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Thème :

Backward Stochastic Differential Equations With Jumps and Applications.

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Dedication

To my dear parents, for all their sacrifices, their love, their tenderness, their support and their prayers throughout my studies.

To my dear sisters Fatima and Khadidja for their constant encouragement and moral support. To my dear little brother Abdellah .

Thank you for always being there for me.

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Contents

ملخص:

الهدف من هذه الأطروحة هو اقتراح نهج جديد للتكامل العشوائي لفئة العمليات العشوائية المستقلة على الفور فيما يتعلق بالحركة البراونية الكسرية على فترة زمنية محدودة. نقطة التقييم هي اكتشاف النظير لنظرية إيتو.

بتعبير أدق ، نعرض بعض النتائج على التكامل العشوائي فيما يتعلق بعدم وجود عمليات تكييف من خلال تعميم النتائج التي حصل عليها عياد و كيوو [4] في الإطار البراوني.

وبالمثل ، نشتق مبادئ قصوى كافية وضرورية لمشكلة التحكم الأمثل العشوائية حيث يتم إعطاء حالة النظام من خلال معادلة تفاضلية عشوائية محكومة مدفوعة بحركة براونية وقفزة واحدة مارتينجال. نطبق الحد الأقصى من المبادئ لحل مشكلة تعظيم أداة السجل مع الوضع الافتراضي .

الكلمات المفتاحية:

الحركة البراونية الكسرية ، التكامل العشوائي ، القياس الغاوسي

مبدأ الحد الأقصى العشوائي ، المعادلات التفاضلية العشوائية المتخلفة مع الافتراضي

Abstract

The objective of this thesis is to propose a new approach to stochastic integration of the class of instantly independent stochastic processes with respect to fractional brownian motion on a finite interval. The appraisal point is to discover the counterpart of the Itô theory. More precisely, we show some results on stochastic integration with respect to no adapted processes by generalizing the results obtained by Ayed and Kuo [4] in the Brownian framework.

Similarly, we derive sufficient and necessary maximum principles for a stochastic optimal control problem where the system state is given by a controlled stochastic differential equation driven by a Brownian motion and a single jump martingale. We apply the maximum principles to solve a log-utility maximization problem with default.

Key words:

Fractional Brownian motion, stochastic integration, gaussian measure, stochastic maximum principle, backward stochastic differential equations with default.

Résumé

L'objectif de cette thèse est de proposer une nouvelle approche d'une intégration stochastique de classe de processus qui sont instantanément indépendants par rapport au mouvement Brownien fractionnaire sur un interval fini. Le point important est de trouver la contrepartie de la théorie d'Itô. Plus précisément, nous montrons des résultats sur l'intégration stochastique par rapport à des processus non adaptés en généralisant les résultats obtenus par Ayed et Kuo [4] dans le cadre Brownien.

De même, on dérive les principes de maximum nécessaires et suffisants pour un control optimal stochastique où l'état du système est donné par une équation différentielle stochastique contrôllée, dirigée par le mouvement Brownien et une martingale à saut. On applique les principes des maximums pour résoudre le problème de maximisation par défaut.

Mots clés:
Mouvement brownien fractionnaire, intégration stochastique, mesure gaussienne, principe de maximum stochastique, équation différentielle stochastique rétrograde avec saut.

List of works

List of research works

- BACHIR CHERIF, K. and KANDOUCI, A. (2021). A New approach to stochastic integration w.r.t fractional brownian motion for no adapted processes, Bulletin of the Institute of Mathematics Academia Sinica New Series , **16**, 321-337.

Presentation

- BACHIR CHERIF, K. AGRAM, N. (November 21-24, 2019). Mean-field reflected delayed backward stochastic differential equations with jumps. Poster presentation at the Eleventh Meeting of Mathematical Analysis and Applications (RAMA11) organized by Djillali Liabes University , Sidi Bel Abbès , Algeria.

Other

1. Participation in the Marrakesh International Conference on Probability and Statistics (MICPS'2016) organized by Cadi Ayyad University, Marrakesh, Morocco, form 25 to 28 april 2016.

2. Participation in the International Workshop on Perspectives on High-dimensional Data Analysis (HDDA-VIII-2018) organized by Cadi Ayyad University, Marrakesh, Morocco, form 09 to 13 April 2018.

3. Participation in the CIMPA Research School (CIMPA' ASA-2019) on Stochastic Analysis and Applications, organized by Dr Moulay Tahar University of Saida, Algeria, from 01 to 09 March 2019.

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Introduction

In the theory of stochastic integration, the mathematician K. Itô [35] introduced the Itô stochastic integral in order to obtain a method to construct diffusion processes as solutions to stochastic differential equations. The main problem of the integration with respect to Brownian motion, is the fact that the Riemann-Stieltjes integration fails. This need inspired Itô to construct a theory of stochastic integration [21].

The Itô stochastic integral has a rage of applications. The most famous are those related to financial modeling, as the Black-Scholes-Merton model. A continuous-time model, which aim is to describe the behavior of stock price. Specifically, it faces the problem of pricing European options. However, this integration theory requires that the stochastic process must be adapted [21].

Fractional Brownian motion (fBm) is a similar to a continuous fractal walk. Nonetheless, fBm has dependant increments contrary to regular brownian motion, which means that the current step of fBm is independent on previous step, The dependance of this process is measured on a scale from zero to one and this measure is appointed(called) the Hurst index $H \in (0,1)$ relative to the hydrologist "Harold Edwin Hurst" for this work in the field of hydrology. Many researchers in recent years were attracted by the stochastic calculus with respect to fBm, this later has been motivated by applications in finance and interest traffic modeling.

In most real-world systems, everyone agrees that uncertainty is inherent. It places many inconvenience (and sometimes surprisingly advantages) on humankind's effort, which are habitually associated with the research for optimal results.

Control theory was originally developed in order to obtain tools for analyzing and tuning a control system. Early development was concerned with centrifuge counters, simple industrial process regulators, electronic amplifiers, and fire control systems. As the theory developed, it became clear that tools could be applied to a large variety of different systems, both technical and non-technical.
Results (findings) from various branches of applied mathematics were exploited during the development of control theory. Control problems have also led (given arise) to new results in applied mathematics.

The story of backward stochastic differential equations (in short BSDEs) was appeared by Bismut [12], whose, he was introduced a linear BSDE in an attempt to solve an optimal stochastic control problem by the maximum principle.

Backward stochastic differential equation can be used to solve stochastic optimal control problems (more precisely, BSDEs can just trait the subclass of standard stochastic control problems with uncontrolled diffusion, and with corresponding semi-linear PDE), define nonlinear expectations and establish probabilistic representations of solutions to partial differential equations. Since many financial problems can be related to stochastic optimization problems and nonlinear expectations, it is not surprising that BSDEs have become a very important tool in financial mathematics. Nowadays, backward stochastic differential equations are an active field of research which is stimulated by new financial and actuarial applications.

Our aim in the third chapter is to derive sufficient and necessary maximum principles of controlled stochastic differential equation driven by a Brownian motion and a pure jump martingale. This type of equations has been studied by Dumitrescu et al. [28]. They proved existence and uniqueness as well as comparison theorems for these types of BSDEs. They also generalized the results to drivers including a singular process. If the driver is λ-linear, they found a representation of the solution of the associated BSDE in terms of a conditional expectation and an adjoint exponential semi-martingale. The framework of Dumitrescu et al. [28] is the same as that of the third chapter. However, in contrast to Dumitrescu et al. [28], we consider a stochastic optimal control problem in this default framework, and derive maximum principles characterizing the optimal solution of this problem.

Several other papers have studied similar frameworks (third chapter): Kharroubi and Lim [45] considered BSDEs with random marked jumps as well as applications to default risk. They connected the BSDEs with random marked jumps to Brownian BSDEs by enlargement of filtrations and proved that the jump BSDEs have solutions if the Brownian BSDEs have solutions. Furthermore, they proved a uniqueness theorem for jump BSDEs via a comparison theorem with Brownian BSDEs. Though the framework of Kharroubi and Lim [45] was similar to that of our paper, we focus on the stochastic control problem instead of the properties of the BSDEs. Lim and Quenez [50] analyzed the exponential utility maximization problem for an incomplete market with a default time which causes a discontinuity in the stock price. They applied dynamic programming to characterize the value function as the maximal subsolution of a BSDE. Lim and Quenez [51] considered a financial market with an asset exposed to a risk which can cause a jump in the asset price. They assume that the asset can be traded after the default time. In this context, they studied the expected utility maximization of terminal wealth for several utility functions. They proved that the value function for the power utility function can be determined as the minimal solution of a BSDE. Though the framework of Lim and Quenez [50] and [51] is similar to ours, they considered the control problem for maximizing the utility of terminal wealth. In contrast, the objective function in the present paper consists of both a terminal time term and an integral term over the whole time period.
For information about stochastic control with default jumps, we refer to Pham [59].

For more on stochastic control for jump processes, see for instance Cohen and Elliott [19] Chapter 21. Note that in Cohen and Elliott [19], the control cannot affect the diffusion coefficient function. In the present work, this is possible.

This thesis is organized in four chapters:

- In the first one, we recall some preliminaries on Malliavin calculus and on the fractional calculus and construct suitable spaces of integrands in order to have a well-defined integral using integral representation. In the second chapter, we introduce a new outcome on stochastic integration w.r.t fractional Brownian motion (fbm) for non adapted process by using idea of Lebovits, and we give a new result on stochastic integration w.r.t. fbm for no adapted processes that are written as the product of two processes, one is adapted, and second is instantly independent.

The next two chapters are related to stochastic controls, we talk about stochastic optimal control , and we go to define our problem of stochastic maximum principle:

- In Chapter 3, we give some preliminaries about jump processes and we shall set up a rigorous mathematical framework for stochastic optimal control problems.
- In the last Chapter, we give the reader a powerful tool of stochastic maximum principle in optimal control.

The appendix include a summary on notions and properties of fractional Brownian motion.

Chapter 1

Stochastic integration for no adapted processes

1.1 Fractional calculus

Let $\tilde{\mathcal{H}}$ be a some class of integrands and complete, and let $\varepsilon \in \tilde{\mathcal{H}}$ be the class of step functions, and $\mathcal{J}^H(f)$ be an integral of $f \in \varepsilon$ w.r.t. fractional brownian motion B_t^H, under this assumptions:

- $\tilde{\mathcal{H}}$ is an inner product space with an inner product $< f, g >_{\tilde{\mathcal{H}}}$, $f, g \in \tilde{\mathcal{H}}$.
- for $f, g \in \varepsilon$, $< f, g >_{\tilde{\mathcal{H}}} = \mathbb{E}[\mathcal{J}^H(f), \mathcal{J}(g)]$.
- The set ε is dense in $\tilde{\mathcal{H}}$.

In this section we give a short summary of fractional calculus for process driven by fractional brownian motion.
Fractional calculus is a branch of mathematical analysis that unifies the integration operator and differentiation operator of classical calculus as one operator. The differintegral is a single operator depending on a real valued parameter α, where positive values of α correspond to differentiation and negative values of α correspond to integration. Fractional calculus is an extension, or generalization, of the well known classical calculus. It was presented by authors and they mentioned the possible approach to fractional-order differentiation in that

sense, that for non-integer values of n the definition has given as

$$\frac{d^n e^{xm}}{dx^n} = m^n e^{mx}, \qquad m > 0.$$

Authors suggested to use this relationship also for negative or non-integer (rational) values of n, and they generalized the notion of differentiation for arbitrary functions.
According to Riemann-Liouville the notion of fractional integral of order $\alpha, (\alpha > 0)$ for a function $f(t)$, is a natural consequence of the well known formula (Cauchy-Dirichlet) that reduces the calculation of the n-fold primitive of a function $f(t)$ to a single integral of convolution type

$$\mathcal{J}_{a+}^{n} f(t) = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau \qquad n \in \mathbb{N}, \tag{1.1}$$

disappear at $t = a$ with its derivative $1, 2, 3, ..., n-1$. Impose $\mathcal{J}_{a+}^{n} f(t)$ and $f(t)$ to be causal function, here, vanishing for $t < 0$. Expand to any positive real value by using the Gamma function $(n-1)! = \Gamma(n)$.

1.1.1 Fractional Integral of order $\alpha > 0$:

(Right-Sided)

$$\mathcal{J}_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad \alpha \in \mathbb{R}. \tag{1.2}$$

We define $\mathcal{J}_{a+}^{0} = I$, $\mathcal{J}_{a+}^{0} f(t) = f(t)$.

Alternatively(left-sided integral)

$$\mathcal{J}_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(n)} \int_t^b (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad \alpha \in \mathbb{R},$$

for$(a = 0, b = +\infty) \longrightarrow$ Riemann derivative. and for $(a = -\infty, b = +\infty)$we have a Liouville derivative. According to [9], we have the following definition

Definition 1.1.1. *[9] Let $f \in L^2[a,b] \subset L^2$, $0 < \alpha < 1$ and $t \in [a,b]$. The fractional derivatives of order α on the interval $[a,b]$ are*

$$(D_{a+}^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_a^b f(u)(u-t)_+^{-\alpha} du,$$

and

$$(D_{b-}^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_a^b f(u)(u-t)_+^{-\alpha} du.$$

Furthermore, this case admits what is known as the Weyl representation of the fractional derivatives:

$$(D^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(t)}{(t-a)^{\alpha}} + \alpha\int_a^t \frac{f(t)-f(u)}{(t-u)^{\alpha-1}}du\right],$$

and

$$(D^{\alpha}_{b-}f)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(t)}{(b-t)^{\alpha}} + \alpha\int_t^b \frac{f(t)-f(u)}{(t-u)^{\alpha-1}}du\right].$$

Notice that $D^{\alpha}_{a+} = I^{-\alpha}_{a+}$ and $D^{\alpha}_{b-} = I^{-\alpha}_{b-}$. Furthermore, the fractional derivatives D^{α}_{a+} and D^{α}_{b-} are called left sided and right sided, respectively.

1.2 The spaces of stochastic test functions and stochastic distributions

In all this work we denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}$ and $\mathcal{S}'(\mathbb{R})$ denote its dual space. Let μ be the standard Gaussian measure on $\mathcal{S}'(\mathbb{R})$. Let $(L^2) = L^2(\mathcal{S}'(\mathbb{R}), \mu)$ and let $(\mathcal{S})$ and $(\mathcal{S})^{\star}$ denote the spaces of test functions and generalized functions on $\mathcal{S}'(\mathbb{R})$, respectively. We have (See: [49])

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^{\star}.$$

1.2.1 $(\mathcal{S})^{\star}$-process, $(\mathcal{S})^{\star}$-derivative and $(\mathcal{S})^{\star}$-integral

Let $\mathcal{B}(\mathbb{R})$ denote the Borelian σ-field on $\mathbb{R}$ and λ a measure on $\mathcal{B}(\mathbb{R})$ such that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is a σ-finite measure space. Through this section, $[0, T]$ denote an element of $\mathcal{B}(\mathbb{R})$.

A measurable function $\Phi : I \longrightarrow (\mathcal{S})^{\star}$ is called a stochastic distribution process ($(\mathcal{S})^{\star}$-process). Φ is said differentiable at t_0 if $\lim\limits_{r\longrightarrow 0} r^{-1}(\Phi_{t_0+r} - \Phi_{t_0})$ exists in $(\mathcal{S})^{\star}$.

one notes $\frac{d\Phi_{t_0}}{dt}$ the $(\mathcal{S})^{\star}$-derivative at t_0 of the stochastic distribution process Φ. If Φ is differentiable at every t_0 of I, we said that Φ to be differentiable over I. Generally, for every $k \in \mathbb{N}$, Φ is $\mathcal{C}^k$ in $(\mathcal{S})^{\star}$ if the process $\Phi : I \longrightarrow (\mathcal{S})^{\star}$ is $\mathcal{C}^k$.

Definition 1.2.1. *[49] Assume that $\Phi : I \longrightarrow (S)^\star$ is weakly in $L^1(I,\lambda)$, i.e. assume that for all φ in $(\mathcal{S})$, the mapping $u \longmapsto \ll \Phi_u, \varphi \gg$, from I to $\mathbb{R}$ belongs to $L^1(I,\lambda)$. Then there exists an unique element in $(\mathcal{S})^\star$, noted $\int_I \Phi_u \lambda(du)$, such that, for all φ in $(\mathcal{S})$.*

$$\ll \int_I \Phi_u \lambda(du), \varphi \gg = \int_I \ll \Phi_u, \varphi \gg \lambda(du)$$

We say in this case that Φ is $(\mathcal{S})^\star$- integrable in I (with respect to the measure λ), in the Pettis sense. In the sequel, when we do not specify a name for the integral (resp. for the measure λ) of an $(\mathcal{S})^\star$- integrable process Φ on I. We always refer to the integral in Petti's sense (resp. to the Lebesgue measure).

1.2.2 $\mathcal{S}$-transform and Wick product

Lemma 1.2.1. *[49] For any $(p,q) \in \mathbb{N}^2$ and $(X,Y) \in (\mathcal{S})_{-p} \times (\mathcal{S})_{-q}$,*

$$|\mathcal{S}(X \diamond Y)(\eta)| \leq \|X\|_{-p} \|Y\|_{-q} e^{|\eta|^2_{\max p;q}}.$$

Where $\eta \in \mathcal{L}(\mathbb{R})$ and "$\diamond$" is notation of wick-product.

Some properties of $\mathcal{S}$-transforms

1. If Φ is deterministic function, then $\Phi \diamond \Psi = \Phi\Psi$, for all $\Psi \in (\mathcal{S})^\star$. Furthermore, let $(X_t)_{t\in\mathbb{R}}$ be a Gaussian process and let $\mathcal{H}$ be the subspace of (L^2) defined by $\mathcal{H} = \overline{vect_{\mathbb{R}}\{X_t; t \in \mathbb{R}\}}^{L^2}$. If X and Y two elements of $\mathcal{H}$, then

$$X \diamond Y = XY - \mathbb{E}[XY].$$

2. We define $\Phi = \sum_k a_k < \cdot, e_k >$ and $\Psi = \sum_n I_n(f_n)$ be in $(\mathcal{S})^\star$. So their $\mathcal{S}$-transform is given by

$$\mathcal{S}(\Phi)(\eta) = \sum_k a_k < \eta, e_k >_{L^2(\mathbb{R})}.$$

and

$$\mathcal{S}(\Psi)(\eta) = \sum_k < f_n, \eta^{\otimes n} > .$$

for every $\eta \in \mathcal{L}(\mathbb{R})$.

3. For every $(f\,\eta\,\xi)$ in $L^2(\mathbb{R})\times\mathcal{L}(\mathbb{R})\times\mathbb{R}$, we have this equality

$$(S)(e^{i\xi<\cdot,f>})(\eta)=e^{\frac{1}{2}(|\eta|_0^2+2i\xi<f,\eta>-\xi^2|f|_0^2)}.$$

Also, the $\mathcal{S}$-transform verifies the following properties: see [49]

Lemma 1.2.2. *[49]*

- *(i) The map $\mathcal{S}:\Phi\longmapsto\mathcal{S}(\Phi)$, from $(\mathcal{S})^\star$ into $\mathcal{F}(\mathcal{L}(\mathbb{R});\mathbb{R})$, is injective.*
- *(ii) Let $\Phi:I\longrightarrow(\mathcal{S})^\star$ be an $(\mathcal{S})^\star$ processes. If Φ is $(\mathcal{S})^\star$ -integrable over I w.r.t λ. then one has, for all η in $\mathcal{L}(\mathbb{R})$, $\mathcal{S}(\int_I\Phi(u)\lambda(du))(\eta)=\int_I\mathcal{S}(\Phi(u))(\eta)\lambda(du)$.*
- *(iii) Let $\Phi:I\longrightarrow(\mathcal{S})^\star$ be an $(\mathcal{S})^\star$ processes. If Φ is $(\mathcal{S})^\star$ -differentiable at t in I. Then, for every η in $\mathcal{L}(\mathbb{R})$the map $u\longmapsto[\mathcal{S}\Phi(u)](\eta)$ is differentiable at t and verifies*

$$\mathcal{S}[\frac{d\Phi}{dt}(t)](\eta)=\frac{d}{dt}[\mathcal{S}\Phi(t)](\eta).$$

Theorem 1.1. *[47] Let $\Phi:I\longrightarrow(\mathcal{S})^\star$ be a stochastic distribution such that, for all η in $\mathcal{L}(\mathbb{R})$. the real- valued map $t\longmapsto\mathcal{S}[\Phi(t)](\eta)$ is measurable and such that there exist a natural integer p, a real a and a function $\mathcal{L}$ in $L^1(I,\lambda)$ such that $|\mathcal{S}(\Phi(t))(\eta)|\leq\mathcal{L}(t)e^{a|\eta|_p^2}$, for all η of $\mathcal{L}(R)$ and for almost every t of I. Then Φ is $(\mathcal{S})^\star$- integrable over I, w.r.t to λ.*

Theorem 1.2. *[8] For any differentiable map $F:I\longrightarrow\mathcal{L}'(\mathbb{R})$, the element $<\cdot,F(t)>$ is a differentiable stochastic distribution process which satisfies the equality:*

$$\frac{d}{dt}<\cdot,F(t)>=<\cdot,\frac{dF}{dt}(t)>.$$

1.2.3 Operators $(M_H)_{H\in(0,1)}$

We now define our fundamental $L^2(\mathbb{R})$-operator M_H for $0<H<1$, in the Fourier domain by:([49])

$$\widehat{M_H(u)}(y)=\frac{\sqrt{2\pi}}{c_H}|y|^{1/2-H}\widehat{u}(y),\forall y\in\mathbb{R}^\star.$$

Where c_H is defined by

$$c_H=\left(\frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}\right)^{\frac{1}{2}}. \tag{1.3}$$

We define the homogeneous Sobolev space of order $1/2 - H$, noted $L_H^2(\mathbb{R})$ as:

$$L_H^2(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) : \widehat{u} = T_\varphi; \varphi \in L_{Loc}^1(\mathbb{R}) \text{ and } \|u\|_H < +\infty\},$$

where the norm $\|\cdot\|_H$ derives from the inner product $\langle \cdot, \cdot \rangle_H$ which is defined on $L_H^2(\mathbb{R})$,

$$\langle u, v \rangle_H := \frac{1}{c_H^2} \int_{\mathbb{R}} |\xi|^{1-2H} \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi.$$

M_H being an isometry from $(L_H^2(\mathbb{R}), \|\cdot\|_H)$ into $(L^2(\mathbb{R}), \|\cdot\|)_{L^2(\mathbb{R})}$, it is clear that, for every $(H,t,s) \in (0,1) \times \mathbb{R} \times \mathbb{R}$, $\langle M_H(\mathbf{1}_{[0,t]}), M_H(\mathbf{1}_{[0,s]}) \rangle_L^2(\mathbb{R}) = R_H(t,s)$.

1.3 Instantly independent processes and Itô isometry (BM case)

The Itô isometry is considered to be the one of the important theorems of Itô calculus. In this section, we present a first step to find an extension of the Itô isometry for the new integral. In the definition of the next section, we based on result of Ayed and Kuo [4].

1.3.1 Anticipating stochastic integrals

A New viewpoint for stochastic integral (Ayed-Kuo,s Idea)

Let $B(t)$ be a Brownian motion $\{B(t)_{t\geq 0}\}$ and let $\{\mathcal{F}_t, 0 \leq a \leq t \leq b \leq T\}$ a filtration satisfying the following conditions:

- (a) B_t is $\{\mathcal{F}\}$-adapted ie: B_t is $\mathcal{F}_t$ measurable for each $t \in [a,b]$.
- (b) $(B_t - B_s)$ and $\mathcal{F}_s$ are independent for any $s \leq t \in [a,b]$.

Definition 1.3.1. *[4] A stochastic process $\phi(t)$ is said to be instantly independent with respect to a filtration $\{\mathcal{F}_t\}$ if $\phi(t)$ and $\mathcal{F}_t$ are independent for each t.*

Lemma 1.3.1. *[4] If a stochastic process $\phi(t)$ is both adapted and instantly independent with respect to a filtration $\{\mathcal{F}_t\}$, then $\phi(t)$ is a deterministic function.*

Proof. $\phi(t)$ is $\mathcal{F}_t$-adapted, we have $E(\phi(t)|\mathcal{F}_t) = \phi(t)$.

Moreover, $\phi(t)$ is an instantly independent process with respect to the filtration $\mathcal{F}_t, \quad a \leq t \leq b$ such that

$$E(\phi(t)|\mathcal{F}_t) = E(\phi(t)).$$

Combining both statements, we have

$$E(\phi(t)) = \phi(t).$$

Thus, $\phi(t)$ is a deterministic function. $\square$

In this sense, we can view instantly independent stochastic processes as a counterpart of the adapted stochastic processes for the Itô integral.

Definition 1.3.2. *(Ayed-Kuo's stochastic integral[4]) For an adapted stochastic process $f(t)$ and an instantly independent stochastic process $\phi(t)$, we define the stochastic integral of $f(t)\phi(t)$ to be the limit*

$$\int_a^b f(t)\phi(t)dW(t) = \lim_{\|\Delta\| \longrightarrow 0} \sum_{i=1}^{n} f(t_{i-1})\phi(t_i)(W(t_i) - W(t_{i-1})), \tag{1.4}$$

provided that the limit in probability exists. Where $\Delta = \{a = t_0, t_1, t_2, \cdots, t_n = b\}$ is a partition of the interval $[a, b]$ and $\|\Delta_n\| = \max_{1 \leq i \leq n}(t_i - t_{i-1})$.

In general, for a stochastic process $F(t) = \sum_{n=1}^{N} \int_a^b f_n(t)\phi_n(t)$ with $f_n(t)$'s being adapted and $\phi_n(t)$'s instantly independent.

This definition follows the argument of Itô integral(can be defined in terms of Riemann sums by evaluating the integrand at the left endpoints of the intervals of the partition) but with adapted processes, and the instantly independent process is evaluated at the right endpoints of the intervals of the partition in order to take advantage of the independence property [21].

Unfortunately, the question of Ayed and Kuo about the new stochastic integral did not have an answer.

We pass now to give some example of stochastic process.

Example 1.3.1. *Consider the stochastic process*

$$\int_0^t B(1)dB(s), \qquad 0 \le t \le 1.$$

We have by linearity

$$\int_0^t B(1)dB(s) = \int_0^t (B(1) - B(s))dB(s) + \int_0^t B(s)dB(s), \qquad 0 \le t \le 1. \tag{1.5}$$

The second integral on the right-hand side of equation (1.5) is an adapted stochastic process, whereas(while) the first integral is an instantly independent stochastic process with respect to filtration $\mathcal{F}_t$, it means anticipating , we start calculating the first integral of the right-hand side of equation (1.5) as follows

$$\begin{aligned}
\int_0^t (B(1) - B(s))dB(s) &= \lim_{\|\Delta_n\| \longrightarrow 0} \sum_{i=1}^{n} (B(1) - B(s_i))(B(s_i) - B(s_{i-1})), \\
&= \lim_{\|\Delta_n\| \longrightarrow 0} \Big(B(1) \sum_{i=1}^{n} (B(s_i) - B(s_{i-1})) \\
&- \sum_{i=1}^{n} (B(s_i) - B(s_{i-1}) + B(s_{i-1}))(B(s_i) - B(s_{i-1})) \Big), \\
&= \lim_{\|\Delta_n\| \longrightarrow 0} \Big(B(1)B(t) - \Big(\sum_{i=1}^{n} (B(s_i) - B(s_{i-1}))^2 \\
&+ B(s_{i-1})(B(s_i) - B(s_{i-1})) \Big) \Big), \\
&= B(1)B(t) - t - \int_0^t B(s)dB(s), \qquad 0 \le t \le 1.
\end{aligned} \tag{1.6}$$

By substituting equation (1.6) into (1.5), we get

$$\begin{aligned}
\int_0^t B(1)dB(s) &= \int_0^1 B(1)dB(s) + \int_1^t B(t)dB(s), \\
&= B(1)B(t) - t - \int_0^t B(s)dB(s) + \int_0^t B(s)dB(s), \\
&= B(1)B(t) - t, \qquad 0 \le t \le 1.
\end{aligned} \tag{1.7}$$

When $t > 1$, we can write the integral from 0 to t as the integral from 0 to 1 plus the integral from 1 to t to obtain the equality, such that

$$\begin{aligned}
\int_0^t B(1)dB(s) &= \int_0^1 B(1)dB(s) + \int_1^t B(1)dB(s), \\
&= B(1)^2 - 1 + B(1)\int_1^t dB(s), \\
&= B(1)^2 - 1 + B(1)B(t) - B(1)^2, \\
&= B(1)B(t) - 1, \qquad t > 1.
\end{aligned}$$

Example 1.3.2. *Let a stochastic process*

$$\int_0^t B(1)B(s)dB(s), \qquad 0 \le t \le 1.$$

We note that the integrand can be decomposed as

$$B(1)B(s) = (B(1) - B(s) + B(s))B(s) = (B(1) - B(s))B(s) + B(s)^2.$$

By integral linearity, we have

$$\int_0^t B(1)B(s)dB(s) = \int_0^t (B(1) - B(s))B(s)dB(s) + \int_0^t B(s)^2 dB(s), \quad 0 \le t \le 1. \tag{1.8}$$

We calculate the first integral of the right-hand side instantly independent stochastic process with respect to the filtration $\mathcal{F}_t$ of Equation (1.8)

$$\begin{aligned}
\int_0^t (B(1)-B(s))B(s)dB(s) &= \lim_{\|\Delta_n\|\longrightarrow 0} \sum_{i=1}^{n} B(s_{i-1})(B(1)-B(s_i))(B(s_i)-B(s_{i-1})), \\
&= \lim_{\|\Delta_n\|\longrightarrow 0} \Big(B(1) \sum_{i=1}^{n} B(s_{i-1})(B(s_i)-B(s_{i-1})) \\
&- \sum_{i=1}^{n} B(s_{i-1})(B(s_i)-B(s_{i-1})+B(s_{i-1}))(B(s_i)-B(s_{i-1})) \Big), \\
&= \lim_{\|\Delta_n\|\longrightarrow 0} \Big(B(1) \sum_{i=1}^{n} B(s_{i-1})(B(s_i)-B(s_{i-1})) \\
&- \sum_{i=1}^{n} B(s_{i-1})(B(s_i)-B(s_{i-1})^2 - \sum_{i=1}^{n} B(s_{i-1})^2(B(s_i)-B(s_{i-1})) \Big), \\
&= \lim_{\|\Delta_n\|\longrightarrow 0} \Big(B(1)\int_0^t B(s)dB(s) - \int_0^t B(s)ds - \int_0^t B(s)^2 dB(s) \Big).
\end{aligned}$$

Also, by classical Itô integration theory, we conclude that

$$B(1)\int_0^t B(s)dB(s) = \frac{1}{2}(B(t)^2 - t),$$

and

$$\int_0^t B(s)^2 dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)ds.$$

We have

$$\begin{aligned}
\int_0^t B(s)(B(1)-B(s))dB(s) &= \tfrac{1}{2}B(1)(B(t)^2-t) - \int_0^t B(s)ds - \Big(\frac{1}{3}B(t)^3 - \int_0^t B(s)ds\Big), \\
&= \tfrac{1}{2}B(1)(B(t)^2-t) - \tfrac{1}{3}B(t)^3, \qquad 0 \le t \le 1.
\end{aligned}$$

(1.9)

Thence, substituting equation (1.8) into equation (1.9), we get

$$\begin{aligned}
\int_0^t B(1)B(s)dB(s) &= \int_0^t B(s)(B(1)-B(s))dB(s) + \int_0^t B(s)^2 dB(s), \\
&= \tfrac{1}{2}B(1)(B(t)^2-t) - \tfrac{1}{3}B(t)^3 + \left(\tfrac{1}{3}B(t)^3 - \int_0^t B(s)ds\right), \\
&= \tfrac{1}{2}B(1)(B(t)^2-t) - \int_0^t B(s)ds, \qquad 0 \le t \le 1.
\end{aligned}$$

If $t > 1$, we get

$$\begin{aligned}
\int_0^t B(1)B(s)dB(s) &= \int_0^1 B(1)B(s)dB(s) + \int_1^t B(1)B(s)dB(s), \\
&= \tfrac{1}{2}B(1)(B(t)^2-t) - \int_0^1 B(s)ds + \frac{1}{2}B(1)(B(t)^2-t) - \frac{1}{2}B(1)(B(t)^2-t), \\
&= \tfrac{1}{2}B(1)(B(t)^2-t) - \int_0^1 B(s)ds, \qquad t \ge 1.
\end{aligned}$$

For more example, see ([21]).

Chapter 2

New approach to stochastic integration w.r.t fractional brownian motion

In this chapter, we propose a new approach to stochastic integration of the class of instantly independent stochastic processes with respect to fractional Brownian motion on a finite interval. The appraisal point is to discover the counterpart of the Ito theory. More precisely, we show some result on stochastic integration with respect to no adapted processes by generalizing the results obtained by Ayed and Kuo [4] in the Brownian framework.

2.1 Introduction

Fractional Brownian motion was introduced by Kolmogorov [46] while studying spiral curves in Hilbert space. Later, its properties were given by Mandelbrot and Van Ness [10]. In fact, in [10], authors considered the fractional brownian motion as a centered and continuous gaussian process, denoted by $B^H = \{B_t^H, t \geq 0\}$, $H \in (0,1)$ with covariance

$$\mathbb{E}(B_t^H B_s^H) = \frac{V_H}{2}(t^{2H} + s^{2H} - (t-s)^{2H}),$$

and V_H is normalizing constant given by

$$V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

This process is starting from 0 with stationary increments,
$\mathbb{E}(B_t^H - B_s^H) = V_H|t-s|^{2H}$, which is self-similar, and, B_{at}^H has the same distribution as $a^H B_t^H$ (See definition A.1.1 in Appendix A). The parameter H determines the sign of the covariance of past increments and the future. This latter is positive when $H > \frac{1}{2}$ and negative when $H < \frac{1}{2}$. Moreover, it exhibits a long-range dependence in the sense that the covariance between increments at a distance u decreases to zero as u^{2H-2}.
The self-similarity and long-range dependence properties make the fractional Brownian motion a suitable driving noise in different applications like economics, finance, and telecommunications especially in internet traffic to hydrology problems via linguistic.

Lin [52] also Dai and Heyde [22] gave a stochastic integral $\int_0^t \phi_s dB_s^H$ as limit of Riemann sums.The propriety $\mathbb{E}(\int_0^t \Phi_s dB_s^H) = 0$ is not satisfying by this integral, Duncan et al.[29] introduced a new stochastic integral with zero measn which is the limit of Riemann sums defined by means of the Wick product [2]. Alo et al.[2] constructed a stochastic integral with respect to the fBm with Hurst parameter lesser than $\frac{1}{2}$, and Ayed and Kuo [4] explained this idea for an extension of the Itô integral.
The main objective in next section is to introduce a new approach of stochastic integration for processes not necessarily adapted with respect to fractional brownian motion. Particulary we are interested in the case when the index H is greater than $\frac{1}{2}$.

2.1.1 A New stochastic integration

In this part, we define our idea for stochastic integration ; we will integrate with respect to Fractional brownian motion a no adapted process, this later is a product of tow processes, one is adapted, and the second is instantly independent.
Our work is based on result of Ayed and Kuo [4], but in our case, we integrate with respect to fractional brownian motion where the Hurst parameter H is upper than $\frac{1}{2}$, and we append this work by a new integration of [48].

Definition 2.1.1. *[4] A stochastic process $\phi(t)$ is called instantly independent process of $\{\mathcal{F}_t\}$ if $\phi(t)$ and $\mathcal{F}_t$ are independent for each $t \in [a,b]$.*

Suppose $f(t)$ is an $\{\mathcal{F}_t\}$-adapted continuous stochastic process which is also L^2- integrable. We present the following definition

Definition 2.1.2. *[4] For an adapted stochastic process $f(t)$ and an instantly independent stochastic process $\phi(t)$, we define the stochastic integral of $f(t)\phi(t)$ with respect to fractional brownian motion B_t^H as follows :*

$$\int_a^b f(t)\phi(t)dB_t^H = \lim_{\|\Delta_n\|\longrightarrow 0} \sum_{i=1}^{n} f(t_{i-1})\phi(t_i)(B_{t_i}^H - B_{t_{i-1}}^H). \tag{2.1}$$

According to the stochastic integration w.r.t fractional Brownian motion introduced in [37], for $t \in [a,b]$ we have the following Wienner's Integration :

$$\int_a^b \phi(t)dB_t^H = \int_a^b \phi(t)K_H^\star dB_t. \tag{2.2}$$

Hence, if $f(t)\phi(t)K_H^\star \in L^2[a,b]$, then this integral is well defined and we have

$$\int_a^b f(t)\phi(t)dB_t^H = \int_a^b f(t)\phi(t)K_H^\star dB_t. \tag{2.3}$$

Here, we integrate the product of two measurable processes with respect to standard brownian motion B_t. Therefore, we find ourselves in the definition presented by Kuo et al.[16].

If we Take $\psi = f(t)K_H^\star$, we verify that this new process is also adapted.

By the above definition of $K_H^\star$, we can show that ψ is adapted. Indeed, we have $f(t)$ is an $\{\mathcal{F}_t\}$-adapted stochastic process with almost all sample paths being in $L^2[a,b]$, and moreover, $K_H^\star$ is an linear operator in $L^2[a,b]$.
Therefore, the product of $f(t)$ and $K_H^\star$ are in $L^2[a,b]$, from it, we conclude that ψ is an $\{\mathcal{F}_t\}$-adapted process with almost all sample paths being in $L^2[a,b]$.

We have given some properties of fractional brownian motion, our aim is to define stochastic integral of the form

$$\int_a^b F(t,\omega)dB_t^H, \tag{2.4}$$

where B^H is fbm(fractional brownian motion) and F is non-adapted stochastic process, which is written as a product of two processes, one is adapted process and the second is instantly independent.

It's necessary to give some notions before explaining our approach.

Since B^H is a Gaussian process, it easier for us to integrate with respect to this process (For more details see [49]). There remains the problem that the integrator is an non-adapted process.

The next definition from [49] is needed for our approach

Definition 2.1.3. *[49] Define for every t in $\mathbb{R}_D$*

$$W_t^G =< \cdot, g_t^{'} >, \tag{2.5}$$

where the equality holds in $(\mathcal{S}^\star)$. Then $(W_t^{(G)})_{t\in\mathbb{R}_D}$ is a $(\mathcal{S}^\star)$-process and is the $(\mathcal{S}^\star)$-derivative of the process $(G_t)_{t\in\mathbb{R}_D}$. We will sometimes $\frac{dG_t}{dt}$ instead of $W_t^{(G)}$.

By proposition 2.2 [49], $W_t^{(G)}$ is defined as

$$W_t^{(G)} = \sum_{k=0}^{+\infty} (\frac{d}{dt} < g_t, e_k >) < \cdot, e_k >, \tag{2.6}$$

$\|W_t^{(G)}\|_{-p}$ is continuous If and only if $|g_t^{'}|_{-p}$ is continuous.

Remark 2.1.1. *On $[0,T]$, in [48] , as well as the fact that $\|W_t^{(B^H)}\|_{-p}$ is continuous on compact set $[0,T]$ are verified for every $p \geq 2$ and $H \in (0,1)$.*

As the fbm B^H is continuous non-derivable, we define its increments for a time interval s. They can be assimilated to a derivative of the process at a resolution s. They are called fractional Gaussian noises, it's define by

$$G_s(i) = B^H(i) - B^H(i-s). \tag{2.7}$$

In this subsection, we introduce the integral of a non-adapted process F w.r.t to fbm . Since the map $s \longrightarrow G_s$ is weakly differentiable on I , we give a Wienner's integral w.r.t G_s as

$$\begin{aligned}
\int_I F(s)dG_s &= \int_I f(s)\phi(s)dG_s; \\
&= \int_I f(s)\phi(s)d[B^H(i) - B^H(i-s)]; \\
&= \int_I f(s)\phi(s)d[\frac{B^H(i) - B^H(i-s)}{ds}]ds; \\
&= \int_I f(s)\phi(s)W^{B_i^H - B_{i-s}^H}ds.
\end{aligned} \tag{2.8}$$

2.1.2 Bochner's integral

Before passing to give assumption about a process $f(t)$ and $\phi(t)$, we need the next definition from [47] ;

Definition 2.1.4. *[47] Let I be a subset of $[0,1]$ endowed with the Lebesgue's measure. One says that $\Phi : I \longrightarrow (\mathcal{S})^\star$ is Bochner's integrable of index p on I if it satisfies the two following conditions:*

1. *Φ is weakly measurable on I, i.e. $t \longrightarrow \ll \Phi_t, \varphi \gg$ is measurable on I for every φ in $(\mathcal{S})$.*

2. *There exists p in $\mathbb{N}$ such that $\Phi_t \in (\mathcal{S}_{-p})$ for almost every t in I and such that $t \longrightarrow \|\Phi_t\|_{-p}$ belongs to $L^1(I, dt)$.*

The Bochner's integral of Φ on I is denoted $\int_I \Phi(t)dt$.

2.1.3 Wick-itô's integral w.r.t gaussian process

According to [49], we have this definition

Definition 2.1.5. *[49] Let $X : \mathbb{R} \longrightarrow (\mathcal{S})^\star$ be a process such that the process $t \longmapsto X_t \diamond W_t^G$ is $(\mathcal{S})^\star$-integrable on $\mathbb{R}$. The process X is then said to be dG-integrable on $\mathbb{R}$ w.r.t the Gaussian* *process G. The Wick-Itô's integral of X w.r.t G, on* $\mathbb{R}$*, is defined by setting:*

$$\int_{\mathbb{R}} X_s d^{\diamond} G_s := \int_{\mathbb{R}} X_s \diamond W_s^{(G)} ds. \tag{2.9}$$

For any Borel set I of $\mathbb{R}$*, we define* $\int_I X_s d^{\diamond} G_s := \int_{\mathbb{R}} \mathbf{1}(s) X_s d^{\diamond} G_s.$

Now, back to our approach ; Let $H \in (0,1)$, $[a,b]$ is borel's subset of $(0,1)$. $B^H = (B_t^H)_{t\in[a,b]}$ is a fbm with Hurst parameter $H \in [0,1]$. Assuming that $f(t)\phi(t)$ is an $(\mathcal{S})^{\star}$-valued process, we have the following result

- (i) There exists $p \in \mathbb{N}$ such that $f(t)\phi(t) \in (\mathcal{S})^{\star}$ a.e. $t \in [a,b]$.
- (ii) the process $f(t)\phi(t) \diamond W_t^H$ is Bochner's integral on $[a,b]$.

Let $\Delta = \{0 = t_0, t_1, t_2, \cdots, t_n = 1\}$ be a partition of the interval $[0,1]$. On the subinterval $[t_{i-1}, t_i]$, we take the $rightendpoint$ t_i as the evaluation point for the integrand $B(1) - B(t)$. We define this integral

$$\begin{aligned}
\int_0^1 (B(1) - B(t)) d^{\diamond} B^H(t) &= \lim_{\|\Delta\| \longrightarrow 0} \sum_{i=1}^{n} (B(1) - B(t_i))(W_{t_i}^{B^H} - W_{t_{i-1}}^{B^H}); \\
&= B(1)^2 - \lim_{\|\Delta\| \longrightarrow 0} \sum_{i=1}^{n} B(t_i)(W_{t_i}^{B^H} - W_{t_{i-1}}^{B^H}); \\
&= B(1)^2 - \lim_{\|\Delta\| \longrightarrow 0} \sum_{i=1}^{n} \{[B(t_i) - B(t_{i-1})] + B(t_{i-1})\}(W_{t_i}^{B^H} - W_{t_{i-1}}^{B^H}); \\
&= B(1)^2 - 1 - \int_0^1 B(t) d^{\diamond} B^H(t),
\end{aligned} \tag{2.10}$$

where the last integral is the Wick-Itô's integral.
We conclude that $\int_0^1 B(1) d^{\diamond} B^H(t) = B(1)^2 - 1.$

In this way, we define an adapted process f and an instantly independent process ϕ on $[0,T]$, with respect to B^H. Since the map $s \longrightarrow B_s^H$ is weakly differentiable on $[0,T]$, one may think the formal definition of the Wienner's integral w.r.t B^H denoted $\int_{[0,T]} f(s)\phi(s)d^{\diamond}B_s^H$, by setting

$$\int_{[0,T]} f(s)\phi(s)d^{\diamond}B_s^H = \int_{[0,T]} f(s)\phi(s)\frac{dB_s^H}{ds}ds = \int_{[0,T]} f(s)\phi(s)W_s^{B^H}ds. \tag{2.11}$$

In general, the problem of non-adaptation remains , in this case, we passe to give this definition

Definition 2.1.6. *For an adapted stochastic process $f(t)$ and an instantly independent stochastic process $\phi(t)$, we define stochastic integral of $f(t)\phi(t)$ as follow*

$$\int_{[0,T]} f(s)\phi(s)d^{\diamond}B_s^H = \lim_{\|\Delta_n\|\longrightarrow 0}\sum_{i=1}^{n} f(t_{i-1})\phi(t_i)(w_{t_i}^{B^H} - w_{t_{i-1}}^{B^H}), \tag{2.12}$$

provided that the limit in probability exists.

2.1.4 Some examples of our approach

Example 2.1.1. *For $t \in [0,1]$, we calculate the following stochastic integral using our approach ;*

$$\int_0^t B(1)B(s)d^{\diamond}B_s^H.$$

The random variable B^1 is independent of B_s^H because s is less than t, but the random variable B_s is adapted to the filtration generated by B_s^H and so the random variable $B(1)B(s)$ is not adapted to the filtration generated by B_s^H and which is written as a product of two variables, one is instantaneously independent of B_s^H and the other is adapted to the filtration generated by B_s^H

By (2.10) this formula is given as

$$\int_0^t B(1)B(s)d^{\diamond}B_s^H = \begin{cases} \frac{1}{2}B(1)(B(t)^2 - t) - \int_0^t W_s^{B^H}ds, \; 0 < t \le 1 \\ \frac{1}{2}B(1)(B(t)^2 - t) - \int_0^1 W_s^{B^H}ds, \; t > 1. \end{cases}$$

Example 2.1.2. *Let $f(t)$ and $g(t)$ be two deterministic functions in $L^2([0,1])$. Then*

$$\int_0^1 g(t)\Big(\int_0^1 f(s)dB(s)\Big)d^{\diamond}B^H(t) = \int_{[0,1]^2} g(t)f(s)dB(s)\otimes d^{\diamond}B^H(t).$$

Note that the Wiener integral of $f(s)$ in the left-hand side has the decomposition

$$\int_0^1 f(s)dB(s) = \int_0^t f(s)dB(s) + \int_t^1 f(s)dB(s).$$

Remarking that the first integral is an the Itô part and the second integral is an the counterpart. *Let* $W_i = W^{B^H}_{t_i} - W^{B^H}_{t_{i-1}}$, *by definition (2.1.6), we have*

$$\begin{aligned}\int_0^1 g(t)\Big(\int_0^1 f(s)dB(s)\Big)d^\diamond B^H(t) = &\lim_{\|\Delta\|\longrightarrow 0}\sum_{i=1}^n g(t_{i-1})\Big[\int_0^{t_{i-1}} f(s)dB(s) + \int_{t_i}^1 f(s)dB(s)\Big]\Delta W_i;\\ &= \lim_{\|\Delta\|\longrightarrow 0}\sum_{i=1}^n g(t_{i-1})\Big[\int_0^1 f(s)dB(s) - \int_{t_{i-1}}^{t_i} f(s)dB(s)\Big]\Delta W_i;\\ &= \int_0^1 f(s)dB(s) - \lim_{\|\Delta\|\longrightarrow 0}\sum_{i=1}^n g(t_{i-1})f(t_{i-1})[B(t_i)-B(t_{i-1})]\Delta W_i;\\ &= \int_0^1 f(s)dB(s)\int_0^1 g(t)dB^H(t)\\ &- \lim_{\|\Delta\|\longrightarrow 0}\sum_{i=1}^n g(t_{i-1})f(t_{i-1})[B(t_i)-B(t_{i-1})]\Delta W_i;\\ &= \int_0^1 f(s)dB(s)\int_0^1 g(t)dB^H(t) - \int_0^1 g(t)f(t)W^{B^H}(t)dt.\end{aligned}$$

Concluded remarks

We conclude that the Itô's integral is perfect for making sense of studying differential systems directed by a semimartingale, so unfortunately we cannot use it in the case of the fractional.

If $H > \frac{1}{2}$, the fractional has sufficiently regular trajectories to be able to use Young's integral, which is defined a bit like the Riemann integral, namely a passage to the limit after discretization. But the problem is going to complicate when the integrands are not adapted processes, in this case we take the wick-Itô's integral. In this paper, we introduced a new approach of stochastic integration for non-adapted processes with respect to fractional Brownian motion $(B^H_s)_{s\geq 0}$ (which is not a semimartingale for $H \neq \frac{1}{2}$). These processes are written as a product of two processes, one is instantaneously independent of B^H_s and the other is adapted to the filtration generated by $B^H_s, s \geq 0$.

Chapter 3

Stochastic maximum principle with jumps

Lévy processes theory is much simpler than the more general theory of processes with independent increment. We recall that Lévy processes have stationary and independent increments.

3.1 Lévy processes

Definition 3.1.1. *[61] An adapted process $X = (X_t)_{t\geq 0}$ with $X_0 = 0$ a.s. is a Lévy process if*

(i) *X has increments independent of the past; ie, $X_t - X_s$ independent $\mathcal{F}_s, 0 \leq s < t$;*

(ii) *X has stationary increments; ie, $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t < \infty$;*

(iii) *X_t is continuous in probability; ie, $\lim\limits_{t \longrightarrow s} X_t = X_s$, where the limit is taken in probability.*

Example 3.1.1. *Brownian motion is one example of Lévy processes.*

Definition 3.1.2. *[61] A stochastic process $W = (W(t))_{t\geq 0}$ on $\mathbb{R}^d$ is a brownian motion if it is a Lévy procees, and*

(i) $\mathbb{P}(\omega \in \Omega : W_t(0) = 0) = 1$;

(ii) $\forall n, \forall t_i, 0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$. $(W_{t_n} - W_{t_{n-1}}, \ldots, W_{t_1} - W_{t_0}, W_{t_0})$ *are independent;*

(iii) *For any $s \leq t$, $W_t - W_s$ is a centered real valued gaussian distributed with variance $t - s$, i.e. $W_t - W_s \sim \mathcal{N}(0, t - s)$;*

(iv)

$$\mathbb{P}\Big(\omega \in \Omega : t \longmapsto W_t(\omega) \text{ is continuous}\Big) = 1.$$

3.2 Enlargement of filtrations

We start with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ is a given filtration satisfying the usual conditions, and $\mathcal{F} = \mathcal{F}_\infty$.

There are mainly two kinds of enlargement of filtration:

- **Initial enlargement of filtrations:** in this case, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L)$ where

  L is a real-valued random variable. (more generally $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{F}}$ is a σ-algebra)

- **Progressive enlargement of filtrations:** where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, with $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$ the natural filtration of right-continuous process $H_t = \mathbf{1}_{\{\tau \leq t\}}$ where τ is a random time.

3.3 Martingale measure

- **Martingale Representation Theorem:** Let W be a Brownian motion on a standard filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and let $\mathcal{G}_t$ be the augmented filtration generated by W. If X is a square integrable random variable measurable with respect to $\mathcal{G}_\infty$, then there exist a predictable process C which is adapted with respect to $\mathcal{G}_t$, such that

$$X = E(X) + \int_0^\infty C_s dW_s.$$

  Consequently,

$$E(X|\mathcal{G}_t) = E(X) + \int_0^t C_s dW_s.$$

Theorem 3.1. *[60]$\Big($**Brownian Martingale Representation**$\Big)$ Let M be a martingale (càdlàg) square integrable for a filtration $\{\mathcal{F}_t^W\}_{t\in[0,T]}$. Then, there exists a unique processes $(H_t)_{t\in[0,T]} \in$ $\mathbf{M}^2(\mathbb{R}^K)$, such that*

$$M_t = M_0 + \int_0^t H_s \cdot dW_s, \mathbb{P}-a.s. \qquad \forall t \in [0,T].$$

Remark 3.3.1. *This result implies that in a brownian filtration, the martingale are continuous.*

Stopping times

Definition 3.3.1. *[58] Let Ω be the set of outcomes and let $\mathcal{F}$ be a filtration on Ω. Let $\tau : \Omega \longrightarrow \Theta \cup \{\infty\}$.*

1. *The function τ is a stopping time if for every $t \in \Theta$*
$$\{\tau \leq t\} \in \mathcal{F}_t.$$

2. *The function τ is a weak stopping time if for every $t \in \Theta$*
$$\{\tau < t\} \in \mathcal{F}_t.$$

Proposition 3.3.1. *[64](**Stopping times and martingales**)*
If $(M(t))_{t\in T}$ be a martingale and let τ be a stopping time. Then the stopped process $(M_{t\wedge\tau})_{t\in T}$ is a martingale. This implies that
$$E(M_{t\wedge\tau}) = E(M_0).$$

3.4 Stochastic optimal control problems

This section is structured as follows: We present strong and weak formulations of stochastic optimal control problems (SOCP) and it also concerned with the existence of SOCP for both strong and weak formulations, a statement of the stochastic maximum principle is given in which the stochastic Hamiltonian system is introduced, after, we discuss about backward stochastic differential equation and we give this application in finance, geometry, and in stochastic controls.

3.4.1 Formulations of stochastic optimal control problems

We present two mathematical formulations (strong and weak formulations) of stochastic optimal control problems

Strong formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions (i.e.$(\Omega, \mathcal{F}, \mathbf{P})$ is complete, $\mathcal{F}_0$ contains all the $\mathbf{P}$-null sets in $\mathcal{F}$, and $\{\mathcal{F}_t\}_{t\geq 0}$ in right continuous.) on which an m-dimensional standard brownian motion $W(\cdot)$ is defined, we consider the following controlled stochastic differential equation:

$$\begin{cases} dX_t &= b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t); \\ X_0 &= x_0 \in \mathbb{R}^n, \end{cases} \quad (3.1)$$

where

$b : [0, T] \times \mathbb{R}^n \times \mathbf{U} \longrightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times \mathbf{U} \longrightarrow \mathbb{R}^{n\times m}$, and $\mathbf{U}$ is given a separable metric space, and $T \in (0, \infty)$ being fixed. The function $u(\cdot)$ is called the control representing the action, policy of the decision-makers (controllers) or decision. The nonanticipative restriction in mathematical terms can be represented as "$u(\cdot)\, is \{\mathcal{F}_t\}_{t\geq 0}$-adapted". The control $u(\cdot)$ is taken from the set

$$\mathcal{U}[0, T] \triangleq \{u : [0, T] \times \Omega \longrightarrow \mathbf{U} | u(\cdot) \quad is \quad \{\mathcal{F}_t\}_{t\geq 0}$$

-adapted}.

Next, we introduce the performance functional as follows:

$$J(u(\cdot)) = E\left\{\int_0^T f(t, x(t), u(t))dt + h(x(T))\right\}. \quad (3.2)$$

The following definition is given in [43].

Definition 3.4.1. *[43] Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be given satisfying the usual conditions and $W(t)$ be a m-dimensional standard $\{\mathcal{F}_t\}_{t\geq 0}$-browniam motion. A control $u(\cdot)$ is called an s- admissible control, and $(x(\cdot), u(\cdot))$ an s-admissible pair, if*

(i) $u(\cdot) \in \mathcal{U}[0, T]$;

(ii) *$x(\cdot)$ is the unique solution of equation (3.1);*

(iii) $f(\cdot, x(\cdot), u(\cdot)) \in \mathbf{L}^1_{\mathcal{F}}(0, T; \mathbb{R})$;

(iv) $h(x(T)) \in \mathbf{L}^1_{\mathcal{F}}(\Omega; \mathbb{R})$.

Our stochastic optimal control problem under strong formulation can be stated as follows:

Problem(PP)

Minimize (3.2) over $\mathcal{U}^s_{ad}[0,T]$.

The purpose is to find $\overline{u} \in \mathcal{U}^s_{ad}[0,T]$, such that

$$J(\overline{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}^s_{ad}[0,T]} J(u(\cdot)). \tag{3.3}$$

Weak formulation

Starting by a definition of [43],

Definition 3.4.2. *[43] A 6-tuple $\pi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P, W(\cdot), u(\cdot))$ is called a w-admissible control, and $(x(\cdot), u(\cdot))$ a w-admissible pair, if*

(i) *$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a filtered probability space satisfying the usual conditions;*

(ii) *$W(\cdot)$ is an m-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$;*

(iii) *$u(\cdot)$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process on $(\Omega, \mathcal{F}, P)$ taking value in $\mathbf{U}$;*

(iv) *$x(\cdot)$ is the unique solution of equation (3.1) on $\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P$ under $u(\cdot)$;*

(v) *some prescribed state constraints (for example, (3.2)) are satisfied;*

(vi) *$f(\cdot, x(\cdot), u(\cdot)) \in \mathbf{L}^1_{\mathcal{F}}(0, T, \mathbb{R})$ and $h(x(T)) \in \mathbf{L}^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$.*

Problem (WS). Minimize (3.2) over $\mathcal{U}^w_{ad}$.

Namely, we search $\overline{\pi} \in \mathcal{U}^w_{ad}[0,T]$, such that

$$J(\overline{\pi}) = \inf_{\pi\in\mathcal{U}^w_{ad}[0,T]} J(\pi). \tag{3.4}$$

3.4.2 Existence of optimal controls

In this section we discuss the existence of optimal control.
The background of the theory is the following: A lower semicontinuous function defined on some compact metric space attains it minimum.
Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfies the usual conditions, and $W(\cdot)$ be a given one-dimensional Brownian motion. Consider the following stochastic linear controlled system:

$$\begin{cases} dx(t) &= [Ax(t)+Bu(t)]dt + [Cx(t)+Du(t)]dW(t), \quad t\in[0,T] \\ \\ x(0) &= x_0, \end{cases} \tag{3.5}$$

where A, B, C and D are matrices of suitable sizes. The states $x(\cdot)$ takes values in $\mathbb{R}^n$, and the control $u(\cdot)$ is in

$$\mathcal{U}^L[0,T] \triangleq \{u(\cdot) \in \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^k) | u(t) \in \mathbf{U}, \quad a.e. t\in[0,T], P-a.s.\},$$

With $\mathbf{U} \subset \mathbb{R}^k$.

Existence under strong formulation

Problem (SL). Minimize (3.5) over $\mathcal{U}^L[0,T]$.
We introduce the following assumptions:

(H1) The set $U \in \mathbb{R}^K$ is convex and closed, and the function f and h are convex and for some $\sigma, K > 0$

$$f(x,u) \geq \sigma|u|^2 - K, \; h(x) \geq -K, \quad \forall\,(x,u) \in \mathbb{R}^n \times U. \tag{3.6}$$

(H2) The set $U \in \mathbb{R}^K$ is convex and compact, and the function f and h are convex.

Theorem 3.2. *[43] Under either (H1) or (H2), if problem* ***SL*** *is finite, then it admits an optimal control.*

Proof. We suppose that $(H1)$ holds. Let $x_j(\cdot), u_j(\cdot)$ be a minimizing sequence. By (3.6), we have

$$E\int_0^T |u_j(t)|^2 dt \leq K, \quad \forall j \geq 1,$$

for some positive constant K. Thus, there is a subsequence which is still labeled by $u_j(\cdot)$, such that

$$u_j(\cdot) \longrightarrow \overline{u}(\cdot), \quad in \quad \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^k). \qquad weakly \quad convergence.$$

By Mazur's theorem, we have a sequence of convex combinations

$$\widetilde{u}_j(\cdot) \triangleq \sum_{i\geq 1} \alpha_{ij} u_{i+j}(\cdot), \qquad \alpha_{ij} \geq 0 \quad and \quad \sum_{i\geq 1} \alpha_{ij} = 1,$$

such that

$$\widetilde{u}_j(\cdot) \longrightarrow \overline{u}(\cdot), \quad in \quad \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^k). \qquad Strongly \quad convergence.$$

Since the set $U \subseteq \mathbb{R}^k$ is convex and closed, it follows that $\overline{u}(\cdot) \in \mathcal{U}^{\mathbf{L}}[0,T]$.
On the other hand, if $\widetilde{x}_j(\cdot)$ is the state under the control $\widetilde{u}_j(\cdot)$, here we have the convergence

$$\widetilde{x}_j(\cdot) \longrightarrow \overline{x}(\cdot), strongly \quad in \quad \mathcal{C}_{\mathcal{F}}([0,T],\mathbb{R}^n).$$

So, $(\overline{x}(\cdot),\overline{u}(\cdot))$ is admissible, and the convexity of f and h implies

$$\begin{aligned} J(\overline{u}(\cdot)) &= \lim_{j\longrightarrow\infty} J(\widetilde{u}_j(\cdot)) \leq \lim_{j\longrightarrow\infty} \sum_{i\geq 1} \alpha_{ij} J(u_{i+j}(\cdot)), \\ &= \inf_{u(\cdot)\in\mathcal{U}^{\mathbf{L}}[0,T]} J(u(\cdot)). \end{aligned}$$

Whence, $(\overline{x}(\cdot),\overline{u}(\cdot))$ is optimal. □

Existence under weak formulation

Let us make the following assumptions:

(SE1) (U,d) is a compact metric space and $T>0$.

(SE2) The maps b,σ,f and h are continuous, and there exists a constant $L>0$ such that for $\phi(t,x,u) = b(t,x,u),\sigma(t,x,u),f(t,x,u),h(x)$,

$$\begin{cases} |\phi(t,x,u)-\phi(t,\widehat{x},u)| & \leq L|x-\widehat{x}|, \forall t\in[0,T],\ x,\widehat{x}\in\mathbb{R}^n, u\in U. \\ |\phi(t,0,u)| & \leq L; \quad \forall(t,u)\in[0,T]\times U. \end{cases}$$

(SE3) For every $(t,x) \in [0,T] \in \mathbb{R}^n$, the set

$$(b,\sigma\sigma^\top,f)(t,x,U) \triangleq \ |(b_i(t,x,u),(\sigma\sigma^\top)^{ij}(t,x,u),f(t,x,u))|u \in U,\{i=1,\cdots,n,j=1,\cdots,m\}$$

is convex in $\mathbb{R}^{n+nm+1}$.

(SE4) $S(t) \equiv \mathbb{R}^n$.

Theorem 3.3. *[43] Under (SE1)-(SE4), if problem* **(WS)** *is finite, then it admits an optimal control.*

Proof. Let $\pi_k \equiv (\Omega_k,\mathcal{F}_k,\{\mathcal{F}_{k_t}\}_{t\geq 0},P_k,W_k(\cdot),u_k(\cdot)) \in \mathcal{U}^w_{a,d}[0,T]$ be a minimizing sequence, namely,

$$\lim_{k\longrightarrow\infty} J(\pi_k) = \inf_{\pi\in\mathcal{U}^w_{a,d}[0,T]} J(\pi). \tag{3.7}$$

Let $x_k(\cdot)$ be the state trajectory corresponding to π_k. Define

$$X_k(\cdot) \triangleq (x_k(\cdot),B_k(\cdot),\Sigma_k(\cdot),F_k(\cdot),W_k(\cdot)), \tag{3.8}$$

where

$$\begin{cases} B_k(t) & \triangleq \displaystyle\int_0^t b(s,x_k(s),u_k(s))ds, \\ \Sigma_k(t) & \triangleq \displaystyle\int_0^t \sigma(s,x_k(s),u_k(s))dW_k(s), \quad (3.9) \\ F_k(t) & \triangleq \displaystyle\int_0^t f(s,x_k(s),u_k(s))ds. \end{cases}$$

By (**SE2**), it is routine to show that there is a constant $K>0$ such that

$$E_k|X_k(t)-X_k(s)|^4 \leq K|t-s|^2, \qquad \forall t,s\in[0,T],\forall k,$$

where E_k is the expectation under P_k. By Skorohod's theorem, we have

$$\begin{cases} \Big\{(\overline{X}_k(\cdot),\overline{\lambda}_k)\Big\} & \equiv \Big\{(\overline{x}_k(\cdot),\overline{B}_k(\cdot),\overline{\Sigma}_k(\cdot),\overline{F}_k(\cdot),\overline{W}_k(\cdot),\overline{\lambda}_k)\Big\}, \\ \\ \Big(\overline{X}(\cdot),\overline{\lambda}\Big) & \equiv \Big(\overline{x}(\cdot),\overline{B}(\cdot),\overline{\Sigma}(\cdot),\overline{F}(\cdot),\overline{W}(\cdot),\overline{\lambda}\Big), \end{cases}$$

on a suitable probability space $(\overline{\Omega},\overline{\mathcal{F}},\overline{P})$ such that

$$law \quad of(\overline{X}_k(\cdot),\overline{\lambda}_k) = law \quad of(X_k(\cdot),\lambda_{uk(\cdot)}), \quad \forall k \geq 1, \tag{3.10}$$

and $\overline{P}$-a.s.,

$$\overline{X}_k(t) \longrightarrow \overline{X}(t) \qquad uniformly \quad on \quad t \in [0,T], \tag{3.11}$$

and

$$\overline{\lambda}_k \longrightarrow \overline{\lambda} \qquad weakly \quad on \quad \Lambda.$$

Set

$$\begin{cases} \overline{\mathcal{F}}_{kt} & \triangleq \Big(\sigma\{\overline{W}_k(s),\overline{x}_k : s \leq t\} \vee \overline{\lambda}_k^{-1}(\beta_t(\Lambda))\Big), \\ \\ \overline{\mathcal{F}}_t & \triangleq \Big(\sigma\{\overline{W}(s),\overline{x} : s \leq t\} \vee \overline{\lambda}^{-1}(\beta_t(\Lambda))\Big). \end{cases}$$

We have

$$E_k\Big\{g(Y_k)(W_k(t)-W_k(s))\Big\} = 0,$$

where

$$Y_k \triangleq \quad \Big\{W_k(t_i),x_k(t_i),\lambda_k(f_{j\alpha}^{t_i})\Big\},$$

$$0 \leq t_1 \leq t_2 \leq \cdots \leq t_l \leq s, \quad \alpha = 1,2,\cdots,\beta.$$

In view of (3.10),

$$\overline{E}_k\Big\{g(Y_k)(\overline{W}_k(t)-\overline{W}_k(s))\Big\} = 0,$$

where

$$\overline{Y}_k \triangleq \Big\{\overline{W}_k(t_i), \overline{x}_k(t_i), \overline{\lambda}_k(f_{j\alpha}^{t_i})\Big\},$$

$$0 \le t_1 \le t_2 \le \cdots \le t_l \le s, \quad \alpha = 1, 2, \cdots, \beta.$$

By (3.10), we have the following Stochastic differential equation on $(\overline{\Omega}, \overline{\mathcal{F}}, \{\overline{F}_{kt}\}_{t\ge 0}, \overline{P})$:

$$\begin{aligned} \overline{x}_k(t) &= x_0 + \overline{B}_k(t) + \overline{\Sigma}_k(t), \\ &= x_0 + \int_0^t \widetilde{b}(s, \overline{x}_k(s), \overline{\lambda}_k(s))ds + \int_0^t \widetilde{\sigma}(s, \overline{x}_k(s), \overline{\lambda}_k(s))d\overline{W}_k(s). \end{aligned} \tag{3.12}$$

Note that all the integrals in (3.12) are well-defined due to the fact that $\overline{W}_k(\cdot)$ is an $\{\overline{F}_{kt}\}_{t\ge 0}$-Brownian motion. Moreover

$$\begin{aligned} \overline{EF}_k(T) &= \overline{E}\Big\{\int_0^T \widetilde{f}(s, \overline{x}_k(s), \overline{\lambda}_k(s)) + h(\overline{x}_k(T))\Big\}, \\ &= J(\pi_k) \longrightarrow \inf_{\pi \in \mathcal{U}_{ad}^w[0,T]} J(\pi), \quad as \quad k \longrightarrow \infty, \end{aligned} \tag{3.13}$$

where $\overline{E}$ is the expectation under $\overline{P}$. Letting $k \longrightarrow \infty$ in (3.12) and (3.13), and noting (3.11), we get

$$\begin{cases} \overline{x} = x_0 + \overline{B}(t) + \overline{\Sigma}(t), \quad t \in [0, T], \quad \overline{P} - a.s., \\ \overline{E}(\overline{\mathcal{F}}(T)) = \inf_{\pi \in \mathcal{U}_{ad}^w[0,T]} J(\pi). \end{cases} \tag{3.14}$$

After several calculations (See the book of [43]), we arrive that

$\overline{\pi} \triangleq \Big(\widehat{\Omega}, \widehat{\mathcal{F}}, \{\widehat{\mathcal{F}}_t\}_{t\ge 0}, \widehat{P}, \widehat{W}(\cdot), \overline{u}(\cdot)\Big) \in \mathcal{U}_{ad}^w[0, T]$ is an optimal control.

□

3.4.3 The pontryagin maximum principle

Statement of the stochastic principle

We consider the following stochastic controlled system:

$$\begin{cases} dX_t &= b(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dW_t, \qquad t \in [0,T] \\ X_0 &= x_0, \end{cases} \tag{3.15}$$

with the cost functional

$$J(u(\cdot)) = E\left\{\int_0^T f(t,x(t),u(t)dt + h(x(T)))\right\}. \tag{3.16}$$

In the above

$$b : [0,T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n,$$

$$\sigma : [0,T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times m},$$

$$f : [0,T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R},$$

$$h : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

We define

$$\begin{cases} b(t,x,u) &= \begin{pmatrix} b^1(t,x,u) \\ . \\ . \\ . \\ b^n(t,x,u) \end{pmatrix}; \\ \sigma(t,x,u) &= \sigma^1(t,x,u), \cdots, \sigma^m(t,x,u); \\ \sigma^j(t,x,u) &= \begin{pmatrix} \sigma^{1j}(t,x,u) \\ . \\ . \\ . \\ b^{nj}(t,x,u) \end{pmatrix}, 1 \leq j \leq n, \end{cases}$$

Let now give the following assumptions:

(S0) $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $W(t)$, augmented by all the **P**-null set in $\mathcal{F}$.

(S1) (U,d) is a separable metric space and $T>0$.

(S2) The maps b,σ,f and h are measurable, and there exist a constant $L>0$ and $\overline{\omega}:[0,\infty)\longrightarrow[0,\infty)$ such that for $\varphi(t,x,u)=b(t,x,u),\sigma(t,x,u),f(t,x,u),h(x)$, we have

$$\begin{cases} |\varphi(t,x,u)-\varphi(t,\widehat{x},\widehat{u}) & | \leq L|x-\widehat{x}|+\overline{\omega}(d(u,\widehat{u})), \\ & \forall t\in[0,T], \quad x,\widehat{x}\in\mathbb{R}^n, \quad u,\widehat{u}\in U, \\ |\varphi(t,0,u)| & \leq L, \quad \forall (t,u)\in[0,T]\times U. \end{cases} \tag{3.17}$$

(S3) The maps b,σ,f and h are C^2 in x. Otherwise, there exist a constant $L>0$ and a modulus continuous $\overline{\omega}$ such that for $\varphi=b,\sigma,f,h$, we have

$$\begin{cases} |\varphi_x(t,x,u)-\varphi_x(t,\widehat{x},\widehat{u})| & \leq L|x-\widehat{x}|+\overline{\omega}(d(u,\widehat{u})), \\ |\varphi_{x,x}(t,x,u)-\varphi_{x,x}(t,\widehat{x},\widehat{u})| & \leq \overline{\omega}(|x-\widehat{x}|+d(u,\widehat{u})), \\ & \forall t\in[0,T], \quad x,\widehat{x}\in\mathbb{R}^n, \quad u,\widehat{u}\in U. \end{cases} \tag{3.18}$$

Problem (S). Minimize (3.16) over $\mathcal{U}[0,T]$.

Any $\bar{u}(\cdot)\in\mathcal{U}[0,T]$ satisfying

$$J(\bar{u}(\cdot))=\inf_{u(\cdot)\in\mathcal{U}[0,T]} J(u(\cdot)) \tag{3.19}$$

is called an optimal control.

Adjoint equations

in this part, we introduce adjoint equations involved in a stochastic maximum principle and the associated stochastic Hamiltonian system. Recall that $S_n=\{A\in\mathbb{R}^{n\times n} \mid A^\top=A\}$ and let $x(\cdot),u(\cdot)$ be a given optimal pair.

We introduce the following terminal value problem for a stochastic differential equation:

$$\begin{cases} dp(t) &= -\{b_x(t,\overline{x}(t),\overline{u}(t))^\top + \sum_{j=1}^{m} \sigma_x^j(t,\overline{x}(t),\overline{u}(t))^\top q_j(t) \\ & \quad -f_x(t,\overline{x}(t),\overline{u}(t))\}dt + q(t)dW(t), \quad t \in [0,T]; \\ p(T) &= -h_x(\overline{x}(T)). \end{cases} \tag{3.20}$$

We have seen that in the deterministic case the adjoint variable $p(\cdot)$ plays a central role in the maximum principle. The adjoint equation that $p(.)$ satisfies is a backward ordinary differential equation (meaning that the terminal value is specified). It is nevertheless equivalent to a forward equation if we reverse the time. In the stochastic case, however, one cannot simply reverse the time, as it may destroy the nonanticipativeness of the solutions. an additional adjoint equation

$$\begin{cases} dP(t) &= -\{b_x(t,\overline{x}(t),\overline{u}(t))^\top P(t) + P(t)b_x(t,\overline{x}(t),\overline{u}(t)) \\ & \quad + \sum_{j=1}^{m} \sigma_x^j(t,\overline{x}(t),\overline{u}(t))^\top P(t)\sigma_x^j(t,\overline{x}(t),\overline{u}(t)) \\ & \quad + \sum_{j=1}^{m} \{\sigma_x^j(t,\overline{x}(t),\overline{u}(t))^\top Q_j(t) + Q_j(t)\sigma_x^j(t,\overline{x}(t),\overline{u}(t))\} \\ & \quad + \mathcal{H}_{xx}(t,\overline{x}(t),\overline{u}(t),p(t),q(t))\}dt + \sum_{j=1}^{m} Q_j(t)dW^j(t), \\ P(T) &= -h_{xx}(\overline{x}(T)). \end{cases} \tag{3.21}$$

Where the *Hamiltonian* $\mathcal{H}$ is defined by

$$\mathcal{H}(t,x,u,p,q) = \langle p, b(t,x,u)\rangle + tr[q^\top \sigma(t,x,u)] - f(t,x,u),$$

$$(t,x,u,p,q) \in [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n\times m}.$$

and $(p(\cdot), q(\cdot))$ is the solution to (3.20).

The maximum principle and stochastic hamiltonian systems

Theorem 3.4. *[43]{**Stochastic Maximum Principle**} Let **(S0)-(S3)** hold. Let $((\overline{x}(\cdot)),(\overline{u}(\cdot)))$ be an optimal pair of problem **(S)**. Then there are pairs of processes*

$$\begin{cases} (p(\cdot),q(\cdot)) \in & \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)\times(\mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^n))^m, \\ \\ (P(\cdot),Q(\cdot)) \in & \mathbf{L}^2_{\mathcal{F}}(0,T;Sn)\times(\mathbf{L}^2_{\mathcal{F}}(0,T;S^n))^m, \end{cases} \tag{3.22}$$

where

$$\begin{cases} q(\cdot)=(q_1(\cdot),\ldots,q_m(\cdot)), \quad Q(\cdot)=(Q_1(\cdot),\ldots,Q_m(\cdot)) \\ \\ \quad q_j \in \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^n), \qquad Q_j \in \mathbf{L}^2_{\mathcal{F}}(0,T;S^n) \end{cases} \tag{3.23}$$

satisfying the first-order and second-order adjoint equations (3.20) and (3.21), respectively, such that

$$\begin{gathered} H(t,\overline{x}(t),\overline{u}(t),p(t),q(t)) - H(t,\overline{x}(t),u(t),p(t),q(t)) \\ \\ \tfrac{1}{2}tr\Big(\{\sigma(t,\overline{x}(t),\overline{u}(t)) - \sigma(t,\overline{x}(t),u(t))\}^\top P(t) \\ \\ \cdot\{\sigma(t,\overline{x}(t),\overline{u}(t)) - \sigma(t,\overline{x}(t),u(t))\}\Big) \geq 0, \\ \\ \forall u \in \mathcal{U}, \quad a.e. t \in [0,T] \quad \mathbf{P}.a.s. \end{gathered} \tag{3.24}$$

Or, equivalently,

$$\mathcal{H}(t,\overline{x}(t),\overline{u}(t)) = \max_{u\in\mathcal{U}} \mathcal{H}(t,\overline{x}(t),u(t)), \quad a.e. t \in [0,T], \quad \mathbf{P}.a.s. \tag{3.25}$$

Theorem 3.5. *[43] Let **(S0)-(S3)** hold. Let problem **(S)** admits an optimal pair $(\overline{x}(\cdot),\overline{u}(\cdot))$. Then the optimal 6-tuple $(\overline{x}(\cdot),\overline{u}(\cdot),p(\cdot),q(\cdot),P(\cdot),Q(\cdot))$ of problem **(S)** solves the stochastic hamiltonian system (3.21) and (3.24).*

3.4.4 Backward stochastic differential equation

We start by simple example (but it's illustrative) to know $f \equiv 0$. Let $m = 1$, $T > 0$, and $\xi \in \mathbf{L}^2_{\mathcal{F}_t}(\Omega;\mathbb{R})$. Consider the following stochastic differential equation:

$$\begin{cases} dY(t) = & 0, \quad t[0,T]; \\ & \qquad\qquad (3.26) \\ Y(T) = & \xi. \end{cases}$$

It's impossible to find an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted solution $Y(\cdot)$, since the only solution of (3.26) is

$$Y(t) = \xi, \quad t \in [0,T]. \tag{3.27}$$

A natural way to making (3.27) $\{\mathcal{F}_t\}_{t\geq 0}$-adapted is to redefine $Y(\cdot)$ as follows:

$$Y(t) = E(\xi|\mathcal{F}_t), \quad t \in [0,T]. \tag{3.28}$$

Then, $Y(\cdot)$ is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and satisfies the terminal condition $Y(T) = \xi$. Noting that the process $Y(\cdot)$ defined by (3.28) is a square integrable $\{\mathcal{F}_t\}_{t\geq 0}$-martingale. By the martingale representation theorem, we can find an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process $Z(\cdot) \in \mathbf{L}^2_{\mathcal{F}_t}([0,T];\mathbb{R})$ such that

$$Y(t) = Y(0) + \int_0^t Z(s)dW(s), \quad \forall t \in [0,T], \quad P-a.s. \tag{3.29}$$

From (3.28)-(3.29), it follows that

$$\xi = Y(T) = Y(0) + \int_0^T Z(s)dW(s). \tag{3.30}$$

Hence,

$$Y(t) = \xi - \int_t^T Z(s)dW(s), \quad \forall t \in [0,T]. \tag{3.31}$$

Here, the role of the process Z is to make the process Y adapted.
We allow f to depend on the process Z, the equation therefore becomes:

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t, \qquad Y_T = \xi, \tag{3.32}$$

or, equivalently, in integral form,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{3.33}$$

Definition 3.4.3. *[60] A solution of BSDE (3.33) is a pair $\{(Y_t, Z_t)\}_{0\leq t\leq T}$ verify:*

1. *Y and Z are progressively measurable at values respectively in $\mathbb{R}^k$ and $\mathbb{R}^{k\times d}$;*
2. *$\int_0^T \{|f(s, Y_s, Z_s)| + \|Z_s\|^2\}ds < \infty$ $\mathbb{P}$.a.s.;*
3. *$\mathbb{P}$.a.s., we are*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T.$$

Proposition 3.4.1. *[60] We suppose that exists a positif process $\{f_t\}_{0\leq t\leq T} \in M^2(\mathbb{R})$, and a constant $\lambda > 0$ such as*

$$\forall (y, t, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}, \quad |f(t, y, z)| \leq f_t + \lambda(|y| + \|z\|).$$

If $\{(Y_t, Z_t)\}_{0\leq t\leq T}$ is a solution of (3.33) such as $Z \in M^2$, then Y belongs to S_c^2.

Lipschitz case

Pardoux-Peng result

We give some assumption

Assumption 3.4.1. *There exists a constant λ $\mathbb{P}$-a.s.,*

1. ***Lipschitz condition in** (y, z): for everything $t, y, \grave{y}, z, \grave{z}$,*

$$|f(t, y, z) - f(y, \grave{y}, \grave{z})| \leq \lambda(|y - \grave{y}| + \|z - \grave{z}\|);$$

2. ***Integrability condition***

$$E\Big[|\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds\Big] < \infty.$$

Theorem 3.6. *[60] (**Pardoux–Peng 90.**)*

Under hypothesis (3.4.1), BSDE (3.33) has a unique solution (Y, Z).

Priori estimate

We give an estimate on the BSDE: it is it is in fact a question of studying the dependence of the solution of the BSDE on the data which is ξ and the process $\{f(t,0,0)\}_{0\leq t\leq T}$

Proposition 3.4.2. *[60] Suppose that* (ξ, f) *(checks 3.4.1). Let* (Y, Z) *be a solution of equation (3.33), then there exists an constant* C_k *such that, for all* $\beta \geq 1 + 2\lambda + 2\lambda^2$*,*

$$E\Big[\sup_{0\leq t\leq T} e^{\beta_t}|Y_t|^2 + \int_0^T e^{\beta_t}\|Z_t\|^2 dt\Big] \leq C_k E\Big[e^{\beta T}|\xi| + \int_0^T e^{\beta t}|f(t,0,0)|^2 dt\Big].$$

3.4.5 BSDEs and stochastic control

In this section, we give a several applications of backward stochastic differential equations.

Applications

we present a results of Etienne Pardoux.[55]

In financial mathematics

Consider a typical model for continuous time asset pricing. Let V_t denote the total wealth of an agent at time t, which he can invest in $n+1$ different assets. one nonrisky asset, whose price per unit P_t^0 is governed by the linear ordinary differential equation (ODE) $dP_t^0 = P_t^0 r_t$, and n risky assets, where the price process for one share of the ith stock is governed by the linear stochastic differential equation (SDE) $dP_t^i = P_t^i[\mu_t^i dt + \sum_{j=1}^n \sigma_t^{ij} dB_t^j]$.

The asset pricing problem is as follows. Given a contingent claim ξ which is an $\mathcal{F}_T$-measurable random variable that we suppose to be square integrable, find an initial wealth V_0 and a portfolio $(\Pi_t^i, 0 \leq t \leq T), 1 \leq i \leq n$ such that the wealth at time T is exactly ξ. Hence, we need to solve the following linear BSDE

$$V_t = \xi - \int_t^T r_s[V_s - \Pi_s^\star \mathbb{1}]ds - \int_0^T \Pi_s^\star[\mu_s ds + \sigma dB_s].$$

This linear BSDE is a very classical model in financial mathematics. It is in particular the starting point of the celebrated Black-Scholes formula for option pricing. No general theory is necessary to study such a linear equation. However, there is at least one unreasonable assumption in our model: $V_t - \Pi_t^\star \mathbb{1}$ represents an amount of money that is deposited in the bank whenever it is positive, but it represents an amount of money that is borrowed from the bank if it is negative. As the interest rate for borrowing is in fact bigger than the bond rate, we should rather write the above equation as a nonlinear BSDE, with some interest rate process $R_t > r_t$

$$V_t = \xi - \int_t^T r_s[V_s - \Pi_s^\star \mathbb{1}]^+ ds + \int_t^T R_s[V_s - \Pi_s^\star \mathbb{1}]^- ds - \int_0^T \Pi_s^\star[\mu_s ds + \sigma dB_s].$$

In stochastic controls

From ([27],[57]), we consider the following applications:

Suppose Now that $k = 1$, and the coefficient f of BSDE is concave in variable y and z We define the following "polar" process:

$$F(t,\beta,\gamma) := \sup_{y\in\mathbb{R}, z\in\mathbb{R}^d} [f(t,y,z) - \beta y - \gamma \cdot z].$$

It follows from a measurable selection theorem that to each progressively measurable process (Y_t, Z_t), one can associate a progressively measurable pair $(\beta_t^\star, \gamma_t^\star)$ such that

$$F(t,\beta_t^\star,\gamma_t^\star) = f(t,Y_t,Z_t) - \beta_t^\star Y_t - \gamma_t^\star \cdot Z_t \qquad 0 \le t \le T.$$

Let $\mathcal{A}$ denote the set of progressively "control" (β_t, γ_t) that satisfy $E\Big[\int_0^T F(t,\beta_t,\gamma_t)^2\Big]dt < \infty$. Consider for each $t \ge 0$ the scalar forward linear SDE

$$\Gamma_{t,s}^{\beta,\gamma} = 1 + \int_t^s \Gamma_{t,r}^{\beta,\gamma}[\beta_r dr + \gamma_r dB_r], \qquad s \ge t.$$

We then have the following.

Theorem 3.7. *[27] Let (Y_t, Z_t) be the unique solution of the BSDE (f, ξ). Then for each $0 \leq t \leq T$, Y_t is the value function of a stochastic control problem, in the sense that*

$$Y_t = \sup_{(\beta,\gamma)\in\mathcal{A}} E\Big[\int_t^T \Gamma_{t,s}^{\beta,\gamma} F(s, \beta_s, \gamma_s) ds + \Gamma_{t,T}^{\beta,\gamma} \xi | \mathcal{F}_t\Big].$$

In stochastic geometry

One can show that the construction of a gamma-martingale (which is a notion of martingale adapted to processes with values in a manifold equipped with a connection Γ) with prescribed final value ξ can be achieved by solving a backward SDE where the coefficient f takes the form

$$f_i(y, z) = \sum_{j,k,q} \Gamma_{j,k}^i(y) z_{j,q} z_{k,q}.$$

One can assume that Γ is bounded and Lipschitz, however, in this case f is not lipschitz in z, hence Theorem (3.6) does not apply directly. However, combining BSDE and gamma-martingale techniques, one can show existence and uniqueness of a solution in this case.

Chapter 4

Stochastic maximum principle with default

In this chapter, We prove existence of a unique solution to the controlled default stochastic differential equation. Furthermore, we prove existence and uniqueness of solution to the adjoint backward stochastic differential equation which appears in connection to the maximum principles. Finally, we apply the maximum principles to solve a utility maximization problem with logarithmic utility functions and exponential utility functions.

4.1 Framework

Let $(\Omega, \mathcal{G}, P)$ be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion W and a single jump process $H_t = \mathbf{1}_{\tau \leq t}$, $t \in [0, T]$, where the random variable τ is positive and may represent a default time in credit- or counterparty risk, or a death time in actuarial issues[1]. We assume that this default can appear at any time, i.e. $P(\tau \geq t) > 0$ for any $t \geq 0$. We denote by $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$ the complete natural filtration of W and H. We assume that W is a $\mathbb{G}$-Brownian motion.

We suppose that the increasing process H admits a predictable compensator Λ. Moreover, the process Λ is assumed to be absolutely continuous w.r.t. Lebesgue's measure, there exists a positive process λ, called the intensity, such that $\Lambda_t = \int_0^t \lambda_s ds$ for each $t \geq 0$. The

[1] If τ is a death time, the control is often stopped at this time. This complicates the problem, see e.g., Bouchard and Pham [13], Choulli and Yansori [17] and Jeanblanc et al. [40].

process M defined as

$$M_t = H_t - \int_0^t \lambda_s ds.$$

is a $\mathbb{G}$-martingale called the compensated martingale of H. If the intensity is $\mathbb{G}$-adapted, it vanishes after tau. This is important for the following. We will state the so-called predictable representation theorem (PRT) (Theorem 3.12 in Aksamit and Jeanblanc [1], reformulated to the current notation.)

Theorem 4.1. *Every $\mathbb{G}$-martingale Y admits a representation*

$$Y_t = Y_0 + \int_0^t \varphi_s dW_s + \int_0^t \gamma_s dM_s,$$

where M is the compensated martingale of H, and $\varphi = (\varphi_t)_{t\in[0,T]}, \gamma = (\gamma_t)_{t\in[0,T]}$ are $\mathbb{G}$-predictable processes, such that the stochastic integrals are well defined.

Throughout this section, we introduce some basic spaces.

- $\mathbb{S}^2$ is the subset of $\mathbb{R}$-valued $\mathbb{G}$-adapted càdlàg processes $(Y_t)_{t\in[0,T]}$, such that

$$\|Y\|^2_{\mathbb{S}^2} := \mathbb{E}[\sup_{t\in[0,T]} |Y_t|^2] < \infty.$$

- $\mathbb{H}^2$ is the subset of $\mathbb{R}$-valued $\mathbb{G}$-predictable processes $(Z_t)_{t\in[0,T]}$, such that

$$\|Z\|^2_{\mathbb{H}^2} := \mathbb{E}[\int_0^T |Z_t|^2 \, dt] < \infty.$$

- $\mathbb{H}^2(\lambda)$ is the subset of $\mathbb{R}$-valued $\mathbb{G}$-predictable processes $(U_t)_{t\in[0,T]}$, such that

$$\|U\|^2_{\mathbb{H}^2(\lambda)} := \mathbb{E}[\int_0^T \lambda_t |U_t|^2 \, dt] < \infty.$$

4.2 Stochastic maximum principles

In this section, we present two stochastic maximum principles which can be used to solve stochastic optimal control problems where the system state is determined by the controlled with default.

We denote by $\mathcal{V}$ a bounded convex subset of $\mathbb{R}$ and the $\mathcal{V}$-valued controls $(u_t)_{t\geq 0}$ which are $\mathbb{G}$-predictable in $L^2(\Omega \times [0,T])$ are called admissible. We denote by $\mathcal{A}$ the set of all admissible controls. Let $X^u_t = X_t$ be the controlled stochastic differential equation (SDE) with default under the filtration $\mathbb{G}$,

$$\begin{cases} dX_t = & b(t,X_t,u_t)dt + \sigma(t,X_t,u_t)dW_t + \gamma(t,X_{t^-},u_t)dM_t; \\ \\ X_0 = & x_0, \end{cases} \tag{4.1}$$

where the coefficient functions are as follows:

$$b : \Omega \times [0,T] \times \mathbb{R} \times \mathcal{V} \to \mathbb{R},$$

$$\sigma : \Omega \times [0,T] \times \mathbb{R} \times \mathcal{V} \to \mathbb{R},$$

$$\gamma : \Omega \times [0,T] \times \mathbb{R} \times \mathcal{V} \to \mathbb{R},$$

and the initial value $x_0 \in \mathbb{R}$.

Note that we often suppress the ω for ease of notation. So, for instance, we write $b(t,X_t,u_t)$ instead of $b(\omega,t,X_t(\omega),u_t(\omega))$.

We make the following set of assumptions on these coefficient functions.

Assumption 4.2.1. (*a*) *The functions $b(\omega,t,\cdot)$, $\sigma(\omega,t,\cdot)$ and $\gamma(\omega,t,\cdot)$ are assumed to be bounded and C^1 for each fixed ω,t with bounded derivatives.*

(*b*) *The functions $b(\cdot,x,u)$ and $\sigma(\cdot,x,u)$ and $\gamma(\cdot,x,u)$ are $\mathbb{G}$–predictable, for each $(x,u) \in \mathbb{R}\times\mathcal{V}$.*

(*c*) Lipschitz condition: *The functions b,σ,γ are uniformly Lipschitz in the variable x for each $u \in \mathcal{V}$, with the Lipschitz constant, $\psi > 0$, independent of the variables t,ω.*

(*d*) Linear growth: *The functions b,σ,γ satisfy the linear growth condition in the variable x, for each $u \in \mathcal{V}$, with the linear growth constant independent of the variables t,ω.*

Theorem 4.2 (Existence of unique solution to the SDE with default). *For every $u \in \mathcal{A}$ and $\omega \in \Omega$, the coefficients b,σ and γ satisfy the assumptions (a-d). Thus there exists a unique solution $X \in \mathbb{S}^2$ of SDE* (4.1).

Proof. Let us first introduce a norm in the Banach space $V := S^2$ for $\beta > 0$, for $X \in V$: $\|X\|_V := \mathbb{E}[\int_0^T e^{-\beta s}|X_s|^2\,ds]$. Setting $\Phi(x) := X$ we define a mapping $\Phi : V \to V$, for fixed $u \in \mathcal{A}$, as follows

$$\mathrm{X}_t := x_0 + \int_0^t b(s,x_s)ds + \int_0^t \sigma(s,x_s)dW_s + \int_0^t \gamma(s,x_s)dM_s$$

We are going to prove that $\Phi : (V,\|\cdot\|_V) \to (V,\|\cdot\|_V)$ is contracting. Indeed, we consider arbitrary $x^i \in V$, $i = 1,2$, and we put $X^i := \Phi\left(x^i\right)$, $i = 1,2$. Let $\bar{x} := x^1 - x^2$ and $\bar{X} := X^1 - X^2$. Then, by applying Itô's formula to $(e^{-\beta t}\left|\bar{X}_t\right|^2)_{t\geq 0}$, and taking the expectation, leads to

$$
\begin{aligned}
E\Big[e^{-\beta t}|\overline{X}_t|^2\Big]+E\Big[\int_0^t e^{-\beta s}|\overline{X}_s|^2 ds\Big] &= 2E\Big[\int_0^t e^{-\beta s}\{b(s,x_s^1)-b(s,x_s^2)\}ds\Big]\\
&+E\Big[\int_0^t e^{-\beta s}|\sigma(s,x_s^1)-\sigma(s,x_s^2)|^2 ds\Big]\\
&+E\Big[\int_0^t e^{-\beta s}\lambda_s|\gamma(s,x_s^1)-\gamma(s,x_s^2)|^2 ds\Big],
\end{aligned}
$$

where we have used the fact that $M_t = H_t - \int_0^{t\wedge\tau}\lambda_s ds$ is a single jump martingale.

that $(\delta M_s)^2 = \delta M_s = \delta H_s$. Thus, the covariation process of M is H :

$$[M]_t = \sum_{0\le s\le t}(\delta M_s)^2 = \sum_{0\le s\le t}(\delta H_s)^2 = H_t,$$

The quadratic variation is:

$$\langle M\rangle_t = \int_0^{t\wedge\tau}\lambda_s ds = \int_0^t \lambda_s(\mathbf{1}_{s<\tau}-H_s)ds, t\ge 0.$$

Consequently,

$$
\begin{aligned}
\int_0^t |\gamma(s,x_s^1)-\gamma(s,x_s^2)|^2 d\langle M\rangle_s &= \int_0^t \lambda_s(\mathbf{1}_{s<\tau}-H_s)|\gamma(s,x_s^1)-\gamma(s,x_s^2)|^2 ds\\
&\le \int_0^t \lambda_s|\gamma(s,x_s^1)-\gamma(s,x_s^2)|^2 ds.
\end{aligned}
$$

Using the assumptions

$$
\begin{aligned}
E\Big[e^{-\beta t}|\overline{X}_t|^2\Big]+E\Big[\int_0^t e^{-\beta s}|\overline{X}_s|^2 ds\Big] &\le CE\Big[\int_0^t e^{-\beta s}|\overline{X}_s||\overline{x}_s|ds\Big]+C^2E\Big[\int_0^t e^{-\beta s}|\overline{x}_s|^2 ds\Big]\\
&\le \tfrac{C^2}{\epsilon}E\Big[\int_0^t e^{-\beta s}|\overline{X}_s|^2 ds\Big]+(\epsilon+C^2)E\Big[\int_0^t e^{-\beta s}|\overline{x}_s|^2 ds\Big],
\end{aligned}
$$

where for the last inequality, we have used the inequality $2ab\le \epsilon a^2+\frac{1}{\epsilon}b^2$ for all $\epsilon>0$ (which follows by basic algebra from $(\epsilon a-b)^2\ge 0$).

Choosing $\beta = \frac{C^2}{\epsilon} + 1$, we have

$$E\Big[e^{-\beta t}|\overline{X}_t|^2\Big] + E\Big[\int_0^t e^{-\beta s}|\overline{X}_s|^2 ds\Big] \le (\epsilon + C^2)E\Big[\int_0^t e^{-\beta s}|\overline{x}_s|^2 ds\Big].$$

Then, $\left\|\bar{X}\right\|_V^2 \le (\varepsilon + C^2)\|\bar{x}\|_V$, i.e.,

$$\left\|\Phi\left(x^1\right) - \Phi\left(x^2\right)\right\|_V \le (\varepsilon + C^2)\left\|x^1 - x^2\right\|_V, \text{ for all } x^1, x^2 \in V.$$

Choosing $\varepsilon > 0$ such that $\varepsilon + C^2 < 1$, we obtain that $\Phi : (V, \|\cdot\|_V) \to (V, \|\cdot\|_V)$ is a contraction on the Banach space $(V, \|\cdot\|_V)$. Hence, from Banach's fixed point theorem, there exists a unique fixed point $X \in H$, such that $X = \Phi(X)$, i.e.,

$$X_t := x_0 + \int_0^t b(s, x_s)ds + \int_0^t \sigma(s, x_s)dW_s + \int_0^t \gamma(s, x_s)dH_s.$$

As a consequence, there also exists a unique solution to the SDE with default (4.1).

□

In fact, the proof of this theorem can be obtained as a consequence of a general result of Theorem 16.3.1 in Cohen and Elliott [19]. Now that we know that there exists a unique solution to the controlled SDE with default (4.1), we can move on to study a stochastic optimal control problem with default.

The performance functional, which we would like to maximize over all strategies $u \in \mathcal{A}$, is defined as

$$J(u) = E\left[\int_0^T h(t, X_t, u_t)dt + g(X_T)\right].$$

We assume that the functions

$$h : \Omega \times [0, T] \times \mathbb{R} \times \mathcal{V} \to \mathbb{R},$$
$$g : \Omega \times \mathbb{R} \to \mathbb{R},$$

are $\mathbb{G}$-predictable, $\mathcal{G}_T$-measurable respectively, C^1 w.r.t. x, u and admits bounded derivatives. Moreover,

$$E\left[\int_0^T h^2(t, X_t, u_t)dt + g^2(X_T)\right] < \infty.$$

We would like to derive stochastic maximum principles for this problem.

The associated Hamiltonian functional is defined by

$$\mathcal{H}(t,x,u,p,q,w) := h(t,x,u) + b(t,x,u)p + \sigma(t,x,u)q + \lambda_t \gamma(t,x,u)w, \tag{4.2}$$

where p,q,w are called the adjoint processes.

Notation 4.1. *For ease of notation, we define the following shorthand for some given control u with corresponding X*

$$b_t := b(t,X_t,u_t), \sigma_t := \sigma(t,X_t,u_t), \gamma_t := \gamma(t,X_{t^-},u_t),$$

$$h_t := h(t,X_t,u_t), \frac{\partial b_t}{\partial x} := \frac{\partial b}{\partial x}(t,X_t,u_t), \frac{\partial \sigma_t}{\partial x} := \frac{\partial \sigma}{\partial x}(t,X_t,u_t),$$

$$\frac{\partial \gamma_t}{\partial x} := \frac{\partial \gamma}{\partial x}(t,X_{t^-},u_t), \frac{\partial h_t}{\partial x} := \frac{\partial h}{\partial x}(t,X_t,u_t), \frac{\partial \mathcal{H}_t}{\partial x} := \frac{\partial \mathcal{H}}{\partial x}(t,X_t,u_t,p_t,q_t,w_t),$$

and we will use the same notations for the partial derivatives w.r.t. u.

The adjoint processes $(p,q,w) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2(\lambda)$ are given as the solution of the adjoint BSDE with default

$$dp_t = -\tfrac{\partial \mathcal{H}}{\partial x}(t,X_t,u_t,p_t,q_t,w_t)dt + q_t dW_t + w_t dM_t; \quad p_T = g'(X_T).$$

Using the definition of the Hamiltonian (4.2), the above adjoint BSDE can be rewritten as:

$$\begin{cases} dp_t = -\Big[\tfrac{\partial h_t}{\partial x} + \tfrac{\partial b_t}{\partial x} p_t + \tfrac{\partial \sigma_t}{\partial x} q_t + \lambda_t \tfrac{\partial \gamma_t}{\partial x} w_t\Big]dt + q_t dW_t + w_t dM_t; \\ p_T = g'(X_T). \end{cases} \tag{4.3}$$

Note that this adjoint equation is a linear BSDE with default, which by our assumptions of the coefficients and Theorem 2.14 in Dumitrescu et al. [28], has the following explicit solution

$$p_t = E\Big[\Gamma_{t,T} g'(X_T) + \int_t^T \Gamma_{t,s} \frac{\partial h_s}{\partial x} ds \Big| G_t\Big], \quad 0 \le t \le T, a.s.$$

where for each $t \in [0,T], (\Gamma_{t,s})_{s \in [t,T]}$ is the unique solution of the following linear SDE

$$d\Gamma_{t,s} = \Gamma_{t,s^-}\Big[\tfrac{\partial b_s}{\partial x} ds + \tfrac{\partial \sigma_s}{\partial x} dW_s + \tfrac{\partial \gamma_s}{\partial x} dM_s\Big]; \quad \Gamma_{t,t} = 1.$$

4.2.1 Sufficient stochastic maximum principle

Now, we are ready to prove the following sufficient maximum principle for optimal control of an SDE with default of the form (4.1).

Theorem 4.3. *Let $\hat{u}$ be an admissible performance strategy with corresponding solution $\hat{X}$ of the SDE* (4.1) *and the triple adjoint solution $(\hat{p}, \hat{q}, \hat{w})$ to equation* (4.3)*. Assume*

(i) The functions $x \to g(x)$ and $(x, u) \to \mathcal{H}(t, x, u, \hat{p}, \hat{q}, \hat{w})$ are concave a.s. for every $t \in [0, T]$.

(ii) For every $v \in \mathcal{V}$,

$$\max_{v \in \mathcal{V}} \mathcal{H}(t, X_t, v, \hat{p}, \hat{q}, \hat{w}) = \mathcal{H}(t, \hat{X}_t, \hat{u}_t, \hat{p}, \hat{q}, \hat{w}), \quad dt \times P \text{ a.s.}$$

Then, $\hat{u}$ is an optimal control for the stochastic optimal control problem with default.

Before we move on to the proof, note that the theorem says that if g and the Hamiltonian are concave, then we may maximize the Hamiltonian instead of the performance functional in order to find the optimal control of our problem. This essentially reduces the stochastic optimal control problem to the problem of solving the SDE (4.1) and the adjoint BSDE (4.3). The idea of the proof is to show that

$$J(u) - J(\hat{u}) \leq 0.$$

From this, the maximum principle follows.

Proof. Fix $\hat{u} \in \mathcal{A}$ with corresponding solutions $\hat{X}_t, \hat{p}_t, \hat{q}_t, \hat{w}_t$. Define the notations

$$\hat{b}_t := b(t, \hat{X}_t, \hat{u}_t), \hat{\sigma}_t := \sigma(t, \hat{X}_t, \hat{u}_t), \hat{\gamma}_t := \gamma(t, \hat{X}_{t^-}, \hat{u}_t),$$

$$\hat{h}_t := h(t, \hat{X}_t, \hat{u}_t), \mathcal{H}_t := \mathcal{H}(t, X_t, u_t, \hat{p}_t, \hat{q}_t, \hat{w}_t), \hat{\mathcal{H}}_t := \mathcal{H}(t, \hat{X}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t, \hat{w}_t),$$

$$\frac{\partial \hat{b}_t}{\partial x} := \frac{\partial b}{\partial x}((t, \hat{X}_t, \hat{u}_t), \frac{\partial \hat{\sigma}_t}{\partial x} := \frac{\partial \sigma}{\partial x}(t, \hat{X}_t, \hat{u}_t), \frac{\partial \hat{\gamma}_t}{\partial x} := \frac{\partial \gamma}{\partial x} t, \hat{X}_{t^-}, \hat{u}_t),$$

$$\frac{\partial \hat{h}_t}{\partial x} := \frac{\partial h}{\partial x}(t, \hat{X}_t, \hat{u}_t), \frac{\partial \hat{\mathcal{H}}_t}{\partial x} := \frac{\partial \mathcal{H}}{\partial x}(t, \hat{X}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t, \hat{w}_t),$$

and we introduce the same notation for their partial derivatives w.r.t. u.

Write

$$J(u) - J(\hat{u}) = A_1 + A_2,$$

where

$$A_1 := E\Big[\int_0^T \{h_t - \hat{h}_t\}dt\Big], A_2 := E\Big[g(X_T) - g(\hat{X}_T)\Big].$$

Then, from the definition of the Hamiltonian

$$\begin{aligned} A_1 &= E\Big[\int_0^T \Big\{\mathcal{H}_t - \hat{\mathcal{H}}_t - (b_t - \hat{b}_t)\hat{p}_t - (\sigma_t - \hat{\sigma}_t)\hat{q}_t - \lambda_t(\gamma_t - \hat{\gamma}_t)\hat{w}_t\Big\}dt\Big] \\ &\leq E\Big[\int_0^T \Big\{\frac{\partial\hat{\mathcal{H}}_t}{\partial x}(X_t - \hat{X}_t) + \frac{\partial\hat{\mathcal{H}}_t}{\partial u}(u_t - \hat{u}_t) - \Big(b_t - \hat{b}_t\Big)\hat{p}_t - (\sigma_t - \hat{\sigma}_t)\hat{q}_t \\ &\quad -\lambda_t(\gamma_t - \hat{\gamma}_t)\hat{w}_t\Big\}dt\Big], \end{aligned} \tag{4.4}$$

where the second equality follows because the Hamiltonian $\mathcal{H}$ is concave. Similarly from the concavity of g, we have

$$A_2 = E\Big[g(X_T) - g(\hat{X}_T)\Big] \leq E\Big[g'(\hat{X}_T)\Big(X_T - \hat{X}_T\Big)\Big] = E\Big[\hat{p}_T\Big(X_T - \hat{X}_T\Big)\Big],$$

where we have used the terminal condition of the BSDE (4.3) in the final equality. From Itô's product rule,

$$d(\hat{p}_t(X_t - \hat{X}_t)) = (X_t - \hat{X}_t)d\hat{p}_t + \hat{p}_t d(X_t - \hat{X}_t) + d[\hat{p},(X - \hat{X})]_t. \tag{4.5}$$

We have

$$d[\hat{p},(X - \hat{X})]_t = (\sigma_t - \hat{\sigma}_t)\hat{q}_t dt + (\gamma_t - \hat{\gamma}_t)\hat{w}_t dH_t.$$

Substituting this into (4.5), integrating between 0 and T and taking the expectation, we get

$$\begin{aligned} E\Big[\hat{p}_T(X_T - \hat{X}_T)\Big] &= E\Big[\int_0^T (X_t - \hat{X}_t)(-\frac{\partial\hat{\mathcal{H}}}{\partial x}(t) + \hat{q}_t dW_t + \hat{w}_t dM_t) \\ &\quad + \int_0^T \hat{p}_t\{\Big(b_t - \hat{b}_t\Big)dt + (\sigma_t - \hat{\sigma}_t)dW_t + (\gamma_t - \hat{\gamma}_t)dM_t\} \\ &\quad + \int_0^T \{\hat{q}_t(\sigma_t - \hat{\sigma}_t) + \lambda_t(\gamma_t - \hat{\gamma}_t)\hat{w}_t\}dt + \int_0^T (\gamma_t - \hat{\gamma}_t)\hat{w}_t dM_t\Big]. \end{aligned} \tag{4.6}$$

Setting

$$\begin{aligned} d\mathcal{M}_t &= \Big[(X_t - \hat{X}_t)\hat{q}_t + (\sigma_t - \hat{\sigma}_t)\hat{p}_t\Big]dW_t \\ &\quad + \Big[(X_t - \hat{X}_t)\hat{w}_t + (\gamma_t - \hat{\gamma}_t)\hat{p}_t + (\gamma_t - \hat{\gamma}_t)\hat{w}_t\Big]dM_t. \end{aligned}$$

Since $X, \hat{X}, \hat{p} \in \mathbb{S}^2, \hat{q} \in \mathbb{H}^2, \hat{w} \in \mathbb{H}^2(\lambda)$ and the conditions on $\sigma, \hat{\sigma}$ and $\gamma, \hat{\gamma}$, we get that the local martingale $\mathcal{M}$ is a martingale which has 0 mean.

By combining the expressions for A_1 and A_2 found in equations (4.4) and (4.6) respectively, we find that

$$A_1 + A_2 \leq E\Big[\int_0^T \frac{\partial\hat{\mathcal{H}}_t}{\partial u}(u_t - \hat{u}_t)dt\Big] \leq 0, \tag{4.7}$$

where the final inequality follows from the Assumption (ii) in theorem 4.3 and the Kuhn-Tucker condition.

Hence, $J(u) \leq J(\hat{u})$, so since $\hat{u} \in \mathcal{A}$ it is an optimal control.

□

4.2.2 Equivalence maximum principle

A problem with the sufficient maximum principle from the previous section is that the concavity condition is quite strict, and may not hold in applications. In this section, we derive an alternative maximum principle, called a necessary maximum principle or equivalence principle for the optimal control of the SDE with default.

In order to do this we need some additional notation and assumptions:

For all $u \in A$ and for all $\beta \in A$ bounded such that there exists $\delta > 0$ satisfying

$$u + y\beta \in A, \text{ for all } y \in [0, \delta]. \tag{4.8}$$

Fix $s \in [0, T]$ and define $\beta_t := \mathbf{1}_{[s,T]}(t)\kappa$, where κ is a bounded and $\mathcal{G}_s$-measurable random variable, the process $\beta_t \in A$.

We denote by $X_t^{u+y\beta}$ and X_t^u the corresponding solutions to $u + y\beta$ and u respectively.

Assume that for all $u, \beta \in A$ the following derivative process exists and belongs to $L^2([0, T] \times \Omega)$:

$$x_t := \left. \tfrac{d}{dy} X_t^{u+y\beta} \right|_{y=0} = \lim_{y \to 0^+} \frac{X_t^{u+y\beta} - X_t^u}{y}. \tag{4.9}$$

Remark 4.2.1. *The existence and L^2-features of these derivative process is a non-trivial issue, and we do not discuss conditions for this in our paper. We refer to Theorem 4.2 in Métivier [53], for a study of this issue in a related setting.*

Notation 4.2. *We will use the following notations:*

$$\frac{\partial b_t}{\partial x} := \frac{\partial b}{\partial x}(t, X_t^{u+y\beta}, u_t + y\beta_t), \frac{\partial \sigma_t}{\partial x} := \frac{\partial \sigma}{\partial x}(t, X_t^{u+y\beta}, u_t + y\beta_t), \frac{\partial \gamma_t}{\partial x} := \frac{\partial \gamma}{\partial x}(t, X_{t^-}^{u+y\beta}, u_t + y\beta_t),$$
$$\frac{\partial h_t}{\partial x} := \frac{\partial h}{\partial x}(t, X_t^{u+y\beta}, u_t + y\beta_t), \frac{\partial \mathcal{H}_t}{\partial x} := \frac{\partial \mathcal{H}}{\partial x}(t, X_t^{u+y\beta}, u_t + y\beta_t, p_t, q_t, w_t),$$

and we will use the same notations for partial derivatives w.r.t. u.

Now, note that from the SDE (4.1), we define the equation of the derivative process

$$dx_t = \left[\tfrac{\partial b_t}{\partial x} x_t + \tfrac{\partial b_t}{\partial u}\beta_t\right]dt + \left[\tfrac{\partial \sigma_t}{\partial x} x_t + \tfrac{\partial \sigma_t}{\partial u}\beta_t\right]dW_t + \left[\tfrac{\partial \gamma_t}{\partial x} x_{t^-} + \tfrac{\partial \gamma_t}{\partial u}\beta_t\right]dM_t; \quad x_0 = 0. \tag{4.10}$$

We remark that this derivative process is a linear SDE, then by assuming that b, σ and γ admit bounded partial derivatives w.r.t. x and u, there is a unique solution $x \in \mathbb{S}^2$ of (4.10). The following equivalence principle says that for a control to be a critical point for the performance functional J is equivalent to the critical point of the Hamiltonian.

Theorem 4.4. *The following two statements are equivalent:*

(i)

$$\frac{dJ(u+y\beta)}{dy}\Big|_{y=0} = 0. \tag{4.11}$$

(ii)

$$\frac{\partial \mathcal{H}_t}{\partial u} = 0.$$

Proof. Note that

$$\frac{dJ(u+y\beta)}{dy}\Big|_{y=0} = \frac{d}{dy}E\Big[\int_0^T h(t, X_t^{u+y\beta}, u_t + y\beta_t)dt + g(X_T^{u+y\beta})\Big]\Big|_{y=0}.$$

Define $I_1 := \frac{d}{dy}E\Big[\int_0^T h(t, X_t^{u+y\beta}, u_t + y\beta_t)dt\Big]\Big|_{y=0}$ and $I_2 := \frac{d}{dy}E\Big[g(X_T^{u+y\beta})\Big]\Big|_{y=0}$.
Since the coefficients have uniformly bounded derivatives, it follows from the dominated convergence theorem that the equality follows

$$I_1 = E\Big[\int_0^T \Big\{\frac{\partial h_t}{\partial x}x_t + \frac{\partial h_t}{\partial u}\beta_t\Big\}dt\Big].$$

Also,

$$I_2 \;=\; E\Big[g'(X_T^u)x_T\Big] = E[p_T x_T],$$

where the first equality follows by changing the order of differentiation and integration (again, using the dominated convergence theorem), the second equality follows from the adjoint equation (4.3). So, by the previous expression for dx_t (4.10) and $d[p,x]_t$, as well as the expression for dp_t from the BSDE (4.3), we obtain

$$\begin{aligned} I_2 \;=\; & E\Big[\int_0^T p_t(\frac{\partial b_t}{\partial x}x_t + \frac{\partial b_t}{\partial u}\beta_t)dt - \int_0^T x_t\frac{\partial \mathcal{H}_t}{\partial x}dt + \int_0^T q_t(\frac{\partial \sigma_t}{\partial x}x_t + \frac{\partial \sigma_t}{\partial u}\beta_t)dt \\ & + \int_0^T \lambda_t w_t(\frac{\partial \gamma_t}{\partial x}x_t + \frac{\partial \gamma_t}{\partial u}\beta_t)dt\Big]. \end{aligned}$$

By collecting the β_t- and x_t-terms and using the definition of the Hamiltonian to cancel all x_t-terms against $x_t\frac{\partial \mathcal{H}_t}{\partial x}$, we find that

$$\begin{aligned} I_1 + I_2 \;=\; & E\Big[\int_0^T \beta_t(\frac{\partial h_t}{\partial u} + p_t\frac{\partial b_t}{\partial u} + q_t\frac{\partial \sigma_t}{\partial u} + w_t\lambda_t\frac{\partial \gamma_t}{\partial u})dt\Big] \\ =\; & E\Big[\int_0^T \beta_t\frac{\partial \mathcal{H}_t}{\partial u}dt\Big], \text{ for all } \beta \in A. \end{aligned}$$

In particular, if we apply this to

$$\beta_t = \mathbf{1}_{[s,T]}(t)\kappa,$$

where κ is bounded and $\mathcal{G}_s$-measurable, we get

$$0 = E\left[\int_s^T \frac{\partial \mathcal{H}_t}{\partial u}\kappa dt\right].$$

Since this holds for all such κ (positive or negative) and all $s \in [0,T]$, we conclude that

$$0 = \frac{\partial \mathcal{H}_t}{\partial u}, \qquad \text{for a.a. } t,$$

and hence the theorem follows.

□

4.3 Application

We shall solve some examples.

4.3.1 Log-utility maximization with default

In this subsection, we illustrate the stochastic maximum principles Theorem 4.3 and Theorem 4.4 by applying them to a logarithmic utility maximization problem. As pointed out by the referee, it is also possible to solve this problem directly by using the formula of X_t and integrating by parts. Consider the cash flow process with default

$$dX_t = X_{t-}\Big[(\alpha_t - c_t)dt + \rho_t dW_t + \mu_t dM_t\Big]; \quad X_0 > 0, \tag{4.12}$$

where the coefficients α, ρ, μ are bounded, $\mathbb{R}$-valued $\mathbb{G}$-predictable processes and we assume that $\mu_t \geq -1$ for all $t \in [0,T]$ a.s. From the so-called, Doléans-Dade formula, we can write the linear SDE (4.12) explicitly, as follows

$$X_t = X_0 exp\Big(\int_0^t \Big\{\alpha_s - c_s - \frac{1}{2}\rho_s^2\Big\}ds + \int_0^t \rho_s dW_s\Big) exp\Big(-\int_0^t \mu_s\lambda_s ds\Big)\Big(1 + \mu_\tau \mathbf{1}_{\tau \leq t}\Big). \tag{4.13}$$

Since $\mu_\tau \geq -1$ and $X_0 > 0$ imply that $X_t > 0$ a.s. for each $t \in [0,T]$. Also, note that in the SDE (4.12), the control $c_t \geq 0$ corresponds to a consumption process because of its negative impact on the cash flow process X_t. The default term $\mu_t dM_t$ implies that the wealth process will grow w.r.t. μ_t until the default time τ. From that point on, μ_t has no impact on the cash flow. This may correspond to investing in a defaultable firm. The performance function we want to maximize is

$$J(c) = E\Big[\int_0^T U^1(X_t, c_t)dt + \theta U^2(X_T)\Big],$$

where U^1, U^2 are some given deterministic utility functions and $\theta := \theta(\omega) > 0$ is a $\mathcal{G}_T$-measurable, square integrable random variable which expresses the importance of the terminal value. To be able to find explicit solutions for our optimal control, we consider logarithmic utilities. Hence, the performance function is

$$J(c) = E\Big[\int_0^T \log(X_t c_t)dt + \theta \log(X_T)\Big].$$

The corresponding Hamiltonian functional, see (4.2), is

$$\mathcal{H}(t, x, c, p, q, w) = \log(xc) + x(\alpha - c)p + x\rho q + \lambda_t x \mu w.$$

The adjoint BSDE, see (4.3), has the form

$$\begin{cases} dp_t = & -\frac{\partial \mathcal{H}_t}{\partial x} dt + q_t dW_t + w_t dM_t; \\ & \\ p_T = & \frac{\theta}{X_T}, \end{cases} \qquad (4.14)$$

such that

$$\frac{\partial \mathcal{H}_t}{\partial x} = \frac{1}{X_t} + (\alpha_t - c_t)p_t + \rho_t q_t + \lambda_t \mu_t w_t. \qquad (4.15)$$

Using the first order necessary condition of Theorem 4.4, we obtain

$$\frac{\partial \mathcal{H}_t}{\partial c} = X_t p_t + \frac{1}{c_t} = 0.$$

Consequently,

$$\hat{c}_t = \frac{1}{X_t p_t}.$$

To derive an explicit expression for the optimal control, we do the following computations:

Note that by the Itô product rule,

$$\begin{aligned}
d(p_tX_t) &= p_t dX_t + X_t dp_t + d[p,X]_t \\
&= X_t[-\tfrac{\partial \mathcal{H}}{\partial x} dt + q_t dWt + w_t dM_t] \\
&+ p_t X_{t^-}[(\alpha_t - c_t)dt + \rho_t dW_t + \mu_t dM_t] \\
&+ dp_t dX_t.
\end{aligned}$$

Hence,

$$\begin{aligned}
p_T X_T - p_t X_t &= \int_t^T X_s dp_s + \int_t^T p_s dX_s + \int_t^T 1 dp_s dX_s \\
p_T X_T - p_t X_t &= \int_t^T X_s\Big(p_s(\alpha_s - c_s) - \frac{\partial \mathcal{H}_s}{\partial x} + w_s \mu_s \lambda_s\Big) ds + X_s\Big(p_s \rho_s + q_s\Big) dW_s \\
&\quad + X_{s-}\Big(p_{s-}\mu_s + w_s(1+\mu_s)\Big) dM_s.
\end{aligned} \tag{4.16}$$

By taking the conditional expectation w.r.t G_t on both sides of equation (4.16), we see that

$$\begin{aligned}
P_t X_t &= E\Big[p_T X_T \Big| G_t\Big] - E\Big[\int_t^T X_s dp_s + \int_t^T p_s dX_s \\
&+ \int_t^T 1 dp_s dX_s \Big| G_t\Big] \\
&= E\Big[p_T X_T \Big| G_t\Big] - E\Big[\int_t^T X_s[-\frac{\partial \mathcal{H}}{\partial x} ds + q_s dWs + w_s dM_s] \\
&+ \int_t^T p_s X_{s^-}[(\alpha_s - c_s)ds + \rho_s dW_s + \mu_s dM_s] \\
&+ \int_t^T 1 dp_s dX_s \Big| G_t\Big] \\
&= E\Big[\theta \Big| G_t\Big] + E\Big[\int_t^T ds \Big| G_t\Big] \\
&= E\Big[\theta + T - t \Big| \mathcal{G}_t\Big],
\end{aligned}$$

where the second to last equality follows by inserting expression for p_T from the adjoint BSDE (4.14).

Hence, an explicit expression for the stochastic optimal consumption is

$$\begin{aligned}\hat{c}_t &= \frac{1}{E\left[\theta+T-t \middle| \mathcal{G}_t\right]} \\ &= \frac{1}{E[\theta|\mathcal{G}_t]+T-t}.\end{aligned}$$

4.3.2 Exponential utility maximization

Consider a financial market which consists of two investment possibilities:

(i) Safe, or risk free asset with unit price

$$S_t^0 = 1.$$

(ii) Risky asset with unit price

$$dS_t^1 = S_{t-}^1[a_t dt + b_t dW_t + d_t dM_t].$$

We assume that a_t, b_t and d_t are $\mathbb{G}$-predictable processes with $b_t > 0$ and we assume that $d_t > -1$ for all $t \in [0,T]$ a.s.

Let π_t be a self-financing portfolio invested in the risky asset at time t which is a $\mathbb{G}$-predictable process. The corresponding wealth process $X^\pi = X$ satisfies

$$dX_t = \pi_t \left[a_t dt + b_t dW_t + d_t dM_t \right]; \quad X_0 = x_0, \tag{4.17}$$

for some given initial state x_0.

We want to maximize a performance functional of the form

$$J(\pi) = E[U(X_T)], \tag{4.18}$$

over the admissible processes $\mathcal{A}$, for some given constant $\gamma > 0$. In particular, we consider the following function

$$U(x) = -exp(-\gamma x).$$

This exponential utility is always negative, but it is increasing and convex, so it is indeed a utility function. The Hamiltonian is given by

$$\mathcal{H}(t,x,\pi,p,q,w) = \pi(ap + bq + \lambda dw), \tag{4.19}$$

and the adjoint equation becomes

$$\begin{cases} dp_t = & q_t dW_t + w_t dM_t; \\ p_T = & \gamma(exp(-\gamma X_T)). \end{cases}$$

such that

$$\frac{\partial \mathcal{H}_t}{\partial x} = 0.$$

Suppose now that π is an optimal control. Then by the necessary maximum principle, it holds for every t, P-a.s. that

$$0 = \frac{\partial \mathcal{H}}{\partial \pi}\Big(X_t, \pi_t, p_t, q_t, w_t\Big) = a_t p_t + b_t q_t + \lambda_t d_t w_t. \tag{4.20}$$

So we search for a candidate $\hat{\pi}$ satisfying

$$a_t p_t + b_t q_t + \lambda_t d_t w_t = 0. \tag{4.21}$$

We start by guessing that p has the form

$$p_t = A_t \gamma exp(-\gamma X_t), \tag{4.22}$$

for some deterministic function $A \in C^1([0,T])$ with

$$A_T = 1. \tag{4.23}$$

Using Itô's formula to find the integral representation of p and comparing with the adjoint equation (4.22), we find that the following three equations need to be satisfied:

$$0 = A'_t \gamma exp(-\gamma X_t) - A_t \gamma^2 exp(-\gamma X_t)\pi_t a_t + \frac{1}{2} A_t \gamma^3 exp(-\gamma X_t)(\pi_t^2 b_t^2 + \lambda_t \pi_t^2 d_t^2),$$

i.e.,

$$0 = A'_t - A_t \gamma \pi_t a_t + \frac{1}{2} A_t \gamma^2 (\pi_t^2 b_t^2 + \lambda_t \pi_t^2 d_t^2), \tag{4.24}$$

$$q_t = -A_t \gamma^2 exp(-\gamma X_t)\pi_t b_t, \tag{4.25}$$

$$w_t = -A_t \gamma^2 exp(-\gamma X_t)\pi_t d_t. \tag{4.26}$$

Now inserting the expressions for the adjoint processes (4.22), (4.25) and (4.26) into (4.21), the following equation need to be satisfied:

$$0 = a_t A_t \gamma exp(-\gamma X_T) - b_t^2 A_t \gamma^2 exp(-\gamma X_T)\pi_t - \lambda_t d_t^2 A_t \gamma^2 exp(-\gamma X_T)\pi_t.$$

This means that the control $\hat{\pi}$ also needs to satisfy

$$\hat{\pi}_t = \frac{a_t}{\gamma(b_t^2 + \lambda_t d_t^2)}. \tag{4.27}$$

Substituting the expressions for $\hat{\pi}$ into (4.24), we find that

$$0 = A_t' - A_t \frac{a_t^2}{2(b_t^2 + \lambda_t d_t^2)}.$$

This is 0 if and only if

$$A_t' = \frac{a_t^2}{2(b_t^2 + \lambda_t d_t^2)} A_t; \quad A_T = 1,$$

i.e.,

$$A_t = exp\Big(\int_t^T \frac{a_s^2}{2(b_s^2 + \lambda_s d_s^2)} ds \Big); \quad A_T = 1.$$

Our computations show that $\hat{\pi}$ given by (4.27) satisfies all the conditions of the sufficient maximum principle (Theorem 3.2) and therefore we have proved the following:

Theorem 4.5. *The optimal portfolio for the problem to maximise the performance* (4.18) *is given by* (4.27).

Conclusion and perspective

In this thesis, we introduce a new method of stochastic integration for no-adapted processes w.r.t fractional Brownian motion by choosing a hurst parameter greater than $\frac{1}{2}$.

Another way, we study a controlled stochastic differential equation driven by Brownian motion and a single jump martingale, we derive sufficient and necessary maximum principles for a stochastic optimal control problem, we apply the maximum principles to solve some examples.
The study established in this thesis offers different perspectives, let's mentioned for instance:

- the role of stochastic maximum principles to solve stochastic optimal control problems.
- In the future, we will study rough path theory to eliminate the problem of irregular trajectories.

Appendix A

Notions and proprieties of fractional brownian motion

A.1 Fractional brownian motion

Definition A.1.1. *A Gaussian process $B_t^{(H)} = \{B_t^{(H)}(t), t \geq 0\}$ is called fractional brownian motion (fbm) of hurst index $H \in (0,1)$ if it has mean zero with covariance function*

$$R_H(t,s) = E[B^{(H)}(t)B^{(H)}(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

By definition, fbm is a Gaussian process, and therefore it is strictly characterized by its mean and covariance. Its mean by definition is zero and covariance given by $R_H(t,s)$ Hence the following three properties are obtained through $R_H(t,s)$ in Definition A.1.1.

1. Self-similarity: The process $\{a^{-H}B^H(at), t \geq 0\}$ has the same law as $\{B^H(t), t \geq 0\}$, i.e. $a^{-H}B^H(at) \sim B^H(t)$.

2. Stationary increments: $B^H(t+s) - B^H(s) \sim B^H(t)$ for $s, t \geq 0$.

3. Variance: $E[B^H(t)^2] = t^{2H}$ for all $t \geq 0$.

There are an other representations of the fbm as a Wienner's integral. taken $[0,T]$. Fbm $(B_t^{(H)})_{0 \leq t \leq T}$ has defined by general formula:

$$B_t^H = \int_0^t K_H(t,s) dB_s, \qquad t \in [0,T],$$

where $(B_t)_{0\leq t\leq T}$ is one-sided standard brownian motion satisfying the conditions (a) and (b)-Chapter 1.

A.2 Lévy-Hida Representation:

Following Decrensfond and $\ddot{U}$stunel in [25], this Kernel is done as

$$K_H(t,s) = \frac{(t-s)_+^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-\frac{t}{s})\, 0 < s < t < \infty,$$

and F is Gauss hypergeometric function.

Generally, remarking that we have:

$$R_H(t,s) = \int_0^{s\wedge t} K_H(t,u)K_H(s,u)du,$$

where, $R_H(t,s)$ is the covariance function.

Case $H \in (\frac{1}{2}, 1)$

Proposition A.2.1. *[37] For $H \in (\frac{1}{2}, 1)$, the Kernel function is given by*

$$K_H(t,s) = C_H s^{\frac{1}{2}-H} \int_s^t |u-s|^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where

$$C_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}},$$

with β is the beta function:

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{1-b} dt.$$

Corollary A.1. *[37] Besides, we have*

$$R_H(t,s) = (\varpi_1(H))^2 \int_0^t \{r^{\frac{1}{2}-H}(I_{T-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} \mathbf{1}_{[0,t)(u)})(r)(r^{\frac{1}{2}-H}(I_{T-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} frm[o]--_{[0,s)(u)})(r))\} dr,$$

ϖ_1 is defined by:

$$\varpi_1 = \left(\frac{\Gamma(H-\frac{1}{2})^2 H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}},$$

hence, this Kernel can be rewritten as

$$K_H(t,s) = \varpi_1(H) s^{\frac{1}{2}-H}(I_{T-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} frm[o]--_{[0,t)}(u))(s).$$

Theorem A.2. *[37] The representation of a fbm for $H \in (\frac{1}{2}, 1)$ over a finite interval is*

$$B_t^{(H)} = \int_0^t K_H(t,s) dWs, \quad s,t \in [0,T],$$

where $(W_t)_{t\in[0,T]}$ is a particular Wienner process.

Lévy-Hida approach:[37]

By the above condition in last proposition, we have

$$\frac{\partial K_H(t,s)}{\partial t} = C_H(\frac{t}{s})^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}.$$

Thus, a linear operator $K_H^* : \mathcal{E} \longrightarrow \mathbb{L}^2[a,b]$ is given by:

$$(K_H^*\phi)(s) := \int_s^T \phi(t) \frac{\partial K_H(t,s)}{\partial t} dt,$$

where, $\phi \in \varepsilon$.

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