République Algérienne Démocratique et Populaire
Ministère de l'enseignement supérieur et de la recherche scientifique

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Mémoire présenté en vue de l'obtention du diplôme de

Master Académique

Filière : MATHEMATIQUES

Spécialité: Analyse Stochastique Statistique des Processus et Applications

par

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**Thème**:

# Estimation Locale Linéaire De La Fonction De Répartition Conditionnelle Pour Des Données Fonctionnelles.

Soutenue le 11/06/2022 devant le jury composé de

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Année univ.: 2021/2022

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# Contents

# Notations

| | |
|---|---|
| r.r.v. | real random variable. |
| f.r.v. | functional random variable. |
| i.i.d. | independent and identically distributed. |
| PCA | Principal Components Analysis. |
| NPFDA | NonParametric Functional Data Analysis. |
| CDF | Cumulative Distribution Function. |
| NWE | Nadaraya-Watson Estimator. |
| LLE | Local Linear Estimator. |
| LCE | Local Constant Estimator. |
| $\mathbb{N}$ | Set of positive natural numbers. |
| $\mathbb{R}$ | Set of real numbers. |
| $\mathfrak{F}$ | Semi-metric space. |
| $d(\cdot,\cdot)$ | Semi-metric on $\mathfrak{F}$. |
| $(\Omega,\mathcal{F},\mathbb{P})$ | Probability space. |
| $X$ | Functional random variable, (f.r.v.) |
| $X_i,\ i=1,\dots,n$ | Sample of f.r.v. |
| $x$ | Observations of f.r.v. |
| $Y$ | Real random variable, r.r.v. |
| $\mathbb{E}(X)$ | Mathematical expectation of the random variable $X$. |
| $Var(X)$ | Variance of the random variable $X$. |
| $\mathbb{E}(Y\|X)$ | Conditional expectation of $Y$ given $X$. |
| $Var(Y\|X)$ | Conditional variance of $Y$ given $X$. |
| $Cov(X,Y)$ | Covariance between random variables $X$ and $Y$. |
| $B(x,h)$ | Ball of center $x$ and radius h in the space $(\mathfrak{F},d)$. |
| $\phi_x(h)$ | Probability measure or concentration probability. |
| $F^x(y)$ | Conditional distribution of r.r.v. $Y$ given f.r.v $X_i=x$. |

| | |
|---|---|
| $\xrightarrow{a.co}$ | Almost complete convergence. |
| $\xrightarrow{P}$ | Convergence in probability. |
| $\xrightarrow{\mathcal{D}}$ | Convergence in distribution. |
| $O_{a.co.}$ | rate of almost complete convergence. |
| $O_p$ | rate of convergence in probability. |
| $:=$ | Definition of a quantity. |
| $C$ and $C'$ | Real positive constants. |
| $1\!\!1_{\{\}}$ | Indicator function. |
| $^t$ | Transpose symbol. |
| $\mathcal{N}(0, \sigma^2)$ | Standard normal random variable ( mean $\mu = 0$ , variance $\sigma^2$). |

# List of Figures

# Dedication

All praise to ALLAH. We tuck the day's effort, the accomplishment represented between the covers of this modest work. Looking back, it was all worth it.

Regret does not come from failure. It comes from giving up.
Mel Robbins

I dedicate this work to:

My parents who planted in me the spirit of perseverance and diligence to reach what I aspire to, who always accompanied me by their blessed prayers and sacrifices over the years in order to climb the ladders of success. May ALLAH bless them.

To my brothers and sisters, nephews and nieces.

To all my friends, my colleagues and to those who share with me my success.

To my professors from elementary school through graduation.

# Acknowledgments


The amount of praise to ALLAH, who had the lead and the journey of knowledge and education and was the source of strength to confront all difficulties.

I would like to experss my sincere thanks and appreciation to my supervisor Dr. Oussama Bouanani, who spared no effort in supporting this project throughout its preparation period. His thoughtful guidance, valuable information and constructive criticism prompted me to raise my work to a higher level and come out with this best picture.

My thankfulness is also extended to the committee members, Pr. Saâdia Rahmani, Dr. Fatima Benziadi for thier contributions to this work for offering me their valuable time and giving me a clear imprints through their directives comments in order to improve my work.

I would like to extend my heartfelt thankfulness to Dr. Rajaa Hazeb for kindly sharing her time and enlightening me to make this manuscript well written.

I also thank all the university stass, laboratory of stochastic models, statistic and applications and his members for their good assistante and facilitation that would give us a comfortable space to seek knowledge.

# Chapter 1

# Basic concepts in functional statistics

## 1 Functional Data

Functional data analysis has recently become an important tool in statistical research. it should be noted that classical statistics treats realizations of real random variables or random vectors for some random phenomena. However, there exists a data that emerge in continuous form (curves, surfaces, etc.) which can be considered as discretized functions (functions observed on a fairly fine discretization scale), so, it is obviously that infinite dimensional data are the so called functional data. This area of modern statistics has attracted the attention of many authors. From a historical point of view, the first work in this field dates back to Deville [16], Besse and Ramsay [5], Besse [6], where they approached factor analysis in functional cases, in particular principal component analysis of curves. More later, Ramsay and Silverman [36] treated these functional data also with factor analysis for regression models. Besse and Cardot [7] showed that functional regression is well fitted and more efficient than a vector approach. From a practical perspective, functional data is found in many applications. Among the most famous, we can cite: Müller et al. [27] on biological data, Chiou et al. [12] on demographic data.

## 1.1 Generation of functional data

In practice, the functional expression of the observed curves is unknown and we only have access to discrete observations measured at specific time.
we generated the functional covariate $X$ on the interval $[0, \pi]$ (see figure 1.1 ) by the following process:

$$X_i(\tau) = cos((A_i - 1) + \pi\tau) - sin(0.8(B_i - 2)\pi\tau), \text{ for } i = 1, 2, \ldots, 100.$$

where $\tau$ the discretization grid of 100 points in the interval $[0, \pi]$, $A_i$ are independent and identically distributed (i.i.d.) and following the normal distribution $N(1, 0.01)$, while the random variables $B_i$ are generated from a uniform distribution on the interval $[\frac{3}{2}, \pi]$ ($B_i \sim U([\frac{3}{2}, \pi])$). All the curves $X_i$'s are generated from 100 equidistant values in $[0, \pi]$.

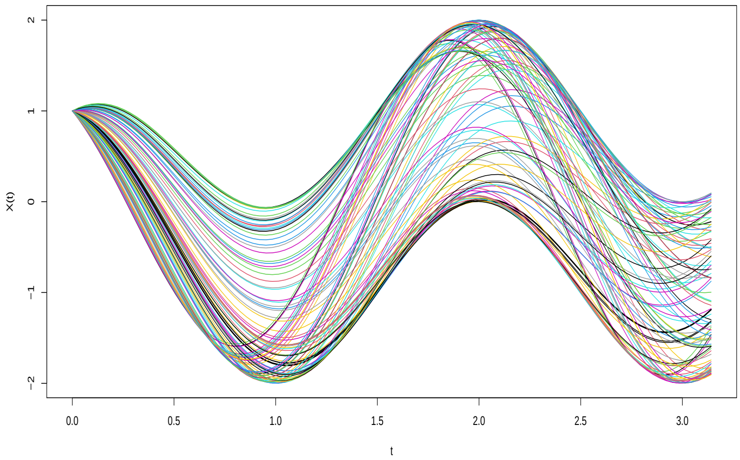



Figure 1.1: A sample of 100 curves

## 1.2 Some examples of applications in functional data

- **In biology**: An individual's (or a plant's) growth curve over time. The heights of girls and boys, between 1 and 18 years, are included in the growth data of the package fda, as an example. Figure (1.2) depicts the related curves for 39 boys and 54 girls.

- **In Chemometric Data**: The original data come from a quality control problem in the food industry and can be found at http://lib.stat.cmu.edu/datasets/tecator. Note that they were first studied by [BT92] using a neural networks approach. This dataset concerns a sample of finely chopped meat. (1.3) displays some units among the original spectrometric data.

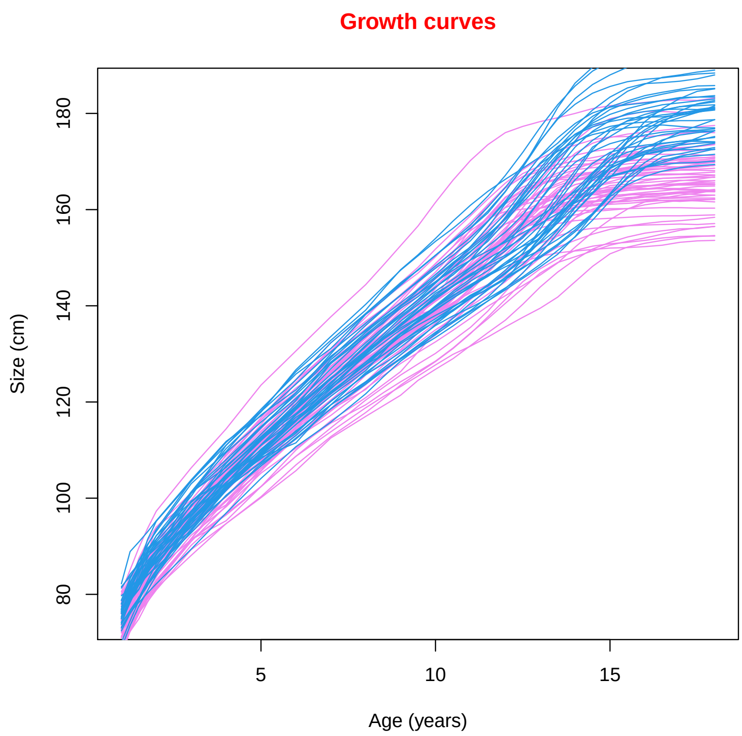



Figure 1.2: Growth charts of 39 boys and 54 girls, with age from 1 to 18.

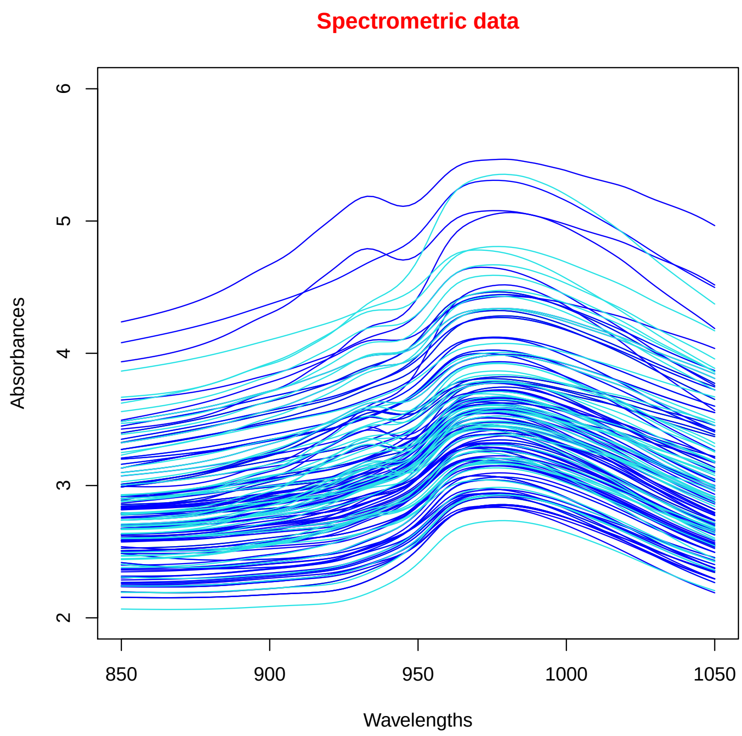



Figure 1.3: Spectrometric curves obtained from 150 pieces of meat.

# 2 Small ball probabilities and semimetrics

The curse of dimensionality is a well-known phenomenon in nonparametric regression on multivariate variable. In multivariate nonparametric regression, convergence rates (for the dispersion part) are expressed in terms of smoothing parameter $h_n^d$. In the functional case we adopt more general concentration notions called small ball probabilities to express our asymptotic results in function of these quantitities. Small ball probabilities are defined by:

$$\phi_x(h) = P(d(X, x) \leq h).$$

Consider the semimetric d. The choice of the $d$ has a direct influence on the topology and consequently on small ball probabilities. The diversity of semimetrics allows us to find a topology that gives a relevant notion of proximity between curves in various situations.

Now, we cite some examples of semimetrics used in functional statistics:

- The Semimetric used is the standard $L^2$ metric $d^{L^2}$ defined for all curves $x_i$ and $x_j$ as:

$$d^{L^2}(x_i, x_j) = \sqrt{\int (x_i(t) - x_j(t))^2 dt}$$

This definition is the natural functional extension of the vectorial $L^2$ metric. In that case, $d^{L^2}$ is a metric because it satisfies the indistinguishability axiom $(d^{L^2}(x_i, x_j) = 0 \Leftrightarrow x_i = x_j)$. $d^{L^2}$ uses the whole information contained in the curves and hence it may suffer from the curse of dimensionality.

- The Semimetric $d^{FPCA}$ is based on a functional principal components analysis. This pseudometric is defined for all curves $x_i$ and $x_j$ as:

$$d_q^{FPCA}(x_i, x_j) = \sqrt{\sum_{k=1}^{q} \left( \int |x_i(t) - x_j(t)| \, v_k(t) dt \right)^2}.$$

with $v_1, v_2, \ldots$ are the associated eigenfunctions of the covariance operator $\Gamma_x(t, t') = E\left[\bar{x}(t)\bar{x}(t')\right]$, with the decreasing eigenvalues $\mu_1 \geq \mu_2 \geq \ldots$ where $\bar{x}$ stands for the centered version of $x$. $q$ is the usual FPCA parameter which controls the dimension of the decomposition.

- The semi-metric of the functional index model: for a functional variable $x$ belongs to a Hilbert space $\mathbb{H}$ equipped with an inner product $\langle \cdot, \cdot \rangle$. We define the functional index model semi-metric by:

$$\forall x, y, \theta \in \mathbb{H} \quad, d_\theta(x, y) = \left| \langle x - y, \theta \rangle \right|.$$

We could estimate $\hat{\theta}$ by the cross-validation type techniques, which gives us the "best" prediction.

# 3 Some tools of probability

The main goal of this section is to provide inequalities and classical tools in non-parametric statistics that facilitate the theorems proofs of the following chapter.

## 3.1 Modes of convergence

### 3.1.1 Almost complete convergence

**Definition 3.1.1.** *[19]*
*We say that $(X_n)_{n\in\mathbb{N}}$ converges almost completely to some r.r.v. $X$, if and only*

$$\forall \varepsilon > 0, \sum_{n\in N} \mathbb{P}\left[|X_n - X| > \varepsilon\right] < \infty,$$

*and the almost complete convergence of $(X_n)_{n\in\mathbb{N}}$ to $X$ is denoted by*

$$\lim_{n\to\infty} X_n = X, a.co. \quad or \quad X_n \underset{n\to\infty}{\xrightarrow{a.co.}} X$$

**Definition 3.1.2.** *[19]*
*We say that the rate of almost complete convergence of $(X_n)_{n\in\mathbb{N}}$ to $X$ is of order $u_n$ if and only if*

$$\exists \varepsilon_0 > 0, \sum_{n\in N} \mathbb{P}\left[|X_n - X| > \varepsilon_0 u_n\right] < \infty,$$

*and we write*

$$X_n - X = \mathrm{O}_{a.co.}(u_n).$$

### 3.1.2 Convergence in probability

**Definition 3.1.3.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables defined on the same probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We say that $X_n$ converges to $X$ in probability if*

$$\forall \varepsilon > 0, \lim_{n\to\infty} \left(\mathbb{P}\left[|X_n - X| \geq \varepsilon\right]\right) = 0.$$

*Sometimes we note:* $\lim_{n\to\infty} X_n = X, p. \quad or \quad X_n \underset{n\to\infty}{\xrightarrow{p}} X.$

**Proposition 3.1.** *[19]*

*i) If* $X_n \xrightarrow[n\to\infty]{a.co.} X$ *and* $Y_n \xrightarrow[n\to\infty]{a.co.} Y$, *where* $X$ *and* $Y$ *are two deterministic real random variables. We have:*

1. $X_n + Y_n \xrightarrow[n\to\infty]{a.co.} X + Y$;
2. $X_n Y_n \xrightarrow[n\to\infty]{a.co.} XY$;
3. $\dfrac{1}{X_n} \xrightarrow[n\to\infty]{a.co.} \dfrac{1}{X}$ *as long as* $X \neq 0$.

*ii) If* $\lim\limits_{n\to\infty} u_n = 0$, $T_n - T = \mathrm{O}_{a.co}(u_n)$, *and* $S_n - S = \mathrm{O}_{a.co}(u_n)$, *we have:*

1. $(T_n + S_n) - (T + S) = \mathrm{O}_{a.co}(u_n)$;
2. $T_n S_n - TS = \mathrm{O}_{a.co}(u_n)$;
3. $\dfrac{1}{T_n} - \dfrac{1}{T} = \mathrm{O}_{a.co}(u_n)$ *as long as* $T \neq 0$.

*iii) If* $\lim\limits_{n\to\infty} u_n = 0$, $S_n = \mathrm{O}_{a.co}(u_n)$, *and* $X_n \xrightarrow[n\to\infty]{a.co.} l_X$, *where* $l_X$ *is a deterministic real number. We have:*

1. $S_n X_n = \mathrm{O}_{a.co}(u_n)$;
2. $\dfrac{S_n}{X_n} = \mathrm{O}_{a.co}(u_n)$ *as long as* $X \neq 0$.

**Theorem 3.1.1.** ***(Slutsky)****[23]*
*Let* $(X_n)_{n\in\mathbb{N}}$ *and* $(Y_n)_{n\in\mathbb{N}}$ *be two sequences of random variables defined on a probability space* $(\Omega; \mathcal{A}; \mathbb{P})$ *and* $X$ *a random variable.*
*If* $X_n$ *converge in law to* $X$, *and if* $Y_n$ *converges in probability to a constant* $c$ *then*

1- $X_n + Y_n \xrightarrow[n\to\infty]{D} X + c$

2- $X_n \times Y_n \xrightarrow[n\to\infty]{D} X \times c$

*3-* $\frac{X_n}{Y_n} \xrightarrow[n\to\infty]{D} \frac{X}{c}$, *with* $c \neq 0$.

## 3.2 Some useful inequalities

**Corollary 3.1.** *[19]*

*i) If* $\forall m \geq 2, \exists C_m > 0, E\left|Z_1^m\right| \leq C_m a^{2(m-1)}$, *we have:*

$$\forall \epsilon \geq 0, \mathbb{P}\left(\left|\sum_{i=1}^{n} Z_i\right| > \epsilon n\right) \leq 2\exp\left\{-\frac{\epsilon^2 n}{2a^2(1+\epsilon)}\right\}.$$

*ii) Assume that the i.i.d real random variables depend on n (i, e* $Z_i = Z_{i,n}$ *).*
*If* $\forall m \geq 2, \exists C_m > 0, E\left|Z_1^m\right| \leq C_m a_n^{2(m-1)}$ *and if* $u_n = n^{-1}a_n^2 \log n$ *verifies* $\lim_{n\to\infty} u_n = 0$, *we have:*

$$\frac{1}{n}\sum_{i=1}^{n} Z_i = O_{a.co.}\left(\sqrt{u_n}\right).$$

**Theorem 3.2.1.** ***(Jensen)*** *[32]*
*Let* $X$ *be a real random variable and* $\varphi$ *a convex function. So*

$$\varphi(\mathbb{E}(X)) \leq \mathbb{E}(\varphi(X)).$$

**Theorem 3.2.2.** ***(Markov)*** *[21]*
*Let* $X$ *be a real random variable. Then for all* $a > 0$ *:*

$$\mathbb{P}(|X| > a) < \frac{\mathbb{E}(|X|)}{a}.$$

**Theorem 3.2.3.** ***(Bienaymé-Tchebychev)*** *[8]*
*Let* $X$ *be a real random variable. Then for all* $a > 0$ *:*

$$\mathbb{P}(|X - \mathbb{E}(X)| > a) < \frac{Var(X)}{a^2}.$$

**Theorem 3.2.4.** ***(Hölder's inequality)[1]***
*Let $X$ and $Y$ be two random variables such that $X \in L^p(\Omega; \mathcal{A}; \mathbb{P})$ and $Y \in L^q(\Omega; \mathcal{A}; \mathbb{P})$ with $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$ and $p \geq 1, q \geq 1$, then:*

$$\mathbb{E}(|XY|^{\frac{1}{r}}) \leq \mathbb{E}(|X|^p)^{\frac{1}{p}} \mathbb{E}(|Y|^q)^{\frac{1}{q}}.$$

*If $p = 2$ and $q = 2$, we get the Cauchy-Schwarz inequality.*

## 3.3 Central limit theorem

**Lindeberg's Theorem**:[13]
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $X_k : \Omega \to \mathbb{R},\ k \in \mathbb{N}$ be independent random variables defined on that space. Assume that the expected values $\mathbb{E}[X_k] = \mu_k$, $\mathrm{Var}[X_k] = \sigma_k^2$ exist and are finite, let $S_n^2 = \sum_{i=1}^{n} \sigma_i^2$, If the sequence of independent random variables $X_k$ satisfies Lindeberg's condition:

$$\text{If, for all } \epsilon > 0,\ \lim_{n \to +\infty} \frac{1}{S_n^2} \sum_{i=1}^{n} \mathbb{E}[(X_i - \mu_i)^2 1_{|X_i - \mu_i| > \epsilon S_n}] = 0,$$

then the central limit theorem holds, i.e. the random variables $Z_n = \dfrac{1}{S_n} \sum_{i=1}^{n} (X_i - \mu_i)$, converge in distribution to a standard normal random variable $\mathcal{N}(0, 1)$.

## 3.4 Landau's notations

$o_P(1)$ ("small oh-P-one") is short notation for a sequence of random vectors that converges to zero in probability. The expression $O_P(1)$ ("big ohP-one") denotes a bounded sequence in probability. More generally, for a given sequence of random variables $R_n$, we have,

1- $X_n = o(R_n)$ means $X_n = Y_n R_n$ and $Y_n \underset{n \to \infty}{\longrightarrow} 0$;

2- $X_n = O(R_n)$ means $X_n < C R_n$, $C$ is a constant;

3- $X_n = o_P(R_n)$ means $X_n = Y_n R_n$ and $Y_n \xrightarrow[n\to\infty]{p} 0$;

4- $X_n = O_P(R_n)$ means $X_n = Y_n R_n$ and $Y_n = O_P(1)$.

This expresses that the sequence $X_n$ converges in probability to zero or is bounded in probability at the "rate" $R_n$. For deterministic sequences $(X_n)_n$ and $R_n$, the stochastic "oh" symbols reduce to the usual $o$ and $O$ from calculus.
There are many rules of calculus with $o$ and $O$ symbols, which we apply without comment, (for more details see [37]) . For instance

1- $o_P(1) + o_P(1) = o_P(1)$.

2- $o_P(1) + O_P(1) = O_P(1)$.

3- $O_P(1) o_P(1) = o_P(1)$.

4- $(1 + o_P(1))^{-1} = O_P(1)$.

# 4 Outline of the manuscript

This manuscript is organized in three chapters as follows:
We started with an introductory chapter, which is divided into three parts. In the first one, we focused on the basics of nonparametric statistics for functional data, this part was followed by some application examples in order to clarify the simulation of functional data using some R software codes and are packages to generate the functional data. Finally, this introduction ends with mathematical tools such as the inequalities and techniques utilized in proving the theorems.

In the second chapter, we presented a brief historical on the local linear method in the functional case. Moreover, we considered a sequence of independent and identically distributed observations to construct an estimator, by the local linear method, of the conditional distribution function. Then we studied, under certain conditions, the almost complete convergence of this estimator by specifying its convergence rate, beside that, we considered the same type of independency of observations as in the previous case, and then, we established under more conditions, the limit law of local linear estimator of the conditional distribution function.

The illustration of the proposed method via real data set and simulation with comparative study between the local linear and constant local method are analyzed in Chapter 3.

# Chapter 2

# Nonparametric functional modeling for the study of asymptotic properties

In this chapter we will present in details, the obtained results in non-parametric statistics in case of functional data using the local linear method. We focus on the estimation of the conditional distribution function of a real random variable Y conditioned by a functional random variable X (valued in an infinite dimensional space).

## 1 A brief history of the local linear method in the infinite dimension

Local linear modeling is an alternative statistical approach to kernel estimation, which has many advantages over the latter. In particular, the biggest advantage of the local linear method over the kernel method (local constant) is the reduction of the bias of the estimator and non-adaptation of the boundary effects spcially in finite dimension. Moreover, the kernel method can be treated as a particular case of the local linear method. Historically, Baìllo and Grané [3] first proposed a local linear estimator of the regression operator when the explanatory variable takes values in a Hilbert space. However, as Barrientos-Marin et al. [4] pointed, the estimator is a bit difficult in computation although it has a better performance than the kernel estimator. When the explanatory variable takes values in a semi-metric space, Barrientos-Marin et al. [4] proposed another alternative version of the local linear estimator of the regression operator in the i.i.d. setting, which was called locally modelled regression estimator.

They found that the estimator made its computation easy and fast while keeping good performance. After that, this method has been used to estimate the conditional density, the conditional distribution and the conditional mode, we mention some recent works. ( Bouanani et al. [10], Rahmani and Bouanani [35])

# 2 Construction of the local linear estimation of the conditional distribution for functional data

Let $(X_i, Y_i)_{1 \leq i \leq n}$ be a sequence of independent and identically distributed (i.i.d.) random vectors that we assume to be drawn from the pair $(X, Y)$ where the random variable (r.v.) $X$ belongs to a semi-metric space $\mathfrak{F}$ equipped with a semi metric $d$ (in most practical applications $\mathfrak{F}$ is a Banach space) and $Y$ is a real-valued r.v. For a fixed $x \in \mathfrak{F}$, we denote the conditional cumulative distribution function (CDF) of $Y_i$ given $X_i = x$, by:

$$\text{for all } y \in \mathbb{R}, \quad F^x(y) = \mathbb{P}(Y_i \leq y \mid X_i = x).$$

We are implicitly supposing that there exists a regular version of this conditional probability. Recall that if $d$ is a metric, existence is insured under general separability conditions. However, as far as we know, for semi-metric space this is still a field of probabilistic researches.

We focus on the estimation of the conditional distribution (CDF) of Y given $X = x$ via the local linear method. For this purpose, it is well known that the main idea, in the local linear smoothing, is based on the fact that the function $F^x(y)$ admits a linear approximation in the neighborhood of the conditioning point and that the (CDF) can be expressed as a regression model with the response variable $H(h_H^{-1}(\cdot - Y_i))$, where $H$ is cumulative distribution function and $(h_H = h_{H,n})$ is a sequence of positive real numbers. This consideration is steered by the following fact:

$$\mathbb{E}[H(h_H^{-1}(y - Y_i))|X_i = x] \to F^x(y) \text{ as } h_H \to 0$$

The approximation of $F^x$ can be expressed, for any $z$ in the neighborhood of $x$ by:

$$F^z(y) = \underbrace{F^x(y)}_{a} + \underbrace{F'^x(y)}_{b} \beta(z, x) + o(\beta(z, x))$$

In our setting, the local linear estimator denoted by $\hat{F}^x(y)$ is defined as the first component of the pair $(\widehat{a}, \widehat{b})$, obtained by the following minimization problem:

$$(\widehat{a}, \widehat{b}^{1}) = \arg \min_{(a,b)\in\mathbb{R}^2} \sum_{i=1}^{n} \big(H(h_H^{-1}(y-Y_i)) - a - b\beta(X_i,x)\big)^2 K(h_K^{-1}\delta(x,X_i)), \quad (2.1)$$

where the locating functions $\delta(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ are defining on $\mathfrak{F}\times\mathfrak{F}$ into $\mathbb{R}$ such that $\mid \delta(.,.) \mid= d(.,.)$, and $\beta$ refers to the local behaviour of our model. $K$ is a kernel, $H$ is a distribution function and $h_K = h_K, n$ (resp. $h_H = h_H, n$) are the smoothing parameters with respect to the kernels $K$ and $H$.
The solution of this minimization in problem 2.1 can be derived by using a matrix notations, from which it is defined by:

$$\widehat{a} = \widehat{F}^x(y) =^t e_1((^t\mathbf{Q}_\beta \mathbf{K} \mathbf{Q}_\beta)^{-1})^t\mathbf{Q}_\beta \mathbf{K}\, \mathbf{H}, \quad (2.2)$$

$^t\mathbf{Q}_\beta$ is the matrix defined by:

$$^t\mathbf{Q}_\beta = \begin{bmatrix} 1 & \cdots & 1 \\ \beta(X_1,x) & \cdots & \beta(X_n,x) \end{bmatrix},$$

$$\mathbf{H} =^t [\ H(h_H^{-1}(y-Y_1)), \ldots, H(h_H^{-1}(y-Y_n))]$$

and

$$\mathbf{K} = \begin{pmatrix} (K(h_k^{-1}\delta(X_1,x)) & 0 & \cdots & 0 \\ 0 & (K(h_k^{-1}\delta(X_2,x)) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & (K(h_k^{-1}\delta(X_n,x)) \end{pmatrix}$$

From Equation (2.2) and by some algebra, the quantity $\widehat{F}^x(y)$ is explicitly defined by

$$\hat{F}^x(y) \quad = \frac{\sum\limits_{i,i\neq j,j=1}^{n} W_{ij}H_j}{\sum\limits_{i,i\neq j,j=1}^{n} W_{ij}} = \frac{\sum\limits_{j=1}^{n} \Delta_j K_j H_j}{\sum\limits_{i,i\neq j,j=1}^{n} W_{ij}} \quad (2.3)$$

[1] $\widehat{b} = \widehat{F}'^x(y) =^t e_2((^t\mathbf{Q}_\beta \mathbf{K} \mathbf{Q}_\beta)^{-1})^t\mathbf{Q}_\beta \mathbf{K}\, \mathbf{H}$ where $e_2 =^t (0,1)$

where

$$W_{ij} = \beta_i\left(\beta_i - \beta_j\right) K_i K_j \text{ and } \Delta_j = K_j^{-1}\left(\sum_{i=1}^{n} W_{ij}\right) = \sum_{i=1}^{n} \beta_i^2 K_i - \left(\sum_{i=1}^{n} \beta_i K_i\right)\beta_j,$$

with $\beta_i = \beta(X_i, x)$, $\quad K_i = K\left(h_K^{-1}\left(\delta(x, X_i)\right)\right)$ $\quad$ and $\quad$ $H_j = H\left(h_H^{-1}(y - Y_j)\right)$.

# 3 Pointwise almost complete convergence

In this section, we establish the first asymptotic property which is the almost complete convergence of our estimator in the case where the observations are independent and identically distributed. This section is divided into three parts. For better readability, we group together in the first part all the assumptions used to establish this convergence. The second part is devoted to the main result of this chapter. The detailed proof of this result is given in the last part.

## 3.1 Hypotheses and notations

We define $\phi_x(r_1, r_2) = \mathbb{P}(r_2 \leq \delta(x, X) \leq r_1)$, where $r_1$ and $r_2$ are two real numbers and we denote the closed-ball in $\mathfrak{F}$ of center x and radius $r$ by $B(x, r) := \{x' \in \mathfrak{F} :\mid \delta(x, x') \mid \leq r\}$. We will denote by $C$ and $C'$ some strictly positive constants.

**(H.1) On the small ball probabilities of the functional variable** For any $h > 0$, $\phi_x(h) := \phi_x(-h, h) = P(X \in B(x, h)) > 0$,

Note that this assumption is very classic in the FDA context. In particular, we focus on the concentration function of the functional variable $X$ in small ball, which allows us to control the effect of the topological structure in the asymptotic results (see [18]).

**(H.2) On the regularity of the model**

The estimation of $F^X$ is related to the regularity condition. It is worth noting that the characterization of the functional space of our model is given by this assumption and it is needed to evaluate the asymptotic bias term. This condition is as follows:

$F^x(y)$ satisfies that there exist some positive constants $b_1$ and $b_2$, such that:

$$\mid F^{x_1}(y_1) - F^{x_2}(y_2) \mid \leq C_x \left(\delta^{b_1}(x_1, x_2) + \mid y_1 - y_2 \mid^{b_2}\right).$$

**(H.3) On the locating operators**

$$\text{for all } z \in \mathfrak{F}, \;\; C \mid \delta(x,z) \mid \leq \mid \beta(x,z) \mid \leq C' \mid \delta(x,z) \mid,$$

Assumption (H3) was introduced and commented for the first time in [4] and it plays an important role in our methodology, especially when we calculate the leading terms involved in our asymptotic result.

**(H.4) On the kernels**

(i) The kernel $K$ is a bounded and positive function which is supported within $[-1, 1]$ and for which the first derivative $K^{(1)}$ satisfies: $K(1) > 0$, $K^{(1)}(u) < 0$, for $u \in [-1, 1]$.

(ii) The kernel function $H$ is a bounded and differentiable function, and such that

$$\int_{\mathbb{R}} H^{(1)}(z)dz = 1, \text{ and } \int_{\mathbb{R}} \mid z \mid^{b_2} H^{(1)}(z)dz < \infty.$$

**(H.5) On the smoothing parameter $h_K$**

The bandwidth satisfies the following conditions: there exists a positive integer $n_0$ such that,

$$-\frac{1}{\phi_x(h_K)} \int_{-1}^{1} \phi_x(zh_K, h_K) \frac{d}{dz}\left(z^2 K(z)\right) dz > C_3 > 0 \text{ for } n > n_0.$$

$$\lim_{n\to\infty} n\phi_x(h_K) = \infty \text{ and } \lim_{n\to\infty} \frac{\log n}{n\phi_x(h_K)} = 0,$$

and

$$h_K \int_{B(x,h_K)} \beta(u,x)dP(u) = o\left(\int_{B(x,h_K)} \beta^2(u,x)\, dP(u)\right),$$

where $P(u)$ is the cumulative distribution of $X$.

## 3.2 Asymptotic property

**Theorem 3.2.1.** *[15] Under assumptions $(H.1)-(H.5)$, we obtain:*

$$|\widehat{F}^x(y) - F^x(y)| = O\left(h_K^{b_1} + h_H^{b_2}\right) + O\left(\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right), \quad a.co.$$

Before proving this result, it is necessary to present the following lemma, this lemma plays a crucial role in the proof of Theorem 3.2.1.

**Lemma 3.2.1.** *(see [4])*

(a) $\mathbb{E}\left[K_1^a\left|\beta_1^b\right|\right] \leq Ch_K^b\phi_x(h_K)$, *for all* $a > 0, b \geq 0$;

(b) $\mathbb{E}\left[K_1\beta_1^2\right] > Ch_K^2\phi_x(h_K)$;

**proof of (a)** One starts by using the assumption $(H.3)$, which implies

$$K_1^a|\beta_1|^b h_K^{-b} \leq CK_1^a|\delta(X_1,x)|^b h_K^{-b}$$

and because the kernel $K$ is bounded, one gets

$$K_1^a|\beta_1|^b h_K^{-b} \leq C' 1_{B(x,h_K)}|\delta(X_1,x)|^b h_K^{-b}$$

and thus, we have

$$\begin{aligned} \mathbb{E}\left[K_1^a|\beta_1|^b h_K^{-b}\right] &\leq C\int_{B(x,h_K)} dP_X(v) \\ \mathbb{E}\left[K_1^a|\beta_1|^b\right] &\leq Ch_K^b\phi_x(h_K) \end{aligned}$$

which is the claimed result.

**proof of (b)** Under assumption (H.3), we have

$$\mathbb{E}\left[K_1\beta_1^2\right] > C\mathbb{E}\left[\delta^2(X_1,x)K_1\right].$$

Moreover, it is clear that:

$$
\begin{aligned}
\mathbb{E}\left[K_1^a \delta_1^b\right] &= h_K^b \int_{-1}^{1} v^b K_1^a(v) dP_X^{\delta(X_1,x)/h_K}(v) \\
&= h_K^b \int_{-1}^{1} \left[K_1^a(1) - \int_{v}^{1} \left(u^b K_1^a(u)\right)^{(1)} du\right] dP_X^{\delta(X_1,x)/h_K}(v) \\
&= h_K^b \left[K_1^a(1)\phi_x(h_K) - \int_{-1}^{1} \left(u^b K_1^a(u)\right)^{(1)} \phi_x(-h_K, uh_K) du\right] \quad (2.4)
\end{aligned}
$$

Finally, under assumption (H.5), with $a = 1$ and $b = 2$ we get:

$$
\mathbb{E}\left[K_1 \beta_1^2\right] > C h_K^2 \phi_x(h_K),
$$

which is the claimed result

## 3.3 proof of Theorem 3.2.1

The proof of this theorem is based on the following decomposition

$$
\begin{aligned}
\widehat{F}^x(y) - F^x(y) &= \frac{1}{\widehat{F}_D^x} \left\{ \left(\widehat{F}_N^x(y) - \mathbb{E}[\widehat{F}_N^x(y)]\right) - \left(F^x(y) - \mathbb{E}[\widehat{F}_N^x(y)]\right) \right\} \\
&+ \frac{F^x(y)}{\widehat{F}_D^x} \left(1 - \widehat{F}_D^x\right),
\end{aligned}
$$

where

$$
\widehat{F}_N^x(y) = \frac{1}{n(n-1)\mathbb{E}[W_{12}]} \sum_{i, i\neq j, j=1}^{n} W_{ij} H(h_H^{-1}(y - Y_j)),
$$

and

$$
\widehat{F}_D^x = \frac{1}{n(n-1)\mathbb{E}[W_{12}]} \sum_{i, i\neq j, j=1}^{n} W_{ij}.
$$

Thus, this theorem is a direct consequence of the following lemmas:

**Lemma 3.3.1.** *Under assumptions* $(H.1), (H.2), (H.4)$ *and* $(H.5)$*, we obtain:*

$$\left| F^x(y) - \mathbb{E}[\hat{F}_N^x(y)] \right| = O\left( h_K^{b_1} + h_H^{b_2} \right).$$

**Lemma 3.3.2.** *Under the assumptions of Theorem 3.2.1, we get:*

$$\left| \hat{F}_N^x(y) - \mathbb{E}[\hat{F}_N^x(y)] \right| = O\left( \sqrt{\frac{\log n}{n\phi_x(h_K)}} \right), a.co$$

**Lemma 3.3.3.** *Under assumptions* $(H.1), (H.3), (H.4)$ *and* $(H.5)$*, we have that:*

**i)** $1 - \widehat{F}_D^x = O\left( \sqrt{\frac{\log n}{n\phi_x(h_K)}} \right), \quad a.co.$

**ii)** $\exists\, \delta > 0 \quad \textit{such as} \quad \sum_{n=1}^{\infty} \mathbb{P}\left( \widehat{F}_D^x < \delta \right) < \infty.$

### 3.3.1 Proof of lemma 3.3.1

Since the pairs $(X_i, Y_i)$ are identically distributed, then:

$$\mathbb{E}[\hat{F}_N^x(y)] = \frac{1}{E[W_{12}]} \mathbb{E}\left[ W_{12} \left[ \mathbb{E}\left[ H_2 \,|X_2 \right] \right] \right]$$

Next, we use an integration by part to show that:

$$\mathbb{E}\left[ H_2 \,|X_2 \right] = h_H^{-1} \int_{\mathbb{R}} H^{(1)}\left( h_H^{-1}(y-z) \right) F^X(z)\, dz$$

Now, the change of variables $t = \dfrac{y-z}{h_H}$ allows us to write:

$$\left| \mathbb{E}\left[ H_2 \,|X_2 \right] - F^x(y) \right| \le \int_{\mathbb{R}} H^{(1)}(t) \left| F^X(y - th_H) - F^x(y) \right| dt$$

Thus, from assumptions $(H.2)$ and $(H.4, ii)$ we get:

$$1_{B(x,h_K)}(X) \left| \mathbb{E}\left[ H_2 \,|X_2 \right] - F^x(y) \right| \le \int_{\mathbb{R}} H^{(1)}(t) \left( h_K^{b_1} + |t|^{b_2} h_H^{b_2} \right) dt.$$

Then,

$$|F^x(y) - \mathbb{E}[\widehat{F}_N^x(y)]| = O\left(h_K^{b1} + h_H^{b2}\right).$$

$\square$

### 3.3.2 Proof of lemma 3.3.2

$$\widehat{F}_N^x(y) = S_1(S_2S_3 - S_4S_5). \tag{2.5}$$

where

$$S_1 = \frac{n^2 h_K^2 \phi_x(h_K)^2}{n(n-1)\mathbb{E}(w_{12})}, \qquad S_2 = \frac{1}{n}\sum_{j=1}^{n}\frac{K_j H_j}{\phi_x(h_K)}, \qquad S_3 = \frac{1}{n}\sum_{i=1}^{n}\frac{K_i\beta_i^2}{h_K^2\phi_x(h_K)},$$

$$S_4 = \frac{1}{n}\sum_{j=1}^{n}\frac{K_j\beta_j H_j}{h_K\phi_x(h_K)} \quad \text{and} \quad S_5 = \frac{1}{n}\sum_{i=1}^{n}\frac{K_i\beta_i}{h_K\phi_x(h_K)}$$

which allows us to write

$$\begin{aligned}
\widehat{F}_N^x(y) - \mathbb{E}[\widehat{F}_N^x(y)] &= S_1[(S_2S_3 - S_4S_5) - \mathbb{E}[S_2S_3 - S_4S_5]], \\
&= S_1[(S_2S_3 - \mathbb{E}[S_2S_3]) - (S_4S_5 - \mathbb{E}[S_4S_5])].
\end{aligned}$$

Moreover, we notice that:

$$\begin{aligned}
S_2S_3 - \mathbb{E}[S_2S_3] &= (S_2 - \mathbb{E}[S_2])(S_3 - \mathbb{E}[S_3]) + (S_3 - \mathbb{E}[S_3])\mathbb{E}[S_2] \\
&+ (S_2 - \mathbb{E}[S_2])\mathbb{E}[S_3] + \mathbb{E}[S_2]\mathbb{E}[S_3] - \mathbb{E}[S_2S_3].
\end{aligned}$$

And similarly:

$$\begin{aligned}
S_4S_5 - \mathbb{E}[S_4S_5] &= (S_4 - \mathbb{E}[S_4])(S_5 - \mathbb{E}[S_5]) + (S_5 - \mathbb{E}[S_5])\mathbb{E}[S_4] \\
&+ (S_4 - \mathbb{E}[S_4])\mathbb{E}[S_5] + \mathbb{E}[S_4]\mathbb{E}[S_5] - \mathbb{E}[S_4S_5].
\end{aligned}$$

So, the claimed result will be obtained as soon as the following assertions have been checked:

$$S_1 = O(1), \quad \mathbb{E}[S_l] = O(1) \quad \text{for} \quad l = 2,3,4,5, \tag{2.6}$$

$$S_i - \mathbb{E}[S_i] = O_{p.co.}\left(\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right), \quad \text{for} \quad i = 2,3,4,5, \tag{2.7}$$

$$Cov(S_2, S_3) = O\left(\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) \tag{2.8}$$

and

$$Cov(S_4, S_5) = O\left(\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right). \tag{2.9}$$

- **Proof of the result(2.6):**

◦ By applying lemma (3.2.1), we have

$$\begin{aligned}\mathbb{E}[W_{12}] &= \mathbb{E}[\beta_1^2 K_1]\mathbb{E}[K_1] \ - \ (\mathbb{E}[\beta_1 K_1])^2, \\ &> Ch_K^2(\phi_x^2(h_K)).\end{aligned}$$

We can deduce $S_1 = O(1)$.

◦ New, we show that $\mathbb{E}[S_l] = O(1) \quad$ for $l = 2, 3, 4, 5$, we have:

$$\begin{aligned}\mathbb{E}[S_l] &= \mathbb{E}\left[\frac{1}{n}\sum_{j=0}^{n}\frac{K_j H_j}{\phi_x(h_K)}\right]. \\ &= \phi_x^{-1}(h_K)\mathbb{E}[K_1 H_1], \\ &= \phi_x^{-1}(h_K)\mathbb{E}[K_1].\end{aligned}$$

According to lemma (3.2.1), we have $\mathbb{E}[K_1] \leq C\phi_x(h_K)$, which shows that $\mathbb{E}[S_2] = O(1)$.

By following the same reasoning, we can show that

$$\begin{aligned}\mathbb{E}[S_3] &= h_K^{-2}\phi_x^{-1}(h_K)\mathbb{E}[\beta_1^2 K_1] &= O(1), \\ \mathbb{E}[S_4] &= h_K^{-1}\phi_x^{-1}(h_K)\mathbb{E}[\beta_1 K_1 H_1] &= O(1), \\ \mathbb{E}[S_5] &= h_K^{-1}\phi_x^{-1}(h_K)\mathbb{E}[\beta_1 K_1] &= O(1).\end{aligned}$$

- **Proof of the result(2.7)**: We have

$$S_{l,k} - \mathbb{E}[S_{l,k}] = \frac{1}{n}\sum_{i=1}^{n} Z_i^{l,k} \quad \text{for} \quad l = 0,1,2 \quad \text{and} \quad k = 0,1,$$

where $\quad Z_i^{l,k} = \dfrac{1}{h_K^l \phi_x(h_K)}\left(K_i H_i^k \beta_i^l - \mathbb{E}[K_i H_i^k \beta_i^l]\right).$

From the expansion of Newton's binomial, we have:

$$\begin{aligned}
\mathbb{E}[|Z_i^{l,k}|^m] &= \mathbb{E}\left|h_K^{-lm}\phi_x(h_K)^{-m}\left(K_i H_i^k \beta_i^l \ - \ \mathbb{E}[K_i H_i^k \beta_i^l]\right)^m\right|, \\
&= h_K^{-lm}\phi_x(h_K)^{-m}\mathbb{E}\left|\sum_{d=0}^{m} C_m^d \left(K_i H_i^k \beta_i^l\right)^d \ \left(\mathbb{E}[K_i H_i^k \beta_i^l]\right)^{m-d} \ (-1)^{m-d}\right|, \\
&\leq h_K^{-lm}\phi_x(h_K)^{-m}\sum_{d=0}^{m} C_m^d \mathbb{E}\left|K_i H_i^k \beta_i^l\right|^d \ \left|\mathbb{E}[K_i H_i^k \beta_i^l]\right|^{m-d}, \\
&\leq h_K^{-lm}\phi_x(h_K)^{-m}\sum_{d=0}^{m} C_m^d \mathbb{E}\left|K_1^d \beta_i^{ld} \ \mathbb{E}[H_1^{dk}/X_1]\right| \ \left|\mathbb{E}[K_1 \beta_1^l \mathbb{E}[H_1^k/X_1]]\right|^{m-d},
\end{aligned}$$

where $C_m^d = \dfrac{m!}{d!(m-d)!}$.

Using the proof of Lemma (3.3.1) and taking $H$ by $H^d$, $\forall\, d \leq m$ :

$$\mathbb{E}[H_1^d/X_1] = \int_R (H^d(t))^{(1)} F^x(y - h_H t)dt.$$

Thus, according to the hypotheses $(H.2)$ and $(H.4)$, we obtain:

$$\mathbb{E}[H_1^{dk}/X_1] = O(1), \quad \forall\, d \leq m \ \text{ and } \ k = 0,1.$$

Then

$$\begin{aligned}
\mathbb{E}[|Z_i^{l,k}|^m] &= O\left(h_K^{-lm}\phi_x(h_K)^{-m}\sum_{d=0}^{m}\mathbb{E}\left[K_1^d \beta_1^{ld}\right] \ \left(\mathbb{E}[K_1 \beta_1^l]\right)^{m-d}\right). \\
&= O\left(\max_{d\in\{0,\dots,m\}} \phi_x(h_K)^{-d+1}\right). \\
&= O\left(\phi_x(h_K)^{-m+1}\right).
\end{aligned}$$

for $m = 2$ we can write $\mathbb{E}\left(Z_i^{l,k^2}\right) \leq \dfrac{C}{\phi_x\left(h_K\right)}$

Finally, it suffices to apply Corollary 3.1, we get:

$$\begin{aligned}
\mathbb{P}\left(|S_{l,k} - \mathbb{E}[S_{l,k}]| > \eta\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) &= \mathbb{P}\left(\frac{1}{n}\left|\sum_{i=0}^{n} Z_i^{l,k}\right| > \eta\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right)\\
&\leq 2\exp\left(-Cn\frac{\eta^2\log n}{n\phi_x(h_K)}\phi_x(h_K)\right)\\
&\leq C'n^{-C\eta^2}.
\end{aligned}$$

By choosing $\eta$ such that $C\eta^2 = 1+\alpha$, we get:

$$\mathbb{P}\left(|S_{l,k} - \mathbb{E}[S_{l,k}]| > \eta\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) \leq C'n^{-1-\alpha} \quad \text{for } \ l = 0,1,2 \quad \text{and} \quad k = 0,1.$$

which gives the result.

$$S_i - \mathbb{E}[S_i] = O_{p.co.}\left(\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right), \quad \text{for} \quad i = 2,3,4,5.$$

- **Proof of the results (2.8) et (2.9)**

For the both equations we use the fact that the pairs $(X_i, Y_i), i = 1, ..., n$ are identically distributed. Thus, we obtain:

$$\begin{cases}
Cov(S_2, S_3) = \dfrac{1}{nh_K^2\phi_x^2\left(h_K\right)}\left[\mathbb{E}[K_1^2H_1\beta_1^2] - \mathbb{E}[K_1H_1]\mathbb{E}[K_1\beta_1^2]\right]\\
and\\
Cov(S_4, S_5) = \dfrac{1}{nh_K^2\phi_x^2\left(h_K\right)}\left[\mathbb{E}[K_1^2H_1\beta_1^2] - \mathbb{E}[K_1H_1\beta_1]\mathbb{E}[K_1\beta_1]\right]
\end{cases}$$

So, for the both results, we have to evaluate:

$$\mathbb{E}[K_iH_i^k\beta_i^l] \ \text{ for } \ l = 0,1,2 \ \text{ and } \ k = 0,1$$

Once again, as $H < 1$ then for all $l = 0,1,2$ and $k = 0,1$ we obtain:

$$\mathbb{E}[K_iH_i^k\beta_i^l] = O\left(\mathbb{E}[K_i\beta_i^l]\right)$$

and by Lemma 3.2.1, we obtain that:

$$\mathbb{E}[K_i H_i^k \beta_i^l] = O\left(h_K^l \phi_x\left(h_K\right)\right)$$

which implies that:

$$Cov(T_2, T_3) \quad = \quad O\left(\frac{1}{n\phi_x\left(h_K\right)}\right) = o\left(\frac{\log n}{n\phi_x\left(h_K\right)}\right)$$

and

$$Cov(T_4, T_5) \quad = \quad O\left(\frac{1}{n\phi_x\left(h_K\right)}\right) = o\left(\frac{\log n}{n\phi_x\left(h_K\right)}\right)$$

□

### 3.3.3 Proof of lemma 3.3.3

**proof of Eq (i))**

The result of this part is established by following the same idea used in the proof of the Lemma 3.3.2

**Proof of Eq (ii)**

We assume that $\delta = \frac{1}{2}$, we have

$$\left\{\widehat{F}_D^x \leq \frac{1}{2}\right\} \quad \Longrightarrow \quad |1 \;-\; \widehat{F}_D^x| \;\geq \frac{1}{2}.$$

Then,

$$\mathbb{P}\left\{\widehat{F}_D^x \leq \frac{1}{2}\right\} \quad \leq \quad \mathbb{P}\left\{|1 - \widehat{F}_D^x| \;\geq \frac{1}{2}\right\}.$$

Therefore,

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{|\widehat{F}_D^x| \leq \frac{1}{2}\right\} \quad \leq \quad \sum_{n=1}^{\infty} \mathbb{P}\left\{|1 \;-\; \widehat{F}_D^x| \;\geq \frac{1}{2}\right\} \leq \infty.$$

□

# 4 Asymptotic normality of the nonparametric local linear estimator of the conditional cumulative distribution

In this section, we study the limit law of the local linear estimator of the conditional distribution function, by specifying the explicit expression of the asymptotically dominant terms of bias and variance. We gather all the assumptions used to establish our asymptotic result. Then, we state the theorem with detailed proof.

## 4.1 Main results

In order to establish the asymptotic convergence of $\widehat{F}^x(y)$ we need some notations and assumption.

- Firstly, for $l \in \{0, 2\}$,

$$\varphi_l(., y) = \frac{\partial^l F^{\cdot}(y)}{\partial y^l} \text{ and } \psi_l(s) = I\!E\left[\varphi_l(X, y) - \varphi_l(x, y) | \beta(X, x) = s\right],$$

- The quantities $M_j$ and $N(a, b)$, which will appear in the bias and variance dominant terms:

$$M_j = K^j(1) - \int_{-1}^{1} (K^j(u))^{(1)} \Psi_x(u) du \text{ where } j = 1, 2,$$

$$\text{and for all } a > 0 \text{ and } b = 1, 2, \quad N(a, b) = K^a(1) - \int_{-1}^{1} (u^b K^a(u))^{(1)} \Psi_x(u)(u) du.$$

- To simplify the proofs of our results let us note

$$\widehat{F}^x(y) = \frac{\widehat{F}^x_N(y)}{\widehat{F}^x_D},$$

where

$$\widehat{F}^x_N(y) = \frac{1}{n\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^{n} \Delta_j K_j H_j \text{ and } \widehat{F}^x_D = \frac{1}{n\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^{n} \Delta_j K_j,$$

We set the following hypotheses which will be needed to enounce our result

(M1) • The hypothesis (H.1) holds and there exists a function $\Psi_x(\cdot)$:

$$\forall u \in [-1,1],\ \lim_{h\to 0} \frac{\phi_x(-h, uh)}{\phi_x(u)} = \Psi_x(u)$$

such that $\Psi_x(\cdot)$ intervenes in all our asymptotic results, it is particulary needed for the calculation of $\mathbb{E}(K^j)$ with $j = 1.2$. We refer to [20], for some examples of calculating $\Psi_x(\cdot)$.

(M2) • For any $l \in \{0, 2\}$, the quantities $\psi_l^{(2)}(0)$ exist, where $F^{(k)}$ denotes the kth order derivative of F.

(M3) • The hypothesis (H.3) holds, and:

$$\sup_{v\in B(x,r)} \mid \beta(v,x) - \delta(x,v) \mid = o(r).$$

(M4) • The kernel $K$ satisfies the assumption (H.4, i) and its first derivative $K^{(1)}$ satisfies:

$$K^2(1) - \int_{-1}^{1} (K^2(u))^{(1)} \Psi_x(u)du > 0.$$

(M5) • The kernel $H$ satisfies (H.4, ii) and its first derivative $H^{(1)}$ is symmetric and

$$\int_{\mathbb{R}} z^2 H^{(1)}(z)dz < \infty.$$

(M6) • On the functional space:

For all $(x_1, x_2, y_1, y_2) \in \mathcal{N}_x \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{N}_y$:

$$\begin{cases} F : \mathfrak{F} \times \mathbb{R} \longrightarrow \mathbb{R}, \lim\limits_{|\delta(x_1,x_2)|\to 0} F^{x_1}(y) = F^{x_2}(y), \\ \qquad \text{and} \\ \lim\limits_{|y_1-y_2|\to 0} F^x(y_1) = F^x(y_2). \end{cases}$$

This assumption is a continuity-type which allow us to get the pointwise convergence. The reason behind this hypothesis lies in the fact that, with an appropriate choice of the semi-metric $d$, our functional space can be identified.

**Theorem 4.1.1.** *[9] Under assumptions (M1),(M3)-(M6) and (H.5) we obtain*

$$\sqrt{n\phi_x(h_K)}(\widehat{F}^x(y) - F^x(y) - B_n(x,y)) \xrightarrow{D} \mathcal{N}(0, V_{HK}(x,y)),$$

where

$$V_{HK}(x,y) = \frac{M_2}{M_1^2} F^x(y)(1 - F^x(y)), \tag{2.10}$$

and

$$B_n(x,y) = \frac{\mathbb{E}(\widehat{F}_N^x(y))}{\mathbb{E}(\widehat{F}_D^x)} - F^x(y). \tag{2.11}$$

**Remark 4.1.** *As mentioned in Demongeot et al. [14], the function $\phi_x(t)$ can be empirically estimated by*

$$\widehat{\phi}_x(t) = \frac{\#\{i : \ |\delta(X_i,x)| \leq t\}}{n},$$

*where $\#(A)$ denotes the cardinality of the set $A$. So, if we take advantage of the following assumption,*

*(M7)* $\lim\limits_{n\to\infty} \sqrt{n\phi_x(h_K)} B_n(x,y) = 0,$

*we can cancel the bias term and obtain the following corollary.*

**Corollary 4.1.** *Under the conditions of the theorme (4.1.1) and by the assumption (M7), we have the following asymptotic result*

$$\sqrt{\frac{n\phi_x(h_K)}{V_{HK}(x,y)}} (\widehat{F}^x(y) - F^x(y)) \xrightarrow{D} \mathcal{N}(0,1).$$

## 4.2 Proofs

***Proof of Theorem 4.1.1.*** Remark that

$$\widehat{F}^x(y) - F^x(y) - B_n(x,y) = \frac{\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x - \widehat{F}_D^x B_n(x,y)}{\widehat{F}_D^x}.$$

If,

$$\begin{aligned} Q_n(x,y) &= \widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x - \mathbb{E}\left(\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x\right) \\ &= \widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x - B_n(x,y), \end{aligned} \tag{2.12}$$

since

$$\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x = Q_n(x,y) + B_n(x,y),$$

then, the proof of this theorem will be completed from the following expression

$$\widehat{F}^x(y) - F^x(y) - B_n(x,y) = \frac{Q_n(x,y) - B_n(x,y)(\widehat{F}_D^x - \mathbb{E}(\widehat{F}_D^x))}{\widehat{F}_D^x}, \tag{2.13}$$

Moreover, in addition to lemma 3.2.1, the following auxiliary lemmas, which play a key role in the proof of our result

**Lemma 4.2.1.** *(see [39] and [38] )*

(c) $\mathbb{E}\left[K_1^a\right] = M_a\phi_x(h_K) + o\left(\phi_x(h_K)\right)$, *for all* $a > 0$;

(d) $\mathbb{E}\left[K_1^a\beta_1\right] = o\left(h_K\phi_x(h_K)\right)$, *for all* $a > 0$;

(e) $\mathbb{E}\left[K_1^a\beta_1^b\right] = N(a,b)\, h_K^b\phi_x(h_K) + o\left(h_K^b\phi_x(h_K)\right)$, *for all* $a > 0, b > 1$;

(f) $\dfrac{\mathbb{E}(K_1\Delta_1)}{n-1} = \mathbb{E}(W_{12}) = N(1,2)M_1h_K^2\phi_x^2(h_K) + o\left(h_K^2\phi_x^2(h_K)\right)$.

**Lemma 4.2.2.** *(see [15]).*

1. $\mathbb{E}(\widehat{F}_N^x(y)) - F^x(y) = B_H(x,y)h_H^2 + B_K(x,y)h_K^2 + o(h_K^2) + o(h_K^2)$,

2. $Var(\hat{F}^x(y)) = \dfrac{V_{HK}(x,y)}{n\phi_x(h_K)} + o\left(\dfrac{1}{n\phi_x(h_K)}\right)$,

3. $Cov(\widehat{F}_N^x(y), \widehat{F}_D^x(y)) = o\left(\dfrac{1}{n\phi_x(h_K)}\right)$,

4. $Var(\widehat{F}_D^x(y)) = o\left(\dfrac{1}{n\phi_x(h_K)}\right)$,

*where*

$$B_H(x,y) = \frac{1}{2}\frac{\partial^2 F^x(y)}{\partial y^2}\int t^2 H^{(1)}(t)dt \ , \ B_K(x,y) = \frac{1}{2}\psi_0^{(2)}(0)\frac{N(1,2)}{M_1},$$

*and*

$$V_{HK}(x,y) = F^x(y)(1-F^x(y))\frac{M_2}{M_1^2}.$$

**Lemma 4.2.3.** *Under assumptions of Theorem 4.1.1, we have*:

$$\widehat{F}_D^x \xrightarrow{P} I\!\!E(\widehat{F}_D^x) = 1.$$

**Lemma 4.2.4.** *Under assumptions (H.1), (H.4, ii), (M6), as $n \to \infty$, we have*:

$$I\!\!E\left(K_1^2\, Var\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right) \to I\!\!E(K_1^2)F^x(y)(1-F^x(y)).$$

***Proof of Lemma 4.2.1.***

**proof of (c)** Under assumptions (M1) and (H.4-i) , we have

$$\begin{aligned}
\mathbb{E}\left[K_1^j\right] &= \int_{-1}^{1} K_1^j(v)dP_X^{\delta(X_1,x)/h_K}(v), \quad \text{for} \quad j=1,2 \\
&= \int_{-1}^{1}\left[K_1^j(1) - \int_v^1 \left(K_1^j(u)\right)^{(1)}du\right]dP_X^{\delta(X_1,x)/h_K}(v) \\
&= K_1^j(1)\phi_x(h_K) - \int_{-1}^{1}\left[\int_v^1 \left(K_1^j(u)\right)^{(1)}du\right]dP_X^{\delta(X_1,x)/h_K}(v) \\
&= K_1^j(1)\phi_x(h_K) - \int_{-1}^{1}\left[\int_{-1}^{u} dP_X^{\delta(X_1,x)/h_K}(v)\right]\left(K_1^j(u)\right)^{(1)}du \\
&= K_1^j(1)\phi_x(h_K) - \int_{-1}^{1}\left(K_1^j(u)\right)^{(1)}\phi_x(-h_K,uh_K)du
\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[K_1^j\right] &= \left[K_1^j(1) - \int_{-1}^{1} \left(K_1^j(u)\right)^{(1)} \Psi_x(u)du\right] \phi_x(h_K) + o(\phi_x(h_K)) \\ &= M_j \phi_x(h_K) + o(\phi_x(h_K)).\end{aligned}$$

**proof of (d)** Under assumption $(H.4, i)$, we obtain that:

$$\mathbb{E}\left[K_1^a \beta_1\right] \leq C \int_{B(x,h_K)} \beta(u,x) dP_X(u)$$

So, by using the assumptions $(H.3)$ and $(H.5)$, we get:

$$h_K \mathbb{E}\left[K_1^a \beta_1\right] = o\left(\int_{B(x,h_K)} \beta^2(u,x) dP_X(u)\right) = o\left(h_K^2 \phi_x(h_K)\right)$$

which allows to write:

$$\mathbb{E}\left[K_1^a \beta_1\right] = o\left(h_K \phi_x(h_K)\right).$$

**proof of (e)** We can write:

$$\mathbb{E}\left[K_1^a \beta_1^b\right] = \mathbb{E}\left[K_1^a \delta^b(X_1,x)\right] + \mathbb{E}\left[K_1^a \left(\beta^b(X_1,x) - \delta^b(X_1,x)\right)\right]$$

Concerning the second term we have:

$$\begin{aligned}\mathbb{E}\left[K_1^a \left(\beta^b(X_1,x) - \delta^b(X_1,x)\right)\right] &= \mathbb{E}\Big[K_1^a 1_{B(x,h_K)} \left(\beta(X_1,x) - \delta(X_1,x)\right) \\ &\times \sum_{l=0}^{b-1} \left(\beta(X_1,x)\right)^{b-l-1} \left(\delta(X_1,x)\right)^l\Big] \\ &\leq \sup_{u \in B(x,h_K)} |\beta(u,x) - \delta(u,x)| \\ &\times \sum_{l=0}^{b-1} \mathbb{E}\left[K_1^a 1_{B(x,h_K)} \, |\beta(X_1,x)|^{b-l-1} |\delta(X_1,x)|^l\right].\end{aligned}$$

Moreover by assumption (H.3), we get

$$1_{B(x,h_K)}\,|\beta(X_1,x)| \le 1_{B(x,h_K)}\,|\delta(X_1,x)|\,.$$

Thus, it follows:

$$\begin{aligned}
\mathbb{E}\left[K_1^a\left(\beta^b(X_1,x)-\delta^b(X_1,x)\right)\right] &\le b \sup_{u\in B(x,h_K)} |\beta(u,x)-\delta(u,x)|\,\mathbb{E}\left[K_1^a|\delta|^{b-1}(X_1,x)\right]\\
&\le b \sup_{u\in B(x,h_K)} |\beta(u,x)-\delta(u,x)|\,h_K^{b-1}\,E\left[K_1^a\right]\\
&\le b \sup_{u\in B(x,h_K)} |\beta(u,x)-\delta(u,x)|\,h_K^{b-1}\,\phi_x(h_K)\\
&= o(h_K^b\phi_x(h_K))
\end{aligned}$$

It suffices to use (c) and the assumption (M3) for obtaining the desired lower bound.

Concerning the term $\mathbb{E}\left[K_1^a\delta_1^b\right]$, which can be evaluated by the same computation as in 2.4, then we can write

$$\mathbb{E}\left[K_1^a\delta_1^b\right] = h_K^b\left[K_1^a(1)\phi_x(h_K) - \int_{-1}^{1}\left(u^bK_1^a(u)\right)^{(1)}\phi_x(-h_K,uh_K)du\right]$$

Finally, under assumption (M1) , we get:

$$\begin{aligned}
\mathbb{E}\left[K_1^a\beta_1^b\right] &= h_K^b\phi_x(h_K)\left[K_1^a(1) - \int_{-1}^{1}\left(u^bK_1^a(u)\right)^{(1)}\Psi_x(u)du\right] + o(h_K^b\phi_x(h_K))\\
&= N(a,b)h_K^b\phi_x(h_K) + o(h_K^b\phi_x(h_K)).
\end{aligned}$$

**proof of (f)** It is clear that:

$$\begin{aligned}
\mathbb{E}(\Delta_jK_j) = \mathbb{E}\left(\sum_{i,i\neq j}^{n} W_{ij}\right) = (n-1)\mathbb{E}[W_{12}] &= (n-1)\left(\mathbb{E}[\beta_1^2K_1K_2] - \mathbb{E}[\beta_1\beta_2K_1K_2]\right)\\
&= (n-1)\left(\mathbb{E}[\beta_1^2K_1]\mathbb{E}[K_1] - (\mathbb{E}[\beta_1K_1])^2\right)
\end{aligned}$$

by using $(c), (d)$ and $(e)$, with $a = 1$ and $b = 2$ we obtain

$$\frac{\mathbb{E}(K_1\Delta_1)}{n-1} = \mathbb{E}(W_{12}) = N(1,2)M_1 h_K^2 \phi_x^2(h_K) + o\left(h_K^2 \phi_x^2(h_K)\right).$$

□

***Proof of Lemma 4.2.3.***
By applying the Bienaymé-Tchebychev's inequality, we obtain, for all $\varepsilon > 0$

$$\mathbb{P}(|\hat{F}_D^x - \mathbb{E}(\hat{F}_D^x)| > \varepsilon) \quad \leq \quad \frac{Var(\hat{F}_D^x)}{\varepsilon^2}.$$

By using Lemma 4.2.2, and with assumption (H.5), we get:

$$\mathbb{P}(|\hat{F}_D^x - \mathbb{E}(\hat{F}_D^x)| > \varepsilon) = O\left(\frac{1}{\varepsilon^2 n \phi_x(h_K)}\right) \longrightarrow 0, \quad \text{as} \quad n \to \infty$$

□

***Proof of Lemma 4.2.4.*** We have

$$\begin{aligned}
& I\!\!E\left(K_1^2\, Var\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right) \\
&\qquad = \quad I\!\!E\left(K_1^2 I\!\!E\left(\left((H\left(\frac{y-Y_1}{h}\right)\right)^2|X_1\right)\right) \qquad (2.14)\\
&\qquad - \quad I\!\!E\left(K_1^2 I\!\!E^2\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right). \qquad (2.15)
\end{aligned}$$

Concerning the term 2.14 under assumptions (M6) and (H.4, ii) and by an integration by parts followed by a change of variable, we get

$$\begin{aligned}
I\!\!E\left(\left(H\left(\frac{y-Y_1}{h}\right)\right)^2|X_1\right) &= \int_{\mathbb{R}} \left(H\left(h_H^{-1}(y-z)\right)\right)^2 f^{X_1}(z)\,dz \\
&= -h_H \int_{\mathbb{R}} \left(H(t)\right)^2 f^{X_1}(y-th_H)\,dt \\
&= -h_H \int_{\mathbb{R}} \left(H(t)\right)^2 dF^{X_1}(y-th_H)
\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(\left(H\left(\frac{y-Y_1}{h}\right)\right)^2 |X_1\right) &= \int_{\mathbb{R}} 2H(t)H^{(1)}(t)\left(F^{X_1}(y-th_H)-F^x(y)\right)dt \\ &+ \int_{\mathbb{R}} 2H(t)H^{(1)}(t)F^x(y)\,dt.\end{aligned}$$

Since $\int_{\mathbb{R}} 2H(t)H^{(1)}(t)F^x(y)\,dt = F^x(y)$, as $n\to\infty$, we deduce that

$$\mathbb{E}\left(K_1^2\mathbb{E}\left(\left((H\left(\frac{y-Y_1}{h}\right)\right)^2 |X_1\right)\right) \to \mathbb{E}\left(K_1^2\right)F^x(y)$$

and

$$\mathbb{E}\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right) - F^x(y) \to 0.$$

So, the term 2.15 tends to $(F^x(y))^2$ as $n$ tends to infinity. Then

$$\mathbb{E}\left(K_1^2\mathbb{E}^2\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right) \to \mathbb{E}\left(K_1^2(F^x(y))^2\right) = \mathbb{E}\left(K_1^2\right)(F^x(y))^2$$

Finally, as $n\to\infty$, we have,

$$\mathbb{E}\left(K_1^2\,Var\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right) \to \mathbb{E}(K_1^2)F^x(y)(1-F^x(y)).$$

$\square$

So, Lemma 4.2.3, implies that $\widehat{F}_D^x \to 1$. Moreover, $B_n(x,y)=o_p(1)$ as $n\to\infty$ due to the continuity of $F^x$. Then, we obtain that

$$\widehat{F}^x(y) - F^x(y) - B_n(x,y) = \frac{Q_n(x,y)}{\widehat{F}_D^x}(1+o_p(1)).$$

So, it suffices to show that

$$\sqrt{n\phi_x(h_K)}Q_n(x,y) \xrightarrow{D} \mathcal{N}\left(0, V_{HK}(x,y)\right), \tag{2.16}$$

where $V_{HK}(x,y)$ is defined by (2.10). For this, notice that on one side, we have

$$\sqrt{n\phi_x(h_K)}Q_n(x,y) = \frac{\sqrt{n\phi_x(h_K)}}{n\mathbb{E}(\Delta_1 K_1)}\left(\sum_{j=1}^n \Delta_j K_j(H_j - F^x(y)) - \mathbb{E}\left(\sum_{j=1}^n \Delta_j K_j(H_j - F^x(y))\right)\right),$$

which, combined with (2.12) implies that

$$\begin{aligned}
\sqrt{n\phi_x(h_K)}Q_n(x,y) &= \frac{1}{n\mathbb{E}(\beta_1^2 K_1)}\sum_{i=1}^n \beta_i^2 K_i \frac{\sqrt{n\phi_x(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)}\sum_{j=1}^n K_j(H_j - F^x(y)),\\
&- \frac{1}{n\mathbb{E}(\beta_1 K_1)}\sum_{i=1}^n \beta_i K_i \frac{\sqrt{n\phi_x(h_K)}\mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Delta_1 K_1)}\sum_{j=1}^n \beta_j K_j(H_j - F^x(y)),\\
&- \mathbb{E}\left(\frac{1}{n\mathbb{E}(\beta_1^2 K_1)}\sum_{i=1}^n \beta_i^2 K_i \frac{\sqrt{n\phi_x(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)}\sum_{j=1}^n K_j(H_j - F^x(y))\right),\\
&+ \mathbb{E}\left(\frac{1}{n\mathbb{E}(\beta_1 K_1)}\sum_{i=1}^n \beta_i K_i \frac{\sqrt{n\phi_x(h_K)}\mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Delta_1 K_1)}\sum_{j=1}^n \beta_j K_j(H_j - F^x(y))\right).
\end{aligned}$$

Denote by

$$T_1 = \frac{1}{n\mathbb{E}(\beta_1^2 K_1)}\sum_{i=1}^n \beta_i^2 K_i \quad , \quad T_2 = \frac{\sqrt{n\phi_x(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)}\sum_{j=1}^n K_j(H_j - F^x(y)),$$

$$T_3 = \frac{1}{n\mathbb{E}(\beta_1 K_1)}\sum_{i=1}^n \beta_i K_i \quad \text{and} \quad T_4 = \frac{\sqrt{n\phi_x(h_K)}\mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Delta_1 K_1)}\sum_{j=1}^n \beta_j K_j(H_j - F^x(y)).$$

Then

$$\begin{aligned}
\sqrt{n\phi_x(h_K)}Q_n(x,y) &= (T_1T_2 - T_3T_4 - \mathbb{E}(T_1T_2 - T_3T_4))\\
&= (T_1T_2 - \mathbb{E}(T_1T_2)) - (T_3T_4 - \mathbb{E}(T_3T_4)).
\end{aligned}$$

Hence, by the Slutsky's theorem, to show (2.16), it suffices to prove the following two claims:

$$T_1T_2 - \mathbb{E}(T_1T_2) \xrightarrow{D} \mathcal{N}(0, V_{HK}(x,y)) \tag{2.17}$$

$$T_3T_4 - \mathbb{E}(T_3T_4) \xrightarrow{P} 0, \tag{2.18}$$

*Proof of (2.17).* We can write that

$$T_1T_2 - \mathbb{E}(T_1T_2) = (T_2 - \mathbb{E}(T_2)) + ((T1-1)T_2 - \mathbb{E}((T_1-1)T_2)).$$

Again by the Slutsky's Theorem, (2.17), we can deduce the two following intermediate results,

$$(T_1-1)T_2 - \mathbb{E}((T_1-1)T_2) \xrightarrow{P} 0, \tag{2.19}$$

and

$$T_2 - E(T_2) \xrightarrow{D} \mathcal{N}(0, V_{HK}(x,y)). \tag{2.20}$$

Concerning the proof of (2.19), by applying the Bienaymé-Tchebychev's inequality, we obtain for all $\varepsilon > 0$

$$\mathbb{P}(|(T_1-1)T_2 - \mathbb{E}((T_1-1)T_2)| > \varepsilon) \leqslant \frac{\mathbb{E}(|(T_1-1)T_2 - \mathbb{E}((T_1-1)T_2)|)}{\varepsilon}.$$

Then, the Cauchy-Schwarz inequality implies that

$$\mathbb{E}|(T_1-1)T_2 - \mathbb{E}((T_1-1)T_2)| \leqslant 2\mathbb{E}|(T_1-1)T_2| \leq 2\sqrt{\mathbb{E}((T_1-1)^2)}\sqrt{\mathbb{E}(T_2^2)}.$$

On one side, by using Lemma 4.2.1's result, we obtain

$$\begin{aligned}
\mathbb{E}\left((T_1-1)^2\right) &= Var(T_1) = \frac{1}{n^2\mathbb{E}^2(\beta_1^2K_1)} n\, Var(\beta_1^2K_1) \\
&\leqslant \frac{1}{n(O(h_K^4\phi_x^2(h_K)))}\mathbb{E}(\beta_1^4K_1^2) = O\left(\frac{1}{n\phi_x(h_K)}\right),
\end{aligned}$$

and on the other side, we obtain

$$\begin{aligned}
\mathbb{E}\left((T_2)^2\right) &= \frac{n\phi_x(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Delta_1K_1)}\mathbb{E}\left(\sum_{j=1}^{n} K_j(H_j - F^x(y))\right)^2 \\
&= \frac{n}{(n-1)^2O(\phi_x(h_K))}\left(nO(\phi_x(h_K)) + n(n-1)o(\phi_x^2(h_K))\right) \\
&= O(1) + o(n\phi_x(h_K)).
\end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}|(T_1-1)T_2-\mathbb{E}((T_1-1)T_2) &\leqslant 2\sqrt{\mathbb{E}\left((T_1-1)^2\right)}\sqrt{\mathbb{E}(T_2^2)} \\ &\leqslant 2\sqrt{O\left(\frac{1}{n\phi_x(h_K)}\right)(O(1)+o(n\phi_x(h_K)))}=o(1), \end{aligned}$$

which implies that

$$(T_1-1)T_2-\mathbb{E}((T_1-1)T_2)=o_p(1),$$

then, as $n\to\infty$, we get:

$$\mathbb{P}(|\ (T_1-1)T_2-\mathbb{E}((T_1-1)T_2)\ |>\varepsilon)\leqslant\frac{\mathbb{E}(|\ (T_1-1)T_2-\mathbb{E}((T_1-1)T_2)\ |)}{\varepsilon}\to 0.$$

Therefore, to prove (2.17), we just need to prove (2.20). For that we denote

$$\begin{aligned} R_n &= T_2-\mathbb{E}(T_2) \\ &= \frac{\sqrt{n\phi_x(h_K)}\mathbb{E}(\beta_1^2K_1)}{\mathbb{E}(\Delta_1K_1)}\sum_{j=1}^{n}(K_j(H_j-F^x(y))-\mathbb{E}(K_j(H_j-F^x(y))) \\ &= \sum_{j=1}^{n}\varepsilon_{nj}(x,y), \end{aligned}$$

where,

$$\varepsilon_{nj}=\frac{\sqrt{n\phi_x(h_K)}E(\beta_1^2K_1)}{\mathbb{E}(\Delta_1K_1)}[K_j(H_j-F^x(y))-\mathbb{E}(K_j(H_j-F^x(y))].$$

By the fact that $\varepsilon_{nj}(x,y)$ are i.i.d., it follows that

$$Var(R_n(x,y)) \quad = \quad n\,Var(\varepsilon_{n1}(x,y)).$$

Thus,

$$\begin{aligned} &Var(R_n(x,y)) \qquad (2.21)\\ &=\frac{n^2\phi_x(h_K)E^2(\beta_1^2K_1)}{\mathbb{E}^2(\Delta_1K_1)}[\mathbb{E}\left((H_1-F^x(y))^2K_1^2\right)-\mathbb{E}\left((H_1-F^x(y))K_1\right)^2]. \end{aligned}$$

Concerning the second term on the right hand side of (2.21), we have

$$\begin{aligned}
\left(\mathbb{E}\left((H_1 - F^x(y))K_1\right)\right)^2 &= \left(\mathbb{E}\left(\mathbb{E}(H_1 - F^x(y))K_1 \big| X_1\right)\right)^2 \\
&= \left(\mathbb{E}\left(K_1 \mathbb{E}((H_1 \big| X_1) - F^x(y))\right)\right)^2,
\end{aligned}$$

where,

$$\mathbb{E}((H_1 \big| X_1) - F^x(y)) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.22}$$

Now, let us return to the first term of the right hand side of (2.21). We have

$$\begin{aligned}
&\frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)}\left(\mathbb{E}\left((H_1 - F^x(y))^2 K_1^2\right)\right. \\
&= \frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{E^2(\Delta_1 K_1)}\left(\mathbb{E}\left(\mathbb{E}((H_1 - F^x(y))^2 \big| X_1\right) K_1^2\right) \\
&= \frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)}\mathbb{E}\left(Var(H_1 \big| X_1) K_1^2\right) \\
&+ \frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)}\left(\mathbb{E}\left(\mathbb{E}((H_1 \big| X_1) - F^x(y))\right)^2 K_1^2\right).
\end{aligned}$$

By using (2.22), we have

$$\frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)}\left(\mathbb{E}\left(\mathbb{E}((H_1 \big| X_1) - F^x(y))\right)^2 K_1^2\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Combining Lemma 4.2.1 with Lemma 4.2.4, we obtain,

$$\mathbb{E}\left(Var(H_1 \big| X_1) K_1^2\right) \underset{n\to\infty}{\longrightarrow} \mathbb{E}(K_1^2) F^x(y)(1 - F^x(y)) = M_2 F^x(y)(1 - F^x(y))\phi_x(h_K).$$

Therefore, by using Lemma 4.2.1's result, equation (2.21) becomes

$$\begin{aligned}
Var(R_n(x,y)) &= \frac{n^2\phi_x(h_K)(N(1,2)h_K^2\phi_x(h_K))^2}{((n-1)N(1,2)M_1 h_K^2 \phi_x^2(h_K))^2} M_2 F^x(y)(1 - F^x(y))\phi_x(h_K) \\
&= \frac{n^2 M_2}{(n-1)^2 M_1^2} F^x(y)(1 - F^x(y)) \\
&\underset{n\to\infty}{\longrightarrow} \frac{M_2}{M_1^2} F^x(y)(1 - F^x(y)) = V_{HK}(x,y).
\end{aligned}$$

Now, in order to end the proof of (2.17), we focus on the central limit theorem. So, the proof of (2.17) is completed if the Lindeberg's condition is verified. In fact, the Lindeberg's condition holds since,

$$\frac{1}{Var(R_n(x,y))}\sum_{j=1}^{n} I\!E\left(\varepsilon_{nj}^2 1\!\!1_{(|\varepsilon_{nj}|>\eta\sqrt{Var(R_n(x,y))})}\right) \to 0, \quad \text{for all} \quad \eta > 0$$

where $1\!\!1$ is the indicator function. Indeed, since $|\varepsilon_{nj}| \leq C(n\phi_x(h_K))^{-1/2}$ and

$$Var(R_n(x,y)) \quad = \quad n\,Var(\varepsilon_{n1}(x,y)),$$

we have

$$\frac{1}{Var(R_n(x,y))}\sum_{j=1}^{n} I\!E\left(\varepsilon_{nj}^2 1\!\!1_{(|\varepsilon_{nj}|>\eta\sqrt{Var(R_n(x,y))})}\right) \leq C_{x,h_K}\sum_{i=1}^{n}\mathbb{P}\left(|\varepsilon_{nj}| > \eta\sqrt{Var(R_n(x,y))}\right).$$

On the other hand, we have

$$\frac{|\varepsilon_{nj}|}{\sqrt{Var(R_n(x,y))}} \leq \frac{C}{(Var(R_n(x,y))n\phi_x(h_K))^{1/2}} \to 0, \quad \text{for } n \to \infty.$$

So, for all $\eta$ and if n is large enough, then $\mathbb{P}\left(|\varepsilon_{nj}| > \eta\sqrt{Var(R_n(x,y))}\right) = 0$.

*Proof of (2.18).* To use same arguments as those invoked to prove (2.17), let us write

$$T_3T_4 - I\!E(T_3T_4) = (T_4 - I\!E(T_4)) + ((T_3-1)T_4 - I\!E((T_3-1)T_4)).$$

By applying the Bienaymé-Tchebychev's inequality we obtain, for all $\varepsilon > 0$

$$I\!P(\mid T_3T_4 - I\!E(T_3T_4)\mid > \varepsilon) \leq \frac{I\!E(\mid T_3T_4 - I\!E(T_3T_4)\mid)}{\varepsilon},$$

and the Cauchy-Schwarz's inequality implies that

$$I\!E|(T_3-1)T_4 - I\!E((T_3-1)T_4)| \leq 2I\!E\mid (T_3-1)T_4 \mid \leq 2\sqrt{I\!E\left((T_3-1)^2\right)}\sqrt{I\!E(T_4^2)}.$$

By the same arguments, and by using Lemma 4.2.1's result, we get

$$\begin{aligned}
\mathbb{E}((T_3-1)^2) &= Var(T_3) = \frac{n}{n^2\mathbb{E}^2(\beta_1K_1)}Var(\beta_1K_1) \\
&\leqslant \frac{1}{n(O(h_K^2\phi_x^2(h_K)))}\mathbb{E}(\beta_1^2K_1^2) = O\left(\frac{1}{n\phi_x(h_K)}\right).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathbb{E}(T_4^2) &= \frac{n\phi_x(h_K)\mathbb{E}^2(\beta_1K_1)}{E^2(\Delta_1K_1)}\mathbb{E}\left(\sum_{j=1}^n \beta_jK_j(H_j-F^x(y))\right)^2 \\
&= \frac{n\phi_x(h_K)O(h_K^2\phi_x^2(h_K))}{(n-1)^2O(h_K^4\phi_x^4(h_K)}\left[n\mathbb{E}(\beta_1K_1(H_1-F^x(y)))^2\right. \\
&+ \left. n(n-1)\mathbb{E}^2(\beta_1K_1(H_1-F^x(y)))\right] \\
&= o(1)+o\left(n\phi_x(h_K)\right).
\end{aligned}$$

Thus,

$$\mathbb{E}|(T_3-1)T_4-\mathbb{E}((T_3-1)T_4)\mid \;\leqslant\; 2\sqrt{\mathbb{E}((T_3-1)^2)}\sqrt{\mathbb{E}(T_4^2)} = o(1),$$

which implies that $(T_3-1)T_4-\mathbb{E}((T_3-1)T_4) = o_p(1)$. Therefore,

$$\mathbb{P}(\mid T_3T_4-\mathbb{E}(T_3T_4)\mid > \varepsilon) \leqslant \frac{\mathbb{E}(\mid T_3T_4-\mathbb{E}(T_3T_4))\mid)}{\varepsilon} \to 0 \text{ as } n\to\infty.$$

So, to prove (2.18), it suffices to show that $T_4-\mathbb{E}(T_4) = o_p(1)$, while

$$\mathbb{E}\left(T_4-\mathbb{E}(T_4)\right)^2 = Var(T_4) = \frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1K_1)}{\mathbb{E}^2(\Delta_1K_1)}Var(\beta_1K_1(H_1-F^x(y))).$$

We arrive finally at

$$Var(\beta_1K_1(H_1-F^x(y))) = F^x(y)(1-F^x(y))\mathbb{E}(\beta_1^2K_1^2).$$

This last result together with Lemma 4.2.1 lead directly to

$$\begin{aligned}
\mathbb{E}\left(T_4-\mathbb{E}(T_4)\right)^2 &= \frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1K_1)}{E^2(\Delta_1K_1)}(F^x(y)(1-F^x(y)))\mathbb{E}(\beta_1^2K_1^2) \\
&= (F^x(y)(1-F^x(y)))o(1),
\end{aligned}$$

which brings us to the end of the proof.

□

# Chapter 3

# Application on simulated and real data

## 1 Simulation

Our main goal of this application is to display the usefulness of the conditional mode in a prediction context. More precisely, we illustrate the performance and the superiority of our estimator by using the criteria of the mean square error (MSE). For this aim , we make comparison with the following two prediction models:

The conditional mode, via the local constant estimation method (classical kernel method) is defined by:

$$\mu(x) = \sup_{y \in S} f^x(y),$$

where

$$\widehat{f}^x_{L.C.E}(y) = \frac{h_H^{-1} \sum_{i=1}^{n} K(h_K^{-1} d(x, X_i)) H^{(1)}(h_H^{-1}(Y_i - y))}{\sum_{i=1}^{n} K(h_K^{-1} d(x, X_i))}. \tag{3.1}$$

The conditional mode via the local linear estimation method is defined by:

$$\mu(x) = \arg\max_y \frac{h_H^{-1} \sum_{i,i\neq j,j=1}^{n} W_{ij} H_j^{(1)}}{\sum_{i,i\neq j,j=1}^{n} W_{ij}}.$$

Next, we generated the functional covarite $X$ on the interval $[0,\pi]$ (see figure 3.1 ) by the following process:

$$X_i(\tau) = 3\,cos(W_i \pi \tau), \text{ for } i = 1, 2, \ldots, 100.$$

where $W_i$ are independent and identically distributed and following the uniform distribution on the interval $[1, \pi/2]$ ($W_i \sim U([1, \pi/2])$).

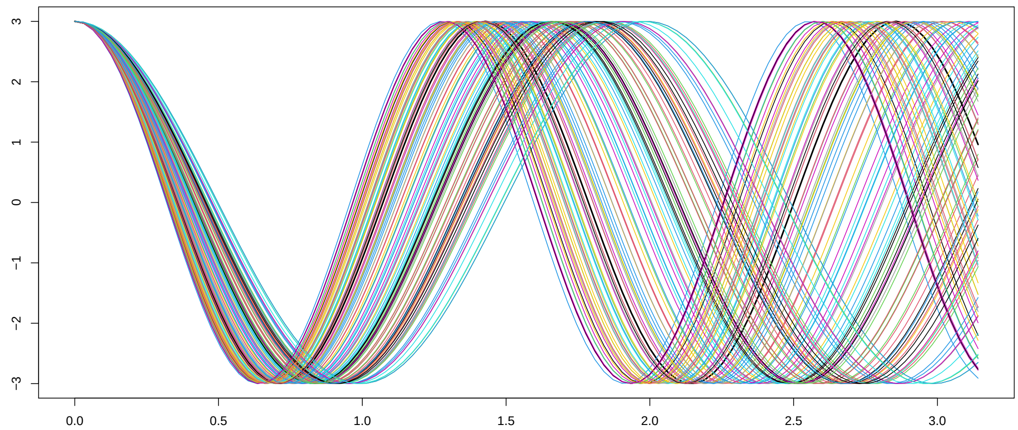



Figure 3.1: The curves $X_i(\tau_j),\ \tau_j \in [0,\pi],$ for $i = 1, \ldots, 100$.

To illustrate the performance of our estimator, we proceed the following algorithm

- Step 1. We generate the response variables $Y_i$ by

$$Y_i = r(X_i) + \epsilon_i, \text{ where } r(x) = \int_0^{\pi} \frac{3}{ln(X_i^2(\tau)+2)} d\tau,$$

  and $\epsilon_i$ simulate independently and follow the normal distribution $N(0,1)$

- Step 2. We divide our observations into two subsets:

  - $(X_i, Y_i)_{i=1,..,80}$, training sample.

  - $(X_j, Y_j)_{j=81,..,100}$, test sample.

- Step 3. We choose $\theta$ the first eigenfunction corresponding to the first higher eigenvalue of the empirical covariance operator by using the same idea as Attaoui, S. and Ling, N [2] and we select $\theta$ as follows:

  - We compute the covariance operator by using the empirical covariance operator in the sample $\Lambda = \{1, \ldots, 100\}$ and $|\Lambda| = 100$.

$$\frac{1}{|\Lambda|}\sum_{i\in\Lambda}(X_i(\tau) - \overline{X}(\tau))^t(X_i(\tau) - \overline{X}(\tau)), \tag{3.2}$$

    where

$$\overline{X}(\tau) = \frac{1}{|\Lambda|}\sum_{i\in\Lambda} X_i(\tau).$$

  - We compute the eigenvectors of (3.2) (empirical covariance operator).

The obtained results are shown in the following graphs, such that:
The covariance operator for $\Lambda = \{1, \ldots, 100\}$ gives the discretization of the eigenfunctions $\theta_i(\tau)$ (The eigenfunction is presented by a continuous curve). The three eigenfunction $\theta_1(\tau)$, $\theta_2(\tau)$ and $\theta_3(\tau)$, are represented in Figure (3.2), ten and all the eigenfunctions are displayed in Figure (3.3) and (3.4)

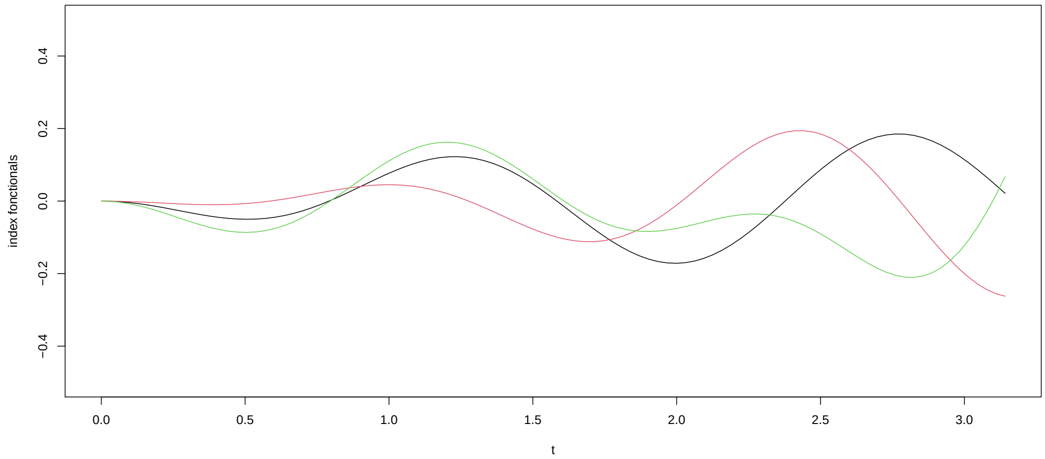



Figure 3.2: The curves $\theta_i(\tau_j),\ \tau_j \in [0, \pi]$, for $i = 1, 2, 3$.

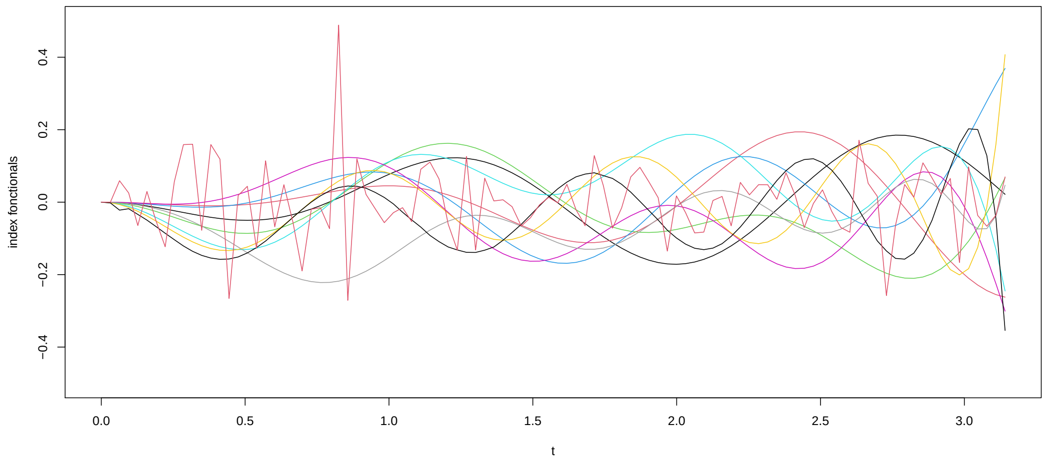



Figure 3.3: The curves $\theta_i(\tau_j),\ \tau_j \in [0, \pi]$, for $i = 1, 2 \ldots, 10$.

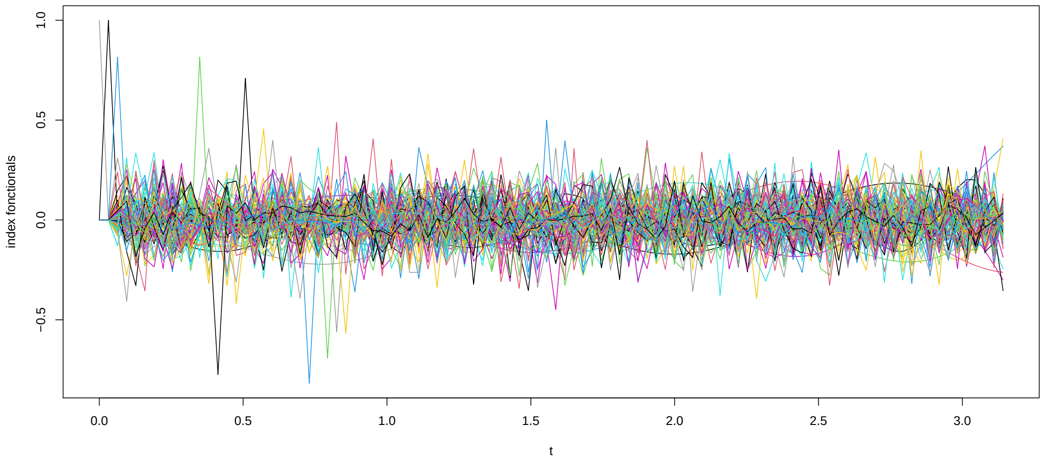



Figure 3.4: The curves $\theta_i(\tau_j),\ \tau_j \in [0, \pi],\ \text{for } i = 1, \ldots, 100.$

- Step 4. Due to the nature of the data (the shape of the curves (3.1), we chose the following family of locating functions:

$$\beta(x_1, x_2) = \int_0^1 \theta(\tau)(x_1(\tau) - x_2(\tau))d\tau,$$

  and

$$\delta(x_1, x_2) = \sqrt{\int_0^1 (x_1(\tau) - x_2(\tau))^2 d\tau},$$

- Step 5. We choose a quadratic kernel $K$ on $[-1, 1]$ and take $K = H^{(1)}$, where the bandwidths $h_K$ and $h_H$ are automatically selected by the procedure of the cross validation

- Step 6. For each $j$ in the test sample, we compute $\widehat{Y}_j = \widehat{\mu}(X_j)$ by using the two approach (L.C.E) and (LLE)

- Step 7. We present our results by plotting the boxplot of the prediction error, which are represented in (Figure 3.5) and we compute the empirical mean square error with LLE (resp. L.C.E):

  - MSE$= \frac{1}{20}\sum_{i=1}^{20}(Y_i - \widehat{\mu}_{LLE}^{X_i}(Y_i))^2 = 1,932$
  - MSE $= \frac{1}{20}\sum_{i=1}^{20}(Y_i - \widehat{\mu}_{LCE}^{X_i}(Y_i))^2 = 2,483$

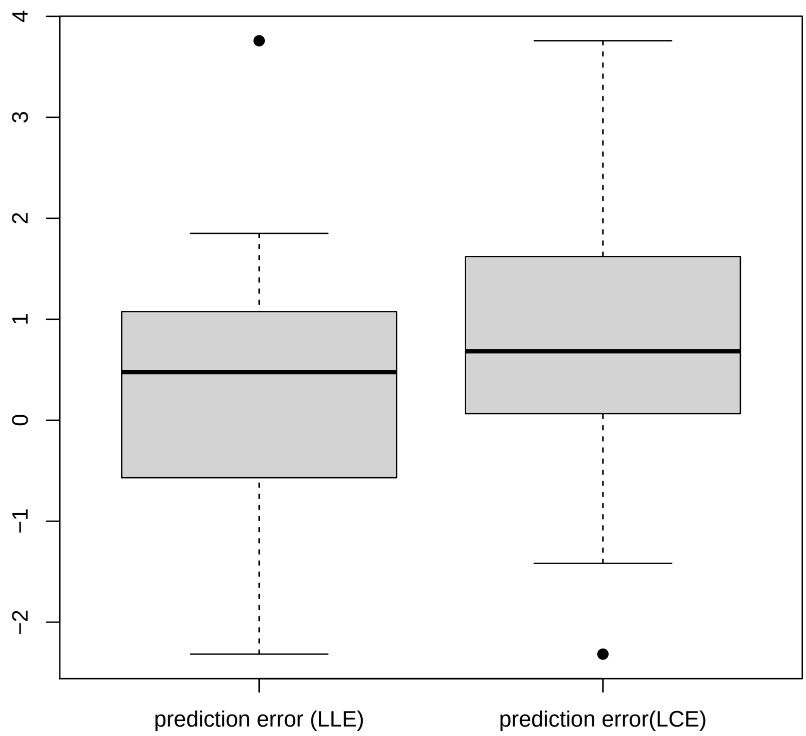
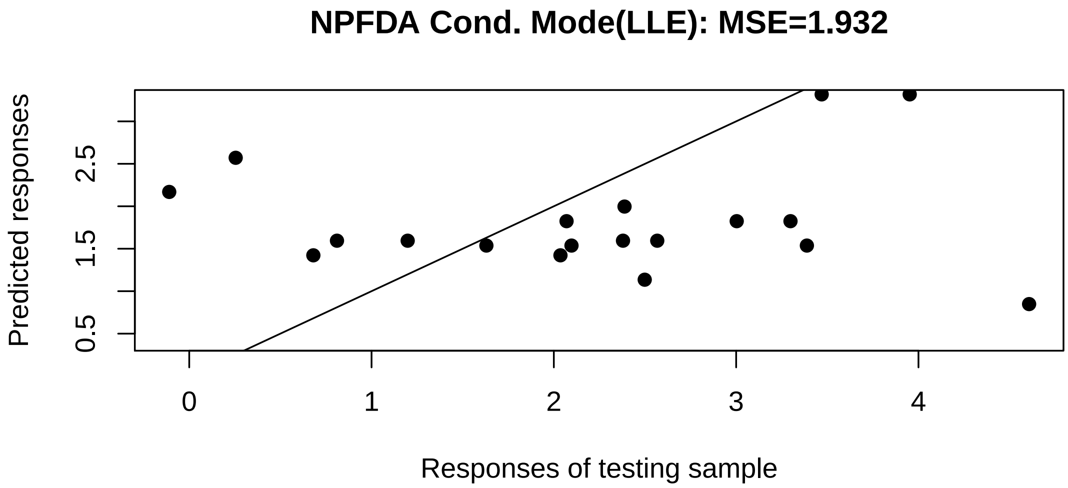



Figure 3.5: Comparison of the prediction error



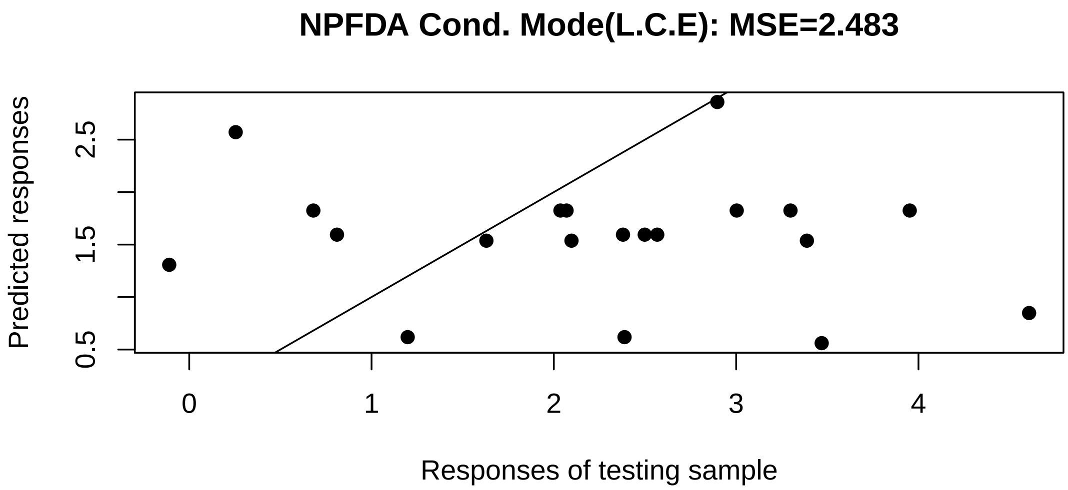




Figure 3.6: Comparison results between the local linear estimator and the classical estimator (LCE)

Based on the figure (3.6), the mean quadratic error presented by the LLE is much improved than the classical estimator (NWE). This is confirmed by the mean squared error MSE(NWE)=2,483 whereas MSE(LLE)=1,932 , which confirms that the proposed method also has a practical advantage.

# 2 Real data application

In many situations, forecasting via predictive regions is much more adequate than point forecasting. On this subject, the existing results in this manuscript should be used to fit a predictive region and/or interval widely known in NPFDA. At this point, we have constructed predictive interval by using the shortest conditional modal interval. This notion was proposed by Lientz [25] for the case of an unconditional distribution. The conditional case was approached by Gooijer and Gannoun [17] In our work, this concept is given via the conditional distribution function on the interval

$$[a-b] = argmin\{\ \mathrm{Leb}[a-b], F^{X_n}(b) - F^{X_n}(a) \geq \alpha\},$$

where Leb(C) denotes the Lebesgue measure of the set C. the SCM interval is the smallest interval among all predictive intervals with coverage probability $\alpha$. The practical determination of this interval is based on the estimation of the conditional distribution function:

$$[a-b] = argmin\{\ \mathrm{Leb}[a-b], \widehat{F^{X_n}}(b) - \widehat{F^{X_n}}(a) \geq \alpha\}. \tag{3.3}$$

We apply our approach to the forage quality data. In practice, the analysis of this kind of data is very important for the food industry. In fact, it intervenes in many food products such as milk quality, dairy products, meat quality, .... Notice that, there are three fundamental parameters in the study of the forage quality that are (i) the concentration of crude protein (CP), (ii) the acid detergent fiber (ADF) and (iii) the neutral detergent fiber (NDF). The classical analytical procedures are time-consuming and very expensive. That is why we propose a new approach based on the functional local linear approach. More Precisely, we use the conditional mode to predict the level of the concentration of crude protein (CP) given the spectrometric curves of the sheepgrass hay. Moreover, we use the conditional distribution function and the conditional density to construct a predictive region of this parameter. Notice that data are collected from sheepgrass fields of the hay factories in Heilongjiang Province of Northeast China. We refer to Chen et al. [11], for a complete description of this data-set. We consider a sample of size 150 and the near-infrared spectra were recorded at 5 $nm$ intervals from 950 to 1650 $nm$. These functional curves are plotted in Fig. 3.7.

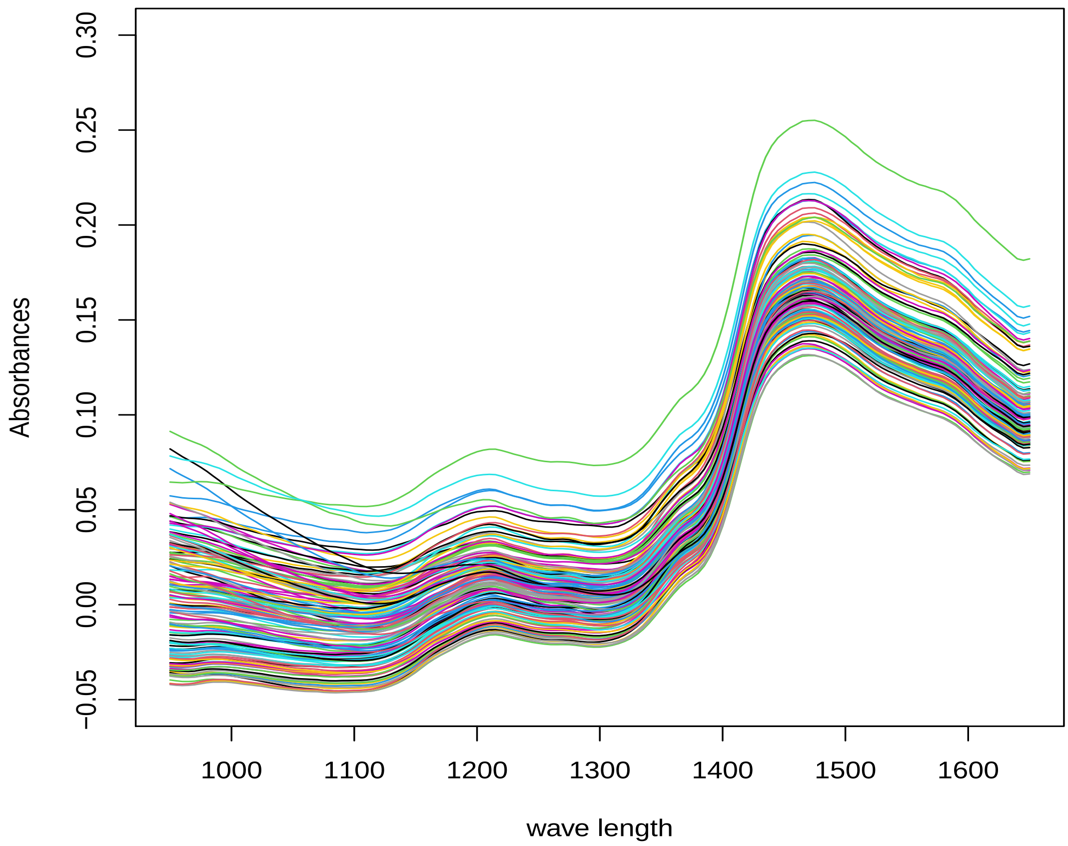



Figure 3.7: Near-infrared spectra curves

In order to highlight our model, we compare the local linear approach method to the kernel method in the prediction by the conditional mode function, while for the second method we propose the predictive region which is defined in the formula 3.3. For a practical purpose, we randomly split our data into two subsets. The first sample, of size $n = 130$, will be used to calculate our estimators on the 20 remaining curves. The estimators are computed by using the $B$-spline semi-metric of the first derivative and the bandwidths parameters are selected by the local cross-validation technique on the number of nearest neighbors. On the other hand, we consider the quadratic kernel $K(x) = 0.75(1 - x^2)$if $x \in [-1, 1]$, which is supported within $[-1, 1]$, and the function $H$ is chosen as the primitive of the kernel $K$. Then, the single-point prediction results are presented in Fig. 3.8.

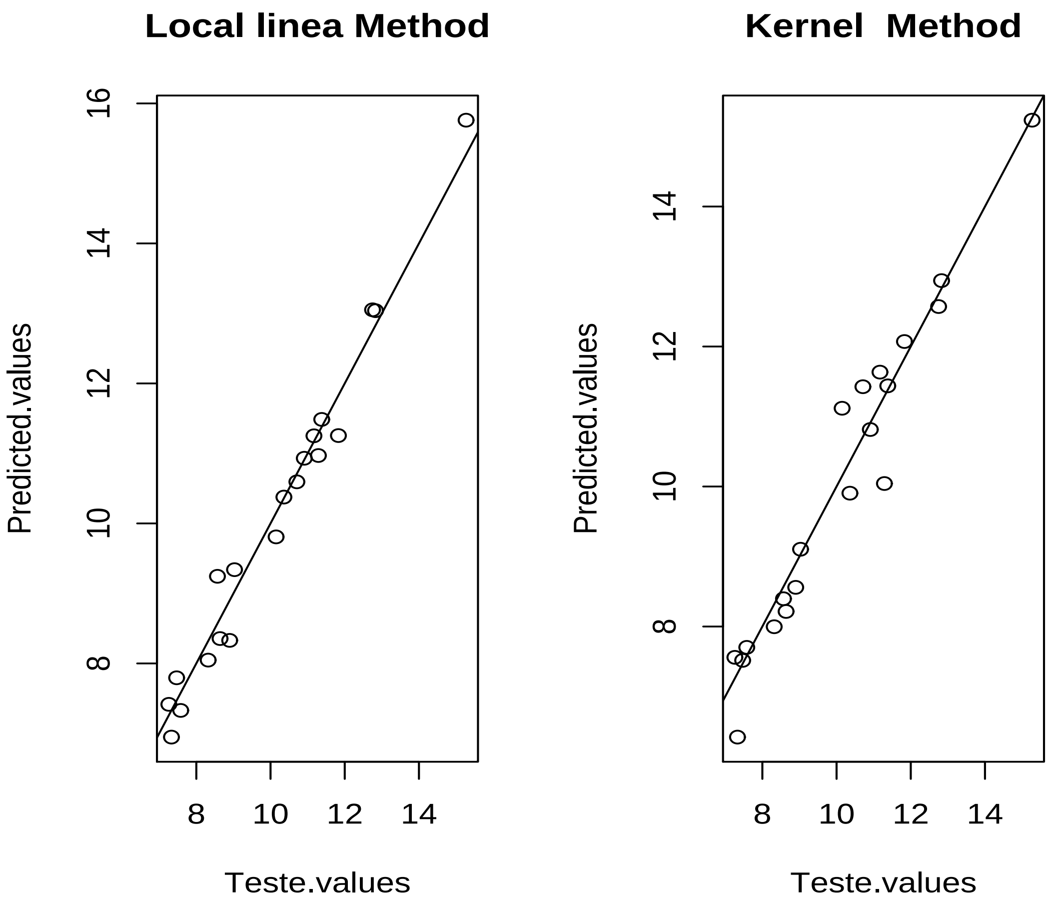



Figure 3.8: Predicted values ($y$-axis) versus Test values ($x$-axis) results: local linear method (left plot) and the kernel smoothing approach (right plot)

The comparison of both scatterplots indicates that the local linear approach (on the left plot) gives better prediction results than the classical kernel method (on the right plot). This is confirmed by the mean squared error which is equal 0.27 versus 0.46. On the other hand, we give in Table 3.1 the 90% predictive intervals of the concentrations of crude protein of the 10 values in the sample test.

| The true value | $[a$ $-$ | $b\,]$ |
|---|---|---|
| 7.26 | [7.42 | 8.56] |
| 7.33 | [6.91 | 8.23] |
| 7.58 | [7.02 | 8.36] |
| 7.47 | [7.36 | 8.55] |
| 15.27 | [13.98 | 17.34] |
| 12.75 | [11.02 | 12.56] |
| 10.15 | [9.71 | 11.16] |
| 10.71 | [10.76 | 11.03] |
| 11.38 | [10.85 | 13.82] |
| 12.83 | [11.99 | 13.51] |

Table 3.1: The 90% predictive intervals

We observe that the results of the SCMI predictive intervals are very satisfactory in regards the average mean length (M.L) which is M.L = 1,61 and the percentage of the true values in the predictive intervals which is 70%.

# Conclusion

In this work, we were interested in the non-parametric estimation, by the local linear method, and we set as objective the conditional distribution function, when the explanatory variable is functional and the response is real.

The main results we obtained are the following: First, we built a local linear estimator of the conditional distribution function, then we established its almost complete convergence by specifying its rate of convergence, when the observations are independent identically distributed. We also established the asymptotic normality of the same estimator by giving an implicit expression of the terms of bias and its the variance.

The advantage of this approach is the its superiority over the kernel method, in the bias part, is always preserved even in functional statistics. Moreover, the classical kernel method is reduced to a particular case of this method.

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