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par

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Théme :

Perov's and Krasnosel'skii Type Fixed Point Results and Application

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Didication

I dedicate this modest work To my dear Parents for their support, their patience, their encouragement during my school career.

And my husband who was by my side, To my sisters and my brother as well to all my family.

And to my little daughter Chahd Malak.

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Contents

Introduction

The Banach contraction principle is a fundamental result of metric fixed point theory. This result has many applications in different branches of mathematics like differential and integral equations, optimization and variational analysis, etc. The simplicity and applicability of this result attracted many researchers, thats why, this result has many generalizations in different settings. One of the worthwhile generalization of this result was given by Perov [33] in 1964. In [33] , Perov extended the Banach contraction principle to a space with vector-valued metric. This result helps to study the existence of solution for different types of differential and integral equations. Some interesting contributions to the development of fixed point theory and its applications in this context are obtained by [8],[10],[15],[32],[36],[38],[2].
It is well Known that Krasnoselskii's theorem may combined with Banach and Schauder's fixed point theorems. In a certain sense, we can interpret this as follows: if a compact operator has the fixed point property, under a small perturbation, then this property can be inherited. The sum of operators is clearly seen in delay integral equations and neutral functional equations,which have been discussed extensively in 916, for example. Krasnoselskii proved that the sum of $A+B$ has a fixed point in M ,(i) A is continuous and compact,(ii) $Ax + By \in M$ for every $x, y \in M$ and (iii) is also quite restrictive. That result combined the Banach contraction principle and fixed point theorem .The existence of fixed points for the sum of two operators has attracted tremendous interest, and their applications are frequent in nonlinear analysis. Many improvements of Krasnosel'skii's theorem have been established in the literature in the course of time by modifying the above assumptions; see,for example, [5],[6] ,[11],[20],[17],[16],[19],[23],[27],[41].

chapter 1, we collect some definitions and facts which will be needed in this master thesis.
chapter2, is a brief expose on development of perov fixed point theory .
chapter3, we first state a simplified version of Krasnoselskii's theorem and discuss several generalizations.

Chapter 1

Preliminaries

1.1 Some Notations and Definitions

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis. In this section, we introduce notations, definitions, and preliminary facts which are used throughout this section. Let $(E, \|\cdot\|)$ be a Banach space.
$C([-r,T],E)$ is the Banach space of all continuous functions from $[-r,T]$ into E with the norm

$$\|x\|_\infty = \sup_{\theta\in[-r,0]} \sup_{t\in[0,T]} \|x(t+\theta)\|.$$

$L^1([0,T],E)$ denotes the Banach space of measurable functions $x : [0,T] \to E$ which are Boche integrable and is normed by

$$\|x\|_{L^1} = \int_0^T \|x(t)\|dt.$$

In a normae space $(X, \|\cdot\|_X)$, the open ball around a point x_0 with radius R is denoted by $B_X(x_0,R)$, i.e., $B_X(x_0,R) := \{x \in X : \|x - x_0\|_X < R\}$, and the corresponding closed ball is denoted by $\bar{B}_X(x_0,R)$.

Let $B(E)$ be the Banach space of bounded linear operators from E into E.

Definition 1.1.1. *A linear map $T : E \to Y$ is said to be compact if for any bounded sequence (x_n) in E , $(T(x_n))$ has a convergent subsequence.*

Definition 1.1.2. *Let E be a real normed space. A mapping $T : D(T) \subset E \to E$ is called compact if T maps every bounded subset of $D(T)$ to a relatively compact subset in E. T is said to be completely continuous if T is continuous and compact.*

Definition 1.1.3. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties :*

(i) $d(u,v) > 0$ *for all* $u, v \in X$ *; if* $d(u,v) = 0$ *then* $u = v$ *;*

(ii) $d(u,v) = d(v,u)$ *for all* $u, v \in X$ *;*

(iii) $d(u,v) \leq d(u,w) + d(w,v)$ *for all* $u, v, w \in X$ *.*

We call the pair (X,d) *a generalized metric space . For* $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$ *, we will denote by*

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}$$

the open ball centered in x_0 *with radius* r *and*

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \geq r\}$$

the closed ball centered in x_0 *with radius* r *. We mention that for generalized metric space, the notation of open subset, closed set convergence Cauchy sequence and completeness are similar to those in unseal metric spaces.*
If, $x, y \in \mathbb{R}^n, x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ *, by* $x \leq y$ *we mean* $x_i \leq y_i$ *for all* $i = 1, \ldots, n$ *. Also* $|x| = (|x_1|, \ldots, |x_n|)$ *and* $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$
If $c \in \mathbb{R}$ *,then* $x \leq c$ *means* $x_i \leq c$ *for each* $i = 1, \ldots, n$ *.*

Definition 1.1.4. *A square matrix of real numbers is said toby convergent to zero if and only if its spectral radius* $p(M)$ *is strictly less than 1. In other words, this means that all the eigenvalues of* M *are in the open unit disc(i.e.* $|\lambda| < 1$ *, for every* $\lambda \in C$ *with deit* $(A - \lambda I) = 0$ *, where* I *denote the unit matrix of* $\mathcal{M}_{n\times n}(\mathbb{R}))$ *.*

Theorem 1.1.1. *[39] Let* $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *. The following assertion are equivalent :*

(i) M *is convergent towards zero;*

(ii) $M^k \to 0$ *as* $K \to \infty$ *;*

(iii) *The matrix* $(I - M)$ *is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iv) *The matrix* $(I - M)$ *is nonsingular and* $(I - M)^{-1}$ *has nonnegative elements .*

Definition 1.1.5. *We say that a non-singular matrix* $A = (a_{ij})_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{R})$ *has the absolute value property if*

$$A^{-1}|A| \leq I.$$

where

$$|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Exemples *Some examples of matrices* $A \in M_{n\times n}(\mathbb{R}_+)$ *convergent to zero , which also satisfies the property* $(I - A)^{-1}|I - A| \leq I$ *are :*

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ *, where* $a, b \in \mathbb{R}_+$ *and* $max\ (a, b) < 1$

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$ *,where* $a, b, c \in \mathbb{R}_+$ *and* $a + b < 1, c < 1$

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$ *, where* $a, b, c \in \mathbb{R}_+$ *and* $|a - b| < 1, a > 1, b > 0.$

Definition 1.1.6. *Let* (X, d) *be a generalized metric space . An operator* $N : X \to X$ *is said to be contractive if there exists* $M \in M_{n\times n}(\mathbb{R}_+)$ *, wich is convergent to zero* $\lim_{K\to\infty} M^k \to 0$ *such that*

$$d(N(x), N(y)) \leq Md(x, y)\, for\, all\, x, y \in X.$$

Definition 1.1.7. *Let* (X, d) *be a generalized metric space . A multivalued operator* $N : X \to \mathcal{P}_{cl}(X)$ *is said to be contractive if there exists a matrix* $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *such that*

$$M^K \to 0\, as\, K \to \infty$$

and

$$H_d(N(u), N(v)) \leq Md(u, v),\ for\, all\, u, v \in X.$$

Remark 1.1.1. *In generalized metric space in sense in Perov's sense ,the notation of convergence sequence ,Cauchy sequence completeness open subset and closed subset are similar for usual metric spaces .*

Lemma 1.1.2. *[18] Let* Y *be a separable metric space and* $F : [a, b] \to \mathrm{Y}$ *a measurable multi-valued map with nonempty closed values . Then* F *has a measurable selection .*

Definition 1.1.8. *A Banach space is a complete normed vector space for the distance induced by the norm*

Definition 1.1.9. *A metric space is a pair* (X, d) *where* E *is a nonempty set and d is a distance on* E *, that is to say a map* $d : E \times E \to \mathbb{R}_+$ *which satisfies the following three properties.*

- *Symmetry*

$$\forall x, y \in E, d(x, y) = d(y, x).$$

- *Separation*

$$\forall x, y \in E, d(x, y) = 0 \Longleftrightarrow x = y.$$

- *Triangular inequality*

$$\forall x, y, z \in E, d(x, y) \geq d(x, z) + d(z, y)$$

By For the sake of simplicity, a metric space will sometimes be denoted only by the only by the set E *and not by the pair* (E, d) *when there is no ambiguity about the underlying distance d*

1.2 Multi-Valued Analysis

Let (X, d) be a metric space and Y be a subset of X. Denote by

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$.
- $\mathcal{P}_p(X) = \{Y \in \mathcal{P}(X) : Y$ has the property "p"$\}$ where p could be: *cl*=closed, *b*=bounded, *cp*=compact, *cv*=convex, etc. Thus,
- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$.
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$.
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$.
- $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$ where X is a Banach space.
- $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$.

Let (X, d_*) be a metric space, we will denote by H_{d_*} the Haustoria pseudo-metric distance on $\mathcal{P}(X)$, defined as

$$H_{d_*}\colon\ \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\},\ H_{d_*}(A, B) = \max\left\{\sup_{a \in A} d_*(a, B), \sup_{b \in B} d_*(A, b)\right\}.$$

where $d_*(A, b) = \inf_{a \in A} d_*(a, b)$ and $d_*(a, B) = \inf_{b \in B} d_*(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_{d_*})$ is a metric space and $(\mathcal{P}_{cl}(X), H_{d_*})$ is a generalized metric space. In particular H_{d_*} satisfies the triangle inequality.

Consider the generalized Hausdorff pseudo-metric distance

$$H_d\colon\ \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$$

defined by

$$H_d(A, B) := \begin{pmatrix} H_{d_1}(A, B) \\ \cdots \\ H_{d_n}(A, B) \end{pmatrix}.$$

Definition 1.2.1. *Let (X, d) and (Y, ρ) be two metric spaces and $F : X \to \mathcal{P}(Y)$ be a multi-valued mapping. Then F is said to be* lower semi-continuous *(l.s.c.) if the inverse image of V by F*

$$F^{-1}(V) = \{x \in X\ :\ F(x) \cap V \neq \emptyset\}$$

is open for any open set V in Y. Equivalently F is l.s.c. if the core of V by F

$$F^{+1}(V) = \{x \in X\ :\ F(x) \subset V\}$$

is closed for any closed set V in Y.

Likewise, the map F is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subseteq Y$. That is, if the set $F^{-1}(V)$ is closed for any closed set V in Y. Equivalently, F is u.s.c. if the set $F^{+1}(V)$ is open for any open set V in Y.

The mapping F is said to be completely continuous if it is u.s.c. and, for every bounded subset $A \subseteq X$, $F(A)$ is relatively compact i.e. there exists a relatively compact set $K = K(A) \subset X$ such that

$$F(A) = \bigcup\{F(x) : x \in A\} \subset K.$$

Also, F is compact if $F(X)$ is relatively compact, and it is called locally compact if for each $x \in X$, there exists an open set U containing x such that $F(U)$ is relatively compact.

We denote the graph of F to be the set $Graph(F) = \{(x, y) \in X \times Y,\ y \in F(x)\}$, and we recall the following facts.

Definition 1.2.2. *A multivalued map $F : [a, b] \to \mathcal{P}(Y)$ is said measurable if for every open $U \subset Y$, the set*

$$F_+^{-1}(U) = \{x \in Y : F(x) \subset U\}$$

is Lebesgue measurable.

Definition 1.2.3. *A multi-map F is called a Carathéodory function if*
(a) the multi-map $t \mapsto F(t, x)$ is measurable for each $x \in X$;
(b) for a.e. $t \in J$, the map $x \mapsto F(t, x)$ is upper semi-continuous.
Furthermore, F is L^1-Carathéodory if it is further locally integrably bounded, i.e., for each positive r, there exists $h_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\|_{\mathcal{P}} \leq h_r(t), \quad \text{for a.e. } t \in J \text{ and all } |x| \leq r.$$

Lemma 1.2.1. ([18]) *The multivalued map $F : [a, b] \to \mathcal{P}_{cl}(Y)$ is measurable if and only if for each $x \in Y$, the function $\zeta : [a, b] \to [0, +\infty)$ defined by*

$$\zeta(t) = \operatorname{dist}(x, F(t)) = \inf\{\|x - y\| \,:\, y \in F(t)\}, \quad t \in [a, b],$$

is Lebesgue measurable.

The following two lemmas are needed. The first one is the celebrated Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 1.2.2. *([18], Theorem 19.7) Let Y be a separable metric space and $F : [a, b] \to \mathcal{P}(Y)$ a measurable multi-valued map with nonempty closed values. Then F has a measurable selection.*

Lemma 1.2.3. *[26] Let I be a compact interval and E be a Banach space. Let F be an L^1-Carathéodory multi-valued map with $S_{F,y} \neq \emptyset$, and let Γ be a linear continuous mapping from $L^1(I, E)$ to $C(I, E)$. Then, the operator*

$$\Gamma \circ S_F : C(I, E) \longrightarrow \mathcal{P}_{cp,c}(E), \quad y \longmapsto (\Gamma \circ S_F)(y) = \Gamma(S_{F,y}),$$

is a closed graph operator in $C(I, E) \times C(I, E)$, where $S_{F,y}$ is known as the selectors set from F and given by
$f \in S_{F,y} = \{f \in L^1(I, E) : f(t) \in F(t, y(t)) \;\; for \; a.e.t \in I\}$.

Lemma 1.2.4. *[26] Let I be a compact interval and E be a Banach space. Let F be an L^1-Carathéodory multi-valued map with $S_{F,y} \neq \emptyset$, and let Γ be a linear continuous mapping from $L^1(I, E)$ to $C(I, E)$. Then, the operator*

$$\Gamma \circ S_F : C(I, E) \longrightarrow \mathcal{P}_{cp,c}(E), \quad y \longmapsto (\Gamma \circ S_F)(y) = \Gamma(S_{F,y}),$$

is a closed graph operator in $C(I, E) \times C(I, E)$, where $S_{F,y}$ is known as the selectors set from F and given by
$f \in S_{F,y} = \{f \in L^1(I, E) : f(t) \in F(t, y(t)) \;\; for \; a.e.t \in I\}$.

Lemma 1.2.5. *[3] If $G : X \to \mathcal{P}_{cp}$ is u.s.c, then for any $x_0 \in X$,*

$$\lim_{x \to x_0} \sup G(x) = G(x_0).$$

Lemma 1.2.6. *(See e.g. [3], Lemma 1.1.9). Let $(k_n)_{n\in\mathbb{N}} \subset k \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{co}(\lim_{n\to\infty} \sup k_n) = \bigcap_{N>0} \overline{co}(\bigcup_{n\geq N} k_n),$$

where $\overline{co}A$ refers to the closure of the convex hull of A.

Lemma 1.2.7. *[3] Every semi-compact sequence $L^1([0, b], E)$ is weakly compact in $L^1([0, b], E)$.*

Lemma 1.2.8. *(Mazur's Lemma, [30], Theorem 21.4). Let E be a normed space and $x_{k k\in\mathbb{N}} \subset E$ be a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^{k=m} \alpha_{mk} x_k$ with $\alpha_{mk} > 0$ for $k = 1, 2, ..., m$ and $\sum_{k=1}^{k=m} \alpha_{mk} = 1$, which converges strongly to x.*

Theorem 1.2.9. *[31] Let (X, d) be a complete generalized metric space and $F : X \to \mathcal{P}_{cl,b}(X)$ a contractive multivalued operator with Lipschitz matrix M. Then N has at least one fixed point.*

Theorem 1.2.10. *[31] Let (X,d) be a complete generalized metric space and $F : X \to \mathcal{P}_{cl}(X)$ be a multivalued map. Assume that there exist $A, B, C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$H_d(F(x), F(y)) \leq Ad(x,y) + Bd(y, F(x)) + Cd(x, F(x)))$$

where $A + C$ converge to zero. Then there exist $x \in X$ such that $x \in F(x)$.

Theorem 1.2.11. *[31] Let $(X, \|\cdot\|)$ be a generalized Banach space and $F : X \to \mathcal{P}_{cp,cv}(X)$ be a completely continuous multivalued mapping and u.s.c. Moreover assume that the set*

$$\mathcal{A} = \{x \in X : x \in \lambda F(x) \quad \textit{for some } \lambda \in (0,1)\}$$

is bounded. Then F has a fixed point.

Chapter 2

Perov Type Fixed Point Theorem

Our first purpose here is to establish a Perov fixed point theorem type for expansive and nonexpansive operators.

Lemma 2.0.1. *[31] Let (X, d) be a generalized metric space .Then there exists a homeomorphism map $h : X \to \overline{X}$.*
proof.*Cosider $h : X \to \overline{X}$ defined by $h(x) = (x, \dots, x) forall x \in X$. Obviously h is bijective.*

- *To prove that h is a continuous map. let $x, y \in X$.Thus*

$$d_*(h(x), h(y)) \leq \sum_{i=1}^{n} d_i(x, y).$$

For $\varepsilon > 0$ we take $\delta = (\frac{\varepsilon}{n}, \dots, \frac{\varepsilon}{n})$,let $x_0 \in X$ be fixed and $B(x_0, \delta) = \{x \in X : d(x_0, x) < \delta\}$,Then for every $x \in B(x_0, \delta)$ we have

$$d_*(h(x_0), h(x)) \leq \varepsilon$$

- *Now , $h^{-1} : \overline{X} \to X$ is a map defined by*

$$h^{-1}(x, \dots, x) = x, (x, \dots, x) \in \overline{X}$$

To show that $h^{-1}$is continuous , Let $(x, \dots, x), (y, \dots, y) \in \overline{X}$,Then

$$d(h^{-1}(x, \dots, x), h^{-1}(y, \dots, y)) = d(x, y).$$

Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) > 0$ .We take $\delta = \frac{1}{n}(\min_{1 \leq i \leq n} \varepsilon_i)$ and we fix $(x_0, \dots, x_0) \in \overline{X}$ .Set

$$B((x_0, \dots, x_0), \delta) = \{(x, \dots, x) \in \overline{X} : d_*((x_0, \dots, x_0), (x, \dots, x)) < \delta\}.$$

For $(x,\dots,x)\in B((x_0,\dots,x_0),\delta)$ *we have*

$$d_*((x_0,\dots,x_0),(x,\dots,x))<\delta\Rightarrow\sum_{i=1}^{n}d_i(x_0,x)<\frac{1}{n}(\min_{1\le i\le n}\varepsilon_i)$$

Then

$$d_i(x_0,x)<\frac{1}{n}(\min_{1\le i\le n}\varepsilon_i,i=1,\dots,n\Rightarrow d(x_0,x)<\varepsilon.$$

Hence h^{-1} *is continuous.*

Theorem 2.0.2. *[31] Every generalized metric space is paracompact.*

Proof. Let X be a generalized metric space , By there exists $\overline{X}$,metric space which is homeomorphic to X .Since every metric space is paracompact hence X is paracompact. □

Theorem 2.0.3. *[31] Let* (X,d) *a generalized metric space .To any locally finite open converging* $(U_i)_{i\in I}$ *of* X *,we can associate locally Lipschitzian partition of unity subordinated .*

Proof. X is paracompact ,there exists a family of locally finite open set ,let us write,

$$\nu=\{V_i\setminus i\in I_*\}$$

convering of X such that

$$\overline{V}_i\subset U_i\,for\,every\,i\in I_*.$$

Let us define for any $i\in I_*$ the function $f_i:X\to\mathbb{R}_+$ by

$$f_i(x)=\sum_{i=1}^{n}d_j(x,X\setminus V_i)$$

For each $x,y\in X$ we have

$$\left|\sum_{j=1}^{n}d_j(X\setminus V_i)-\sum_{j=1}^{n}d_j(y,X\setminus V_i)\right|\le\sum_{j=1}^{n}d_j(x,y)\,for\,each\,x,y\in X$$

hence

$$\left|\sum_{j=1}^{n}d_j(x,X\setminus V_i)-\sum_{j=1}^{n}(y,X\setminus V_i)\right|\le Ad(x,y)\,for\,each\,x,y\in X$$

where $A=(1,\dots,1)\in\mathcal{M}_{1\times n}(\mathbb{R}_+)$.Then for every $i\in I_*$, f_i is Lipschitzian and verifies

$$supp(f_i)=\overline{V}_i\subset U_i.$$

Let us introduce for any $i\in I_*$ the following function $\psi_i:X\to[0,1]$ defined by

$$\psi_i(x)=\frac{f_i(x)}{\sum_{i\in I_*}f_i(x)}\,for\,all\,x\in X.$$

1. Firstly ,we prove that ψ_i is locally on X ,Indeed ,let $x \in X$,then there exists neighborhood V_x of x with meets only a finite number of $\{\overline{V}_i \setminus i \in I_*\}$ .That is there is $\{i_1, \ldots, i_m\}$ such that $V_x \cap V_i = \emptyset$ for each $i \in I_* \setminus \{1, \ldots, i_p\} \Rightarrow \sum_{i\in I_*} f_i(y) = \sum_{k=1}^{p} f_{ik}(y) > 0, y \in V_x$.

By the continuity of $\sum_{k=1}^{n} f_{ik}$ there exists a neighborhood $W_x \subset V_x$ of x and $m, \overline{M} > 0$ Such that

$$m \leq \sum_{i\in I_*} f_{ik}(y) \leq \overline{M}\, for\, any\, y \in W_x.$$

Thus for $y, z \in W_x$,we get

$$\begin{aligned}
|\psi_i(z) - \psi_i(y)| &= \left| \frac{f_i(y)}{\sum_{i\in I_*} f_i(y)} - \frac{f_i(z)}{\sum_{i\in I_*} f_i(z)} \right| \\
&= \left| \frac{\sum_{k=1}^{p} f_{ik}(z) f_i(y) - \sum_{k=1}^{p} f_{ik}(y) f_i(z)}{\sum_{k=1}^{p} f_{ik}(y) \sum_{k=1}^{p} f_{ik}(z)} \right| \\
&\leq \frac{1}{m^2} \left|\sum_{k=1}^{p} f_{ik}(z) f_i(y) - \sum_{k=1}^{p} f_{ik}(z)\right| \\
&\leq \frac{1}{m^2} \sum_{k=1}^{p} |f_{ik}(z) f_i(y) - f_{ik}(y) f_i(z)| \\
&\leq \frac{1}{m^2} \sum_{k=1}^{p} |f_{ik}(z) - f_{ik}(y)||f_i(y)| + \sum_{k=1}^{p} |f_{ik}(y)||f_i(y) - f_i(z)|
\end{aligned}$$

Therefore

$$|\psi(z) - \psi_i(y)| \leq \frac{2\overline{M}p}{m^2} Ad(y, z)\, for\, any\, y, z \in W_x$$

Now,we show that ψ_i is continuous .Let $x_0 \in X$ .then there exists a neighborhood V_x of x which meets only a finite number of $\{\overline{V} \setminus i \in I_*\}$ .That is there is $\{i_1, \ldots, i_m\}$ such that

$$V_{x_0} \cap V_i = \emptyset\, for\, each\, i \in I_* \setminus \{i_1, \ldots, i_p\}.$$

This implies that ,for every $i \in I_* \setminus \{i_1, \ldots, i_p\}$ we have

$$V_{x_0} \subset X \setminus V_i \Rightarrow f_i(V_{x_0}) = 0,$$

and

$$V_{x_0} \cap supp(f_i) = \emptyset\, for\, each\, x \in I_* \setminus \{i_1, \ldots, i_p\}.$$

Form 1) we obtain

$$\sum_{i\in I_*} f_i(x) = \sum_{i=1}^{p} f_i(x)\, for\, each\, x \in V_{x_0}.$$

Therefore,

$$\psi_i(x) = \frac{f_i(x)}{\sum_p^{k=1} f_{ik}(x)} \, for\, every\, x \in V_{x_0}$$

It is clear that $\sum_{k=1}^{p} f_{ik}(x_0) \neq 0$,since for each $i \in I_*$,f_i is continuous function .Hence ψ_i is a continuous on X . □

Theorem 2.0.4. *Let $(X, |.|)$ be a generalized normd space , $(Y, \|.\|)$ be a generalized Banch space and $F : X \to \rho_{cv}(Y)$ be an u.s.c. multivalued map.Then,for every $\varepsilon \in \mathbb{R}_+^n$,there exists a locally Lipschitzian function $f_\varepsilon : X \to Y$ such that*

$$f_\varepsilon(x) \subseteq coF(x)$$

and

$$\text{graph}(f_\varepsilon) \subseteq \text{Graph}(F) + B(F(x), \varepsilon)$$

Proof. Fix $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) > 0$ for every $x \in X$ there exists $B(x, \delta(x)) \subset X$ such that

$$F(y) \subseteq F(B(x, \delta(x)) \subset F(x) = B(0, \varepsilon) \, for\, each\, y \in B(x, \delta(x))$$

were $\delta(x) = (\delta_1(x), \dots, \delta_n(x)) > 0$ .We family $B(x, \delta(x))_{x \in X}$ cover X .From theorem 2.0.4 ,X is paracompact ,Let $U_{i \in I_*}$ be a locally Lipschtizian partition of unity subordunate to it .Chose for each $i \in I_*$ an $x_i \in U_i$ and define f_ε by

$$f_\varepsilon(x) = \sum f_i z_i \, for\, each\, x \in X$$

Now we show that f_ε is an approximation of F,let $x \in X$and $I_*(x)$the subset of all $i \in I_*$ such that $f_i(x) \neq 0$ let $i, j \in I_*$

$$d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) \leq \delta_i + \delta_j < \in$$

let $k \in I_*(x)$ be such that

$$\delta_k = \max \delta_i$$

$$F(x_i) \subset F(B(x_k, 2\delta_k)) \subset F(x_k) + B(0, \varepsilon), \; for\, all\, i \in I_*(x)$$

Using the fast that $F(x) + B(0, \varepsilon)$ is convex then

$$f_\varepsilon(x) \in F(x) + B(0, \varepsilon)$$

$$|\psi_i(z) - \psi_i(y)| \qquad = \left| \frac{f_i(y)}{\sum_{i \in I} f_i(y)} - \frac{f_i(z)}{\sum_{i \in I} f_i(z)} \right|$$

$$= \left| \frac{\sum_{k=1}^{n} f_{ik}(z) f_i(y) - \sum_{k=1}^{p} f_{ik}(y) f_i(z)}{\sum_{k=1}^{p} f_{ik}(z) \sum_{i=1}^{p} f_{ik}(z)} \right|$$

□

Theorem 2.0.5. *Let X be a generalized Banach ,C be a nonempty convex subset of $X, G : C \to \mathrm{P}_{cp,cv}(c)$ be an u.s.c/ multivalued map ,then the operator inclusion G has at leat one fixed point, that is there exists $x \in C$ such that $x \in G(x)$*

Definition 2.0.1. *Let E be a vector space on R or C .By a vector-valued norm on E we mean a map $\|.\| :\to \mathbb{R}^n$ with the following properties:*

1. $\|x\| \geq 0 \, for \, all \, x \in E; if \|x\| = 0 then x = 0$
2. $\|\lambda x\| = |\lambda| \|x\| \, forall \, x \in E and \lambda \in \mathbb{K}$
3. $\|x + y\| = \leq \|x\| + \|y\| \, for \, all \, x \in E$

The pair $(E, \|.\|)$ is called a generalized normed space .If the generalized by $\|.\|(i.e.d(x, y) = \|x - y\|)$ is complete then the space $(E, \|.\|)$ is called a generalized Banach space.

Theorem 2.0.6. *[18] Let E be a generalized Banach space ,let $C \in \mathrm{P}_{cv}(E)$ and $f : C \to C$ be a continuous operator with relatively compact range .Then f has at leat fixed point in C .*

Definition 2.0.2. *Let (X, d) be a generalized metric space and C be a subset of X .The mapping $B : C \to X$ is said to be expansive, if there exists a constant $k \in \mathbb{R}, k > 1$ such that*

$$d(B(x), B(y)) \geq kd(x, y) \, forall \, x, y \in C.$$

Lemma 2.0.7. *let $B : X \to X$ be a map such that B^m ($m-$power) is an expansive for some $m \in \mathbb{N}$, Assume further that there exist a closed subset C of X such that C is contained $B(C)$. There exists a unique fixed point of B.*

Lemma 2.0.8. *Let X be a generalized metric space and $C \subseteq X$. Assume the mapping $B : C \to X$ is expansive with constant $k > 1$ .Then the inverse of $B : C \to B(c)$ exists and*

$$d(B^{-1}(x), B^{-1}(y)) \leq \frac{1}{k} d(x, y) \, , x, y \in B(C)$$

Proof. Let $x, y \in C$ and $B(x) = B(y)$,then

$$d(B(x, y), B(y)) \geq kd(x, y) \Rightarrow d(x, y) = 0 \Rightarrow x = y.$$

Thus $B : C \to B(C)$ is invertible. Let $x, y \in B(C)$, then there exist $a, b \in C$ such that

$$B(a) = x, B(b) = y.$$

Hence

$$d(a, b) = d(B^{-1}(x), B^{-1}(y)) \, and \, d(x, y) = d(B(a), B(b)) \geq kd(a, b).$$

Therefore

$$d(B^{-1}(x), B^{-1}(y)) \leq \frac{I}{K} d(x, y) \, for \, all \, x, y \in C.$$

□

Theorem 2.0.9. *Let X be a complete generalized metric space and C be a closed subset of X. Assume $B : C \to X$ is expansive and $C \subseteq B(C)$,Then there exists a unique point $x \in C$ such that $x = B(x)$*

Proof. Since B is expansive the there exists $K > 1$ such that

$$d(B(x), B(y)) \geq kd(x, y) forall x, y \in C$$

From the operator $B : C \to C$ is invertible and

$$d(B^{-1}(x); B^{-1}(y)) \leq \frac{I}{K} d(x, y), x, y \in C$$

Hence B^{-1} is contractive

$$B^{-1}(x) = x \Rightarrow x = B(x)$$

□

Theorem 2.0.10. *[33] Let (X, d) be a complete generalized metric space and $N : X \to X$ a contractive operator with Lipschitz matrix M. Then N has a unique fixed point x_* and for each $x_0 \in X$ we have*

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1} d(x_0, N(x_0)) \quad \text{for all } k \in \mathbb{N}.$$

Theorem 2.0.11. *[35] Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that for every $\omega \in \Omega$ the matrix, $M(\omega)$ converge to 0 and*

$$d(F(\omega, x_1), F(\omega, x_2)) \leq M(\omega) d(x_1, x_2) \text{ for each } x_1, x_2 \in X, \ \omega \in \Omega.$$

then there exists any random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Theorem 2.0.12. *Let X be a separable generalized Banach space and let $F : \Omega \times X \to X$ be a completely continuous random operator. Then, either*

(i) the random equation $F(\omega, x) = x$ has a random solution, i.e, there is a measurable function $x : \Omega \to X$ such that $F(\omega, x(\omega)) = x(\omega)$ for all $\omega \in \Omega$, or

(ii) the set $\mathcal{M} = \{x : \Omega \to X \text{ is measurable} | \lambda(\omega) F(\omega, x) = x\}$ is unbounded for some measurable $\lambda : \Omega \to X$ with $0 < \lambda(\omega) < 1$ on Ω.

Theorem 2.0.13. *Let X be a separable metric space and $G : \Omega \times X \to X$ be a mapping such that $G(., x)$ is measurable for all $x \in X$ and $G(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to G(\omega, x)$ is jointly measurable.*

As consequence of above theorem we can easily prove the following result.

Lemma 2.0.14. *[35] Let X be a separable generalized metric space and $G : \Omega \times X \to X$ be a mapping such that $G(., x)$ is measurable for all $x \in X$ and $G(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to G(\omega, x)$ is jointly measurable.*

2.1 Application

Application of theorem 2.0.11 Using topological degree methods, we give some existence results for functional differential equations, we study the following systems

$$\begin{cases} x'(t,\omega) = f(t, x_t(.,\omega), y_t(.,\omega), \omega), & t \in J := [0,T] \\ y'(t,\omega) = g(t, x_t(.,\omega), y_t(.,\omega), \omega), & t \in J := [0,T] \\ x(\theta,\omega) = \varphi(\theta,\omega), & \theta \in [-r,0] \\ y(\theta,\omega) = \psi(\theta,\omega), & \theta \in [-r,0]. \end{cases} \tag{2.1.1}$$

where $f, g : J \times C([-r,0]\times\Omega,\mathbb{R}) \times C([-r,0]\times\Omega,\mathbb{R}) \times \Omega \to \mathbb{R}$, $(\Omega, \mathcal{A})$ is a measurable space.
For any function x defined on $[-r,T]\times\Omega$ and any $t \in J$ we denote by $x_t(.,\omega)$ the element of $C([-r,0]\times\Omega,\mathbb{R})$ defined by

$$x_t(\theta,\omega) = x(t+\theta,\omega),\ \theta \in [-r,0].$$

Here $x_t(.,\omega)$ represents the history of the state from time $t-r$, up to the present time t.

2.1.1 Existence and uniqueness of random solutions

In this section we shall use a random version of the Perov type and study the nonlinear initial value problems of random functional differential equations.

Set $C_r := C([-r,0]\times\Omega,\mathbb{R})$ and $C := C([-r,T]\times\Omega,\mathbb{R})$.

Theorem 2.1.1. *$f, g : J \times C_r \times C_r \times \Omega \to \mathbb{R}$ are two Carathéodory functions. Assume that the following condition hold:*

(H_1) *There exist $p_1, p_2, p_3, p_4 : \Omega \to \mathbb{R}_+$ are random variable such that*

$$|f(t,x,y,\omega) - f(t,\widetilde{x},\widetilde{y},\omega)| \le p_1(\omega)|x-\widetilde{x}| + p_2(\omega)|y-\widetilde{y}|$$

and

$$|g(t,x,y,\omega) - g(t,\widetilde{x},\widetilde{y},\omega)| \le p_3(\omega)|x-\widetilde{x}| + p_4(\omega)|y-\widetilde{y}|,$$

for each $t \in J$, $x, y, \widetilde{x}, \widetilde{y} \in C_r$ and $\omega \in \Omega$.

Suppose that, for every $\omega \in \Omega$, the matrix

$$M(\omega) = \begin{pmatrix} Tp_1(\omega) & Tp_2(\omega) \\ Tp_3(\omega) & Tp_4(\omega) \end{pmatrix}$$

has converge to 0, then the problem (2.1.1) has a unique random solution.

Proof. Consider the operator $N : C\times C\times\Omega \to C\times C$, $(x, y, \omega) \to (L_1(x, y, \omega), L_2(x, y, \omega))$ where

$$L_1(x(t,\omega), y(t,\omega), \omega) = \varphi(0,\omega) + \int_0^t f(s, x_s(.,\omega), y_s(.,\omega), \omega)ds$$

and

$$L_2(x(t,\omega), y(t,\omega), \omega) = \psi(0,\omega) + \int_0^t g(s, x_s(.,\omega), y_s(.,\omega), \omega)ds.$$

First we show that N is a random operator on $C \times C \times \Omega$. Since f and g are Carathéodory functions, then $\omega \to f(t, x, y, \omega)$ and $\omega \to g(t, x, y, \omega)$ are measurable maps in view of lemma 2.0.14. Further, the integral is a limit of a finite sum of measurable functions, therefore, the maps

$$\omega \to L_1(x(t,\omega), y(t,\omega), \omega), \quad \omega \to L_2(x(t,\omega), y(t,\omega), \omega)$$

are measurable. As a result, N is a random operator on $N : C \times C \times \Omega$ into $C \times C$. We show that N satisfies all the conditions of Theorem 2.0.11 on $C \times C \times \Omega$.
Let $(x, y), (\widetilde{x}, \widetilde{y}) \in C \times C$ then

$$\begin{aligned}
&|L_1(x(t,\omega), y(t,\omega), \omega) - L_1(\widetilde{x}(t,\omega), \widetilde{y}(t,\omega), \omega)| = \\
&\qquad |\int_0^t (f(s, x_s(.,\omega), y_s(.,\omega), \omega) - f(s, \widetilde{x}_s(.,\omega), \widetilde{y}_s(.,\omega), \omega))ds| \\
&\leq \int_0^t |f(s, x_s(.,\omega), y_s(.,\omega), \omega) - f(s, \widetilde{x}_s(.,\omega), \widetilde{y}_s(.,\omega), \omega)|ds \\
&\leq \int_0^t p_1(\omega)|x_s(.,\omega) - \widetilde{x}_s(.,\omega)|ds \\
&\qquad + \int_0^t p_2(\omega)|y_s(.,\omega) - \widetilde{y}_s(.,\omega)|ds.
\end{aligned}$$

Then

$$\|L_1(x, y, \omega) - L_1(\widetilde{x}, \widetilde{y}, \omega)\|_\infty \leq Tp_1(\omega)\|x - \widetilde{x}\|_\infty + Tp_2(\omega)\|y - \widetilde{y}\|_\infty.$$

Similarly, we obtains

$$\|L_2(x, y, \omega) - L_2(\widetilde{x}, \widetilde{y}, \omega)\|_\infty \leq Tp_3(\omega)\|x - \widetilde{x}\|_\infty + Tp_4(\omega)\|y - \widetilde{y}\|_\infty.$$

Hence

$$d(N(x, y, \omega), N(\widetilde{x}, \widetilde{y}, \omega)) \leq M(\omega)d((x, y), (\widetilde{x}, \widetilde{y})),$$

where

$$d(x, y) = \begin{pmatrix} \|x - y\|_\infty \\ \|\widetilde{x} - \widetilde{y}\|_\infty \end{pmatrix}.$$

From theorem 2.0.11 there exists unique random solution of problem (2.1.1). □

Lemma 2.1.2. *[9] Let* $I = [p, q]$ *and let* $u, g : I \to \mathbb{R}$ *be positive continuous functions. Assume there exist* $c > 0$ *and a continuous nondecreasing function* $h : [0, \infty) \to (0, +\infty)$ *such that*

$$u(t) \leq c + g(s)h(u(s))ds, \quad \forall t \in I.$$

Then

$$u(t) \leq H^{-1}\Big(\int_p^t g(s)ds\Big), \quad \forall t \in I,$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_p^q g(s)ds,$$

where H^{-1} *refers to inverse of the function* $H(u) = \int_c^u \frac{dy}{h(y)}$ *for* $u \geq c$.

We consider the following set of hypotheses in what follows:

(H_2) The functions f and g are random Carathéodory on $[0, T] \times C_r \times C_r \times \Omega$.

(H_3) There exist a measurable and bounded functions $\gamma_1, \gamma_2 : \Omega \to L^1([0, T], \mathbb{R}_+)$ and a continuous and nondecreasing function $\psi_1, \psi_2 : \mathbb{R}_+ \to (0, \infty)$ such that

$$|f(t, x, y)| \leq \gamma_1(t, \omega)\psi_1(|x| + |y|), \quad |g(t, x, y)| \leq \gamma_2(t, \omega)\psi_2(|x| + |y|) \quad \text{a.e. } t \in [0, T]$$

for all $\omega \in \Omega$ and $x, y \in C_r$.

Now, we give prove of the existence result of problem (2.1.1) by using Schaefer's random fixed point theorem type in generalized Banach space.

Theorem 2.1.3. *Assume that the hypotheses* (H_2) *and* (H_3) *hold. If*

$$\int_0^T (\gamma_1(s, \omega) + \gamma_2(s, \omega))ds < \int_{\|\varphi(.,\omega)\|_\infty + \|\psi(.,\omega)\|_\infty}^{\infty} \frac{du}{\psi_1(u) + \psi_2(u)}, \quad \textit{for all } \omega \in \Omega$$

Then the problem (2.1.1) has a random solution.
moreover the set

$$S = \{(x; y) \in C \times C : \textit{(x,y) is solution of the problem (2.1.1)} \}$$

is compact.

Proof. Let $N : C \times C \times \Omega \to C \times C$ a random operator defined in Theorem 1.1.1. Clearly, the random fixe point of N are solutions to (2.1.1), where N is defined in Theorem (1.1.1) . In order to apply Theorem (2.0.12) , we first show that N is completely continuous. The proof will be given in several steps.

Step 1: $N(., ., \omega) = (L_1(., ., \omega), L_2(., ., \omega))$ is continuous.

Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x, y)$ in $C \times C$ as $n \to \infty$. Then

$$\begin{aligned} &|L_1(x_n(t,\omega), y_n(t,\omega), \omega) - L_1(x(t,\omega), y(t,\omega), \omega)| \\ \leq\ & \int_0^t |f(s, x_{ns}(.,\omega), y_{ns}(.,\omega), \omega) - f(s, x_s(.,\omega), y_s(.,\omega), \omega)| ds, \end{aligned}$$

and so

$$\begin{aligned} &\|L_1(x_n(.,\omega), y_n(.,\omega), \omega) - L_1(x(.,\omega), y(.,\omega), \omega)\|_\infty \\ \leq\ & \int_0^T |f(s, x_{ns}(.,\omega), y_{ns}(.,\omega), \omega) - f(s, x_s(.,\omega), y_s(.,\omega), \omega)| ds. \end{aligned}$$

Since f is an L^1-Carath$\acute{e}$odory function, we have by the Lebesgue dominated convergence theorem, we have

$$\|L_1(x_n(.,\omega), y_n(.,\omega), \omega) - L_1(x(.,\omega), y(.,\omega), \omega)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Similarly

$$\|L_2(x_n(.,\omega), y_n(.,\omega), \omega) - L_2(x(.,\omega), y(.,\omega), \omega)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Thus N is continuous.
Step 2: N maps bounded sets into bounded sets in $C \times C$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $(x, y) \in B_q = \{(x, y) \in C \times C : \|x\|_\infty \leq q, \|y\|_\infty \leq q\}$, we have

$$\|N(x, y, \omega)\|_\infty \leq l = (l_1, l_2).$$

Then for each $t \in [0, T]$, we get

$$\begin{aligned} |L_1(x(t,\omega), y(t,\omega), \omega)| &= |\varphi(0,\omega) + \int_0^t f(s, x_s(.,\omega), y_s(.,\omega), \omega) ds| \\ &\leq |\varphi(0,\omega)| + \int_0^t |f(s, x_s(.,\omega), y_s(.,\omega), \omega)| ds. \end{aligned}$$

From (H_3), we have

$$\|L_1(x(.,\omega), y(.,\omega), \omega)\|_\infty \ \leq \ \|\varphi(0,\omega)\| + \psi_1(2q)\int_0^T \gamma_1(s,\omega)ds := l_1.$$

Similarly, we have

$$\|L_2(x(.,\omega), y(.,\omega), \omega)\|_\infty \ \leq \ \|\psi(0,\omega)\| + \psi_2(2q)\int_0^T \gamma_2(s,\omega)ds := l_2.$$

Step 3: N maps bounded sets into equicontinuous sets of $C \times C$.

Let $0 < \tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and B_q be a bounded set of $C \times C$ as in Step 2. Let $(x, y) \in B_q$ then for each $t \in J$ we have

$$|L_1(x(\tau_2,\omega), y(\tau_2,\omega), \omega) - L_1(x(\tau_1,\omega), y(\tau_1,\omega), \omega)| \ \leq \ \int_{\tau_1}^{\tau_2} |f(s, x_s(.,\omega), y_s(.,\omega), \omega)|ds.$$

Hence

$$|L_1(x(\tau_2,\omega), y(\tau_2,\omega), \omega) - L_1(x(\tau_1,\omega), y(\tau_1,\omega), \omega)| \ \leq \ \psi_1(2q)\int_{\tau_1}^{\tau_2} \gamma_1(s,\omega)ds$$

and

$$|L_2(x(\tau_2,\omega), y(\tau_2), \omega) - L_2(x(\tau_1), y(\tau_1,\omega), \omega)| \ \leq \ \psi_2(2q)\int_{\tau_1}^{\tau_2} \gamma_2(s,\omega)ds.$$

he right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$.

As a consequence of Steps 2, 3 and the Arzela á Ascot theorem we can conclude that we conclude that N maps B_q into a precompact set in $C \times C$.

Step 4: *(A priori bounds on solutions.)*

Now, it remains to show that the set

$$\Sigma = \Big\{ \ (x, y) \in C \times C : (x, y) = \lambda(\omega)N(x, y), \lambda(\omega) \in (0, 1) \ \Big\} \text{is bounded.}$$

Let $(x, y) \in \Sigma$. Then $x = \lambda(\omega)L_1(x, y)$ and $y = \lambda(\omega)L_2(x, y)$ for some $0 < \lambda(\omega) < 1$. Thus, for $t \in [0, T]$, we have

$$|x(t,\omega)| \ \leq \ |\varphi(0,\omega)| + \int_0^t |\gamma_1(t,\omega)\psi_1(|x_s(.,\omega)| + |y_s(.,\omega)|)|ds$$

and

$$|y(t,\omega)| \leq |\psi(0,\omega)| + \int_0^t |\gamma_2(t,\omega)\psi_2(|x_s(.,\omega)| + |y_s(.,\omega)|)|ds.$$

Therefore

$$|x(t,\omega)| + |y(t,\omega)| \leq c + \int_0^t p(s)\phi(|x_s(.,\omega)| + |y_s(.,\omega)|)ds,$$

where

$$c = |\varphi(0,\omega)| + |\psi(0,\omega)|, \ \phi = \psi_1 + \psi_2 \text{ and } p = \gamma_1 + \gamma_2.$$

By Lemma 2.0.1, we have

$$|x(t,\omega)| + |y(t,\omega)| \leq \Gamma^{-1}\big(\int_0^t p(s)ds\big) := K_*, \text{ for each } t \in [0,T],$$

where

$$\Gamma(z) = \int_c^z \frac{du}{\phi(u)}.$$

Consequently

$$\|x\|_\infty \leq K_* \text{ and } \|y\|_\infty \leq K_*.$$

This shows that Σ is bounded. As a consequence of Theorem 2.0.12 we deduce that N has a random fixed point (x,y) which is a solution to the problem (2.1.1).
Step 5: It remains to show that the set S is compact.

Let the sequence $(x_n, y_n)_{n\in\mathbb{N}} \subset S$, then

$$x_n(t,\Omega) = \begin{cases} \varphi(t,\omega), & \text{t} \in [-r,0] \\ \varphi(0,\omega) + \int_0^t f(s, x_{ns}(.,\omega), y_{ns}(.,\omega),\omega)ds\ , & \text{t} \in J \end{cases}$$

and

$$y_n(t,\omega) = \begin{cases} \psi(t,\omega), & \text{t} \in [-r,0] \\ \psi(0,\omega) + \int_0^t f(s, x_{ns}(.,\omega), y_{ns}(.,\omega),\omega)ds\ , & \text{t} \in J. \end{cases}$$

Let $B = \{(x_n, y_n) : n \in \mathbb{N}\} \subseteq C \times C$.
Then from earlier parts of the proof of this theorem, we conclude that B is bounded and equicontinuous. Then from the Ascoli-Arzelà theorem we can conclude that B is compact, then there exists a subsequence $(x_{nm}, y_{nm}) \subset S$; $(x_{nm}, y_{nm}) \to (x,y)$ as $n_m \to \infty$. Consider

$$z(t,\Omega) = \begin{cases} \varphi(t,\omega), & \text{t} \in [-r,0] \\ \varphi(0,\omega) + \int_0^t f(s, z_s(.,\omega), j_s(.,\omega),\omega)ds\ , & \text{t} \in J \end{cases}$$

and

$$j(t,\omega)=\begin{cases}\psi(t,\omega), & \text{t} \in [-r,0]\\ \psi(0,\omega)+\displaystyle\int_0^t f(s,z_s(.,\omega),j_s(.,\omega),\omega)ds\ , & \text{t} \in J,\end{cases}$$

then

$$|x_{nm}(t,\omega)-z(t,\omega)| \ \leq \ \int_0^t |f(s,x_{ns}(.,\omega),y_{ns}(.,\omega),\omega)-f(s,z_s(.,\omega),j_s(.,\omega),\omega)|ds$$

and

$$|y_{nm}(t,\omega)-j(t,\omega)| \ \leq \ \int_0^t |f(s,x_{ns}(.,\omega),y_{ns}(.,\omega),\omega)-f(s,z_s(.,\omega),j_s(.,\omega),\omega)|ds$$

$(x_{nm}(t,\omega),y_{nm}(t,\omega))\to(z(t,\omega),j(t,\omega))$ as $n_m\to\infty$. Thus

$$x(t,\Omega)=\begin{cases}\varphi(t,\omega), & \text{t} \in [-r,0]\\ \varphi(0,\omega)+\displaystyle\int_0^t f(s,x_s(.,\omega),y_s(.,\omega),\omega)ds\ , & \text{t} \in J\end{cases}$$

and

$$j(t,\omega)=\begin{cases}\psi(t,\omega), & \text{t} \in [-r,0]\\ \psi(0,\omega)+\displaystyle\int_0^t f(s,x_s(.,\omega),y_s(.,\omega),\omega)ds\ , & \text{t} \in J,\end{cases}$$

□

2.1.2 An example

Let $\Omega=\mathbb{R}$ be equipped with the usual $\sigma-$ algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$ and $J:=[0,1]$.
Consider the following random differential equation system.

$$\begin{cases}x'(t,\omega)=\frac{t\omega^2}{(2+\omega^2)(1+x_t^2(.,\omega)+y_t^2(.,\omega))}, & t\in J\\ y'(t,\omega)=\frac{t^2\omega^2}{(2+\omega^2)(x_t^2(.,\omega)+y_t^2(.,\omega))}, & t\in\\ x(\theta,\omega)=\varphi(\theta,\omega), & \theta\in[-r,0]\\ y(\theta,\omega)=\psi(\theta,\omega), & \theta\in[-r,0].\end{cases} \tag{2.1.2}$$

here

$$f(t,x,y,\omega)=\frac{t\omega^2}{(2+\omega^2)(1+x^2+y^2)}$$

$$g(t,x,y,\omega)=\frac{t^2\omega^2}{(2+\omega^2)(1+x^2+y^2)}$$

Clearly, the map $(t,\omega)\mapsto f(t,x,y,\omega)$ is jointly continuous for all $x,y\in[1,\infty)$. The same for the map g. Also the maps $x\mapsto f(t,x,y,\omega)$ and $y\mapsto f(t,x,y,\omega)$ are continuous for all $t \in J$ and $\omega \in \Omega$. Similarly for the maps corresponding to function g. Thus the functions f and g are Carathéodory on $J \times [1,\infty) \times [1,\infty) \times \Omega$. Firstly, we show that f and g are Lipschitz functions. Indeed, let $x, y \in \mathbb{R}$, then

$$\begin{aligned}
|f(t,x,y,\omega) - f(t,\widetilde{x},\widetilde{y},\omega)| &= \left|\frac{t\omega^2}{(2+\omega^2)(1+x^2+y^2)} - \frac{t\omega^2}{(2+\omega^2)(1+\widetilde{x}^2+\widetilde{y}^2)}\right| \\
&= \left|\frac{t\omega^2[(1+\widetilde{x}^2+\widetilde{y}^2)-(1+x^2+y^2)]}{2(1+\omega^2)(1+x^2+y^2)(1+\widetilde{x}^2+\widetilde{y}^2)}\right| \\
&= \frac{t\omega^2}{(2+\omega^2)(1+x^2+y^2)(1+\widetilde{x}^2+\widetilde{y}^2)}\left|\widetilde{x}^2+\widetilde{y}^2-x^2-y^2\right| \\
&\leq \frac{\omega^2}{(2+\omega^2)}|x-\widetilde{x}| + \frac{\omega^2}{(2+\omega^2)}|y-\widetilde{y}|.
\end{aligned}$$

Then

$$\|f(t,x,y,\omega) - f(t,\widetilde{x},\widetilde{y},\omega)\|_\infty \leq \frac{\omega^2}{(2+\omega^2)}\|x-\widetilde{x}\|_\infty + \frac{\omega^2}{(2+\omega^2)}\|y-\widetilde{y}\|_\infty.$$

Analogously for the function g, we get

$$\|g(t,x,y,\omega) - g(t,\widetilde{x},\widetilde{y},\omega)\|_\infty \leq \frac{\omega^2}{(2+\omega^2)}\|x-\widetilde{x}\|_\infty + \frac{\omega^2}{(2+\omega^2)}\|y-\widetilde{y}\|_\infty.$$

We take,

$$p_1(\omega) = p_2(\omega) = p_3(\omega) = p_4(\omega) = \frac{\omega^2}{(2+\omega^2)}$$

and

$$M(\omega) = \begin{pmatrix} \frac{\omega^2}{(2+\omega^2)} & \frac{\omega^2}{(2+\omega^2)} \\ \frac{\omega^2}{(2+\omega^2)} & \frac{\omega^2}{(2+\omega^2)} \end{pmatrix}.$$

We remark that

$$|\rho(M(\omega))| = \frac{\omega^2}{(2+\omega^2)} < 1,$$

then

$$M(\omega), \text{ converge to } 0.$$

Therefore, all the conditions of Theorem 2.1.1 are satisfied. Hence the problem (2.1.2) has a unique random solution.

Chapter 3

Krasnoselskii's Theorem Type

In this section we present the Krasnosel'skki fixed point theorem by using the expansive operator combined with continuous operator.

Lemma 3.0.1. *Let E be generalized normed space and $C \subseteq E$ .Assume the mapping $B : C \to X$ is expansive with constant $K > 1$ .Then the inverse of $I - B : C \to (I - B)(c)$ exists and*

$$d((I-B)^{-1})(x),(I-B)^{-1}(y) \leq \frac{1}{K-1} d(x,y), x, y \in (I-B)(C).$$

Proof : *Let $x, y \in C$ and $x - B(x) = y - B(y)$,then*

$$\begin{aligned} d(x-B(x), y-B(y)) &= \begin{pmatrix} \|x-B(x)-y+B(y)\|_1 \\ \cdots \\ \|x-B(x)-y+B(y)\|_n \end{pmatrix} \\ &\geq \begin{pmatrix} \|B(y)-B(x)\|_1 - \|x-y\|_1 \\ \cdots \\ \|B(y)-B(x)\|_n - \|x-y\|n \end{pmatrix} . \\ &\geq \begin{pmatrix} K\|y-x\|_1 - \|x-y\|_1 \\ \cdots \\ K\|y-x\|_n - \|x-y\|_n \end{pmatrix} \\ &= (K-1)Id(x,y) \end{aligned}$$

Thus $I - B : C \to (I - B)(C)$ is invertible . let $x, y \in (I - B)(C)$.then there exist $a, b \in C$ such that

$$a - B(a) = x, b - B(b) = y.$$

Hence

$$d(a,b) = d(I-B)^{-1}(x), (I-B)^{-1}(y)) and d(x,y) \geq Kd(a,b) - d(a,b).$$

Therefore

$$d((I-B)^{-1}(x),(I-B)^{-1}(y)) \leq \frac{I}{K-1}d(x,y)\, for\, all\, x,y \in (I-B)(C).$$

Theorem 3.0.2. *Let E be a generalized Banach space and C be a compact convex subset of E .Assume that $A : M \to X$ is continuous and $B : C \to E$ is continuous expansive map satisfy*
$(\mathcal{H}_1)$ for each $x, y \in C$ such that

$$x = B(x) + A(y) \Rightarrow x \in C.$$

Then there exists $y \in C$ such that $y = By + A(y)$.

Proof : Let $y \in C$ .let $F_y : C \to X$ be a operator defined by

$$F_y(x) = B(x) + A(y),\ x \in C.$$

From theorem 2.0.6 there exist unique $x(y) \in C$ such that

$$x(y) + B(x(y)) + A(y),\ x \in C.$$

By lemma(3.0.1) $I - B$ is invertible .Moreover, $(I - B)^{-1}$ is continuous .Let us define $N : C \to C$ by

$$y \to N(y) = (I-B)^{-1}A(y).$$

Let $x \in C$ and $N(x) = (I - B)^{-1}(A(x))$ .Then

$$N(x) = (I-B)^{-1}(A(x)) \Rightarrow N(x) = B(N(x)) + A(x).$$

and thus $(\mathcal{H}_1)$ implies that $N(x) \in C$. Let $\{y_m : m \in \mathbb{N}\} \subseteq C$ be a sequence converging to y in C we show that $N(y_m)$ converge to $y \in C$ we show that $N(y_m)$ converge to $N(y)$ .Set $x_m(I - B)^{-1}A(y_m)$,then

$$(I-B)(x_m) = A(y_m), m \in \mathbb{N}.$$

Since C is compact , there exists a subsequence of $\{x_m\}$ converging for some $x \in C$. Then

$$(I-B)(x_m) \to (I-B)(x)\, as\, m \to \infty.$$

Hence

$$A(y_m) \to (I-B)(x)\, as\, m \to \infty.$$

Therefore

$$N(y_m) \to N(y)\, as\, m \to \infty.$$

Hence from theorem prove there exist $y \in C$ such that $y = (I - B)^{-1}A(y)$, and we deduce that $B + G$ has a fixed point in C .

Theorem 3.0.3. *Let (X, d) be a complete generalized metric space and $F : X \to \mathcal{P}_{cl,b}(X)$ a contractive multivalued operator with Lipschitz matrix M . Then N has at least one fixed point .*

Theorem 3.0.4. *Let (X, d) be a generalized complete metric space , and let $F : X \to \mathcal{P}_{cl}(x)$ be a multivalued map . Assume that there exist $A, B, C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$H_d(F(x), F(y)) \leq Ad(x, y) + Bd(y, F(x)) + Cd(x, F(x)) \tag{3.0.1}$$

where $A + C$ converge to zero . Then there exist $x \in X$ such that $x \in F(x)$.

Proof. Let $x \in X$ and

$$D(x, d(x, F(x))) = \{y \in X : d(x, y) \leq d(x, F(x))\}.$$

Since $F(x)$ is closed , then

$$D(x) \cap F(x) \neq \varnothing.$$

So we can select $x_1 \in F(x)$ such that

$$d(x, x_1) \leq d(x, F(x)) \leq Ad(x, x1) + Bd(x_1, F(x)) + Cd(x, F(x)).$$

thus

$$d(x, x_1) \leq (A + C)d(x, F(x)). \tag{3.0.2}$$

For $x_2 \in F(x_1)$ we have

$$\begin{aligned} d(x_2, x_1) &\leq d(x_1, F(x)) + H_d(F(x), F(x_1)) \\ &\leq Ad(x, x_1) + Cd(x, F(x)) \\ &\leq (A + C)d(x, x_1), \end{aligned}$$

then

$$d(x_2, x_1) \leq (A + C)^2 d(x, F(x)). \tag{3.0.3}$$

Continuing this procedure we can find a sequence $(x_n)_{n\in\mathbb{N}}$ of X such that

$$d(x_n, x_{n+p}) \leq (A + C)^{n+1} d(x, F(x)), n \in \mathbb{N}.$$

Hence , for all $n, p \in \mathbb{N}$, the following estimation holds

$$d(x_n, x_{n+p}) \leq (A + C)^{n+1}(I + (A + C) + (A + C)^2 + \ldots + (A + C)^{p-1} d(x, F(x))$$

Therefore

$$d(x_n, x_{n+p}) \to 0 \, as \, n \to \infty,$$

So $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete generalized metric space X . Then there exists $x_* \in X$ such that

$$d(x_n, x_*) \to 0\, as\, n \to \infty.$$

Form 3.0.1 we obtain

$$\begin{aligned} d(x_*, F(x_*)) &\leq d(x_*, x_n) + H_d(F(x_{n+1}), F(x_*)) \\ &\leq d(x_n, x_*) + Ad(x_{n+1}, x_*) + Bd(x_*, F(x_{n+1})) \\ &+Cd(x_{n+1}, F(x_{n+1})) \\ &\leq d(x_n, x_*) + Ad(x_{n+1}, x_*) + Bd(x_*, F(x_{n+1})) \\ &+Cd(x_{n+1}, F(x_{n+1})) \\ &\leq d(x_n, x_*) + Ad(x_{n+1}, x_*) + Bd(x_\times, x_n) \\ &+Cd(x_{n+1}, x_n) \to as \qquad n \to \infty. \end{aligned}$$

This implies that $x_* \in F(x_*)$. □

Lemma 3.0.5. *Let (X, d) be a generalized Banach space and $F : X \to \mathcal{P}_{cl}(Y)$ be a multivalued map . Assume that there exist $p \in \mathbb{N}$ and $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ converge to zero such that*

$$H_d(F^p(x), F^p(y)) \leq Md(x, y),\ foreach\, x, y \in X$$

and

$$\sup_{a\in F^{p+1}(y)} d(a, F(x)) \leq d(y, F(x)).$$

Then there exists $x \in X$,such that $x \in F(x)$.

Proof. By Theorem (3.0.4) , there exists $x \in X$such that $x \in F^p(x)$. Now we show that $x \in F(x)$.

$$\begin{aligned} d(x, F(x)) &\leq d(x, F^{p+1}(x)) + H_d(F^{p+1}(x), F(x)) \\ &\leq H_d(F^p(x), F^{p+1}(x)) \\ &\leq Md(x, F(x)) \end{aligned}$$

.

Hence

$$d(x, F(x)) \leq M^k d(x, F(x)) \to 0\, as\, k \to \infty \Rightarrow d(x, F(x)) = 0.$$

□

Theorem 3.0.6. *Let (X,d) be a complete generalized metric space and $B(x_0,r_0) = \{x \in X : d(x,x_0) < r_0\}$ be the open ball in X with radius r_0 and centred at some point $x_0 \in X$. Assume that $F : B(x_0,r_0) \to \mathcal{P}_{cl}(X)$ be a contractive multivalued map such that*

$$H_d(x_0, F(x_0)) < (I-M)r_0,$$

where $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is the matrix contractive for F .Then F has at least one fixed point .

Proof. Let $r_1 \in \mathbb{R}^n_+$ such that

$$d(x_0, F(x_0)) \leq (I-M)r_1 < (I-M)r_0.$$

set

$$K(x_0,r_1) = \{x \in X : d(x, x_0 \leq r_1\}.$$

It is clear that $K(x_0,r_1)$ is complete generalized metric space .Let us define a multi-valued map

$$F_*(x) = F(x) for all x \in K(x_0,r_1).$$

In view of Theorem (3.0.4) , for the Proof it is sufficient to show that

$$F_*(K(x_0,r_1)) \subseteq K(x_0,r_1).$$

Let $x \in K(x_0,r_1)$;then we have :

$$d(x_0,y) \leq \sup_{z\in F(x)} d(x_0,z) = H_d(x_0,F(x)), for all y \in F(x).$$

Thus

$$\begin{aligned} d(x_0,y) &\leq H_d(x_0,F(x_0)) + H_d(F(x_0),F(y)) \\ &\leq (I-M)r_1 + Md(x_0,y) \\ &\leq (I-M)r_1 + r_1 M = r_1 \end{aligned}$$

and the Proof is completed. □

Lemma 3.0.7. *Let E be a generalized Banach space , $Y \subseteq E$ nonempty convex compact subset of E and $F : X \to \mathcal{P}_{cl}(y)$ be a multivalued map such that*

$$H_d(F(x),F(y)) \leq d(x,y),\ for\, each\, x,y \in X.$$

Then there exists $x \in X$, such that $x \in F(x)$.

Proof. For every $m \in \mathbb{N}$,We have $\dfrac{I}{2^m} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ and

$$\frac{I}{2^{mk}} \to 0\, as\, k \to \infty.$$

Thus ,for some $x_0 \in Y$ the mapping $f_m : Y \to Y$ defined by

$$F_m(x) = (1 - \frac{I}{2^m})F(x) + \frac{1}{2^m}x_0 \in Y, \; for\, all\, x, y \in Y.$$

Then

$$H_d(F_m(x), F_m(y)) \leq \frac{I}{2^m}d(x, y), \; for\, all\, x, y \in Y.$$

From Theorem (3.0.4) there exists $x_m \in Y$ such that

$$x_m \in F_m(x_m), m \in \mathbb{N}.$$

Since Y is compact , then there exists subsequence of $(x_m)_{m\in\mathbb{N}}$ converge to $x \in Y$. Now we show that $x \in F(x)$.

$$\begin{aligned} d(x, F(x)) &= \begin{pmatrix} d_1(x, F(x)) \\ \dots \\ d_n(x, F(x)) \end{pmatrix} \\ &\leq d(x, x_m) + d(x_m, F(x_m)) + H_d(F(x_m), F(x)) \\ &\leq 2Id(x, x_m) + d(x_m, F(x_m)) \end{aligned}$$

and

$$\begin{aligned} d(x_m, F(x_m)) &= \begin{pmatrix} d_1(x_m, F(x_m)) \\ \dots \\ d_n(x_m, F(x_m)) \end{pmatrix} \\ &= \begin{pmatrix} \|x_m - F(x_m\|_1 \\ \dots \\ \|x_m, F(x_m\|_n \end{pmatrix} \\ &\leq d(x_m, F(x)) + H_{d(F(x), F(x_m)} \\ &\leq d(x_m, z_m) + d(x, x_m) \\ &\leq d(x_m, x) + \frac{1}{2^m}d(z_m, x_0) \end{aligned}$$

Where

$$z_m \in F(x)\, and\, x_m = (1 - \frac{1}{2^m})z_m + \frac{1}{2^m}x_0.$$

Since $x_m \to x$ as $m \to \infty$,then

$$z_m \to x\, as\, m \to \infty.$$

Hence

$$d(x, F(x)) \leq 3Id(x, x_m) + \frac{1}{2^m} d(z_m, x_0) \to 0 \, as \, m \to \infty.$$

□

3.1 Application

Application for theorem 3.0.4 Differential equations with impulses were considered for the first time by Milman and Myshkis [29] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [22]. the dynamics of many processes in physics, population dynamics, biology, medicine may be subject to abrupt changes such that shocks, perturbation (see for instance [1],[24] and the references therein). These perturbation may be seen as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions. Important contribution to the study of the mathematical aspects of such equations have been undertaken in [4],[7],[14],[21],[25],[37] among others. In this section we consider the following system of differential inclusions with impulse effects

$$x'(t) \in F_1(t, x(t), y(t)), \; y'(t) \in F_2(t, x(t), y(t)), \; a.e. \, t \in [0, 1] \tag{3.1.1}$$

$$x(\tau^+) - x(\tau^-) = I_1(x(\tau), y(\tau)), \; y(\tau^+) - y(\tau^-) = I_2(x(\tau), y(\tau)) \tag{3.1.2}$$

$$x(0) = x_0, \; y(0) = y_0, \tag{3.1.3}$$

where $0 < \tau < 1$, $i = 1, 2$, $J = [0, 1]$, $F_i : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are a multifunction, $I, \overline{I} \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}$, The notations $x(\tau^+) = \lim_{h \to 0^+}(x(\tau + h) \, and \, x(\tau) = \lim_{h \to 0^+}(x(\tau - h)$ stand for the right and the right and the left limits of the function y at $t = \tau$, respectively.

In order to define a solutions for problem (3.1.1)-(3.1.3),consider the space of piecewise continuous functions :

$$PC([0, 1], \mathbb{R}) = \{y : [0, 1] \to \mathbb{R}, y \in C(J \setminus \{\tau\}, \mathbb{R}); \, such \, that$$

and

$$y(\tau^-) \, and \, y(\tau^+) \, exist \, and \, satisfy \; y(\tau^-) = y(\tau)\}.$$

Endowed with the norm

$$\|y\|_{pc} = \sup\{|y(t)| : t \in J\},$$

PC is a Banach space.

In the proof the existence result for the problem we can easily proof the following auxiliary lemma.

Lemma 3.1.1. *Let $f_1, f_2 \in L^1(J,\mathbb{R})$ .Then y solution of the impulsive system*

$$x'(t) = f_1(t), y'(t) = f_2(t),\ a..\, t \in [0,1] \tag{3.1.4}$$

$$x(\tau^+) - x(\tau^-) = I_1(x(\tau),\ y(\tau)), y(\tau^+) - y(\tau^-) = I_2(x(\tau), y(\tau)) \tag{3.1.5}$$

$$x(0) = x_0,\ y(0) = y_O, \tag{3.1.6}$$

if and only if y is a solution of the impulsive integral equation

$$\left\{ \begin{array}{ll} x(t) & = x_0 + g_1(t) + I_1(x(\tau), y(\tau)) \quad a.e.\, t \in [0,1] \\ y(t) & = y_0 + g_2(t) + I_2(x(\tau), y(\tau)) \quad a.e.\, t \in [0,1] \end{array} \right\} \tag{3.1.7}$$

where $g_i(t) = \int_0^t f_i(s)ds,\ i = 1,2.$
In this section we assume the following condition:

($\mathcal{H}_1$) $F_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R}); t \to F_i(t,u,v)$ *are measurable for each* $u, v \in \mathbb{R},\ i = 1,2.$

($\mathcal{H}_2$) *There exist a functions* $l_i \in L^1(J,\mathbb{R}^+),\ i = 1,\dots,3$ *such that*

$$H_d(F_i(t,u,v), l_i(t,\overline{u},\overline{v})) \le l_i(t)|u-\overline{u}| + l_i(t)|v-\overline{v}|, t \in J\, for\, all\, u, \overline{u}, v, \overline{v} \in \mathbb{R}$$

and

$$H_d(0, F_i(t,0,0)) \le l_i(t)\, for a.e.\, t \in J, i = 1,2,$$

Theorem 3.1.2. *Assume that $(\mathcal{H}_1) - (\mathcal{H}_2)$ are satisfied and the matrix*

$$M = \begin{vmatrix} \|l_1\|_{L^1} + a_1 & \|l_2\|_{L^1} + a_2 \\ \|l_3\|_{L^1} + b_1 & \|l_4\|_{L^1} + b_2 \end{vmatrix}$$

converge to zero,then the problem has at least one solution

Proof. Consider the operator $N : PC \times PC \to PC \times PC$ defined by $N(x,y) = \left\{ (h_1,h_2) \in PC \times PC : \left\{ \begin{array}{ll} h_1(t) & = x_0 + \int_0^t f_1(s)ds + I_1(x(\tau),y(\tau)), t \in J \\ h_2(t) & = y_0 + \int_0^t f_2(s)ds + I_2(x(\tau),y(\tau)), t \in J \end{array} \right. \right\}$
where $f_i \in S_{F_i,x,y} = \{f \in L^1(J,\mathbb{R}) : f(t) \in F_i(t,x(t),y(t)),\ a.e.\, t \in J\}$ .Clearely,fixed points of the operator N are solution of problem (3.1.1)-(3.1.3).Let

$$N_i(x,y)=\left\{h\in PC: h(t)=x_i+\int_0^t f_i(s)ds+I_i(x(\tau),y(\tau)),\ t\in J\right\},$$

were $x_1=x0,\ x_2=y_0,\ f_i\in S_{F_i,x,y}=\{f\in L^1(J,\mathbb{R}): f(t)\in F_i(t,x(t),y(t)),\ a.e.\, t\in J\}$.we show N satisfies the assumption of theorem 3.0.4 .
Let $(x,y)=(\overline{x},\overline{y})\in PC\times PC$ and $(h_1,h_2)\in N(x,y)$. Then there exists $f_i\in S_{F_i,x,y},\ i=1,2$ such that

$$(h_1(t),h_2(t))=\left\{\begin{array}{ll} h_1(t) & =x_0+\int_0^t f_1(s)ds+I_1(x(\tau),y(\tau)),\ t\in J\\ \\ h_2(t) & =y_0+\int_0^t f_2(s)ds+I_2(x(\tau),y(\tau)),\ t\in J\end{array}\right\}$$

$(\mathcal{H}_2)$ implies that

$$H_{d1}(F_1(t,x(t),y(t)),F_1(t,\overline{x}(t),\overline{y}(t)))\leq l_1(t)|x(t)-\overline{x}(t)|+l_2(t)|y(t)-\overline{y}(t)|,\ t\in J$$

and

$$H_{d2}(F_2(t,x(t),y(t)),F_2(t,\overline{x}(t),\overline{y}(t)))\leq l_3(t)|x(t)-\overline{x}(t)|+l_4(t)|y(t)-\overline{y}(t)|,\ t\in J;$$

Hence , there is some $(w,\overline{w})\in F_1(t,\overline{x}(t),\overline{y}(t))\times F_1(t,\overline{x}(t),\overline{y}(t))$ such that

$$|f_1(t)-w|\leq l_1(t)|x(t)-\overline{x}(t)|+l_2(t)|y(t)-\overline{y}(t)|,\ t\in J,$$

and

$$|f_2(t)-w|\leq l_3(t)|x(t)-\overline{x}(t)|+l_4(t)|y(t)-\overline{y}(t)|,\ t\in J,$$

Consider the multi-valued maps $U_i: J\to\mathcal{P}(\mathbb{R}),\ i=1,2$ defined by

$$U_1=\{v\in F_1(t,x(t),y(t)): |f_i(t)-w|\leq l_1(t)|x(t)-\overline{x}(t)|+l_2(t)|y(t)-\overline{y}(t)|,\ a.e.\, t\in J\}$$

and

$$U_2=\{v\in F_2(t,x(t),y(t)): |f_2(t)-w|\leq l_3(t)|x(t)-\overline{x}(t)|+l_4(t)|y(t)-\overline{y}(t)|,\ a.e.\, t\in J\}$$

In[12] tells us that U_i are measurable .Moreover ,the multi-valued intersection operator $V_i(.)=U_i(.)\bigcap F_i(.,\overline{x}(.),\overline{y}(.))$ are measurable.

$$|f_1(t) - \overline{f}_1(t)| \leq l_1(t)|x(t) - \overline{x}(t)| + l_2(t)|y(t) - \overline{y}(t)|, a.e.t \in J.$$

and

$$|f_2(t) - \overline{f}_2(t)| \leq l_3(t)|x(t) - \overline{x}(t)| + l_4(t)|y(t) - \overline{y}(t)|, a.e.t \in J.$$

Define $\overline{h}_1, \overline{h}_2$ by

$$\overline{h}_1(t) = x_0 + \int_0^t \overline{f}_1(s)ds + I_1(x(\tau), y(\tau)), t \in J.$$

and

$$\overline{h}_2(t) = y_0 + \int_0^t \overline{f}_2(s)ds + I_2(x(\tau), y(\tau)), t \in J.$$

Then we have ,for $t \in J$,

$$|h_1(t) - \overline{h}_1(t)| \leq (\|l_1\|_{L^1} + a_1)\|x - \overline{x}\|_{PC} + (\|l_2\|_{L^1} + a_2)\|_{L^1}\|y - \overline{y}\|_{PC}.$$

Thus

$$\|h_1 - \overline{h}_1\|_{PC} \leq (\|l_1\|_{L^1} + a_1)\|x - \overline{x}\|_{PC} + (\|l_2\|_{L^1} + a_2)\|_{L^1}\|y - \overline{y}\|_{PC}.$$

By an analogous relation ,obtained by interchanging the roles of y and $\overline{y}$,we finally arrive at the estimate

$$H_{d1}(N_1(x, y), N_2(\overline{x}, \overline{y})) \leq (\|l_3\|_{L^1} + a_1)\|x - \overline{x}\|_{PC} + (\|l_2\|_{L^1} + a_2)\|y - \overline{y}\|_{PC}.$$

Similarly we have

$$H_{d2}(N_2(x, y), N_2(\overline{x}, \overline{y})) \leq (\|l_3\|_{L^1} + b_1)\|x - \overline{x}\|_{pc} + (\|l_2\|^{L^1} + b_2)\|_{L^1}\|y - \overline{y}\|_{pc}.$$

Therefore

$$H_d(N(x, y), N(\overline{x}, \overline{y})) \leq M \left\| \begin{matrix} \|x - \overline{x}\|_{PC} \\ \|y - \overline{y}\|_{PC} \end{matrix} \right\|, \; for\, all\, (x, y), (\overline{x}, \overline{y}) \in PC \times PC.$$

Hence ,by Theorem , the operator N has at least one fixed point which is solution of (3.1.1)-(3.1.3). □

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