisfies the following integral equation

$$u(t) = c_r + a_r h(t) + b_r \int_0^t h(s)ds, \tag{5.4}$$

where

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)}, \ c_r = u_0 - a_r h(0).$$

As in the prove of the above Lemma, we can show the following one:

Lemma 77. *A function u is a random solution of problem (5.1)-(5.2), if and only if u satisfies the following integral equation*

$$u(t) = c_r + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds, \tag{5.5}$$

where $c_r = u_0 - a_r f(0, u(0))$.

Let $\emptyset \neq \Omega \subset BC$, and let $\Lambda : \Omega \to \Omega$, and consider the solutions of equation

$$(\Lambda u)(t) = u(t). \tag{5.6}$$

We introduce the following concept of attractivity of solutions for equation (5.6).

Definition 78. *A solutions of equation (5.6) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that, for arbitrary solutions $v = v(t)$ and $w = w(t)$ of equations (5.6) belonging to $B(u_0, \eta) \cap \Omega$, we have*

$$\lim_{t\to\infty} (v(t) - w(t)) = 0. \tag{5.7}$$

When the limit (5.7) is uniform with respect to $B(u_0, \eta) \cap \Omega$, solutions of equation (5.6) are said to be uniformly locally attractive (or equivalently that solutions of (5.6) are locally asymptotically stable).

Lemma 79. *([62], p. 62). Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold:*
(a) D is uniformly bounded in BC,
(b) The functions belonging to D are almost equicontinuous on $\mathbb{R}_+$,
i.e. equicontinuous on every compact of $\mathbb{R}_+$,
(c) The functions from D are equiconvergent, that is, given $\epsilon > 0$ there corresponds $T(\epsilon) > 0$ such that $|u(t) - \lim_{t\to\infty} u(t)| < \epsilon$ for any $t \geq T(\epsilon)$ and $u \in D$.

The following hypotheses will be used in the sequel.

(H_1) For any bounded set $B \subset C$, the set:

$$\{t \mapsto f(t, u(t)) : u \in B\};$$

is equicontinuous in C.

(H_2) There exists a continuous function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, u)| \leq \frac{p(t)}{1 + |u|}, \ \textit{for } t \in \mathbb{R}_+, \ \textit{and each } u \in \mathbb{R}.$$

Moreover, assume that

$$\lim_{t\to\infty} p(t) = 0 \ \textit{and} \ \lim_{t\to\infty} \int_0^t p(s)ds = 0.$$

Set

$$p^* = \sup_{t\in\mathbb{R}_+} p(t) \ \textit{and} \ p_* = \sup_{t\in\mathbb{R}_+} \int_0^t p(s)ds.$$

Now, we shall prove the following theorem concerning the existence and the attractivity of solutions of our problem (5.1)-(5.2).

Theorem 80. *Assume that the hypotheses (H_1) and (H_2) hold. Then the problem (5.1)-(5.2) has at least one solution defined on $\mathbb{R}_+$. Moreover, solutions of problem (5.1)-(5.2) are uniformly locally attractive.*

Proof. Consider the operator N such that, for any $u \in BC$,

$$(Nu)(t) = c_r + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds. \tag{5.8}$$

The operator N maps BC into BC. Indeed the map $N(u)$ is continuous on $\mathbb{R}_+$ for any $u \in BC$, and for each $t \in \mathbb{R}_+$; we have

$$\begin{array}{rcl} |(Nu)(t)| & \leqslant & |c_r| + a_r|f(t, u(t))| + b_r \int_0^t |f(s, u(s))|ds \\ & \leqslant & |c_r| + a_r p(t) + b_r \int_0^t p(s)ds \\ & \leqslant & |c_r| + a_r p^* + b_r p_*. \end{array}$$

Thus

$$\|N(u)\|_{BC} \leqslant |c_r| + a_r p^* + b_r p_* := R. \tag{5.9}$$

Hence, $N(u) \in BC$. This proves that the operator N maps BC into itself. Furthermore, N transforms the ball

$$B_R := B(0, R) = \{w \in BC : \|w\|_{BC} \leqslant R\}$$

into itself.

We shall show that the operator N satisfies all the assumptions of Theorem 40. The proof will be given in several steps.

Step 1. N is continuous.
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R.
Then, for each $t \in \mathbb{R}_+$, we have

$$\begin{array}{rcl} |(Nu_n)(t) - (Nu)(t)| & \leqslant & a_r|f(t, u_n(t)) - f(t, u(t))| \\ & + & b_r \displaystyle\int_0^t |f(s, u_n(s)) - f(s, u(s))|ds. \end{array} \tag{5.10}$$

Case 1. If $t \in [0, T]$; $T > 0$, then, since $u_n \to u$ as $n \to \infty$ and f is continuous, then by the Lebesgue dominated convergence theorem, equation (5.10) implies

$$\|N(u_n) - N(u)\|_{BC} \to 0 \; as \; n \to \infty.$$

Case 2. If $t \in (T, \infty)$; $T > 0$, then from our hypotheses and (5.10), we get

$$|(Nu_n)(t) - (Nu)(t)| \leqslant 2a_r p(t) + 2b_r \int_0^t p(s)ds. \tag{5.11}$$

Since $u_n \to u$ as $n \to \infty$ and $p(t) \to 0$ and $\displaystyle\int_0^t p(s)ds \to 0$ as $t \to \infty$, then (5.11) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \; as \; n \to \infty.$$

Step 2. $N(B_R)$ is uniformly bounded.
This is clear since $N(B_R) \subset B_R$ and B_R is bounded.

Step 3. $N(B_R)$ is equicontinuous on every compact subset $[0, T]$ of $\mathbb{R}_+$; $T > 0$.
Let $t_1, t_2 \in [0, T]$, $t_1 < t_2$ and let $u \in B_R$. Thus we have

$$\begin{aligned}
|(Nu)(t_2) - (Nu)(t_1)| &\leqslant a_r|f(t_2, u(t_2)) - f(t_1, u(t_1))| \\
&+ b_r\{\int_0^{t_2} |f(s, u(s))|ds - \int_0^{t_1} |f(s, u(s))|ds\} \\
&\leqslant a_r|f(t_2, u(t_2)) - f(t_1, u(t_1))| \\
&+ b_r \int_{t_1}^{t_2} |f(s, u(s))|ds \\
&\leqslant a_r|f(t_2, u(t_2)) - f(t_1, u(t_1))| \\
&+ b_r \int_{t_1}^{t_2} p(s)ds.
\end{aligned}$$

Thus, from the continuity of the function p and by letting $\overline{p} = \sup\limits_{t \in [0,T]} p(t)$; we get

$$\begin{aligned}
|(Nu)(t_2) - (Nu)(t_1)| &\leqslant a_r|f(t_2, u(t_2)) - f(t_1, u(t_1))| \\
&+ b_r\overline{p}(t_2 - t_1).
\end{aligned}$$

As $t_1 \to t_2$ and the continuity of the function f; the right hand side of the above inequality tends to zero.

Step 4. $N(B_R)$ is equiconvergent.

Let $t \in \mathbb{R}_+$ and $u \in B_R$, then we have

$$\begin{aligned}
|(Nu)(t)| &\leqslant |c_r| + a_r|f(t, u(t))| + b_r \textstyle\int_0^t |f(s, u(s))|ds \\
&\leqslant |c_r| + a_r p(t) + b_r \textstyle\int_0^t p(s)ds.
\end{aligned}$$

Since $p(t) \mapsto 0$ and $\int_0^t p(s) \mapsto 0$ as $t \mapsto \infty$, then we get

$$|(Nu)(t) - (Nu)(\infty)| \mapsto 0 \ as \ t \mapsto \infty.$$

As a consequence of steps 1 to 4 together with the Lemma 79, we can conclude that $N : B_R \to B_R$ is continuous and compact. From an application of Theorem 40, we deduce that N has a fixed point u which is a solution of the problem (5.1)-(5.2) on $\mathbb{R}_+$.

Step 5. The uniform local attractivity of solutions.

let us assume that u_0 is a solution of problem (5.1)-(5.2) with the conditions of this theorem.

Taking $u \in B(u_0, \overline{R})$ with $\overline{R} := 2a_r p^* + 2b_r p_*$; we have

$$\begin{array}{rcl} |(Nu)(t) - u_0(t)| & = & |(Nu)(t) - (Nu_0)(t)| \\ & \leqslant & a_r |f(t, u(t)) - f(t, u_0(t))| + b_r \int_0^t |f(s, u(s)) - f(s, u_0(s))| ds \\ & \leqslant & 2a_r p(t) + 2b_r \int_0^t p(s) ds \\ & \leqslant & 2a_r p^* + 2b_r p_* \\ & := & \overline{R}. \end{array}$$

Thus, we get

$$\|N(u) - u_0\|_{BC} \leqslant \overline{R}.$$

Hence, we obtain that N is a continuous function such that

$$N(B(u_0, \overline{R})) \subset B(u_0, \overline{R}).$$

Moreover, if u is a solution of problem (5.1)-(5.2), then

$$\begin{array}{rcl} |u(t) - u_0(t)| & = & |(Nu)(t) - (Nu_0(t))| \\ & \leqslant & a_r |f(t, u(t)) - f(t, u_0(t))| + b_r \int_0^t |f(s, u(s) - f(s, u_0(s))| ds. \end{array}$$

Thus

$$|u(t) - u_0(t)| \leqslant 2a_r p(t) + 2b_r \int_0^t p(s) ds. \tag{5.12}$$

By using (5.12) and the fact that $\lim_{t\to\infty} p(t) = 0$ and $\lim_{t\to\infty} \int_0^t p(s) ds = 0$, we deduce that

$$\lim_{t\to\infty} |u(t) - u_0(t)| = 0.$$

Consequently, all solutions of problem (5.1)-(5.2) are uniformly locally attractive.

5.3 An Example

Consider the following problem of Caputo-Fabrizio fractional differential equations

$$\begin{cases} ({}^{CF}D_0^{\frac{1}{2}}u)(t) = f(t, u(t)); \ t \in \mathbb{R}_+ \\ u(0) = 2, \end{cases} \tag{5.13}$$

where

$$f(t, u) = \frac{(t-1)e^{-t}\sin t}{1 + t^2 + |u|} \ ; \ t \in \mathbb{R}_+, \ u \in \mathbb{R}.$$

Clearly, the function f is continuous.

The hypothesis (H_2) is satisfied with

$$p(t) = |t-1|e^{-t}|\sin t| \ ; \ t \in \mathbb{R}_+.$$

Also, for $t > 1$, we have

$$|(t-1)e^{-t}\sin t| \leq (t-1)e^{-t} \to 0 \ as \ t \to \infty,$$

and

$$\begin{aligned} \int_0^t p(s)ds &\leq \int_0^t (s-1)e^{-s}ds \\ &= -te^{-t} \to 0 \ as \ t \to \infty. \end{aligned}$$

All conditions of Theorem 80 are satisfied. Hence, the problem (5.13) has at least one solution defined on $\mathbb{R}_+$; and solutions of this problem are uniformly locally attractive.

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Conclusion and Perspectives

In this thesis; we have considered the following Caputo-Hadamard fractional defferential equation

$$({}^{Hc}D_1^{\alpha}u)(t) = f(t, u(t));\ t \in [1, T],$$

and the implicit fractional differential equation

$$({}^{Hc}D_1^{\alpha}u)(t) = f(t, u(t), ({}^{Hc}D_1^{\alpha}u)(t));\ t \in [1, T],$$

with Four-point boundary conditions.

After that, we have considered the following fractional differential equation

$$({}^{CF}D_1^{\alpha}u)(t) = f(t, u(t));\ t \in \mathbb{R}_+$$

Here ${}^{CF}D_1^{\alpha}$ is the **Caputo-Fabrizio** fractional derivative.
We discussed and established the existence, the uniqueness and the attractivity of the solution with initial condition and non-local condition.

A similar work will be there in; the existence and uniqueness of solutions and Ulam-type stability and the attractivity of some classes of differential equations with fractional derivatives of Caputo, Hadamard and Fabrizio in **b-metric space**.

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