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par

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Thème :

Functional Differential Equations with State-Dependent Delays

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Dedication

To those who raised me when I was young, and they were for me a beacon that lights up my mind with advice and guidance in my old age, To the dear ones: My mother and my father, may Allah protect them.
To those who helped me and motivate me to move forward my best friend Hayet Djelaila, my dear sisters and my brother. To all my dear family.
To everyone who has supported me. To everyone who was responsible for my learning even one letter. To everyone who helped me with my knowledge.
To everyone who helped me in my academic career.
To all of them and all of you, I dedicate the fruit of my efforts, my humble theme.
And praise be to Allah who enabled me to complete this work.

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Abstract

The objective of this thesis is to establish stability, existence, uniqueness and model results for various classes of functional differential equations, with delay which may be finite or state-dependent in Banach space.

Our results are based upon very recently fixed point theorems.

Key words and phrases:

Mild solution, stability, existence and uniquneness, state-dependent delays, fixed point, functional differential equations.

AMS (MOS) Subject Classifications: 34A37, 4B15, 34B37, 34K45, 39A12, 34G20, 34K20,

ملخص

الهدف من هذه الاطروحة هو دراسة الاستقرار ووجود ووحدانية نتائج ونموذج لفئات مختلفة من المعادلات التفاضلية الوظيفية ، مع تأخر قد يكون محدودا او مرتبطا بالحل في فضاء بناخ.

تستند نتائجنا الى نظريات حديثة للنقطة الصامدة.

الكلمات و العبارات المفتاحية:

حل ضعيف، استقرار، وجود ووحدانية، تأخرات مرتبطة بالحل، نقطة صامدة، معادلات تفاضلية وظيفية.

AMS (MOS) Subject Classifications: 34A37, 4B15, 34B37, 34K45, 39A12, 34G20, 34K20,

Résumé

L'objectif de cette thèse est d'établir la stabilité, l'existence, l'unicité et les résultats du modèle pour différentes classes d'équations différentielles fonctionnelles, avec un retard qui peut être fini ou dépendant de l'état dans l'espace de Banach.

Nos résultats sont basés sur des théorèmes de point fixe très récents.

Mots et expressions clés :

Solution faible, stabilité, existence et unicité, retards dépendants de l'état, point fixe, équations différentielles fonctionnelles.

AMS (MOS) Subject Classifications: 34A37, 4B15, 34B37, 34K45, 39A12, 34G20, 34K20,

Introduction

Functional differential equations emerge in all fields of biology, engineering and physical applications, and such equations have granted much attention in recent years. A good directory to the literature for functional differential equations is the books by Hale and Verduyn Lunel [33], Kolmanovskii and Myshkis [45], and the references therein. During the last decades, existence and uniqueness of mild solutions.
functional differential equations has been studied extensively by many authors using fixed point argument, measures of non-compactness, Semigroup of Linear Operator. We mention, for instance, the books by Ahmed [3], Kamenskii et Al. [43], Pazy [51], Wu [54], and the references therein. Hernandez in [40] proved the existence of mild solutions for neutral equations. Studying the possibility of controlling a certain period of a set of neutral functional differential equations by Fu in [24, 25]. Existence of moderate and climacteric solutions of a class of neutral partial functional differential Equations with non-local terms as I consider it Fu and Ezzinbi [26]. The existence and regularity of solutions to functional and neutral functional differential equations with unbounded delay studied by Henriquez [39] and Hernandez [40, 41]. Neutral functional differential equations with infinite delay and Various classes of partial functional have been studied by Adimy and Al. Balachandran and Dauer have considered various classes of first and second order semi-linear ordinary, functional and neutral functional differential equations on Banach spaces in [9].
Abstract neutral differential equations are found in the fields of applied mathematics, that is why it has been largely studied over the past few decades. There is a reference to literature the relevance to ordinary neutral differential equations is very broad, for which we refer the reader to [33] only, which contains a comprehensive description of such equations. Such as, for more on partial neutral functional differential equations and related issues we refer to Adimy and Ezzinbi [1], Hale [32], Wu and Xia [55] and [54] for finite delay equations.
A functional differential equation with state-dependent delays is frequently found in applications Like typical equations (see, e.g., [4, 7, 12, 49]). The study of such equations is an active research area (see, e.g., [13, 1, 25, 35, 34, 36, 37, 41, 46, 47, 48, 52, 53].

This work consists of five chapters and each chapter contains more sections.
They are arranged as follows:
In **Chapter** 1, we introduce definitions, theories, and notations preliminary facts that

will be used through this work.
In **Chapter** 2, we prove the existence of mild solutions of nonlinear neutral time varying multiple delay differential equations in Banach space.
In **Chapter** 3, we study and prove the existence and uniqueness of solutions for neutral differential equations with state-dependent delays.
In **Chapter** 4, we solve the stability problem of neural networks equipped with state-dependent state delay.
In **Chapter** 5, we introduce applications and models of neutral differential equations with state-dependent delays.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis. In this section, we introduce notations, definitions, and preliminary facts which are used throughout this section.

1.1 Measure of Noncompactness

First, we define in this Section the Kuratowski (1896-1980) and Hausdorf measures of noncompactness (**MNC** for short) and give their basic properties.

Definition 1.1.1. *([44]) Let (X, d) be a complete metric space and $\mathcal{B}$ the family of bounded subsets of X. For every $B \in \mathcal{B}$ we define the Kuratowski measure of noncompactness $\alpha(B)$ of the set B as the infimum of the numbers d such that B admits a finite covering by sets of diameter smaller than d.*

Remark 1.1.1. *The diameter of a set B is the number $\sup\{d(x, y) : x \in B, y \in B\}$ denoted by diam(B), with diam$(\emptyset) = 0$.*
It is clear that $0 \leq \alpha(B) \leq diam(B) < +\infty$ for each nonempty bounded subset B of X and that $diam(B) = 0$ if and only if B is an empty set or consists of exactly one point.

Definition 1.1.2. ([*11*]) *Let E be a Banach space and Ω_E the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty)$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 \,:\, B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\},$$

where

$$diam(B_i) = sup\{||x - y|| : x, y \in B_i\}.$$

The Kuratowski measure of noncompactness satisfies the following properties:

Lemma 1.1.1. *([11]) Let A and B bounded sets.*

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is relatively compact), where* $\overline{B}$ *denotes the closure of* B.

(b) *nonsingularity :* α *is equal to zero on every one element-set.*

(c) *If* B *is a finite set, then* $\alpha(B) = 0$.

(d) $\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$, *where* $convB$ *is the convex hull of* B.

(e) *monotonicity :* $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(f) *algebraic semi-additivity :* $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, *where*

$$A + B = \{x + y : x \in A, \quad y \in B\}.$$

(g) *semi-homogencity :* $\alpha(\lambda B) = |\lambda|\alpha(B)$; $\lambda \in \mathbb{R}$, *where* $\lambda(B) = \{\lambda x : x \in B\}$.

(h) *semi-additivity :* $\alpha(A \bigcup B) = max\{\alpha(A), \alpha(B)\}$.

(i) $\alpha(A \bigcap B) = min\{\alpha(A), \alpha(B)\}$.

(j) *invariance under translations :* $\alpha(B + x_0) = \alpha(B)$ *for any* $x_0 \in E$.

Remark 1.1.2. *The* α*-measure of noncompactness was introduced by Kuratowski in order to generalize the Cantor intersection theorem*

Theorem 1.1.2. *([44]) Let* (X, d) *be a complete metric space and* $\{B_n\}$ *be a decreasing sequence of nonempty, closed and bounded subsets of* X *and* $\lim\limits_{n\to\infty} \alpha(B_n) = 0$. *Then the intersection* B_∞ *of all* B_n *is nonempty and compact.*

In [42], Horvath has proved the following generalization of the Kuratowski theorem:

Theorem 1.1.3. *([44]) Let* (X, d) *be a complete metric space and* $\{B_i\}_{i\in I}$ *be a family of nonempty of closed and bounded subsets of* X *having the finite intersection property. If* $\inf\limits_{i\in I} \alpha(B_i) = 0$ *then the intersection* B_∞ *of all* B_i *is nonempty and compact.*

Lemma 1.1.4. *([31]) If* $V \subset C(J, E)$ *is a bounded and equicontinuous set, then*

(i) *the function* $t \to \alpha(V(t))$ *is continuous on* J, *and*

$$\alpha_c(V) = \sup_{0\leq t\leq T} \alpha(V(t)).$$

(ii) $\alpha\left(\int_0^T x(s)ds : x \in V\right) \leq \int_0^T \alpha(V(s))ds,$

where

$$V(s) = \{x(s) : x \in V\}, \; s \in J.$$

In the definition of the Kuratowski measure we can consider balls instead of arbitrary sets. In this way we get the definition of the Hausdorff measure:

Definition 1.1.3. *([44]) The Hausdorff measure of noncompactness $\chi(B)$ of the set B is the infimum of the numbers r such that B admits a finite covering by balls of radius smaller than r.*

Theorem 1.1.5. *([44]) Let $B(0,1)$ be the unit ball in a Banach space X. Then $\alpha(B(0,1)) = \chi(B(0,1)) = 0$ if X is finite dimensional, and $\alpha(B(0,1)) = 2$, $\chi(B(0,1)) = 1$ otherwise.*

Theorem 1.1.6. *([44]) Let $S(0,1)$ be the unit sphere in a Banach space X. Then $\alpha(S(0,1)) = \chi(S(0,1)) = 0$ if X is finite dimensional, and $\alpha(S(0,1)) = 2$, $\chi(S(0,1)) = 1$ otherwise.*

Theorem 1.1.7. *([44]) The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(B) \leq \alpha(B) \leq 2\chi(B).$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

Example 1.1.1. *Let l^∞ be the space of all real bounded sequences with the supremum norm, and let A be a bounded set in l^∞. Then $\alpha(A) = 2\chi(A)$.*

For further facts concerning measures of noncompactness and their properties we refer to [5, 11, 44, 8] and the references therein.

1.2 Semigroup of Linear Operator

Definition 1.2.1. *A one-parameter family $S(t)$ for of bounded linear operators on a Banach space X is a C_0-semigroup (or strongly continuous) on X if*

(i) $S(t) \circ S(s) = S(t+s)$, *for* $t, s \geq 0$, *(semigroup property),*

(ii) $S(0) = I$, *(the identity on X);*

(iii) the map $t \to S(t)x$ is strongly continuous, for each $x \in X$, i.e;

$$\lim_{t\to 0} S(t)(x) = x, \, \forall x \in X.$$

Remark 1.2.1. *A semigroup of bounded linear operators $(S(t))_{t\geq 0}$ is uniformly continuous if*

$$\lim_{t\to 0} \|S(t) - I\| = 0.$$

Here I denotes the identity operator in E. A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0 semigroup. If only (i) and (ii) are satisfied we say that the family $(S(t))_{t\geq 0}$ of bounded linear operators is a semigroup.

Definition 1.2.2. *Let $S(t)$ be a semigroup of class (C_0) defined on X. The infinitesimal generator A of $S(t)$ is the linear operator defined by*

$$A(x) = \lim_{h\to 0} = \frac{S(h)(x) - x}{h}, \text{ for } x \in D(A),$$

where $D(A) = \{x \in X \mid \lim_{h\to 0} = \dfrac{S(h)(x) - x}{h}$ *exists in* $X\}$.

Let us recall the following property:

Theorem 1.2.1. *[51] If $S(t)$ is a C_0-semigroup, then there exist $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\|_{B(E)} \leq M \exp(\omega t), \text{ for } 0 \leq t < \infty \tag{1.2.1}$$

Proof. We show first there is $\eta \in (0, 1]$ such that

$$\sup_{t\in[0,\eta]} \|S(t)\| < +\infty.$$

Assume the contrary, i.e $\forall \eta = \frac{1}{n} \in (0, 1]$ with $n \in \mathbb{N} : \sup_{t\in[0,\eta]} \|S(t)\| = +\infty$. It follows that

$$\forall n \in \mathbb{N}, \exists t_n \in [0, \tfrac{1}{n}] \text{ such that } \sup_{n\geq 1} \|S(t_n)\| = +\infty.$$

By uniform boundedness principle $\exists x \in X : \sup_{n\geq 1} \|S(t_n)x\| = +\infty$ that $\|S(t_n)x\|$ is unbounded.
On the other hand $\forall x \in X, \mathbb{R} \ni t \to S(t)x \in X$ is continuous at 0; that is $\forall \epsilon > 0$, $\exists \delta > 0$: $|t| < \delta \Rightarrow \|S(t)x - x\| < \epsilon$.
In particular, let $\epsilon = 1$.
Then,

$$\|S(t)x - x\| < 1.$$

Hence we obtain the estimates:

$$\|S(t)x\| - \|x\| \leq |\|S(t)x\| - \|x\|| \leq \|S(t)x - x\| < 1.$$

This implies that

$$\|S(t)x\| \leq 1 + \|x\|.$$

But one has $0 \leq t_n \leq \frac{1}{n}$ and then $t_n \to 0$ as $n \to +\infty$ i.e take $\epsilon = \delta$,

$$\exists n_0 \in \mathbb{N} : |t_n| < \delta; \ \forall n > n_0,$$

then
$$\|S(t_n)x\| \leq 1 + \|x\|,\ n > n_0;$$
it follows that
$$\sup_{n \geq n_0} \|S(t_n)x\| \leq 1 + \|x\|, n > n_0. \tag{1.2.2}$$
Now let $n = 1, 2, ..., n_0 - 1$ there is only a finite number of $S(t_n)x$.
Let $M^* = \max \|S(t_n)x\|, n = 1, 2, ..., n_0 - 1$. Then for these,
$$\sup \|S(t_n)x\| \leq M^* \text{ for } n = 1, 2, ..., n_0 - 1. \tag{1.2.3}$$
So from (1.2.2) and (1.2.3) we have $\sup\limits_{n \geq 1} \|S(t_n)x\| \leq 1 + \|x\| + M^*$.
Hence we get the contradiction,
Thus,
$$\exists \eta \in (0, 1] : \sup_{t \in [0, \eta]} \|S(t)\| < +\infty.$$
Let $M := \sup\limits_{t \in [0, \eta]} \|S(t)\|$, since $\|S(0)\| = 1$ then $M \geq 1$.
Let $\omega = \eta^{-1} \log M$. Given $t \geq 0$ with $t > \eta$ we have $t = n(t)\eta + \delta$, where $0 \leq \delta < \eta$ and $n(t) \in \mathbb{N}$.
By semigroup property
$$\begin{aligned}
\|S(t)\| &= \|S(\eta)^{n(t)} S(\delta)\| \\
&\leq \|S(\eta)^{n(t)}\| \|S(\delta)\| \\
&\leq M M^{n(t)} = M M^{\frac{t-\delta}{\eta}} \\
&\leq M M^{\frac{t}{\eta}} = M \exp(\omega \eta \frac{t}{\eta}) = M \exp(\omega t).
\end{aligned}$$
This completes the proof. □

Remark 1.2.2. *If, $M = 1$ and $\omega = 0$, i.e; $\|S(t)\|_{B(E)} \leq 1$, for $t \geq 0$, then the semigroup $S(t)$ is called a contraction semigroup (C_0)*

Theorem 1.2.2. *If $(S(t))_{t \geq 0}$ is a C_0 semigroup then $t \to S(t)x$ is continuous, for each $x \in X$ is continuous from $\mathbb{R}^+$ (the positive real line) into X.*

Proof. Let $t_0 > 0$, $x \in X$.
We want to show that $\lim\limits_{t \to t_0} S(t)(x) = S(t_0)x$.
Case 1: $t > t_0$
$$\begin{aligned}
S(t)(x) - S(t_0)x &= S(t_0)[S(t - t_0)x - x] \\
\|S(t)(x) - S(t_0)x\| &\leq \|S(t_0)\| \|S(t - t_0)x - x\| \to 0 \text{ as } t \to t_0.
\end{aligned}$$

Therefore, $\lim_{t\to t_0^+} S(t)(x) = S(t_0)x$.

Case 2: $t < t_0$

$$\begin{aligned}\|S(t)(x) - S(t_0)x\| &= S(t)[S(t_0 - t)x - x] \\ \|S(t)(x) - S(t_0)x\| &\leq \|S(t)\|\|S(t_0 - t)x - x\| \to 0 \text{ as } t \to t_0.\end{aligned}$$

Therefore, $\lim_{t\to t_0^-} S(t)(x) = S(t_0)x$.

□

Theorem 1.2.3. *Let $S(t)_{t\geq 0}$ be a C_0 semigroup and A be its infinitesimal generator. Then*

(*a*) *For $x \in X$,*

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} S(s)xds = S(t)x. \tag{1.2.4}$$

(*b*) *For $x \in X$, $\int_0^t S(s)xds \in D(A)$ and*

$$A(\int_0^t S(s)xds) = S(t)x - x. \tag{1.2.5}$$

(*c*) *For $x \in D(A)$, $S(t) \in D(A)$ and*

$$\frac{d}{dt}S(t)(x) = A(S(t)(x)) = S(t)(A(x)). \tag{1.2.6}$$

(*d*) *For $x \in D(A)$*

$$S(t)x - S(s)x = \int_s^t S(\tau)Axd\tau = \int_s^t AS(\tau)xd\tau. \tag{1.2.7}$$

$$\lim_{t\to 0} S(t)(x) = x, \forall x \in X.$$

Proof. (a) Let $x \in X$ and $h > 0$; let's write the estimates

$$\begin{aligned}\|\frac{1}{h}\int_t^{t+h} S(s)xds - S(t)x\| &= \|\frac{1}{h}\int_t^{t+h} S(s)xds - \frac{1}{h}\int_t^{t+h} S(t)xds\| \\ &\leq \frac{1}{h}\int_t^{t+h} \|S(s)x - S(t)x\|ds. \end{aligned} \tag{1.2.8}$$

Changing the variable, set $u + t = s$, $du = ds$; if $s = u$ then $u = 0$ and $s = t + h$ then $u = h$.

$$\frac{1}{h}\int_t^{t+h} S(s)xds\|S(s)x - S(t)x\|ds = \frac{1}{h}\int_0^h \|S(t+u)x - S(t)x\|du.$$

Since u is a dummy variable one can write

$$\frac{1}{h}\int_t^{t+h} S(s)xds\|S(s)x - S(t)x\|ds = \frac{1}{h}\int_0^h \|S(s+t)x - S(t)x\|ds.$$

Since $t \mapsto S(t)x$ is a continuous function from $\mathbb{R}^+$ to X i.e, given $\epsilon > 0$, $\exists\delta > 0$ such that $|t-t_0| < \delta$ then $\|S(t)x - S(t_0x\| < \epsilon$. Take $h = t_0 - t$, we can write the continuity of $t \mapsto S(t)x$ equivalentely as follows given $\epsilon > 0$, $\exists\delta > 0$ such that $|h| < \delta$ then $\|S(t)x - S(t+h)x\| < \epsilon$.

$$\frac{1}{h}\int_0^h \|S(s+t)x - S(t)x\|ds < \frac{1}{h}\int_0^h \epsilon ds = \epsilon.$$

It is then natural to write

$$\frac{1}{h}\int_t^{t+h} \|S(s)x - S(t)x\|ds = \frac{1}{h}\int_0^h \|S(s+t)x - S(t)x\|ds < \frac{1}{h}\int_0^h \epsilon ds = \epsilon.$$

Therefore

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} S(s)xds = S(t)x.$$

(b) Let $x \in X$ and $h > 0$;

$$\begin{aligned}\frac{S(h)-I}{h}\int_0^t S(s)xds &= \frac{1}{h}\int_0^t S(s+h)xds - \frac{1}{h}\int_0^t S(s)xds \\ &= \frac{1}{h}\int_h^{t+h} S(s)xds - \frac{1}{h}\int_0^t S(s)xds.\end{aligned}$$

In the right hand side we have set for the first integral $u = s + h$; $du = ds$; if $s = 0$ then $u = h$ and if $s = t$ then $u = t + h$.

$$\begin{aligned}\frac{1}{h}\int_h^{t+h} S(s)xds - \frac{1}{h}\int_0^t S(s)xds &= \frac{1}{h}\int_h^t S(s)xds + \frac{1}{h}\int_t^{t+h} S(s)xds - \frac{1}{h}\int_0^t S(s)xds \\ &= \frac{1}{h}\int_h^0 S(s)xds + ac1h\int_0^t S(s)xds \\ &\quad +\frac{1}{h}\int_t^{t+h} S(s)xds - \frac{1}{h}\int_0^t S(s)xds \\ &= \frac{1}{h}\int_t^{t+h} S(s)xds - \frac{1}{h}\int_0^t S(s)xds.\end{aligned}$$

and letting $h \to 0$ the right-hand side tends to $S(t)x - x \in X$, which proves (b).

(c) Let $x \in D(A)$ and $h > 0$; then

$$\begin{aligned} \frac{S(h) - I}{h} S(t)x &= \frac{S(t+h) - S(t)}{h} x \\ &= S(t)\frac{S(h) - I}{h} x \to S(t)A(x) \text{ as } \ h \to 0. \end{aligned} \tag{1.2.9}$$

Thus, $S(t)x \in D(A)$ and $AS(t)x = S(t)x$. (1.2.9) implies also that

$$\frac{d+}{dt} S(t)x = AS(t)x = S(t)Ax,$$

i.e the right derivative of $S(t)x$ is $S(t)Ax$, to prove (1.2.6) we have to show that for $t > 0$ the left derivative of $S(t)x$ exists and equals $S(t)Ax$.
This follows from,

$$\lim_{h\to 0}[\frac{S(t)x - S(t-h)x}{h} - S(t)x] = \lim_{h\to 0} S(t-h)[\frac{S(h)x - x}{h} - S(t)x] + \lim_{h\to 0}(S(t-h)Ax - S(t)Ax)$$

and the fact that both terms on the right-hand side are zero, the first since $x \in D(A)$ and $\|S(t-h)\|$ is bounded on $0 \le h \le t$ and the second by continuity of $S(t)$. This conclude the proof of (c).
(d) Integrating (1.2.6) from s to t we obtain (d). □

Corollary 1.2.4. *If A is the infinitesimal generator of a C_0 semigroup $(S(t)_{t\ge 0})$ then $D(A)$ the domain of A, is dense in X and A is closed linear operator.*

Proof. Let $x \in X$, set $x_t = \frac{1}{t}\int_0^t S(t)xds$. By part ($c$) of Theorem 1.2.3, $x_t \in D(A)$ for $t > 0$ and by part (a) of the same theorem $x_t \to x$ as $t \to 0$. Thus $\overline{D(A)} = X$.
Let $(x, y) \in \overline{A}$ then there exist $(x_n)_{n\ge 1} \subset D(A)$ such that $(x_n, Ax_n) \to (x, y)$ i.e $x_n \to x$ and $Ax_n \to y$.
By part (b) of Theorem 1.2.3, we have

$$S(t)x_n - x_x = \int_0^t S(s)Ax_n ds \tag{1.2.10}$$

Claim: $\int_0^t S(s)Ax_n ds \to \int_0^t S(s)yds$ uniformly on bounded interval. Let $t \in [0, a]$ with $a > 0$;

$$\begin{aligned} \| \int_0^t S(s)Ax_n ds - \int_0^t S(s)yds\| &\le \int_0^t \|S(s)(Ax_n - y\|ds \\ &\le \int_0^t \|S(s)\|\|Ax_n - y\|ds \\ &\le Me^{\omega t}\|Ax_n - y\|. \end{aligned}$$

Since $Ax_n \to y$, it follows that

$$\lim_{n\to+\infty} \sup_{t\in[0,a]} \| \int_0^t S(s)Ax_n ds - \int_0^t S(s)yds\| = 0,$$

therefore our claim is true. Using the previous claim and letting $n \to +\infty$ in (1.2.10) yields

$$S(t)x - x = \int_0^t S(s)ydy. \tag{1.2.11}$$

Dividing (1.2.11) by $t > 0$ and letting $t \to 0$, we see, using part (a) of Theorem 1.2.3 that $x \in D(A)$ and $Ax = y$. $\square$

Theorem 1.2.5. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Proof. (a) It is known that the series $\sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$ in norm for every $t \geq 0$ and defines for each such t a bounded linear operator $S(t)$. It is easy to see that

- $S(0) = I$,
- S(t+s)=S(t)S(s), for all $t, s \geq 0$,
- $$\begin{aligned} e^{tA} &= \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = I + \sum_{n=1}^{\infty} \frac{(tA)^n}{n!} \\ e^{tA} - I &= tA \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{(n-1)!}. \end{aligned}$$

Taking the norm of both side, one has

$$\begin{aligned} \|e^{tA} - I\| &\leq \|tA\| \| \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{(n-1)!} \| \\ &\leq |t| \|A\| \| \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{(n-1)!} \| \\ &\leq |t| \|A\| e^{t\|A\|}, \end{aligned}$$

$\|e^{tA} - I\| \leq t\|A\|e^{t\|A\|}$ which goes to 0 as t goes to 0. Now, we claim that A is the infinitesimal generator of $S(t)$. Let us prove our claim, let $t > 0$. We have

$$\begin{aligned} e^{tA} - I &= tA\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!} \\ \frac{e^{tA}-I}{t} &= A\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!} \\ \frac{e^{tA}-I}{t} - A &= A\Big[\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!} - I\Big]. \end{aligned}$$

Taking the norm of both side, one has

$$\begin{aligned} \Big\|\frac{e^{tA}-I}{t} - A\Big\| &\leq \|A\|\Big\|\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!} - I\Big\|. \\ &\leq \|A\|\Big\|\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!} - I\Big\| = \|A\|\|e^{tA} - I\|. \end{aligned}$$

That is $\|\frac{e^{tA}-I}{t} - A\| \leq \|A\|\|S(t) - I\|$. Which implies as $t \to 0^+$ that $\lim\limits_{t\to 0^+}\dfrac{e^{tA}-I}{t} = A$.
We have have established that $S(t)$ is a uniformly continuous semgroup of bounded linear operators on X and that A is its infinitesimal generator.
(b) Let $S(t)$ be a C_0 semigroup of bounded linear operator on X.
Fix $\rho > 0$, small enough such that $\|I - \rho^{-1}\displaystyle\int_0^{\rho} S(s)ds\| \leq 1$ this implies that $\rho^{-1}\displaystyle\int_0^{\rho} S(s)ds$ is invertible and therefore $\displaystyle\int_0^{\rho} S(s)ds$ is invertible.

Now, let $h > 0$,

$$h^{-1}(S(h) - I)\int_0^{\rho} S(s)ds = h^{-1}\int_0^{\rho} S(h+s)ds - h^{-1}\int_0^{\rho} S(s)ds$$

and therefore

$$\begin{aligned} h^{-1}\big(S(h) - I\big)\int_0^{\rho} S(s)ds &= h^{-1}\Big(\int_0^{\rho} S(s+h)ds - \int_0^{\rho} S(s)ds\Big) \\ &= h^{-1}\Big(\int_h^{h+\rho} S(s)ds - \int_0^{\rho} S(s)ds\Big) \end{aligned}$$

$h^{-1}(S(h) - I) = h^{-1}\Big(\displaystyle\int_h^{h+\rho} S(s)ds - \int_0^{\rho} S(s)ds\Big)\Big(\int_0^{\rho} S(s)ds\Big)^{-1}$ and letting $h \to 0$ it follows that $h^{-1}(S(h) - I)$ converges in norm to a bounded linear operator $(S(\rho - I)\Big(\displaystyle\int_0^{\rho} S(s)ds\Big)^{-1}$ which is the infinitesimal generator of $S(t)$. $\square$

Theorem 1.2.6. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two C_0 semigroup on X, generated respectively by A and B. If $A = B$ then $T(t) = S(t)$, $t \geq 0$.*

Proof. Assume $A = B$ and let $x \in D(A) = D(B)$.
Define $\alpha(s) := T(t-s)S(s)x$, $s \in [0,t]$.
From Theorem 1.2.3 part (c) it follows that α is differentiable and that
$\alpha'(s) = \frac{d}{ds}T(t-s)S(s)x = -T(t-s)AS(s)x + T(t-s)BS(s)x = 0$, since $A = B$.
It follows $\alpha(s) = constant$. In particular, its values at $s = 0$ and $s = t$ are the same that is $T(t)x = S(s)x$ $\forall x \in D(A)$. By Corollary 1.2.4 $D(A)$ is dense in X and $T(t)$, $S(s)$ are closed;
therefore $T(t)x = S(s)x$; $\forall x \in X$.

□

For more details see [2, 6, 28, 29, 30, 38, 44, 56]

1.3 Some Fixed Point Theorems

Theorem 1.3.1 (Banach's fixed point theorem (1922) [16])**.** *Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 1.3.2 (**Schaefer's** fixed point theorem [16])**.** *Let X be a Banach space, and $N : X \longrightarrow X$ completely continuous operator.*
If the set $\mathcal{E} = \{y \in X : \ y = \lambda Ny, \ forsome\ \lambda \in (0,1)\}$ is bounded, then N has fixed points.

Theorem 1.3.3 (Darbo's Fixed Point Theorem [27, 16])**.** *Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $T : C \to C$ is such that for all closed subsets D of C,*

$$\alpha(T(D)) \leq k\alpha(D), \tag{1.3.1}$$

where $0 \leq k < 1$, and α is the Kuratowski measure of noncompactness. Then T has a fixed point in C.

Remark 1.3.1. *Mappings satisfying the Darbo-condition (1.3.1) have subsequently been called k-set contractions.*

Theorem 1.3.4 (Mönch's Fixed Point Theorem [2, 50])**.** *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.
Here α is the Kuratowski measure of noncompactness.

Chapter 2

Existence of Mild Solution for Neutral Functional Equations

In this chapter, we study the existence of solutions for neutral differential equations of the form:

$$\begin{cases} \frac{d}{dt}[x(t) + F(t, x(t), x(b_1(t)), ..., x(b_m(t)))] = \\ Ax(t) + G(t, x(t), x(a_1(t)), ..., x(a_n(t))), t \in J = [0, a], \\ x(0) = x_0. \end{cases} \tag{2.0.1}$$

Where A is the infinitesimal generator of a compact analytic semigroup of bounded linear operators $T(t)$ in a Banach space X, $F : [0, a] \times X^{m+1} \longrightarrow X, G : [0, a] \times X^{n+1} \longrightarrow X$ are continuous functions. The delays $a_i(t), b_j(t)$ are continuous scalar valued functions defined on J such that $a_i(t) \leq t, b_j(t) \leq t$. The purpose of this paper is to prove the existence of mild solutions for the same class of neutral equations with mild solutions by applying Schaefer's theorem instead of Sadovskii's theorem.

2.1 Main Result

Let $A : D(A) \longrightarrow X$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator $T(t)$ defined on a Banach space X with norm $\| \cdot \|$. Let $0 \in \rho(A)$ then define the fractional power A^α, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$ which is dense in X. Further $D(A^\alpha)$ is a Banach space under the norm

$$\| x \|_\alpha = \| A^\alpha x \|, \text{ for } x \in D(A^\alpha),$$

and is denoted by X_α imbedding $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta < \alpha \leq 1$ is compact whenever the resolvent operator of A is compact. For semigroup $\{T(t)\}$ the following properties will be used.

(a) there is a $M_1 > 1$ such that $\| T(t) \| \leq M_1$, for all $0 \leq t \leq a$.

(b) for any $\alpha > 0$, there exists a positive constant $M_2 > 0$ such that

$$\| A^{\alpha}T(t) \| \leq M_2 t^{-\alpha}, 0 < t \leq a. \tag{2.1.1}$$

Definition 2.1.1. *A function $x(\cdot)$ is called a mild solution of the system (2.0.1) if $x(0) = x_0$, the restriction of $x(\cdot)$ to the interval $[0, a]$ is continuous and for each $0 \leq t \leq a$ the function $AT(t-s)F(s, x(s), x(b_1(s)), \ldots, x(b_m(s)))$, $s \in [0, t)$, is integrable, and the following integral equation*

$$\begin{aligned} x(t) = \;& T(t)[x_0 + F(0, x_0, x(b_1(0)), \ldots, x(b_m(0))] \\ & - F(t, x(t), x(b_1(t)), \ldots, x(b_m(t))) \\ & - \int_0^t AT(t-s)F(s, x(s), x(b_1(s)), \ldots, x(b_m(s)))ds \\ & + \int_0^t T(t-s)G(s, x(s), x(a_1(s)), \ldots, x(a_n(s)))ds, \end{aligned} \tag{2.1.2}$$

is satisfied.

Assume that the following conditions hold:

(H1) For each $t \in J$, the function $G(t, \cdot) : X^{n+1} \longrightarrow X$ is continuous, and for each $(x_0, x_1, \ldots, x_n) \in X^{n+1}$ the function $G(\cdot, x_0, x_1, \ldots, x_n) : [0, a] \longrightarrow X$ is strongly mesurable.

(H2) For each positive integer k there exists $\alpha_k \in L^1[0, a]$ such that

$$\sup_{\|x_0\| \ldots \|x_n\| \leq k} \| G(t, x_0, x_1, \ldots, x_n) \| \leq \alpha_k(t) \quad \text{for} \quad t \in J.$$

(H3) The function $F : [0, a] \times X^{m+1} \longrightarrow X$ is completely continuous and for any bounded set $\mathbf{Q}$ in $C([-r, a], X)$ the set

$$\{t \longrightarrow F(t, x(t), x(a_1(t)), \ldots, x(a_m(t))) : x \in \mathbf{Q}\},$$

is equicontinuous.

(H4) There exist $\beta \in (0, 1)$ and a constant $c_1 \geq 0$such that
$\| (A)^{\beta}F(t, u(t)) \| \leq M_3, t \in J.$

(H5) There exists an integrable function $m : [0, a] \longrightarrow [0, \infty)$ such that

$$\| G(t, x(t), x(a_1(t)), \ldots, x(a_n(t))) \| \leq (n+1)m(t)\Omega(\| x(t) \|),$$

where $\Omega : [0, \infty) \longrightarrow (0, \infty)$ is a continuous nondecreasing function.

(H6)

$$\int_0^a \widehat{m}(s)ds < \int_0^\infty \frac{ds}{s+\Omega(s)},$$

where

$$c = M_1[\| x_0 \| + M_3M_4] + M_3M_4 + \frac{M_3M_2a^\beta}{\beta},$$

$$M_4 = \| (A)^{-\alpha} \|,$$

$$m(t) = M_1 m(t)(n+1)^2.$$

Now let us take

$$(t, x(t), x(b_1(t)), \ldots, x(b_m(t))) = (t, u(t)),$$

$$(t, x(t), x(a_1(t)), \ldots, x(a_n(t))) = (t, v(t)).$$

Theorem 2.1.1. *If the above assumptions are satisfied then the problem (2.0.1) has a mild solution on $J = [0, a]$.*

Proof. Consider the Banach space $Z = C(J, X)$ with norm

$$\| x \| = \sup\{| x(t) | : t \in J\}.$$

To prove the existence of mild solution of (2.0.1) we have to apply Schaefer theorem for the following operator equation

$$x(t) = \lambda\Psi x(t), 0 < \lambda < 1. \qquad (2.1.3)$$

Balachandran et al. Where $\Psi : Z \longrightarrow Z$ is defined as

$$(\Psi x)(t) = T(t)[x_0 + F(0, u(0))] - F(t, u(t)) - \int_0^t AT(t-s)F(s, u(s))ds + \int_0^t T(t-s)G(s, v(s))ds$$

. Then from (2.1.2) we have

$$\begin{aligned} \| x(t) \| &\leq M_1[\| x_0 \| + M_3M_4] + M_3M_4 + M_2 \textstyle\int_0^t M_3^{\beta-1}(t-s)ds \\ &+ M_1 \textstyle\int_0^t (n+1)m(s)\Omega(\| v(s) \|)ds \\ &\leq M_1[\| x_0 \| + M_3M_4] + M_3M_4 + \tfrac{M_3M_2a^\beta}{\beta} + M_1 \textstyle\int_0^t (n+1)m(s)\Omega(\| v(s) \|)ds. \end{aligned}$$

Denoting the right hand side of above inequality as $\mu(t)$ then

$$\| x(t) \| \leq \mu(t) \quad \text{and}\, \mu(0) = c = M_1[\| x_0 \| + M_3M_4] + M_3M_4 + \frac{M_3M_2a^\beta}{\beta},$$

$$\mu'(t) = M_1(n+1)m(t)\Omega(\| v(t) \|) \leq M_1(n+1)^2 m(t)\Omega(\mu(t)) \leq m(t)[\Omega(\mu(t))].$$

This implies

$$\int_{\mu(0)}^{\mu(t)} \frac{ds}{\Omega(s)} \leq \int_0^a \widehat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega(s)}, \quad 0 \leq t \leq a \tag{2.1.4}$$

Inequality (2.1.4) implies that there is a constant K such that $\mu(t) \leq K, t \in [0,a]$ and hence we have $\| x \| = \sup\{| x(t) | : t \in J\} \leq K$, where K depends only on a and on the functions $\widehat{m}$ and Ω.
We shall now prove that the operator $\Psi : Z \longrightarrow Z$ is a completely continuous operator. Let $B_k = \{x \in Z : \| x \|_1 \leq k\}$ for some $k \geqslant 1$. We first show that Ψ maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in [0,a]$. Then if $0 < t_1 < t_2 < a$,

$$\begin{aligned}
&\| (\Psi x)(t_1) - (\Psi x)(t_2) \| \\
&\leq \| (T(t_1) - T(t_2))[x_0 + F(0,u(0))] \| + \| F(t_1,u(t_1)) - F(t_2,u(t_2)) \| \\
&+ \| \textstyle\int_0^{t_1} A[T(t_1-s) - T(t_2-s)]F(s,u(s))ds \| + \| \int_{t_1}^{t_2} AT(t_2-s)F(s,u(s))ds \| \\
&+ \| \textstyle\int_0^{t_1} [T(t_1-s) - T(t_2-s)]G(s,v(s))ds \| + \| \displaystyle\int_{t_1}^{t_2} T(t_2-s)G(s,v(s))ds \| \\
&\leq \| (T(t_1) - T(t_2))[x_0 + F(0,u(0))] \| + \| F(t_1,u(t_1)) - F(t_2,u(t_2)) \| \\
&+ \int_0^{t_1} \| A[T(t_1-s) - T(t_2-s)] \| M_3 M_4 ds + \int_{t_1}^{t_2} \| AT(t_2-s) \| M_3 M_4 ds \\
&+ \int_0^{t_1} \| T(t_1-s) - T(t_2-s) \| \alpha_k(s)ds + \int_{t_1}^{t_2} \| T(t_2-s) \| \alpha_k(s)ds
\end{aligned}$$

.

The right hand side is independent of $x \in B_K$ and tends to zero as $t_2 - t_1 \longrightarrow 0$, since F is completely continuous and the compactness of $T(t)$ for $t > 0$ implies continuity in the uniform operator topology. Thus Ψ maps B_k into an equicontinuous family of functions. It is easy to see that ΨB_k is uniformly bounded. Next, we show $\overline{\Psi B_k}$ is compact. Since we have shown ΨB_k is equicontinuous collection, by the Arzela-Ascoli theorem it suffices to show that Ψ maps B_k into a precompact set in X.
Let $0 < t \leq a$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_k$,

we define

$$
\begin{aligned}
(\Psi_\epsilon x)(t) &= T(t)[x_0 + F(0,u(0))] - F(t,u(t)) - \int_0^{t-\epsilon} AT(t-s)F(s,u(s))ds \\
&+ \int_0^{t-\epsilon} T(t-s)G(s,v(s))ds \\
&= T(t)[x_0 + F(0,u(0))] - F(t,u(t)) - T(\epsilon)\int_0^{t-\epsilon} AT(t-s-\epsilon)F(s,u(s))ds \\
&+ T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)G(s,v(s))ds
\end{aligned}
$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = (\Psi_\epsilon x)(t) : x \in B_k$ is precompact in X for every $\epsilon, 0 < \epsilon < t$. Moreover, for every $x \in B_k$ we have

$$
\begin{aligned}
&\parallel (\Psi x)(t) - (\Psi_\epsilon x)(t) \parallel \\
&\qquad \leq \int_t^{t-\epsilon} \parallel AT(t-s)F(s,u(s)) \parallel ds + \int_t^{t-\epsilon} \parallel T(t-s)G(s,v(s)) \parallel ds \\
&\qquad \leq \int_t^{t-\epsilon} \parallel AT(t-s)F(s,u(s)) \parallel ds + \int_t^{t-\epsilon} \parallel T(t-s) \parallel \alpha_k(s)ds
\end{aligned}
$$

. Therefore there are precompact sets arbitrarily close to the set $\{(\Psi x)(t) : x \in B_k\}$. Hence, the set $\{(\Psi x)(t) : x \in B_k\}$ is precompact in X. It remains to show that $\Psi : Z \longrightarrow Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \longrightarrow x$ in Z. Then there is an integer q such that $\parallel x_n(t) \parallel \leq q$ for all n and $t \in J$, so $x_n \in B_r$ and $x \in B_r$. by (H2)

$$
G(t, v_n(t)) \longrightarrow G(t, v(t)),
$$

for each $t \in J$ and since

$$
\parallel G(t, v_n(t)) - G(t, v(t)) \parallel \leq 2\alpha_q(t),
$$

Balachandran et al. we have, by the dominated convergence theorem, that

$$\begin{aligned}
\| \Psi x_n - \Psi x \| &= \sup_{t\in J} \| [F(t,u_n(t)) - F(t,u(t))] \\
&+ \int_0^t AT(t-s)[F(s,u_n(s)) - F(s,u(s))]ds \\
&+ \int_0^t T(t-s)[G(s,u_n(s)) - G(s,u(s))]ds \| \\
&\leq \| F(t,u_n(t)) - F(t,u(t)) \| \\
&+ \int_0^t \| AT(t-s) \| \| F(s,u_n(s)) - F(s,u(s)) \| ds \\
&+ \int_0^t \| T(t-s) \| \| G(s,u_n(s)) - G(s,u(s)) \| ds \\
&\longrightarrow 0,
\end{aligned}$$

as $n \longrightarrow \infty$. Thus Ψ is continuous. This completes the proof that Ψ is completely continuous.
Finally the set $\varsigma(\Psi) = \{x \in Z : x = \lambda\Psi x, \lambda \in (0,1)\}$ is bounded, as we proved in the first step. Consequently, by Schaefer theorem, the operator Ψ has a fixed point in Z. This means that any fixed point of Ψ is a mild solution of (2.0.1) on J satisfying $(\Psi x)(t) = x(t)$. □

2.2 Application

As an application of Theorem (4.1.1), we shall consider the system (5.1.1) with a control parameter such as

$$\begin{aligned}
\tfrac{d}{dt} \quad & [c(t) + F(t,x(t),x(b_1(t),\ldots,x(b_m(t))))] = Ax(t) + B\omega(t) \\
& + G(t,x(t),x(a_1(t)),\ldots\ldots,x(a_n(t))), t \in J = [0,a], \qquad (2.2.1) \\
& x(0) = x_0
\end{aligned}$$

Where B is a bounded linear operator from U, a Banach space, to X and $\omega \in L^2(J,U)$. In this case the mild solution of (2.2.1) is given by

$$\begin{aligned} x(t) &= T(t)[x_0 + F(0,u(0))] - F(t,u(t)) - \int_0^t AT(t-s)F(s,u(s))ds \\ &+ \int_0^t T(t-s)[B\omega(s)ds + G(s,v(s))]ds \end{aligned}$$

We say the system (2.2.1) is locally controllable on the interval J if for any subset $Y \subset X$ and for every $x_0, x_1 \in Y$, there exists a control $\omega \in L_2(J,U)$ such that the solution $x(\cdot)$ of (2.2.1) satisfies $x(a) = x_1$. Let $X_r = \{x \in X : |x| \leq r\{$ for some $r > 0$ and $Z_r = C^1(J, X_r)$.
Controllability of nonlinear systems of various types in Banach spaces has been studied by several authors by means of fixed point principles . Recently Balachandran and Anandhi and Fu investigated the controllability problem for neutral systems. To establish the controllability result for the system (2.2.1) we need the following additional hypotheses.

(H7) The linear operator $W : L^2(J,U) \longrightarrow X$ defined by

$$Wu = \int_0^a T(a-s)B\omega(s)ds$$

has an induced inverse operator $\tilde{W}^{-1}$ which takes values in $L^2(J,U)/\ker W$ and there exists a positive constant M_5 such that $\| B\tilde{W}^{-1} \| \leq M_5$.

(H8)

$$\int_0^a m(s)ds < \int_0^\infty \frac{ds}{\Omega(s)},$$

where

$$\begin{aligned} c = & \quad M^1[\| x_0 \| + M_3 c_1] + M_3 M_4 + \tfrac{M_3 M_4 a^\beta}{\beta} + M_1 N a, \\ N = & \quad M_5\{\| x_1 \| + M_1 \| x_0 \| + M_3 M_4 + \tfrac{M_3 M_4 a^\beta}{\beta} \\ & + M_1 \int_0^a m(s)(n+1)\Omega(r)ds\}. \end{aligned}$$

Theorem 2.2.1. *If the hypotheses (H1)-(H8) are satisfied, then the system (2.2.1) is controllable.*

Proof. Using the hypotheses (H7), for an arbitrary function $x(.)$, define the control

$$\begin{aligned} \omega(t) &= W^{-1}\{x_1 - T(a)[x_0 + F(0,u(0))] + F(a,u(a)) \\ &+ \int_0^a AT(a-s)F(s,u(s))ds - \int_0^1 T(a-s)G(s,v(s))ds\}(t) \end{aligned}$$

We shall show that when using this control the operator $\Phi : Z_r \longrightarrow Z_r$ defined by

$$\begin{aligned}(\Phi x)(t) \quad &= T(t)[x_0 + F(0, u(0))] - F(t, u(t)) - \int_0^t AT(t-s)F(s, u(s))ds \\ &\int_0^t T(t-s)[B\omega(s) + G(s, v(s))]ds, t \in J,\end{aligned}$$

has a fixed point. This fixed point is then a solution of (2.2.1). Substituting $\omega(t)$ in the above equation we get $(\Phi x)(a) = x_1$, which means that the control ω steers system (2.2.1)from the given initial condition $x_0$tox_1 in time a. Thus the system (2.2.1) is controllable. The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted. $\square$

Chapter 3

Functional Differential Equations with State-Dependent Delays

3.1 Uniqueness of Mild Solutions

We study the uniqueness of solutions for neutral differential equations with state-dependent delays of the following form, on $J := [0, T]$

$$\frac{d}{dt}(x(t) - g(t, x(t - \eta(t)))) = A(x(t) - g(t, x(t - \eta(t)))) + f(t, x_t, x(t - \tau(t, x_t))),\ t \in J, \tag{3.1.1}$$

with initial condition

$$x(t) = \varphi(t),\ t \in [-r, 0], \tag{3.1.2}$$

where A generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on a Banach space E, $f : J \times C([-r, 0], E) \times E \to E$, $g : J \times E \to E$ are given functions, and $\varphi : [-r, 0] \to E$, $\tau : [0, T] \times C([-r, 0], E) \to [0, r]$ and $\eta : J \to [0, r]$ are also given continuous functions.

This chapter is organized as follows: in Section 3.1, we give one of our main Uniqueness results for solutions of (3.1.1)-(3.1.2), with the proof based on Banach's fixed point theorem 1.3.1. In Section 3.2, we give two other existence results for solutions of (3.1.1)-(3.1.2). Their proofs involve the measure of noncompactness paired in one result with a Mönch fixed point theorem 1.3.4 and paired in the other result with a Darbo fixed point theorm 1.3.3.

Lemma 3.1.1. *We say that a continuous function $x : [-r, T] \to E$ is a mild solution of problem (3.1.1), (3.1.2) if $x(t) = \varphi(t),\ t \in [-r, 0]$ and*

$$\begin{aligned} x(t) &= S(t)[\varphi(0) - g(0, x(-\eta(0)))] + g(t, x(t - \eta(t))) \\ &+ \int_0^t S(t-s) f(s, x_s, x(s - \tau(s, x_s)))ds,\ t \in J. \end{aligned}$$

Proof. we pose

$$
\begin{aligned}
g(s) = &\quad T(t-s)x(s), \quad \text{then}\\
g(0) = &\quad T(t)x_0 = T(t)\varphi(0),\\
g(t) = &\quad T(t-t)x(t) = x(t)\\
&\quad \text{and}\\
g'(s) = &\quad -AT(t-s)x(s) + T(t-s)x'(s)\\
&\quad = T(t-s)(-Ax(s) + x'(s))\\
= &\quad T(t-s)(g'(t, x(t-\eta(t))) - Ag(t, x(t-\eta(t))) + f(t, x_t, x(t-\eta(t, x_t))))\\
= &\quad T(t-s)g'(t, x(t-\eta(t))) - AT(t-s)g + T(t-s)f(t, x_t, x(t-\tau(t, x_t))\\
= &\quad (T(t-s)g(t, x(t-\eta(t))))' + T(t-s)f(t, x_t, x(t-\tau(t, x_s))
\end{aligned}
$$

An integration from 0 to t , we have:

$$
\begin{aligned}
g(t) - g(0) = &\quad [T(t-s)g]_0^t + \int_0^t T(t-s)f ds\\
g(t) = x(t) = &\quad g(0) + g(t, x(t-\eta(t))) - T(t)g(0, x(0, x(-\eta(0))) + \int_0^t T(t-s)ds\\
x(t) = &\quad T(t)\varphi(0) + g(t, x(t-\eta(t))) - T(t)g(0, x(-\eta(0))) + \int_0^t T(t-s)f ds\\
x(t) = &\quad T(t)\varphi(0) - \varphi(0, x(-\eta(0))) + g(t, x(t-\eta(t))) + \int_0^t T(t-s)f ds
\end{aligned}
$$

□

Lemma 3.1.2. *(See [37]) Let $a > 0$, $b \geq 0$, $r_1 > 0$, $r_2 \geq 0$, $r = \max\{r_1, r_2\}$, and $v : [0, \sigma] \to [0, \infty)$ be continuous and nondecreasing. Let $u : [-r, \sigma] \to [0, \infty)$ be continuous and satisfy the inequality*

$$u(t) \leq v(t) + bu(t - r_1) + a\int_0^t u(s - r_2)ds,\ t \in [0, \sigma].$$

Then $u(t) \leq d(t)e^{ct}$ for $t \in [0, \sigma]$, where c is the unique positive solution of $cbe^{-cr_1} + ae^{-cr_2} = c$, and

$$d(t) = \max\Big\{\frac{v(t)}{1 - be^{-cr_1}}, \max_{-r \leq s \leq 0} e^{-cs}u(s)\Big\},\ t \in [0, \sigma].$$

Let $\Omega_1 \in C$, $\Omega_2 \in E$ and $\Omega_3 \in E$ be open subsets of their respective spaces. Let $T > 0$ be finite, or $T = \infty$, in which case $[0, T]$ denotes the interval $[0, \infty)$.
We define the set

$$\Pi = \{\varphi \in C : \ \varphi \in \Omega_1, \ \varphi(-\tau(0, \varphi)) \in \Omega_2, \ \varphi(-\eta(0)) \in \Omega_3\}.$$

Let us introduce the following hypotheses:

(H_1) A is the generator of a strongly continuous semigroup $S(t)$, $t \in J$ which is compact for $t > 0$ in the Banach space E. Let $M > 0$ be such that

$$\|S(t)\| \leq M \quad \text{for all } t \in J.$$

(H_2) (i) $f : J \times \Omega_1 \times \Omega_2 \to E$ is continuous;
(ii) $f(t, \psi, u)$ is locally Lipschitz continuous in ψ and u in the following sense: for every finite $\sigma \in (0, T]$, for every closed and bounded subset $M_1 \subset \Omega_1$ of C and closed and bounded subset $M_2 \subset \Omega_2$ of E, there exists a constant $L_1 > 0$ such that

$$\|f(t, \psi_1, u_1)) - f(t, \psi_2, u_2))\| \leq L_1(\sup_{\zeta \in [-r, -r_0]} \|\psi_1(\zeta) - \psi_2(\zeta)\| + \|u_1 - u_2\|),$$

for every $t \in [0, \sigma]$, ψ_1, $\psi_2 \in M_1$ and u_1, $u_2 \in M_2$,

(H_3) (i) $g : J \times \Omega_3 \to E$ is continuous;
(ii) $g(t, u)$ is locally Lipschitz continuous in u in the following sense: for every finite $\sigma \in (0, T]$, for every closed and bounded subset $M_3 \subset \Omega_3$ of E, there exists a constant $0 < L_2 < 1$ such that

$$\|g(t, u_1) - g(t, u_2)\| \leq L_2\|u_1 - u_2\|,$$

for every $t \in [0, \sigma]$ and u_1, $u_2 \in M_3$,

(H_4) there exists a constant $r_0 > 0$, such that $r_0 \leq \tau(t, \psi) \leq r, \ t \in [0, T]$, and $\psi \in \Omega_1$.

(H_5) there exists a constant $L_3 > 0$, such that

$$\|\varphi(\zeta) - \varphi(\bar{\zeta})\| \leq L_3\|\zeta - \bar{\zeta}\|,$$

for ζ, $\bar{\zeta} \in [-r, 0]$.

Theorem 3.1.3. *Assume that assumptions* $(H_1) - (H_4)$ *hold and let* $\gamma \in \Pi$. *Then, there exist* $\delta > 0$ *and* $0 < \sigma \leq T$ *finite numbers such that*

(i) $P = \bar{B}_C(\gamma, \delta) \subset \Pi$;

(ii) the problem (3.1.1)-(3.1.2) has a unique mild solution on a maximal interval of existence $[-r, T)$ *for all* $\gamma \in P$.

Proof. We define the following constants $K_1, K_2, K_3 > 0$ such that:

$$\| f(t, \psi, \psi(-\tau(t, \psi))) \| \leq K1,$$

$$\| g(t, \psi(-\eta(t))) \| \leq K_2$$

and

$$K_3 = \|\varphi(0)\|.$$

Let $\delta > 0$ and

$$E_0 = \{u \in C([-r, \sigma], E),\ u(t) = \varphi(t) \text{ if } t \in [-r, 0] \text{ and } \sup_{t\in[0,\sigma]} \|u(t) - \varphi(0)\| \leq \delta\}.$$

It is clear that E_0 is a closed set of $C([-r, \sigma], E)$, for more details see [15]. Transform the problem (3.1.1)-(3.1.2) into a fixed point problem. Consider the operator

$$N : E_0 \to C([-r, \sigma], E)$$

defined by

$$Nx(t) = \begin{cases} \varphi(t), & \text{t} \in [-r, 0]. \\ \\ S(t)[\varphi(0) - g(0, x(-\eta(0)))] + g(t, x(t - \eta(t))) & \\ + \displaystyle\int_0^t S(t - s) f(s, x_s, x(s - \tau(s, x_s)))ds\ , & \text{t} \in J. \end{cases} \tag{3.1.3}$$

Note that a fixed point of N is a mild solution of (3.1.1)-(3.1.2). We will show that

$$N(E_0) \subseteq E_0.$$

Let $v \in E_0$ and $t \in [0, \sigma]$. We have

$$\begin{aligned} \|N(v)(t) - \varphi(0)\| &\leq \|S(t)[\varphi(0) - g(0, v(-\eta(0)))] - \varphi(0)\| + \|g(t, v(t - \eta(t)))\| \\ &\quad + \| \int_0^t S(t - s) f(s, v_s, v(s - \tau(s, v_s)))ds\| \\ &\leq (M + 1)\|\varphi(0)\| + M\|g(0, v(-\eta(0)))\| + \|g(t, v(t - \eta(t)))\| \\ &\quad + M\| \int_0^t f(s, v_s, v(s - \tau(s, v_s)))ds\| \\ &\leq (M + 1)K_3 + MK_2 + K_2 + MK_1 \int_0^t ds \\ &\leq (M + 1)K_3 + (M + 1)K_2 + M\sigma K_1 \\ &\leq 3\beta \leq \delta. \end{aligned}$$

Hence,

$$N(E_0) \subseteq E_0.$$

On the other hand, let v, $w \in E_0$. Then for $t \in [0, \sigma]$, we have

$$\begin{aligned}\|N(v)(t) - N(w)(t)\| \;&\leq\; \|S(t)[g(0, v(-\eta(0))) - g(0, w(-\eta(0)))]\| \\ &+\|g(t, v(t-\eta(t))) - g(t, w(t-\eta(t)))\| \\ &+\| \int_0^t S(t-s)[f(s, v_s, v(s-\tau(s, v_s))) \\ &-f(s, w_s, w(s-\tau(s, w_s)))]ds\| \\ &\leq ML_2\|v(-\eta(0)) - w(-\eta(0))\| \\ &+L_2\|v(t-\eta(t)) - w(t-\eta(t))\| \\ &+ML_1 \int_0^t \sup_{\zeta\in[-r,-r_0]} \|v_s(\zeta) - w_s(\zeta)\| \\ &+ML_1 \int_0^t \|v(s-\tau(s, v_s)) - w(s-\tau(s, w_s))\|ds \\ &\leq ML_2\|v(-\eta(0)) - w(-\eta(0))\|\end{aligned}$$

$$L_2\|v(t-\eta(t)) - w(t-\eta(t))\|$$

$$\begin{aligned}&+ML_1 \int_0^t \sup_{\zeta\in[-r,-\sigma]} \|v_s(\zeta) - w_s(\zeta)\| \\ &+ML_1 \int_0^t \|v(s-\tau(s, v_s)) - w(s-\tau(s, w_s))\|ds.\end{aligned}$$

Since $u_t(\zeta) = u(t+\zeta) = \varphi(t+\zeta) = \varphi_t(\zeta)$ for $t \in [0, \sigma]$ and $\zeta \in [-r, -\sigma]$. We have $t - \tau(t, \varphi_t) \leq t - r_0 \leq t - \sigma \leq 0$ for $t \in [0, \sigma]$, so $u_t(-\tau(t, \varphi_t)) = \varphi_t$ for $t \in [0, \sigma]$, and $v(-\eta(0)) = w(-\eta(0)) = \varphi(-\eta(0))$. Then

$$\begin{aligned}\|N(v)(t) - N(w)(t)\| \;&\leq\; ML_2\|\varphi(-\eta(0)) - \varphi(-\eta(0))\| \\ &+L_2\|v(t-\eta(t)) - w(t-\eta(t))\| \\ &+ML_1 \int_0^t [\|\varphi_s - \varphi_s\| + \|\varphi(s-\tau(s, \varphi_s)) - \varphi(s-\tau(s, \varphi_s))\|]ds. \\ &\leq\; L_2\|v(t-\eta(t)) - w(t-\eta(t))\| \\ &\leq\; L_2 \sup_{\theta\in[-r,0]} \sup_{t\in[0,\sigma]} \|v(t+\theta) - w(t+\theta)\| \\ &\leq\; L_2\|v - w\|_\infty.\end{aligned}$$

Consequently,

$$\|N(v) - N(w)\|_\infty \leq L_2\|v - w\|_\infty.$$

Since $L_2 < 1$, N is a contraction. By the Banach fixed point theorem 1.3.1 we conclude that N has a unique fixed point in E_0 and the problem (3.1.1)-(3.1.2) has a unique mild solution on $[-r, \sigma]$.

Let $u(t)$ be the unique mild solution of problem (3.1.1)-(3.1.2) defined on its maximal interval of existence $[0,T)$, $T>0$. Assume that $T<\infty$ and

$$\lim_{t\to T^-}\|u(t)\|<\infty.$$

Then, there exists a constant $\rho>0$ such that $\|u(t)\|\leq\rho$, for $t\in[-r,T)$.
Note that (H_2) and (H_3) imply that

$$\begin{aligned}\|f(t,\psi,\psi(-\tau(t,\psi)))-f(0,\varphi,\varphi(-\tau(0,\varphi)))\| &\leq L_1(\|\psi-\varphi\|+\|\psi(-\tau(t,\psi))\\ &\quad -\varphi(-\tau(0,\varphi))\|)\end{aligned}$$

for $t\in[0,\sigma]$, $\psi\in\bar{B}_C(\widehat{\varphi},\delta)$. Similarly,

$$\|g(t,\psi(-\eta(t)))-g(0,\varphi(-\eta(0)))\|\leq L_2\|\psi(-\eta(t))-\varphi(-\eta(0))\|$$

for $t\in[0,\sigma]$, $\psi\in\bar{B}_C(\widehat{\varphi},\delta)$.
We define the following constants

$$c_1=\|f(0,\varphi,\varphi(-\tau(0,\varphi)))\|+L_1(\|\varphi\|+\|\varphi(-\tau(0,\varphi))\|),$$

$$c_2=\|g(0,\varphi(-\eta(0)))\|+L_2\|\varphi(-\eta(0))\|.$$

Let $t\in[0,T)$. We obtain

$$\begin{aligned}\|u(t)\| &\leq \|S(t)[\varphi(0)-g(0,u(-\eta(0)))]\|\\ &\quad +\|g(t,u(t-\eta(t)))\|\\ &\quad +\|\int_0^t S(s)f(s,u_s,u(s-\tau(s,u_s)))ds\|\\ &\leq M[\|\varphi(0)\|+\|g(0,u(-\eta(0)))\|]+L_2\|u(t-\eta(t))\|+c_2+Mc_1t\\ &\quad +ML_1\int_0^t[\|u_s\|+\|u(s-\tau(s,u_s))\|]ds\\ &\leq M[\|\varphi(0)\|+\|g(0,u(-\eta(0)))\|]+L_2\|u(t-\eta(t))\|+c_2+tMc_1\\ &\quad +tML_1\|u\|_\infty+ML_1\int_0^t\|u(s-\tau(s,u_s))\|ds\\ &\leq v(t)+L_2\|u(t-r_1)\|+ML_1\int_0^t\|u(s-r_2)\|ds\end{aligned}$$

where $r_1=\eta$, $r_2=\tau$ and

$$v(t)=M[\|\varphi(0)\|+\|g(0,u(-\eta(0)))\|]+c_2+tMc_1+tML_1\|u\|_\infty.$$

By Lemma 3.1.2, it follows that

$$\|u(t)\| \leq d(t)e^{ct}$$

for $t \in [0,T)$, where c is the unique positive solution of $cL_2e^{-cr_1} + ML_1e^{-cr_2} = c$, and

$$d(t) = \max\left\{ \tfrac{v(t)}{1-L_2e{-cr_1}}, \max_{-r\le s\le 0} e^{-cs}u(s), \right\}, \ t \in [0,T).$$

It follows that $\lim_{t\to T^-} u(t)$ exists. Consequently, $u(t)$ can be extended to T, which contradicts the maximality of $[0,T)$. □

3.2 Existence of Mild Solutions

In this section we apply a technique based on noncompactness measure assumption on the nonlinear term in proving an existence result for problem (3.1.1)-(3.1.2).
We introduce some additional hypotheses:

(H_5) The function $f : J \times C \times E \to E$ is continuous.

(H_6) (i) There exist constants $c_1 \ge 0$ and $c_2 \ge 0$ such that

$$\|g(t,u)\| \le c_1\|u\| + c_2, \text{ a.e. } t \in J,\ u \in E;$$

(ii) the function g is completely continuous and for any bounded set B in Ω, the set $\left\{ t \to g(t, x(t-\eta(t))) : \ \mathrm{x} \in B \right\}$ is equicontinuous in Ω.

(H_7) There exist $c_3 > 0$, $p \in L^1(J,\mathbb{R}_+)$ and a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\|f(t,u,v)\| \le p(t)\psi(\|u\|) + c_3\|v\|, \text{ for each } u \in C, v \in E \text{ and } t \in J.$$

(H_8) For each bounded $B \subset E$, $B' \subset E$ and $t \in J$ we have

$$\alpha(f(t,B,B')) \le p(t)\alpha(B) + c_3\alpha(B').$$

(H_9) For each $t \in J$ and bounded $B \subset E$ we have

$$\alpha(g(t,B)) \le c_1\alpha(B).$$

(H_{10}) There exists $q > 0$ such that

$$M\|\varphi\|_\infty + (M+1)[c_1q + c_2] + M[\|p\|_{L^1}\psi(q) + Tc_3q] \le q.$$

Theorem 3.2.1. *Assume that (H_1), (H_5), (H_6), (H_7), (H_8), (H_9) and (H_{10}) hold. Suppose that*

$$\big[c_1 + M\big(c_1 + \|p\|_{L^1} + c_3T\big)\big] < 1. \tag{3.2.1}$$

Then the problem (3.1.1)-(3.1.2) has at least one mild solution on $[-r,T]$.

Proof. Transform the problem (3.1.1)-(3.1.2) into a fixed point problem. Consider the operator

$$N : \Omega \to \Omega$$

defined by

$$(Nx)(t) = \begin{cases} \varphi(t), & \mathrm{t} \in [-r, 0]. \\ S(t)[\varphi(0) - g(0, x(-\eta(0)))] + g(t, x(t - \eta(t))) & \\ + \displaystyle\int_0^t S(t-s) f(s, x_s, x(s - \tau(s, x_s))) ds \ , & \mathrm{t} \in J. \end{cases} \tag{3.2.2}$$

Note that a fixed point of N is a mild solution of (3.1.1)-(3.1.2).

We will show that N satisfies the assumptions of the Mönch fixed point theorem 1.3.4.

Consider the set

$$B_q = \{u \in \Omega : \|u\|_\infty \leq q\},$$

where q is the constant defined in (H_{10}). Clearly, the subset B_q is closed, bounded, and convex.

The proof will be given in several steps.

Step 1: N is continuous.

Using (H_6), it suffices to show that the operator $N_1 : \Omega \to \Omega$ defined by

$$N_1(x)(t) = \begin{cases} \varphi(t), & \mathrm{t} \in [-r, 0]. \\ S(t)\varphi(0) + \displaystyle\int_0^t S(t-s) f(s, x_s, x(s - \tau(s, x_s))) ds \ , & \mathrm{t} \in J. \end{cases} \tag{3.2.3}$$

Is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in Ω. Then

$$\begin{aligned} \|N_1(u_n)(t) - N_1(u)(t)\| &\leq \| \int_0^t S(t-s)[f(s, u_{ns}, u_n(s - \tau(s, u_{ns}))) \\ &\quad - f(s, u_s, u(s - \tau(s, u_s)))] ds \| \\ &\leq M \int_0^t \|f(s, u_{ns}, u_n(s - \tau(s, u_{ns}))) \\ &\quad - f(s, u_s, u(s - \tau(s, u_s)))\| ds \\ &\leq M \int_0^t \sup_{\theta \in [-r,0]} \sup_{s \in [0,T]} \|f(s, u_{ns}, u_n(s + \theta)) \\ &\quad - f(s, u_s, u(s + \theta))\| ds \\ &\leq MT \|f(\cdot, u_{n\cdot}, u_n(\cdot)) - f(\cdot, u_{\cdot}, u(\cdot))\|_\infty. \end{aligned}$$

Since f is a continuous function, we have

$$\|N_1(u_n) - N_1(u)\|_\infty \leq MT\|f(\cdot, u_{n\cdot}, u_n(\cdot)) - f(\cdot, u_{\cdot}, u(\cdot))\|_\infty \to 0 \text{ as } n \to \infty \ .$$

Thus N_1 is continuous.

Step 2: N maps B_q into itself.

For each $u \in B_q$, by (H_6), (H_7) and (H_{10}), we have for each $t \in [0, T]$

$$\begin{aligned}
\|N(u)(t)\| &\leq \|S(t)[\varphi(0) - g(0, u(-\eta(0)))]\| + \|g(t, u(-\eta(0)))\| \\
&\quad + \|\int_0^t S(t-s)f(s, f(s, u_s, u(s - \tau(s, u_s)))ds\| \\
&\leq M\|\varphi(0)\| + (M+1)(c_1 q + c_2) + M[\psi(q)\int_0^t p(s)ds + c_3 q\int_0^t ds] \\
&\leq M\|\varphi\|_\infty + (M+1)[c_1 q + c_2] + M\psi(q)\|p\|_{L^1} + MTc_3 q.
\end{aligned}$$

Thus

$$\|N(u)\|_\infty \ \leq \ M\|\varphi\|_\infty + (M+1)[c_1 q + c_2] + M[\psi(q)\|p\|_{L^1} + Tc_3 q] \leq q.$$

Step 3: $N(B_q)$ is bounded and equicontinuous.

By Step 2, it is obvious that $N(B_q) \subset B_q$ is bounded. Using (H_6), it suffices to show that the operator N_1 defined in (3.1.3) is equicontinuous.

Let $0 < \tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and B_q be a bounded set of Ω as in Step 2. Let $u \in B_q$ then for each $t \in J$ we have

$$\begin{aligned}
\|N_1(u)(\tau_2) - N_1(u)(\tau_1)\| &\leq \|S(\tau_2)\varphi(0) - S(\tau_1)\varphi(0)\| \\
&\quad + \int_0^{\tau_1-\epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|[p(s)\psi(q) + c_3 q]ds \\
&\quad + \int_{\tau_1}^{\tau_1-\epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|[p(s)\psi(q) + c_3 q]ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|[p(s)\psi(q) + c_3 q]ds.
\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small, since $S(t)$ is a strongly continuous operator and the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

Now let V be a subset of B_q such that $V \subset \overline{conv}(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on J. By

(H_8), (H_9), and the properties of the measure α we have for each $t \in J$,

$$\begin{aligned}
v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\
&\leq \alpha(N(V)(t)) \\
&\leq c_1\big[M\alpha(V(-\eta(0))) + \alpha(V(t-\eta(t)))\big] \\
&\quad + M\int_0^t \big[p(s)\alpha(V_s) + c_3\alpha(V(s-\tau(s,V_s)))\big]ds \\
&\leq c_1\big[Mv(-\eta(0)) + v(t-\eta(t))\big] + M\int_0^t \big[p(s)v_s + c_3 v(s-\tau(s,V_s))\big]ds \\
&\leq c_1(M+1)\|v\|_\infty + M\big[\|p\|_{L^1}\|v\|_\infty + c_3T\|v\|_\infty\big] \\
&\leq \big[c_1 + M(c_1 + \|p\|_{L^1} + c_3T\big]\|v\|_\infty.
\end{aligned}$$

Then

$$\|v\|_\infty\big(1 - \big[c_1 + M\big(c_1 + \|p\|_{L^1} + c_3T\big)\big]\big) \leq 0.$$

Since $\big[c_1 + M\big(c_1 + \|p\|_{L^1} + c_3T\big)\big] < 1$ it follows that $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_q. As a consequence of the Mönch fixed theorem 1.3.4 we deduce that N has a fixed point which is a mild solution of problem (3.1.1)-(3.1.2). □

□

For the next theorem we replace the condition (3.2.1) by

$$c_1(M+1) < 1. \tag{3.2.4}$$

Now, consider the Kuratowski measure of noncompactness α_C defined on the family of bounded subsets of the space $C(J,E)$ by

$$\alpha_C(H) = \sup_{\theta\in[-r,0]}\sup_{t\in J} e^{-\tau L(t)}\alpha(H(t+\theta)),$$

where $L(t) = \displaystyle\int_0^t \tilde{l}(s)ds$, $\quad \tilde{l}(t) = M(p(t)+c_3)$, $\quad \tau > \frac{1}{1-c_1(M+1)}$.

Our next result is based on the Darbo fixed point theorem 1.3.3.

Theorem 3.2.2. *Assume that $(H_1), (H_5), (H_6), (H_8), (H_9)$ and (3.2.4) are satisfied. Then the problem (3.1.1)-(3.1.2) has at least one mild solution on $[-r,T]$.*

Proof. As in Theorem 3.2.1, we can prove that the operator $N : B_q \to B_q$ defined in that theorem is continuous and $N(B_q)$ is bounded.
Now, we show that the operator $N : B_q \to B_q$ is a strict set contraction, i.e., there is a constant $0 \leq \lambda < 1$ such that $\alpha(N(H)) \leq \lambda\alpha(H)$ for any $H \subset B_q$. In particular, we

need to prove that there exists a constant $0 \leq \lambda < 1$ such that $\alpha_C(N(H)) \leq \lambda \alpha_C(H)$ for any $H \subset B_q$. For each $t \in J$ we have

$$\begin{aligned}
\alpha((N(H)(t)) &\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] \\
&\quad + M\int_0^t [p(s)\alpha(H_s) + c_3\alpha(H(s-\tau(s,H_s)))]ds \\
&\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] \\
&\quad + M\int_0^t e^{\tau L(s)}e^{-\tau L(s)}[p(s)\alpha(H_s) + c_3\alpha(H(s-\tau(s,H_s)))]ds \\
&\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] \\
&\quad + M\sup_{\theta\in[-r,0]}\sup_{s\in J} e^{-\tau L(s)}\alpha(H(s+\theta))\int_0^t e^{\tau L(s)}[p(s)+c_3]ds \\
&\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] + \alpha_C(H)\int_0^t \tilde{l}(s)e^{\tau L(s)}ds \\
&\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] + \alpha_C(H)\int_0^t \Big(\frac{e^{\tau L(s)}}{\tau}\Big)'ds \\
&\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] + \alpha_C(H)\frac{1}{\tau}e^{\tau L(t)}.
\end{aligned}$$

Then

$$\begin{aligned}
e^{-\tau L(t)}\alpha((N(H)(t)) &\leq c_1 e^{-\tau L(t)}[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] + \alpha_C(H)\frac{1}{\tau} \\
&\leq c_1[M+1]\sup_{\theta\in[-r,0]}\sup_{s\in J} e^{-\tau L(s)}\alpha(H(s+\theta)) + \alpha_C(H)\frac{1}{\tau}.
\end{aligned}$$

Consequently,

$$\alpha_C(NH) \leq \big[c_1(M+1) + \frac{1}{\tau}\big]\alpha_C(H).$$

So, the operator N is a set contraction. By the Darbo fixed point theorem 1.3.3 we deduce that N has a fixed point which is a mild solution of problem (3.1.1)-(3.1.2). $\square$ $\square$

Chapter 4

Stability of Differential Equations with State-Dependent Delay

1. Let $\mathscr{R}$ and $\mathscr{R}^+$ represent the sets of real numbers and nonnegative real numbers, respectively. $\mathscr{R}^n$ denotes the n-dimension Euclidean space. For a matrix E, $\lambda_{max}(E)$ is used to denote its maximum eigenvalue. $\mathscr{P}(E)$ stands for the minimum value of all elements of matrix E. The vector 1-norm and 2-norm are severally expressed by $\|\cdot\|_1$ and $\|\cdot\|_2$.

2. Based on the work of Li and Yang, we put forward the following neural network model with SDSD, which is described by

$$\begin{aligned} \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) \\ &+ \sum_{j=1}^{n} d_{ij} f_j(x_j(t - \tau(t, \mathscr{X}))), i = 1, 2, \ldots, n, t \geq t_0, \end{aligned} \tag{4.0.1}$$

   for the sake of presentation; we also give the compact form of system (4.0.1) as follows:

$$\dot{\mathscr{X}}(t) = -A\mathscr{X}(t) + Bg(\mathscr{X}(t)) + Df(\mathscr{X}(t - \tau(t, \mathscr{X}))), \tag{4.0.2}$$

where n stands for the number of neurons in the network, $\dot{\mathscr{X}}(t)$ denotes the upper right derivative of $\mathscr{X}(t), \mathscr{X} = \mathscr{X}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, and $x_i(t)$ represents the state of the ith neuron. A is a diagonal matrix, for $i = 1, 2, \ldots, n, a_i > 0$ and B and D are constant matrices with corresponding dimensions.
$g(\mathscr{X}(t)) = (g_1(x_1)(t), g_2(x_2(t)), \ldots, g_n(x_n(t)))^T$ and $f(\mathscr{X}(t - \tau(t, \mathscr{X}))) = (f_1(x_1(t - \tau(t, \mathscr{X}))),$
$f_2(x_2(t - \tau(t, \mathscr{X}))), \ldots, f_n(x_n(t - \tau(t, \mathscr{X}))))^T$ are the excitation functions of the ith neuron at time $t and t - \tau(t, \mathscr{X})$, respectively.
Furthermore, we use $\mathscr{X}(s) = \Psi(s), s \in [t_0 - \eta, t_0]$ to denote the initial value of system

(4.0.2), where $\Psi = \Psi(s) = (\Psi_1(s), \Psi_2(s), \ldots, \Psi_n(s))^T \in \mathscr{C}([t_0-\eta, t_0], \mathscr{R}^n)$.
$\mathscr{C}([t_0-\eta, t_0], \mathscr{R}^n)$ is a Banach space whose elements are continuous vector-valued functions. These continuous functions map the interval $[t_0-\eta, t_0]$ into $\mathscr{R}^n$. Let $\|\Psi\|_\alpha = \sup_{t_0-\eta\le s\le t_0} \| \Psi(s)\|$ stand for the norm of a function $\Psi(.) \in \mathscr{C}([t_0-\eta, t_0], \mathscr{R}^n)$, where $\|.\|$ is the vector norm matching with the content of the paper.

Remark 4.0.1. *$\mathscr{X}(t)$ is right-upper derivable, which implies that the solution of system (4.0.2) can be continuous but not smooth. The state delay $\tau(t, \mathscr{X})$ is related to the state of each neuron. For subsequent analysis, we need the following assumptions for system (4.0.1) and (4.0.2).*

assumption 4.0.1. *Functions $g(.), f(.) \in \mathscr{R}^n$ satisfy $f(0) = 0, g(0) = 0$. Through Assumption (4.0.1), this ensures that $\mathscr{X} = 0$ is a constant solution of systems (4.0.1) and (4.0.2).*

assumption 4.0.2. *$g(.), f(.) \in \mathscr{R}^n$ are locally Lipschitz continuous; in other words, $\forall \beta_1, \beta_2 \in \mathscr{R}$, and we have*

$$\begin{aligned} |g_i(\beta_1) - g_i(\beta_2)| &\le \ell_i|\beta_1-\beta_2|, \forall i \in \{1,2,\ldots,n\}, \\ |f_i(\beta_1) - f_i(\beta_2)| &\le \overline{\ell}_i|\beta_1-\beta_2|, \forall i \in \{1,2,\ldots,n\}, \end{aligned} \tag{4.0.3}$$

where $\ell_i > 0$ and $\overline{\ell}_i > 0$.

According to assumption (4.0.2), we can get two constant sets
$\{\ell_1, \ell_2, \ldots, \ell_n\}$ and $\{\overline{\ell}_1, \overline{\ell}_2, \ldots, \overline{\ell}_n\}$.
Let $\{\ell_1, \ell_2, \ldots, \ell_n\}$ and $l_f = \max\{\overline{\ell}_1, \overline{\ell}_2, \ldots, \overline{\ell}_n\}$.

assumption 4.0.3. *The state delay $\tau(t, \mathscr{X})_i n\mathscr{C}(\mathscr{R}^+ \times \mathscr{R}^n, [0, \eta])$ is locally Lipschitz continuous, namely, for any $\Gamma_1, \Gamma_2 \in \mathscr{R}^n$, there always exists a constant $\ell_\tau > 0$ such that*

$$|\tau(t, \Gamma_1) - \tau(t, \Gamma_2)| \le \ell_\tau \|\Gamma_1 - \Gamma_2\|. \tag{4.0.4}$$

assumption 4.0.4. *When $\mathscr{X} = 0, \tau(t, \mathscr{X})$ has supremum equipped with $\tau_0(\le \eta)$,* $\sup\{\tau(t, 0), t \ge t_0\} = \tau_0$.
For ease of expression, let

$$\begin{aligned} \pi_1 &= \max_{1\le i\le n}\left(-a_i + \sum_{j=1}^{n} \mid b_{ji} \mid \ell_i\right), \\ \pi_2 &= \max_{1\le i\le n} \sum_{j=1}^{n} \mid b_{ji} \mid \overline{\ell}_i. \end{aligned} \tag{4.0.5}$$

Definition 4.0.1. *(see [58]) The zero solution of system (4.0.2) is said to be locally exponentially stable (LES) in region $\mathscr{M}$; if there exist constants $\gamma > 0$ and Lyapunov exponent $\zeta > 0$, for any $t \ge t_0$, we have*

$$\| \mathscr{X}(t; t_0, \Psi) \| \le \gamma \| \Psi \|_\alpha e^{-\zeta(t-t_0)}, \tag{4.0.6}$$

where $\mathscr{X}(t;t_0,\Psi)$ is a solution of system (4.0.2) with the initial condition $\Psi \in \mathscr{C}([t_0-\eta,t_0],\mathscr{M})$, $\mathscr{M} \subset \mathscr{R}^n$, and $\mathscr{M}$ called a local exponential attraction set of the zero solution.

Lemma 4.0.1. *Let $\Gamma_1, \Gamma_2 \in \mathscr{R}^n$ and we have*

$$\Gamma_1^T\Gamma_2 + \Gamma_2^T\Gamma_1 \leq \varpi\Gamma_1^T\Gamma_1 + \varpi^{-1}\Gamma_2^T\Gamma_2, \tag{4.0.7}$$

for any $\varpi > 0$.

4.1 Main Result

Theorem 4.1.1. *Under Assumptions 4.0.1-4.0.4, the zero equilibrium of system (4.0.2) is LES if*

$$\pi_1 + \pi_2 < 0 \tag{4.1.1}$$

and Lyapunov exponent $\zeta > 0$ satisfies

$$\zeta + \pi_1 + \pi_2 e^{\zeta(\ell_\tau\|\Psi\|_\alpha+\tau_0} \leq 0, \tag{4.1.2}$$

Proof. . We assume that $\mathscr{X}(t;t_0,\Psi)$ is a trajectory of system (4.0.2) with initial value (t_0,Ψ), where $\Psi \in \mathscr{C}([t_0-\eta,t_0],\mathscr{R}^n$ and $\Psi \neq 0$. For the sake of convenience, let $V(t) = V(t,\mathscr{X}) = \parallel \mathscr{X}(t) \parallel_1 = \sum_{i=1}^n \mid x_i(t) \mid$ and $V_0 = \{\sup V(s), s \in [t_0-\eta,t_0]\}$. Then, for any $\epsilon \in (0,\zeta)$, we claim that

$$e^{(\zeta-\epsilon)(t-t_0)}V(t) \leq V_0, \forall t \geq t_0. \tag{4.1.3}$$

Firstly, when $t = t_0$, (4.1.3) holds. Next, we prove that (4.1.3) holds on $(t_0,+\infty)$. In contrast to (4.1.3), there are some instants on $(t_0,+\infty)$ to make (4.1.3) untenable, and then we can find an instant $t_q \geq t_0$; the following three events will happen:

1. $e^{(\zeta-\epsilon)(t-t_0)}V(t_q) = V_0$.

2. $e^{(\zeta-\epsilon)(t-t_0)}V(t) \leq V_0$, for $\forall t \in (t_0-\eta,t_q]$.

3. There exists a right neighbor of $t_q(U_+^0(t_q,\xi))$ such that $\forall t^\xi \in U_+^0(t_q,\xi)$ and $e^{(\zeta-\epsilon)(t-t_0)}V(t^\xi) > V_0$.

On the contrary, by Assumptions 4.0.1-4.0.4 and combining (4.0.2), the derivative of

$e^{(\zeta-\epsilon)(t-t_0)}V(t)$ at time t_q is as follows:

$$\begin{array}{rl} & \frac{d}{dt}(e^{(\zeta-\epsilon)(t-t_0)}V(t))|_{t=t_q} \\ = & (\zeta-\epsilon)e^{(\zeta-\epsilon)(t_q-t_0)}V(t_q)+[e^{(\zeta-\epsilon)(t-t_0)}\dot{V}(t)]|_{t=t_q} \\ \leq & (\zeta-\epsilon)V_0+e^{(\zeta-\epsilon)(t_q-t_0)}(\sum_{i=1}^{n} sgn(x_i(t_q))(\dot{x}_i(t_q))) \\ = & (\zeta-\epsilon)V_0+e^{(\zeta-\epsilon)(t_q-t_0)}(\sum_{i=1}^{n} sgn(x_i(t_q))[-a_ix_i(t_q) \\ + & \sum_{j=1}^{n} b_{ij}g_j(x_j(t_q))+\sum_{j=1}^{n} d_{ij}f_j(x_j(t_q-\tau(t_q,\mathscr{X}(t_q))))]) \\ \leq & (\zeta-\epsilon)V_0+e^{(\zeta-\epsilon)(t_q-t_0)}(\sum_{i=1}^{n} -a_i|x_i(t_q)| \\ + & e^{(\zeta-\epsilon)(t_q-t_0)}\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij})|\ell_i|x_j(t_q)| \\ + & e^{(\zeta-\epsilon)(t_q-t_0)}(\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij})|\ell_i|x_j(t_q-\tau(t_q,\mathscr{X}(t_q)))| \\ & \leq (\zeta-\epsilon)V_0 \\ + & e^{(\zeta-\epsilon)(t_q-t_0)}\max_{1\leq i\leq n}(-a_i+\sum_{j=1}^{n}|b_{ji}|\ell_i)\|\mathscr{X}(t_q)\| \\ + & e^{(\zeta-\epsilon)(t_q-t_0)}\max_{1\leq i\leq n}(\sum_{j=1}^{n}|d_{ji}|\bar{\ell}_i)\parallel\mathscr{X}(t_q-\tau(t_q,\mathscr{X}))\parallel \\ = & (\zeta-\epsilon+\pi_1)V_0+e^{(\zeta-\epsilon)(t_q-\tau(t_q,\mathscr{X})-t_0)}\parallel\mathscr{X}t_q-\tau(t_q,\mathscr{X})\parallel \\ & \times\pi_2e^{(\zeta-\epsilon)\tau(t_q,\mathscr{X})} \\ & \leq(\zeta-\epsilon+\pi_1+\pi_2e^{(\zeta-\epsilon)\tau(t_q,\mathscr{X})})V_0 \\ = & (\zeta-\epsilon+\pi_1+\pi_2e^{(\zeta-\epsilon)[\tau(t_q,\mathscr{X})-\tau(t_q,0)]}e^{(\zeta-\epsilon)\tau(t_q,0)})V_0 \\ & \leq\left(\zeta-\epsilon+\pi_1+\pi_2e^{(\zeta-\epsilon)\ell_\tau\|\mathscr{X}(t_q)}\ \|_1\right)e^{(\zeta-\epsilon)\tau(t_q,0)}V_0 \\ & \leq(\zeta-\epsilon+\pi_1+\pi_2e^{(\zeta-\epsilon)(\ell_\tau\|\mathscr{X}(t_q)\|_1+\tau_0)})V_0. \end{array} \quad (4.1.4)$$

Together with the definition of $V_0, V(t), t_q$ and condition (1), we have

$$\parallel\mathscr{X}(t_q)\parallel_1=V(t_q)\leq V_0=\parallel\Psi\parallel_\alpha, \quad (4.1.5)$$

and then from (4.1.2) and (4.1.4), we obtain

$$\begin{array}{l} \frac{d}{dt}(e^{(\zeta-\epsilon)(t-t_0)}V(t))|_{t=t_q} \\ \leq(\zeta-\epsilon+\pi_1+\pi_2e^{(\zeta-\epsilon)(\ell_\tau\|\Psi\|_\alpha+\tau_0)})\parallel\Psi\parallel<0, \end{array} \quad (4.1.6)$$

which is a contradiction with condition (9), and thus, (10) holds. Consider the arbitrariness of ϵ, let $\epsilon\longrightarrow 0$, and then we obtain

$$e^{\zeta(t-t_0)}V(t)\leq V_0, \forall t\geq t_0, \quad (4.1.7)$$

i.e.,

$$\| \mathscr{X}(t) \|_1 = V(t) \leq \| \Psi \|_\alpha e^{-\zeta(t-t_0)}, \forall t \geq t_0, \tag{4.1.8}$$

The reasoning process of , theorem 4.1.1 is completed. $\square$

Example 4.1.1. *Consider a 2-dimensional neural network with SDSD, which is described by*

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} &= - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\
&+ \begin{pmatrix} 0.25 & 0.25 \\ 0.02 & 0.01 \end{pmatrix} \begin{pmatrix} g_1(x_1(t)) \\ g_2(x_2(t)) \end{pmatrix} \\
&+ \begin{pmatrix} x_1(t) \\ g_1(x_1(t)) \end{pmatrix} \begin{pmatrix} f_1(x_1(t-\tau(t,\mathscr{X}))) \\ f_2(x_2(t-\tau(t,\mathscr{X}))) \end{pmatrix}
\end{aligned} \tag{4.1.9}$$

where $t_0 = 0$ and

$$\begin{aligned}
&g_i(\cdot) = |x_i(t)+1| + |x_i - 1|, \qquad i = 1,2, \\
&f_i(\cdot) = \sin(x_i(t - |\sin(x_1(t)+x_2(t))|)), \qquad i = 1,2, \\
&\tau(t,\mathscr{X}) = |\sin(x_1(t)+x_2(t))|.
\end{aligned} \tag{4.1.10}$$

Evidently, $\ell_1 = \ell_2 = 2, \dot{\ell}_1 = \dot{\ell}_2 = 1, \ell_\tau = 1, \tau(t,0) = 0, \tau(t,\mathscr{X}) \in [0,1]$. By calculating,

$$\begin{aligned}
\pi_1 &= \max_{1\leq i\leq n} \left\{ (-a_i + \sum_{j=1}^{n} | b_{ji} | \ell_i) \right\} = -0.46, \\
\pi_2 &= \max_{1\leq i\leq n} \sum_{j=1}^{n} | b_{ji} | \overline{\ell}_i = 0.31.
\end{aligned} \tag{4.1.11}$$

Then, from theorem 4.1.1, system (4.1.8) is LES. The trajectories of the solution from a random initial value are shown in Figure 1. As shown in Figure 1, $x_1(t)$ and $x_2(t)$ in neural network model (4.1.8) are convergent. Figure 2 shows the phase diagram of system (4.1.8) evolving with time.

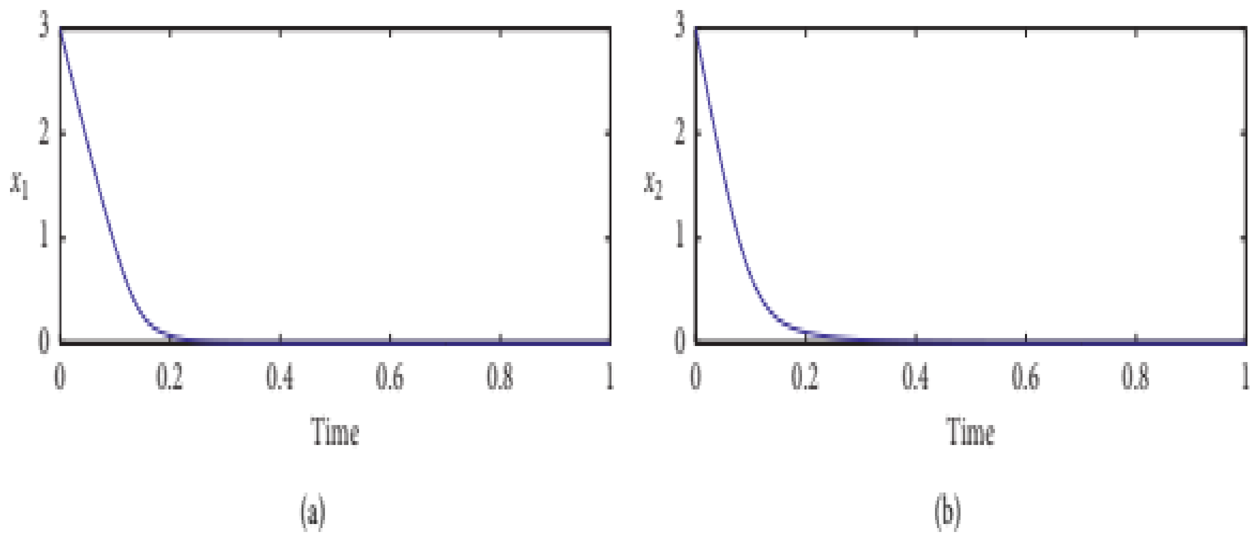

Figure 4.1: Transient behavior of (a) $x_1(t)$ and (b) $x_2(t)$ in system (4.1.9).

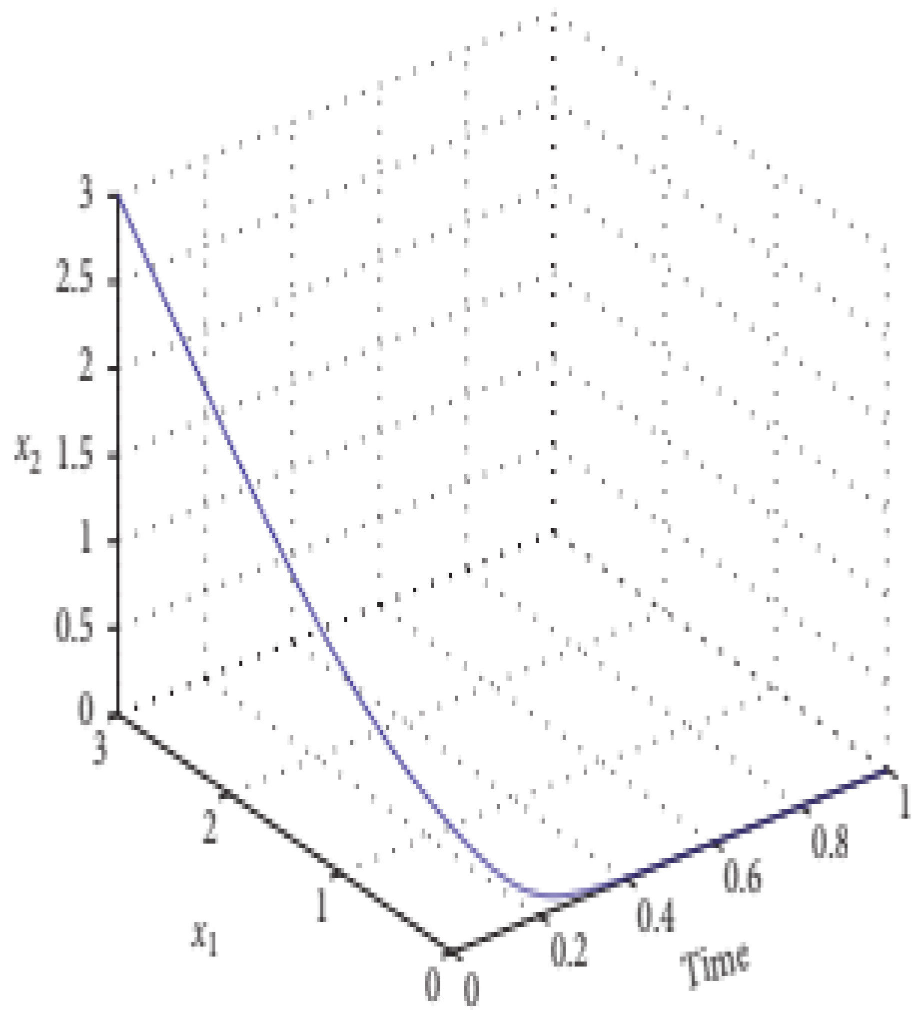

Figure 4.2: Transient behavior of $(x_1(t), x_2(t))$ in system (4.1.9).

Chapter 5

Model and Application

A remark in [57] says that state-dependent delays arise in various circumstances, but it seems not obvious how to single out a tractable class of equations which contains a large set of examples which are well motivated. The difficulty of singling out a tractable class of equations to include many interesting models may prove to be an extremely valuable source to stimulate new mathematical techniques and theories. In this section we describe differential equations with state-dependent delay that arise from electrodynamics, automatic and remote control, machine cutting, neural networks, population biology, mathematical epidemiology and economics.

5.1 A Two Body Problem of Classical Electrodynamics

In Driver [21] (see also [17, 23]), a mathematical model for a two-body problem of classical electrodynamics incorporating retarded interaction is proposed and analyzed. He considers the motion for two charged particles moving along the x-axis and substituted the expressions for the field of a moving charge, calculated from the Linéard-Wiechert potential, into the Lorentz-Abraham force law. Radiation reaction is omitted, but time delays are incorporated due to the finite speed of propagation, c, of electrical effects. As a result, the model is a system of delay differential equations involving time delays, which depend on the unknown trajectories. From this model and after some analysis, he obtains a system of six delay-differential equations for the evolution of the states, the velocities and the time delays. To describe his model, we denote by $x_i(t)(i = 1, 2)$ the positions of the two point charges on the axis in a given inertial system at time t , the time of an observer in that system. Let $v_i(t) = x_i'(t)(i = 1, 2)$ be the velocities of the charges. As mentioned above, we omit radiation reaction but allow an external electric field, $E_{ext}(t, x)$, in the $x - direction$, that is assumed to be continuous over some open set D in the $(t, x) - plane$. Then the equation of motion of charge i is

$$\frac{m_i v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} = q_i E_j(t, x_i(t)) + q_i E_{ext}(t, x_i(t)), \quad i, j \in \{1, 2\}, j \neq i, \tag{5.1.1}$$

where m_i is the rest mass and q_i is the magnitude of charge i, c is the speed of light, and $E_j(t, x)$ is the electric field at (t, x) due to other charge $j \neq i$. The magnetic field of charge j is not involved in this one-dimensional case. The field at time t and at the point $x_i(t)$ produced by charge j is assumed to be that computed from the Linéard-Wiechert potentials. The expression for this field involves a time lag, $t - \tau_{ji}$, representing the instant at which a light signal would have to leave charge j in order to arrive at $x_i(t)$ at the instant t . Therefore, the delay $\tau_{ji}(t)$ must be a solution of the functional equation

$$\tau_{ji}(t) = \mid x_i(t) - x_j(t - \tau_{ji}(t)) \mid /c. \tag{5.1.2}$$

Clearly, $\tau_{ji}(t)$ cannot be written explicitly. Because of the occurrence of time delays in the model equation (5.1.1), one needs to specify initial trajectories of the two charges over some appropriate interval $[\alpha, t_0]$. Consider now those initial trajectories and their extensions $(x_1(t), x_2(t))$ defined on some interval $[\alpha, \beta)$, where $\beta > t_0$, such that

(a) each $_i(t)$ is continuous and $\mid x_i'(t) \mid < c$ for all $t \in [\alpha, \beta)$;

(b) $x_2(t) > x_1(t) \text{and} and (t, x_i(t)) \in D \text{for all} t \in [t_0, \beta)$;

(c) the two functional equations $\tau_{ji}^0 = \mid x_i(t_0) - x_j(t_0 - \tau_{ji}^0) \mid /c$ have solutions $\tau_{ji}^0, i \neq j, i, j \in \{1, 2\}$.

Then Driver proves that $(x_1(t), x_2(t))$ is a solution of (5.1.1).(5.1.2) if and only if it satisfies the following system of six delay differential equations for $t \in (t_0, \beta)$:

$$\begin{cases} x_i'(t) = v_i(t), \\ \\ \tau_{ji}'(t) = \frac{(-1)^i v_i(t) - (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))}, \\ \\ \frac{v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} = \frac{(-1)^i a_i c}{\tau_{ji}^2(t)} \cdot \frac{c + (-1)^i v_j(t - \tau_{ji}(t))}{c} + q_i E_{ext}(t, x_i(t))/m_i, \end{cases} \tag{5.1.3}$$

where $\tau_{ji}(t_0) = \tau_{ji}^0, a_i = q_1 q_2/(4\pi\epsilon_0 m_i c^3)$ (a constant, and in particular, ϵ_0 is the dielectric constant of free space), and $(i, j) = (1, 2) or (2, 1)$. It is shown in Driver [21] that if given initial trajectories satisfy condition (a) for $\alpha \leqslant t \leqslant t_0$, condition (b) at t_0, and condition (c), and if $E_{ext}(t, x)$ is Lipschitz continuous with respect to x in each compact subset of D and if the initial velocity of each particle is Lipschitz continuous, then a unique solution does exist. This solution can be continued as long as the charges do not collide ($\lim x_1(t) = \lim x_2(t)$ as t approaches the right endpoint of the maximal interval for existence) and neither $(t, x_1(t)) nor (t, x_2(t))$ approaches the boundary$D$.

We remark here that in Driver and Norris [22], the above Lipschitz continuity for the initial velocities is relaxed to the integrability of the initial velocity on $[\alpha, t_0]$. In Driver [18], one special case was given where the positions and velocities of the particles at some instant will determine the state of the system. More precisely, in this example of electrodynamic equations of motion, instantaneous values of positions and velocities of the particles will determine their trajectories, if the solutions are defined for all future time. This property was frequently conjectured, asserted, or implicitly assumed, as in Newtonian mechanics and as indicated by the long list of related references in Driver [18], but this property should not be expected for general electrodynamic equations. In the case where $E_{ext}(t,x) = 0$for all$(t,x) \in \mathbb{R}^2$and if$q_1q_2 > 0$ (two point charges of like sign), then $\lim_{t\to\infty}[x_2(t) \cdot x_1(t)] = \infty$and $\mid v_i(t) \mid \leqslant \bar{c} < c$for all$t \geqslant \alpha$. This is a quite interesting result as it indicates that the delay $\tau_{ji}(t)$ may become unbounded, as such, one obtains a system of functional differential equation with unbounded state-dependent delays. It is noted that if three-dimensional motions are considered, then one obtains a functional differential system of neutral type where the delays are dependent on the states, and the change rate of vi at the current time also depends on its historical value $v_j(t - \tau_{ji})$. More precisely, if we introduce a unit vector

$$u_i = \frac{x_i - x_j(t - \tau_{ji})}{c\tau_{ji}}$$

and a scalar quantity

$$\gamma_{ji} = 1 - \frac{1}{c} v_j(t - \tau_{ji}) \cdot u_i$$

as Driver [20] does, where indicates the dot or scalar product in $\mathbb{R}_3$ (note, of course, x_1, x_2 are now vectors in $\mathbb{R}^3$), then the Lorentz force law yields

$$v_i'(t) = \frac{q_i(1- \mid v_i \mid^2 /c^2)^{1/2}}{m_i}[E_j + (v_i/c \cdot E_j)(u_i - v_i/c) - (v_i/c \cdot u_i)E_j], \tag{5.1.4}$$

where E_j is the retarded (vector-valued) electric field arriving at x_i at the instant t from particle j . This field, in $\mathbb{R}^3$, can be found from the Liénard-Weichert potentials as

$$\begin{aligned} E_j = & \tfrac{kcq_j}{\tau_{ji}^2\gamma_{ij}^3}[u_i - v_i(t - \tau_{ji})/c][1- \mid v_j \mid^2 (t - \tau_{ji})] \\ & + \tfrac{kq_j}{\tau_{ji}\gamma_{ij}^3} u_i \times ([u_i - v_j(t - \tau_{ji})/c] \times v_j'(t - \tau_{ji})), \end{aligned} \tag{5.1.5}$$

where $k > 0$ is a constant depending on the units, and $\times$ indicates the vector cross product in $\mathbb{R}^3$. The dynamical adaptation for τ_{ji} is given by

$$\tau_{ji}'(t) = \frac{u_i \cdot [v_i - v_j(t - \tau_{ji})]}{c\gamma_{ij}}. \tag{5.1.6}$$

In the above discussions, the motion of each particle is influenced by the electromagnetic fields of the others, and due to the finite speed of the propagation of these fields, the model equations describing the motion of charged particles via action at a distance will involve time delays which depends on the state of the whole system. In Driver [20] and in Hoag and Driver [59], it is noted that if one considers that the basic laws of physics are symmetric with respect to time reversal, then the existence of these delays implies that there should also be advanced terms in the equations, and thus one is led to a system of functional differential equations with mixed arguments (Hoag and Driver [59]), and of neutral type (Driver [19]). In summary and in conclusion, despite the fact that much of the work by Driver and his collaborators on electrodynamics was published nearly 40 years ago, many interesting questions related to the fundamental issues of electrodynamics remain unsolved mathematically and Driver?s models remain as a source of inspiration for the theoretical development and a testing tool for new results.

Conclusion

The main goals of this thesis is to establish stability, existence, uniqueness and model results for various classes of functional differential equations, with delay which may be finite or state-dependent in Banach space.
In chapter 2, we prove the existence of mild solutions of nonlinear neutral time varying multiple delay differential equations in Banach space. The purpose of this chapter is to prove the existence of mild solutions for the same class of neutral equations with mild solutions by applying Schaefer's theorem instead of Sadovskii's theorem.
In **Chapter** 3, we study and prove the existence and uniqueness of solutions for neutral differential equations with state-dependent delays. with the proof based on Banach's fixed point theorem, Their proofs involve the measure of noncompactness paired in one result with a Mönch fixed point theorem and paired in the other result with a Darbo fixed point theorem.
In **Chapter** 4, we solve the stability problem of neural networks equipped with state-dependent state delays, and example for that.
In **Chapter** 5, we introduce applications and models of neutral differential equations with state-dependent delays, a mathematical model for a two-body problem of classical electrodynamics incorporating retarded interaction is proposed and analyzed.

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