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par

Mohamed alaa eddine Guennoun[1]

Sous la direction de

Pr.A.Kandouci

Titre de mémoire:

A study of stochastic differential equations driven by the Rosenblatt processes

Soutenu le 13/06/2022 devant le jury composé de :

M.Dr.KADI Mokhtar	Université de Saïda - Dr Moulay Tahar	Président
M.Pr.KANDOUCI Abdeldjebbar	Université de Saïda - Dr Moulay Tahar	Encadreur
Mlle. Dr.IDRISSI Soumia	Université de Saïda - Dr Moulay Tahar	Examinatrice

[1]e-mail:guennoun.m.aleddine@gmail.com

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DEDICATION

I dedicate my humble work to my parents, I know that i was not a righteous son, I know that I did not live up to your expectations, sorry for that .Next time if there is a next time, I promise that i will do my best.

Introduction

Self-similar stochastic processes, that are processes whose distributions are invariant under suitable scaling, can be used as mathematical models of various physical phenomena. These processes have been used for modeling in hydrology, biophysics, geophysics, telecommunication, turbulence, cognition, and finance. Typically, these self-similar processes exhibit long-range dependence, that is to say, their autocorrelations decay slower than exponentially.

The family of fractional Brownian motions is among the most studied self-similar stochastic processes. There are at least two reasons why fractional Brownian motions are of interest. First, these processes are self-similar, have stationary increments, and exhibit long-range dependence for $1/2 < H < 1$. These properties make them very attractive for practical modeling and applications. The second reason is the fact that they are Gaussian processes which make some mathematical models using fractional noise feasible for analysis.

However, In some practical applications the stochastic processes show Non-Gaussian properties, these processes have received considerable attention recently because of their importance in many diverse fields like structural engineering [12](the accurate representation of material, geometric properties, soil properties, wind, wave and earthquake loads) and in Finance [4][14] (Markets dynamics, Option pricing with Non-Gaussian returns, Portfolio Allocation, Risk management) and also since the non-Gaussian data with fractal features have been observed empirically. Domanski[9] has shown from the data of some physical systems that the Gaussian assumption is not always appropriate. In such cases, it does not seem reasonable to use a Gaussian process like fractional Brownian motion as a model for these physical phenomena and to use other type of processes like the Hermite processes as an example. The Hermite processes of order k is an H-sssi process with $0 < H < 1$, which is represented with the aid of a multiple stochastic integral called the

Wiener-Itô integral. For $k = 1$ the Hermite process is nothing else then the fractional Brownian motion with Hurst parameter $H \in (0, 1)$. For $k = 2$ the Hermite process is not Gaussian and it is known as the Rosenblatt process. In our case the Rosenblatt process can provide a useful alternative. A Rosenblatt process with the Hurst parameter $1/2 < H < 1$, denoted here as Z^H, can arise as a non-Gaussian limit of suitably normalized sums of long-range dependent random variables in a non-central limit theorem. Unlike the family of fractional Brownian motions, the family of Rosenblatt processes is not Gaussian.

The Rosneblatt process is not a semi-martingale. So, we cannot apply the classical stochastic calculus developed by Itô, since Itô calculus is used just for the semi-martingale cases. Different approaches have been proposed in order to build an integral with respect to it. The most important contributions which are:

1. **Pathwise calculus**: the stochastic integral is defined pathwise with Rieman-Stieltjes methods i.e. path by path integration. Since the Rosenblatt process has an Hölder continuous paths, this approach can be applied naturally on it. For more details consult Ciprian A. Tudor in 2006 [27].

2. **Malliavin calculus**: also known as the stochastic calculus of variation . This is the base of the modern approach to the Skorohod integral with respect to the Rosenblatt process. This calculus has been introduced also by Ciprian A. Tudor in 2006 [27] and in 2020 by Petr Coupek , Tyrone E. Duncana, Bozenna Pasik-Duncan [5].

3. **A white noise approach:** this approach defines a stochastic calculus with respect to the Rosenblatt process by means of white noise distribution theory. The Rosenblatt process is not diffrentiable but by using this approach we can define its derivative . This approach was introduced by Benjamin Arras in 2015 [1].

The study of stochastic differential equations driven by the Rosenblatt process with arbitrary Hurst parameter $H \in (\frac{1}{2}, 1)$ is very limited. There are few works in the literature done concerning SDE's with respect to the Rosenblatt process. In this work, we are interested with the dissipative systems driven by the Rosenbaltt process.

This thesis is divided into three chapters. In Chapter one, we cover some background and preliminaries about the Rosenblatt process providing its definitions, properties and representations. In chapter two, we cover the stochastic calculus with respect to Rosenblatt processes, we introduce the Wiener integration with respect to the Rosenblatt process, then we investigate the different stochastic integration approaches and construct the different Itô formula for each approach. Finally we investigate the existence of the local time and we give its representation. In chapter three, we introduce a dissipative system driven by the Rosenblatt process and we investigate the existence and the uniqueness of the solution, then we give an application which is a mathematical model in neuro-biology (A network model for a neuronal cell).

Chapter 1

Generalities on the Rosenblatt processes

1.1 Basic notions

Definition 1.1.1. *Let* $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ *be an integrable process, then* X *is a:*

i) ***Martingale*** *if and only if* $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ *a.s. for* $0 \leq s \leq t < \infty$.

ii) ***Supermartingale*** *if and only if* $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s$ *a.s. for* $0 \leq s \leq t < \infty$.

iii) ***Submartingale*** *if and only if* $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$ *a.s. for* $0 \leq s \leq t < \infty$.

Definition 1.1.2. $M = \{M_t, \mathcal{F}_t, t \geq 0\}$ *is a local martingale if and only if there exists a sequence of stopping times* T_n *tending to infinity, such that* M^{T_n} *are martingales for all* n*. The space of local martingales is denotes* $\mathcal{M}_{loc}$*, and the subspace of continuous local martingales is denotes* $\mathcal{M}^c_{loc}$.

Definition 1.1.3. $A = (A_t)_{t\geq 0}$ *is a finite variation process if it is an adapted continuous process whose trajectories is almost certainly have a finite variation with* $A_0 = 0$.

Definition 1.1.4. $X = (X_t, t \geq 0)$ *is a continuous Semi-martingale if*

$$X_t = X_0 + M_t + A_t, \tag{1.1}$$

with M *is local martingale and* A *is a finite variation process and* $M_0 = A_0$.

Remark 1.1.1. *If X is continuous Semi-martingale and $M_0 = A_0$, so the decomposition (1.1) is unique.*

Lemme 1.1. *(Kolmogorov's continuity criterion)[3]*

Consider a stochastic process $(X_t)_{t\in T}$ where $T \subset \mathbb{R}$ is a compact set. Suppose that there exist constants $p, C > 0$ and $\beta > 1$ such that for every $t, s \in T$:

$$\mathbb{E}|X_t - X_s|^p \leq C|t-s|^\beta, \tag{1.2}$$

then X has a continuous modification Y. Moreover for every $0 < \gamma < \frac{\beta-1}{p}$

$$\mathbb{E}\left(\sup_{s,t\in T; s\neq t} \frac{|X_t - Y_s|}{|t-s|^\gamma}\right)^p < \infty.$$

In particular X admits a modification which is Hölder continuous of any order $\alpha \in (0, \frac{\beta-1}{p})$.

1.2 Hermite processes

Definition 1.2.1. *The Hermite process $(X_H^k(t))_{t\in\mathbb{R}}$ of order $k \geq 1$, $k \in \mathbb{Z}$ with Hurst parameter $H \in (0,1)$ is defined by a multiple Wiener-Itô integral of order k with respect to the standard Brownian motion $(B(y))_{y\in\mathbb{R}}$*

$$X_H^k(t) = c(H,k)\int_{\mathbb{R}^k}\left(\int_0^t \prod_{i=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{K})} ds\right) dB(y_1)...dB(y_k), \tag{1.3}$$

with $y_+ = max(y, 0)$, and the constant $c(H,k)$ to make sure that $Var(X_1^H) = 1$.

Remark 1.2.1. *If $k = 2$, we get a Rosenblatt process (Defined below).*

Remark 1.2.2. *If $k > 1$, the process $X_H^k(t)$ is not Gaussian.*

1.2.1 Some properties of the Hermite processes

1. $X_H^k(t)$ is H self-similar, so that $\forall c > 0$

$$X_H^k(ct) \overset{d}{=} c^H X_H^k(t), \tag{1.4}$$

with stationary increments and finite moments.

Proof. Let $c > 0$, we have:

$$\begin{aligned}
X_H^k(ct) &= c(H,k)\int_{\mathbb{R}^k}\left(\int_0^{ct}(\prod_{i=1}^{k}(s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})})ds\right)dB(y_1)...dB(y_k)\\
&= c\times c(H,k)\int_{\mathbb{R}^k}\left(\int_0^{t}(\prod_{i=1}^{k}(cs-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})})ds\right)dB(y_1)...dB(y_k)\\
&= c\times c(H,k)\int_{\mathbb{R}^k}\left(\int_0^{t}(\prod_{i=1}^{k}(cs-cy_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})})ds\right)dB(cy_1)...dB(cy_k)\\
&= c\times c^{-k(\frac{1}{2}+\frac{1-H}{k})}c(H,k)\int_{\mathbb{R}^k}\left(\int_0^{t}(\prod_{i=1}^{k}(s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})})ds\right)dB(cy_1)...dB(cy_k)\\
&= c\times c^{-k(\frac{1}{2}+\frac{1-H}{k})}c^{\frac{k}{2}}c(H,k)\int_{\mathbb{R}^k}\left(\int_0^{t}(\prod_{i=1}^{k}(s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})})ds\right)dB(y_1)...dB(y_k)\\
&\stackrel{d}{=} c^H X_H^k(t).
\end{aligned}$$

□

2. The Hermite processes have a null expectation with moments

$$\mathbb{E}(|X_H^k(t)|)^p = t^{pH}\mathbb{E}(|X_H^k(1)|)^p \quad p > 1. \tag{1.5}$$

Proof. The Hermite processes, as we already motioned, is H self-similar process with stationary increments. By that we get:

$$\begin{aligned}
\mathbb{E}(|X_H^k(t)|)^p &= \mathbb{E}(|t^H X_H^k(1)|)^p\\
&= t^{pH}\mathbb{E}(|X_H^k(1)|)^p.
\end{aligned}$$

□

3. The covariance of the Hermite processes is

$$R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{1.6}$$

Proof. For all $t, s \in \mathbb{R}^+$ and $H \in (0,1)$, by using Itô theorem and funbini we have:

$$\begin{aligned}
R_H(t,s) &= \mathbb{E}(X_H^k(t)X_H^k(s)) \\
&= 2c(H,k)^2 \int_{\mathbb{R}^k} \left(\int_0^t \int_0^s \prod_{i=1}^k (u-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} (v-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} dudv \right) dy_1 ... dy_k \\
&= 2c(H,k)^2 \int_0^t \int_0^s \left(\int_{\mathbb{R}^k} \prod_{i=1}^k (u-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} (v-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} dy_1 ... dy_k \right) dudv \\
&= 2c(H,k)^2 \int_0^t \int_0^s \left[\int_{\mathbb{R}} (u-y)_+^{-(\frac{1}{2}+\frac{1-H}{k})} (v-y)_+^{-(\frac{1}{2}+\frac{1-H}{k})} dy \right]^k dudv,
\end{aligned}$$

let the beta function be

$$\beta(p,q) = \int_0^1 z^{p-1}(1-z)^{q-1}dz = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad (p,q>0), \tag{1.7}$$

by using the identity

$$\int_{\mathbb{R}} (u-y)_+^{a-1}(v-y)_+^{a-1}dy = \beta(a, 2a-1)|u-v|^{2a-1}, \tag{1.8}$$

we have

$$\begin{aligned}
R_H(t,s) &= 2c(H,k)^2 \beta\left(\tfrac{1}{2} - \tfrac{1-H}{k}, \tfrac{2H-2}{k}\right)^k \int_0^t \int_0^s (|u-v|^{\frac{2H-2}{k}})^k dvdu \\
&= 2c(H,k)^2 \frac{\beta(\frac{1}{2} - \frac{1-H}{k}, \frac{2H-2}{k})^k}{H(2H-1)} \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),
\end{aligned}$$

by choosing

$$c(H,k)^2 = \left(\frac{\beta(\frac{1}{2} - \frac{1-H}{k}, \frac{2H-2}{k})^k}{2H(2H-1)} \right)^{-1},$$

in order to have $\mathbb{E}(X_H^k(t))^2 = 1$. So by that we get:

$$R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

□

4. The Hermite process has Hölder continuous paths of order $0 < \delta < H$.

Proof. The previous property follows from Kolmogorov's continuity criterion and the fact that for any $p > 0$

$$\begin{aligned}\mathbb{E}(|X_H^k(t) - X_H^k(s)|^p) &= \mathbb{E}(|X_H^k(t-s)|)^p \\ &= \mathbb{E}(|(t-s)^H X_H^k(1)|)^p \\ &= \mathbb{E}(|X_H^k(1)|)^p |t-s|^{pH}.\end{aligned}$$

□

1.3 Rosenblatt process

Definition 1.3.1. *The Rosenblatt process is a Hermite process with $k = 2$ and Hurst index $H \in (\frac{1}{2}, 1)$, represented as follows:*

$$Z_H(t) = c(H,2) \int_{\mathbb{R}^2} \left(\int_0^t (s-y_1)_+^{\frac{H}{2}-1} (s-y_2)_+^{\frac{H}{2}-1} ds \right) dB(y_1) dB(y_2), \tag{1.9}$$

with $x_+ = max(0,x)$ and $(B(y), y \in \mathbb{R})$ is a standard Brownian motion on $\mathbb{R}$. The constant of normalizing $c(H,2)$ is chosen to ensure that $\mathbb{E}(Z_H(1)^2) = 1$, by

$$c(H,2) = \left(\frac{\beta(\frac{H}{2}, 1-H)^2}{2H(2H-1)} \right)^{-\frac{1}{2}}. \tag{1.10}$$

1.3.1 Representations of the Rosenblatt process

The Rosenblatt process has other representations which are the spectral representation and the finite time interval representation.

The Spectral representation of the Rosenblatt process

Definition 1.3.2. *The spectral representation of the Rosenblatt process is*

$$Z_H(t) = A_2(H) \int_{\mathbb{R}^2} \frac{\exp(it(y_1+y_2)) - 1}{i(y_1+y_2)} \frac{1}{|y_1^{H/2} y_2^{H/2}|} dB(y_1) dB(y_2), \tag{1.11}$$

where

$$A_2(H) = \left(\frac{H(2H-1)}{2[2\Gamma(1-H)\sin(H\pi/2)]^2} \right)^{1/2}, \quad (1.12)$$

to ensure that $\mathbb{E}(Z(1)^2) = 1$.

The finite time interval representation of the Rosenblatt process

Definition 1.3.3. *The corresponding representation of the Rosenblatt process is*

$$Z_H(t) \stackrel{d}{=} A_3(H) \int_{[0,t]^2} \left(\int_{x_1 \vee x_2}^{t} \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_2) du \right) dB(x_1) dB(x_2), \quad (1.13)$$

with

$$A_3(H) = \frac{1}{H+1} \left(\frac{2(2H-1)}{H} \right)^{\frac{1}{2}}, \quad (1.14)$$

and K is the self-similar kernel which is defined by: $\forall x \in [0,t]$

$$\begin{aligned} K^{H_0}(t,x) &= C_2(H_0) x^{\frac{1}{2}-H_0} \int_x^t (u-x)^{H_0-\frac{3}{2}} u^{H_0-\frac{1}{2}} du \quad x \in [0,t] \\ &= C_2(H_0) x^{\frac{1}{2}-H_0} \int_0^t (u-x)_+^{H_0-\frac{3}{2}} u^{H_0-\frac{1}{2}} du \quad x \in [0,t], \end{aligned}$$

(1.15)

where

$$C_2(H_0) = \left(\frac{\beta(2-2H_0, H_0-\frac{1}{2})}{H_0(2H_0-1)} \right)^{-\frac{1}{2}}, \quad (1.16)$$

and with

$$H_0 = \frac{H+1}{2}. \quad (1.17)$$

1.3.2 Some proprieties on the Rosenblatt process

In this subsection, we will prove some bacis properties on the rosenblatt process but before we start, we shall intoduce the kernel of the rosenblatt process and give its properties .

Some proprieties on the kernel of the Rosenblatt process

Proposition 1.3.1. *The kernel of the Rosenblatt process defined previously verifies the following properties:*

1. *K is a self-similar kernel such that for any $a > 0$:*

$$K^{H_0}(at, ax) = a^{H_0-\frac{1}{2}} K^{H_0}(t, x), \tag{1.18}$$

$$\frac{\partial K^{H_0}(au, ax)}{\partial u} = a^{H_0-\frac{1}{2}} \frac{\partial K^{H_0}(u, x)}{\partial u}. \tag{1.19}$$

Proof. Let $a > 0$, we get:

$$\begin{aligned}
K^{H_0}(at, ax) &= C_2(H_0)(ax)^{\frac{1}{2}-H_0} \int_{ax}^{at} (u-x)^{\frac{3}{2}} u^{H_0-\frac{1}{2}} du \\
&= a^{\frac{1}{2}-H_0} C_2(H_0)(x)^{\frac{1}{2}-H_0} \int_{ax}^{at} (u-x)^{H_0-\frac{3}{2}} u^{H_0-\frac{1}{2}} du \\
&= a^{\frac{1}{2}-H_0} C_2(H_0)(x)^{\frac{1}{2}-H_0} \int_{x}^{t} (au-ax)^{H_0-\frac{3}{2}} (au)^{H_0-\frac{1}{2}} dau \\
&= a^{\frac{1}{2}-H_0} a^{H_0-\frac{3}{2}} a^{H_0-\frac{1}{2}} a C_2(H_0)(x)^{\frac{1}{2}-H_0} \int_{x}^{t} (u-x)^{H_0-\frac{3}{2}} (u)^{H_0-\frac{1}{2}} du \\
&= a^{H_0-\frac{1}{2}} K^{H_0}(t, x) \\
&= a^{\frac{H}{2}-1} K^{H}(t, x).
\end{aligned}$$

□

2. *The kernel K verifies:*

$$\int_0^{u \wedge v} \frac{\partial K^{H_0}}{\partial u}(u, x) \frac{\partial K^{H_0}}{\partial v}(v, x) dx = H_0(2H_0 - 1)|u - v|^{2H_0-2}. \tag{1.20}$$

Proof. In this proof we will use

$$K^{H_0}(t, x) = C_2(H_0) x^{\frac{1}{2}-H_0} \int_x^t (u-x)^{H_0-\frac{3}{2}} u^{H_0-\frac{1}{2}} du,$$

and its partial derivative is

$$\frac{\partial K^{H_0}(u,x)}{\partial u} = C_2(H_0)\left(\frac{x}{t}\right)^{\frac{1}{2}-H_0}(u-x)_+^{H_0-\frac{3}{2}},$$

by that we get:

$$\begin{aligned}\int_0^{u\wedge v}\frac{\partial K^{H_0}}{\partial u}(u,x)\frac{\partial K^{H_0}}{\partial v}(v,x)dx &= C_2(H_0)^2\int_0^{u\wedge v}\left(\frac{x}{u}\right)^{\frac{1}{2}-H_0}(u-x)_+^{\frac{3}{2}-H_0}\left(\frac{x}{v}\right)^{\frac{1}{2}-H_0}(v-x)_+^{\frac{3}{2}-H_0}d\\ &= C_2(H_0)^2\beta(2-2H_0,H_0-\tfrac{1}{2})|u-v|^{2H_0-2}\\ &= H_0(2H_0-1)|u-v|^{2H_0-2}.\end{aligned}$$

□

3. *For any $u,v>0$ we have the next relation:*

$$|u-v|^{2H_0-2} = \frac{(uv)^{H_0-\frac{1}{2}}}{\beta(2-2H_0,H_0-\frac{1}{2})}\int_0^{u\wedge v}(u-x)_+^{H_0-\frac{3}{2}}(v-x)_+^{H_0-\frac{3}{2}}x^{1-2H_0}du. \quad (1.21)$$

Proof. This relation is a direct consequence of the previous property of the kernel K by making the change of variable $x=\frac{u-z^2}{1-z}$, $z\neq 1$ we get:

$$\begin{aligned}H_0(2H_0-1)|u-v|^{2H_0-2} &= C_2(H_0)^2\int_0^{u\wedge v}\left(\frac{x}{u}\right)^{\frac{1}{2}-H_0}(u-x)_+^{H_0-\frac{3}{2}}\left(\frac{x}{v}\right)^{\frac{1}{2}-H_0}(v-x)_+^{H_0-\frac{3}{2}}du\\ |u-v|^{2H_0-2} &= \tfrac{C_2(H_0)^2}{H_0(2H_0-1)}\int_0^{u\wedge v}\left(\frac{x}{u}\right)^{\frac{1}{2}-H_0}(u-x)_+^{H_0-\frac{3}{2}}\left(\frac{x}{v}\right)^{\frac{1}{2}-H_0}(v-x)_+^{H_0-\frac{3}{2}}du\\ &= \tfrac{1}{\beta(2-2H_0,H_0-\frac{1}{2})}\int_0^{u\wedge v}\left(\frac{x}{u}\right)^{\frac{1}{2}-H_0}(u-x)_+^{H_0-\frac{3}{2}}\left(\frac{x}{v}\right)^{\frac{1}{2}-H_0}(v-x)_+^{H_0-\frac{3}{2}}du\\ &= \tfrac{1}{\beta(2-2H_0,H_0-\frac{1}{2})}\int_0^{u\wedge v}\left(\frac{\frac{u-z^2}{1-z}}{u}\right)^{\frac{1}{2}-H_0}(u-x)_+^{H_0-\frac{3}{2}}\left(\frac{\frac{u-z^2}{1-z}}{v}\right)^{\frac{1}{2}-H_0}\\ &\quad (v-x)_+^{H_0-\frac{3}{2}}du\\ &= \tfrac{(uv)^{\frac{1}{2}-H_0}}{\beta(2-2H_0,H_0-\frac{1}{2})}\int_0^{u\wedge v}\left(\frac{u-z^2}{1-z}\right)^{\frac{1}{2}-H_0}(u-x)_+^{H_0-\frac{3}{2}}\left(\frac{u-z^2}{1-z}\right)^{\frac{1}{2}-H_0}\\ &\quad (v-x)_+^{H_0-\frac{3}{2}}du\\ &= \tfrac{(uv)^{\frac{1}{2}-H_0}}{\beta(2-2H_0,H_0-\frac{1}{2})}\int_0^{u\wedge v}(u-x)_+^{H_0-\frac{3}{2}}(v-x)_+^{H_0-\frac{3}{2}}(x)^{1-2H_0}du.\end{aligned}$$

□

Basic proprerties of the Rosenblatt process

Proposition 1.3.2. *Let Z_H be the Rosenblatt process of the Hurst parameter $H \in (\frac{1}{2}, 1)$. Then:*

1. *The Rosenblatt process $Z_H(t)$ is H-self-similar with $1/2 < H < 1$, so that $\forall C > 0$:*

$$Z_H(Ct) \stackrel{d}{=} C^H Z_H(t). \tag{1.22}$$

Proof. This proof depends on the self similarity of the kernel that we have already proved in property one of the proposition 1, by using (1.19):

$$\begin{aligned} Z_H(Ct) &= \int_{[0,Ct]^2} \left(\int_{x_1 \vee x_2}^{Ct} \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_2) du \right) dB(x_1) dB(x_2) \\ &= \int_{[0,Ct]^2} \left(\int_{\frac{x_1}{C} \vee \frac{x_2}{C}}^{t} \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(Cu, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(Cu, x_2) C du \right) dB(x_1) dB(x_2) \\ &= \int_{[0,t]^2} \left(\int_{x_1 \vee x_2}^{t} \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(Cu, Cx_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(Cu, Cx_2) C du \right) dB(Cx_1) dB(Cx_2), \end{aligned}$$

and let us not forget that we have $B(Cx) = C^{\frac{1}{2}} B(x)$. So we get finally that:

$$Z_H(Ct) = C^H Z_H(t).$$

□

2. *The Rosenblatt process has zero mean with covariance:*

$$R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \tag{1.23}$$

Proof. In this proof we need to use (1.21):

$$\begin{aligned} &= \mathbb{E}(Z_H(t) Z_H(s)) \\ &= 2A(H_0)^2 \int_0^{t \wedge s} \int_0^{t \wedge s} \left[\int_{x_1 \vee x_2}^{t} \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_2) du \right] \\ &\quad \left[\int_{x_1 \vee x_2}^{s} \frac{\partial K^{\frac{H+1}{2}}}{\partial v}(v, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial v}(v, x_2) du \right] dx_1 dx_2 \\ &= 2A(H_0)^2 \int_0^t \int_0^s \left[\int_0^{u \wedge v} \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x) \frac{\partial K^{\frac{H+1}{2}}}{\partial v}(v, x) dx \right]^2 du dv \\ &= 2A(H_0)^2 [H_0(2H_0 - 1)]^2 \int_0^t \int_0^s |u - v|^{2H-2} dv du \\ &= R_H(t, s). \end{aligned}$$

□

3. *The Rosenblatt process $Z_H(t)$ has a zero quadratic variation .*

Proof. By fixing $t > 0$, let $\Delta_n = \{0 = t_0^n < t_1^n < ... < t_N^n = t\}$ a sequence of subdivisions of [0,t] of lags tends to 0, which means $sup_i(t_i^n - t_{i-1}^n) \longrightarrow_{n\longrightarrow+\infty} 0$ so:

$$\begin{aligned}
&= \lim_{n\longrightarrow\infty}\sum_{i=1}^{N}(Z_H(t_i^n) - Z_H(t_{i-1}^n))^2 \\
&= \lim_{n\longrightarrow\infty}\sum_{i=1}^{N}((t_i^n)^H Z_H(1) - (t_{i-1}^n)^H Z_H(1))^2 \\
&= \lim_{n\longrightarrow\infty}\sum_{i=1}^{N}(Z_H(1))^2((t_i^n)^H - (t_{i-1}^n)^H)^2 \\
&= \lim_{n\longrightarrow\infty}(Z_H(1))^2\sum_{i=1}^{N}((t_i^n)^H - (t_{i-1}^n)^H)^2 \\
&= (Z_H(1))^2\lim_{n\longrightarrow\infty}\sum_{i=1}^{N}((t_i^n)^H - (t_{i-1}^n)^H)((t_i^n)^H - (t_{i-1}^n)^H) \\
&= (Z_H(1))^2\lim_{n\longrightarrow\infty}\sum_{i=1}^{N}((t_i^n)^H - (t_{i-1}^n)^H)((t_i^n)^H - (t_{i-1}^n)^H) \\
&\leq (Z_H(1))^2\lim_{n\longrightarrow\infty}\Delta_n \\
&\underset{n\longrightarrow+\infty}{\longrightarrow} 0.
\end{aligned}$$

□

4. *The Rosenblatt process $(Z_H(t), t > 0)$ has Hölder continuous paths of order γ, with $0 < \gamma < H$.*

Proof. This folows from the Kolmogorov's continuity criterion, and the fact that for any $\gamma > 0$, we have:

$$\mathbb{E}(|Z_H(t) - Z_H(s)|)^\gamma = \mathbb{E}(|Z_H(1)|^\gamma)|t - s|^{\gamma H}. \tag{1.24}$$

The previous result is a direct consequence of the stationary of increments and the self-similarity of the process, and we have already did this proof for the Hermite processes (property4). □

5. *The Rosenblatt process is not differentiated in mean square for all $t \in [0, \infty[$ and $H \in (\frac{1}{2}, 1)$ since:*

$$\lim_{t\longrightarrow 0}\mathbb{E}\left(\frac{Z_H(t)}{t}\right)^2=\infty. \tag{1.25}$$

Proof. For $t\in\mathbb{R}^+$, we have

$$\begin{aligned}
\lim_{t\longrightarrow 0}\mathbb{E}\left(\frac{Z_H(t)-Z_H(0)}{t-0}\right)^2 &= \lim_{t\longrightarrow 0}\mathbb{E}\left(\frac{Z_H(t)}{t}\right)^2\\
&= \lim_{t\longrightarrow 0}\mathbb{E}\left((t)^{-2}Z_H(t)^2\right)\\
&= \lim_{t\longrightarrow 0}\mathbb{E}\left((t)^{2H-2}Z_H(1)^2\right)\\
&= \lim_{t\longrightarrow 0}(t)^{2H-2}\mathbb{E}\left(Z_H(1)^2\right)\\
&= \lim_{t\longrightarrow 0}(t)^{2H-2}\\
&= \lim_{t\longrightarrow 0}(t)^{2H-2}\\
&= \lim_{t\longrightarrow 0}\frac{1}{(t)^{2-2H}}\\
&= \infty.
\end{aligned}$$

□

Long and Short-Range Dependence.

The notion of long range dependence has, clearly, something to do with memory in a stochastic process. Memory is, by definition, somemthing that lasts. The Modern real world can't be modeliezid by a markovienne model it requires process with long range of memory , processes with long-range dependence have many applications, such as in telecommunication, specially in Internet traffic problems. Basically, the notion of long-range dependence is that the variance of the sum of stationary sequence grows non-linearly with respect to n.

Definition 1.3.4. *A stationary sequence $(X_n)_{n\in N}$ exhibits a long-range dependence if $\rho(n)=cov(X_n,X_{k+n})$ satisfies:*

$$\lim_{n\longrightarrow\infty}\frac{\rho(n)}{cn^{-\alpha}}=1, \tag{1.26}$$

for $\alpha\in(0,1)$ and some constant c.

Remark 1.3.1. *If a stationary sequence $(X_n)_{n\in\mathbb{N}}$ is long-range dependent, then the dependence between X_k and X_{k+n} decays slowly as n tends to infinity and $\sum_{n=1}^{\infty}\rho(n)=\infty$.*

Proposition 1.3.3. *The Rosenblatt process has stationary increments such that $(Z_H(t+s)-Z_H(s), t\geq 0)$ does not depend on $s\geq 0$, and with long-range dependence such that $\forall n\in\mathbb{N}$:*

$$\sum_{n=0}^{\infty}\mathbb{E}(X_{k+n}X_k)=\infty, \tag{1.27}$$

with

$$X_{k+n}=Z_{k+n}-Z_{k+n-1} \quad \textit{and} \quad X_k=Z_k-Z_{k-1}, \tag{1.28}$$

and the covariance

$$\mathbb{E}(X_{k+n}X_k)=\frac{1}{2}\{|n+1|^2-2|n|^{2H}+|n-1|^{2H}\}\sim H(2H-1)n^{2H-2} \quad n\longrightarrow\infty. \tag{1.29}$$

Proof. In this proof we have two parts:

Part.1 The Rosenblatt process has stationary increments such that $\forall h>0$, we have

$$\begin{aligned}
\mathbb{E}((Z^H_{t+h}-Z^H_h)(Z^H_{s+h}-Z^H_h)) &= \mathbb{E}(Z^H_{t+h}Z^H_{s+h})-\mathbb{E}(Z^H_{t+h}Z^H_h)-\mathbb{E}(Z^H_{s+h}Z^H_h)+\mathbb{E}(Z^H_h)^2\\
&= \tfrac{1}{2}[((t+h)^{2H}+(s+h)^{2H}-|t-s|^{2H})\\
&\quad -((t-h)^{2H}+h^{2H}-t^{2H})-((s+h)^{2H}+h^{2H}-s^{2H})+2h^{2H}]\\
&= \tfrac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})\\
&= \mathbb{E}(Z^H_tZ^H_s).
\end{aligned}$$

Therefore the Rosenblatt has a stationary increments.

Part.2 Before we prove the long-range dependency property we will provide a proof of the formula (1.29):

$$\begin{aligned}
\mathbb{E}(X_{k+n}X_k) &= \mathbb{E}((Z_{k+n}-Z_{k+n-1})(Z_k-Z_{k-1}))\\
&= \mathbb{E}(Z_1(Z_{n+1}-Z_n))\\
&= R_H(1,n+1)-R_H(1,n)\\
&= \tfrac{1}{2}\{|n+1|^{2H}-2|n|^{2H}+|n-1|^{2H}\},
\end{aligned}$$

for $n\neq 0$:

$$\begin{aligned}
&= \tfrac{1}{2}n^{2H-2}(\tfrac{n^2}{n^{2H}}(n+1)^{2H}+\tfrac{1}{n^{2H}}(n-1)^{2H}-2)\\
&= \tfrac{1}{2}n^{2H-2}(n^2(\tfrac{n+1}{n})^{2H}+(\tfrac{n-1}{n})^{2H}-2)\\
&= \tfrac{1}{2}n^{2H-2}\underbrace{(n^2(1+\frac{1}{n})^{2H}+(1-\frac{1}{n})^{2H}-2)}_{\longrightarrow 2H(2H-1), as \quad n\longrightarrow\infty}\\
&= \tfrac{1}{2}n^{2H-2}2H(2H-1),
\end{aligned}$$

by using the previous result, it follows that

$$\rho(n) > 0 \quad and \quad \sum_{n}^{\infty} |\rho_H(n)| = \infty.$$

Therefore, long-range dependency property is verified by the Rosenblatt process since $H > \frac{1}{2}$. □

Lack of Semi-martingale Property

The Rosenblatt process is not a semi-martingale but in this subsection, we will show that it can be approximated by a sequence of semi-martingales (since $H > \frac{1}{2}$, by a sequence of bounded variation processes).

Proposition 1.3.4. *The sequence of semi-martingales $Z_H^{\varepsilon}(t)$ is defined by replacing $K^{\frac{H+1}{2}}(u, x)$ in (1.13) by $K^{\frac{H+1}{2}}(u + \varepsilon, x)$ to get:*

$$Z_H^{\varepsilon}(t) =^d A_3(H) \int_{[0,t]^2} \left(\int_{x_1 \vee x_2}^{t} \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u + \varepsilon, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u + \varepsilon, x_2) du \right) dB(x_1) dB(x_2), \tag{1.30}$$

by that, the Rosenblatt process can be approximated by that sequence of semi-martingales

.

Proof. In this proof we have two parts:

Part.1. In this part, we prove that $Z_H^{\varepsilon}(t)$ is a semi-martingale $\forall \varepsilon > 0$. The basic observation is that, if one interchanges formally the stochastic and Lebesque integrals in (1.13), one gets

$$Z(t)' =' \int_0^t \left(\int_0^u \int_0^u \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2)\, dB(y_1)\, dB(y_2) \right) du,$$

but the above expression cannot hold because the kernel $\frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2)$ does not belong to $L^2([0,T]^2)$ since the partial derivative $\frac{\partial K^{H'}}{\partial u}(u, y_1)$ behaves on the diagonal as $(u- \; y_1)^{\frac{H-2}{2}}$. Let us define, for every $\varepsilon > 0$,

$$\begin{aligned} Z_H^{\varepsilon}(t) &=^d A_3(H) \int_{[0,t]^2} \left(\int_{x_1 \vee x_2}^{t} \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u + \varepsilon, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u + \varepsilon, x_2) du \right) dB(x_1) dB(x_2) \\ &= A_3(H) \int_{x_1 \vee x_2}^{t} \left(\int_{[0,u]^2} \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u + \varepsilon, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u + \varepsilon, x_2) dB(x_1) dB(x_2) \right) du \\ &= \int_0^t H_{\varepsilon}(u) du, \end{aligned}$$

we have that H_ε is $L^2[0,T]$ adapted ($H_\varepsilon \in L^2[0,T]\times\Omega$), which makes $Z_H^\varepsilon(t)$ a semi-martingale.

Part.2. In this part we prove that $\forall t \in [0,T], \quad Z_H(t) \longrightarrow Z_H^\varepsilon(t)$ in $L^2(\Omega)$:

$$\begin{aligned} Z_H^\varepsilon(t) - Z_H(t) &= \int_0^t \int_0^t \left(\int_{x_1 \vee x_2}^t \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u+\varepsilon, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u+\varepsilon, x_2) \right. \\ &\quad - \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_2) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_1) du dB(x_1) dB(x_2), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}(Z_H^\varepsilon(t) - Z_H(t))^2 &= \int_0^t \int_0^t \int_{x_1 \vee x_2}^t \int_{x_1 \vee x_2}^t \\ &\left(\frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u+\varepsilon, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u+\varepsilon, x_2) - \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_2) \right) dv du \\ &\left(\frac{\partial K^{\frac{H+1}{2}}}{\partial v}(v+\varepsilon, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial v}(v+\varepsilon, x_2) - \frac{\partial K^{\frac{H+1}{2}}}{\partial v}(v, x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial v}(v, x_2) \right) dx_1 dx_2, \end{aligned}$$

we can now see that the previous quantity converge to zero when ε converge to 0, which makes $\mathbb{E}(Z_H^\varepsilon(t) - Z_H(t))^2$ tend to 0, so by that we are sure that in $L^2[0,T]$:

$$Z_H^\varepsilon(t) \underset{\varepsilon \longrightarrow 0}{\longrightarrow} Z_H(t).$$

□

1.3.3 On the Rosenblatt distribution

The Rosenblatt distribution is the simplest non-Gaussian distribution which arises in a non-central limit theorem involving long range dependent random variables .

Definition 1.3.5. *The Rosenblatt distribution is the law of the Rosenblatt process at time 1 with the characteristic function:*

$$\phi(\theta) = \exp\left(\frac{1}{2} \sum_{k=2}^{\infty} \frac{(2i\theta\sigma(D))^k}{k} c_k \right), \tag{1.31}$$

where

$$c_k = \int_0^1 ... \int_0^1 |x_1 - x_2|^{-D} |x_2 - x_3|^{-D} ... |x_k - x_{k-1}|^{-D} |x_1 - x_k|^{-D} dx_1 ... dx_k, \tag{1.32}$$

and by Cauchy-Schwartz,

$$c_k \leq \left(\int_0^1 \int_0^1 |x_1 - x_2|^{-2D} dx_1 dx_2\right)^{\frac{k}{2}} = \left(\frac{1}{(1-2D)(1-D)}\right)^{\frac{k}{2}} = \left(\frac{1}{2\sigma(D)}\right)^{\frac{k}{2}}, \qquad (1.33)$$

which ensures that the series (1.31) converges around the origin.

Here, $\sigma(D)$ is a normalizing constant so that $\forall D \in]0, \frac{1}{2}[$ is given by:

$$\sigma(D) = \left[\frac{1}{2}(1-2D)(1-D)\right]^{\frac{1}{2}}, \qquad (1.34)$$

and

$$H = 1 - D. \qquad (1.35)$$

Proof. Before we start the proof, we shall begin by motivating the Rosenblatt distribution using Rosenblatt's famous counter-example.

Counter-example

Consider a stationary Gaussian sequence $X_i, i = 1, 2, ...$ which has a covariance structure of the form $\mathbb{E}(X_0 X_K) \sim k^{-D}$ as $k \longrightarrow \infty$ with $0 < D < 1/2$. Using the transformation

$$Y_i = X_i^2 - 1,$$

one can define a sequence of normalized sums

$$Z_n = \frac{\sigma(D)}{n^{1-D}} \sum_{i=1}^{N} Y_i. \qquad (1.36)$$

The sequence Z_n tends in distribution to a non-Gaussian limit $Z(1)$ as $n \longrightarrow \infty$ with mean 0 and variance 1 [28]. This limiting distribution has been named the Rosenblatt distribution . Now we need to define the characteristic function of Z_n.

Let $d_n = n^{1-D}$ and let R_n denote the covariance matrix of the Gaussian vector $(X_1, ..., X_n)$. Each component has mean 0 and unit variance. Let $x' = (x_1, ..., x_n)$ denote the row vector and $|.|$ a determinant. Then the characteristic function of Z_n is

$$\begin{aligned}
\mathbb{E}(\exp(i\theta Z_n)) &= \int_{\mathbb{R}^n} \exp\{i\theta\sigma d_n^{-1}\sum_{j=1}^{n}(x_j^2-1)\}\frac{1}{\sqrt{2\pi|R_n|}}e^{-\frac{1}{2}x'R_n^{-1}}d^n x \\
&= \exp\{-i\theta\sigma d_n^{-1}n\}\frac{1}{\sqrt{2\pi|R_n|}}\int_{\mathbb{R}^n}\exp\{-\frac{1}{2}x'[R_n^{-1}-2i\theta\sigma d_n^{-1}I]x\}d^n x \\
&= \exp\{-i\theta d_n^{-1}n\}|R_n|^{-\frac{1}{2}}|R_n^{-1}-2i\theta\sigma d_n^{-1}I|^{-\frac{1}{2}} \\
&= \exp\{-i\theta\sigma d_n^{-1}n\}|I-2i\theta\sigma d_n^{-1}R_n|^{-\frac{1}{2}} \\
&= \exp\{\tfrac{1}{2}\sum_{j=1}^{n}[-2i\theta\sigma d_n^{-1}-ln(1-2i\theta\sigma d_n^{-1}\lambda_{j,n})]\} \\
&= \exp\{\tfrac{1}{2}[-2i\theta\sigma d_n^{-1}n-\sum_{j=1}^{n}ln(1-2i\theta\sigma d_n^{-1}\lambda_{j,n})]\},
\end{aligned}$$

where the $\lambda_{j,n}, j=1,...,n$ denote the eigenvalues of R_n. let $Tr(a)$ denote the trace of a. Expanding the logarithm, we get

$$ln(1-2i\theta\sigma d_n^{-1}\lambda_{j.n}) = -\sum_{k=1}^{\infty}\frac{(2i\theta\sigma d_n^{-1}\lambda_{j,n})^k}{k},$$

and

$$-\sum_{j=1}^{n}ln(1-2i\theta\sigma d_n^{-1}\lambda_{j.n}) = 2i\theta\sigma d_n^{-1}\sum_{j=1}^{n}\lambda_{j.n}+\sum_{j=1}^{n}\sum_{k=2}^{\infty}\frac{(2i\theta\sigma d_n^{-1}\lambda_{j.n})^k}{k}.$$

Since R_n has 1 in its diagonals, $\sum_{j=1}^{n}\lambda_{j.n}=n$ and thus

$$\mathbb{E}(e^{i\theta Z_n}) = \exp\left\{\frac{1}{2}\sum_{k=2}^{\infty}\frac{(2i\theta\sigma)^k}{k}d_n^{-k}\sum_{j=1}^{n}\lambda_{n.j}^k\right\},$$

where

$$d_n^{-k}\sum_{j=1}^{n}\lambda_{n.j}^k = d_n^{-k}Tr(R_n^k),$$

$$= d_n^{-k}\sum_{i_1,i_2....,i_k=1}^{n} r(|i_1-i_2|)r(|i_2-i_3|)....r(|i_{k-1}-i_k|)r(|i_k-i_1|) \longrightarrow c_k.$$

As $n\longrightarrow\infty$, the previous expansion converges absolutely for $|\theta|<\varepsilon$, ε small enough. By this we have been proved the formula (1.32) and (1.31). □

Remark 1.3.2. *The distribution $Z(1)$ can be given in terms of a weighted sum of chi-squared distributions, for $k\geq 2$*

$$Z(1) \stackrel{d}{=} \sum_{i=1}^{\infty}\lambda_n(\varepsilon_n^2-1), \tag{1.37}$$

where $\varepsilon_n \quad i.i.d \sim N(0,1)$ *and*

$$\sum_{i=1}^{\infty} \lambda_n^k = \sigma^k(D)c_k,$$

because they have the same characteristic function, and since it converge in L^2,

$$\begin{aligned} \sum_{i=1}^{\infty} var[\lambda_n(\varepsilon_n^2 - 1)] &= \mathbb{E}[(\varepsilon_1^2 - 1)^2] \sum_{i=1}^{\infty} \lambda_n^2 \\ &= 2\sum_{i=1}^{\infty} \lambda_n^2 < \infty, \end{aligned}$$

in fact $\sum_{i=1}^{\infty} \lambda_n^2 = \frac{1}{2}$ *by (1.32) and (1.34). The weights* λ_n *are given as the eigenvalues of an integral operator which we will discuss in more detail in definition 1.3.7 .*

The behavior of the Rosenblatt distribution with respect to D

The characteristic function of $Z(1)$ is distinguished by a parameter D such that $D \in]0, \frac{1}{2}[$. So let's consider the extremes when $D \longrightarrow 0^+$ and $D \longrightarrow \frac{1}{2}^-$.

1. When $D \longrightarrow 0^+$, $c_k \longrightarrow 1$ for all k, $\sigma(D) \longrightarrow \frac{1}{\sqrt{2}}$ and thus for θ small enough, the characteristic function approaches to the characteristic function of $\frac{1}{\sqrt{2}}(\varepsilon^2 - 1)$ where ε is $N(0,1)$.

   Proof. Let us denote the characteristic function when $D \longrightarrow 0^+$ by:

$$\begin{aligned} \phi(\theta) &= \exp\left(\frac{1}{2}\sum_{k=2}^{\infty}\frac{(\sqrt{2}i\theta)^k}{k}\right) \\ &= \exp\left(\frac{1}{2}(log(1-\sqrt{2}i\theta) - \sqrt{2}i\theta)\right) \\ &= \left(\frac{1}{1-\sqrt{2}i\theta}\right)^{\frac{1}{2}} \exp\left(\frac{-i\theta}{\sqrt{2}}\right), \end{aligned}$$

   which is the the characteristic function of $\frac{1}{\sqrt{2}}(\varepsilon^2 - 1)$.

   Hence when $D = 0$, the Rosenblatt distribution is simply a chi-squared distribution standardized to have mean 0 and variance 1. □

2. When $D \longrightarrow \frac{1}{2}^-$, the limit is $N(0,1)$. This is expected since the scaling term in (1.36) approaches $\sqrt{n}$.

The cumulants of the Rosenblatt distribution

The cumulants κ_k of the Rosenblatt distribution are given by $\kappa_1 = 0$ and

$$\kappa_k = 2^{k-1}(k-1)!(\sigma(D))^k c_k, \tag{1.38}$$

where the c_k are given by the multiple integrals (1.32). In order to compute any cumulant, it is necessary to compute the multiple integrals c_k.

The first two can be computed directly:

$$\begin{aligned} c_2 &= \int_0^1 \int_0^1 |x_1 - x_2|^{-2D} dx_1 dx_2 \\ &= 2\int_0^1 \int_0^{x_2} (x_1 - x_2)^{-2D} dx_1 dx_2 \\ &= \frac{1}{(1-2D)(1-D)}, \end{aligned}$$

$$\begin{aligned} c_3 &= \int_0^1 \int_0^1 \int_0^1 |x_1 - x_2|^{-D} |x_2 - x_3|^{-D} |x_3 - x_1|^{-D} dx_1 dx_2 dx_3 \\ &= 3\int_0^1 x_3^{-3D} \int_0^{x_3} \int_0^{x_3} |\frac{x_1}{x_3} - \frac{x_2}{x_3}|^{-D} \left(1 - \frac{x_1}{x_3}\right)^{-D} \left(1 - \frac{x_2}{x_3}\right)^{-D} dx_1 dx_2 dx_3 \\ &= 3\left(\int_0^1 x_3^{-3D+2} dx_3\right)\left(\int_0^1 \int_0^1 |u_1 - u_2|^{-D}(1-u_2)^{-D}(1-u_1)^{-D} du_1 du_2\right) \\ &= \tfrac{2}{1-D}\int_0^1 \omega_2^{-D} \int_0^{\omega_2} (\omega_2 - \omega_1)^{-D} \omega_1^{-D} d\omega_1 d\omega_2 \\ &= \tfrac{2}{1-D}\left(\int_0^1 \omega_2^{-3D+1} d\omega_2\right)\left(\int_0^1 \nu^{-D}(1-\nu)^{-D} d\nu\right) \\ &= \tfrac{2}{(1-D)(2-3D)}\beta(1-D, 1-D), \end{aligned}$$

where β is the beta function that we motioned previously (1.7), and by using the following variable changes $u_1 = x_1/x_3, u_2 = x_2/x_3, w_1 = 1-u_1, w_2 = 1-u_2$ and $v = w_1/w_2$.

c_k could not be found for $k \geq 4$ which means we must be computed numerically; so for this reason, we will use the next definition as a sophisticated method for computing c_k.

Definition 1.3.6. *Let $L^2[0,1]$ denote the Hilbert space of all real-valued measurable functions $f(x), 0 < x < 1$, such that $\|f\|_2 = (\int_0^1 f(x)^2 dx)^{\frac{1}{2}} < \infty$ with the inner product $\langle f, g\rangle \equiv \int_0^1 f(x)g(x)dx$. For $0 < D < \frac{1}{2}$, we define the integral operator $\mathcal{K}_D : L^2[0,1] \longrightarrow L^2[0,1]$ as*

$$\langle \mathcal{K}_D, f\rangle(x) = \int_0^1 |x-u|^{-D} f(u) du.$$

Finally, we define the sequence of functions $G_{k,D} \in L^2[0,1]$, $k \geq 1$, recursively as follows:

$$G_{1,D}(x) = \frac{(1-x)^{-D}}{\sqrt{1-D}}; \quad G_{k,D} = \langle \mathcal{K}_D, G_{k-1,D}\rangle(x), \quad k \geq 2.$$

As we already mentioned, by using this definition we can for sure calculate c_k for $k \geq 4$, such that $\forall \mu, \nu$ be any two positive integers such that $\mu + \nu = k$. Then

$$c_k = \langle G_{\mu,D}, G_{\nu,D}\rangle. \tag{1.39}$$

Proof. Let ν, μ be a positive integer and by using the circular symmetry of the integrand in c_k, if we take x_k as the largest of the $x_i, i = 1, 2, ...,$ and then factor an x_k out of all the terms, we can rewrite c_k as

$$c_k = k\int_0^1 x_k^{-kD} \int_{(0,x_k)^{k-1}} \left(1 - \frac{x_1}{x_k}\right)^{-D} |\frac{x_1}{x_k} - \frac{x_2}{x_k}|^{-D} ... \times |\frac{x_{k-2}}{x_k} - \frac{x_{k-1}}{x_k}|^{-D} \left(1 - \frac{x_{k-1}}{x_k}\right)^{-D} dx_1..dx_k,$$

with the change of variables $u_i = x_i/x_k, i = 1, 2, ..., k-1$, one of the k integrals can be separated out, and we obtain

$$\begin{aligned}
c_k &= k\left(\int_0^1 x_k^{-kD+(k-1)} dx_k\right)\left(\int_{(0,1)^{k-1}} (1-u_1)^{-D}|u_1 - u_2|^{-D}.|u_{k-1} - u_{k-2}|^{-D}(1-u_{k-1})^{-D}\right. \\
&\qquad \left. du_1...du_{k-1}\right) \\
&= \tfrac{1}{1-D}\left(\int_{(0,1)^{k-1}} (1-u_1)^{-D}|u_1-u_2|^{-D}..|u_{k-1}-u_{k-2}|^{-D}(1-u_{k-1})^{-D} du_1..du_{k-1}\right) \\
&= \int_{(0,1)^{k-1}} G_{1,D}(u_1)G_{1,D}(u_{k-1}) \underbrace{|u_1-u_2|^{-D}..|u_{k-1}-u_{k-2}|^{-D}}_{k-2 \quad terms} du_1..du_{k-1} \\
&= \int_{(0,1)^{k-3}} \left[\int_0^1 G_{1,D}(u_1)|u_1-u_2|^{-D} du_1\right] [|u_3-u_2|^{-D}...|u_{k-2}-u_{k-3}|^{-D}] \\
&\quad \times \left[\int_0^1 G_{1,D}(u_{k-1})|u_{k-1}-u_{k-2}|^{-D} du_{k-1}\right] du_3..du_{k-2} \\
&= \int_{(0,1)^{k-3}} G_{2,D}(u_2) \underbrace{|u_3-u_2|^{-D}...|u_{k-2}-u_{k-3}|^{-D}}_{k-4 \quad terms} G_{2,D}(u_{k-2}) du_2..du_{k-2} \\
&. \\
&. \\
&= \int_0^1 G_{\mu,D}(u_\mu)G_{\nu,D}(u_{k-\nu}) du_\mu \\
&= \langle G_{\mu,D}, G_{\nu,D}\rangle.
\end{aligned}$$

By that we finish our proof. □

Remark 1.3.3. *To minimize the number of integrals one needs to compute, it makes sense to choose $\mu = \nu = \frac{k}{2}$ if k is even, and $\mu = \frac{k+1}{2}$ and $\nu = \frac{k-1}{2}$ if k is odd.*

Thorin class

We next define the Thorin class of probability distributions on $\mathbb{R}$. Originally this class was studied by Thorin (1977a,b) and the Thorin class on $\mathbb{R}_+$, denoted by $T(\mathbb{R}_+)$, is the smallest class of distributions on $\mathbb{R}_+$ that contains all gamma distributions and is closed under convolution and weak convergence. A probability distribution in $T(\mathbb{R}_+)$ is called generalized gamma convolution .

The main goal of this part is to derive new results related to the Rosenblatt distribution. For example, we show that a random variable that follows the Rosenblatt distribution can be represented in law as a Wiener integral with respect to some Levy process. We also obtain new properties of the density of the Rosenblatt distribution.

Definition 1.3.7. *For every $t \geq 0$, A_t define an integral operator by:*

$$A_t h(x) = C(D) \int_{-\infty}^{+\infty} \frac{e^{it(y_1-y_2)} - 1}{i(y_1 - y_2)} |y|^{D-1} h(y) dy \quad , h \in \mathcal{H}_D, \tag{1.40}$$

where

$$\mathcal{H}_D = \{h : h \quad is \quad a \quad complex \quad valued \quad function \quad on \quad \mathbb{R}, h(x) = \overline{h(x)} = \int_R h(x)^2 |x|^{D-1} dx < \infty\},$$

which is a self-adjoint Hilbert Schmidt operator and all eigenvalues $\lambda_n(t), n = 1, 2, ...,$ are real and satisfy $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$.

Theorem 1.3.1. *For every $t_1, ..., t_d \geq 0$,*

$$(Z_D(t_1), ..., Z_D(t_d)) \stackrel{d}{=} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1), .., \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1) \right),$$

where $\{\varepsilon_n\}$ are i.i.d $N(0,1)$ random variables.

Proof. Before we start the proof, we need to give other representation in density for the Rosenblatt process.

Let consider the Rosenblatt process defined as follows:

$$Z_D(t) = c(D, 2) \int_{\mathbb{R}^2} \int_0^t (s - y_1)_+^{-(\frac{D+1}{2})} (s - y_2)_+^{-(\frac{D+1}{2})} ds) dB(y_1) dB(y_2).$$

Let

$$f_t(y_1, y_2) = c(D, 2) \int_0^t (s - y_1)_+^{-(\frac{D+1}{2})} (s - y_2)_+^{(\frac{D+1}{2})} ds.$$

Then
$$Z_D(t) = \int_{\mathbb{R}^2} f_t(s_1, s_2) dB(y_1) dB(y_2).$$
By using definition (1.3.2), we can give this following representation:

$$Z_D(t) =^d \int_{\mathbb{R}^2} \frac{e^{it(y_1+y_2)} - 1}{i(y_1 + y_2)} |y_1|^{\frac{D-1}{2}} |y_2|^{\frac{D-1}{2}} dB(y_1) dB(y_2), \tag{1.41}$$
where $\int_{\mathbb{R}^2}$ is the integral over $\mathbb{R}^2$ except the hyper planes $y_1 \neq \pm y_2$.

Let $\alpha_1, ..., \alpha_1 \in \mathbb{R}$. It is sufficient to show that

$$\alpha_1 Z_D(t_1) + .. + \alpha_d Z_D(t_d) \stackrel{d}{=} \alpha_1 \sum_{n=1}^{\infty} \lambda_n^2(t_1)(\varepsilon_n^2 - 1)(t) + ... + \alpha_d \sum_{n=1}^{\infty} \lambda_n^2(t_d)(\varepsilon_n^2 - 1).$$

But, by (1.41), we have

$$\alpha_1 Z_D(t) + .. + \alpha_d Z_D(t_d)$$

$$\stackrel{d}{=}=^d \int_{\mathbb{R}^2} \left(\alpha_1 \frac{e^{it_1(y_1+y_2)} - 1}{i(y_1 + y_2)} |y_1|^{\frac{D-1}{2}} |y_2|^{\frac{D-1}{2}} + ... + \alpha_d \frac{e^{it_d(y_1+y_2)} - 1}{i(y_1 + y_2)} |y_1|^{\frac{D-1}{2}} |y_2|^{\frac{D-1}{2}} \right) dB(y_1) dB(y_2), \tag{1.42}$$
$$= \int_{\mathbb{R}^2} H_{t_1,...t_d}(y_1, y_2)) B(dy_1) B(dy_2),$$
where $H_{t_1,...,t_d}(y_1, y_2) = H_{t_1,...,t_d}(y_2, y_1) = \overline{H_{t_1,...,t_d}(-y_1, -y_2)}, y_1, y_2 \in \mathbb{R}$, and

$$\int_{\mathbb{R}^2} |H_{t_1,...,t_d}|^2 dy_1 dy_2 < \infty.$$

By Proposition 2 of Dobrushin and Major (1979) (See [21]), we see that (1.42) can be represented in law as
$$\sum_{n=1}^{\infty} \lambda_n(t_1, .., t_d)(\varepsilon_n^2 - 1),$$
where $\lambda_n(t_1, .., t_n)$ are the eigenvalues of the integral operator

$$A_{t_1,..,t_d} h(x) = C(D) \int_{-\infty}^{+\infty} \left(\alpha_1 \frac{e^{it_1(x+y)} - 1}{i(x + y)} + ... + \alpha_d \frac{e^{it_d(x+y)} - 1}{i(x + y)} \right) |y|^{D-1} h(y) dy, h \in \mathbb{H}_D.$$

On the other hand, it is clear that the eigenvalues of $A_{t_1,..,,...,t_d}$ are $\alpha_1 \lambda_n(t_1) + ... + \alpha_d \lambda_n(t_d)$. This concludes the statement of the theorem. $\square$

Theorem 1.3.2. *For every $t_1, ..., t_d \geq 0$, the law of $(Z_D(t_1), ..., Z_D(t_d))$ belongs to $T(\mathbb{R}^d)$.*

Proof. by using the previous theorem we get

$$(Z_D(t_1), ..., Z_D(t_d)) =^d \sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1), .., \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1)$$

$$= \sum_{n=1}^{\infty} (\lambda_n(t_1)....\lambda_n(t_d))(\varepsilon_n^2 - 1),$$

where $(\lambda_n(t_1)....\lambda_n(t_d))(\varepsilon_n^2 - 1)$, $n = 1, 2, ...$, are the elementary gamma random variables in $\mathbb{R}^d$. Since they are independent, by the properties of the class $T(\mathbb{R}^d)$ that the class is closed under convolution and weak convergence, we see, by the definition of $T(\mathbb{R}^d)$, that $(Z_D(t_1), ..., Z_D(t_d))$ belongs to $T(\mathbb{R}^d)$. This completes the proof. $\square$

Chapter 2

Stochastic Calculus on the Rosenblatt processes

2.1 Stochastic Integration with respect to the Rosenblatt process

2.1.1 Wiener Integration for the Rosenblatt process

In the previous Section, we have seen that the Rosenblatt process is not a semi-martingale. But the classical stochastic integration namely the Itô calculus, is valid to semi-martingales as an integrator. Therefore, we cannot apply directly this type of calculus. Moreover, the Lebesgue-Stieltjes integration cannot be used since the paths of the Rosenblatt have unbounded variation as we mentioned previously, so we need to build other types of integrals.

In general, these generalized methods are essentially of two types: the first is the pathwise type calculus and (here we included the rough path analysis ([29]) and the stochastic calculus via regularization [11]) and the second type is Malliavin calculus and the Skorohod integration theory ([8]). In general the pathwise type calculus is connected to the trajectorial regularity and/or to the covariance structure of the integrator process, the Malliavin calculus instead is very related to the Gaussian character of the driven process. One of the methods developed to deal with the stochastic integral with respect to the Rosenblatt process is the white noise distribution theory by Hida and al.[26] , this approach succeed to define the derivative of the Rosenblatt process denoted by the

Rosenblatt noise dissipate the fact that it is not differentiated and also to define the Rodenblatt noise integral as a result.

Riemann-Stieltjes Integral

In order to understand the notion of the stochastic-integration, it's very useful to understand the Riemann-Stieltjes integral, so let us recall the basic Riemann integral first.

Definition 2.1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. We define the Riemann integral over $[a, b] \subset \mathbb{R}$ by*

$$\int_a^b f(t)dt = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1}),$$

where $\Delta_n = \{t_0, t_1, ..., t_n\}$ is a partition of $[a, b]$ such that $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$, $\|\Delta_n\| = \max_{1 \le i \le n}(t_i - t_{i-1})$ and τ_i is an evaluation point in the interval $[t_{i-1}, t_i]$.

Definition 2.1.2. *The p-variation of a function $f : [a, b] \to \mathbb{R}$ is defined as*

$$\sum_{i=1}^{n} (f(t_k^n) - f(t_{k-1}^n))^p,$$

where $a = t_0^n < ... < t_n^n = b$ is a partition of the interval with lags tends to 0 as $n \to \infty$.

Definition 2.1.3. *A function of bounded variation is a function $g : [a, b] \to \mathbb{R}$ such that $\forall t > 0$,*

$$\sup_{\pi \in \mathcal{P}} \sum_{i=1}^{np} |g(t_i) - g(t_{i-1})| < \infty,$$

where the supremum is taken over the set $\mathcal{P} = \{\pi = \{t_0, ..., t_{np}\}, \pi$ is a partition of $[a, b]\}$.

Definition 2.1.4. *Let $f : [a, b] \to \mathbb{R}$ continuous and $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. We define the Riemann-Stieltjes integral as follows:*

$$\int_a^b f(t)dg(t) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(\tau_i)(g(t_i) - g(t_{i-1})),$$

where $\Delta_n = \{t_0, t_1, ..., t_n\}$ is a partition of $[a, b]$ such that $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$, $\|\Delta_n\| = \max\limits_{1 \le i \le n}(t_i - t_{i-1})$ and τ_i is an evaluation point in the interval $[t_{i-1}, t_i]$.

Remark 2.1.1. *Note that if $g(t) = t$ then the Riemann-Stieltjes integral is the Riemann integral.*

Proposition 2.1.1. *[16] If f is continuous and $g \in \mathscr{C}^1$, then*

$$\int_a^b f(t)dg(t) = \int_a^b f(t)g\prime(t)dt,$$

and if f, g have bounded variations then

$$\int_a^b f(t)dg(t) = f(b)g(b) - f(a)g(a) - \int_a^b g(t)df(t). \tag{2.1}$$

Wiener Integral

The Wiener integral is basically an integration of a deterministic function with respect to a Gaussian stochastic process. It is a generalization of the Riemann-Stieltjes integral theory. Now Let us define the integral :

$$I(f) = \int_a^b f(t)dZ_t. \tag{2.2}$$

Since (2.1) defines a Riemann-Stieltjes integral, we can apply the integration by parts formula to get :

$$\int_a^b f(t)dZ_t = f(b)Z_b - f(a)Z_a - \int_a^b Z_t df(t), \tag{2.3}$$

but this is not valid since the Rosenblatt process has unbounded variation, so the integral (2.2) is not well defined as a Riemann-Stieltjes integral in this case. Therefore, we need new approaches to define it.

Integrands as step functions

Let us denote by $\mathcal{E}$ the set of step functions. For $f \in \mathcal{E}$, i.e. $f = \sum_{i=1}^{n} a_i 1_{]t_{i+1},t_i]}$, we define the Wiener Integral as follows :

Definition 2.1.5. *We naturally define the Wiener integral of f with respect to $Z(t)$ as :*

$$\int_0^T f(u)dZ(t) = \sum_{i=0}^{n-1} a_i(Z_{t_{i+1}} - Z_{t_i}) = \int_0^T \int_0^T I(f)(y_1, y_2)dB(y_1)dB(y_2), \tag{2.4}$$

where

$$f(t) = \sum_{i=0}^{n-1} a_i 1_{]t_i,t_{i+1}]}(t), \qquad t_i \in [0, T], \tag{2.5}$$

and

$$I(f)(y_1, y_2) = A_3(H) \int_{y_1 \vee y_2}^T f(u) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, y_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, y_2)du. \tag{2.6}$$

General integrands

Definition 2.1.6. *Let us denote $\mathcal{H}$ the space of functions such that*

$$\|f\|_{\mathcal{H}}^2 = 2\int_0^T \int_0^T I(f)(y_1, y_2)^2 dy_1 dy_2 < \infty, \tag{2.7}$$

and it holds that

$$\|f\|_{\mathcal{H}}^2 = H(2H-1)\int_0^T \int_0^T f(u)f(v)|u-v|^{2H-2} dudv. \tag{2.8}$$

Theorem 2.1.1. *[27] The mapping*

$$f \longrightarrow \int_0^T f(u)dZ(u), \tag{2.9}$$

defines an isometry from the space of step functions $\mathcal{E}$ to $L^2(\Omega)$[see [27]], it can also be expend to an isometry from $\mathcal{H}$ to $L^2(\Omega)$ by continuity, since $\mathcal{E}$ is dense in $\mathcal{H}$.

Proposition 2.1.2. *[18]*

1. *The space $\mathcal{H}$ may not contain just functions but also distributions; it is therefore more practical to define the subspaces of $\mathcal{H}$ that are sets of functions. Such a subspace is*

$$|\mathcal{H}| = \{f : \mathbb{R} \longrightarrow \mathbb{R} | \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2} dudv < \infty\}. \tag{2.10}$$

The space $|\mathcal{H}|$ is a strict subset of $\mathcal{H}$ and we have

$$L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^{\frac{1}{H}}(\mathbb{R}) \subset |\mathcal{H}| \subset \mathcal{H}. \tag{2.11}$$

2. *The space $|\mathcal{H}|$ is not complete with respect to this norm $\|.\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm*

$$\|f\|_{|\mathcal{H}|}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2} dudv < \infty. \tag{2.12}$$

3. *The spectral domain included in $\mathcal{H}$ is defined by*

$$\overset{\curlywedge}{\mathcal{H}} = \{f \in L^2(\mathbb{R}) | \int_{\mathbb{R}} |\overset{\curlywedge}{f}(x)|^2 |x|^{-2H+1} dx < \infty\}, \tag{2.13}$$

where $\overset{\curlywedge}{f}$ denotes the Fourier transform of f. We have again $\overset{\curlywedge}{\mathcal{H}}$ as a strict subspace of H and the inclusion

$$L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^{\frac{1}{H}}(\mathbb{R}) \subset \overset{\curlywedge}{\mathcal{H}} \subset \mathcal{H}, \tag{2.14}$$

and we define

$$\|f\|^2_{\overset{\curlywedge}{\mathcal{H}}} = \int_{\mathbb{R}} |\overset{\curlywedge}{f}(x)|^2 |x|^{-2H+1} dx. \tag{2.15}$$

Proposition 2.1.3. *[27] Let $g, f \in \mathcal{H}$. Then, the Wiener integrals $\int_0^T f(u)dZ(u)$ and $\int_0^T g(u)dZ(u)$ are not necessarily independent when the functions f and g are orthogonal in $\mathcal{H}$; they are independent if and only if*

$$\langle f(.)\frac{\partial K^{H_0}}{\partial u}(., y_1), g(.)\frac{\partial K^{H_0}}{\partial u}(., y_2)\rangle_{\mathcal{H}'} = 0, \quad (y_1, y_2) \in [0, T]^2, \tag{2.16}$$

where $\mathcal{H}'$ is the space analogous to $\mathcal{H}$ corresponding to the Hurst parameter $H' = \frac{H+1}{2}$.

Corollary 2.1.1. *[27] The construction of Wiener integrals with respect to the Rosenblatt process allows to consider associated Ornstein-Uhlenbeck processes which are the solutions of the equation*

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma Z(t), \quad t \geq 0,$$

where $\sigma, \lambda > 0$ and the initial condition ξ is a random variable in $L^0(\Omega)$ has an unique solution that can be represented as

$$X_t^\xi = e^{-\lambda t}\left(\xi + \sigma \int_0^t e^{-\lambda u} dZ(u)\right), \quad t \geq 0,$$

where the stochastic integral above exists in the Wiener sense. When the initial condition is $\xi = \sigma \int_{-\infty}^0 e^{\lambda u} dZ(u)$, the solution of can be written as

$$X_t = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dZ(u),$$

and it is called the stationary Rosenblatt Ornstein-Uhlenbeck process.

2.1.2 Pathwise stochastic calculus

The Rosenblatt process with $H > \frac{1}{2}$; as we have already mentioned; has zero quadratic variation and regular paths (Hölder continuous paths)[See property 3 and 4]. The pathwise calculus can be naturally applied to our process to construct stochastic integrals with respect to it. Here we choose to use the approach of Russo and Vallois.

Definition 2.1.7. *Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be continuous processes. We introduce, for every t :*

$$I^-(\varepsilon, Y, dX) = \int_0^t Y_s \frac{X_{\varepsilon+s} - X_s}{\varepsilon} ds, \quad I^+(\varepsilon, Y, dX) = \int_0^t Y_s \frac{X_s - X_{(s-\varepsilon)_+}}{\varepsilon} ds, \tag{2.17}$$

$$I^0(\varepsilon, Y, dX) = \int_0^t Y_s \frac{X_{s+\varepsilon} - X_{(s-\varepsilon)_+}}{2\varepsilon} ds, \tag{2.18}$$

and

$$C_\varepsilon(X, Y)(t) = \int_0^t \frac{(X_{s+\varepsilon} - X_{(s-\varepsilon)_+})(Y_{s+\varepsilon} - Y_{(s-\varepsilon)_+})}{\varepsilon} ds. \tag{2.19}$$

.

Now we are ready to give the forward, backward and symmetric integrals of Y with respect to X, it is given by :

$$\int_0^t Y d^- X = \lim_{\varepsilon \longrightarrow 0^+} I^-(\varepsilon, Y, dX), \int_0^t Y d^+ X = \lim_{\varepsilon \longrightarrow 0^+} I^+(\varepsilon, Y, dX). \tag{2.20}$$

$$\int_0^t Y d^0 X = \lim_{\varepsilon \longrightarrow 0^+} I^0(\varepsilon, Y, dX). \tag{2.21}$$

The covariation of X and Y is defined as

$$[X, Y]_t = ucp \lim_{\varepsilon \longrightarrow 0^+} C_\varepsilon(X, Y)(t).$$

If $X = Y$ we denote $[X, X] = [X]$ and when $[X]$ exists then X is said to be a finite quadratic variation process. When $[X] = 0$, then X is called a zero quadratic variation process.

The Rosenblatt process is clearly a zero quadratic variation process since

$$\mathbb{E} C_\varepsilon(Z, Z)(t) = \mathbb{E} \int_0^t \frac{1}{\varepsilon}(Z_{s+\varepsilon} - Z_s)^2 ds = t\varepsilon^{2H-1} \underset{\varepsilon \longrightarrow 0}{\rightarrow} 0.$$

Therefore the stochastic calculus via regularization can be directly applied to it.

Theorem 2.1.2. *[27] For every $f \in \mathscr{C}^2(\mathbb{R})$, the integrals*

$$\int_0^t f^{'}(Z) d^- Z, \int_0^t f^{'}(Z) d^+ Z \quad and \quad \int_0^t f^{'}(Z) d^0 Z, \tag{2.22}$$

exist and they are equal and we have the Itô formula

$$f(Z_t) = f(Z_0) + \int_0^t f^{'}(Z) d^0 Z. \tag{2.23}$$

Corollary 2.1.2. *[27] An immediate consequence of the existence of the quadratic variation of the Rosenblatt process is the existence and uniqueness of the solution of a Stratonovich stochastic differential equation driven by Z. Concretely, if* $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ *and* $b : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ *satisfy some regularity assumptions and* V *is a locally bounded variation process with* $X(0) = G$ *where* G *is an arbitrary random variable, then the equation*

$$dX(t) = \sigma(X(t))d^{\circ}Z(t) + b(t, X(t))dV(t),$$

has an unique solution.

The first and second-order fractional integral

Definition 2.1.8. *The first-order fractional integral is defined by :*

$$I_{+}^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(u)(x-u)^{\alpha-1}du, \tag{2.24}$$

and

$$I_{-}^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} f(u)(u-x)^{\alpha-1}du, \tag{2.25}$$

where $\alpha \in (0,1)$ *and* $1 \leq p < \frac{1}{\alpha}$.

The second-order fractional integral is defined by :

$$(I_{+,+}^{\alpha_1,\alpha_2} f)(x_1, x_2) \overset{Def}{=} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(u,v)(x_1-u)^{\alpha_1-1}(x_2-v)^{\alpha_2-1}dudv, \tag{2.26}$$

with $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *and* $\alpha_i \in (0,1), i = 1, 2, \ldots$ *, and we can define also*

$$(I_{-,tr}^{\alpha_1,\alpha_2} f)(x_1, x_2) \overset{Def}{=} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1 \vee x_2}^{\infty} f(u)(u-x_1)^{\alpha_1-1}(u-x_2)^{\alpha_2-1}du, \tag{2.27}$$

for $f : \mathbb{R} \longrightarrow \mathbb{R}$*. The operator* $I_{-,tr}^{\frac{H}{2},\frac{H}{2}}$ *plays the role of the transfer operator in the following definition of the Skorokhod integral with respect to a Rosenblatt process.*

2.1.3 Skorohod integral with respect to the Rosenblatt process

The Skorohod integral is stochastic integral, introduced for the first time by A. Skorohod in 1975, may be regarded as an extension of the Itô integral to integrands that are not necessarily $\mathcal{F}$*-adapted.*

This part will be dedicated to define an integral with respect to $(Z_t)_{t\in[0,t]}$ *in the divergence sense and to build generalized Skorohod integrals with respect to processes which* *are not a Gaussian process or a semi-martingales process. But before we start let use mention some basic elements of the Malliavin calculus with respect to a Wiener process* $(W_t)_{t\in[0,t]}$.

Definition 2.1.9. *We denote* $\mathcal{S}$ *by the class of smooth random variables of the form*

$$F = f(W_{t_1}.....W_{t_n}), \qquad t_1, ..., t_n \in [0,t], \tag{2.28}$$

where $f \in C_b^\infty(\mathbb{R}^n)$.

If F *is of the form (2.28) , we have*

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{t_1}.....W_{t_n}) 1_{[0,t_i]}(t), \quad t \in [0,T], \tag{2.29}$$

will be it's Malliavin derivative, with the operator D *as an unbounded closable operator which can be extended to the closure of* $\mathcal{S}$ *(denoted* $\mathbb{D}^{k,p}$, $k \geq 1$ *integer,* $p \geq 2$*) with respect to the norm*

$$\|F\|_{k,p}^p = \mathbb{E}|F|^p + \sum_{j=1}^{k} \mathbb{E}\|D^{(j)}F\|_{L^2([0,T])^j}^p, \quad F \in \mathcal{S}, k \geq 1, p \geq 2, \tag{2.30}$$

where the j*-th derivative* $D^{(j)}$ *is defined by iteration.*

Definition 2.1.10. *The Skorohod integral* δ *is the adjoint of* D *and its domain is*

$$Dom(\delta) = \{u \in L^2[0,T] \times \Omega / |\mathbb{E}\int_0^T u_s D_s F ds| \leq C\|F\|_2\}, \tag{2.31}$$

with D *and* δ *satisfing the duality relationship*

$$\mathbb{E}(F\delta(u)) = \mathbb{E}\int_0^T D_s F u_s ds, \qquad F \in \mathcal{S}, u \in Dom(\delta), \tag{2.32}$$

with

$$\delta(u) = \int_0^T u_s \delta W_s. \tag{2.33}$$

We denote $\mathbb{L}^{k,p} = L^p([0,T] \times \Omega; \mathbb{D}^{k,p})$ *and* $\mathbb{L}^{k,p} \subset Dom(\delta)$.

Definition 2.1.11. *For* $F \in \mathbb{D}^{1,2}$ *and* $u \in \mathbb{L}^{1,2}$, *the integration by parts formula is defined by*

$$F\delta(u) = \delta(Fu) + \int_0^T D_s F u_s. \tag{2.34}$$

Definition 2.1.12. *Let consider the a square integrable stochastic process $(g_s)_{s\in[0,T]}$, we define its Skorohod integral with respect to Z by:*

$$\begin{aligned}\int_0^T g_s dZ(s) &= \int_0^T\int_0^T\left(\int_{y_1\vee y_2}^T g(u)\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du\right)dB(y_1)dB(y_1)\\ &= \int_0^T\int_0^T I(g)(y_1,y_2)dB(y_1)dB(y_2),\end{aligned}$$

the process g is Skorohod integrable with respect to Z if the process $I(g)\in Dom(\delta^{(2)})$, where $\delta^{(2)}$ is the double Skorohod integral with respect to the Brownian motion B.

Remark 2.1.2. *The Skorohod integral coincide with the Wiener integral if the integrand g is a deterministic function in $\mathcal{H}$.*

Lemme 2.1. *If $g\in L^2(\Omega;\mathcal{H})$ such that $g\in\mathbb{L}^{2,2}$ and*

$$\mathbb{E}\int_0^T\int_0^T\|D_{x_1,x_2}g\|^2_{\mathcal{H}}dx_1dx_2<\infty. \tag{2.35}$$

Then g is Skorohod integrable with respect to Z and

$$\mathbb{E}|\int_0^T g_s dZ(s)|^2\leq c.\left[\mathbb{E}\|g\|^2_{\mathcal{H}}+\mathbb{E}\int_0^T\int_0^T\|D_{x_1,x_2}g\|^2_{\mathcal{H}}dx_1dx_2\right]. \tag{2.36}$$

Before we prove this lemma, we shall introduce "Meyer's inequality" for the double Skorohod integral.

Lemme 2.2. *[6] One can deduce the following estimations for the L^p norm of the generalized multiple integral:*

$$\begin{aligned}E\left(\left|\delta^k u\right|^p\right)\leq & C_{p,k}\left\{\left(E\int_{T^k}u_{\mathbf{t}}^2\mu^k(d\mathbf{t})\right)^{p/2}\right.\\ &\left.+E\left(\left|\int_{T^k}\int_{T^k}\left(D^k u_{\mathbf{t}}\right)^2_{\mathbf{s}}\mu^k(d\mathbf{t})\mu^k(d\mathbf{s})\right|^{p/2}\right)\right\},\end{aligned}$$

for all $1<p<\infty$, and for any process $u\in L^2\left(T^k\times\Omega\right)$ such that $u_t\in Dom(D^k)$ for every t and $\left(D^k\mu_t\right)_s$ belongs to $L^2\left(T^k\times T^k\times\Omega\right)$.

Proof. This lemma guaranties the convergence the Skorohod integrability, and to prove it we need to use the equality of meyer for the double Skorohod integral and we obtain :

$$\begin{aligned}
\mathbb{E}|\int_0^T g_s dZ(s)|^2 &\leq c.\left[\mathbb{E}\int_0^T\int_0^T I(g)(y_1,y_2)^2 dy_1 dy_2\right] \\
&\quad +c\left[\mathbb{E}\int_0^T\int_0^T\int_0^T\int_0^T (D_{x_1,x_2}I(g)(y_1,y_2))^2 dx_1 dx_2 dy_1 dy_2\right] \\
&= c.\left[\mathbb{E}\left(H(2H-1)\int_0^T\int_0^T g(u)g(v)|u-v|^{2H-2}dudv\right)\right] \\
&+ c.\left[\mathbb{E}\left(\int_0^T\int_0^T\left(\int_0^T\int_0^T D_{x_1,x_2}g(u)D_{x_1,x_2}g(v)|u-v|^{2H-2}dvdu\right)dx_1dx_2\right)\right] \\
&= c.\left[\mathbb{E}\|g\|_{\mathcal{H}}^2+\mathbb{E}\int_0^T\int_0^T \|D_{x_1,x_2}g\|_{\mathcal{H}}^2 dx_1dx_2\right].
\end{aligned}$$

□

Corollary 2.1.3. *The process g is Skorohod integrable with respect to Z and verifies*

$$\mathbb{E}|\int_0^T g_s dZ(s)|^2 \leq c.\|g\|^2, \tag{2.37}$$

where

$$\|g\|^2 = \left[\mathbb{E}\|g\|_{|\mathcal{H}|}^2+\mathbb{E}\int_0^T\int_0^T \|D_{x_1,x_2}g\|_{|\mathcal{H}|}^2 dx_1dx_2\right], \tag{2.38}$$

if $g \in L^2(\Omega;|\mathcal{H}|)$ such that $g\in\mathbb{L}^{2,2}$ and

$$\mathbb{E}\int_0^T\int_0^T \|D_{x_1,x_2}g\|_{|\mathcal{H}|}^2 dx_1dx_2<\infty. \tag{2.39}$$

Example *Here we try to give an example of the previous corollary. Let us consider a Skorohod integral of the Rosenblatt process with respect to itself ; by that we get*

$$\mathbb{E}|\int_0^T Z_t\delta Z_t|^2 \leq c.\int_0^T\int_0^T R(u,v)|u-v|^{2H-2}dudv.$$

Proof. In this proof we will treat the formula (2.36), so we will compute the two terms of its right side. From what we have seen previously, we can see clearly that :

$$\mathbb{E}\|Z\|_{\mathcal{H}}^2 = c.\int_0^T\int_0^T R(u,v)|u-v|^{2H-2}dudv,$$

then for the next term we have, $\forall x_1,x_2\in[0,T]$,

$$D_{x_1,x_2}Z(u) = 2A_3(H)1_{[0,u]^2}(x_1,x_2)\int_{x_1\vee x_2}^u \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u,x_1)\frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u,x_2)du,$$

then we have

$$\begin{aligned}
\mathbb{E} \quad \int_0^T \int_0^T \quad & \|D_{x_1,x_2} g\|^2_{|\mathcal{H}|} dx_1 dx_2 = \int_0^T \int_0^T \int_{x_1 \vee x_2}^T \int_{x_1 \vee x_2}^T |u-v|^{2H-2} dudv \\
\| \quad & \int_{x_1 \vee x_2}^u \frac{\partial K^{\frac{H+1}{2}}}{\partial u'}(u', x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u'}(u', x_2) \int_{x_1 \vee x_2}^v \frac{\partial K^{\frac{H+1}{2}}}{\partial v'}(v', x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial v'}(v', x_2) \| du' dv' dx_1 dx_2 \\
= \quad & \int_0^T \int_0^T |u-v|^{2H-2} \int_0^u \int_0^v \left(\int_0^{u' \wedge v'} \frac{\partial K^{\frac{H+1}{2}}}{\partial u'}(u', x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial v'}(v', x_1) dx_1 \right)^2 dudv \\
= \quad & c. \int_0^T \int_0^T R(u,v) |u-v|^{2H-2} dudv.
\end{aligned}$$

□

Definition 2.1.13. *Let $H \in (\frac{1}{2}, 1)$, and let $M \subseteq \mathbb{R}$ be an interval. Define*

$$\Lambda_{Z_H}(M) \stackrel{Def}{=} \{g : \mathbb{R} \longrightarrow L^2(\Omega) \quad such \quad that \quad I_{-,tr}^{\frac{H}{2},\frac{H}{2}}(1_M g) \in Dom(\delta^2)\}.$$

We say that the stochastic process g is Skorokhod integrable with respect to the Rosenblatt process Z_H on M if $g \in \Lambda_{Z_H}(M)$. As g as an integrand, the Skorokhod integral is defined by

$$\int_M g_s \delta Z_H(s) \stackrel{Def}{=} c_H^Z (\delta^2 \circ I_{-,tr}^{\frac{H}{2},\frac{H}{2}})(1_M g). \tag{2.40}$$

Lemme 2.3. *[5] Let $H \in (\frac{1}{2}, 1)$. The linear operator $I_{-,tr}^{\frac{H}{2},\frac{H}{2}}$ is bounded from $L^{\frac{1}{H}}(\mathbb{R})$ to $L^2(\mathbb{R}^2)$.*

The following lemma provides a mapping property of the Skorokhod integral with respect to a Rosenblatt process. It ensures that stochastic processes from the space $L^{\frac{1}{H}}(M; \mathbb{D}^{2,2})$ are Skorokhod integrable with respect to the Rosenblatt process Z_H and the stochastic integral is a square-integrable random variable.

Lemme 2.4. *[5] Let $H \in (\frac{1}{2}, 1)$ and $M \subseteq \mathbb{R}$ be an interval. The linear operator $\int_M (..) \delta Z_H$ is bounded from $L^{\frac{1}{H}}(M; \mathbb{D}^{k,p})$ to $\mathbb{D}^{k-2,p}$ for every integer $k \geq 2$ and every p such that $1 \leq pH < \infty$.*

The following theorem relates the Skorokhod integrals with respect to the fractional Brownian motion B_H and the Rosenblatt process Z_H to the fractional stochastic derivatives $\nabla^{H-\frac{1}{2}}$ and $\nabla^{\frac{H}{2},\frac{H}{2}}$, respectively.

Theorem 2.1.3. *[5] Let $H \in (\frac{1}{2}, 1)$ and M is a subset of $\mathbb{R}$.*

1. *If* $g \in L^{\frac{1}{H}}(M;\mathbb{D}^{2,2})$*, then for all* $G \in \mathbb{D}^{2,2}$ *we have :*

$$\mathbb{E}\left[G\int_M g_u \delta Z_H(u)\right] = c_H^Z \int_M \mathbb{E}\left[(\nabla^{\frac{H}{2},\frac{H}{2}}G)(u,u)g_u\right] du. \tag{2.41}$$

2. *If* $g \in L^{\frac{1}{H}}(M;\mathbb{D}^{1,2})$ *with* $G \in \mathbb{D}^{1,2}$*, then we have :*

$$\mathbb{E}\left[G\int_M g_u \delta B_H(u)\right] = c_H^B \int_M \mathbb{E}\left[(\nabla^{H-\frac{1}{2}}G)(u)g_u\right] du, \tag{2.42}$$

with

$$\nabla^{\alpha,\alpha} \stackrel{Def}{=} I_{+,+}^{\alpha,\alpha} \circ D^2 \quad \text{and} \quad \nabla^{\alpha} \stackrel{Def}{=} I_{+}^{\alpha} \circ D.$$

The following two theorems can be used to compute the first and second-order fractional stochastic derivatives of the Skorokhod integral with respect to a Rosenblatt process.

Theorem 2.1.4. *[5] Let* $g \in L^{\frac{1}{H}}([0,T],\mathbb{D}^{3,2})$*. The following equality is satisfied for all* $x \in \mathbb{R}$*:*

$$\begin{aligned} \nabla^{\frac{H}{2}}\left(\int_0^T g_s \delta Z_H\right)(x) &= \int_0^T (\nabla^{\frac{H}{2}} g_s)(x)\delta Z_H \\ &+ 2c_H^{B,Z}\frac{\beta(\frac{H}{2},1-H)}{\Gamma(\frac{H}{2})^2}\int_0^T g_s|s-x|^{H-1}\delta B^{\frac{H}{2}+\frac{1}{2}}. \end{aligned} \tag{2.43}$$

Theorem 2.1.5. *[5] Let* $g \in L^{\frac{1}{H}}([0,T],\mathbb{D}^{4,2})$*. The following equality is satisfied for all* $x, y \in \mathbb{R}$*:*

$$\nabla^{\frac{H}{2},\frac{H}{2}}\left(\int_0^T g_s\delta Z_H\right)(x,y) = \int_0^T (\nabla^{\frac{H}{2},\frac{H}{2}} g_s)(x,y)\delta Z_H + 2c_H^{B,Z}\frac{\beta(\frac{H}{2},1-H)}{\Gamma(\frac{H}{2})^2}$$
$$\left(\int_0^T (\nabla^{\frac{H}{2}} g_s)(x)|s-y|^{H-1}\delta B^{\frac{H}{2}+\frac{1}{2}} + \int_0^T (\nabla^{\frac{H}{2}} g_s)(y)|s-x|^{H-1}\delta B^{\frac{H}{2}+\frac{1}{2}}\right)$$
$$+2c_H^Z\frac{\beta(\frac{H}{2},1-H)^2}{\Gamma(\frac{H}{2})^4}\int_0^T g_s|s-x|^{H-1}|s-y|^{H-1}ds.$$

2.1.4 The relation between the pathwise and the Skorohod integrals

In this subsection we will cite an important relation between the pathwise and the Skorohod integrals given by Tudor in [27]. This plays a major role in the construction of the Itô formula for the functionals of the Rosenblatt process.

Definition 2.1.14. *We say that a stochastic process $g \in \mathbb{L}^{1,2}$ admits a trace of order 1 if*

$$\frac{1}{\varepsilon}\int_0^T\int_0^T D_\alpha g_s \delta(f_{d+\varepsilon}(.,\alpha) - f_s(.,\alpha))d\alpha ds, \tag{2.44}$$

converges in probability as $\varepsilon \longrightarrow 0$. The limit will be denoted by $Tr^{(1)}(D^{(1)}g)$.

We say that a stochastic process $g \in \mathbb{L}^{2,2}$ admits a trace of order 2 if

$$\frac{1}{\varepsilon}\int_0^T\int_0^T\int_0^T D^{(2)}_{\alpha,\beta} g_s \delta(f_{d+\varepsilon}(\beta,\alpha) - f_s(\beta,\alpha))d\beta d\alpha ds, \tag{2.45}$$

converges in probability as $\varepsilon \longrightarrow 0$. The limit will be denoted by $Tr^{(2)}(D^{(2)}g)$.

Theorem 2.1.6. *[27] Let $g \in \mathbb{L}^{2,2}$ such that*

$$\mathbb{E}\|g\|^2_{|\mathcal{H}|} + \mathbb{E}\int_0^T\int_0^T \|D_{x_1,x_2}g\|^2_{|\mathcal{H}|}dx_1dx_2 < \infty. \tag{2.46}$$

Assuming that the process g has trace of order 1 and 2. Then

$$\int_0^T g_s d^- Z(s) = \int_0^T g_s \delta Z(s) + 2Tr^{(1)}(D^{(1)}g) - Tr^{(2)}(D^{(2)}g). \tag{2.47}$$

2.1.5 A white noise approach

In this subsection we will define a stochastic integral with respect to the Rosenblatt process using white noise distribution theory, which leads to an Itô formula for a certain class of functionals of this process. Before we start let us introduce the tools from the white noise distribution theory needed in order to define a stochastic calculus with respect to the Rosenblatt process. For a good introduction to the theory of white noise, we refer the reader to the book of Kuo [13].

Definition 2.1.15. *Let $\Phi \in (S)^*$. For every function $\xi \in S(\mathbb{R})$, we define the S-transform of Φ by:*

$$S(\Phi)(\xi) = \langle\langle \Phi; e^{<;\xi>}\rangle\rangle,$$

where $e^{<;\xi>} = e^{<;\xi> - \frac{\|\xi\|^2_{L^2(\mathbb{R})}}{2}} = \sum_{n=0}^{\infty}\frac{I_n(\xi^{\otimes n})}{n!} \in (S)$ and $\otimes$ is defined below .

Remark 2.1.3.

1. *(S) is the stochastic space of test functions and its dual is the space of generalized functions $(S)^*$ or Hida distributions, and $\langle\langle . \rangle\rangle$ denote the duality bracket between elements of (S) and $(S)^*$, which reduces to the classical inner product on (L^2) for two elements in (L^2).*

2. *The space $S(\mathbb{R})$ is a Schwartz space and its dual $S^{'}(\mathbb{R})$ the space of tempered distributions.*

3. $(S) \subset L^2(\Omega, \mathcal{G}, \mathbb{P}) \subset (S)^*$.

Theorem 2.1.7. *[1]*

1. *The S-transform is injective. If $\forall \xi \in S(\mathbb{R}), S(\Phi)(\xi) = S(\Psi)(\xi)$ then $\Phi = \Psi \in (S)^*$.*

2. *For $\Phi, \Psi \in (S)^*$ there is a unique element $\Phi \diamond \Psi \in (S)^*$ such that for all $\xi \in S(\mathbb{R}), S(\Psi)(\xi)S(\Phi)(\xi) = S(\Psi \diamond \Phi)(\xi)$. It is called the Wick product of Φ and Ψ.*

3. *. Let $\Phi_n \in (S)^*$ and $F_n = S(\Phi_n)$. Then Φ_n converges strongly in $(S)^*$ if and only if the following conditions are satisfied:*

   (a) $\lim\limits_{n \longrightarrow \infty} F_n(\xi)$ exists for each $\xi \in S(\mathbb{R})$.

   (b) There exist strictly positive constants K, a and p independent of n such that:

$$\forall n \in \mathbb{N}, \forall \xi \in S(\mathbb{R}) |F_n(\xi)| \leq K e^{(a \|A^p \xi\|^2_{L^2(R)})}.$$

Definition 2.1.16. *The trace operator τ is the element of $\hat{S}' (\mathbb{R}^2)$ and is uniquely defined by*

$$\forall \psi, \phi \in S(\mathbb{R}) \quad < \tau; \phi \otimes \psi > = < \phi; \psi > .$$

Definition 2.1.17. *The Wick tensors of any elements $\omega \in S^{'}(\mathbb{R})$ are defined by:*

$$\omega^{\otimes n} = \sum_{k=0}^{\left[\frac{n}{2}\right]} C^n_{2k} (2k-1)!!(-1)^k \omega^{\otimes(n-2k)} \hat{\otimes} \tau^{\otimes k},$$

where $C^n_{2k} = \frac{n!}{(2k)!(n-2k)!}$, $(2k-1)!! = (2k-1)(2k-3)....3.1$ and $\hat{\otimes}$ is the symmetric tensor product .

Definition 2.1.18. *Let $y \in S^{'}(\mathbb{R})$ and $\Phi \in (S)$. The operator D_y is continuous from (S) into itself and we have:*

$$\forall \omega \in S^{'}(\mathbb{R}) \quad D_y(\Phi)(\omega) = \sum_{n=1}^{\infty} n < \omega^{\otimes n-1}; y \otimes_1 \phi_n >,$$

where we denote by $\otimes_1$ the contraction of order 1.

Definition 2.1.19. *Let $y \in S^{'}(\mathbb{R})$ and $\Psi \in (S)^*$. The adjoint operator D^*_y is continuous from $(S)^*$ into itself and we have:*

$$\forall \xi \in S(\mathbb{R}) \quad S(D^*_y(\Psi))(\xi) = < y; \xi > S(\Psi)(\xi) = S(I_1(y) \diamond \Psi)(\xi),$$

where $I_1 = (y)$ *is a generalized Wiener- Itô integral in* $(S)^*$. *Moreover, we have the following generalized Wiener- Itô decomposition for* $D_y^*(\Psi)$:

$$D_y^*(\Psi)(.) = \sum_{n=0}^{\infty} < .^{\otimes n+1}; y\hat{\otimes}\psi_n > .$$

Definition 2.1.20.

1. *Let* $I \subset \mathbb{R}$ *be an interval. A mapping* $X : I \longrightarrow (S)^*$ *is called a stochastic distribution process.*
2. *A stochastic distribution process* X *is said to be differentiable;* $\lim_{h \longrightarrow 0} \frac{X_{t+h} - X_t}{h}$ *exists in* $(S)^*$.

Theorem 2.1.8. *[1] The Rosenblatt process is* $(S)^*$ *differentiable, and its derivative, the Rosenblatt noise, admits the following generalized double Wiener- Itô integral representation (as an element of* $(S)^*$*):*

$$\forall t > 0 \quad \dot{Z}_t^H = c(H,2) I_2 \left(\delta_t^{\otimes 2} \circ (I_+^{\frac{H}{2}})^{\otimes 2} \right), \tag{2.48}$$

and its S-transform is equal to:

$$\forall \xi \in S(\mathbb{R}) \quad S(\dot{Z}_t^H)(\xi) = c(H,2)(I_+^{\frac{H}{2}}(\xi)(t))^2, \tag{2.49}$$

and we have

$$\forall \xi \in S(\mathbb{R}) \quad S(Z_t^H)(\xi) = c(H,2) \int_0^t (I_+^{\frac{H}{2}}(\xi)(t))^2 ds. \tag{2.50}$$

Definition 2.1.21. *Let* $k \geq 2$. *For any* $t > 0$, *we define the following sequence of stochastic processes (belonging to the second Wiener chaos):*

$$X_t^{H,k} = \int_{\mathbb{R}} \int_{\mathbb{R}} \underbrace{...((f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H) ... \otimes_1 f_t^H)(x_1, x_2) dB_{x_1} dB_{x_2}}_{k-1 \times \otimes_1} . \tag{2.51}$$

Where

$$f_t^{H(x_1, x\_2)} = c(H,2) \int_0^t \frac{(s-x_1)_+^{\frac{H}{2}-1} (s-x_2)_+^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})\Gamma(\frac{H}{2})} ds.$$

Moreover, the S-transform of this process is given by:

$$\forall \xi \in S(\mathbb{R}) \quad S(X_t^{H,k})(\xi) = c(H,2) \sqrt{\frac{H(2H-1)^{k-1}}{2}} \int_0^t \int_0^t I_+^{\frac{H}{2}}(\xi)(s) I_+^{\frac{H}{2}}(\xi)(r) K_t^{k-2}(s,r) ds dr. \tag{2.52}$$

Theorem 2.1.9. *[1] For any $k \geq 2$, the process $(X_t^{H,K}, t > 0)$ is $(S)^*$ differentiable and its derivative, $\dot{X}_t^{H,K}$, is uniquely defined by the following S-transform:*

$$\begin{aligned} \forall \xi \in S(\mathbb{R}) \quad S(\dot{X}_t^{H,K})(\xi) &= c(H,2)\sqrt{\frac{H(2H-1)^{k-1}}{2}}\Big[(H(k-1)+1)t^{H(k-1)} \\ &\int_0^1\int_0^1 I_+^{\frac{H}{2}}(\xi)(tu)I_+^{\frac{H}{2}}(\xi)(tv)K_1^{k-2}(u,v)dudv \\ &+ t^{H(k-1)+1}\int_0^1\int_0^1 \frac{d}{dt}\left[I_+^{\frac{H}{2}}(\xi)(tu)I_+^{\frac{H}{2}}(\xi)(tv)\right]K_1^{k-2}(u,v)dudv\Big]. \end{aligned} \tag{2.53}$$

Definition 2.1.22. *A stochastic distribution process $X : I \longrightarrow (S)^*$ is integrable if:*

1. *$\forall \xi \in S(\mathbb{R}), S(X)(\xi)$ is measurable on I.*
2. *$\forall \xi \in S(\mathbb{R}), S(X)(\xi) \in L^1(I)$.*
3. *$\int_I S(X_t)(\xi)dt$ is the S-transform of a certain Hida distribution.*

Theorem 2.1.10. *[1] Let $X : I \longrightarrow (S)^*$ be a stochastic distribution process satisfying:*

1. *$\forall \xi \in S(\mathbb{R}), S(X)(\xi)$ is measurable on I.*
2. *There is a $p \in \mathbb{N}$, a strictly positive constant a and a non-negative function $L \in L^1(I)$ such that:*

$$\forall \xi \in S(\mathbb{R}) \quad |S(X_t)(\xi)| \leq L(t)e^{a\|A^p\xi\|_2^2},$$

Then X is $(S)^*$-integrable.

Definition 2.1.23. *Let $\{\phi_t; t \in I\}$ be a $(S)^*$ stochastic process satisfying the assumptions of the previous theorem. Then $\phi_t \diamond \dot{Z}_t^H$ is $(S)^*$ integrable over I and we define the Rosenblatt noise integral of $\{\psi_t\}$ by:*

$$\int_I \phi_t dZ_t^H = \int_I \phi_t \diamond \dot{Z}_t^H dt. \tag{2.54}$$

Moreover, we have the following representation

$$\int_I \phi_t dZ_t^H = \int_I \left(D^*_{\sqrt{c(H,2)}\delta_t \circ I_+^{\frac{H}{2}}}\right)^2 (\phi_t)dt. \tag{2.55}$$

2.1.6 The relation between the white noise and the Skorohod approach

We have introduced two definitions of the stochastic integral with respect to the Rosenblatt process. The fisrt one is a pathwise integral using the method of russo-vallois and the second one is a Skorohod integral using elements of the Malliavin calculus and we gave the relation between them . Now we will show that the Skorohod integral coincides with the Rosenblatt noise integral using the finite interval representation.

Definition 2.1.24. *The Rosenblatt process $(Z_t^H)_{t\in\mathbb{R}}$ is equal in distribution to the process $(Y_t^H)_{t\in[0,T]}$ defined by*

$$\forall t \in [0,T] \quad Y_t^H = c(H,2)\int_{[0,t]^2}\int_0^t \prod_{j=1}^{2}\left(\frac{s}{x_j}\right)^{\frac{H}{2}}(s-x_j)_+^{\frac{H}{2}-1} ds dBx_1 dBx_2.$$

Theorem 2.1.11. *[1] The process (Y_t^H) is $(S)^*$ -differentiable and the S-trasnform of its derivative is equal to:*

$$\forall \xi \in S(\mathbb{R}), \forall t \in [0,T] \quad S(\dot{Y}_t^H)(\xi) = c(H,2)\left(\int_0^t \xi(x)\left(\frac{t}{x}\right)^{\frac{H}{2}}(t-x)_+^{\frac{H}{2}-1}dx\right)^2. \tag{2.56}$$

Theorem 2.1.12. *[1] Let $(\phi_t; t \in [0,T])$ be an $(S)^*$ stochastic process satisfying the assumptions of Theorem 3.9. Then, $\phi_t \diamond \dot{Y}_t^H$ is $(S)^*$ integrable over $[0,T]$ and :*

$$\int_{[0,T]^2}\phi_t dY_t^H = \int_{[0,T]^2}\phi_t \diamond \dot{Y}_t^H dt. \tag{2.57}$$

Proposition 2.1.4. *[1] Let $(\phi_t; t \in [0;T])$ be a stochastic process such that $\phi \in L^2(\Omega;\mathcal{H})\cap L^2([0,T];\mathbb{D}^{2,2})$ verifies*

$$\mathbb{E}\left[\int_0^T\int_0^T \|D^2_{s_1,s_2}\phi\|^2_{\mathcal{H}} ds_1 ds_2\right] < \infty,$$

Then, (ϕ_t) is Skorohod integrable and $(S)^*$-integrable with respect to the Rosenblatt process, $(Y_t^H)_{t\in[0;T]}$, and we have:

$$\int_0^T \phi_t \delta Y_t^H = \int_0^T \phi_t \diamond \dot{Y}_t^H dt. \tag{2.58}$$

In order to introduce the Itô formula for functional of the Rosenblatt process, we need to define one of the most known hermite processes which is the fractional Brownian motion.

Definition 2.1.25. *Let $H \in (1/2, 1)$. The fractional Brownian motion $(B_t^H)_{t\in\mathbb{R}}$ of the Hurst parameter H is defined by*

$$B_t^H \overset{Def}{=} C_H^B \int_{\mathbb{R}} \left(\int_0^t (u-y)_+^{H-\frac{3}{2}} du \right) dB_y, \quad t \in \mathbb{R}$$

where C_H^B is a normalizing constant such that $\mathbb{E}(B_1^H)^2 = 1$.

Remark 2.1.4. *The normalizing constant is defined by*

$$C_H^B = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}.$$

It will be also convenient to denote

$$c_H^B = C_H^B \Gamma\left(H - \frac{1}{2}\right) \quad and \quad c_H^Z = c(H,2)\Gamma\left(\frac{H}{2}\right)^2,$$

and

$$c_H^{B,Z} \overset{Def}{=} \frac{c_H^Z}{c_{\frac{H}{2},\frac{1}{2}}^B} = \sqrt{\frac{(2H-1)\Gamma(1-\frac{H}{2})\Gamma(\frac{H}{2})}{(H+1)\Gamma(1-H)}}.$$

2.2 Itô formula for functionals of Skorokhod integrals with respect to the Rosenblatt process

Since the Itô formula for functionals of Skorokhod integrals with respect to the fractional Brownian motion is well-known, functionals of the Skorokhod integral with respect to a Rosenblatt process are considered. Moreover, in this case, it is possible to formulate sufficient conditions for the Itô formula in terms of the integrand rather than in terms of the integral.

Proposition 2.2.1. *[5] Let f be a function in $\mathscr{C}^3(\mathbb{R})$ such that for all $c \geq 0$ and $\alpha \geq 0$ we have*

$$|f'''(x)| \leq c(1+|x|^\alpha), \quad x \in \mathbb{R}.$$

Let the stochastic process $(\psi_s)_{s\in[0,T]}$ which satisfies the following assumptions :

1. *The process ψ is in $L^\infty([0,T]; \mathbb{D}^{4,p})$ for some*

$$p > max\{\frac{2}{(2H-1)}, 8(\alpha+1)\}.$$

2. *For almost every $r, v \in [0,T]$, the following equalities are satisfied*

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \operatorname*{ess\,sup}_{u\in(v,v+\varepsilon)} \|(\nabla^{\frac{H}{2}}\psi_r)(u) - (\nabla^{\frac{H}{2}}\psi_r)(v)\|_{\mathbb{D}^{3,8}} &= 0,\\
\lim_{\varepsilon\downarrow 0} \operatorname*{ess\,sup}_{u\in(v,v+\varepsilon)} \|(\nabla^{\frac{H}{2},\frac{H}{2}}\psi_r)(u,u) - (\nabla^{\frac{H}{2},\frac{H}{2}}\psi_r)(v,v)\|_{\mathbb{D}^{2,8}} &= 0,\\
\lim_{\varepsilon\downarrow 0} \operatorname*{ess\,sup}_{u\in(v,v+\varepsilon)} \|(\nabla^{\frac{H}{2},\frac{H}{2}}\psi_r)(r,u) - (\nabla^{\frac{H}{2},\frac{H}{2}}\psi_r)(r,v)\|_{\mathbb{D}^{2,8}} &= 0.
\end{aligned}$$

3. *There exist functions $p_1 \in L^{\frac{6}{1+H}}[0,T]$, $p_2 \in L^{\frac{1}{H}}[0,T]$, $p_3 \in L^{\frac{1}{H}}[0,T]$ such that for almost every $r \in [0,T]$, the estimates*

$$\begin{aligned}
\|(\nabla^{\frac{H}{2}}\psi_r)(v)\|_{\mathbb{D}^{3,8}} &\leq p_1(r),\\
\|(\nabla^{\frac{H}{2},\frac{H}{2}}\psi_r)(v,v)\|_{\mathbb{D}^{2,8}} &\leq p_2(r),\\
\|(\nabla^{\frac{H}{2},\frac{H}{2}}\psi_r)(r,v)\|_{\mathbb{D}^{2,8}} &\leq p_3(r),
\end{aligned}$$

are satisfied for almost every $v \in [0,T]$.

Define the process $(R_t)_{t\in[0,T]}$ by

$$R_t \overset{Def}{=} \int_0^t \psi_r \delta Z_r^H. \tag{2.59}$$

Then the following equality is satisfied for every $t \in [0,T]$ almost surely

$$\begin{aligned}
f(R_t^H) - f(0) &= \int_0^t f^{'}(R_s^H)\psi_s \delta Z_r^H\\
&+ 2c_H^{B,Z}\int_0^t f^{''}(R_s^H)(\nabla^{\frac{H}{2}}R_s^H)(s)\psi_s\delta B^{\frac{H}{2}+\frac{1}{2}}\\
&+ c_H^Z\int_0^t \left(f^{''}(R_s^H)(\nabla^{\frac{H}{2},\frac{H}{2}}R_s^H)(s,s) + f^{'''}(R_s^H)[(\nabla^{\frac{H}{2}}R_s^H)(s)]^2\right)\psi_s ds.
\end{aligned} \tag{2.60}$$

The following corollary is a direct consequence of the previous proposition. It provides an Itô-type formula for functionals of Wiener integrals with respect to the Rosenblatt process, i.e. when the integrand is deterministic.

Corollary 2.2.1. *[5] Let $f \in \mathscr{C}^3(\mathbb{R})$ such that its third derivative has at most polynomial growth and let $(\psi_t)_{t\in[0,T]}$ be a bounded deterministic function. Let $(Z_t)_{t\in[0,T]}$ be the integral* *process defined by (2.59). Then the formula*

$$\begin{aligned} f(R_t^H) &= f(0) + \int_0^t f'(R_t^H)\psi_s \delta R_s^H \\ &+ H(2H-1)\int_0^t f''(R_s)\psi_s \int_0^s \psi_r (s-r)^{2H-2} dr ds \\ &+ c_1(H)\int_0^t f''(R_s^H)\psi_s \left(\int_0^s \psi_r (s-r)^{H-1}\delta B^{\frac{H}{2},\frac{1}{2}}\right)\delta B^{\frac{H}{2}+\frac{1}{2}} \\ &+ (\sqrt{2H(2H-1)})^3 \int_0^t f'''(R_s^H)\psi_s \int_0^s \psi_u (s-u)^{H-1}\int_0^u \psi_v (s-v)^{H-1}(u-v)^{H-1} dv du ds \\ &+ c_2(H)\int_0^t f'''(R_s^H)\psi_s \left(\int_0^s \psi_u (s-u)^{H-1}\left(\int_0^u \psi_v (s-v)^{H-1}\delta B^{\frac{H}{2}+\frac{1}{2}}\right)\delta B^{\frac{H}{2}+\frac{1}{2}}\right) ds, \end{aligned}$$

is satisfied almost surely for every $t \in [0,T]$ with the constants

$$c_1(H) \overset{Def}{=} \frac{4(2H-1)}{H+1}, \quad c_2(H) \overset{Def}{=} \frac{8(2H-1)}{H+1}\sqrt{\frac{H(2H-1)}{2}}.$$

Next, we will introduce two corollaries considered as an applications of the previous obtained Itô formula. In the first corollary, we will use the Itô-type formula in the proposition 2.2.1 to compute the second moment of the stochastic integral with respect to a Rosenblatt process and in the second, we will give an estimation for higher absolute moments of the stochastic integral with respect to the Rosenblatt process.

Corollary 2.2.2. *Let ψ be a stochastic process which satisfies the the first and the third assumption (1-3) of the proposition 2.2.1 and R_t^H is defined in (2.59). Then we have*

$$\begin{aligned} \mathbb{E}(R_t)^2 &= H(2H-1)\int_0^t\int_0^t \mathbb{E}[\psi_r\psi_s]|s-r|^{2H-2} dr ds \\ &+2H(2H-1)c_3(H)\int_0^t\int_0^t \mathbb{E}\left[\nabla^{\frac{H}{2}}\psi_r(s)\nabla^{\frac{H}{2}}\psi_s(r)\right]|s-r|^{H-1} dr ds \\ &+\tfrac{1}{2}H(2H-1)c_3(H)^2\int_0^t\int_0^t \mathbb{E}\left[\nabla^{\frac{H}{2},\frac{H}{2}}\psi_r(s,s)\nabla^{\frac{H}{2},\frac{H}{2}}\psi_s(r,r)\right] dr ds, \end{aligned} \tag{2.61}$$

with the constant $c_3(H)$ as given by

$$c_3(H) \overset{Def}{=} \frac{\Gamma(\frac{H}{2})\Gamma(1-\frac{H}{2})}{\Gamma(1-H)}.$$

Proof. In this proof we will need to use the previous definition (proposition 2.2.1) and by taking the function $f(x) = x^2$, we have

$$\mathbb{E}(R_t^2) = 2c_H^Z \mathbb{E}\int_0^t \left[(\nabla^{\frac{H}{2},\frac{H}{2}} R_s)(s)\psi_s\right] ds,$$

since the stochastic integral has a zero mean. Now we will use theorem 2.1.5 to have the following

$$\begin{aligned}\mathbb{E}\left[(\nabla^{\frac{H}{2},\frac{H}{2}}R_s)(s)\psi_s\right] &= \mathbb{E}\left[\psi_s\left(\int_0^s(\nabla^{\frac{H}{2},\frac{H}{2}}\psi_r)(s)\delta Z_r^H\right)\right]\\ &+\frac{4c_H^{B,Z}B(\frac{H}{2},1-H)}{\Gamma(\frac{H}{2})^2}\mathbb{E}\left[\psi_s\left(\int_0^s(\nabla^{\frac{H}{2}}\psi_r)(s)(s-r)^{H-1}\delta B_r^{\frac{H}{2}+\frac{1}{2}}\right)\right]\\ &+\frac{2c_H^Z B(\frac{H}{2},1-H)^2}{\Gamma(\frac{H}{2})^4}\int_0^s\mathbb{E}[\psi_r\psi_s](s-r)^{2H-2}dr.\end{aligned}$$

By using 1 of the theorem 2.1.3 we get

$$\mathbb{E}\left[\psi_s\left(\int_0^s(\nabla^{\frac{H}{2},\frac{H}{2}}\psi_r)(s,s)\delta Z_r^H\right)\right]=c_H^Z\int_0^s\mathbb{E}\left[(\nabla^{\frac{H}{2},\frac{H}{2}}\psi_s)(r,r)(\nabla^{\frac{H}{2},\frac{H}{2}}\psi_r)(s,s)\right]dr,$$

and by 2 of the same theorem, it follows that

$$\mathbb{E}\left[\psi_s\left(\int_0^s(\nabla^{\frac{H}{2}}\psi_r)(s)(s-r)^{H-1}\delta B_r^{\frac{H}{2}+\frac{1}{2}}\right)\right]=c^B_{\frac{H}{2}+\frac{1}{2}}\int_0^s\mathbb{E}\left[(\nabla^{\frac{H}{2}}\psi_s)(r)(\nabla^{\frac{H}{2}}\psi_r)(s)\right](s-r)^{H-1}dr,$$

which is satisfied for every $s\in[0,t]$. □

Remark 2.2.1. *The formula (2.61) holds under weaker assumptions: it is sufficient if the integrand ψ belongs to the space $L^{\frac{1}{H}}(0,T;\mathbb{D}^{4,2})$. This follows because the duality formula from Lemma 4 can be used instead of the Itô formula; in which case the assumptions (1) - (3) are not needed.*

Corollary 2.2.3. *Let $q\geq 3$ and let ψ be a stochastic process that satisfies the first and the second condition of proposition 2.2.1 with $\alpha=q-2$. Let $(R_t)_{t\in[0,T]}$ be the stochastic process defined by (2.59). Then the estimate*

$$\|R_t\|^3_{L^q(\Omega)}\leq 3(q-1)c_H^Z\int_0^t\|\psi_s\left(|R_s|(\nabla^{\frac{H}{2},\frac{H}{2}}R_s)(s,s)+sgn(R_s)(q-2)\left[(\nabla^{\frac{H}{2}}R_s)(s)\right]^2\right)\|_{L^{\frac{q}{3}}(\Omega)}ds, \tag{2.62}$$

is satisfied for every $t\in[0,T]$.

Proof. Initially, we assume that $q>3$. Using proposition 2.2.1 with $f(x)=|x|^q$ (f is $\mathscr{C}^3$

since $q > 3$), it follows that

$$\begin{aligned} |R_t|^q = \int_0^t \quad q \quad & |R_s|^{q-1} sgn(R_s)\psi_s \delta R_s^H \\ + \quad 2c_H^{B,Z} & \int_0^t q(q-1)|R_s|^{q-2}(\nabla^{\frac{H}{2}} R_s)\psi_s \delta B^{\frac{H}{2}+\frac{1}{2}} \\ + \quad c_H^R & \int_0^t \psi_s q(q-1)|R_s|^{q-2}(\nabla^{\frac{H}{2},\frac{H}{2}} R_s(s,s))ds \\ & + c_H^R \int_0^t \psi_s q(q-1)(q-2)|R_s|^{q-3} sgn(R_s) \left[(\nabla^{\frac{H}{2}} R_s)(s)\right]^2 ds, \end{aligned} \tag{2.63}$$

where sgn denotes the sign function. Taking the expectation of both sides of (2.63), it follows that

$$\mathbb{E}|R_t|^q = q(q-1)c_H^Z \int_0^t \mathbb{E}\left[|R_s|^{q-3}\psi_s \left(|R_s|(\nabla^{\frac{H}{2},\frac{H}{2}} R_s)(s,s) + (q-2)sgn(R_s)[(\nabla^{\frac{H}{2}} R_s)(s)]^2\right)\right] ds,$$

because the stochastic integrals have zero expectation. Thus, Hölder inequality yields

$$\mathbb{E}|R_t|^q \le q(q-1)c_H^Z \int_0^t (\mathbb{E}|R_t|^q)^{\frac{q-3}{q}} \|\psi_s \left(|R_s|(\nabla^{\frac{H}{2},\frac{H}{2}} Z_s)(s,s) + (q-2)sgn(R_s)[(\nabla^{\frac{H}{2}} R_s)(s)]^2\right) \|_{L^{\frac{q}{3}}(\Omega)} ds.$$

The desired inequality is proved by using Bihariâs inequality, see [10], Theorem 3, p. 135], which gives

$$\mathbb{E}|R_t|^q \quad \le \quad \left(\tfrac{3}{q}(q-1)c_H^Z \int_0^t \|\psi_s \left(|R_s|(\nabla^{\frac{H}{2},\frac{H}{2}} R_s)(s,s) + (q-2)sgn(R_s)[(\nabla^{\frac{H}{2}} R_s)(s)]^2\right) \|_{L^{\frac{q}{3}}(\Omega)} ds\right)^{\frac{q}{3}}.$$

For the case $q = 3$, proposition 2.2.1 cannot be used directly, since the function $f(x) = |x|^3$ does not belong to $\mathscr{C}^3(\mathbb{R})$. Instead, for $\varepsilon > 0$, consider the function

$$f_\varepsilon(x) \overset{Def}{=} (x^2 + \varepsilon^2)^{\frac{3}{2}}, \quad x \in \mathbb{R}.$$

The function f_ε is a smooth approximation of $f(x) = |x|^3$ with a bounded third derivative. Hence, by proposition 2.2.1 it follows that $\mathbb{E}f_\varepsilon(Z_t)$ satisfies the formula

$$\mathbb{E}f_\varepsilon(R_t) = \varepsilon^3 + c_H^Z \int_0^t \mathbb{E}\left[\psi_s \left(f_\varepsilon^{''}(R_s)(\nabla^{\frac{H}{2},\frac{H}{2}} R_s)(s,s) + f_\varepsilon^{'''}(R_s)[(\nabla^{\frac{H}{2}} R_s)(s)]^2\right)\right] ds, \tag{2.64}$$

similarly as in the case $q > 3$. Since

$$\lim_{\varepsilon\downarrow 0} f_\varepsilon^{'''}(x) = 6|x| \quad and \quad \lim_{\varepsilon\downarrow 0} f_\varepsilon^{'''}(x) = 6sgn(x),$$

taking the limit $\varepsilon \downarrow 0$ in equality (2.64) and using Lebesgueâs dominated convergence theorem to interchange the limit, the integrals yields

$$\mathbb{E}|R_t|^3 = 6c_H^Z \int_0^t \mathbb{E}\left[\psi_s \left(|R_s|(\nabla^{\frac{H}{2},\frac{H}{2}} R_s)(s,s) + sgn(R_s)[(\nabla^{\frac{H}{2}} R_s)(s)]^2\right)\right] ds,$$

which concludes the proof.

□

2.3 Itô formula for the Rosenblatt process

The Itô formula for the Rosenblatt process has been studied by several authors starting by [Ciprian A. Tudor] in 2006 [27] and ending with [Petr Coupek , Tyrone E. Duncana, , Bozenna Pasik-Duncana] in 2020 [5]; all the pre-mentioned authors introduced the Itô formula using different approaches and conditions .

In this section we will introduce two versions of the Itô formula using the skorohod approach then we will generate the Itô formula using the white noise approach introduced by Benjamin Arras in 2015[1].

2.3.1 Itô formula in the Skorohod sense

We study the Itô formula for the Rosenblatt process in the divergence sense, We will deduce the Skorohod Itô formula by using the pathwise Itô formula.

Theorem 2.3.1. *[27] Recall that for any function $f \in \mathscr{C}^2(\mathbb{R})$.*

$$
\begin{aligned}
f(Z_t^H) &= f(0) + \int_0^t f'(Z_s^H) d^- Z_s^H \\
&= f(0) + \int_0^t f'(Z_s^H) \delta Z_s^H + 2Tr^{(1)}(D^{(1)} f'(Z_s^H)) - Tr^{(2)}(D^{(2)} f''({Z^H}_s)).
\end{aligned}
$$

Remark 2.3.1. *We are actually able to prove Skorohod Itô formula only in two particular cases: when $f(x) = x^2$ and for $f(x) = x^3$, since the two trace term exists and is proved only in the previous two particular cases . For more details [see [27]].*

For the sake of completeness, it is noted that from proposition 2.2.1, the Itô formula for functionals of the Rosenblatt process itself can be obtained. It is only required that f be $\mathscr{C}^3$ and to set $\psi \equiv 1$.

Theorem 2.3.2. *[5] Let $f \in \mathscr{C}^3$ be such that its third derivative has at most a polynomial growth. Then the equality:*

$$\begin{aligned} f(Z_t^H) = f(0) &+ \int_0^t f'(Z_t^H)\delta Z_t^H \\ &+ H\int_0^t f''(Z_t^H)s^{2H-1}ds \\ &+ c_1(H)\int_0^t f''(Z_t^H)\left(\int_0^t (s-u)^{H-1}\delta B_u^{\frac{H}{2}+\frac{1}{2}}\right)\delta B_s^{\frac{H}{2}+\frac{1}{2}} \\ &+ \tfrac{H}{2}\kappa_3(Z_1^H)\int_0^t f'''(Z_s^H)s^{3H-1}ds \\ &+ c_2(H)\int_0^t f'''(Z_s^H)\left(\int_0^s (s-u)^{H-1}\left(\int_0^u (s-v)^{H-1}\delta B_u^{\frac{H}{2}+\frac{1}{2}}\right)\delta B_u^{\frac{H}{2}+\frac{1}{2}}\right)ds, \end{aligned}$$

is satisfied for $t \in [0,T]$ almost surely with

$$\kappa_3(Z_1^H) = \frac{4\sqrt{2H(2H-1)^3}}{3H-1}\beta(H,H),$$

where β is the beta function.

2.3.2 Itô formula using the white noise approach

In this subsection, we will derive an Itô formula for a certain class of functionals of the Rosenblatt process, in the framework of white noise distribution theory. Firstly we will start by getting an Itô formula for x^2 and x^3 like what tuder did but in the $(S)^*$ sense, then we identify the class of functionals for which this Itô formula is true.

Theorem 2.3.3. *[1] Let $(a,b) > 0$ such that $a < b$. Then in $(S)^*$:*

$$(Z_b^H)^2 - (Z_a^H)^2 = 2\int_a^b Z_s^H dZ_s^H + b^{2H} - a^{2H} + 4\int_a^b dX_s^{H,2},$$

and

$$\begin{aligned} (Z_b^H)^3 - (Z_a^H)^3 = 3\int_a^b (Z_s^H)^2 dZ_s^H &+ 6H\int_a^b s^{2H-1}Z_s^H ds + 12\int_a^b Z_s^H dX_s^{H,2} \\ &+ \kappa_3(Z_1^H)(b^{3H} - a^{3H}) + 24\int_a^b dX_s^{H,3}. \end{aligned}$$

Theorem 2.3.4. *[1] Let $(a,b) \in \mathbb{R}_+^*$ such that $a \leq b < \infty$. Let F be an entire analytic function of the complex variable verifying:*

$$\exists N \in \mathbb{N}, \exists C > 0, \forall z \in \mathbb{C} \quad |F(z)| \leq C(1+|z|)^N e^{\frac{1}{\sqrt{2}b^H}|\Im(z)|}.$$

Then, we have in $(S)^*$:

$$\begin{aligned} F(Z_b^H) - F(Z_a^H) &= \int_a^b F^{(1)}(Z_t^H) \diamond \dot{Z}_t^H dt \\ &+ \sum_{k=2}^{\infty} \left(H\kappa_k(Z_1^H) \int_a^b \frac{t^{Hk-1}}{(k-1)!} F^{(k)}(Z_t^H) dt + 2^k \int_a^b F^{(k)}(Z_t^H) \diamond \dot{X}_t^{H,k} dt \right). \end{aligned}$$

Remark 2.3.2.

1. *The Itô formula from the previous theorem holds for an infinitely differentiable function as condition.*
2. *The non-zero cumulants of the Rosenblatt distribution seem to be the ones responsible for this result.*

2.4 Local times and their properties for the Rosenblattt process

Itô lemma is one of the most important and useful results in the theory of stochastic calculus, which differs from the classical deterministic formulas by the presence of a quadratic variation term. One drawback which can limit the applicability of Itô lemma in some situations, is that it only applies for twice continuously differentiable functions. However, the quadratic variation term can alternatively be expressed using local times, which relaxes the differentiability requirement.

In this section, we will prove the existence of the local time for the Rosenblatt process and more than that we will give it's representation and some of it's properties .

Definition 2.4.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a deterministic function, the occupations measure is defined by*

$$\nu(A, B) = \mu(B \cap f^{-1}(A)), \tag{2.65}$$

where $A \subset \mathbb{R}$ and $B \subset \mathbb{R}^+$ are Borel sets and μ is the Lebesgue measure on $\mathbb{R}^+$.

Then, when $\nu(., B)$ is absolutely continuous with respect to μ, the occupation density (or local time) is given by the Radon-Nikodym derivative:

$$L(x, B) = \frac{d\nu}{d\mu}(x, B). \tag{2.66}$$

For a fixed trajectory of a process, the Fourier transform

$$F(u) = \int_{\mathbb{R}} e^{iux} L(t, x) dx, \tag{2.67}$$

of L(t, x) with respect to the variable x can be represented with the help of the local time as follows:

$$F(\xi) = \int_0^t e^{i\xi Z_s}. \tag{2.68}$$

Then the local time can be represented via the inverse Fourier transform of this function, that is,

$$L(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{i\xi(Z_s - x)} ds d\xi. \tag{2.69}$$

Proposition 2.4.1. *[17] The function f has an occupation density $L(x, B)$ for $x \in \mathbb{R}, B \in \mathcal{B}([u, U])$ which is square integrable in x for every fixed B if*

$$\int_{\mathbb{R}} |\int_u^U e^{(i\xi f(t))} dt|^2 d\xi < \infty, \tag{2.70}$$

and more than that , in this case, the occupation density can be represented as

$$L(x, B) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_B e^{(i\xi(x - f(t))} d\xi ds. \tag{2.71}$$

The deterministic function $f(t)$ can be chosen to be the single path of a stochastic process $(X_t)_{t \geq 0}$. To make sure that the existence and the square integrability of $L(x, B)$, it is enough to show that :

$$\mathbb{E}\left[\int_{\mathbb{R}} |\int_u^U e^{i\xi X_t} dt|^2 d\xi\right] < \infty. \tag{2.72}$$

If the process is Gaussian its enough to make sure that :

$$\int_{\mathbb{R}} \int_u^U \int_u^U \mathbb{E}\left[e^{i\xi(X_t - X_s)} dt\right] dsdtd\xi < \infty, \tag{2.73}$$

then one can evaluate $\mathbb{E}\left[\exp\left(i\xi\left(X_s - X_t\right)\right)\right]$ explicitly to establish (2.73). It leads to the well-known Gaussian criterion:

Proposition 2.4.2. *[17] Let X be a centered Gaussian stochastic process, X has an occupation density $L = L(x, B, \omega)$ which, for B fixed, is Pa.s. square integrable in x if*

$$\int_{[u,U]^2} \Delta(s, t)^{-\frac{1}{2}} dsdt < \infty, \tag{2.74}$$

where $\Delta(s,t) = \mathbb{E}[(X_s - X_t)^2]$.

In our case, the Rosenblatt process is not Gaussian, so we can't apply the previous proposition and we need other analysis to prove the existence of the local time, which we have done in the next theorem.

Theorem 2.4.1. *The Rosenblatt process has the square integrable local time in each finite interval [0, T].*

Proof. In this proof, we will verify the expression (2.73) for $[u, U] = [0, T]$. By using the self-similarity and the homogeneous increments of the process we have

$$\mathbb{E}\left[e^{i\xi(Z_t^H - Z_s^H)}\right] = \mathbb{E}\left[e^{i\xi Z_{|t-s|}^H}\right] = \mathbb{E}\left[e^{i\xi Z_1^H |t-s|^H}\right].$$

In addition, $Z_1 = I_2(\phi)$, where

$$\phi(x,y) = \phi_H(x,y) = c(H,2)\int_0^1 f_H(s,x,y)ds,$$

and

$$f_H(s,x,y) = (t-x)_+^{\frac{H}{2}-1}(t-y)_+^{\frac{H}{2}-1}.$$

By the fact that we got this bound

$$\begin{aligned} \left|\mathbb{E}\left[e^{i\alpha I_2(f)}\right]\right| &= \left(\prod_{m\geq 1}(1+4\alpha^2\lambda_{k,f}^2)\right)^{-\frac{1}{4}} \\ &\leq \left(1+4\alpha^2\sum_{k\geq 1}\lambda_{k,f}^2 + 16\alpha^4\sum_{j<k}\lambda_{j,f}^2\lambda_{k,f}^2 + 64\alpha^6\sum_{j<k<l}\lambda_{j,f}^2\lambda_{k,f}^2\lambda_{l,f}^2\right)^{-\frac{1}{4}}, \end{aligned} \tag{2.75}$$

where $\lambda_{k,f}, k \geq 1$, be the eigenvalues of the operator A_f which defined by (1.40) we get :

$$\mathbb{E}\left[e^{i\xi Z_1^H|t-s|^H}\right] = \mathbb{E}\left[e^{i\xi I_2(\phi)|t-s|^H}\right] \leq (1+64\xi^6|t-s|^{6H}\lambda_{1,\phi}^2\lambda_{2,\phi}^2\lambda_{3,\phi}^2)^{-\frac{1}{4}}.$$

Below we prove that $rkA_\phi > 2$, whence $\lambda = \lambda_{1,\phi}^2\lambda_{3,\phi}^2\lambda_{3,\phi}^2 > 0$ (recall that the eigen numbers λ, ϕ are ordered according to their absolute values).

Then we get

$$\begin{aligned} \int_{\mathbb{R}}\int_0^T\int_0^T \mathbb{E}\left[e^{i\xi(Z_t^H - Z_s^H)}\right]dsdtd\xi &\leq \int_0^T\int_0^T\int_{\mathbb{R}}(1+64\lambda\xi^6|t-s|^{6H})^{-\frac{1}{4}}d\xi dsdt \\ &= \int_0^T\int_0^T\frac{1}{2}|t-s|^{-H}\int_{\mathbb{R}}(1+\lambda z^6)^{-\frac{1}{4}}dzdsdt < \infty, \end{aligned}$$

and this is what was to be proved.

Proof of the inequality rk $A_\phi > 2$. Assume that rk $A_\phi \leq 2$. Let $f_1(x) = 1_{[0,1]}(x)$, $f_2(x) = 1_{[0,2/3]}(x)$, and $f_3(x) = 1_{[1/3,1]}(x)$. Since rk $A_\phi \leq 2$, there exist numbers α_1, α_2, and α_3, not all being zero, such that $\alpha_1 A_\phi f_1 + \alpha_2 A_\phi f_2 + \alpha_3 A_\phi f_3 = 0$. Let $f = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$. Then

$$\begin{aligned} 0 &= (A_\phi f, f) = c(H,2) \int_{\mathbb{R}^2} \int_0^1 (s-x)_+^{\frac{H}{2}-1} (s-y)_+^{\frac{H}{2}-1} f(x) f(y) ds dx dy \\ &= c(H,2) \int_0^1 \left(\int_{\mathbb{R}} (s-x)_+^{\frac{H}{2}-1} f(x) dx \right)^2 ds, \end{aligned}$$

whence $a(s) = \displaystyle\int_{\mathbb{R}} (s-x)_+^{\frac{H}{2}-1} f(x) dx = 0$ for almost all $x \in [0,1]$ (in fact, for all $x \in [0,1]$, since a is continuous as a linear combination of continuous functions). For $s \in [0, \frac{1}{3}]$, we have $a(s) = \frac{2(\alpha_1+\alpha_2)s^{\frac{H}{2}}}{H}$, whence $\alpha_1 + \alpha_2 = 0$. If $s \in (\frac{1}{3}, \frac{2}{3})$, then(according to what we proved above) $a(s) = \frac{2\alpha_3 (s-\frac{1}{3})^{\frac{H}{2}}}{H}$, whence $\alpha_3 = 0$. Finally, $a(s) = \frac{2\alpha_1 (s-\frac{2}{3})^{\frac{H}{2}}}{H}$ for $s \in [\frac{2}{3}, 1]$, whence $\alpha_1 = \alpha_2 = 0$. □

Remark 2.4.1. *The bound (2.75) is so important such that the existence of the local time of the Rosenblatt depend on it as we saw in the previous proof. That bound is direct consequence of the remark (1.3.2) since the Rosenblatt process can be approximated by (1.37), and for more details see [24].*

The following proposition is the key for the next lemmas and theorems on the local time of the Rosenblatt process.

Proposition 2.4.3. *[17] Let $n \in \mathbb{N}$ and $0 \leq \eta < \frac{1-H}{2H}$. Then, for any times $0 \leq u < U$, the Rosenblatt process satisfies*

$$\int_{[u,U]^n} \int_{\mathbb{R}^n} \prod_{j=1}^{n} |\xi_j|^\eta |\mathbb{E} exp\left(i \sum_{j=1}^{n} \xi_j Z_{t_j} \right) | d\xi dt \leq C^n n^{2nH(1+\eta)} (U-u)^{(1-H(1+\eta))n}, \quad (2.76)$$

where the constant $C > 0$ depends only on H and η. This proposition can be applied to obtain the next Hölder condition on $L(x, B)$.

Theorem 2.4.2. *[17] Let $(Z_t)_{t\geq 0}$ be a Rosenblatt process with $H \in (\frac{1}{2}, 1)$. The local time $(x,t) \longrightarrow L(x,[0,t])$ is almost surely jointly continuous and has finite moments. For a finite closed interval $I \subset (0,\infty)$, let $L^*(I) = \sup_{x\in\mathbb{R}} L(x,I)$. There exist constants C_1 and C_2 such that, almost surely,*

$$\lim\sup_{r\longrightarrow 0} \frac{L^*([s-r, s+r])}{r^{1-H}(log \quad log \quad r^{-1})^{2H}} \leq C_1, \quad (2.77)$$

or any $s \in I$ and

$$\lim_{r \longrightarrow 0} \sup \sup_{s \in I} \frac{L^*([s-r, s+r])}{r^{1-H}(log \quad r^{-1})^{2H}} \leq C_1. \tag{2.78}$$

In particular, the local time $L(x, I)$ is well defined for any fixed x and interval $I \subset (0, \infty)$.

Corollary 2.4.1. *[17] For any finite closed interval $I \subset (0, \infty)$ there exists constants C_1 and C_2, independent of x and t, such that for almost surely, for every $t \in I$ and every $x \in \mathbb{R}$*

$$\lim_{r \longrightarrow 0} \sup \frac{L(x, [t-r, t+r])}{r^{1-H}(log \quad log \quad r^{-1})^{2H}} \leq C_1, \tag{2.79}$$

or any $x \in \mathbb{R}$ and

$$\lim_{r \longrightarrow 0} \sup \sup_{t \in I} \frac{L(x, [t-r, t+r])}{r^{1-H}(log \quad r^{-1})^{2H}} \leq C_1. \tag{2.80}$$

The next result is on the behavior of the trajectories of Z.

Corollary 2.4.2. *[17] Let $I \subset (0, \infty)$ be a finite closed interval. There exists a constant $C > 0$ such that for every $s \in I$ we have, almost surely,*

$$\lim_{r \longrightarrow 0} \inf \sup_{s-r<t<s+r} \frac{|Z_t - Z_s|}{r^H(log \quad log \quad r^{-1})^{-2H}} \geq C, \tag{2.81}$$

and

$$\lim_{r \longrightarrow 0} \inf \inf_{t \in I} \sup_{s-r<t<s+r} \frac{|Z_t - Z_s|}{r^H(log \quad log \quad r^{-1})^{-2H}} \geq C. \tag{2.82}$$

In particular, Z is almost surely nowhere differentiable.

2.4.1 Joint continuity of the local times and moment estimates

Definition 2.4.2. *the local time $L(x,t) := L(x, [0,t])$ for the Rosenblatt process Z exists and admits the representation*

$$L(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^t e^{i\xi(x - Z_s)} ds d\xi. \tag{2.83}$$

Theorem 2.4.3. *[17] The local time $L(x,t)$ is Hölder-continuous both in time and space, such that for every $0 \leq s < t$ and $x \in \mathbb{R}$,*

$$\mathbb{E}|L(x,t) - L(x,s)|^n \leq c^n n^{n2H} |t-s|^{(1-H)n}. \tag{2.84}$$

Moreover, for any $0 \leq \gamma < \frac{1-H}{2H}$ and $y \in \mathbb{R}$, we have

$$|\mathbb{E}(L(x+y,[s,t]) - L(x,[s,t]))^n| \leq c^n n^{n2H(1+)}|t-s|^{(1-H-\gamma H)n}|y|^{\gamma n}. \tag{2.85}$$

In both inequalities the constant c depends only on γ and H.

Corollary 2.4.3. *[17] The local time $L(x,t)$ is jointly Hölder continuous in t and x for Almost surely. This corollary is a modification of the previous Theorem, where we will shift the process in the x-direction by the value Z_a, such that for all $s < t$ and $a > 0$ satisfy $a \leq s$ or $a \leq t$. Then*

$$\mathbb{E}|L(x+Z_a,t) - L(x+Z_a,s)|^n \leq c^n n^{n2H}|t-s|^{(1-H)n}. \tag{2.86}$$

Moreover, for any $0 \leq \gamma < \frac{H^{-1}-1}{2}$,

$$|E(L(x+y+Z_a,[s,t]) - L(x+Z_a,[s,t]))^n| \leq c^n n^{n2H(1+)}|t-s|^{(1-H-\gamma H)n}|y|^{\gamma n}. \tag{2.87}$$

In both cases the constant $c > 0$ depends only on γ and H.

Corollary 2.4.4. *[17] The moment bounds obtained above translate into the following tail estimates:*

1. *For any finite closed interval $I \subset (0,\infty)$,*

$$\mathbb{P}(L(x,I) \geq |I|^{1-H}u^{2H}) \leq C_1 e^{-c_1 u}, \tag{2.88}$$

and

$$\mathbb{P}(|L(x,I) - L(y,I)| \geq |I|^{1-H-\gamma H}|x-y|^{\gamma}u^{2H(1+\gamma)}) \leq C_2 e^{-c_2 u}. \tag{2.89}$$

2. *For $I = [a, a+r]$ or $I = [a-r, a]$, we have*

$$\mathbb{P}(L(x+Z_a,I) \geq r^{1-H}u^{2H}) \leq C_1 e^{-c_1 u}, \tag{2.90}$$

and

$$\mathbb{P}(|L(x+Z_a,I) - L(y+Z_a,I)| \geq r^{1-H-\gamma H}|x-y|^{\gamma}u^{2H(1+\gamma)}) \leq C_2 e^{-c_2 u}. \tag{2.91}$$

Chapter 3

Stochastic differential equations driven by the Rosenblatt process

3.1 Strongly continuous semi-groups and their generators

Definition 3.1.1. *A map $T(.) : \mathbb{R}^+ \longrightarrow \mathcal{B}(X)$ is called a strongly continuous operator semi-group or just C_0-semi-group if it satisfies*

1. $T(0) = I$ *and* $T(t+s) = T(t)T(s)$ *for all* $t, s \in \mathbb{R}^+$ *(the semi-group property),*
2. *for each* $x \in X$ *the orbit* $T(.)x : \mathbb{R}_{\geq 0} \longrightarrow X; t \longmapsto T(t)x$ *is continuous (the strong continuity).*

   The generator A of $T(.)$ which is an operator is given by

1. $D(A) = \{x \in X | \lim\limits_{t \longrightarrow 0, t \in \mathbb{R}^*} \frac{1}{t}(T(t)x - x) \quad exists\},$
2. $Ax = \lim\limits_{t \longrightarrow 0, \quad t \in \mathbb{R}^*} \frac{1}{t}(T(t)x - x) \quad for \quad x \in D(A).$

Remark 3.1.1.

1. *X is non-zero complex Banach space and $\mathcal{B}(X)$ denote by the space of all bounded linear maps $T : X \longrightarrow X$, where $\mathcal{B}(X, X) = \mathcal{B}(X)$.*
2. *I denote the identity operator.*

Remark 3.1.2.

1. *Let A generate a C_0-semi-group. Then its domain $D(A)$ is a linear subspace and A is a linear map.*

2. *Let $T(.) : \mathbb{R}^+ \longrightarrow \mathcal{B}(X)$ be a semi-group. Then we have*

   (a) $T(t)T(s) = T(t+s) = T(s+t) = T(s)T(t)$.

   (b) $T(nt) = T(\sum_{j=1}^{n} t) = \prod_{j=1}^{n} T(t) = T(t)^n$.

Definition 3.1.2. *Let A be a linear operator on X with domain $D(A)$ and let $x \in D(A)$. A function $u : \mathbb{R}^+ \longrightarrow X$ solves the homogeneous evolution equation (or Cauchy problem)*

$$u'(t) = Au(t), \quad t \geq 0, \quad u(0) = x, \tag{3.1}$$

if u belongs to $\mathscr{C}^1(\mathbb{R}^+, X)$ and satisfies $u(t) \in D(A)$ and (3.1) for all $t \geq 0$.

Proposition 3.1.1. *[23] Let A generate the C_0-semi-group $T(.)$ and $x \in D(A)$. Then $T(t)x$ belongs to $D(A)$ and $T(.)x$ belongs to $\mathscr{C}^1(\mathbb{R}^+, X)$, and we have*

$$\frac{d}{dt}T(t)x = AT(t)(x) = T(t)Ax \quad for \quad all \quad t \geq 0.$$

Moreover, the function $u = T(.)x$ is the only solution of (3.1).

Definition 3.1.3. *The operator A is called closed if for every sequence (x_n) in $D(A)$ possessing the limits*

$$\lim_{n \longrightarrow \infty} x_n = x \qquad \lim_{n \longrightarrow \infty} Ax_n = y,$$

we obtain

$$x \in D(A) \quad and \quad Ax = y.$$

Proposition 3.1.2. *[23] We define the resolvent set of a closed operator A by*

$$\rho(A) = \{\lambda \in \mathbb{C} | \lambda I - A : D(A) \longrightarrow X \quad is \quad bijective\}.$$

If $\lambda \in \rho(A)$, we note $\mathcal{R}(\lambda, A)$ for $(\lambda I - A)^{-1}$ and call it resolvent. The spectrum of A is the set

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

The point spectrum

$$\sigma_p(A) = \{\lambda \in \mathbb{C} | \exists v \in D(A) \setminus \{0\} \quad with \quad Av = \lambda v\},$$

is a subset of $\sigma(A)$ which can be empty if $\dim X = \infty$.

Remark 3.1.3. *For a linear operator A in X the following assertions hold.*

1. *The operator A is closed if and only if its graph $Gr(A) = \{(x, Ax)|x \in D(A)\}$ is closed in $X \times X$ (endowed with the product metric) and if and only if $D(A)$ is a Banach space with respect to the graph norm $\|.\|_A$.*
2. *If A is closed with $D(A) = X$, then A is continuous (closed graph theorem).*
3. *Let A be injective. Set $D(A^{-1}) = \mathcal{R}(A) = \{Ax|x \in D(A)\}$. Then A is closed if and only if A^{-1} is closed.*

Definition 3.1.4. *Let $\omega \in \mathbb{R}$. A ω-contraction semi-group is a C_0-semi-group $T(.)$ satisfying $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$. Such a semi-group is also said to be quasi-contractive. If $\omega = 0$, we call $T(.)$ a contraction semi-group.*

3.2 Mild solution and extrapolation

Definition 3.2.1. *Let A generate the C_0-semi-group $T(.)$, $u_0 \in X$, and $f \in \mathscr{C}(J, X)$ where $J \subseteq \mathbb{R}$ satisfy*

$$\int_0^\delta \|f(s)\|ds < \infty \quad for \quad some \quad \delta \in J \setminus \{0\}.$$

The function $u \in C(J', X)$ given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad t \in J'$$

is called mild solution (on $J' = J \cup \{0\}$) of the equation

$$u'(t) = Au(t) + f(t), \quad t \in J, \quad u(0) = u_0. \tag{3.2}$$

Definition 3.2.2. *Let A be a closed operator, $u_0 \in X$, $0 \in J$, and $f \in \mathscr{C}(J, X)$. A function $u \in \mathscr{C}(J, X)$ is called an integrated solution (on J) of (3.2) if the integral $\int_0^t u_s ds$ belongs to $D(A)$ and satisfies*

$$u(t) = u_0 + A\int_0^t u(s)ds + \int_0^t f(s)ds, \tag{3.3}$$

for all $t \in J$.

3.3 Analytic semi-groups and sectorial operators

Definition 3.3.1. *Let $\phi \in (0, \pi]$. We write $\Sigma_\phi = \{\lambda \in \mathbb{C}\backslash\{0\} || \arg\lambda |< \phi\}$ for the open sector with (half) opening angle ϕ. Observe that $\Sigma_{\pi/2} = \mathbb{C}_+$is the open right halfplane and $\Sigma_\pi = \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ is the plane with cut $\mathbb{R}_{\leq 0}$. A closed operator A is called sectorial of type (K, ϕ) if for some constants $\phi \in (0, \pi]$ and $K > 0$ the sector Σ_ϕ belongs to $\rho(A)$ and the resolvent satisfies the inequality*

$$\|R(\lambda, A)\| \leq \frac{K}{|\lambda|} \quad for \quad all \quad \lambda \in \sum_\phi, \tag{3.4}$$

the supremum $\varphi(A) = \varphi \in (0, \pi]$ of all such ϕ is called the angle of A.

Theorem 3.3.1. *[23] Let A be sectorial of type (K, φ) with $\varphi > \frac{\pi}{2}$, $t > 0$, $\theta_0 \in (\frac{\pi}{2}, \phi)$, $\theta \in [\theta_0, \phi]$, $r > 0$ and $\Gamma = \Gamma(r, \theta)$ be defined below. Then the integral*

$$e^{tA} = \lim_{R \longrightarrow \infty} \frac{1}{2\pi i} \int_{\Gamma_R} e^{t\lambda} R(\lambda, A) d\lambda, \tag{3.5}$$

with

$$\Gamma_1 = \Gamma_1(r, \theta) = \{\lambda = \gamma_1(s) = -se^{-i\theta} | s \in (-\infty, -r]\},$$

$$\Gamma_2 = \Gamma_2(r, \theta) = \{\lambda = \gamma_2(s) = re^{i\alpha} | \alpha \in (-\theta, \theta]\},$$

$$\Gamma_3 = \Gamma_3(r, \theta) = \{\lambda = \gamma_3(s) = se^{i\theta} | s \in (r, \infty]\},$$

$$\Gamma = \Gamma(r, \theta) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \quad \Gamma_R = \Gamma \cap \overline{B}(0, R), \quad 0 < r < R,$$

where $\gamma : J \subseteq \mathbb{R} \longrightarrow Y$ piecewise $\mathscr{C}^1$ (Y in a Banach space).

Theorem 3.3.2. *[23] Let A be sectorial of angle $\varphi > \frac{\pi}{2}$. Define e^{tA} as in (3.5) for $t > 0$, and set $e^{0A} = I$. Then the following assertions hold.*

1. *$e^{tA}e^{sA} = e^{sA}e^{tA} = e^{(t+s)A}$ for all $t, s \geq 0$.*
2. *The map $t \longrightarrow e^{tA}$ belongs to $\mathscr{C}^1(\mathbb{R}_+, \mathcal{B}(X))$. Moreover, $e^{tA}X \subseteq D(A)$, $\dfrac{d}{dt}e^{tA} = Ae^{tA}$ and $\|Ae^{tA}\| \leq C/t$ for a constant $C > 0$ and all $t > 0$. We also have $Ae^{tA}x = e^{tA}Ax$ for all $x \in D(A)$ and $t \geq 0$.*
3. *Let $D(A)$ be dense. Then $(e^{tA})_{t \geq 0}$ is a C_0-semi-group generated by A.*

Definition 3.3.2. *A densely defined linear operator T from one topological vector space, X, to another one, Y is a linear operator that is defined on a dense linear subspace* $\mathrm{dom}(T)$ *of X and takes values in Y, written $T : \mathrm{dom}(T) \subseteq X \to Y$. Sometimes this is abbreviated as $T : X \to Y$ when the context makes it clear that X might not be the set-theoretic domain of T.*

Definition 3.3.3. *Let $\vartheta \in (0, \pi/2]$. An analytic C_0-semigroup on Σ_ϑ is a family of operators $\{T(z)|z \in \Sigma_\vartheta \cup 0\}$ such that*

1. *$T(0) = I$ and $T(w)T(z) = T(w + z$ for all $z, w \in \Sigma_\vartheta$.*
2. *The map $T : \Sigma_\vartheta \longrightarrow \mathcal{B}(X); z \longrightarrow T(z)$, is (complex) differentiable.*
3. *$T(z)x \longrightarrow x$ in X as $z \longrightarrow 0$ in $\Sigma_{\vartheta'}$ for all $x \in X$ and each $\vartheta' \in (0, \vartheta)$.*

The generator of $T(.)$ is defined as the generator of the C_0-semigroup $(T(t))_{t\geq 0}$, and its angle $\psi \in (0, \pi/2]$ is the supremum of possible ϑ. If $\|T(z)\|$ is bounded for all $z \in \Sigma_{\psi'}$ and each $\psi' \in (0, \psi)$, the analytic C_0-semigroup is called bounded.

Theorem 3.3.3. *[23] Let $x \in X$, $b > 0$, $f \in \mathscr{C}([0, b], X)$ and $A - \omega I$ be densely defined and sectorial of angle $\varphi > \frac{\pi}{2}$ for some $\omega \in \mathbb{R}$. We study the inhomogeneous evolution equation*

$$u'(t) = Au(t) + f(t), \quad t \in [0, b] = J, \quad u(0) = x. \tag{3.6}$$

This equation has the mild solution

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds = T(t)x + v(t), \quad t \in [0, b], \tag{3.7}$$

where A generates the analytic C_0-semi-group $T(.)$.

Remark 3.3.1. *If A generates an analytic semigroup, then the inhomogeneous problem exhibits better regularity properties than in the general case. The mild solution is 'almost' differentiable in X for continuous inhomogeneities f, and one needs very little extra regularity of f to obtain the differentiability of the solution.*

Theorem 3.3.4. *[23] Let $x \in X$, $b > 0$, $f \in \mathscr{C}([0, b], X)$, and $A - \omega I$ be densely defined sectorial of angle $\varphi > \frac{\pi}{2}$ for some $\omega \in \mathbb{R}$. Then the mild solution u of (3.6) satisfies the following assertions*

1. *We have $u \in \mathscr{C}^\beta([\varepsilon, b], X)$ for all $\beta \in (0, 1)$ and $\varepsilon \in (0, b)$. If also $x \in D(A)$, we can even take $\varepsilon = 0$ here.*

2. *If* $f \in \mathscr{C}^\alpha([0,b],X)$ *for some* $\alpha \in (0,1)$, *then* u *solves* (3.5) *on* $(0,b]$. *If also* $x \in D(A)$, *then* u *solves* (3.6) *on* $[0,b]$.

3.4 Dissipative operators

Definition 3.4.1. *The duality set J(x) of a vector* $x \in X$ *is defined by*

$$J(x) = \{x^* \in X^* | \langle x, x^* \rangle = \|x\|^2, \quad \|x\| = \|x^*\|\},$$

where $\langle x, x^* \rangle = x^*(x)$ *for all* $x \in X$ *and* $x^* \in X^*$.

Definition 3.4.2. *A linear operator* A *is called dissipative if for each vector* $x \in D(A)$ *there is a functional* $x^* \in J(x)$ *such that* $Re\langle Ax, x^* \rangle \leq 0$. *The operator* A *is called accretive if* $-A$ *is dissipative.*

Proposition 3.4.1. *[23] A linear operator* A *is dissipative if and only if it satisfies* $\|\lambda x - Ax\| \geq \lambda \|x\|$ *for all* $\lambda > 0$ *and* $x \in D(A)$. *If* A *generates a contraction semi-group, then we have* $Re\langle Ax, x^* \rangle \leq 0$ *for every* $x \in D(A)$ *and all* $x^* \in J(x)$.

Definition 3.4.3. *Let* X *and* Y *Banach. We say that the linear operator* $A : D(A) \subseteq X \longrightarrow Y$ *admits a closure if there's a linear operator* $B : D(B) \subseteq X \longrightarrow Y$ *such that* $D(A) \subseteq D(B)$, $B|_{D(A)} = A$ *and* $\mathcal{G}(B) = \overline{\mathcal{G}(A)}$, *where* $\mathcal{G}(Z)$ *is the graph of* Z.

Proposition 3.4.2. *[23] Let* A *be dissipative. The following assertions hold.*

1. *Let* $\lambda > 0$. *Then the operator* $\lambda I - A$ *is injective and for* $y \in \mathcal{R}, (\lambda I - A) = (\lambda I - A)(D(A))$ *we have* $\|(\lambda I - A)^{-1} y\| \leq \frac{1}{\lambda}\|y\|$.
2. *Let* $\lambda_0 I - A$ *be surjective for some* $\lambda_0 > 0$. *Then* A *is closed,* $(0, \infty) \subseteq \rho(A)$, *and* $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$ *for all* $\lambda > 0$.
3. *Let* $D(A)$ *be dense in* X. *Then* A *is closable and* $\overline{A}$ *is also dissipative.*

Theorem 3.4.1. *[23] Let* A *be a linear and densely defined operator. The following assertions hold.*

1. *Let* A *be dissipative and* $\lambda_0 > 0$ *such that* $\lambda_0 I - A$ *has dense range. Then* $\bar{A}$ *generates a contraction semigroup.*
2. *Let* A *be dissipative and* $\lambda_0 > 0$ *such that* $\lambda_0 I - A$ *is surjective. Then* A *generates a contraction semigroup.*

3. Let A generate a contraction semigroup. Then A is dissipative, $\mathbb{C}_+ \subseteq \rho(A)$, and $\|R(\lambda, A)\| \leq 1/\operatorname{Re}(\lambda)$ for $\lambda \in \mathbb{C}_+$.

One can replace 'contraction' by 'ω-contraction' and A by $A - \omega I$ for $\omega \in \mathbb{R}$. Operators satisfying the assumptions in assertion b) are called maximally dissipative or m-dissipative.

Now, we finely reached the main goal and objective of this work, which is the stochastic analysis for certain classes of stochastic differential equations, namely the dissipative systems.

3.5 Dissipative Stochastic Evolution Equations Driven by the Rosenblatt process

Definition 3.5.1. *The stochastic evolution equations driven by the Rosenblatt process which is not Gaussian in the Hilbert space $\mathbb{H}$ is defined by the following equation*

$$\begin{aligned} du(t) &= [\mathbb{A}u(t) + F(u(t))]dt + dZ(t) \\ u(0) &= u_0, \end{aligned} \tag{3.8}$$

where $\mathbb{A}$ and F satisfy some dissipativity condition on $\mathbb{H}$ and Z is a general $\mathbb{H}$-valued Rosenblatt process that satisfies some specific conditions on the covariance operator.

Assumption 1.1.

The operator $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{H} \longrightarrow \mathbb{H}$ is associated with a form $(\mathfrak{a}, \mathbb{V})$ that is densely defined, coercive and continuous; the operator $\mathbb{A}$ generates a strongly continuous, analytic semigroup $(S(t))_{t\geq 0}$ on the Hilbert space $\mathbb{H}$ that is uniformly exponentially stable: there exist $M \geq 1$ and $\omega > 0$ such that $\|S(t)\|_{L(\mathbb{H})} \leq Me^{-\omega t}$ for all $t \geq 0$.

Assumption 1.2.

F is an m-dissipative mapping with $\mathbb{V} \subset D(F)$ and $F : \mathbb{V} \longrightarrow \mathbb{H}$ is continuous with polynomial growth.

Let us introduce the class of noises that we are concerned with. We define the mean of a $\mathbb{H}$ valued process $(Z_t)_{t\in[0,T]}$ by $m_Z : [0, T] \longrightarrow \mathbb{H}, \quad m_Z(t) = \mathbb{E}(Z_t)$ and the covariance $C_Z : [0, T]^2 \longrightarrow L_1(\mathbb{H})$ by

$$\langle C_Z(t, s)u, v\rangle_{\mathbb{H}} = \mathbb{E}[\langle Z_t - m_Z(t), v\rangle_{\mathbb{H}} \langle Z_s - m_Z(s), u\rangle_{\mathbb{H}}],$$

for every $s, t \in [0, T]$ *and for every* $u, v \in \mathbb{H}$.

Let Q *be a nuclear self-adjoint operator on* $\mathbb{H}$*(i.e.* $Q \in L_1(\mathbb{H})$ *and* $Q = Q^* > 0$*). It is well-known that* Q *admits a sequence* $(\lambda_j)_{j\geq 1}$ *of eigenvalues such that* $0 < \lambda_j \downarrow 0$ *and* $\sum_{j\geq 1} \lambda_j < \infty$*. Moreover, the eigenvectors* $(e_j)_{j\geq 1}$ *of* Q *form an orthonormal basis of* $\mathbb{H}$*. Let* $(x(t))_{t\in[0,T]}$ *be a centered square integrable one-dimensional process with a given covariance* R*. We define its infinite dimensional counterpart by*

$$Z_t = \sum_{j=1}^{\infty} \sqrt{\lambda_j} x_j(t) e_j \quad t \in [0, T],$$

where x_j *are independent copies of* x*. It is trivial to see that the above series is convergent in* $L^2(\Omega; \mathbb{H})$ *for every fixed* $t \in [0, T]$ *and*

$$\mathbb{E}\|Z_t\|^2_{\mathbb{H}} = (TrQ)R(t, t).$$

Remark 3.5.1. *The process* Z_t *is a* $\mathbb{H}$*-valued centered process with covariance* $R(t, s)Q$*.*

Assumption 1.3.

We will assume that the covariance of the process Z_t *satisfies the following condition*

$$(s, t) \longrightarrow \frac{\partial^2 R}{\partial s \partial t} \in L^1([0, T]^2). \tag{3.9}$$

Assumption 1.4.

Let Z_t *be given in the form*

$$Z_t = \sum_{j\geq 1} \sqrt{\lambda_j} x_j(t) e_j,$$

where λ_j*,* e_j *and* $x_j(t)$ *have been defined above. Suppose that the covariance* R *of the process* $(Z_t)_{t\in[0,T]}$ *satisfies the following condition:*

$$\left|\frac{\partial^2 R}{\partial s \partial t}(s, t)\right| \leq c_1 |t - s|^{2H-2} + g(s, t),$$

for every $s, t \in [0, T]$*, where* $|g(s, t)| \leq c_2 (st)^{\beta}$ *with* $\beta \in (-1, 0)$*,* $H \in (\frac{1}{2}, 1)$ *and* c_1*,* c_2 *are strictly positive constant.*

Remark 3.5.2.

The previous assumptions on the operator $\mathbb{A}$ *,* F *and the driven process are necessary conditions in order to have a mild solution for the equation (3.8).*

3.5.1 The Stochastic Convolution Process

Definition 3.5.2. *The stochastic convolution process is the weak solution of the linear stochastic evolution equation*

$$dY(t) = \mathbb{A}Y(t)dt + dZ(t),$$

and it is given by

$$W_{\mathbb{A}}(t) = \int_0^t S(t-s)dZ_s. \tag{3.10}$$

Proposition 3.5.1. *[2] Assume that the covariance function R satisfies (3.9). Then, for every $t \in [0,T]$, the stochastic convolution given by (3.10) exists in $L^2([0,T];\mathbb{H})$ and it is $(\mathcal{F}_t)_{t\geq 0}$ adapted.*

Proposition 3.5.2. *[2] Suppose that Z_t satisfies Assumption 1.4 and fix $\alpha \in (0,H)$. Let $W_{\mathbb{A}}$ be given by (3.10). Then for every $\gamma < \alpha$ and $\varepsilon < \alpha - \gamma$ it holds that*

$$W_{\mathbb{A}} \in \mathscr{C}^{\alpha-\gamma-\varepsilon}([0,T]; D((-\mathbb{A})^{\gamma})).$$

In particular for any fixed $t \in [0,T]$ the random variable $W_{\mathbb{A}}(t)$ belongs to $D(-\mathbb{A})^{\gamma})$.

Proposition 3.5.3. *[2] Fix $\alpha \in (0, H \wedge (\beta+1))$. Then the process $W_{\mathbb{A}}(.)$ has α Hölder continuous paths.*

3.5.2 Existence and Uniqueness of the Solution

Let $L^2_{\mathcal{F}}(\Omega;\mathscr{C}([0,T];\mathbb{H}))$ denote the Banach space of all $\mathcal{F}_t$-measurable, pathwise continuous processes, taking values in $\mathbb{H}$, endowed with the norm

$$\|X\|_{L^2_{\mathcal{F}}(\Omega;\mathscr{C}([0,T];\mathbb{H}))} = \left(\mathbb{E}\sup_{t\in[0,T]} \|X(t)\|^2_{\mathbb{H}}\right)^{\frac{1}{2}},$$

while $L^2_{\mathcal{F}}(\Omega; L^2([0,T];\mathbb{V}))$ denotes the Banach space of all mappings $X : [0,T] \longrightarrow \mathbb{V}$ such that $X(t)$ is $\mathcal{F}_t$-measurable, endowed with the norm

$$\|X\|_{L^2_{\mathcal{F}}(\Omega;L^2([0,T];\mathbb{V}))} = \left(\mathbb{E}\int_0^T \|X(t)\|^2_{\mathbb{V}}dt\right)^{\frac{1}{2}}.$$

Theorem 3.5.1. *[2] Let $u_0 \in D(F)$ (resp. $u_0 \in \mathbb{H}$). Then there exists a unique mild (resp. generalized) solution to the equation (3.8) which is an $\mathbb{H}$-valued continuous and adapted process*

$$u \in L^2_{\mathcal{F}}(\Omega; \mathscr{C}([0,T]; \mathbb{H})) \cap L^2_{\mathcal{F}}(\Omega; L^2([0,T]; \mathbb{V})),$$

which depends continuously on the initial condition:

$$\mathbb{E}|u(t;u_0) - u(t;u_1)|^2_{\mathbb{H}} \leq C|u_0 - u_1|^2_{\mathbb{H}}. \tag{3.11}$$

Definition 3.5.3. *A process $u \in L^2_{\mathcal{F}}(\Omega; \mathscr{C}([0,T]; \mathbb{H})) \cap L^2_{\mathcal{F}}(\Omega; L^2([0,T]; \mathbb{V}))$ is a solution to equation 3.8 if it satisfies $\mathbb{P}$-a.s. the integral equation*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + W_{\mathbb{A}}(t), \quad t \in [0,T]. \tag{3.12}$$

The proof of this theorems is very classical. By setting $y(t) = u(t) - W_{\mathbb{A}}(t)$, this reduces to a problem interduced in the next definition. In order to prove the existence and the uniqueness of the mild solution for equation(3.8) we will prove the existence and the uniqueness of the mild solution for this problem. This strategy of proof was used in [7] section 5.5.4 .

Definition 3.5.4. *Let us consider the following evolution equation*

$$\begin{aligned} y^{'}(t) &= \mathbb{A}y(t) + F(z(t) + y(t)) \quad t \geq 0 \\ y(0) &= u_0, \end{aligned} \tag{3.13}$$

where $\mathbb{A}$ and F satisfy the dissipativity condition on $\mathbb{H}$ stated in Assumptions 1.1 and 1.2 and z is a trajectory of the stochastic convolution process, which satisfies the regularity conditions stated in Theorem (3.5.2)

$$z \in \mathscr{C}^{\alpha-\gamma-\varepsilon}([0,T]; D(-A)^{\gamma}).$$

Remark 3.5.3.

1. *The key point in the following construction is the observation that $\mathbb{V} = D((-A)^{\frac{1}{2}})$, compare Remark 3.5.6 . Further, in this case we impose the following bound: $\frac{1}{2} < H$. Therefore, we can and do assume that*

$$z \in \mathscr{C}^{H-\frac{1}{2}-\varepsilon}([0,T]; D(-A)^{\frac{1}{2}}),$$

for arbitrary $\varepsilon > 0$.

2. *the assumption on F implies that $F : \mathbb{V} \longrightarrow \mathbb{H}$ is continuous, hence the process $(F(z(t)))_{t\in[0,T]}$ is continuous and satisfies $\sup\limits_{t\in[0,T]} \|F(z(t))\|_{\mathbb{H}} < +\infty$.*

Definition 3.5.5.

1. *The resolvent $J_\lambda : X \longrightarrow X$ of a maximal monotone operator A is defined by $J_\lambda x = x_\lambda$, where x_λ is the unique solution to*

$$0 \in J(x_\lambda x) + \lambda A x_\lambda.$$

2. *The Yosida approximation $A_\lambda : X \longrightarrow 2^{X^*}$ is given by*

$$A_\lambda x = \frac{1}{\lambda} J(x - J_\lambda x), \quad \lambda > 0, \quad x \in X$$

where X^* is the dual space of Banach space X.

Definition 3.5.6. *Let us consider the following approximation of the evolution equation (3.13)*

$$\begin{aligned} y^{'}(t) &= \mathbb{A} y_\alpha(t) + F_\alpha(z(t) + y_\alpha(t)) \quad t \geq 0 \\ y(0) &= u_0, \end{aligned} \tag{3.14}$$

where F_α Yosida approximations of F. It is known that F_α are Lipschitz continuous, dissipative mappings such that, for all $u \in \mathbb{V}$, it holds $F_\pm(u) \longrightarrow F(u)$ in $\mathbb{H}$, as $\alpha \longrightarrow 0$.

Lemme 3.1. *[15] Let $T > 0$ and g be a positive bounded measurable function on $[0, T]$. Suppose that $a \geq 0, b \geq 0$ are constants, such that for all $t \in [0; T]$, we have*

$$g(t) \leq a + b \int_0^t g(s) ds,$$

so we get $g(t) \leq a \exp(bt)$ for all $t \in [0, T]$.

Lemme 3.2. *Let $x \in \mathbb{H}$. Then, for any $\alpha > 0$ there exists a unique mild solution $y_\alpha(t, x)$ to equation (3.14) such that*

$$y_\alpha \in \mathscr{C}([0, T]; \mathbb{H}) \cap L^2([0, T]; \mathbb{V}).$$

Proof. Since F_α are Lipschitz continuous, the existence of the solution to (3.14) is standard. It remains to prove the existence of an estimate that is uniform in α. By the assumptions on $\mathbb{A}$ there exists $\omega > 0$ such that $\langle \mathbb{A}u, u\rangle \leq -\omega\|u\|^2_{\mathbb{V}}$, using the dissipativity of F we have

$$\begin{aligned}
\tfrac{1}{2}\|y_\alpha(t)\|^2_{\mathbb{H}} &= \tfrac{1}{2}\|u_0\|^2_{\mathbb{H}} + \int_0^t \langle \mathbb{A}y_\alpha(s), y_\alpha(s)\rangle_{\mathbb{H}} ds + \int_0^t \langle F_\alpha(z(s)+y_\alpha(s)), y_\alpha(s)\rangle_{\mathbb{H}} ds \\
&\leq \tfrac{1}{2}\|u_0\|^2_{\mathbb{H}} - \omega \int_0^t \|y_\alpha(s)\|^2_{\mathbb{V}} ds + \int_0^t \langle F_\alpha(z(s)), y_\alpha(s)\rangle_{\mathbb{H}} ds \\
&\leq \tfrac{1}{2}\|u_0\|^2_{\mathbb{H}} - \omega \int_0^t \|y_\alpha(s)\|^2_{\mathbb{V}} ds + T \sup_{t\in[0,T]} \|F(z(t))\|^2_{\mathbb{H}} + \int_0^1 \|y_\alpha(s)\|^2_{\mathbb{H}} ds,
\end{aligned}$$

which implies, by an application of Gornwall's lemma, that

$$\sup_{t\in[0,T]} \left(\frac{1}{2}\|y_\alpha(t)\|^2_{\mathbb{H}} + \omega \int_0^t \|y_\alpha(s)\|^2_{\mathbb{V}} ds \right) \leq C(T, u_0, z),$$

with the term on the right-hand side is independent of α.

□

Lemme 3.3. *For every $\alpha > 0$, $u_0, u_1 \in \mathbb{H}$, it holds*

$$\sup_{t\in[0,T]} \|y^{u_0}_\alpha(t) - y^{u_1}_\alpha(t)\|^2_{\mathbb{H}} \leq C\|u_0 - u_1\|^2_{\mathbb{H}}. \tag{3.15}$$

Proof. Let us consider the difference $y^{u_0}_\alpha(t) - y^{u_1}_\alpha(t)$, for $x, \overline{x} \in H$:

$$\frac{d}{dt}[y^{u_0}_\alpha(t) - y^{u_1}_\alpha(t)] = \mathbb{A}[y^{u_0}_\alpha(t) - y^{u_1}_\alpha(t)] + [F_\alpha(z(t) + y^{u_0}_\alpha(t)) - F_\alpha(z(t) + y^{u_1}_\alpha(t))],$$

hence

$$\begin{aligned}
\|y^{u_0}_\alpha(t) - y^{u_1}_\alpha(t)\|^2_{\mathbb{H}} &= \|u_0 - u_1\|^2_{\mathbb{H}} + 2\int_0^t \langle \mathbb{A}(y^{u_0}_\alpha(s) - y^{u_1}_\alpha(s)), y^{u_0}_\alpha(s) - y^{u_1}_\alpha(s)\rangle ds \\
&+ 2\int_0^t \langle F_\alpha(y^{u_0}_\alpha(s)) - F_\alpha(y^{u_1}_\alpha(s)), y^{u_0}_\alpha(s) - y^{u_1}_\alpha(s)\rangle ds,
\end{aligned}$$

and therefore

$$\|y^{u_0}_\alpha(t) - y^{u_1}_\alpha(t)\|^2_{\mathbb{H}} \leq \|u_0 - u_1\|^2_{\mathbb{H}} - 2\omega \int_0^t \|y^{u_0}_\alpha(s) - y^{u_1}_\alpha(s)\|^2_{\mathbb{H}} ds.$$

Applying Gronwall's lemma we obtain

$$\|y^{u_0}_\alpha(t) - y^{u_1}_\alpha(t)\|^2_{\mathbb{H}} \leq e^{-2\omega t}\|u_0 - u_1\|^2_{\mathbb{H}}.$$

□

Lemme 3.4. *[2] The sequence $(y_\alpha)_{\alpha>0}$ is a Cauchy sequence in $\mathscr{C}([0,T];\mathbb{H}) \cap L^2([0,T];\mathbb{V})$.*

Theorem 3.5.2. *For any $z \in \mathscr{C}([0,T];\mathbb{V})$ there exists a unique solution $(y(t))_{t\in[0,T]}$ to equation (3.13)*

$$y \in \mathscr{C}([0,T];\mathbb{V}) \cap L^2([0,T];\mathbb{H}),$$

and it depends continuously on the initial condition $u_0 \in \mathbb{H}$.

Proof. Since y_α is a Cauchy sequence in $\mathscr{C}([0,T];\mathbb{H}) \cap L^2([0,T];\mathbb{V})$ it converges to a unique function y in the same space; it remains to show that $(y(t))_{t\in[0,T]}$ actually solves (3.13). Also, the continuous dependence on the initial condition follows from the same property proved for the approximating functions y_α, since the estimate in (3.15) does not depend on α and it is conserved at the limit. By the claimed convergence of y_α, since J_α is a sequence of continuous mapping that converges to the identity, it holds that $J_\alpha(y_\alpha(s)) \longrightarrow y(s) \in \mathbb{V}$ a.s. on $[0,T]$. Therefore, by the continuity of F, it follows that

$$F_\alpha(z(s) + y_\alpha(s)) \longrightarrow F(z(s) + y(s)) \in \mathbb{H} \quad a.s. \quad on \quad [0,T].$$

Now we use Vitali's theorem (the Uniform Integrability Convergence Theorem, compare [[22], Theorem 9.1.6]), to conclude that

$$\int_0^t S(t-s) F_\alpha(z(s) + y_\alpha(s)) ds \longrightarrow \int_0^t S(t-s) F(z(s) + y(s)) ds.$$

□

3.5.3 A Network Model for a Neuronal Cell

In this part, we will introduce an application for what has been done in the previous subsection, so this part is considered as an extension to the previous one. In 2009 S. Bonaccorsi C. and A. Tudor in their paper [2] investigated a mathematical model of a complete neuron which was subject to stochastic perturbations. Thier model was based on the deterministic one for the whole neuronal network that has been recently introduced in [25]. They treated the neuron as a simple graph with different kinds of (stochastic) evolutions on the edges and with a dynamic Kirchhoff-type condition on the central node (the soma) and schematized a neuron as a network by considering

1. *a FitzHugh-Nagumo (nonlinear) system on the axon, coupled with*
2. *a (linear) Rall model for the dendritical tree, complemented with*

3. *Kirchhoff-type rule in the soma .*

The Wiener process was and is still the perfect idol used in neurobiological models due to it important and significant properties, but it is considerable to apply different kinds of noises: for example long-range dependence processes or self-similar processes, and this is exactly what S. Bonaccorsi C. and A. Tudor did in their paper which we will introduce in this subsection .

The Abstract Formulation

In the following, as long as we allow for variable coefficients in the diffusion operator, we can let the edges of the neuronal network to be described by the interval $[0,1]$. *The general form of the equation we are concerned with can be written as a system in the space* $\mathbb{H} = (L^2(0,1))^2 \times \mathbb{R} \times L^2(0,1)$ *for the unknowns* (u, u_d, d, v)*:*

$$\begin{aligned}
\frac{\partial}{\partial t}u(t,x) &= \frac{\partial}{\partial x}\left(c(x)\frac{\partial}{\partial x}u(t,x)\right) - p(x)u(t,x) - v(t,x) + \theta(u(t,x)) + \frac{\partial}{\partial t}\xi^u(t,x) \\
\frac{\partial}{\partial t}u_d(t,x) &= \tfrac{\partial}{\partial x}\left(c_d(x)\frac{\partial}{\partial x}u_d(t,x)\right) - p_d(x)u_d(t,x) + \frac{\partial}{\partial t}\xi^d(t,x) \\
\frac{\partial}{\partial t}d(t) &= -\gamma d(t) - \left(c(0)\frac{\partial}{\partial x}u(t,0) - c_d(1)\frac{\partial}{\partial x}u_d(t,1)\right) \\
\frac{\partial}{\partial t}v(t,x) &= u(t,x) - \varepsilon v(t,x) + \frac{\partial}{\partial t}\xi^v(t,x),
\end{aligned} \tag{3.16}$$

under the following continuity, boundary and initial conditions

$$\begin{aligned}
d(t) &= u(t,0) = u_d(t,1), \qquad t \geq 0 \\
\frac{\partial}{\partial x}u(t,1) &= 0, \quad \frac{\partial}{\partial x}u_d(t,0) = 0, \quad t \geq 0 \\
u(0,x) &= u_0(x), \quad v(0,x) = v_0(x), \quad u_d(0,x) = u_{d;0}(x).
\end{aligned} \tag{3.17}$$

We will assume that the coefficient of the system (3.16) satisfy the following assumption

Assumption 1.5

1. *The function* $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies some dissipativity conditions: there exists* $\lambda \geq 0$ *such that for* $h(u) = -\lambda u + \theta(u)$ *it holds*

$$[h(u) - h(v)](u - v) \leq 0 \quad \forall u, v \in \mathbb{R}; \quad |h(u)| \leq c(1 + |u|^{2\rho+1}) \quad \rho \in \mathbb{N}. \tag{3.18}$$

2. $c, c_d, p, p_d \in \mathscr{C}^1([0,1])$ *are continuous, positive functions such that, for some* $C > 0$ *we have*

$$C \leq c(x), \quad c_d(x) \leq \frac{1}{C}, \quad C' \leq p(x) - \lambda \text{ and } \quad p_d(x) \leq \frac{1}{C'}.$$

3. $\gamma > 0$, $\varepsilon > 0$ *are given constants.*

Remark 3.5.4. *The function* $\theta : \mathbb{R} \longrightarrow \mathbb{R}$*, in the classical model of FitzHugh, is given by* $\theta(u) = u(1-u)(u-\xi)$ *for some* $\xi \in (0,1)$*; it satisfies (3.18) with* $\lambda = \frac{1}{3}(\xi^2 - \xi + 1)$.

The main goal is to write the equation (3.16) for this model, which satisfies the conditions in the assumptions 1.5, in an abstract form in the Hilbert space $\mathbb{H} = (L^2(0,1))^2 \times \mathbb{R} \times L^2(0,1)$. *We also introduce the Banach space* $\mathbb{Y} = (\mathscr{C}([0,1]))^2 \times \mathbb{R} \times L^2(0,1)$ *that is continuously (but not compactly) embedded in* $\mathbb{H}$. *In order to solve the abstract problem , we will establish the basic framework. To this aim, we need to prove that the linear part of the system defines a linear, unbounded operator* A *that generates on* X *an analytic semi-group. We shall also study the dissipativity of* $\mathbb{A}$ *and of the nonlinear term* $\mathbb{F}$.

On the domain

$$D(\mathbb{A}) := \left\{ \begin{array}{l} \mathfrak{v} := (u, v, d, u_d)^\top \in (L^2(0,1))^2 \times \mathbb{R} \times L^2(0,1) \quad s.\ th.\ u(0) = u_d(1) = d, \\ u'(1) = 0, \quad u_d'(0) = 0, \quad c(0)u'(0) + c_d(1)u_d'(1) = 0 \end{array} \right\}, \tag{3.19}$$

with the operator $\mathbb{A}$ *is defined by*

$$\mathbb{A}\mathfrak{v} := \begin{pmatrix} (cu')' - pu + \lambda u - v \\ (c_d u_d')' - p_d u_d \\ -\gamma d - (c(0)u'(0) - c_d(1)u_d'(1)) \\ u - \varepsilon v \end{pmatrix}. \tag{3.20}$$

In order to treat the non-linearity in our system, we introduce the Nemitsky operator Θ *on* $L^2(0,1)$ *such that* $\Theta(u)(x) = h(u(x))$ *for all* $u \in \mathscr{C}([0,1]) \subset L^2(0,1)$. *Then we define* $\mathbb{F}$ *on* $\mathbb{H}$ *by setting*

$$\mathbb{F}(\mathfrak{v}) = (\Theta(u), 0, 0, 0)^\top,$$

on the domain

$$D(\mathbb{F}) = \left\{ (u, v, d, u_d)^\top \in \mathbb{H} : u \in \mathscr{C}([0,1]) \right\}. \tag{3.21}$$

Remark 3.5.5. *In the above setting, the function* $\mathbb{F}$ *satisfies the conditions in Assumption 1.2 .*

Finally, setting $Z(t) = (\xi^u(t), \xi^v(t), 0, \xi^d(t))^\top$*, we obtain that the initial value problem associated with (3.16-3.17) can be equivalently formulated as an abstract stochastic Cauchy problem*

$$\begin{cases} \mathrm{d}\mathfrak{v}(t) = [\mathbb{A}\mathfrak{v}(t) + \mathbb{F}(\mathfrak{v}(t))]\mathrm{d}t + \mathrm{d}Z(t), \quad t \geq 0 \\ \mathfrak{v}(0) = \mathfrak{v}_0, \end{cases} \tag{3.22}$$

where the initial value is given by $\mathfrak{v}_0 = (u_0, v_0, u_0(0), u_{d;0})^\top \in \mathbb{H}$.

Theorem 3.5.3. *[2] The proposed model for a neuron cell, endowed with a stochastic input that satisfies the conditions in Assumption 1.4, has a unique solution on the time interval* $[0, T]$*, for arbitrary* $T > 0$*, which belongs to*

$$L^2_{\mathcal{F}}(\Omega; \mathscr{C}([0, T]; \mathbb{H})) \cap L^2_{\mathcal{F}}(\Omega; L^2([0, T]; \mathbb{V})),$$

and depends continuously on the initial condition.

The Well-posedness of the Linear System

The first remark is that, neglecting the recovery variable v*, the linear part of the system for the unknown* (u, u_d, d) *is a diffusion equation on a network with dynamical boundary conditions:*

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} c(x) u(t,x)\right) - p(x)u(t,x) + \lambda u(t,x) \\ \frac{\partial}{\partial t} u_d(t,x) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} c_d(x) u_d(t,x)\right) - p_d(x)u_d(t,x) \\ \frac{\partial}{\partial t} d(t) &= -\gamma d(t) - \left(c(0)\frac{\partial}{\partial x}u(t,0) - c_d(1)\frac{\partial}{\partial x}u_d(t,1)\right). \end{aligned}$$

This kind of systems are already existed in the literature.So let us define $\mathcal{X} = (L^2(0,1))^2 \times \mathbb{R}$ *and present the operator*

$$\mathcal{A}\begin{pmatrix} u \\ u_d \\ d \end{pmatrix} = \begin{pmatrix} (cu')' - pu + \lambda u \\ (c_d u_d')' - p_d u_d \\ -\gamma_1 d - (c(0)u'(0) - c_d(1)u_d'(1)) \end{pmatrix},$$

with coupled domain

$$D(\mathcal{A}) = \left\{ (u, u_d, d)^\top \in \left(L^2(0,1)\right)^2 \times \mathbb{C} : u(0) = u_d(1) = d \right\}.$$

Then, by quoting for instance the papers [19],[20] , we can state the following result

Proposition 3.5.4. *[2] The operator $(\mathcal{A}, D(\mathcal{A}))$ is self-adjoint and dissipative and it has compact resolvent; by the spectral theorem, it generates a strongly continuous, analytic and compact semi-group $(\mathcal{S}(t))_{t\geq 0}$ on the Hilbert space $\mathcal{X}$.*

Next, we will introduce the operator $\mathbb{A}$ on the space $\mathbb{H} = \mathcal{X} \times L^2(0,1)$. We can think $\mathbb{A}$ as a matrix operator in the form

$$\mathbb{A} = \begin{pmatrix} \mathcal{A} & -P_1 \\ P_1^\top & -\varepsilon \end{pmatrix},$$

where P_1 is the immersion on the first coordinate of $\mathcal{X}$: $P_1 v = (v,0,0)^\top$, while $P_1^\top (u, u_d, v)^\top = u$.

In order to prove the generation property of the operator $\mathbb{A}$, we introduce the Hilbert space

$$\mathbb{V} := \left\{ \begin{array}{ll} \mathfrak{v} := (u, u_d, d, v)^\top \in (H^1(0,1))^2 \times \mathbb{R} \times L^2(0,1) & s.\ th. \\ u(0) = u_d(1) = d & \end{array} \right\},$$

and the sesquilinear form $\mathfrak{a} : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ defined by

$$\begin{aligned} \mathfrak{a}\left(\mathfrak{u}^{(1)}, \mathfrak{u}^{(2)}\right) := & \int_0^1 p(x) u^{(1)}(x) \overline{u^{(2)}(x)} + c(x) \left(u^{(1)}\right)'(x) \overline{\left(u^{(2)}\right)'(x)} \mathrm{d}x \\ & + \int_0^1 p_d(x) u_d^{(1)}(x) \overline{u_d^{(2)}(x)} + c_d(x) \left(u_d^{(1)}\right)'(x) \overline{\left(u_d^{(2)}\right)'(x)} \mathrm{d}x \\ & + \int_0^1 u^{(1)}(x) \overline{v^{(2)}(x)} - v^{(1)}(x) \overline{u^{(2)}(x)} + \varepsilon v^{(1)}(x) \overline{v^{(2)}(x)} \mathrm{d}x + \gamma d^{(1)} \overline{d^{(2)}}. \end{aligned}$$

Proposition 3.5.5. *[2] The operator $\mathbb{A}$ generates a strongly continuous, analytic semi-group $(S(t))_{t\geq 0}$ on the Hilbert space $\mathbb{H}$ that is uniformly exponentially stable: there exist $M \geq 1$ and $\omega > 0$ such that $\|S(t)\|_{L(\mathbb{H})} \leq Me^{-\omega t}$ for all $t \geq 0$.*

Notice that the operator $\mathbb{A}$ is not self-adjoint, as the corresponding form $\mathfrak{a}$ is not symmetric; also, since $\mathbb{V}$ is not compactly embedded in $\mathbb{H}$, it is easily seen that the semi-group generated by $\mathbb{A}$ is not compact hence it is not Hilbert-Schmidt. For our purposes, they are of fundamental importance the following observations.

Remark 3.5.6. *The form domain $\mathbb{V}$ is isometric to the fractional domain power $D\left((-A)^{1/2}\right)$.*

Remark 3.5.7. *The form $\mathfrak{a}$ is real-valued and coercive, hence*

$$\langle -\mathbb{A}u, u \rangle = \mathfrak{a}(u,u) \geq \omega \|u\|_{\mathbb{V}}^2,$$

for some $\omega > 0$.

Conclusion

The main objective of this memoir was to study a dissipative systems driven by the Rosenblatt processes and to give certain conditions and assumptions to insure the existence and uniqueness of the mild solution. In addition to that, we introduced an application; a network model for a neuronal cell which is a mathematical model of a complete neuron which can be modeled by the previous system.

As in many researches, difficulties was encountered especially in the stochastic differential equations and application part. The work done in this part was very limited and few authors had an interest with, due to the fact that the process is neither Gaussian nor a semi-martingale. This fact makes it hard for the authors to investigate more and more properties and application for this process.

As a perspective, we would be able to study dynamic systems defined by the backward stochastic differential equations (BSDE's) and the forward-backward stochastic differential equations (FBSDE's), based on the work done on the levy processes as non-Gaussian processes. In our knowledge and until this day, nothing has been done concerning the stochastic differential equations driven by the Rosenblatt processes in the existing literature.

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