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Thème:

An Introduction to Stochastic Fractional Calculus

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Dedication

To you reader, I dedicate this work.

Acknowledgments

الحمد لله

I want to start by thanking my classmates for their encouragement and support throughout this master thesis.

Above all, I would like to thank my supervisor, Dr. S. Idrissi, for the time she has set aside for me, for her seemingly endless patience, and for the wealth of knowledge that she has shared with me during the preparation of this master thesis.

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Contents

Abstract

Over the last recent years, the deterministic fractional calculus has been well known and commonly used. It has become very popular and already attracted many scientists and researchers from around the world. Its main advantage is characterized mostly in modeling several complex phenomena, with the best results, in numerous areas of application in science and engineering.

On the other side, mean-square calculus is a suitable calculus for use when dealing with second-order stochastic processes, and based upon a strong stochastic type-convergence, termed *mean square convergence*, whose main advantage is that the results established in mean square are also valid in other important types of stochastic convergence, namely, *convergence in probability* and *convergence in distribution.*

It seems natural to combine mean square and fractional calculus, given important concepts and basic properties of both, and then to link them to get *fractional mean-square calculus*, that is the main goal of this master thesis: We aim to give an extension for the fractional calculus of deterministic functions to a mean-square setting, by including the development of some theoretical aspects of mean-square fractional calculus.

The work accomplished in this master thesis collects and summarizes the existing works of this framework but is still only the beginning of a topic that will become as valuable as deterministic fractional calculus. More work can be devoted to deciphering the other complex properties of deterministic fragmentary calculus at a mean-square setting.

Key words: Mean-square calculus, Fractional calculus, second-order stochastic processes.

List Of Notations And Symbols & Acronyms

Throughout this thesis we will use the following terminology and notation:

$\mathbb{N}$	=	$\{1, 2, 3, \dots\}$
$\mathbb{N}_0$	=	$\{0, 1, 2, \dots\}$
$\mathbb{R}$	=	The set of all real numbers
$[a, b]$	=	The set of all real numbers between a and b
$\beta > 0$	=	All positive real values of β
$\beta \in [a, b]$	=	All real values of β between a and b
$\triangleq$	=	By definition, equals to

List of Acronyms

a.s.	=	Almost surely
IBP	=	Integration by parts
iff	=	If, and only if
m.s.	=	Mean-square
psd	=	Power spectral density
r.m.s	=	Root mean-square
r.v.	=	Random variable
s.p.	=	Stochastic process
w.r.t.	=	With respect to
w.s.s.	=	Wide-sense stationary

Introduction

Fractional Calculus is considered a branch of mathematical analysis which deals with integrals and derivatives of arbitrary order. Therefore, fractional calculus is an extension of the integer-order calculus that considers integrals and derivatives of any real or complex order. The first note about this idea of differentiation, for non-integer numbers, dates back to 1695, with a famous correspondence between Leibniz and L'Hôpital, about the possibility of the order n in the notation $d^n f/dx^n$, for the n^{th} derivative of the function f, to be a non-integer. Since then, several mathematicians investigated this approach, like Lacroix, Fourier, Liouville, Riemann, Letnikov, Grünwald, Caputo, and contributed to the grown development of this field. Currently, this is one of the most intensively developing areas of mathematical analysis as a result of its numerous applications such as finance, viscoelasticity, electromagnetism, signal processing, control theory, and the biomedical field.

On the other hand, many applications involve passing a random process through a system, either dynamic (i.e., one with memory that is described by a differential equation) or one without memory. In the case of dynamic systems, we must deal with derivatives and integrals of stochastic processes. Hence, we need a stochastic calculus, a calculus specialized to deal with random processes. One might ask if "ordinary" calculus can be applied to the sample functions of a random process. The answer is yes, so, developing a calculus associated with a class of stochastic processes in terms of a system of probability distribution has become an indispensable necessity.

In this master thesis we shall study first a calculus relevant to this approach, namely, the calculus in mean square or m.s. calculus. This approach is based upon a strong stochastic type-convergence, termed mean square convergence, whose main advantage is that the results established in mean square are also valid in other important types of stochastic convergence, namely, convergence in probability and convergence in distribution. Additionally, the reason to choose m.s. setting is that important information about a stochastic process can be found from its first and second moments. Moreover this calculus is important for several practical reasons:

- First of all, its importance lies in the fact that simple yet powerful and well-developed methods exist.

- Secondly, the development of m.s. calculus and its application to physical problems follows in broad outline the same steps used in considering calculus of ordinary (deterministic) functions.

- Furthermore, the m.s. calculus is attractive because it is defined in terms of distributions and moments which are our chief concern. Another important reason for studying m.s. calculus is that, for the important case of Gaussian processes, m.s. properties lead to properties on the sample function level.

We aim to do this by translating the basic deterministic fractional calculus and properties to a mean square (m.s.) setting. That m.s. calculus is a well-developed subject with methods that follow, in a general way, those of ordinary calculus only makes it more attractive. There is currently a small body of work dedicated to m.s. fractional calculus (see Hafiz[3], Hafiz[4] and El-Sayed[1]).

In Chapter 1, we will give some preliminary definitions, results and notations from mean square calculus. We begin by looking at the notion of second order random variable, various concepts of convergence for the second order random variable, stochastic processes of second order and some of their properties, then the basic theory of mean-square limit, continuity, m. s. differentiation and mean-square integration.

In Chapter 2, we will introduce fractional derivatives and fractional integrals, shortly differintegrals. After a short introduction, we shall give some basic formulas on the basic special functions. Then the two most frequently used differintegrals definitions for a differintegral will be given, namely the Grünwald-Letnikov and the Riemann-Liouville approach. Finally some basic properties of these differintegrals will be given and proved, such as linearity, composition and the Leibniz rule. Thereafter the definitions of the differintegrals will be applied to a few examples.

In Chapter 3, we will introduce several definitions for the mean square fractional integral and derivative based on some of the common definitions from the deterministic fractional calculus, and consider various properties of m.s. fractional integrals and derivatives: such as the m.s. continuity of the integrals and derivatives.

Chapter 1

Mean-Square Calculus

In this chapter we introduce the major "pillars" of the mean-square calculus which are the notions of mean-square limit, mean-square continuity, mean square differentiation and mean-square integration to provide an extension of the sample function stochastic integral and derivative of stochastic processes. For additional reading on the subject, the reader is referred to (Loève, [5]). Examples and proofs of some theorems in the development of m.s. calculus follow (Stark and Woods[11], Soong [10]).

1.1 Preliminary Background

For the sake of clarity, in this section we introduce notations, definitions, and preliminary facts that will be used. Notations and definitions from mean-square calculus, definitions and properties may be found in Soong[[9], chap.4], and in Stark and Woods [[11], chap.10].

Definition 1.1.1. ***Second-order random variable.***
A random variable, X, is called a second-order random variable if its second moment, $\mathbb{E}[X^2]$, is finite.

Remark.

1. The class of all second order r.v.'s on a probability space constitute a linear vector space.
2. Let us use the notation
$$\mathbb{E}[X_1 X_2] = \langle X_1, X_2 \rangle.$$
With $\langle X_1, X_2 \rangle$ satisfies the inner product properties.

3. Define
$$\|X\| = \langle X, X\rangle^{\frac{1}{2}}.$$
It follows from 2. that $\|X\|$ possesses the norm properties.

4. Define the distance between X_1 and X_2, by
$$d(X_1, X_2) = \|X_1 - X_2\|.$$
The distance $d(X_1, X_2)$ possesses the usual distance properties.

Proposition 1.1.1. [9] The linear vector space of second-order random variables with the inner product, the norm, and the distance defined above is called a L_2-space.

We state without proof an important theorem associated with L_2-spaces (for proof, see Loève [5]).

Theorem 1.1.1. [9] **$\mathbf{L_2}$-Completeness**
L_2-spaces are complete in the sense that any Cauchy sequence in L_2, has a unique limit in L_2.

Remark. We thus see that L_2-spaces are complete normed linear spaces (Banach spaces) and complete inner product spaces (Hilbert spaces).

1.1.1 Convergence in Mean-Square

The development of the m.s. calculus is based upon the concept of convergence in mean square or m.s. convergence. This concept shall be explored first.

Convergence of a Sequence of Random Variables

Definition 1.1.2. A sequence of r.v.'s $\{X_n\}$ converges in m.s. to a r.v. X as $n \to +\infty$ if
$$\lim_{n\to+\infty} \|X_n - X\| = 0.$$
This type of convergence is often expressed by
$$X_n \xrightarrow{m.s} X \qquad \text{or} \qquad \underset{n\to+\infty}{l.i.m.}\ X_n = X$$
The symbol "l.i.m." denotes the limit in the mean.

Remark. Other commonly used names are *convergence in quadratic mean* and *second-order convergence.*

Theorem 1.1.2. [9] Let $\{X_n\}$ be a sequence of second-order r.v.'s. If

$$X_n \xrightarrow{m.s} X$$

then

$$\lim_{n\to\infty} \mathbb{E}\{X_n\} \to \mathbb{E}\{X\}$$

In words, "l.i.m." and "expectation" commute.

Theorem 1.1.3. [9] Limits in m.s. convergence are unique, that is, if

$$X_n \xrightarrow{m.s} X \qquad \text{and} \qquad X_n \xrightarrow{m.s} Y,$$

then X and Y are equivalent, that is, $\mathbb{P}(X \neq Y) = 0$.

Convergence in Probability

Definition 1.1.3. A sequence of r.v.'s $\{X_n\}$ converges in probability or converges i. p. to a r.v. X as $n \to \infty$ if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\{|X_n - X| > \epsilon\} = 0.$$

This type of convergence is often expressed by

$$X_n \xrightarrow{i.p} X, \qquad \underset{n\to\infty}{l.i.p} X_n = X,$$

where "l.i.p." reads "limit in probability." Convergence in probability is sometimes called stochastic convergence or convergence in measure.

Theorem 1.1.4. [9] Convergence in mean square implies convergence in probability.

Convergence Almost Surely

Referred to as "strong " convergence in probability theory, convergence almost surely plays an important role in the limit theorems concerning laws large numbers.

Definition 1.1.4. A sequence $\{X_n\}$ of r.v.'s is said to converge almost surely or converge a.s to a r.v. X as $n \to \infty$ if

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = 1.$$

This type of convergence is sometimes called *almost certain convergence*, *convergence almost everywhere*, or *convergence with probability one*. It is often expressed by

$$X_n \xrightarrow{a.s} X$$

Theorem 1.1.5. [9] Convergence almost surely implies convergence in probability.

Convergence in Distribution

Consider a sequence $\{X_n\}$ of r.v.'s which converges to a r.v. X as $n \to +\infty$ in some sense. Since all three modes of convergence deal with probability statements concerning the sequence X_n and X, one may ask if they are related in any way to a condition related to their associated distribution functions in the form

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

at every continuity point of $F_X(x)$.

1.1.2 Second-order stochastic processes

In the development of m.s. calculus, We shall deal exclusively with second-order random variables and stochastic processes.

Definition 1.1.5. ***Second-order stochastic processes.***
Let a stochastic process $\{X(t), t \in T.\}$ $X(t)$ is called a second-order stochastic process (second order s.p.) if, for every set $t_1, t_2, \ldots, t_n$, the r.v's $X(t_1), X(t_2), \ldots, X(t_n)$ are elements of L_2-space.

Remark. A second order s.p. $X(t)$ is characterized by

$$\|X(t)\|^2 = \mathbb{E}\{X^2(t)\} < \infty, \qquad t \in T$$

Let us recapitulate some of the properties of second-order stochastic processes which are useful in what follows. These properties can be deduced directly from that of second-order random variables.

1. If $X(t), t \in T$, is second-order, its expectation $\mathbb{E}[X(t)]$ is finite on T. Consider a new s.p. $Y(t), t \in T$, defined by

$$Y(t) = X(t) - \mathbb{E}\{X(t)\}$$

It follows that $Y(t)$ is also second-order and $\mathbb{E}[Y(t)] = 0$. Hence, there is no loss of generality in assuming that the mean values of second-order s.p.'s are zero.

2. $X(t), t \in T$, is a second-order s.p. if, and only if, its correlation function $\Gamma(t,s)$ exists and is finite on $T \times T$.

Theorem 1.1.6. [9] Assume that n and n' vary over some index set $\mathbb{N}$, and that n_0, and n'_0 are limit points of $\mathbb{N}$. Let X_n, and $X'_{n'}$ be two sequences of second-order r.v.'s. If

$$\underset{n \to n_0}{l.i.m.}\, X_n = X \qquad \text{and} \qquad \underset{n' \to n'_0}{l.i.m.}\, X'_{n'} = X'$$

then

$$\underset{\substack{n \to n_0 \\ n' \to n'_0}}{l.i.m.}\, \mathbb{E}[X_n X'_{n'}] = \mathbb{E}[XX']$$

Theorem 1.1.7. (Convergence in Mean Square Criterion, [9]). Let $\{X_n(t)\}, t \in T$, be a sequence of second-order s.p.'s. $\{X_n(t)\}$ converges to a second-order process $X(t)$ on T if, and only if, the functions $\mathbb{E}[X_n(t)X_{n'}(t)]$ converge to a finite function on T as $n, n' \to n_0$, in any manner whatever. Then

$$\Gamma_{X_nX_n}(t,s) \to \Gamma_{XX}(t,s), \qquad n \to n_0$$

on $T \times T$.

1.2 Mean-Square Continuity

The concept of continuity for random processes relies on the concept of limit for random processes just the same as in the case of ordinary functions. The most useful and tractable concept of continuity turns out to be a mean-square-based definition. This is the concept that we will use almost exclusively.

Definition 1.2.1. A random process $X(t)$ is continuous in the mean-square sense at the point t if

$$\text{as } \varepsilon \to 0, \text{ we have} \qquad \mathbb{E}\left[|X(t+\varepsilon) - X(t)|^2\right] \to 0$$

If the above holds for all t, we say $X(t)$ is mean-square (m.s.) continuous.

Remark. One advantage of this definition is that it is readily expressible in terms of correlation functions. By expanding out the expectation of the square of the difference, it is seen that we just require a certain continuity in the correlation function.

Theorem 1.2.1. (Continuity in Mean Square Criterion, [9]). The random process $X(t)$ is m.s. continuous at t if $\Gamma_{XX}(t_1, t_2)$ is continuous at the point $t_1 = t_2 = t$.

Proof.

Expand the expectation in Definition 1.2.1 to get an expression involving Γ_{XX},

$$\begin{aligned}\mathbb{E}\left[|X(t+\varepsilon) - X(t)|^2\right] &= \Gamma_{XX}(t+\varepsilon, t+\varepsilon) - \Gamma_{XX}(t, t+\varepsilon) \\ &\quad - \Gamma_{XX}(t+\varepsilon, t) + \Gamma_{XX}(t, t)\end{aligned}$$

Clearly the right-hand side goes to zero as $\varepsilon \to 0$ if the two-dimensional function Γ_{XX} is continuous at $t_1 = t_2 = t$. ■

Example 1.2.1. (**Standard Wiener process.**)

We investigate the m.s. continuity of the Wiener process

$$\Gamma_{XX}(t_1, t_2) = \min(t_1, t_2), \quad t_1, t_2 \geq 0$$

The problem thus reduces to whether the function $\min(t_1, t_2)$ is continuous at the point (t, t). The value of the function $\min(t_1, t_2)$ at (t, t) is t, so we consider

$$|\min(t_1, t_2) - t|$$

for $t_1 = t + \varepsilon_1$ and $t_2 = t + \varepsilon_2$,

$$|\min(t + \varepsilon_1, t + \varepsilon_2) - t|$$

But

$$|\min(t + \varepsilon_1, t + \varepsilon_2) - t| \leq \max(\varepsilon_1, \varepsilon_2)$$

so this magnitude can be made arbitrarily small by choice of $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Thus the Wiener process is m.s. continuous.

Definition 1.2.2. If a second-order s.p. $X(t), t \in T$ is m.s. continuous at every $t \in [t_1, t_2] \subset T$, then $X(t)$ is m.s. continuous on the interval $[t_1, t_2]$.

Corollary 1.2.1. [11] A second-order s.p. $X(t), t \in T$, is m.s. continuous on an interval $t \in [t_1, t_2] \subset T$, if, and only if, $\mathit{\Gamma}_{XX}(t, s)$ is continuous at (t, t) for every $t \in [t_1, t_2]$.

Many random processes are stationary or approximately so. In the stationary case, we have seen that we can write the correlation function as a one dimensional function $\mathit{\Gamma}_{XX}(\tau)$.

We next look at a special case of Theorem 1.2.1 for w.s.s. random processes.

Corollary 1.2.2. A wide-sense stationary random process $X(t)$ is m.s. continuous for all t if $\mathit{\Gamma}_{XX}(\tau)$ is continuous at $\tau = 0$.

Proof.

By Theorem 1.2.1, we need continuity on $t_1 = t_2$, but this is the same as $\tau = 0$. Hence, $\mathit{\Gamma}_{XX}(\tau)$ must be continuous at $\tau = 0$. ■

We note that in the stationary case we get m.s. continuity of the random process for all time after verifying only the continuity of a one-dimensional function $\mathit{\Gamma}_{XX}(\tau)$ at the origin, $\tau = 0$.

1.3 Mean-Square Derivative

Continuity is a necessary condition for the existence of the derivative of an ordinary function. However, considering the case where the difference

$$x(t + \varepsilon) - x(t) = O(\sqrt{\varepsilon}), \tag{1.3.1}$$

we see that it is not a sufficient condition for a derivative to exist, even in the ordinary calculus. Similarly, in the mean-square calculus, we find that m.s. continuity is not a sufficient condition for the existence of the m.s. derivative.

Definition 1.3.1. The random process $X(t)$ has a mean-square derivative at t if the mean-square limit of $[X(t + \varepsilon) - X(t)]/\varepsilon$ exists as $\varepsilon \to 0$.

If it exists, we denote this m.s. derivative by X', $X^{(1)}$, dX/dt, or $\dot{X}$. Generally, we do not know X' when we are trying to determine whether it exists, so we turn to the Cauchy convergence criterion. In this case the test becomes

$$\begin{aligned} &\mathbb{E}\left(\left| [X(t + \varepsilon_1) - X(t)]/\varepsilon_1 - [X(t + \varepsilon_2) - X(t)]/\varepsilon_2 \right|^2\right) \\ &\qquad \to 0 \quad \text{as } \varepsilon_1 \text{ and } \varepsilon_2 \to 0 \end{aligned} \tag{1.3.2}$$

As was the case for continuity, we can express this condition in terms of the correlation function, making it easier to apply see[9, 10].

Theorem 1.3.1. A random process $X(t)$ with auto-correlation function $\Gamma_{XX}(t_1, t_2)$ has a m.s. derivative at time t if $\dfrac{\partial^2 \Gamma_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}$ exists at $t_1 = t_2 = t$.

Proof.

Expand the square inside the expectation in Equation 1.3.2 to get three terms, the first and last of which look like

$$\mathbb{E}\left[|\, (X(t+\varepsilon) - X(t))/\varepsilon \,|^2 \right]$$

$$= \left[\Gamma_{XX}(t+\varepsilon, t+\varepsilon) - \Gamma_{XX}(t, t+\varepsilon) - \Gamma_{XX}(t+\varepsilon, t) + \Gamma_{XX}(t,t) \right] / \varepsilon^2$$

which converges to

$$\frac{\partial^2 \Gamma_{XX}(t_1,t_2)}{\partial t_1 \partial t_2},$$

if the second mixed partial derivative exists at the point $(t_1, t_2) = (t, t)$. The middle or cross-term is

$$-2\mathbb{E}(\, [X(t+\varepsilon_1) - X(t)] / \varepsilon_1 \cdot [X(t+\varepsilon_2) - X(t)]^* / \varepsilon_2 \,)$$

$$= -2[\Gamma_{XX}(t+\varepsilon_1, t+\varepsilon_2) - \Gamma_{XX}(t, t+\varepsilon_2) - \Gamma_{XX}(t+\varepsilon_1, t) + \Gamma_{XX}(t,t)] / \varepsilon_1 \varepsilon_2$$

$$= -2(\, [\Gamma_{XX}(t+\varepsilon_1, t+\varepsilon_2) - \Gamma_{XX}(t+\varepsilon_1, t)] / \varepsilon_2 - [\Gamma_{XX}(t, t+\varepsilon_2) - \Gamma_{XX}(t,t)] / \varepsilon_2 \,) / \varepsilon_1 \varepsilon_2$$

$$\to -2 \frac{\partial}{\partial t_1} \left(\frac{\partial \Gamma_{XX}(t_1,t_2)}{\partial t_2} \right) \Bigg|_{(t_1,t_2)=(t,t)}$$

$$= -2 \left(\frac{\partial^2 \Gamma_{XX}(t_1,t_2)}{\partial t_1 \partial t_2} \right) \Bigg|_{(t_1,t_2)=(t,t)}$$

if this second mixed partial derivative exists at the point $(t_1, t_2) = (t, t)$. Combining all three of these terms, we get convergence to

$$2 \left(\frac{\partial^2 \Gamma_{XX}}{\partial t_1 \partial t_2} \right) - 2 \left(\frac{\partial^2 \Gamma_{XX}}{\partial t_1 \partial t_2} \right) = 0$$

■

Example 1.3.1. (**Derivative of Wiener process**)

Let $W(t)$ be a Wiener process with correlation function $\Gamma_{WW}(t_1, t_2) = \sigma^2 \min(t_1, t_2)$ [1]. We enquire about the behavior of $\mathbb{E}\left[|(W(t+\varepsilon) - W(t))/\varepsilon|^2 \right]$ when ε is near zero.

[1] While it is conventional to use σ^2 as the parameter of the Wiener process, please note that σ^2 is not the variance! Earlier we used α for this parameter.

Assuming that ε is positive, we have by calculation that $\mathbb{E}\,[\,|(W(t+\varepsilon)-W(t))|^2 = \sigma^2\varepsilon]$, which shows that the Wiener process is mean-square continuous, as we already found in Example 1.2.1. But now when we divide by ε^2 inside the expectation, as required by $\mathbb{E}\,[\,|(W(t+\varepsilon)-W(t))/\varepsilon|^2]$, we end up with σ^2/ε which goes to infinity as ε approaches zero. So the mean-square derivative of the Wiener process does not exist, at least in an ordinary sense.

Example 1.3.2. (**Gaussian shaped correlation**)

We look at another random process $X(t)$ with mean function $\mu_X = 5$ and correlation function,

$$\Gamma_{XX}(t_1,t_2) = \sigma^2 e^{-\alpha(t_1-t_2)^2} + 25.$$

The first partial with respect to t_2 is

$$\frac{\partial \Gamma_{XX}}{\partial t_2} = 2\alpha(t_1-t_2)\sigma^2 e^{-\alpha(t_1-t_2)^2}.$$

Then, the second mixed partial becomes

$$\frac{\partial^2 \Gamma_{XX}}{\partial t_1 \partial t_2} = 2\alpha\sigma^2[1-2\alpha(t_1-t_2)^2]e^{-\alpha(t_1-t_2)^2},$$

which evaluated at $t_1 = t_2 = t$ becomes

$$\left.\frac{\partial^2 \Gamma_{XX}}{\partial t_1 \partial t_2}\right|_{(t_1,t_2)=(t,t)} = 2\alpha\sigma^2$$

so that in this case the m.s. derivative $X'(t)$ exists for all t.

1.3.1 Properties of Mean Square Derivatives

1. Mean square differentiability of $X(t)$ at $t \in T$ implies m.s. continuity of $X(t)$ at t.

2. The m.s. derivative $X'(t)$ of $X(t)$ at $t \in T$, if it exists, is unique.

3. If $X(t)$ and $Y(t)$ are m.s. differentiable a $t \in T$, then the m.s. derivative of $aX(t)+bY(t)$ exists at t and

   $$\frac{d}{dt}\,[aX(t)+bY(t)] = aX'(t)+bY'(t)$$

   where a and b are constants, moreover

Corollary 1.3.1. The m.s. derivative is a linear operator, that is,

$$\frac{d}{dt}[aX(t) + bY(t)] = aX'(t) + bY'(t). \qquad \text{(m.s.)}$$

Proof.

See [11] ∎

4. If an ordinary function $f(t)$ is differentiable at $t \in T$ and $X(t)$ is m.s. differentiable at $t \in T$, then $f(t)X(t)$ is m.s. differentiable at t and
$$\frac{d}{dt}\left[f(t)X(t)\right] = \frac{df(t)}{dt}X(t) + f(t)\frac{dX(t)}{dt}.$$

1.3.2 Means and Correlation Functions of Mean Square Derivatives

Given the existence of the m.s. derivative, the next question we might be interested in is: What is its probability law? Or more simply: (a) What are its mean and correlation function?; and (b) How is X' correlated with X? To answer (a) and (b), we start by considering the expectation

$$\mathbb{E}[X'(t)] = \mathbb{E}\left[\frac{dX(t)}{dt}\right] = d\left[\frac{\mathbb{E}X(t)}{dt}\right].$$

To calculate the correlation function of the mean-square derivative process, we first define

$$\Gamma_{X'X'}(t_1, t_2) = \mathbb{E}[X'(t_1)X'^*(t_2)],$$

and formally compute, interchanging the order of expectation and differentiation,

$$\begin{aligned}
\Gamma_{X'X'}(t_1, t_2) = \quad & \lim_{m,n\to\infty} \mathbb{E}\left[n\left(X\left(t_1 + \frac{1}{n}\right) - X(t_1)\right) \cdot m\left(X\left(t_2 + \frac{1}{m}\right) - X(t_2)\right)^*\right] \\
&= \lim_{n\to\infty} n\left[\frac{\partial \Gamma_{XX}(t_1 + \frac{1}{n}, t_2)}{\partial t_2} - \frac{\partial \Gamma_{XX}(t_1, t_2)}{\partial t_2}\right] \\
&= \frac{\partial^2 \Gamma_{XX}(t_1, t_2)}{\partial t_1 \partial t_2}
\end{aligned}$$

Thus, we have finally obtained the following theorem.

Theorem 1.3.2. [11] If a random process $X(t)$ with mean function $\mu_X(t)$ and correlation function $\Gamma_{XX}(t_1, t_2)$ has an m.s. derivative $X'(t)$, then the mean and correlation functions of $X'(t)$ are given by

$$\mu_{X'}(t) = \frac{d\mu_X(t)}{dt}$$

and

$$\Gamma_{X'X'}(t_1,t_2) = \frac{\partial^2 \Gamma_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}.$$

Example 1.3.3. (**Mean-square derivative process**)

We now continue to study the m.s. derivative of the process $X(t)$ of Example 1.3.2 by calculating its mean function and its correlation function. We obtain

$$\mu_{X'}(t) = \frac{d\mu_X(t)}{dt} = 0$$

and

$$\begin{aligned}\Gamma_{X'X'}(t_1,t_2) &= \frac{\partial^2 \Gamma_{XX}(t_1,t_2)}{\partial t_1 \partial t_2} \\ &= 2\alpha\sigma^2[1-2\alpha(t_1-t_2)^2]e^{-\alpha(t_1-t_2)^2}.\end{aligned}$$

We note that in the course of calculating $\Gamma_{X'X'}$, we are effectively verifying the existence of the m.s. derivative by noting whether this deterministic partial derivative exists at $t_1 = t_2$.

Remark. The generalized m.s. derivative of the Wiener process is called white Gaussian noise. It is not a random process in the conventional sense since, for example, its mean-square value $\Gamma_{X'X'}(t,t)$ at time t is infinite.

For the special case of a stationary or a wide-sense stationary random process, we get the following:

Theorem 1.3.3. [11] The m.s. derivative of a WSS random process $X(t)$ exists at time t if the auto correlation $\Gamma_{XX}(\tau)$ has derivatives up to order two at $\tau = 0$.

Proof.

By the previous result we need $\left.\frac{\partial^2 \Gamma_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{(t_1,t_2)=(t,t)}$. Now

$$\Gamma_{XX}(\tau) = \Gamma_{XX}(t+\tau,t), \text{ functionally independent of } t,$$

so

$$\left.\frac{\partial \Gamma_{XX}(t_1,t_2)}{\partial t_1}\right|_{(t_1,t_2)=(t+\tau,t)} = \frac{d\Gamma_{XX}(\tau)}{d\tau}$$

since $t_2 = t$ is held constant, and

$$\begin{aligned}\left.\frac{\partial \Gamma_{XX}(t_1,t_2)}{\partial t_2}\right|_{(t_1,t_2)=(t,t-\tau)} &= \frac{\partial \Gamma_{XX}(t,t-\tau)}{\partial -\tau} \\ &= -\frac{d\Gamma_{XX}(\tau)}{d\tau},\end{aligned}$$

since $t_1 = t$ is held constant here; thus

$$\left.\frac{\partial^2 \Gamma_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{(t_1,t_2)=(t+\tau,t)} = -\frac{d^2\Gamma_{XX}(\tau)}{d\tau^2}$$

■

Calculating the second-order properties of X', we have

$$\mathbb{E}[X'(t)] = \mu_{X'}(t) = 0$$

$$\mathbb{E}[X'(t+\tau)X^*(t)] = \Gamma_{X'X}(\tau) = +\frac{d\Gamma_{XX}(\tau)}{d\tau},$$

$$\mathbb{E}[X(t+\tau)X'^*(t)] = \Gamma_{XX'}(\tau) = -\frac{d\Gamma_{XX}(\tau)}{d\tau},$$

and

$$\mathbb{E}[X'(t+\tau)X'^*(t)] = \Gamma_{X'X'}(\tau) = -\frac{d^2\Gamma_{XX}(\tau)}{d\tau^2},$$

which follow from the formulas used in the above proof.

One can also derive these equations directly, for example,

$$\begin{aligned}\mathbb{E}[X(t+\tau)X'^*(t)] &= \lim_{\varepsilon\to 0}\left(\frac{\mathbb{E}[X(t+\tau)X^*(t+\varepsilon) - \mathbb{E}[X(t+\tau)X^*(t)]]}{\varepsilon}\right) \\ &= \lim_{\varepsilon\to 0}\left(\frac{\Gamma_{XX}(\tau+\varepsilon) - \Gamma_{XX}(\tau)]}{\varepsilon}\right) \\ &= -\frac{d\Gamma_{XX}(\tau)}{d\tau} = \Gamma_{XX'}(\tau)\end{aligned}$$

and similarly for $\Gamma_{X'X}(\tau)$.

Example 1.3.4. (Mean-square derivative of W.S.S. process)

Let $X(t)$ have zero-mean and correlation function

$$\Gamma_{XX}(\tau) = \sigma^2 e^{-\alpha^2\tau^2}.$$

Here the m.s. derivative exists because $\Gamma(\tau)$ is infinitely differentiable at $\tau = 0$. Computing the first and second derivatives, we get

$$\frac{d\Gamma_{XX}}{d\tau} = -2\alpha^2\tau\sigma^2 e^{-\alpha^2\tau^2}.$$

Then

$$\Gamma_{X'X'}(\tau) = -\frac{d^2\Gamma_{XX}}{d\tau^2} = 2\alpha^2\sigma^2(1-2\alpha^2\tau^2)e^{-\alpha^2\tau^2}.$$

1.4 Mean-Square Stochastic Integrals

In this section, we will be interested in the mean-square stochastic integral. It is defined as the mean-square limit of its defining sum as the partition of the integration interval gets finer and finer. We first look at the integration of a random process $X(t)$ over a finite interval (T_1, T_2). The operation of the integral is then just a simple "averager". First we create a partition of (T_1, T_2) consisting of n points using $(t_1, t_2, \ldots, t_n)$. Then the approximate integral is the sum

$$I_n \triangleq \sum_{i=1}^{n} X(t_i)\Delta t_i. \tag{1.4.1}$$

On defining the m.s. limit random variable as I, we have the following definition of the mean-square integral.

Definition 1.4.1. The mean-square integral of $X(t)$ over the interval (T_1, T_2) is denoted I. It exists when

$$\lim_{n\to\infty} \mathbb{E}\left[\left|I - \sum_{i=1}^{n} X(t_i)\Delta t_i\right|^2\right] = 0 \tag{1.4.2}$$

We give I the following symbol:

$$I = \int_{T_1}^{T_2} X(t)dt \triangleq \lim_{n\to\infty} I_n \qquad \text{(m.s.)}$$

Because the m.s. limit is a linear operator, it follows that the integral just defined is linear, that is, it obeys

$$\int_{T_1}^{T_2} [aX_1(t) + bX_2(t)]dt = a\int_{T_1}^{T_2} X_1(t)dt + b\int_{T_1}^{T_2} X_2(t)dt$$

whenever the integrals on the right exist.

To study the existence of the mean-square integral, as before we make use of the Cauchy criterion for the existence of a limit. Thus, the integral I exists iff $\lim_{m,n\to\infty} \mathbb{E}[|I_n - I_m|^2] = 0$. If we expand this expression into three terms, we get $\mathbb{E}[|I_n|^2] - 2Re\mathbb{E}[I_n I_m^*] + \mathbb{E}[|I_m|^2]$. Without loss of generality, we concentrate on the cross-term and evaluate

$$\mathbb{E}[I_n I_m^*] = \sum_{i,j} \mathit{\Gamma}_{XX}(t_i, t_j)\Delta t_i \Delta t_j,$$

where the sums range over $1 \le i \le n$ and $1 \le j \le m$. As $m, n \to +\infty$, this converges to the deterministic integral

$$\int_{T_1}^{T_2}\int_{T_1}^{T_2} \mathit{\Gamma}_{XX}(t_1, t_2)dt_1 dt_2, \tag{1.4.3}$$

if this deterministic integral exists. Clearly, the other two terms $\mathbb{E}[|I_n|^2]$ and $\mathbb{E}[|I_m|^2]$ converge to the same double integral. Thus, we see that the m.s. integral exists whenever the double integral of Equation 1.4.3 exists in the sense of the ordinary calculus.

The mean and mean-square (power) of I are directly computed as

$$\mathbb{E}[I] = \mathbb{E}\left[\int_{T_1}^{T_2} X(t)dt\right] = \int_{T_1}^{T_2} \mathbb{E}[X(t)]dt,$$

and

$$\begin{aligned}\mathbb{E}[|I|^2] &= \mathbb{E}\left[\int_{T_1}^{T_2}\int_{T_1}^{T_2} X(t_1)X^*(t_2)dt_1dt_2\right] \\ &= \int_{T_1}^{T_2}\int_{T_1}^{T_2} \Gamma_{XX}(t_1,t_2)dt_1dt_2.\end{aligned} \tag{1.4.4}$$

The variance of I is given as

$$\sigma_I^2 = \int_{T_1}^{T_2}\int_{T_1}^{T_2} K_{XX}(t_1,t_2)dt_1dt_2.$$

Example 1.4.1. (Integral of white noise over $(0,t]$, [11])

Let the random process $X(t)$ be zero mean with covariance function $K_{XX}(\tau) = \sigma^2(\delta(\tau))$ and define the running m.s. integral as

$$Y(t) \triangleq \int_0^t X(s)ds, \quad t \geq 0.$$

For fixed t, $Y(t)$ is a random variable; therefore, when indexed by t, $Y(t)$ is a stochastic process. Its mean is given as $\mathbb{E}[Y(t)] = \int_0^t \mu_X(s)ds$ and equals 0 since $\mu_X = 0$. The covariance is calculated for $t_1 \geq 0$, $t_2 \geq 0$ as

$$\begin{aligned}K_{YY}(t_1,t_2) &= \int_0^{t_1}\int_0^{t_2} K_{XX}(s_1,s_2)ds_1ds_2 \\ &= \sigma^2\int_0^{t_1}\int_0^{t_2} \delta(s_1-s_2)ds_1ds_2 \\ &= \sigma^2\int_0^{t_2} u(t_1-s_2)ds_2 \\ &= \sigma^2\int_0^{\min(t_1,t_2)} ds_2 \\ &= \sigma^2\min(t_1,t_2)\end{aligned}$$

which we recognize as the covariance of the Wiener process. Thus, the m.s. integral of white Gaussian noise is the Wiener process. Note that $Y(t)$ must be Gaussian if $X(t)$ is Gaussian, since $Y(t)$ is the m.s. limit of a weighted sum of samples of $X(t)$.

We can generalize this integral by including a weighting function f multiplying the second-order random process $X(t)$ defined on $[a,b] \subset T$. Let $f(t,u)$ be an ordinary function defined on the same interval for t and Riemann integrable for every $u \in U$. We form the random variable

$$I_n(u) = \sum_{k=1}^{n} f(t_k^{'}, u) X(t_k^{'})(t_k - t_{k-1}), \qquad t_k^{'} \in [t_{k-1} - t_k), k = 0, 1, 2, \ldots, n$$

Since L_2 -space is linear, $I_n(u)$ is an element of the L_2- space. It is a random variable defined for each partition p_n and for each $u \in U$.

Definition 1.4.2. If, for $u \in U$,

$$\underset{\substack{n\to\infty \\ \Delta_n \to 0}}{l.i.m} I_n(u) = I(u)$$

exists for some sequence of subdivisions p_n, the s.p. $I(u), u \in U$, is called the mean square Riemann integral of $f(t,u)X(t)$ over the interval $[a,b]$, and it is denoted by

$$I(u) = \int_a^b f(t,u) X(t) dt \tag{1.4.5}$$

It is independent of the sequence of subdivisions as well as the positions of $t_k^{'} \in [t_{k-1}, t_k)$.

Theorem 1.4.1. (Integration in Mean Square Criterion, [11])

The s.p. $I(u), u \in U$, defined by Eq. 1.4.5 exists if, and only if, the ordinary double Riemann integral

$$\int_a^b \int_a^b f(t,u) f(s,u) \Gamma_{XX}(t,s) dt\, ds$$

exists and is finite.

We note again the desirable features associated with the m.s. calculus. The m.s. Riemann integral properties of a stochastic process are determined by the ordinary Riemann integral properties of its correlation function. For example, on the basis of Theorem 1.4.1, it is shown that the Wiener process and the Poisson process are m.s. integrable over any finite interval. The binary noise and the random telegraph signal are also m.s. integrable.

Example 1.4.2. (**application to linear systems**)

A linear system L has response $h(t,s)$ at time t to an impulse applied at time s. Then for a deterministic function $x(t)$ as input, we have the output $y(t)$ given as

$$y(t) = L\{x(t)\} = \int_{-\infty}^{+\infty} h(t,s) x(s) ds, \tag{1.4.6}$$

whenever the foregoing integral exists. If $x(t)$ is bounded and integrable, one condition for the existence of Equation 1.4.6 would be

$$y(t) = L\{x(t)\} = \int_{-\infty}^{+\infty} |h(t,s)| ds < +\infty \qquad , \text{for all} - \infty < t < +\infty. \tag{1.4.7}$$

We can generalize this integral to an m.s. stochastic integral if the following double integral exists:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(t_1, s_1) h^*(t_2, s_2) \Gamma_{XX}(s_1, s_2)\, ds_1\, ds_2,$$

in the ordinary calculus. A condition for this, in the case where Γ_{XX} is bounded and integrable, is Equation 1.4.7. Given the existence of such a condition, $Y(t)$ exists as an m.s. limit and defines a random process,

$$Y(t) = \int_{-\infty}^{+\infty} h(t,s) X(s) ds,$$

whose mean and covariance functions are

$$\mu_Y(t) = \int_{-\infty}^{+\infty} h(t,s) \mu_X(s) ds,$$

and

$$\Gamma_{YY}(t_1, t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(t_1, s_1) h^*(t_2, s_2) \Gamma_{XX}(s_1, s_2)\, ds_1\, ds_2.$$

1.4.1 Properties of mean-square Riemann Integrals

As in the case of m.s. differentiation, many useful rules in ordinary integration are valid with in the framework of m.s. integration. In what follows, it is understood that the integral sign stands for either a m.s. integral or an ordinary integral depending upon whether the integrand is random or deterministic.

1. Mean square continuity of $X(t)$ on $[a, b]$ implies m.s. Riemann integrability of $X(t)$ on $[a, b]$. We see from Theorem 1.2.1 that, if $X(t)$ is m.s. continuous on $[a, b]$, its correlation function $\Gamma_{XX}(t, s)$ is continuous on $[a, b] \times [a, b]$. Hence, existence theorems on ordinary integrals state that

$$\int_a^b \int_a^b \Gamma_{XX}(t, s)\, dt\, ds$$

exists. It is also finite as $X(t)$ is of second order. Hence, in view of Theorem 1.4.1, property 1 is established.

2. The m.s. integral of $X(t)$ on an interval $[a, b]$, if it exists, is unique. This property follows immediately from the m.s. convergence uniqueness property.

3. If $X(t)$ is m.s. continuous on $[a, b]$, then

$$\left\| \int_a^b X(t)dt \right\| \leq \int_a^b \|X(t)\|dt \leq M(b-a), \tag{1.4.8}$$

where

$$M = \max_{t \in [a,b]} \|X(t)\|.$$

Proof.

It follows from Property 1 above that the first integral in Eq. 1.4.8 exists. The second integral exists because $\|X(t)\|$ is a real-valued, continuous function of $t \in [a, b]$ (Theorem 1.2.1). Now

$$\|I_n\| = \left\| \sum_{k=1}^{n} X(t_k^{'})(t_k - t_{k-1}) \right\| \to \left\| \int_a^b X(t)dt \right\| \tag{1.4.9}$$

as $n \to \infty$ and $\Delta_n \to 0$. We have, from the norm property (triangle inequality)

$$\|I_n\| \leq \sum_{k=1}^{n} \|X(t_k^{'})\|(t_k - t_{k-1}) \to \int_a^b \|X(t)\|dt. \tag{1.4.10}$$

The above sum also gives

$$\sum_{k=1}^{n} \|X(t_k^{'})\|(t_k - t_{k-1}) \leq M \sum_{k=1}^{n} (t_k - t_{k-1}) = M(b-a). \tag{1.4.11}$$

Equations 1.4.9 and 1.4.10 yield the first inequality; Equations 1.4.10 and 1.4.11 give the second. ■

4. If the m.s. integrals of $X(t)$ and $Y(t)$ exist on $[a, c]$, then

$$\int_a^c [\alpha X(t) + \beta Y(t)]dt = \alpha \int_a^c X(t)dt + \beta \int_a^c Y(t)dt,$$

$$\int_a^c X(t)dt = \int_a^b X(t)dt + \int_b^c X(t)dt, \qquad a \leq b \leq c.$$

5. If $X(t)$ is m.s. continuous on $[a, t] \subset T$, then

$$I(t) = \int_a^t X(s)ds$$

is m.s. continuous on T; it is also m.s. differentiable on T with

$$\dot{I}(t) = X(t).$$

We state without proof two important corollaries to the above result. They are, respectively, the m.s. counterparts of the Leibniz rule and integration by parts in ordinary integral calculus.

Corollary 1.4.1. (Leibniz Rule).

If $X(t)$ is m.s. integrable on T and if the ordinary function $f(t,s)$ is continuous on $T \times T$ with a finite first partial derivative $\dfrac{\partial f(t,s)}{\partial t}$, then the m.s. derivative of

$$I(t) = \int_a^t f(t,s)X(s)\,ds$$

exists at all $t \in T$, and

$$\dot{I}(t) = \int_a^t \frac{\partial f(t,s)}{\partial t} X(s)\,ds + f(t,t)X(t).$$

Corollary 1.4.2. (Integration by Parts).

Let $X(t)$ be m.s. differentiable on T and let the ordinary function $f(t,s)$ be continuous on $T \times T$ whose partial derivative $\dfrac{\partial f(t,s)}{\partial t}$ exists. If

$$I(t) = \int_a^t f(t,s)\dot{X}(s)ds, \tag{1.4.12}$$

then

$$I(t) = f(t,s)X(s)\Big|_a^t - \int_a^t \frac{\partial f(t,s)}{\partial s} X(s)ds. \tag{1.4.13}$$

6. Let $f(t,s) \equiv 1$ in Eqs. 1.4.12 and 1.4.13. We have the useful result that, if $\dot{X}(t)$ is m.s. Riemann integrable on T, then

$$X(t) - X(a) = \int_a^t \dot{X}(s)ds, \qquad [a,t] \in T$$

This property is seen to be m.s. counterpart of the fundamental theorem of ordinary calculus. It may be properly called the **fundamental theorem of mean square calculus**.

1.4.2 Means and Correlation Functions of Mean Square Riemann Integrals

Consider first the mean of a m.s. Riemann integral. If

$$I(u) = \int_a^b f(t,u)X(t)dt \tag{1.4.14}$$

exists, then

$$\mathbb{E}[I(u)] = \int_a^b f(t,u)\mathbb{E}[X(t)]\,dt,$$

as, with the aid of Theorem 1.4.1,

$$\begin{aligned}\mathbb{E}[I(u)] &= \mathbb{E}\left[\underset{\substack{n\to\infty \\ \Delta_n\to 0}}{l.i.m}\sum_{k=1}^{n} f(t_k^{'},u)X(t_k^{'})(t_k - t_{k-1})\right] \\ &= \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{l.i.m}\sum_{k=1}^{n} f(t_k^{'},u)\mathbb{E}[X(t_k^{'})](t_k - t_{k-1}) \\ &= \int_a^b f(t,u)\mathbb{E}[X(t)]dt.\end{aligned}$$

The determination of the correlation function of $I(u)$ in terms of that of $X(t)$ is also simple. In fact, if we wrote out the proof of Theorem 1.4.1, it would result that the correlation function $\Gamma_{II}(u,v)$ of $I(u)$ in Eq. 1.4.14 is given by

$$\Gamma_{II}(u,v) = \int_a^b\int_a^b f(t,u)f(s,v)\Gamma_{XX}(t,s)\,dt\,ds. \tag{1.4.15}$$

Appropriate changes should be made when the integration limits a and b are functions of u. If a and b in Eq. 1.4.14 are replaced by, respectively, $a(u)$ and $b(u)$, Eq. 1.4.15 then takes the form

$$\Gamma_{II}(u,v) = \int_{a(v)}^{b(v)}\int_{a(u)}^{b(u)} f(t,u)f(s,v)\Gamma_{XX}(t,s)\,dt\,ds. \tag{1.4.16}$$

Example 1.4.3. Let us consider the "integrated Wiener process", and compute its correlation function. In this case,

$$I(u) = \int_0^u X(t)dt$$

and

$$\Gamma_{XX}(t,s) = 2D\min(t,s)$$

Hence, from Eq. 1.4.16 we have, for $t \geq s \geq 0$,

$$\begin{aligned}\Gamma_{II}(t,s) &= 2D\int_0^s\int_0^t \min(\tau,\tau')\,d\tau\,d\tau' \\ &= 2D\int_0^s(\tau' t - \frac{\tau'^2}{2})\,d\tau' = \frac{Ds^2}{3}(3t-s)\end{aligned}$$

1.4.3 Mean Square Riemann-Stieltjes integrals

In the development of the following section, there will be the need of considering stochastic Riemann-Stieltjes integrals of the types

$$V_1 = \int_a^b f(t)\, dX(t), \tag{1.4.17}$$

and

$$V_2 = \int_a^b X(t)\, df(t). \tag{1.4.18}$$

Riemann-Stieltjes integrals in the mean square sense can be developed parallel to the development of m.s. Riemann integrals. As expected, the existence of the m.s. integrals V_l and V_2 hinges upon the existence of ordinary double Riemann-Stieltjes integrals involving the ordinary function $f(t)$ and the correlation function of $X(t)$. They also possess the formal properties of ordinary Ricmann-Stieltjes integrals. Finally, the means and the correlation functions of these integrals can also be determined in a straightforward manner.

Because of these similarities, we will bypass many of the details and only point out some pertinent features associated with the integrals of the types represented by V_1 and V_2.

Definition 1.4.3. Keeping all the definitions given above, let us form the random variables

$$V_{1n} = \sum_{k=1}^{n} f(t_k^{'})[X(t_k) - X(t_{k-1})]$$

and

$$V_{2n} = \sum_{k=1}^{n} X(t_k^{'})[f(t_k) - f(t_{k-1})].$$

If

$$\underset{\substack{n\to\infty \\ \Delta n\to 0}}{l.i.m.}\ V_{1n} = V_1 \qquad \left[\underset{\substack{n\to\infty \\ \Delta n\to 0}}{l.i.m.}\ V_{2n} = V_2\right]$$

exists for some sequences of subdivisions p_n, the r.v. V_1 $[V_2]$ is called the m.s. Riemann-Stieltjes integral of $f(t)$ $[X(t)]$ on the interval $[a, b]$ with respect to $X(t)$ $[f(t)]$. It is denoted by Eq. (1.4.17) [Eq. (1.4.18)].

Theorem 1.4.2. [9] The m.s. Riemann-Stieltjes integral V_1 $[V_2]$ exists if, and only if, the ordinary double Riemann-Stieltjes integral

$$\int_a^b \int_a^b f(t)f(s)\, dd\Gamma_{XX}(t,s)$$
$$\left[\int_a^b \int_a^b \Gamma_{XX}(t,s)\, df(t)df(s)\right] \tag{1.4.19}$$

exists and is finite.

The proof follows directly from the convergence in mean square criterion after appropriate substitutions.

Remark. The existence of V_l and V_2 is determined by the existence of appropriate ordinary integrals. From the existence theorems of ordinary calculus, the double integral (1.4.19) exists if $\Gamma_{XX}(t,s)$ is of bounded variation on $[a,b]\times[a,b]$ and if $f(t)$ is continuous on $[a,b]$; the double integral (1.4.19) exists if $\Gamma_{XX}(t,s)$ is continuous on $[a,b]\times[a,b]$ and if $f(t)$ is of bounded variation on $[a,b]$.

An important as well as practical property of m.s. Riemann-Stieltjes integrals is contained in the following theorem.

Theorem 1.4.3. If either V_1 or V_2 exists, then both integrals exist, and

$$\int_a^b X(t)df(t) = [f(t)X(t)]_a^b - \int_a^b f(t)\, dX(t) \tag{1.4.20}$$

Equation (1.4.20) shows that integration by parts is valid for m.s. Riemann-Stieltjes integration, a very useful result.

Without going into details, we mention that, as expected, other properties of ordinary Riemann-Stieltjes integrals are also valid in this setting.

If V_1 and V_2 exist, the following formulas for the means and the second moments of V_1 and V_2 can be easily verified.

$$\mathbb{E}[V_1] = \int_a^b f(t)\, d\mathbb{E}[X(t)]$$
$$\mathbb{E}[V_1^2] = \int_a^b \int_a^b f(t)f(s)\, dd\Gamma(t,s)$$
$$\mathbb{E}[V_2] = \int_a^b \mathbb{E}[X(t)]\, df(t)$$
$$\mathbb{E}[V_1^2] = \int_a^b \int_a^b \Gamma(t,s)\, df(t)df(s)$$

1.4.4 Distributions of Mean Square Derivatives and Integrals

This development of the mean square calculus is now fairly complete. Mean square derivatives and integrals of a stochastic process are defined based upon its second order properties, which in turn determine the second-order properties of its m.s. derivatives and integrals.

For practical reasons, however, it is often desirable that we develop a technique for determining the joint probability distributions of the m.s. derivatives or of the m.s. integrals of $X(t)$ when the same information about $X(t)$ is known.

Assuming that the m.s. derivative $\dot{X}(t)$ exists, we shall give a formal relation between the joint characteristic functions of $\dot{X}(t)$ and that of $X(t)$.

Let the n^{th} characteristic function of $\dot{X}(t)$ be denoted by $\phi_{n\dot{X}}(u_1,t_1;u_2,t_2;\ldots;u_n,t_n)$.

Theorem 1.4.4. [10] If $\dot{X}(t)$ exists at all $t\in T$, then for every finite set $t_l,t_2,\ldots,t_n\in T$,

$$\phi_{n\dot{X}}(u_1,t_1;u_2,t_2;\ldots;u_n,t_n)=\lim_{\tau_1,\tau_2,\ldots,\tau_n\to 0}\phi_{2nX}\left(\frac{u_1}{\tau_1},t_1+\tau_1;-\frac{u_1}{\tau_1},t_1;\ldots;\frac{u_n}{\tau_n},t_n+\tau_n;-\frac{u_n}{\tau_n},t_n\right) \tag{1.4.21}$$

for $t_1+\tau_1,t_2+\tau_2,\ldots,t_n+\tau_n\in T$.

We observe that the n^{th} characteristic function of $\dot{X}(t)$ is determined by the $2n^{\text{th}}$ characteristic function of $X(t)$. The limiting operations in Eq. (1.4.21) are difficult to carry out in general. However, it leads to the following important result.

Theorem 1.4.5. [9] If the m.s. derivative $\dot{X}(t)$ of a Gaussian process $X(t)$ exists, then $\dot{X}(t)$ is a Gaussian process.

Turning now to mean square integrals and consider the simple case

$$I(t)=\int_a^b X(\tau)\,d\tau,\qquad [a,t]\subset T$$

.

Theorem 1.4.6. [10] If the m.s. integral $I(t),t\in T$ exists, then for every $t_l,t_2,\ldots,t_m\in T$,

$$\phi_{nI}(u_1,t_1;\ldots;u_m,t_m)$$

$$=\lim_{\substack{n\to\infty\\ \Delta_n\to 0}}\phi_{mnX}(u_1(\tau_1-\tau_0),\tau_1^{'};\ldots;u_1(\tau_n-\tau_{n-1}),\tau_n^{'};$$

$$\ldots;u_m(\tau_1-\tau_0),\tau_1^{'};\ldots;u_m(\tau_n-\tau_{n-1}),\tau_n^{'}),\qquad \tau_j^{'}\in[\tau_{j-1},\tau_j)\subset T.$$

Theorem 1.4.7. [9] If the m.s. integral $I(t)$ of a Gaussian process $X(t)$, defined by

$$I(t) = \int_a^t f(t,\tau)X(\tau)\,d\tau$$

exists, then $I(t), t \in T$ is a Gaussian process.

Remark. Theorems 1.4.5 and 1.4.7 establish the fact that the Gaussianity of a Gaussian stochastic process is preserved under m.s. integration and differentiation. In view of the derivations given above, it is not difficult to see that this property of Gaussian processes holds under all linear transformations.

Chapter 2

Fractional Calculus

In this chapter we will introduce the basic notions of fractional calculus. The area of mathematics that allows non-integer order integrals and derivatives. Since its to begin with appearance within the late 17^{th} century it has ended up well known (particularly among scientists and engineers) because numerous issues are depicted by, and can be fathomed utilizing fractional calculus. Further details on fractional calculus can be found in [6, 7] and references therein.

2.1 Special Functions

We will first discuss some useful mathematical definitions that are inherently tied to fractional calculus and will commonly be encountered. These include the Gamma function, the Beta function and the Mittag-Leffler function.

2.1.1 The Gamma Function

The most basic interpretation of the Gamma function is simply the generalization of the factorial for all real numbers.

Definition 2.1.1.

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad x \in \mathbb{R}^+.$$

The Gamma function has some properties.

$$\begin{aligned} \Gamma(x+1) &= x\Gamma(x), \qquad x \in \mathbb{R}^+. \\ \Gamma(x) &= (x-1)!, \qquad x \in \mathbb{R}^+. \end{aligned}$$

Example 2.1.1.

$$\begin{aligned} \Gamma(1) &= \Gamma(2) = 1. \\ \Gamma(1/2) &= \sqrt{\pi}. \\ \Gamma(n+1/2) &= \frac{\sqrt{\pi}}{2^n}(2n-1)!, \qquad n \in \mathbb{N}. \end{aligned}$$

2.1.2 The Beta Function

Like the Gamma function, the Beta function is defined by a definite integral.

Definition 2.1.2. It's given by:

$$\mathbf{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad x, y \in \mathbb{R}^+$$

The Beta function can also be defined in terms of the Gamma function:

$$\mathbf{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x, y \in \mathbb{R}^+ \tag{2.1.1}$$

2.1.3 The Mittag-Leffler Function

The Mittag-Leffler function named after a Swedish mathematician who defined and studied it in 1903, is a direct generalization of the exponential function.

Definition 2.1.3. The standard definition of the Mittag-Leffler is given by :

$$\mathrm{E}_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}, \quad \alpha > 0$$

The Mittag-Leffler function with two parameters α and β, is defined as:

$$\mathrm{E}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \quad \beta > 0 \quad \alpha > 0 \tag{2.1.2}$$

As a result of the definition given in (2.1.2), the following relations hold:

$$\mathrm{E}_{\alpha,\beta}(x) = \frac{1}{\Gamma(\beta)} + x\mathrm{E}_{\alpha,\alpha+\beta}(x).$$

and

$$\mathrm{E}_{\alpha,\beta}(x) = \beta\mathrm{E}_{\alpha,\beta+1}(x) + \alpha x \frac{d}{dx}\mathrm{E}_{\alpha,\beta+1}(x).$$

Example 2.1.2.

$$\begin{aligned}
\mathrm{E}_{\alpha,\beta}(0) &= 1. \\
\mathrm{E}_{1,1}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \mathbf{e}^x. \\
\mathrm{E}_{1,2}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)} = \frac{1}{x}\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} = \frac{\mathbf{e}^x - 1}{x}.
\end{aligned}$$

2.2 Fractional Derivatives and Integrals

This section is devoted to review the most important definitions of fractional derivatives and Integrals.

2.2.1 Basic definitions of fractional derivatives and Integrals

Grünwald-Letnikov, 1867-1868.

Grünwald-Letnikov derivative or also named Grünwald-Letnikov differintegral is a basic extension of the natural derivative to fractional one. It was introduced by A. Grünwald in 1867, and then by A. Letnikov in 1868. Hence, it is written as

$$f'(x) = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h},$$

Applying this formula again, we can find the second derivative:

$$\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{f'(x) - f'(x-h)}{h}, \\
&= \lim_{h_1\to 0} \frac{\lim_{h_2\to 0} \frac{f(x+h_2)-f(x)}{h_2} - \lim_{h_2\to 0} \frac{f(x-h_1-h_2)-f(x-h_1)}{h_2}}{h_1}.
\end{aligned}$$

By choosing the same value of h, i.e. $h_1 = h_2 = h$, the expression simplifies to

$$f''(x) = \lim_{h\to 0} \frac{f(x-2h) - 2f(x-h) + f(x)}{h^2},$$

For the n^{th} derivative, this procedure can be consolidated into the following summation

$$f^n(x) = D^n f(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x - mh).$$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

This expression can be generalized for non-integer values for n with $\alpha \in \mathbb{R}$ provided that the binomial coefficient be understood as using the Gamma Function as $frac\Gamma(\alpha+1)m!\Gamma(\alpha-m+1)$ in place of the standard factorial. Also, the upper limit of the summation (no longer the integer, n) goes to infinity as $\dfrac{t-a}{h}$ (where t and a are the upper and lower limits of differentiation, respectively).

We are left with the generalized form of the **Grünwald-Letnikov** fractional derivative.

$$_aD^{\alpha}f(x) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{m=0}^{[\frac{x-a}{h}]}(-1)^m\frac{(\alpha-1)!}{m!(\alpha-m+1)!}f(x-mh).$$

For negative α, the process will be integration. Therefore, for integration we write

$$_aD^{-\alpha}f(x) = \lim_{h\to 0}h^{\alpha}\sum_{m=0}^{[\frac{x-a}{h}]}\frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)}f(x-mh),$$

or equivalently,

$$_aD^{-\alpha}f(x) = \lim_{n\to\infty}\left(\frac{n}{x-a}\right)^{\alpha}\sum_{m=0}^{n}\frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)}f\left(x-m\left(\frac{x-a}{n}\right)\right).$$

Riemann-Liouville, 1832-1847.

The Riemann-Liouville operator is still the most frequently used when fractional integration is performed. It is considered a direct generalization of Cauchy's formula.

We begin by introducing a fractional integral of integer order n in the form of Cauchy formula.

$$_aD_x^{-n}f(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1}f(t)\,dt,$$

It will be shown that the above integral can be expressed in terms of n-multiple integral, that is

$$_aD_x^{-n}f(x) = \int_0^x dx_1\int_a^{x_1}dx_2\int_a^{x_2}dx_3\ldots\int_a^{x_{n-1}}f(t)\,dt. \tag{2.2.1}$$

When $n=2$, by using the well-known Dirichlet formula, namely

$$\int_a^b dx\int_a^x f(x,y)dy = \int_a^b dy\int_y^b f(x,y)\,dx, \tag{2.2.2}$$

(2.2.1) becomes

$$\begin{aligned}\int_a^x dx_1\int_a^{x_1}f(t)\,dt &= \int_a^x dt f(t)\int_t^x dx_1\\ &= \int_a^x (x-t)f(t)\,dt.\end{aligned} \tag{2.2.3}$$

This shows that the two-fold integral can be reduced to a single integral with the help of Dirichlet formula. For $n = 3$, the integral in (2.2.1) gives

$$\begin{aligned} {}_aD_x^{-3} f(x) &= \int_a^x dx_1 \int_a^{x_1} dx_2 \int_a^{x_2} f(t)\, dt, \\ &= \int_a^x dx_1 \left[\int_a^{x_1} dx_2 \int_a^{x_2} f(t)\, dt \right]. \end{aligned} \tag{2.2.4}$$

By using the result in (2.2.3), the integrals within big brackets simplify to yield

$${}_aD_x^{-3} f(x) = \int_a^x dx_1 \left[\int_a^{x_1} (x_1 - t) f(t)\, dt \right].$$

If we use (2.2.2), then the above expression reduces to

$${}_aD_x^{-3} f(x) = \int_a^x dt f(t) \int_x^t (x_1 - t) dx_1 = \int_a^x \frac{(x-t)^2}{2!} f(t)\, dt.$$

Continuing this process, we finally obtain

$${}_aD_x^{-n} f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt. \tag{2.2.5}$$

It is evident that the integral in (2.2.5) is meaningful for any number n provided its real part is greater than zero.

Definition 2.2.1. (Riemann-Liouville fractional integrals)

Let $f(x) \in \mathbb{L}(a, b)$, $\alpha > 0$, then

$${}_aI_x^\alpha f(x) = {}_aD_x^{-\alpha} f(x) = I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \tag{2.2.6}$$

and

$${}_xI_b^\alpha f(x) = {}_xD_b^{-\alpha} f(x) = I_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt. \tag{2.2.7}$$

for $x > a$ is called Riemann-Liouville left-sided and right-sided fractional integral, respectively, of order α.

Theorem 2.2.1. Let $f \in \mathbb{L}_1[a, b]$ and $\alpha > 0$. Then, the integral $I_a^\alpha f(x)$ exists for almost every $x \in [a, b]$. Moreover, the function $I_a^\alpha f$ itself is also an element of $\mathbb{L}_1[a, b]$.

Proof.

We write the integral in question as

$$\int_a^x (x-t)^{\alpha-1} f(t) dt = \int_{-\infty}^{+\infty} \phi_1(x-t) \phi_2(t) dt,$$

where

$$\phi_1(u) = \begin{cases} u^{\alpha-1}, & for \quad 0 < u \leq b-a \\ \\ 0, & else \end{cases}$$

and

$$\phi_2(u) = \begin{cases} f(u), & for \quad a < u \leq b \\ \\ 0, & else \end{cases}$$

By construction, $\phi_j \in \mathbb{L}(\mathbb{R})$ for $j \in \{1, 2\}$ and thus by a classical result on Lebesgue integration. ■

Example 2.2.1. If $f(x) = (x-a)^{\beta-1}$, then find the value of ${}_aI_x^\alpha f(x)$.

We have

$${}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1} dt.$$

If we substitute $t = a + y(x-a)$ in the above integral, it reduces to

$$\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}$$

where $\beta > 0$. Thus

$${}_aI_x^\alpha f(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}$$

Having established these fundamental properties of Riemann-Liouville integral operators, we now come to the corresponding differential operators.

Definition 2.2.2. (Riemann-Liouville Fractional Derivative)
Let $(n-1) \leq \alpha < n$. The operator ${}_aD_x^\alpha$, defined by

$${}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt,$$

and

$${}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}} dt.$$

is called the Riemann-Liouville left-sided and right-sided fractional differential operator, respectively, of order α.
For $\alpha = 0$, we set $D^0 := I$, the identity operator.

Caputo Fractional Derivative, 1967

The Caputo fractional derivative is considered to be an alternative definition for Riemann-Liouville definition, it is introduced by the Italian Mathematician Caputo in 1967.

Definition 2.2.3. Let $\alpha > 0$, the Caputo left-sided and right-sided fractional differential operator of order α is given by:

$$ {}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, $$

and

$$ {}_x^C D_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, $$

and

$$ {}_a^C D_x^\alpha f(x) = I_a^{n-\alpha} f^{(n)}(x). $$

Other definitions of fractional derivative

In the recent years, new definitions of fractional derivative have been introduced in the literature. Interesting examples are Marchaud, Hilfer and Canavati fractional derivatives.

Definition 2.2.4. (***Marchaud derivative:1927***) For a function defined on $\mathbb{R}$ and for every $\alpha \in (0,1)$ distinguishing two types of derivatives, respectively from the right and from the left one :

$$ D_+^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} + \int_0^\infty \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt, $$

and

$$ D_-^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} + \int_{-\infty}^0 \frac{f(x) - f(x+t)}{t^{1+\alpha}} dt. $$

These fractional derivatives are well defined when f is a bounded, locally Hölder continuous function in $\mathbb{R}$.

Remark. If we compare the Marchaud derivative with respect to the Riemann-Liouville one, we immediately realize that, in the latter one, the classical derivative operator appears, while, in the first one, it does not. This is one of the key points that Marchaud's definition makes evident. That is, Marchaud derivative avoids applying the classical derivative after an integration in order to define the fractional operator.

Definition 2.2.5. (***Hilfer derivative:2000***) Let $\mu \in (0,1), \nu \in [0,1]$, and $f \in L^1[a,b], a < t < b$. The Hilfer derivative is defined as

$$(D^{\mu,\nu}_{a+}f)(t) = \left(I^{\nu(1-\mu)}_{a+}\frac{d}{dt}\left(I^{(1-\nu)(1-\mu)}_{a+}f\right)\right)(t);$$

$$(D^{\mu,\nu}_{b-}f)(t) = \left(I^{\nu(1-\mu)}_{b-}\frac{d}{dt}\left(I^{(1-\nu)(1-\mu)}_{b-}f\right)\right)(t).$$

Remark. Notice that Hilfer derivatives coincide with Riemann-Liouville derivatives for $\nu = 0$ and with Caputo derivatives for $\nu = 1$.

Definition 2.2.6. (***Canavati derivative:2009***) Let $n-1 < \alpha < n, f \in \mathcal{C}^{\alpha}([a,b])$. Then, the Canavati derivative of order α is defined as

$${}^{Can}_{a}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(n-\alpha)} + \frac{d}{dt}\int_a^t \frac{f^{(n-1)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau.$$

2.2.2 The basic properties of fractional operator

Representation

Lemma 2.2.1. [7]

- The Riemann Liouville fractional derivative is equivalent to the composition of the same operator ($(n-\alpha)$-fold integration and $n-th$ ordre differentiation) but in reverse ordre i.e

$${}_{a}D^{\alpha}_{x}f(x) = D^n I^{n-\alpha}_{a}f(x)$$

- Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $f(x)$ be such that ${}^{C}D^{\alpha}_{a}f(x)$ exists. Then,

$${}^{C}_{a}D^{\alpha}_{x}f(x) = I^{n-\alpha}_{a}D^n f(x).$$

Proposition 2.2.1. In general the two operators, Riemann-Liouville and Caputo, do not coincide, i.e,

$${}_{a}D^{\alpha}_{x}f(x) \neq {}^{C}_{a}D^{\alpha}_{x}f(x)$$

Proof.

The well-known Taylor series expansion about the point 0 is

$$\begin{aligned} f(x) &= f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \frac{x^3}{3!}f^{(3)}(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{x^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1} \\ R_{n-1} &= \int_0^x \frac{f^{(n)}(s)(x-s)^{n-1}}{(n-1)!} ds = \frac{1}{\Gamma(n)} \int_0^x f^{(n)}(s)(x-s)^{n-1} ds \\ &= I^n f^{(n)}(x). \end{aligned}$$

Using the linearity property of R-L and representation property of Caputo

$$ {}_a^C D_x^\alpha f(x) = I^{n-\alpha} D^n f(x).$$

and

$$\begin{aligned} {}_a D_x^\alpha f(x) &= {}_a D_x^\alpha \left(\sum_{k=0}^{n-1} \frac{x^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1} \right) \\ &= \sum_{k=0}^{n-1} \frac{{}_a D_x^\alpha x^k}{\Gamma(k+1)} f^{(k)}(0) + {}_a D_x^\alpha R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} f^{(k)}(0) + {}_a D_x^\alpha I^n f^{(n)}(x) \\ &= \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} f^{(k)}(0) + I^{n-\alpha} f^{(n)}(x) \\ &= \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} f^{(k)}(0) + {}_a^C D_x^\alpha f(x). \end{aligned}$$

This means that

$$ {}_a D_x^\alpha f(x) \neq {}_a^C D_x^\alpha f(x)$$

■

Proposition 2.2.2. The relation between the Riemann-liouville and Caputo fractional derivatives is given by:

$$ {}_a^C D_x^\alpha f(x) = {}_a D_x^\alpha \left(f(x) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right).$$

Proof.

The proof result of Proposition 2.2.1 is

$$ {}_a D_x^\alpha f(x) = \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} f^{(k)}(0) + {}_a^C D_x^\alpha f(x)$$

This means that

$$ {}_a^C D_x^\alpha f(x) = {}_a D^\alpha \left(f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f^{(k)}(0) \right). $$

∎

Interpolation

Lemma 2.2.2. • Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $f(t)$ be such that $D^\alpha f(t)$ exists. Then the following properties for the R-L operator hold:

$$ \begin{aligned} \lim_{\alpha \longrightarrow n} D^\alpha f(t) &= f^{(n)}(t), \\ \lim_{\alpha \longrightarrow n-1} D^\alpha f(t) &= f^{(n-1)}(t). \end{aligned} $$

• Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $f(t)$ be such that ${}^C D^\alpha f(t)$ exists. Then the following properties for the Caputo operator hold:

$$ \begin{aligned} \overset{C}{\lim_{\alpha \longrightarrow n}} D^\alpha f(t) &= f^{(n)}(t), \\ \overset{C}{\lim_{\alpha \longrightarrow n-1}} D^\alpha f(t) &= f^{(n-1)}(t) - f^{(n-1)}(0). \end{aligned} $$

Proof.

The proof uses integration by parts.

$$ \begin{aligned} {}^c D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds \\ &= \frac{1}{\Gamma(n-\alpha)} \left(-f^{(n)}(s) \frac{(t-s)^{n-\alpha}}{n-\alpha} \Bigg|_{s=0}^{t} - \int_0^t -f^{(n-1)}(s) \frac{(t-s)^{n-\alpha}}{n-\alpha} ds \right) \\ &= \frac{1}{\Gamma(n-\alpha+1)} \left(f^{(n)}(0) t^{n-\alpha} + \int_0^t f^{(n+1)}(s)(t-s)^{n-\alpha} ds \right). \end{aligned} $$

Now, by taking the limit for $\alpha \longrightarrow n$ and $\alpha \longrightarrow n-1$, respectively, it follows

$$ \overset{C}{\lim_{\alpha \longrightarrow n}} D^\alpha f(t) = \left(f^{(n)}(0) + f^{(n)}(s) \right) \Bigg|_{s=0}^{t} = f^{(n)}(t) $$

and

$$ \begin{aligned} \overset{C}{\lim_{\alpha \longrightarrow n-1}} D^\alpha f(t) &= \left(f^{(n)}(0) + f^{(n)}(s)(t-s) \right) \Bigg|_{s=0}^{t} - \int_0^t -f^{(n)}(s) ds \\ &= f^{(n-1)}(s) \Bigg|_{s=0}^{t} \\ &= f^{(n-1)}(t) - f^{(n-1)}(0). \end{aligned} $$

For the Riemann-Liouville fractional differential operator the corresponding interpolation property reads

$$\lim_{\alpha \longrightarrow n} D^{\alpha} f(t) = f^{(n)}(t),$$
$$\lim_{\alpha \longrightarrow n-1} D^{\alpha} f(t) = f^{(n-1)}(t).$$

■

Non-commutation

Lemma 2.2.3. • Let $n-1 < \alpha < n$, $m, n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and the function $f(x)$ is such that ${}_aD_x^{\alpha} f(x)$ exists. Then, in general, Riemann Liouville operator is also non-commutative and satisfies

$$D^m \left({}_aD_x^{\alpha} f(x) \right) = {}_aD_x^{\alpha+m} f(x) \neq {}_aD_x^{\alpha} \left(D^m f(x) \right)$$

- Let $n-1 < \alpha < n$, $m, n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and the function $f(x)$ is such that ${}_a^CD_x^{\alpha} f(x)$ exists. Then in general

$${}_a^CD_x^{\alpha} \left(D^m f(x) \right) = {}_a^CD_x^{\alpha+m} f(x) \neq D^m \left({}_a^CD_x^{\alpha} f(x) \right)$$

Proof.

Let $\alpha = \dfrac{1}{2}$, $f(x) = 1$, and $m = 1$. using the definition of D_x^{α},

$$D_x^{\frac{1}{2}} D^1(1) = D^{\frac{1}{2}}(0) = 0,$$
$$D_x^{\frac{3}{2}}(1) = -\frac{1}{2\sqrt{(\pi)}} x^{-\frac{3}{2}},$$
$$D_x^{\frac{1}{2}} D^1(1) = 0 \neq D_x^{-\frac{3}{2}}.$$

That means

$$D^{\frac{1}{2}} D^1(1) \neq D^1 D^{\frac{1}{2}}(1).$$

The same proof of Caputo. ■

Composition

- **Fractional integration of a fractional integral**

The Riemann-Liouville fractional integral has the following important property

$$ {}_aD_t^{-p}({}_aD_t^{-q}f(t)) = {}_aD_t^{-q}({}_aD_t^{-p}f(t)) = {}_aD_t^{-p-q}f(t), \tag{2.2.8} $$

which is called the composition rule for the Riemann-Liouville fractional integrals. Using the definition the proof is quite straightforward

$$ \begin{aligned} {}_aD_t^{-p}\left({}_aD_t^{-q}f(t)\right) &= \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}({}_aD_t^{-q}f(\tau))d\tau \\ &= \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\left(\frac{1}{\Gamma(q)}\int_a^\tau (\tau-\xi)^{q-1}f(\xi)d\xi\right)d\tau \\ &= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^\tau (t-\tau)^{p-1}(\tau-\xi)^{q-1}f(\xi)\,d\xi\,d\tau. \end{aligned} $$

Changing the order of integration we obtain

$$ {}_aD_t^{-p}\left({}_aD_t^{-q}f(t)\right) = \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\int_a^\tau (t-\tau)^{p-1}(\tau-\xi)^{q-1}\,d\tau\,d\xi. $$

We make the substitution $\dfrac{\tau-\xi}{t-\xi} = \zeta$ from which it follows that $d\tau = (t-\xi)\,d\zeta$ and the new interval of integration is $[0,1]$. Now we are able to rewrite the last expression as

$$ \begin{aligned} {}_aD_t^{-p}\left({}_aD_t^{-q}f(t)\right) &= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\left((t-\xi)^{p+q-1}\int_0^1 (1-\zeta)^{p-1}\zeta^{q-1}d\zeta\right)d\xi \\ &= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\left((t-\xi)^{p+q-1}B(p,q)\right)d\xi, \end{aligned} $$

Using identity (2.1.1) to express the Beta function in terms of the Gamma function we obtain

$$ \begin{aligned} {}_aD_t^{-p}\left({}_aD_t^{-q}f(t)\right) &= \frac{1}{\Gamma(p)\Gamma(q)}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\int_a^t f(\xi)(t-\xi)^{p+q-1}\,d\xi \\ &= \frac{1}{\Gamma(p+q)}\int_a^t (t-\xi)^{p+q-1}f(\xi)\,d\xi \\ &= {}_aD_t^{-p-q}f(t). \end{aligned} $$

- **Fractional differentiation of a fractional integral**

An important property of the Riemann-Liouville fractional derivative is

$$ {}_aD_t^{p}\left({}_aD_t^{-q}f(t)\right) = {}_aD_t^{p-q}f(t), \tag{2.2.9} $$

where f has to be continuous and if $p \geq q \geq 0$, the derivative ${}_aD_t^{p-q}f$ exists. This property is called the composition rule for the Riemann-Liouville fractional derivatives. We shall prove this property, but first we need another property which actually is a special case of the previous one with $q = p$

$$ {}_aD_t^p \left({}_aD_t^{-p} f(t) \right) = f(t), \tag{2.2.10} $$

where $p > 0$ and $t > a$. This implies that the Riemann-Liouville fractional differentiation operator is the left inverse of the Riemann-Liouville fractional integration of the same order p. We prove this in the following way.

- First we consider the case $p = n \in \mathbb{N}^*$, then we have

$$ \begin{aligned} {}_aD_t^n \left({}_aD_t^{-n} f(t) \right) &= \frac{d^n}{dt^n} \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau \\ &= \frac{d}{dt} \int_a^t f(\tau) d\tau = f(t). \end{aligned} $$

- For the non-integer case we take $k - 1 \leq p < k$ and use (2.2.8) to write

$$ {}_aD_t^{-k} f(t) = {}_aD_t^{-(k-p)} \left({}_aD_t^{-p} f(t) \right). $$

  Now using the definition of the Riemann-Liouville differintegral we obtain

$$ \begin{aligned} {}_aD_t^p \left({}_aD_t^{-p} f(t) \right) &= \frac{d^k}{dt^k} \left[{}_aD_t^{-(k-p)} \left({}_aD_t^{-p} f(t) \right) \right] \\ &= \frac{d^k}{dt^k} \left[{}_aD_t^{-k} f(t) \right] = f(t). \end{aligned} $$

- Now we are able to prove (2.2.9). We consider two cases. First we'll deal with $q \geq p \geq 0$. Then we have

$$ {}_aD_t^p \left({}_aD_t^{-q} f(t) \right) = {}_aD_t^p \left[{}_aD_t^{-p} \left({}_aD_t^{-(q-p)} f(t) \right) \right] = {}_aD_t^{p-q} f(t). $$

  This follows directly from (2.2.8) and (2.2.10). Now we will consider the second case in which we have $p > q \geq 0$. Using (2.2.8) we see that

$$ \begin{aligned} {}_aD_t^p \left({}_aD_t^{-q} f(t) \right) &= \frac{d^k}{dt^k} \left[{}_aD_t^{-(k-p)} \left({}_aD_t^{-q} f(t) \right) \right] \\ &= \frac{d^k}{dt^k} \left({}_aD_t^{p-q-k} f(t) \right) = \frac{d^k}{dt^k} \left({}_aD_t^{-(k-(p-q))} f(t) \right) \\ &= {}_aD_t^{p-q} f(t). \end{aligned} $$

  So in both cases we proved equation (2.2.9).

Remark. The converse of (2.2.10) is not true, so ${}_aD_t^{-p} \left({}_aD_t^p f(t) \right) \neq f(t)$. The proof for this can be found in [?].

Semigroup

Theorem 2.2.2. For any $f \in C([a,b])$ the Riemann-Liouville fractional integral satisfies

$$I^{\alpha}_{a+}I^{\beta}_{a+}f(x) = I^{\alpha+\beta}_{a+}f(x),$$

for $\alpha > 0, \beta > 0$.

Proof.

The proof is rather direct, we have by definition:

$$I^{\alpha}_{a+}I^{\beta}_{a+}f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{dt}{(x-t)^{1-\alpha}} \int_a^t \frac{f(u)}{(t-u)^{1-\beta}} du,$$

and since $f(x) \in C([a,b])$ we can, by Fubini's theorem, interchange order of integration and by setting $t = u + s(x-u)$, we obtain

$$I^{\alpha}_{a+}I^{\beta}_{a+}f(x) = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{f(u)}{(x-u)^{1-\alpha-\beta}} du = I^{\alpha+\beta}_{a+}f(x).$$

■

Linearity

Let f and g are functions for which the given derivatives or integrals operator are defined and $\lambda, \mu \in \mathbb{R}$ are real constants.

$${}_aD^p_t(\lambda f(t) + \mu g(t)) = \lambda \; {}_aD^p_t f(t) + \mu \; {}_aD^p_t g(t).$$

Proof.

- For Grüunwald-Letnikov fractional derivative, we have:

$$\begin{aligned} {}_aD^p_t(\lambda f(t) + \mu g(t)) &= \lim_{h \to 0} h^{-p} \sum_{r=0}^{m} (-1)^r \binom{p}{r} (\lambda f(t-rh) + \mu g(t-rh)) \\ &= \lambda \lim_{h \to 0} h^{-p} \sum_{r=0}^{m} (-1)^r \binom{p}{r} f(t-rh) \\ &+ \mu \lim_{h \to 0} h^{-p} \sum_{r=0}^{m} (-1)^r \binom{p}{r} g(t-rh) \\ &= \lambda \; {}_aD^p_t f(t) + \mu \; {}_aD^p_t g(t). \end{aligned}$$

- For Riemann-Liouville differintegral:

$$\begin{aligned}
{}_aD_t^{-p}(\lambda f(t) + \mu g(t)) &= \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1}(\lambda f(t) + \mu g(t))d\tau \\
&= \lambda \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau)d\tau + \mu \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} g(\tau)d\tau \\
&= \lambda \; {}_aD_t^{-p} f(t) + \mu \; {}_aD_t^{-p} g(t).
\end{aligned}$$

■

Zero Rule

It can be proved that if f is continuous for $t \geq a$ then we have

$$\lim_{p \to 0} {}_aD_t^{-p} f(t) = f(t).$$

Proof.

The proof can be found in [7]. Hence, we define

$${}_aD_t^0 f(t) = f(t).$$

■

Product Rule & Leibniz's Rule

If f and g are functions, We know the derivative of their product is given by the product rule

$$(f.g)' = f'.g + f.g'.$$

This can be generalized to

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)},$$

which is also known as the Leibniz rule. In the last expression f and g are n-times differentiable functions.

Corollary 2.2.1. (Leibniz Rule, [7])

- Let $t > 0$, $\alpha \in \mathbb{R}$, $n-1 < \alpha < n$, $n \in \mathbb{N}$. If $f(\tau)$ and $g(\tau)$ are $\mathcal{C}^{\infty}([0,x])$. The Riemann-Liouville fractional derivative of Leibniz rule is given by

$$ {}_aD_x^{\alpha}(f(x)g(x)) = \sum_{k=0}^{\infty} \binom{k}{\alpha} \left({}_aD_x^{\alpha-k} f(x) \right) g^{(k)}(x) \tag{2.2.11}$$

- Let $t > 0$, $\alpha \in \mathbb{R}$, $n-1 < \alpha < n$, $n \in \mathbb{N}$. If $f(\tau)$ and $g(\tau)$ are $\mathcal{C}^{\infty}([0,x])$. The Caputo fractional derivative of Leibniz rule is given by

$$ {}_a^CD_x^{\alpha}(f(x)g(x)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left({}_aD_x^{\alpha-k} f(x) \right) g^{(k)}(x) \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1-\alpha)} \left(\left(f(x)g(x)\right)^{(k)}(0) \right). \tag{2.2.12}$$

Chapter 3

The mean-square fractional calculus

This chapter presents the elements of the theory of fractional calculus in the mean square framework, in which we will introduce several definitions for the mean square fractional integral and derivative based on some of the common definitions from the deterministic fractional calculus, and consider various properties of m.s. fractional integrals and derivatives: such as the m.s. continuity of the integrals and derivatives. This chapter focuses also on the concept of stochastic fractional operators, a new definition applied to second-order stochastic processes and differs from those defined only for the mean square continuous second-order stochastic process, which is a short family of operators, these new stochastic fractional operators provide a rich calculus with interesting properties for more information and detailed results see [8, 4, 12], and references therein.

3.1 Basic approaches of mean-square fractional integral and derivative

3.1.1 Mean-square Riemann-Liouville fractional integral

As in the deterministic case, below we find an expression for the n^{th} integral of $X(t), t \in T$, over the interval $[a, t]$ where $t \in [a, b]$ by showing that Cauchy's formula for repeated integrals holds for mean-square integrals.

Suppose $\int_a^t \int_a^s X(u)du\, ds$ exists in a m.s. sense. Using mean-square IBP we have

$$\begin{aligned}\int_a^t \left[\int_a^s X(u)du\right] ds &= s\int_a^s X(u)du\Big|_a^t - \int_a^t sX(s)ds \\ &= t\int_a^t X(u)du - \left[s\int_a^s X(u)du\right]_{s=a} - \int_a^t sX(s)ds.\end{aligned}$$

Using the fact that

$$\left[t\int_a^t X(u)du\right]_{t=a} = \left[t\int_a^t \frac{(t-a)^{1-1}}{\Gamma(1)} X(u)du\right]_{t=a} = 0.$$

Thus

$$\begin{aligned}\int_a^t \left[\int_a^s X(u)du\right] ds &= \int_a^t tX(u)du - \int_a^t uX(u)du \\ &= \int_a^t (t-u)X(u)du \\ &= \int_a^t \frac{(t-u)^{2-1}}{\Gamma(2)} X(u)du.\end{aligned}$$

Continuing in this manner we are able to show that

$$\begin{aligned}\int_a^t \int_a^{s_{n-1}} \dots \int_a^{s_3} \int_a^{s_2} X(s_1)\, ds_1\, ds_2 \dots ds_{n-2}\, ds_{n-1} &= \int_a^t \frac{(t-s)^{n-1}}{\Gamma(n)} X(s)ds \\ &\triangleq I_a^n X(t).\end{aligned} \tag{3.1.1}$$

Equation (3.1.1) leads us to the first definition for the m.s. fractional integral. The m.s. Riemann-Liouville fractional integral is found by replacing $n \in \mathbb{N}$ by $\beta > 0$ in equation (3.1.1).

Definition 3.1.1. Let $X(t), t \in T$, be a second-order stochastic process and let $\beta > 0$. The mean-square Riemann-Liouville (R-L) fractional integral to order β of $X(t)$ is given by

$$I_a^\beta X(t) \triangleq \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds, \qquad t \in [a,b] \subset T. \tag{3.1.2}$$

Remark. 1. In m.s. theory, the above integral will exist in a m.s. sense iff the following ordinary double Riemann integral exists and is finite:

$$\int_a^t \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(t-\theta)^{\beta-1}}{\Gamma(\beta)} \Gamma_{XX}(s,\theta)ds\, d\theta. \tag{3.1.3}$$

2. If $\mathbb{E}[X(t)X(t)] < \infty$ then $X(t)$ is a second-order s.p. and noting that

$$\int_a^t \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(t-\theta)^{\beta-1}}{\Gamma(\beta)} \Gamma_{XX}(s,\theta)ds\, d\theta = \mathbb{E}[I_a^\beta X(t) I_a^\beta X(t)],$$

we see that if $I_a^\beta X(t)$ exists in a m.s. sense then it is a second-order stochastic process.

Example 3.1.1. Let Φ be a second-order random variable with $E(\Phi) = 0$ and finite variance $Var(\Phi) = \sigma^2$. Then the s.p. $X(t)$ defined by $X(t) = \Phi t$ will be a second-order s.p. for finite t and will have

$$\Gamma_{XX}(s, \theta) = \sigma^2 s\theta.$$

Then

$$\begin{aligned}
&\int_a^t \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(t-\theta)^{\beta-1}}{\Gamma(\beta)} \Gamma_{XX}(s,\theta)\, ds\, d\theta \\
=&\sigma^2 \left[\int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} s\, ds\right] \left[\int_a^t \frac{(t-\theta)^{\beta-1}}{\Gamma(\beta)} \theta\, d\theta\right] \\
=&\sigma^2 \left[\frac{a(t-a)^{\beta}}{\Gamma(\beta+1)} + \frac{(t-a)^{\beta+1}}{\Gamma(\beta+2)}\right]^2
\end{aligned}$$

where ordinary integration by parts was used to get the last line. Clearly $I_a^{\beta} X(t)$ will exist and be finite for $0 < \beta < \infty$.

Remark. Note that under some circumstances it is not necessary to evaluate (3.1.3) in order to check the existence of the m.s. integral. For example, when $X(t)$ is m.s. continuous for $t \in [a, b]$ and $1 \le \beta < \infty$, the integral in (3.1.2) will exist in a m.s. sense.

Recall that from Definition 1.3.1, the $n^{\text{th}}, n \in N$, m.s. derivative of the second-order s.p. $X(t)$ at t is given by

$$X^{(n)}(t) = \underset{\tau \to 0}{l.i.m.} \left[\frac{1}{\tau^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} X(t - j\tau)\right] \tag{3.1.4}$$

provided this limit exists.

3.1.2 Mean-square Riemann-Liouville fractional derivative

Below we give the definitions for the Right-Hand and Left-Hand definitions of the m.s. fractional derivative.

Definition 3.1.2. Let $X(t), t \in T$, be a second-order stochastic process and let $\beta > 0$ be such that $\beta \in (m-1, m), m \in \mathbb{N}$. The mean square Left-Hand (LH) fractional derivative of $X(t)$ at $t, t \in [a, b] \subset T$, is given by

$${}_*D_a^{\beta} X(t) = \begin{cases} \dfrac{d^m}{dt^m} I_a^{m-\beta} X(t), & \beta \in (m-1, m) \\[2ex] \dfrac{d^m}{dt^m} X(t), & \beta = m \end{cases} \tag{3.1.5}$$

Definition 3.1.3. Let $X(t), t \in T$, be a second-order stochastic process and let $\beta > 0$ be such that $\beta \in [m-1, m], m \in \mathbb{N}$. The mean square Right-Hand (RH) fractional derivative of $X(t)$ at $t, t \in [a, b] \subset T$, is given by

$$D_a^{\beta} X(t) = \begin{cases} I_a^{m-\beta} X^{(m)}(t), & \beta \in (m-1, m) \\ \dfrac{d^m}{dt^m} X(t), & \beta = m \end{cases}$$

When $\beta = m \in \mathbb{N}$, both ${}_*D_a^{\beta} X(t)$ and $D_a^{\beta} X(t)$ give us ordinary repeated m.s. derivatives and so will exist if a suitable generalized derivative holds.

Remark. For (3.1.5) to exist, $I_a^{m-\beta} X(t)$ must exist as a m.s. Cauchy-Riemann integral i.e. the (deterministic) repeated Cauchy-Riemann integral

$$\int_a^t \int_a^t \frac{(t-s)^{m-\beta-1}}{\Gamma(m-\beta)} \frac{(t-\theta)^{m-\beta-1}}{\Gamma(m-\beta)} \Gamma_{XX}(s, \theta) ds\, d\theta,$$

must exist.

3.1.3 The mean-square Grünwald fractional approach

We will now consider a third definition; this one based on the form given in (3.1.4) for the n^{th} m.s. derivative of $X(t)$. Using the notation

$$(-1)^j \binom{n}{j} = \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)} = \frac{\psi(j,n)}{\Gamma(-n)},$$

we can re-write equation (3.1.4) as follows:

$$\begin{aligned} X^{(n)}(t) &= \underset{\delta t_N \to 0}{l.i.m.} \left[\frac{1}{(\delta t_N)^n} \sum_{j=0}^{N-1} (-1)^j \binom{n}{j} X(t - j\delta t_N) \right] \\ &= \underset{N \to \infty}{l.i.m.} \left[\frac{(\delta t_N)^{-n}}{\Gamma(-n)} \sum_{j=0}^{N-1} \psi(j,n) X(t - j\delta t_N) \right] \end{aligned}$$

Definition 3.1.4. (Grünwald fractional derivative)
Let $X(t), t \in T$, be a second-order stochastic process and let $a \in T$. The mean-square Grünwald fractional derivative of order $\beta > 0$ of a second-order stochastic process $X(t)$, is given by

$$X_a^{(\beta)}(t) = \underset{N \to \infty}{l.i.m.} \left[\frac{(\delta t_N)^{-\beta}}{\Gamma(-\beta)} \sum_{j=0}^{N-1} \psi(j,\beta) X(t - j\delta t_N) \right] \tag{3.1.6}$$

where, for $N = 1, 2, 3, \ldots$,

$$\delta t_N = \frac{(t-a)}{N} \quad \text{and} \quad \psi(j,\beta) = \frac{\Gamma(j-\beta)}{\Gamma(j+1)}.$$

Remark. 1. Using the convergence in mean-square criterion we see that the above limit will exist iff

$$\frac{(\delta t_N \delta s_{N'})^{-\beta}}{\Gamma^2(-\beta)} \sum_{j=0}^{N-1} \sum_{k=0}^{N'-1} \psi(j,\beta)\psi(k,\beta)\Gamma_{XX}(t - j\delta t_N \, , \, s - k\delta s_{N'})$$

tends to a finite limit as N and N' tend to infinity in any manner whatever.

2. It is important to note that the limit in (3.1.6) is a restricted limit so that for $\beta = m \in \mathbb{N}$, (3.1.6) will give us integer order derivatives only when those derivatives exist i.e. when the unrestricted limit in (3.1.4) exists.

A second definition for the m.s. fractional integral can be found by allowing negative values of β in the formula for the Grünwald m.s. fractional derivative. Doing so we have the following definition:

Definition 3.1.5. (Grünwald fractional integral)
Let $X(t), t \in T$, be a second-order stochastic process and let $a \in T$. The mean-square Grünwald fractional integral of order $\beta > 0$ of a second-order stochastic process $X(t)$, is given by

$$[I_a^{(\beta)} X(t)]_G = \underset{N\to\infty}{l.i.m.} \left[\frac{\left(\frac{(t-a)}{N}\right)^{\beta}}{\Gamma(\beta)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\beta)}{\Gamma(j+1)} X\left(t - j\left[\frac{(t-a)}{N}\right]\right) \right].$$

By replacing β by $-\beta$ in the existence condition for the Grünwald m.s. fractional derivative, we have the existence condition for the Grünwald m.s. fractional integral.

3.2 Properties of mean-square fractional integrals and derivatives

This section reviews some basic properties of fractional integrals and derivatives in the mean square framework.

Theorem 3.2.1. Let $X(t)$ and $Y(t)$ be second-order stochastic processes for which $I_a^{\beta} X(t)$ and $I_a^{\beta} Y(t), \beta > 0$, exist for $t \in [a,b] \subset T$. Then

(a) (Linearity)

$$I_a^{\beta}[X(t) + Y(t)] = I_a^{\beta} X(t) + I_a^{\beta} Y(t).$$

(b) (Homogeneity)

$$I_a^\beta[cX(t)] = c \cdot I_a^\beta X(t)$$

where c is a constant.

(c) $I_a^\beta\ X(t)|_{t=a} = 0$.

Proof.

Letting: $a = s_0 < s_1 < s_2 < \ldots < s_n = t \le b$, $s_k^* \in [s_{k-1}, s_k]$ for $k = 1, 2, \ldots, n$ and $\Delta_n = \max_k(s_k - s_{k-1})$. We have

(a)

$$\begin{aligned}
I_a^\beta[X(t)+Y(t)] &= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}[X(s)+Y(s)]ds \\
&= \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{l.i.m.} \sum_{k=1}^n \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}[X(s_k^*)+Y(s_k^*)](s_k - s_{k-1}) \\
&= \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{l.i.m.} \sum_{k=1}^n \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}X(s_k^*)(s_k - s_{k-1}) \\
&\quad + \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{l.i.m.} \sum_{k=1}^n \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}Y(s_k^*)(s_k - s_{k-1}) \\
&= \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}X(s)ds + \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}Y(s)ds \\
&= I_a^\beta X(t) + I_a^\beta Y(t).
\end{aligned}$$

(b) Similarly,

$$\begin{aligned}
I_a^\beta[cX(t)] &= \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{l.i.m.} \sum_{k=1}^n \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}\ c\ X(s_k^*)(s_k - s_{k-1}) \\
&= c\ \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{l.i.m.} \sum_{k=1}^n \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}X(s_k^*)(s_k - s_{k-1}) \\
&= c \cdot I_a^\beta X(t).
\end{aligned}$$

(c) Since $I_a^\beta X(t)$ is defined as follows

$$I_a^\beta X(t) \triangleq \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}X(s)ds,$$

■

Corollary 3.2.1. Let $\beta \in [m-1, m]$ where $m \in \mathbb{N}$ and let $X(t)$ be a second-order stochastic process such that $D_a^\beta X(t)$ exists for $t \in [a, b] \subset T$. Then

$$D_a^\beta.X(t)\Big|_{t=a} = 0.$$

Proof.

For $\beta \in [m-1, m]$, $D_a^\beta X(t)$ is defined as follows

$$D_a^\beta X(t) \triangleq I_a^{m-\beta} X^{(m)}(t).$$

This is of the form $I_a^\alpha Y(t)$ where $\alpha = m-\beta$ and $Y(t)$ is a second-order stochastic process. Thus, using Part (c) of Theorem 3.2.1 we have

$$D_a^\beta X(t)\Big|_{t=a} = I_a^{m-\beta} X^{(m)}(t)\Big|_{t=a} = 0$$

■

Remark. Later in this chapter we will come across terms like $I_a^\beta X(a)$. By this we mean $I_a^\beta [X(a)]$ and not $I_a^\beta X(t)\Big|_{t=a}$.

In the following theorem we consider the continuity of $I_a^\beta X(t)$.

Theorem 3.2.2. Let $\beta < 0$ and let $X(t)$ be a second-order stochastic process such that $I_a^\beta X(t)$ exists for $t \in [a,b] \subset T$. Then $I_a^\beta X(t)$ is mean-square continuous provided that, for $h > 0$,

$$\int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} \|X(s)\| \, ds$$

and

$$\int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} \|X(s)\| \, ds$$

exist as Riemann integrals when $\beta \geq 1$ and as CR integrals when $\beta \in (0,1)$.

Proof.

To show that $I_a^\beta X(t)$ is m.s. continuous we must show that, for $t, t+h \in T$ where $h > 0$,

$$\|I_a^\beta X(t+h) - I_a^\beta X(t)\| \to 0 \quad \text{as } h \to 0.$$

Now,

$$\begin{aligned}
\|I_a^\beta X(t+h) - I_a^\beta X(t)\| &= \Big\| \int_a^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds \\
&\quad - \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds \Big\| \\
&= \Big\| \int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds \\
&\quad + \int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds \Big\| \\
&\leq \Big\| \int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds \Big\| \\
&\quad + \Big\| \int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds \Big\| \\
&= \|\mathfrak{J}_1\| + \|\mathfrak{J}_2\|
\end{aligned}$$

where

$$\mathfrak{J}_1 = \int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds \quad \text{and}$$

$$\mathfrak{J}_2 = \int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds.$$

Consider $\mathfrak{J}_1$.

$$\|\mathfrak{J}_1\| \leq \int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} \|X(s)\| ds$$

provided the integral on the right exists as a Riemann integral when $\beta \geq 1$ and as a CR integral when $\beta \in (0,1)$. Then, letting $M = \max_{s\in[t,t+h]} \|X(s)\|$, we have

$$\|\mathfrak{J}_1\| \leq M \int_t^{t+h} \frac{(t+h-s)^{\beta-1}}{\Gamma(\beta)} ds \to 0 \quad \text{as } h \to 0.$$

Similarly,

$$\|\mathfrak{J}_2\| \leq \int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)} \|X(s)\| ds$$

provided the integral on the right exists as a Riemann integral when $\beta \geq 1$ and as a CR integral when $\beta \in (0,1)$. Then, letting $N = \max_{s\in[a,t]} \|X(s)\|$, we have

$$\|\mathfrak{J}_2\| \leq N \int_a^t \frac{(t+h-s)^{\beta-1} - (t-s)\beta - 1}{\Gamma(\beta)} ds \to 0 \quad \text{as } h \to 0.$$

Thus

$$\|I_a^\beta X(t+h) - I_a^\beta X(t)\| \leq \|\mathfrak{J}_1\| + \|\mathfrak{J}_2\| \to 0 \quad \text{as } h \to 0.$$

So we see that $I_a^\beta X(t)$ will be m.s. continuous. ∎

Theorem 3.2.3. Let $X(t)$ be a second-order stochastic process such that $I_a^\gamma X(t), t \in [a,b] \subset T$ exists for all $\gamma > 0$. Then, for $\alpha > 0$ where $\alpha \neq \beta$,

$$\underset{\beta\to\alpha}{l.i.m.}\ I_a^\beta X(t) = I_a^\alpha X(t).$$

Proof.

Using the convergence in mean-square criterion to prove that

$$\underset{\beta\to\alpha}{l.i.m.}\ I_a^\beta X(t) = I_a^\alpha X(t),$$

we need to show that $\mathbb{E}[I_a^\beta X(t) I_a^{\beta'} X(t)]$ tends to a finite limit as β and β' tend to α (in any manner whatever) and that the limit is $\mathbb{E}[I_a^\alpha X(t) I_a^\alpha X(t)]$. Now,

$$\begin{aligned}
&\lim_{\beta,\beta'\to\alpha} \mathbb{E}[I_a^\beta X(t) I_a^{\beta'} X(t)]\\
&= \lim_{\beta,\beta'\to\alpha} \int_a^t \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(t-\theta)^{\beta'-1}}{\Gamma(\beta')} \Gamma_{XX}(s,\theta)\, ds\, d\theta\\
&= \int_a^t \int_a^t \left[\lim_{\beta,\beta'\to\alpha} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \frac{(t-\theta)^{\beta'-1}}{\Gamma(\beta')}\right] \Gamma_{XX}(s,\theta)\, ds\, d\theta\\
&= \int_a^t \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \Gamma_{XX}(s,\theta)\, ds\, d\theta < \infty\\
&= \mathbb{E}[I_a^\alpha X(t) I_a^\alpha X(t)].
\end{aligned}$$

■

By looking at the proof of Theorem 3.2.3 it is clear that we could not consider the case $\beta \to \alpha$ where $\alpha = 0$. In the following theorem we consider this case.

Theorem 3.2.4. Let $X(t)$ be a second-order stochastic process such that $\dot{X}(t)$ exists and is mean-square continuous on $[a,b] \subset T$. Then

$$\underset{\beta\to 0}{l.i.m.}\ I_a^\beta X(t) = X(t).$$

Proof.

Using integration by parts we have for $\beta > 0$

$$\begin{aligned}
I_a^{\beta+1}\dot{X}(t) &= \left[\frac{(t-s)^\beta}{\Gamma(\beta+1)} X(s)\right]_a^t + \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds\\
&= -\frac{(t-a)^\beta}{\Gamma(\beta+1)} X(a) + I_a^\beta X(t).
\end{aligned}$$

So

$$\begin{aligned}
\underset{\beta\to 0}{l.i.m.}\ I_a^{\beta}X(t) &= \underset{\beta\to 0}{l.i.m.}\left[I_a^{\beta+1}\dot{X}(t) + \frac{(t-a)^{\beta}}{\Gamma(\beta+1)}X(a)\right] \\
&= \underset{\beta\to 0}{l.i.m.}\ I_a^{\beta+1}\dot{X}(t) + \underset{\beta\to 0}{l.i.m.}\left[\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}X(a)\right] \\
&= \underset{\gamma\to 1}{l.i.m.}\ I_a^{\gamma}\dot{X}(t) + X(a) \\
&= I_a^{1}\dot{X}(t) + X(a) \\
&= X(t) - X(a) + X(a) \\
&= X(t).
\end{aligned}$$

■

Theorem 3.2.5. Let $\beta \in (m-1, m)$ where $m \in \mathbb{N}$. Let $X(t)$ be a second-order stochastic process such that $X^{(m)}(t)$ exists and is mean square continuous on $[a, b] \subset T$. Then

(a)

$$\underset{\beta\to m}{l.i.m.}\ D_a^{\beta}X(t) = X^{(m)}(t).$$

(b)

$$\underset{\beta\to (m-1)}{l.i.m.}\ D_a^{\beta}X(t) = X^{(m-1)}(t) - X^{(m-1)}(a).$$

Proof.

(a)

$$\begin{aligned}
\underset{\beta\to m}{l.i.m.}\ D_a^{\beta}X(t) &= \underset{\beta\to m}{l.i.m.} I_a^{m-\beta}X^{(m)}(t) \\
&= \underset{\gamma\to 0}{l.i.m.}\ I_a^{\gamma}X^{(m)}(t) \\
&= X^{(m)}(t)
\end{aligned}$$

where Theorem 3.2.4 has been used in the last step.

(b)

$$\begin{aligned}
\underset{\beta\to (m-1)}{l.i.m.}\ D_a^{\beta}X(t) &= \underset{\beta\to (m-1)}{l.i.m.}\ I_a^{m-\beta}X^{(m)}(t) \\
&= \underset{\gamma\to 1}{l.i.m.}\ I_a^{\gamma}X^{(m)}(t) \\
&= I_a^{1}X^{(m)}(t) \\
&= X^{(m-1)}(t) - X^{(m-1)}(a)
\end{aligned}$$

where IBP has been used in the last step.

■

From Part (a) of Theorem 3.2.5 we know that

$$\underset{\beta\to m}{l.i.m.}\ D_a^{\beta}X(t) = X^{(m)}(t)$$

and from Part (b) we see that we will have

$$\underset{\beta+1\to(m)}{l.i.m.}\ D_a^{\beta+1}X(t) = X^{(m)}(t) - X^{(m)}(a).$$

The Composition Rule

Since $I_a^{\beta}X(t)$ and $D_a^{\beta}X(t)$, when they exist, are second-order stochastic processes, we can consider expressions such as $I_a^{\alpha}I_a^{\beta}X(t)$ and $D_a^{\alpha}I_a^{\beta}X(t)$. These expressions, and those like it, make up what we will call the composition rule ([3], [4]).

Theorem 3.2.6. Let $\beta > 0$ and $\alpha > 0$ and let $X(t)$ be a second-order stochastic process such that $I_a^{\alpha}I_a^{\beta}X(t)$ exists for $t \in [a,b] \subset T$. Then

$$I_a^{\alpha}I_a^{\beta}X(t) = I_a^{\alpha+\beta}X(t).$$

Proof.

$$\begin{aligned} I_a^{\alpha}I_a^{\beta}X(t) &= \int_a^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \int_a^{\theta} \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)} X(s)\,ds\,d\theta \\ &= \int_a^t X(s) \left[\int_s^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)}\,d\theta \right] ds. \end{aligned}$$

Consider

$$\mathfrak{I} = \int_s^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)}\,d\theta.$$

Letting $v = \theta - s$ we have

$$\mathfrak{I} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{t-s} (t-v-s)^{\alpha-1} v^{\beta-1}\,dv.$$

Now letting $v = (t-s)u$ we have

$$\begin{aligned}
\mathfrak{J} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (t-s)^{\alpha-1}(1-u)^{\alpha-1}(t-s)^{\beta-1}u^{\beta-1}(t-s)\,du \\
&= \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-u)^{\alpha-1}u^{\beta-1}\,du \\
&= \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
&= \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}.
\end{aligned}$$

So

$$\begin{aligned}
I_a^\alpha I_a^\beta X(t) &= \int_a^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} X(s)\,ds \\
&= I_a^{\alpha+\beta} X(t).
\end{aligned}$$

■

Theorem 3.2.7. Let $\alpha > 0$ and $\beta \in (m-1, m], m \in \mathbb{N}$. Let $X(t)$ be a second-order stochastic process such that $D_a^\beta X(t)$ exists for $t \in [a,b] \subset T$. Then

$$I_a^\alpha D_a^\beta X(t) = I_a^{\alpha-\beta} X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^{\alpha-\beta+j}}{\Gamma(\alpha-\beta+j+1)} X^{(j)}(a), \qquad \alpha \geq \beta.$$

Proof.

Using integration by parts we have

$$\begin{aligned}
I_a^1 X^{(1)}(t) &= \int_a^t X^{(1)}(s)\,ds \\
&= [X(s)]_a^t + 0 \\
&= X(t) - X(a),
\end{aligned}$$

$$\begin{aligned}
I_a^2 X^{(2)}(t) &= \int_a^t \frac{(t-s)}{\Gamma(2)} X^{(2)}(s)\,ds \\
&= \left[\frac{(t-s)}{\Gamma(2)} X^{(1)}(s)\right]_a^t + \underbrace{\int_a^t X^{(1)}(s)\,ds}_{I_a^1 X^{(1)}(t)} \\
&= -\frac{(t-a)^1}{\Gamma(2)} X^{(1)}(a) - X(a) + X(t),
\end{aligned}$$

and

$$\begin{aligned}
I_a^3 X^{(3)}(t) &= \int_a^t \frac{(t-s)^2}{\Gamma(3)} X^{(3)}(s)\,ds \\
&= \left[\frac{(t-s)^2}{\Gamma(3)} X^{(2)}(s)\right]_a^t + \underbrace{\int_a^t \frac{(t-s)}{\Gamma(2)} X^{(2)}(s)\,ds}_{I_a^2 X^{(2)}(t)} \\
&= -\frac{(t-a)^2}{\Gamma(3)} X^{(2)}(a) - \frac{(t-a)^1}{\Gamma(2)} X^{(1)}(a) - X(a) + X(t).
\end{aligned}$$

Continuing in the same manner we have, for $\alpha = \beta = m$,

$$I_a^m X^{(m)}(t) = X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a). \tag{3.2.1}$$

For $\alpha = \beta \in (m-1, m)$

$$\begin{aligned}
I_a^\beta D_a^\beta X(t) = I_a^\beta I_a^{m-\beta} X^{(m)}(t) &= I_a^m X^{(m)}(t) \\
&= X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a)
\end{aligned}$$

where we have used equation (3.2.1) to find the last line.

Let $n \in \mathbb{N}$. When $\alpha = n$ and $\beta = m$

$$\begin{aligned}
I_a^n D_a^m X(t) &= I_a^{n-m} I_a^m X^{(m)}(t) \\
&= I_a^{n-m}\left[X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a)\right] \\
&= I_a^{n-m} X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^{n-m+j}}{\Gamma(n-m+j+1)} X^{(j)}(a).
\end{aligned}$$

When $\alpha \in (n-1, n)$ and $\beta = m$

$$\beta = m \le n-1 < \alpha < n$$

so that $\alpha - m > 0$. Under these conditions we have

$$\begin{aligned}
I_a^\alpha X^{(m)}(t) &= I_a^{m-m} I_a^\alpha X^{(m)}(t) \\
&= I_a^{\alpha-m} I_a^m X^{(m)}(t) \\
&= I_a^{\alpha-m}\left[X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a)\right] \\
&= I_a^{\alpha-m} X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^{\alpha-m+j}}{\Gamma(\alpha-m+j+1)} X^{(j)}(a).
\end{aligned}$$

For $\alpha \in (n-1, n]$ and $\beta \in (m-1, m)$

$$\begin{aligned}
I_a^\alpha D_a^\beta X(t) &= I_a^\alpha I_a^{m-\beta} X^{(m)}(t) \\
&= I_a^{\alpha-\beta} I_a^m X^{(m)}(t) \\
&= I_a^{\alpha-\beta}\left[X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a)\right] \\
&= I_a^{\alpha-\beta} X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^{\alpha-\beta+j}}{\Gamma(\alpha-\beta+j+1)} X^{(j)}(a).
\end{aligned}$$

■

Theorem 3.2.8. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ and let $0 < \alpha < \beta$ where $\alpha \in (n-1, n]$ and $\beta \in (m-1, m]$. Let $X(t)$ be a second-order stochastic process such that $D_a^\beta X(t)$ exists for $t \in [a, b] \subset T$. Then

(a)

$$I_a^\alpha D_a^\beta X(t) = D_a^{\beta-\alpha} X(t)$$

when $\beta - \alpha \in (m-1, m)$ and $\beta \neq m$.

(b)

$$I_a^\alpha D_a^\beta X(t) = D_a^{\beta-\alpha} X(t) - \frac{(t-a)^{m-\beta-1+\alpha}}{\Gamma(m-\beta+\alpha)} X^{(m-1)}(a)$$

when $\alpha \in (0, 1), \beta - \alpha \in (m-2, m-1)$ and $\beta \neq m$.

(c)

$$I_a^n D_a^\beta X(t) = D_a^{\beta-n} X(t) - \sum_{j=0}^{n-1} \frac{(t-a)^{m-\beta+j}}{\Gamma(m-\beta+j+1)} X^{(m-n+j)}(a)$$

for $\beta \in (m-1, m]$.

(d)

$$I_a^\alpha D_a^\beta X(t) = D_a^{\beta-\alpha} X(t) - \sum_{j=1}^{n-1} \frac{(t-a)^{m-\beta-n+\alpha+j}}{\Gamma(m-\beta-n+\alpha+j+1)} X^{(m-n+j)}(a)$$

when $\alpha \in (n-1, n), \alpha > 1$.

Proof.

(a)

$$\begin{aligned}
I_a^\alpha D_a^\beta X(t) &= I_a^\alpha I_a^{m-\beta} X^{(m)}(t) \\
&= I_a^{m-(\beta-\alpha)} X^{(m)}(t) \\
&= D_a^{\beta-\alpha} X(t).
\end{aligned}$$

(b) Let $\beta - \alpha = m - 1$.

$$
\begin{aligned}
I_a^\alpha D_a^\beta X(t) &= I_a^\alpha I_a^{m-\beta} X^{(m)}(t) \\
&= I_a^{m-(\beta-\alpha)} X^{(m)}(t) \\
&= I_a^1 X^{(m)}(t) \\
&= X^{(m-1)}(t) - X^{(m-1)}(a) \\
&= D_a^{\beta-\alpha} X(t) - \frac{(t-a)^{m-1-\beta+\alpha}}{\Gamma(m-\beta+\alpha)} X^{(m-1)}(a).
\end{aligned}
$$

Now let $\beta - \alpha \in (m-2, m-1)$.

$$
\begin{aligned}
I_a^\alpha D_a^\beta X(t) &= I_a^{m-(\beta-\alpha)} X^{(m)}(t) \\
&= I_a^{(m-1)-(\beta-\alpha)} I^1 X^{(m)}(t) \\
&= I_a^{(m-1)-(\beta-\alpha)} X^{(m)}(t) - I_a^{(m-1)-(\beta-\alpha)} X^{(m-1)}(a) \\
&= I_a^{(m-1)-(\beta-\alpha)} X^{(m-1)}(t) - \frac{(t-a)^{(m-1)-(\beta-\alpha)}}{\Gamma(m-\beta+\alpha)} X^{(m-1)}(a) \\
&= D_a^{\beta-\alpha} X(t) - \frac{(t-a)^{m-\beta-1+\alpha}}{\Gamma(m-\beta+\alpha)} X^{(m-1)}(a).
\end{aligned}
$$

(c) Using equation (3.2.1) and recalling that we are working under the assumption that $\beta > \alpha$, we have, for $\beta = m$ and $\alpha = n$,

$$
\begin{aligned}
I_a^n D_a^m X(t) &= I_a^n \frac{d^n}{dt^n} \left[X^{(m-n)}(t) \right] \\
&= X^{(m-n)}(t) - \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(m-n+j)}(a).
\end{aligned}
\tag{3.2.2}
$$

Using equation (3.2.2) we have, for $\beta \in (m-1, m)$ and $\alpha = n$,

$$
\begin{aligned}
I_a^n D_a^\beta X(t) &= I_a^n I_a^{m-\beta} X^{(m)}(t) \\
&= I_a^{m-\beta} I_a^n X^{(m)}(t) \\
&= I_a^{m-\beta} \left[X^{(m-n)}(t) - \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(m-n+j)}(a) \right] \\
&= I_a^{m-\beta} X^{(m-n)}(t) - \sum_{j=0}^{n-1} \frac{(t-a)^{m-\beta+j}}{\Gamma(m-\beta+j+1)} X^{(m-n+j)}(a) \\
&= D_a^{\beta-n} X(t) - \sum_{j=0}^{n-1} \frac{(t-a)^{m-\beta+j}}{\Gamma(m-\beta+j+1)} X^{(m-n+j)}(a).
\end{aligned}
$$

(d) Using equation (3.2.2) we have, for $\beta = m$ and $\alpha \in (n-1, n)$ where $\alpha > 1$,

$$\begin{aligned} I_a^{\alpha} X^{(m)}(t) &= I_a^{\alpha-n+1} I_a^{n-1} X^{(m)}(t) \\ &= I_a^{\alpha-n+1} \left[X^{(m-n+1)}(t) - \sum_{j=0}^{n-2} \frac{(t-a)^j}{\Gamma(j+1)} X^{(m-n+1+j)}(a) \right] \\ &= I_a^{\alpha-n+1} X^{(m-n+1)}(t) - \sum_{j=0}^{n-2} \frac{(t-a)^{\alpha-n+1+j}}{\Gamma(\alpha-n+j+2)} X^{(m-n+1+j)}(a) \\ &= D_a^{m-\alpha} X(t) - \sum_{j=1}^{n-1} \frac{(t-a)^{\alpha-n+j}}{\Gamma(\alpha-n+j+1)} X^{(m-n+j)}(a). \end{aligned}$$

Let $\beta \in (m-1, m)$ and $\alpha \in (n-1, n)$ where $\alpha > 1$. If $n = m$, then $\beta - \alpha \in (0, 1)$. Under these conditions we have

$$\begin{aligned} I_a^{\alpha} D_a^{\beta} X(t) &= I_a^{\alpha+m-\beta} X^{(m)}(t) \\ &= I_a^{1-(\beta-\alpha)} I_a^{m-1} X^{(m)}(t) \\ &= I_a^{1-(\beta-\alpha)} \left[X^{m-m+1}(t) - \sum_{j=0}^{m-2} \frac{(t-a)^j}{\Gamma(j+1)} X^{(m-m+1+j)}(a) \right] \\ &= I_a^{1-(\beta-\alpha)} X^{1}(t) - \sum_{j=0}^{m-2} \frac{(t-a)^{1-\beta+\alpha+j}}{\Gamma(j+2-\beta+\alpha)} X^{(j+1)}(a) \\ &= D_a^{\beta-\alpha} X(t) - \sum_{j=1}^{m-1} \frac{(t-a)^{m-\beta-n+\alpha+j}}{\Gamma(m-\beta-n+\alpha+j+1)} X^{(m-n+j)}(a). \end{aligned}$$

Let $\beta \in (m-1, m)$ and $\alpha \in (n-1, n)$ where $\alpha > 1$. If $n = m$, then $\beta - \alpha \in (0, 1)$ and $n \neq m$. Using equation (3.2.2) we have

$$\begin{aligned} I_a^{\alpha} D_a^{\beta} X(t) &= I_a^{\alpha} I_a^{m-\beta} X^{(m)}(t) \\ &= I_a^{m-\beta+\alpha-n+1} I_a^{n-1} X^{(m)}(t) \\ &= I_a^{m-\beta+\alpha-n+1} \left[X^{m-n+1}(t) - \sum_{j=0}^{n-2} \frac{(t-a)^j}{\Gamma(j+1)} X^{(m-n+j+1)}(a) \right] \\ &= I_a^{m-\beta+\alpha-n+1} X^{m-n+1}(t) - \sum_{j=0}^{n-2} \frac{(t-a)^{m-\beta-n+\alpha+1+j}}{\Gamma(m-\beta+\alpha-n+j+2)} X^{(m-n+j+1)}(a) \\ &= D_a^{\beta-\alpha} X(t) - \sum_{j=1}^{n-1} \frac{(t-a)^{m-\beta-n+\alpha+j}}{\Gamma(m-\beta-n+\alpha+j+1)} X^{(m-n+j)}(a). \end{aligned}$$

■

Theorem 3.2.9. Let $\beta > 0, n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $X(t)$ be a second-order stochastic process such that $I_a^\beta X(t)$ exists for $t \in [a,b] \subset T$. Then

$$D_a^\alpha I_a^\beta X(t) = I_a^{\beta-\alpha} X(t)$$

when $\alpha \in (n-1, n]$ and

(a) $\beta = m$ and $n \leq \beta$, or

(b) $\beta \in (m-1, m)$ and $n < \beta$.

Proof.

(a) Let $\alpha = n$.

Using Leibniz's rule for $\beta = 1$ we have

$$\frac{d}{dt} I_a^1 X(t) = X(t).$$

For $\beta = 2$ we have

$$\begin{aligned}
\frac{d}{dt} I_a^2 X(t) &= \frac{d}{dt} \int_a^t \frac{t-s}{\Gamma(2)} X(s)\, ds = \int_a^t X(s)\, ds + 0 = I_a^1 X(t). \\
\frac{d^2}{dt^2} I_a^2 X(t) &= \frac{d}{dt} \frac{d}{dt} I_a^2 X(t) = \frac{d}{dt} I_a^1 X(t) = X(t).
\end{aligned}$$

For $\beta = 3$ we have

$$\begin{aligned}
\frac{d}{dt} I_a^3 X(t) &= \frac{d}{dt} \int_a^t \frac{(t-s)^2}{\Gamma(3)} X(s)\, ds \\
&= \int_a^t \frac{t-s}{\Gamma(2)} X(s) ds + 0 \\
&= I_a^2 X(t). \\
\frac{d^2}{dt^2} I_a^3 X(t) &= \frac{d}{dt} \frac{d}{dt} I_a^3 X(t) = \frac{d}{dt} I_a^2 X(t) = I_a^1 X(t) \\
\frac{d^3}{dt^3} I_a^3 X(t) &= \frac{d}{dt} \frac{d^2}{dt^2} I_a^3 X(t) = \frac{d}{dt} I_a^1 X(t) = X(t).
\end{aligned}$$

Continuing in this manner we have, for $\beta = m$ and $\alpha = n$

$$D_a^n I_a^m X(t) = I_a^{m-n} X(t) \quad \text{for } n \leq m. \tag{3.2.3}$$

Now let $\alpha \in (n-1, n)$ and $\beta = m$. Using equation (3.2.3) we have for $n = m$

$$\begin{aligned} D_a^\alpha I_a^\beta X(t) &= I_a^{n-\alpha} \frac{d^n}{dt^n} I_a^m X(t) \\ &= I_a^{m-\alpha} \frac{d^m}{dt^m} I_a^m X(t) \\ &= I_a^{m-\alpha} X(t). \end{aligned}$$

and for $n < m$

$$\begin{aligned} D_a^\alpha I_a^\beta X(t) &= I_a^{n-\alpha} \frac{d^n}{dt^n} I_a^m X(t) \\ &= I_a^{n-\alpha} I_a^{m-n} X(t) \\ &= I_a^{m-\alpha} X(t). \end{aligned}$$

(b) Here we have the conditions

$$n - 1 < \alpha \leq n \leq m - 1 < \beta < m.$$

Since $n \geq 1$ we have $\beta > 1$.

Let $\alpha = n$. Using Leibniz's rule we have

$$\begin{aligned} \frac{d}{dt} I_a^\beta X(t) &= \frac{d}{dt} \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \\ &= \int_a^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} X(s)\, ds + \left[\frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s) \right]_{s=t} \\ &= I_a^{\beta-1} X(t), \quad \beta > 1. \end{aligned} \tag{3.2.4}$$

Taking the derivative of (3.2.4) we have

$$\begin{aligned} \frac{d^2}{dt^2} I_a^\beta X(t) &= \frac{d}{dt} \frac{d}{dt} I_a^\beta X(t) \\ &= \frac{d}{dt} I_a^{\beta-1} X(t) \\ &= I_a^{\beta-2} X(t), \quad \beta > 2 \end{aligned} \tag{3.2.5}$$

where (3.2.4) has been used - with β replaced by $(\beta - 1)$ - in the last step.

Taking the derivative of (3.2.5) we have

$$\begin{aligned} \frac{d^3}{dt^3} I_a^\beta X(t) &= \frac{d}{dt} \frac{d^2}{dt^2} I_a^\beta X(t) \\ &= \frac{d}{dt} I_a^{\beta-2} X(t) \\ &= I_a^{\beta-3} X(t), \quad \beta > 3 \end{aligned}$$

where (3.2.4) has been used - with β replaced by $(\beta - 2)$ - in the last step.

Continuing in this manner we have

$$\frac{d^n}{dt^n} I_a^\beta X(t) = I_a^{\beta-n} X(t), \qquad \beta > n. \tag{3.2.6}$$

Now let $\alpha \in (n-1, n)$ and $\beta \in (m-1, m)$ where $\beta > n$. Using equation (3.2.6) we have

$$\begin{aligned} D_a^\alpha I_a^\beta X(t) &= I_a^{n-\alpha} \frac{d^n}{dt^n} I_a^\beta X(t) \\ &= I_a^{n-\alpha} I_a^{\beta-n} X(t) \\ &= I_a^{\beta-\alpha} X(t). \end{aligned}$$

■

Corollary 3.2.2. Let $\beta \in (m-1, m), m \in \mathbb{N}$, and let $X(t)$ be a second-order stochastic process such that $I_a^\beta X(t)$ exists for $t \in [a, b] \subset T$. Then for $j \in \{0, 1, 2, \dots, m-1\}$

$$\left. \frac{d^j}{dt^j} I_a^\beta X(t) \right|_{t=a} = 0.$$

Proof.

Using the definition of $D_a^0 X(t)$ we have $\left. D_a^0 I_a^\beta X(t) \right|_{t=0} = \left. I_a^\beta X(t) \right|_{t=0} = 0.$ Using Parts (a) and (b) of Theorem 3.2.9 we have

$$\frac{d^j}{dt^j} I_a^\beta X(t) = I_a^{\beta-j} X(t), \qquad j \in \{1, 2, \dots, m-1\}.$$

Using Part (c) of Theorem 3.2.1 we have

$$\left. \frac{d^j}{dt^j} I_a^\beta X(t) \right|_{t=a} = \left. I_a^{\beta-j} X(t) \right|_{t=a} = 0, \qquad j \in \{1, 2, \dots, m-1\}.$$

■

Using Corollary 3.2.2 we see that the result in Theorem 3.2.7 becomes $I_a^\alpha D_a^\beta X(t) = I_a^{\alpha-\beta} X(t)$ if $X(t)$ is itself a mean-square fractional integral or derivative.

Theorem 3.2.10. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ where $m < n$. Let $X(t)$ be a second-order stochastic process such that $X^{(n-m)}(t)$ exists and is mean-square continuous on $[a, b] \subset T$. Then, for $\alpha \in (n-1, n]$,

$$D_a^\alpha I_a^m X(t) = D_a^{\alpha-m} X(t).$$

Proof.

Let $\alpha = n$. Using Leibniz's rule we have for $m = 1$

$$\begin{aligned}
\frac{d^2}{dt^2} I_a^1 X(t) &= \frac{d}{dt}\frac{d}{dt} I_a^1 X(t) = \dot{X}(t) \\
\frac{d^3}{dt^3} I_a^1 X(t) &= \frac{d}{dt}\frac{d^2}{dt^2} I_a^1 X(t) = \frac{d}{dt}\dot{X}(t) = \ddot{X}(t) \\
\frac{d^4}{dt^4} I_a^1 X(t) &= \frac{d}{dt}\frac{d^3}{dt^3} I_a^1 X(t) = \frac{d}{dt}\ddot{X}(t) = X^{(3)}(t) \\
\vdots \quad & \quad \vdots \\
\frac{d^n}{dt^n} I_a^1 X(t) &= X^{(n-1)}(t), \quad n \geq 1.
\end{aligned}$$

Using equation (3.2.3) we have for $m = 2$

$$\begin{aligned}
\frac{d^3}{dt^3} I_a^2 X(t) &= \frac{d}{dt}\frac{d^2}{dt^2} I_a^2 X(t) = \frac{d}{dt} X(t) = \dot{X}(t) \\
\frac{d^4}{dt^4} I_a^2 X(t) &= \frac{d}{dt}\frac{d^3}{dt^3} I_a^2 X(t) = \frac{d}{dt}\dot{X}(t) = \ddot{X}(t) \\
\frac{d^5}{dt^5} I_a^2 X(t) &= \frac{d}{dt}\frac{d^4}{dt^4} I_a^2 X(t) = \frac{d}{dt}\ddot{X}(t) = X^{(3)}(t) \\
\vdots \quad & \quad \vdots \\
\frac{d^n}{dt^n} I_a^2 X(t) &= X^{(n-2)}(t), \quad n \geq 2.
\end{aligned}$$

Continuing in this manner we have for $n \geq m$

$$\frac{d^n}{dt^n} I_a^m X(t) = X^{(n-m)}(t). \tag{3.2.7}$$

When $\alpha \in (n-1, n)$ and $n > m$ we have

$$\begin{aligned}
D_a^\alpha I_a^m X(t) &= I_a^{n-\alpha} \frac{d^n}{dt^n} I_a^m X(t) \\
&= I_a^{n-\alpha} X^{(n-m)}(t)
\end{aligned}$$

where we have used equation 3.2.7 to find the last line.

Since $\alpha - m \in (n-m-1, n-m)$ we have

$$I_a^{n-\alpha} X^{(n-m)}(t) = I_a^{(n-m)-(\alpha-m)} X^{(n-m)}(t) = D_a^{\alpha-m} X(t)$$

and so $D_a^\alpha I_a^m X(t) = D_a^{\alpha-m} X(t)$ when $\alpha \in (n-1, n)$. ■

Let $X(t)$ be a stochastic process such that $X^{(n)}(t), n \in \mathbb{N}$, exists and is mean-square continuous. Since $X^{(n)}(t)$ exists, all $X^{(j)}(t), j \in \{0, 1, 2, \ldots, n-1\}$, exist and are mean-square continuous. Thus $I_a^\gamma X(t)$ and $I_a^\gamma X^{(j)}(t)$ exist and are mean square continuous for all $\gamma \geq 1$ and $j \in \{0, 1, 2, \ldots, n-1\}$.

Let $\beta > 1$. Using IBP we have

$$\begin{aligned} I_a^\beta \dot{X}(t) &= \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s) \Big|_a^t + \int_a^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} X(s)\, ds \\ &= I_a^{\beta-1} X(t) - \frac{(t-a)^{\beta-1}}{\Gamma(\beta)} X(a). \end{aligned} \tag{3.2.8}$$

Using Leibniz's rule we have

$$\begin{aligned} \frac{d}{dt} I_a^\beta X(t) &= \frac{d}{dt} \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds \\ &= \int_a^t \frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)} X(s)\, ds + \left[\frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s) \right]_{s=t} \\ &= I_a^{\beta-1} X(t). \end{aligned} \tag{3.2.9}$$

Substituting 3.2.8 into 3.2.9 we have

$$\frac{d}{dt} I_a^\beta X(t) = I_a^\beta \dot{X}(t) + \frac{(t-a)^{\beta-1}}{\Gamma(\beta)} X(a). \tag{3.2.10}$$

Taking the derivative of 3.2.10 we have

$$\begin{aligned} \frac{d^2}{dt^2} I_a^\beta X(t) &= \frac{d}{dt} I_a^\beta \dot{X}(t) + \frac{d}{dt} \left[\frac{(t-a)^{\beta-1}}{\Gamma(\beta)} X(a) \right] \\ &= I_a^\beta \ddot{X}(t) + \frac{(t-a)^{\beta-1}}{\Gamma(\beta)} \dot{X}(a) + \frac{(t-a)^{\beta-2}}{\Gamma(\beta-1)} X(a) \end{aligned} \tag{3.2.11}$$

where we have used 3.2.10 - with $X(t)$ replaced by $\dot{X}(t)$ - to find an expression for $\frac{d}{dt} I_a^\beta \dot{X}(t)$.
Taking the derivative of 3.2.11 we have

$$\begin{aligned} \frac{d^3}{dt^3} I_a^\beta X(t) &= \frac{d}{dt} I_a^\beta \ddot{X}(t) + \frac{d}{dt} \left[\frac{(t-a)^{\beta-1}}{\Gamma(\beta)} \dot{X}(a) + \frac{(t-a)^{\beta-2}}{\Gamma(\beta-1)} X(a) \right] \\ &= I_a^\beta X^{(3)}(t) + \frac{(t-a)^{\beta-1}}{\Gamma(\beta)} \ddot{X}(a) + \frac{(t-a)^{\beta-2}}{\Gamma(\beta-1)} \dot{X}(a) + \frac{(t-a)^{\beta-3}}{\Gamma(\beta-2)} X(a) \end{aligned}$$

where we have used 3.2.10 - with $X(t)$ replaced by $\ddot{X}(t)$ - to find an expression for $\frac{d}{dt} I_a^\beta \ddot{X}(t)$.

Continuing in this manner we have the following result.

Theorem 3.2.11. Let $n \in \mathbb{N}$ and let $X(t)$ be a second-order stochastic process such that $X^{(n)}(t)$ exists and is mean-square continuous on $t \in [a,b] \subset T$. Then, for $\beta > 1$,

$$I_a^\beta X^{(n)}(t) = \frac{d^n}{dt^n} I_a^\beta X(t) - \sum_{j=0}^{n-1} \frac{(t-a)^{\beta-n+j}}{\Gamma(\beta-n+j+1)} X^{(j)}(a).$$

Theorem 3.2.12. Let $n \in \mathbb{N}$ and let $X(t)$ be a second-order stochastic process such that $X^{(n)}(t)$ exists and is mean-square continuous on $[a,b] \subset T$. Then, for $\alpha \in (n-1, n)$

$$D_a^\alpha I_a^\alpha X(t) = X(t).$$

Proof.

$$\begin{aligned}
D_a^\alpha I_a^\alpha X(t) &= I_a^{n-\alpha} \frac{d^n}{dt^n} I_a^\alpha X(t) \\
&= I_a^{n-\alpha} \left[I_a^\alpha X^{(n)}(t) + \sum_{j=0}^{n-1} \frac{(t-a)^{\alpha-n+j}}{\Gamma(\alpha-n+j+1)} X^{(j)}(a) \right] \\
&= I_a^n X^{(n)}(t) + \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) \\
&= \left[X(t) - \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) \right] + \sum_{j=0}^{n-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a) \\
&= X(t).
\end{aligned}$$

■

In the following two theorems we find expressions for $D_a^\alpha D_a^\beta X(t)$. In Theorem 3.2.13 we will consider the case when $\beta = m, m \in \mathbb{N}$, and in Theorem 3.2.14 we will consider the case when $\beta \in (m-1, m)$.

Theorem 3.2.13. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ and let $X(t)$ be a second-order stochastic process such that $X^{(m+n)}(t)$ exists for $t \in [a,b] \subset T$. Then, for $\alpha > 0$

$$D_a^\alpha X^{(m)}(t) = D_a^{\alpha+m} X(t).$$

Proof.

When $\alpha = n$, the result clearly holds.

$$\text{If} \quad n-1 < \alpha < n \quad \text{then} \quad n-1+m < \alpha+m < n+m$$

and so

$$D_a^{\alpha+m} X(t) = I_a^{(n+m)-(\alpha+m)} X^{(n+m)}(t) = I_a^{n-\alpha} X^{(n+m)}(t).$$

Using this, we have for $n-1<\alpha<n$

$$D_a^\alpha X^{(m)}(t) = I_a^{n-\alpha}\frac{d^n}{dt^n}X^{(m)}(t) = I_a^{n-\alpha}X^{(n+m)}(t) = D_a^{\alpha+m}X(t).$$

■

Theorem 3.2.14. Let $\alpha \in (n-1,n]$ and $\beta \in (m-1,m)$ where $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $X(t)$ be a second-order stochastic process such that $X^{(m+n)}(t)$ exists and is mean-square continuous on $[a,b] \subset T$. Then

(a) $D_a^\alpha D_a^\beta X(t) = D_a^{\alpha+\beta}X(t)$

for $\alpha \in (0,1)$ and $\alpha+\beta \in (m-1,m]$.

(b) $D_a^\alpha D_a^\beta X(t) = D_a^{\alpha+\beta}X(t) + \dfrac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)}X^{(m)}(a)$

for $\alpha \in (0,1)$ and $\alpha+\beta \in (m-1,m]$.

(c) $D_a^\alpha D_a^\beta X(t) = D_a^{\alpha+\beta}X(t) + \displaystyle\sum_{j=0}^{n-1}\frac{(t-a)^{m-\beta-n+j}}{\Gamma(m-\beta-n+1+j)}X^{(m+j)}(a)$

for $\alpha = n$.

Proof.

(a) For $\alpha \in (0,1)$

$$\begin{aligned}
D_a^\alpha D_a^\beta X(t) &= I_a^{1-\alpha}\left[\frac{d}{dt}I_a^{m-\beta}X^{(m)}(t)\right] \\
&= I_a^{1-\alpha}\left[I_a^{m-\beta}X^{(m+1)}(t) + \frac{(t-a)^{m-\beta-1}}{\Gamma(m-\beta)}X^{(m)}(a)\right] \\
&= I_a^{m-(\alpha+\beta)}I_a^1X^{(m+1)}(t) + \frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)}X^{(m)}(a). \qquad (3.2.12)
\end{aligned}$$

Thus, when $\alpha+\beta=m$, we have

$$\begin{aligned}
D_a^\alpha D_a^\beta X(t) &= I_a^1X^{(m+1)}(t) + X^{(m)}(a) \\
&= X^{(m)}(t) - X^{(m)}(a) + X^{(m)}(a) \\
&= X^{(m)}(t) \\
&= D_a^{\alpha+\beta}X(t).
\end{aligned}$$

When $\alpha+\beta\in(m-1,m)$, the first term on the right hand side of 3.2.12 becomes

$$\begin{aligned} I_a^{1-\alpha+m-\beta}X^{(m+1)}(t) &= I_a^{m-(\alpha+\beta)}[X^{(m)}(t)-X^{(m)}(a)] \\ &= I_a^{m-(\alpha+\beta)}X^{(m)}(t)-\frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)}X^{(m)}(a) \\ &= D_a^{\alpha+\beta}X(t)-\frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)}X^{(m)}(a). \end{aligned}$$

Substituting this into equation 3.2.12 we have, for $\alpha+\beta\in(m-1,m)$,

$$\begin{aligned} D_a^{\alpha}D_a^{\beta}X(t) &= D_a^{\alpha+\beta}X(t)-\frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)}X^{(m)}(a)+\frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)}X^{(m)}(a) \\ &= D_a^{\alpha+\beta}X(t). \end{aligned}$$

(b) If $\alpha\in(0,1)$ and $\alpha+\beta\in(m-1,m)$ then

$$\begin{aligned} D_a^{\alpha}D_a^{\beta}X(t) &= I_a^{1-\alpha}\left[\frac{d}{dt}I_a^{(m-\beta)}X^{(m)}(t)\right] \\ &= I_a^{1-\alpha}\left[I_a^{m-\beta}X^{(m+1)}(t)+\frac{(t-a)^{m-\beta-1}}{\Gamma(m-\beta)}X^{(m)}(a)\right] \\ &= I_a^{(m+1)-(\alpha+\beta)}X^{(m+1)}(t)+\frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)}X^{(m)}(a) \\ &= D_a^{\alpha+\beta}X(t)+\frac{(t-a)^{m-\beta-\alpha}}{\Gamma(m-\beta-\alpha+1)}X^{(m)}(a). \end{aligned}$$

(c) When $\alpha=n$ we have

$$\begin{aligned} D_a^{\alpha}D_a^{\beta}X(t) &= \frac{d^n}{dt^n}I_a^{(m-\beta)}X^{(m)}(t) \\ &= I_a^{(m-\beta)}\frac{d^n}{dt^n}X^{(m)}(t)+\sum_{j=0}^{n-1}\frac{(t-a)^{m-\beta-n+j}}{\Gamma(m-\beta-n+j+1)}X^{(m+j)}(a) \\ &= I_a^{(m+n)-(\beta+n)}X^{(m+n)}(t)+\sum_{j=0}^{n-1}\frac{(t-a)^{m-\beta-n+j}}{\Gamma(m-\beta-n+1+j)}X^{(m+j)}(a) \\ &= D_a^{n+\beta}X(t)+\sum_{j=0}^{n-1}\frac{(t-a)^{m-\beta-n+j}}{\Gamma(m-\beta-n+1+j)}X^{(m+j)}(a). \end{aligned}$$

■

3.3 The New Stochastic Fractional Operators

In this Section, we introduce the new stochastic fractional operators. Their fundamental properties are then given next. for more detail the about this operators see (Zine and all [12])

3.3.1 The New Stochastic Fractional Operators Definitions

We introduce the stochastic fractional operators by composing the classical fractional operators with the expectation $\mathbb{E}$. The new stochastic operators add to the standard notations an 's' for "stochastic".

Definition 3.3.1. Let X be a stochastic process on $[a,b] \subset I, \alpha > 0, n = [\alpha]+1$, such that $\mathbb{E}(X(t)) \in AC^n([a,b] \to \mathbb{R})$ with AC the class of absolutely continuous functions. Then,

(D1) the left stochastic Riemann-Liouville fractional derivative of order α is given by

$$\begin{aligned} {}_a^sD_t^\alpha X(t) &= {}_aD_t^\alpha[\mathbb{E}(X_t)] \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-1-\alpha}\mathbb{E}(X_\tau)d\tau, \quad t>a; \end{aligned}$$

(D2) the right stochastic Riemann-Liouville fractional derivative of order α by

$$\begin{aligned} {}_t^sD_b^\alpha X(t) &= {}_tD_b^\alpha[\mathbb{E}(X_t)] \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{-d}{dt}\right)^n \int_t^b (\tau-t)^{n-1-\alpha}\mathbb{E}(X_\tau)d\tau, \quad t<b; \end{aligned}$$

(D3) the left stochastic Riemann-Liouville fractional integral of order α is given by

$$\begin{aligned} {}_a^sI_t^\alpha X(t) &= {}_aI_t^\alpha[\mathbb{E}(X_t)] \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1}\mathbb{E}(X_\tau)d\tau, \quad t>a; \end{aligned}$$

(D4) the right stochastic Riemann-Liouville fractional integral of order α by

$$\begin{aligned} {}_t^sI_b^\alpha X(t) &= {}_tI_b^\alpha[\mathbb{E}(X_t)] \\ &= \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1}\mathbb{E}(X_\tau)d\tau, \quad t<b; \end{aligned}$$

(D5) the left stochastic Caputo fractional derivative of order α is given by

$$\begin{aligned} {}_a^{sC}D_t^\alpha X(t) &= {}_a^CD_t^\alpha[\mathbb{E}(X_t)] \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-1-\alpha}\mathbb{E}(X_\tau)^{(n)}d\tau, \quad t>a; \end{aligned}$$

(D6) and the right stochastic Caputo fractional derivative of order α by

$$\begin{aligned} {}^{sC}_{\;t}D^{\alpha}_{b}X(t) &= {}^{C}_{t}D^{\alpha}_{b}[\mathbb{E}(X_t)] \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-1-\alpha}\mathbb{E}(X_\tau)^{(n)}d\tau, \quad t<b; \end{aligned}$$

Remark. The stochastic processes $X(t)$ used can be of any type satisfying the announced conditions of existence of the novel stochastic fractional operators. For example, we can consider Levy processes as a particular case, provided one considers some intervals where $\mathbb{E}(X(t))$ is sufficiently smooth.

3.3.2 Basic properties of the stochastic fractional operators

Several properties of the classical fractional operators, like boundedness or linearity, also hold true for their stochastic counterparts.

Proposition 3.3.1. If $t \to \mathbb{E}(X_t) \in L_1([a,b])$, then ${}^{s}_{a}I^{\alpha}_{t}(X_t)$ is bounded.

Proof.

The property follows from definition **(D3)**:

$$|{}^{s}_{a}I^{\alpha}_{t}(X_t)| = \left|\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}\mathbb{E}(X_\tau)d\tau\right| \leq k\|\mathbb{E}(X_t)\|,$$

which shows the intended conclusion. ■

Proposition 3.3.2. The left and right stochastic Riemann-Liouville and Caputo fractional operators given in Definition 3.3.1 are linear operators.

Proof.

Let c and d be real numbers and assume that ${}^{s}_{a}D^{\alpha}_{t}X_t$ and ${}^{s}_{a}D^{\alpha}_{t}Y_t$ exist. It is easy to see that ${}^{s}_{a}D^{\alpha}_{t}(c\cdot X_t + d\cdot Y_t)$ also exists. From Definition 3.3.1 and by linearity of the expectation and the linearity of the classical/deterministic fractional derivative operator, we have

$$\begin{aligned} {}^{s}_{a}D^{\alpha}_{t}(c\cdot X_t + d\cdot Y_t) &= {}_{a}D^{\alpha}_{t}[\mathbb{E}(c\cdot X_t + d\cdot Y_t)] \\ &= c\cdot {}_{a}D^{\alpha}_{t}\mathbb{E}(X_t) + d\cdot {}_{a}D^{\alpha}_{t}\mathbb{E}(Y_t) \\ &= c\cdot {}^{s}_{a}D^{\alpha}_{t}(X_t) + d\cdot {}^{s}_{a}D^{\alpha}_{t}(Y_t) \end{aligned}$$

The linearity of the other stochastic fractional operators is obtained in a similar manner.

■

The next proposition involves both stochastic and deterministic operators.

Proposition 3.3.3. Assume that ${}^s_aI^\beta_t X_t$, ${}^s_tI^\beta_b X_t$, ${}^s_aI^\alpha_t X_t$, ${}_aD^\alpha_t\left[{}^s_aI^\alpha_t X_t\right]$, ${}_aI^\alpha_t\left[{}^s_aI^\beta_t X_t\right]$, and ${}_tI^\alpha_b\left[{}^s_tI^\beta_b X_t\right]$ exist. The following relations hold:

$$
\begin{aligned}
{}_aI^\alpha_t\left[{}^s_aI^\beta_t X_t\right] &= {}^s_aI^{\alpha+\beta}_t X_t,\\
{}_tI^\alpha_b\left[{}^s_tI^\beta_b X_t\right] &= {}^s_tI^{\alpha+\beta}_b X_t,\\
{}_aD^\alpha_t\left[{}^s_aI^\alpha_t X_t\right] &= \mathbb{E}(X_t).
\end{aligned}
$$

Proof.

Using Definition 3.3.1 and well-known properties of the deterministic Riemann-Liouville fractional operators see (Zine and all [12]), one has

$$
\begin{aligned}
{}_aI^\alpha_t\left[{}^s_aI^\beta_t X_t\right] &= {}_aI^\alpha_t\left[{}_aI^\beta_t \mathbb{E}(X_t)\right]\\
&= {}_aI^{\alpha+\beta}_t \mathbb{E}(X_t)\\
&= {}^s_aI^{\alpha+\beta}_t X_t.
\end{aligned}
$$

The second and third equalities are proved in a similar manner. ∎

Proposition 3.3.4. Let $\alpha > 0$. If $\mathbb{E}(X_t) \in L_\infty(a,b)$, then

$$
{}^C_aD^\alpha_t\left[{}^s_aI^\alpha_t X_t\right] = \mathbb{E}(X_t)
$$

and

$$
{}^C_tD^\alpha_b\left[{}^s_tI^\alpha_b X_t\right] = \mathbb{E}(X_t).
$$

Proof.

Using Definition 3.3.1 and well-known properties of the deterministic Caputo fractional operators, we have

$$
\begin{aligned}
{}^C_aD^\alpha_t\left[{}^s_aI^\alpha_t X_t\right] &= {}^C_aD^\alpha_t\left[{}_aI^\alpha_t \mathbb{E}(X_t)\right]\\
&= \mathbb{E}(X_t).
\end{aligned}
$$

The second formula is shown with the same argument. ∎

Lemma 3.3.1 (Stochastic fractional formulas of integration by parts)**.** Let $\alpha > 0, p, q \geq 1$, and $\dfrac{1}{p} + \dfrac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case where $\dfrac{1}{p} + \dfrac{1}{q} = 1 + \alpha$).

(i) If $\mathbb{E}(X_t) \in L_p(a,b)$ and $\mathbb{E}(Y_t) \in L_q(a,b)$ for every $t \in [a,b]$, then

$$\mathbb{E}\left(\int_a^b (X_t)\ {}_a^sI_t^\alpha Y_t dt\right) = \mathbb{E}\left(\int_a^b (Y_t)\ {}_t^sI_b^\alpha X_t dt\right).$$

(ii) If $\mathbb{E}(Y_t) \in\ {}_tI_b^\alpha(L_p)$ and $\mathbb{E}(X_t) \in\ {}_aI_t^\alpha(L_q)$ for every $t \in [a,b]$, then

$$\mathbb{E}\left(\int_a^b (X_t)\,({}_a^sD_t^\alpha Y_t) dt\right) = \mathbb{E}\left(\int_a^b (Y_t)({}_t^sD_b^\alpha X_t) dt\right).$$

(iii) For the stochastic Caputo fractional derivatives, one has

$$\mathbb{E}\left[\int_a^b (X_t)({}_a^{sC}D_t^\alpha Y_t) dt\right] = \mathbb{E}\left[\int_a^b (Y_t)({}_t^sD_b^\alpha X_t) dt\right] + \mathbb{E}\left[({}_t^sI_b^{1-\alpha} X_t)\cdot Y_t\right]_a^b$$

and

$$\mathbb{E}\left[\int_a^b (X_t)({}_t^{sC}D_b^\alpha Y_t) dt\right] = \mathbb{E}\left[\int_a^b (Y_t)({}_a^sD_t^\alpha X_t) dt\right] - \mathbb{E}\left[({}_a^sI_t^{1-\alpha} X_t)\cdot Y_t\right]_a^b$$

for $\alpha \in (0,1)$.

Proof.

(i) We have

$$\begin{aligned}
\mathbb{E}\left(\int_a^b (X_t)\ {}_a^sI_t^\alpha Y_t dt\right) &= \int_a^b \mathbb{E}((X_t)\ {}_a^sI_t^\alpha Y_t) dt \quad \text{(by Fubini-Tonelli's theorem)}\\
&= \int_a^b \mathbb{E}((X_t)\ {}_aI_t^\alpha \mathbb{E}(Y_t)) dt \quad \text{(by (D3))}\\
&= \int_a^b \mathbb{E}(X_t)\ {}_aI_t^\alpha \mathbb{E}(Y_t) dt \quad \text{(the expectation is deterministic)}\\
&= \int_a^b {}_tI_b^\alpha \mathbb{E}(X_t)\cdot \mathbb{E}(Y_t) dt \quad \text{(by fractional integration by parts)}\\
&= \mathbb{E}\left(\int_a^b {}_t^sI_b^\alpha (X_t)(Y_t) dt\right) \quad \text{(by Fubini-Tonelli's theorem [12]).}
\end{aligned}$$

(ii) With similar arguments as in item (i), we have

$$\begin{aligned}
\mathbb{E}\left(\int_a^b (X_t)\,({}_a^sD_t^\alpha Y_t) dt\right) &= \int_a^b \mathbb{E}((X_t)\ {}_a^sD_t^\alpha Y_t) dt\\
&= \int_a^b \mathbb{E}((X_t)\ {}_aD_t^\alpha \mathbb{E}(Y_t)) dt \quad \text{(by (D1))}\\
&= \int_a^b \mathbb{E}(X_t)\ {}_aD_t^\alpha \mathbb{E}(Y_t) dt\\
&= \int_a^b {}_tD_b^\alpha \mathbb{E}(X_t)\cdot \mathbb{E}(Y_t) dt\\
&= \mathbb{E}\left(\int_a^b {}_t^sD_b^\alpha (X_t)(Y_t) dt\right).
\end{aligned}$$

(iii) By using Caputo's fractional integration by parts formula we obtain that

$$\begin{aligned}
\mathbb{E}\left[\int_a^b (X_t)({}_a^{sC}D_t^{\alpha} Y_t)dt\right] &= \int_a^b \mathbb{E}[X_t]({}_a^{C}D_t^{\alpha}\mathbb{E}[Y_t])dt \\
&= \int_a^b ({}_tD_b^{\alpha}\mathbb{E}[X_t]\cdot\mathbb{E}[Y_t])dt + \left[{}_tI_b^{1-\alpha}\mathbb{E}[X_t]\cdot\mathbb{E}[Y_t]\right]_a^b \\
&= \int_a^b ({}_t^{s}D_b^{\alpha}(X_t)\cdot\mathbb{E}[Y_t])dt + \left[{}_tI_b^{1-\alpha}\mathbb{E}[X_t]\cdot\mathbb{E}[Y_t]\right]_a^b \\
&= \mathbb{E}\left[\int_a^b ({}_t^{s}D_b^{\alpha}(X_t)\cdot(Y_t))dt\right] + \mathbb{E}\left[{}_tI_b^{1-\alpha}\mathbb{E}[X_t]\cdot(Y_t)\right]_a^b .
\end{aligned}$$

The first equality of **(iii)** is proved. By using a similar argument and applying the integration by parts formula associated with the right Caputo fractional derivative, we get the second equality of (**(iii)**, Zine and all [12]).

■

Conclusion

The point of this master thesis was to ponder the deterministic fractional calculus and to display too the fractional calculus within the mean square setting. By extending some key concepts and results from deterministic fractional calculus to the random framework using the so-called mean square approach, by furnishing first extensions of important ideas from ordinary calculus such as continuity, differentiation, integration, and also to stochastic processes.

These extensions are known as the mean-square calculus. This enabled us to define some useful integral and derivative operators for a wide class of second-order random processes. In so doing, we derived further results on m.s. convergence, including the notion of random variables with finite mean-square value, that is, second-order random variables. Secondly, fractional derivatives and fractional integrals, shortly differintegrals are given, some preliminaries and definitions such as the Grünwald-Letnikov and Riemann-Liouville approaches for defining a differintegral are presented; then some of their important properties are proved. Finally, definitions for the mean square fractional integral and derivative based on some of the common definitions from the deterministic fractional calculus are presented , more over properties of m.s. fractional integrals and derivatives are demonstrated. At the same line of taught, other new stochastic fractional operators and their fundamental properties are obtained.

In conclusion, we hope and predict that research in this subject will be active and promising for quite some time to come as many questions remain unanswered. For example, it is possible to extend this proposed definition for many classes of stochastic fractional differential equations which will be considered by others for future work.

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