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Par :

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Intégration stochastique par rapport à des processus Gaussiens et application aux modèles de volatilité stochastique fractionnaire mixte.

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This modest work is dedicated to:

The memory of my dear grandmother.

My loving parents.

My sisters and my brother.

All my family.

All my friends.

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Abstract

The objective of this thesis is to model volatility by fractional Gaussian processes with long memory and irregular trajectories. We use high-frequency data to estimate the regularity of the log-volatility paths. We consider the mixed fractional Brownian motion with $H < \frac{1}{2}$ as a stochastic volatility model and construct a stationary mixed fractional Ornstein-Uhlenbeck process as a stationary model of log-volatility. Further, we establish a fundamental result on the integration of non-adapted processes with respect to fractional type processes (sub-fractional Brownian motion, mixed fractional Brownian motion) when $H > \frac{1}{2}$ as a Riemann sum with an appropriate choice of sub-interval evaluation points by decomposing the integrand processes to a linear combination of adapted and instantly independent processes. This study is considered as a generalization of what has been given in the Brownian framework. We prove that our anticipating integrals are near-martingales under some conditions.

Keywords: Gaussian processes, mixed fractional Brownian motion, sub-fractional Brownian motion, non-adapted process, near martingale, stochastic volatility.

Résumé

L'objectif de cette thèse est de modéliser la volatilité par des processus Gaussiens fractionnaires à mémoire longue et à trajectoires irrégulières. Nous utilisons des données à haute fréquence pour estimer la régularité des trajectoires de la log-volatilité. Nous proposons le mouvement Brownien fractionnaire mixte avec $H < \frac{1}{2}$ comme un modèle de volatilité stochastique et nous construisons un processus d'Ornstein-Uhlenbeck fractionnaire mixte stationnaire comme modèle stationnaire de la log-volatilité. Par ailleurs, nous avons démontré un résultat fondamental sur l'intégration des processus non adaptés par rapport aux processus de type fractionnaire (mouvement Brownien sous fractionnaire, mouvement Brownien fractionnaire mixte) lorsque $H > \frac{1}{2}$ comme une somme de Riemann avec un choix approprié de points d'évaluation de sous-intervalles tout en décomposant le processus intégrant à une combinaison linéaire des processus adaptés et instantanément indépendants. Cette étude est considérée comme une généralisation du celle établie dans le cadre Brownien. De plus, sous certaines conditions, nous prouvons que nos intégrales anticipées sont des près-martingales.

Mots clés : Processus gaussiens, mouvement Brownien fractionnaire mixte, mouvement Brownien sous-fractionnaire, processus non-adapté, près-martingale, volatilité stochastique.

الملخص

الهدف من هذه الأطروحة هو نمذجة التقلب العشوائي من خلال عمليات غوسيان الكسرية ذات الذاكرة الطويلة والمسارات غير المنتظمة. نستخدم بيانات عالية التردد لتقدير انتظام مسارات لوغاريتم التقلب العشوائي. نقترح الحركة البراونية الكسرية المختلطة كنموذج تقلب عشوائي ونقدم عملية أورنشتاين أولنبك لمّا $H<1/2$ كنموذج ثابت للوغاريتم التقلب العشوائي. علاوة على ذلك، أظهرنا نتيجة أساسية لتكامل العمليات غير المكيفة بالنسبة لعمليات غوسيان من النوع الكسري (الحركة البراونية الكسرية المختلطة، الحركة البراونية الكسرية الجزئية) لمّا $H > ½$ كمجموع ريمان مع الاختيار المناسب لقيم النقاط في المجالات الفرعية وهذا من خلال تفكيك عملية الدمج إلى مجموعة خطية من العمليات المتكيفة والمستقلة على الفور. تعتبر هذه الدراسة بمثابة تعميم لتلك التي تم وضعها في الإطار البراوني. بالإضافة إلى ذلك، في ظل شروط معينة، نثبت أن التكاملات المتوقعة لدينا قريبة من مارتينغال.

الكلمات المفتاحية: عمليات غوسيان، الحركة البراونية الكسرية المختلطة، الحركة البراونية الكسرية الفرعية، العملية غير المكيفة، القرب من مارتينغال، التقلب العشوائي.

List of works

Publications:

1. A. Belhadj, A. Kandouci and A.A. Bouchentouf, Stochastic integral for non-adapted processes related to sub-fractional Brownian motion when $H > 1/2$, Bulletin of the Institute of Mathematics Academia Sinica (New Series), pp. 165-176, Vol. 16,No. 2, 2021.

2. A. Belhadj, A. Kandouci and A.A. Bouchentouf, An anticipating stochastic integral with respect to mixed fractional Brownian motion, Acta Univ. Sapientiae, Mathematica, accepted, 2022.

3. A. Belhadj, A. Kandouci and A.A. Bouchentouf, Stochastic volatility Modeling via mixed fractional Brownian motion, submitted, 2021.

Communications:

1. Stochastic integration with respect to Gaussian processes that are not semi-martingales: case of sub-fractional Brownian motion,1^{st} edition of the National Doctorials of Mathematics, Constantine, October 27 to 31, 2017.

2. Fractional stochastic volatility models,congress of algerian mathematicians, M'Hamed Bougara university of Boumerdes, May 12 and 13, 2018.

3. A new approach of stochastic integral with respect to sub-fractional Brownian motion, 9th edition of the conference Trends in Mathematical Applications in Tunisia Algeria Morocco, Tlemcen, February, 23 to 27, 2019.

4. Stochastic volatility model driven by a sub-fractional Brownian Motion, International conference on Financial Mathematics: Tools and Applications, Bejaia, October 28 and 29, 2019.

5. Numerical approximation of stochastic differential equations driven by a sub-fractional Brownian motion, Meeting of Mathematical Analysis and Applications RAMA11, Sidi Bel Abbès, November 21 to 24, 2019.

6. Simulations of fractional and sub-fractional stochastic differential equations, International colloquium on stochastic and statistical modeling MSS'19, USTHB-Algiers, November 24 to 26, 2019.

7. Stochastic integration of non adapted process related to sub-fractional Brownian motion, Potential Theory Seminar, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania, October 12, 2021.

8. Mixed fractional Brownian motion as a stochastic volatility model, SPSR Young Researchers Workshop, Bucharest, Romania, November 19, 2021.

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Research events

1. Stochastic modeling and applications, the eighth spring school of the Euro-Mediteranean Research Center for Mathematics and its Applications (CREMMA8), National School of Engineers of Tunis, Tunisia, April 23 to 27, 2018.

2. Workshop EMAD, Redaction: methods and tools, workshop on latex and zotero, M'Hamed Bougara University of Boumerdes, May 28, 2018.

3. Statistical inference and information theory for Markovian and semi-Markovian processes, Laboratory of Stochastic Models, Statistics and Applications, Saida, October 23 to 28, 2018.

4. Support days for doctoral students, Colonel Akli Mohand Oulhadj University of Bouira, December 09 and 10, 2018.

5. Doctorial of operational research, LAMOS, Abderrahmane Mira University of Bejaia, December 12 and 13, 2018.

6. CIMPA'ASA-2019, Stochastic Analysis and Applications, Dr. Moulay Tahar University of Saida, March 01 to 09, 2019.

7. IMAR monthly lectures conference series, Spatial evolution of an epidemic and social networks, November 04, 2021.

Presentations in Laboratory

1. On the sub-mixed fractional Brownian motion, May 13, 2017.

2. Stochastic volatility models in financial Markets, December 12, 2017.

3. Thesis progress state, Jun 13, 2018.

4. Thesis progress state, October 8, 2019.

5. Anticipating stochastic integral with respect to mixed fractional Brownian motion and application in mixed fractional stochastic volatility models, February 19, 2022.

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Introduction

Volatility modeling has received a lot of attention in the literature since the introduction of Black and Scholes model in 1973 [13]. The model describes the dynamic of price of derivative instrument in financial markets as a diffusion stochastic process under the form:

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t), \qquad t \in [0, T].$$

where μ_t is a drift term and W_t is a standard Brownian motion. The term σ_t denotes the volatility process and is the most important parameter in the model. In this model, the volatility is considered as a constant, However, such specification of volatility σ_t is inadequate with observed prices for European options. To overcome this problem, few researchers introduced several volatility models. Dupire [35], Derman and Kani [31] proposed the so-called local volatility $\sigma(t, S_t)$ which considered that volatility is a deterministic function of time and asset price but still insufficient to describe the real dynamic of price. Stochastic volatility models are the most realistic way to describe the behavior of the volatility process of an underlying asset as a diffusion process. Hull and White [47] where the first authors who proposed the use of stochastic volatility models, this model describes the dynamic of asset price S_t as follows:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t,$$

$$d\sigma_t^2 = \theta \sigma_t^2 dt + \gamma \sigma_t^2 dB_t,$$

where σ_t denotes the stochastic volatility of price S_t and W_t and B_t, $t \in [0,T]$ are two standards Bm with correlation coefficient $\rho \in (-1,1)$ and μ, θ, γ are constants.

Scott [95] assumed that the volatility follows a mean-reverting Ornstein-Uhlenbeck process with an independence between the asset price and the volatility in order to compute the prices of options. Stein and Stein [98] adopted the same model but with a correlation between the asset price and the volatility. Hagan et al.[43] introduced SABR model which is a stochastic version of elasticity of variance (CEV) model. Heston [44] suggested that the variance follows a Cox-Ingersoll-Ross (CIR) interest model. This model enjoys success since it is possible to deduce a closed formula for the price of an European call (put) options[1]. Notice that all this models are diffusion processes driven by a standard noises.

Andersen and Bollerslev [2], Andersen et al. [4] and Ding et al. [33] observed the presence of long memory in volatility process through statistical analysis, this feature became later a stylized fact in financial mathematics. Many authors proposed long memory model based on a fractional Brownian motion, indeed the fractional Brownian motion $B_t^H, t \geq 0$ with Hurst index $H \in (0,1)$ offers a good formulation of this long memory property. A few studies related to fractional stochastic volatility models have been published (Comte et al. [24], Comte and Renault [25], Rosenbaum [93] and so on), where the volatility is modeled by a stochastic differential equation driven by a fractional Brownian motion. Specifically, Comte and Renault [25] suggested that the log-volatility is a fractional Ornstein-Uhlenbeck process with Hurst index $H > \frac{1}{2}$ to capture the long memory. However, the analysis of Fukasawa [39] show that the choice of $H > \frac{1}{2}$ is inconsistent with the term of volatility skew for short expirations. Another extension has been studied in Corlay et al.[27] where the volatility process is driven by a multifractional Brownian motion. Recently, Gatheral et al. [42] proposed a rougher fractional volatility model with

[1] An option is a contract that gives its holder (buyer or seller) the right but not the obligation) to buy (call option) or sell (put option) a quantity of assets at a specified price K called the strike price (or exercise price) and at a specified date T called the maturity. An option which may be exercised at any time before the maturity is an European option while an option which may be exercised only at the expiration date of the option is an American option.

a Hurst index $H < \frac{1}{2}$. This model permits to generate the term of observed volatility skew and also to reproduce the observed regularity of the volatility process. We note that even if this model has not long memory property, the authors proved that it is consistent with time series data of volatility, what solidified the stylized fact that volatility is long memory.

The so-called fractional Brownian motion was introduced as an appropriate generalization of standard Brownian motion parameterized by a Hurst index $H \in (0,1)$ with features that provide its utility as a suitable model in many applications including finance, traffic internet, turbulence, geophysics, etc. This process is considered as the most known process exhibiting the properties of self-similarity, long-range dependence and stationarity of increments, it was introduced first by Kolmogorov [58] in 1940, in order to study spiral curves in Hilbert space, and studied later in the famous paper of Mandelbrot and Van Ness [73] in 1968 that shed light on its properties such as its integral representation with respect to standard Brownian motion. Note that for $H = \frac{1}{2}$, the fBm coincides with the standard Brownian motion.

An extension of Brownian motion is the so-called mixed fractional Brownian motion (mfBm), this process is is a linear combination of fractional Bm and independent standard Bm, such mixture has been firstly introduced by Cheridito [20] to present an interesting stochastic model of the discount stock price in some arbitrage-free and complete financial markets under the form $M = B + \alpha B^H$, where B is a standard Brownian motion independent of the fBm B^H. Zili [109] considered a more general form of mixed fBm as $M^H = aB^H + bB$ and studied some stochastic properties and characteristic of this process. Note that, for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$, the process M^H is not a semimartingale, when $H > \frac{3}{4}$ is equivalent in distribution to bB (Cheridito [20]), and for $H < \frac{1}{4}$, M^H is equivalent in distribution to aB^H (Van Zanten [108]).

A more general self-similar Gaussian process, considered as an intermediate between standard Brownian motion and fractional Brownian motion, has been introduced in Bojdecki et al.[14], and appeared in Dzhaparidze and Van Zanten [36] as the even part of fractional Brownian motion. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. The so called sub-fractional Brownian motion $(S_t^H; t \geq 0)$ with a Hurst parameter $H \in (0,1)$ is a centered Gaussian process. The sub-fBm preserves many properties of fBm (self-similarity, long-range dependence, and Hölder paths) unless the stationarity of increments and its long range dependence decays faster than that of fBm. Similarly to fBm, the case $H = \frac{1}{2}$ also corresponds to standard Brownian motion.

Note that the fBm and sub-fBm are neither Markov processes nor a semimartingales for $H \neq \frac{1}{2}$. This fact is considered as a limitation for applying classical stochastic calculus developed by Itô; i.e, to give a sense for $\int_0^t u_s dX_s$, the integrator X must be a semimartingale which is not the case for B^H and S^H. Therefore, many techniques from classical stochastic analysis are not available when dealing with this processes, for this reason various approaches have been proposed.

In the case of fractional Brownian motion, the simplest approach is the wiener approach which discusses the problem of integration of deterministic function with respect to fractional Brownian motion. Pipiras and Taqqu [86] defined the families of integrand corresponding to the Wiener integral via fractional integrals and derivatives. For $0 < H < \frac{1}{2}$, the entire domain is defined whereas for $\frac{1}{2} < H < 1$, a class of functions that are subset of the integral domain are determined due to the non-completeness of the whole domain. In [87], the authors discussed the question of completeness of the class of deterministic integrand on an interval. After that, Jolis [54] treated the problem of Wiener integral with respect to fBm for every $H \in (0,1)$.

When the integrand is a not deterministic function, various approaches have been proposed in order to define $\int_0^t X_s dB_s^H, s \in [0,t]$. Due to the results of Young [106] applied to fractional Brownian motion, the pathwise Riemann-Stieltjes integral $\int_0^t X_s(\omega) dB_s^H(\omega), s \in$

$[0, t]$ exists for all $\omega \in \Omega$ as a Riemann sums, specifically, in the case where $H > \frac{1}{2}$, the pathwise integral $\int_0^t f(B_s^H)dB_s^H$ exists and the change of variables formula $f(B_t^H) = f(0) + \int_0^t f'(B_s^H)dB_s^H$ holds for all continuously differentiable function f. We refer to Zähle [107], Dudley and Norvaisa [34], Feyel and Pradelle [38] and Mikosch and Norvaisa [77] for more details about pathwise integral. In the case where $H < \frac{1}{2}$, the pathwise integral may not exists. Lyons [70] introduced the theory of rough paths analysis to define a pathwise approach to the stochastic integrals of the form $\int_0^t f(B_s^H)dB_s^H$ in the case where $\frac{1}{4} < H < \frac{1}{2}$ (see also Coutin and Qian [28]). The inconvenient of these pathwise integrals is that the mean is not null and it is not easy to get a formula for the variance.

In the case of sub-fractional Brownian motion, Tudor [101] characterized the Wiener integral's domain with respect to S^H for all $H \in (0, 1)$. After that, Shen and Chen [96] defined a stochastic integral with respect to sub-fBm S^H with $H < \frac{1}{2}$. This extends the divergence integral from Malliavin calculus. In addition, like the fBm, the pathwise Riemann-Stieltjes integral $\int_0^t u_s(\omega)dS_s^H(\omega)$ exists for a stochastic process u with β-Hölder continuous trajectories, where $\beta > 1 - H$ and the sub-fBm is Hölder continuous of order γ for any $\gamma < H$ on any finite interval (see Young [106]).

As we mentioned above, the fact that the fractional type Gaussian process are not semimartingales limited the use of the classical itô stochastic calculus. Other limitation of classical Itô integral is the adaptedness of the integrand to the natural filtration of integrator ie, $\int_0^t Y_s dX_s$ is well defined if and only if Y_t is adapted to the filtration $\mathcal{F}_t$ $=\sigma\{X_t, t \geq 0\}$. Then, the integral is defined like Riemann sums at which the evaluation points are the left endpoints of subintervals. The problem of non-adaptedness of the integrand has been discussed by Itô [52], where he raised the question how to define

$$\int_0^t B(1)dB(s), \qquad 0 \leq t \leq 1, \tag{1}$$

since $B(1)$ is not $\mathcal{F}_t$-adapted. Itô proposed to enlarge the filtration by considering $\mathcal{G}_t$ the field generated by $\mathcal{F}_t$ and $B(1)$, ie $\mathcal{G}_t = \sigma\{\mathcal{F}_t, B(1)\}$, $B(1)$ is adapted to $\mathcal{G}_t$ and B_t is a $\mathcal{G}_t$-quasimartingale. Therefore, the integral (1) may be defined as a stochastic integral with respect to quasimartingale.

There have been several extensions in the literature on anticipating stochastic integration. Let's cite for instance Hitsuda [46], Skorokhod [97] and serval works treating the anticipating integrals and their applications, see Buckdahn [18], León and Protter [67], Pardoux and Protter [85], and the references therein.

Ayed and Kuo [6] proposed a new viewpoint for defining this kind of integrals by decomposing the anticipating stochastic integrand into a linear combination of the products of instantly independent and adapted stochastic processes. Then, authors defined a stochastic integral of the product of an adapted process and instantly independent process as a Riemann sum thanks to the classical definition of stochastic integral proved in Kuo [60]. Notice that the evaluation points are the left endpoints of subintervals for the first process and the right endpoints for the second. Motivated by this new approach, many authors developed different studies. Itô formula of anticipating integral proved in Kuo and Ayed [6], was generalized to different cases in Kuo et al.[62, 61, 49]. The study of a class of stochastic differential equations with anticipating initial conditions was treated in Khalifa et al.[57]. The Itô isometry based on the new integral for anticipating processes was discussed by Kuo et al. [64]. The near-martingale property of anticipating stochastic integral introduced in Kuo et al.[63] was studied recently in Hwang et al. [50] and Hibino et al. [45].

Contribution of the thesis

In this thesis, we aim to define a stochastic integral of the two anticipating integral $\int_0^T U_t dS^H(t)$ and $\int_0^T U_t dM^H(t)$, $t \in [0,T]$, $H > \frac{1}{2}$, to be the limit of the corresponding Riemann sum, where the anticipating process U_t is a product of an instantly independent process $g(B(T) - B(t))$ and an adapted process $f(B(t))$. Furthermore, we show that our stochastic integral admits the near-martingale property under some conditions. In the case of mixed fBm, we give some examples of anticipating integral in the specific case when $H > \frac{3}{4}$.

In addition, we propose a mixed fractional volatility model which extends the rough stochastic volatility model given in Gatheral el al. [42], We show through empirical experiments that the measured smoothness of log-volatility is similar to that of mixed fractional Brownian motion when $0.09 \leq H \leq 0.2$. We propose to model the log-volatility by a mixed fractional Brownian motion with $H < \frac{1}{2}$ and we construct a stationary mixed fractional Ornstein-Uhlenbeck process:

$$dX_t = -\lambda(X_t - \mu)dt + \gamma dM_t^H, t \in [0,T],$$

as a stationary model of log-volatility X_t, and we show that it tends in distribution to the mixed fractional Brownian motion when $\lambda \to 0$.

Outline of thesis

This thesis is organized as follows: we recall in the first chapter the basic background on fractional, sub-fractional and mixed fractional Brownian motions, as well as, the fractional-Ornstein Uhlenbeck process. We present in the second chapter the anticipating stochastic integral introduced by Ayed and Kuo [6] based on the decomposition of the integrand, we present also the near martingale property and we show that the anticipating integral admits this property. Then, we give our main result by defining anticipating stochastic integrals with respect to sub-fractional and mixed fractional Brownian motions, we show that our stochastic integral admits the near-martingale property under some conditions. In the case of mixed fBm, we give some examples of anticipating integral in the specific case when $H > \frac{3}{4}$.

The third chapter is consecrated to the financial modeling, where we give a review of the volatility notion in financial markets and its different classes: constant volatility, local volatility and stochastic volatility. Finally, we present the details of the construction of our mixed fractional stochastic volatility model. We show via empirical studies that the smoothness of the increment of log-volatility and those of mixed fBm are similar when $H < \frac{1}{2}$. We suggest to model the volatility process with a mixed fractional Ornstein-Uhlenbeck process, where $H < \frac{1}{2}$.

Chapter 1

On fractional Gaussian processes: Background and definitions

Gaussian processes enjoy success as a crucial tool various probabilistic, statistical, financial or machine learning phenomena, because of their useful properties derived from the Gaussian distribution. These latters can be seen as an infinite generalizations of multivariate normal variables distributions. A Gaussian process is characterized by a mean vector and covariance matrix. These are key elements that controls its properties.

Motivated by its interest in applications specifically in internet traffic modeling and finance, stochastic integration with respect to Gaussian processes has attracted considerable attention since last century. The classical theory of stochastic integration was developed by Wiener [104] in 1923, Itô [51] in 1944 and Lévy [68] in 1948. The classical Brownian motion (Bm in short) plays a crucial role in this theory, due to its tractability and the easiness of making the stochastic calculus when dealing with classical Bm, as well as its considerable success in modeling several random problems in various fields. We note that these theories are reduced by the use of stochastic process which are semi-martingales as integrators. Indeed, we say that a process $(X_t)_{t\in[0,T]}$ is a semi-martingale for a given filtration $(\mathcal{F}_t)_{t\in[0,T]}$ if it is a càdlàg (right continuous with left limits), adapted process and it can be decomposed as:

$$X_t = X_0 + M_t + A_t,$$

where M_t is a local martingale and A_t is an adapted càdlàg process with finite variation.

In recent years, several random phenomena are not compatible with the criteria of Brownian motion. This fact provides the use of largest class of stochastic process that are not semi-martingales. For instance, in finance lots of evidence proposed that many financial products like asset returns, interest rates and volatility have a long memory. This feature cannot be described using a classical Bm that is a Markov process.
Starting form this fact, the interest in fractional Gaussian processes has mostly increased due to applications in such fields as hydrology, economics, telecommunications and finance. The so-called fractional Brownian motion (fBm) is the best known and most used fractional Gaussian process. The fBm is an appropriate generalization of standard Brownian motion. Though, unlike regular Brownian motion, fBm has dependent increments, which means that the current "step" of a fBm is dependent on previous "steps". This dependence is measured on a scale from zero to one and this measure is called the Hurst index, $H \in (0, 1)$, named in honor of the hydrologist Harold Edwin Hurst, for his work in the field of hydrology. In 1951, Hurst [48] studied the yearly variance in levels of the Nile river and applied this to the so called R/S statistic, where R is the range of partial sums of the data and S is the sample standard deviation. The R/S statistic should grow like $n^{\frac{1}{2}}$ under normal assumptions of independent and identically distributed observations and finite variance, where n is the sample size. Interestingly enough, the Nile data indicated growth of n^H, where $H \in (\frac{1}{2}, 1)$. Random walk typically yields a growth of $n^{\frac{1}{2}}$, and the scaling limit of random walk in dimension one is Brownian motion.

Hence, it must be the case that the growth n^H, with $H \in (\frac{1}{2}, 1)$ corresponds to something else. Mandelbrot [74] noticed that while Brownian motion has standard deviation $t^{\frac{1}{2}}$, fractional Brownian motion has a standard deviation t^H, where $H \in (0, 1)$, and thus fBm might be a more appropriate fit for this behavior. The Hurst index describes the self-similarity, the long range dependence and the smoothness of the path of the fBm.

In 1940, it was Andrei Kolmogorov [58], while studying spiral curves in Hilbert space, who first introduced fractional Brownian motion. In 1968, Mandelbrot and Van Ness recognized fBm's significance. They derived many of its important properties in their famous paper [73]. In that paper, fractional Brownian motion was named. This comes from its representation as a fractional stochastic integral with respect to Brownian motion.

Since then, several authors proposed some extensions of this process which preserve many properties of fBm. For instance, Bojdecki et al.[14] introduced a rather special class of fractional Gaussian processes preserving many of its properties. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. This process is called the sub-fractional Brownian motion. Cheridito [20] introduced another extension to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets as a linear combination of independent standard and fractional Brownian motions. We present in this chapter a background about of the fractional, sub-fractional, mixed fractional Brownian motions and fractional Ornstein-Uhlenbeck process.

1.1 Fractional Brownian motion

The fractional Brownian motion (fBm) is a suitable generalization of standard Brownian motion, it is the most known process which is not a semi-martingale. It is the only Gaussian self similar stationary process with long-range dependance property. Due to these interesting properties it enjoyed success as a modeling tool in many field of applications including telecommunications, turbulence and finance, the demand to stochastic calculus with respect to fBm are raised. This process was introduced by Kolmogorov [58] and studied later by Mandelbrot and Van Ness [73] who provided an integral representation of fBm with respect to a standard Brownian motion over a real line time interval.

Definition 1.1.1. *The fractional Brownian motion* $B^H = \{B_t^H, t \geq 0\}$ *with Hurst index* $H \in (0,1)$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a continuous centered Gaussian* *process, starting from zero, with covariance*

$$R^H(s,t) = \frac{1}{2}[t^{2H} + s^{2H} + |t-s|^{2H}], \qquad s,t \geq 0,$$

verifying:

- $B_0^H = 0, a.s,$
- $\mathbb{E}(B_t^H)^2 = t^{2H},$
- B^H *has a stationary increments.*

Remark 1.1.1. *We have:*

- *For* $H = \frac{1}{2}$, $R^{\frac{1}{2}}(s,t) = t \wedge s$, *then* B^H *restricts to a standard Brownian motion.*
- *For* $H = 1$, B_t^H *and* ηt *have the same probability distribution, where* $\eta \to \mathcal{N}(0,1)$.

We state some main properties of the fBm, Their proofs can be found in many sources such as Mishura [79], Nourdin [82], Mishura and Zili [78], Biagini el al. [11], Nualart [84] and other references.

1.1.1 Basic properties

1. B^H is self-similar with index of similarity equal to H. It means that for all $a > 0$, $\{B_{at}^H, t \geq 0\}$ and $\{a^H B_t^H, t \geq 0\}$ have the same probability distribution.

2. The increments of B^H are stationary. It means that $\forall t > 0$, $\{B_t^H - B_s^H, s \geq 0\}$ has same distribution as $\{B_s^H, s \geq 0\}$, and the second moment of increments is given by
$$E[|B_t^H - B_s^H|^2] = |t-s|^{2H}, \quad s,t > 0.$$

3. The sample paths of B^H are almost surely Hölder continuous of order γ for all $\gamma < H$. This fact follows from Kolmogorov-Centsov criteria and the fact that for any $\alpha > 0$, we have
$$E[|B_t^H - B_s^H|^\alpha] = C_\alpha |t-s|^{2H},$$

where $C_\alpha = \mathbb{E}(|B_1^H|^\alpha)$.

4. The sample paths of B^H are nowhere differentiable. Indeed for every $t_0 \geq 0$, we have

$$\mathbb{P}\left(\limsup_{t_0 \to 0} \left|\frac{B_t^H - B_0^H}{t - t_0}\right| = \infty\right) = 1.$$

5. S^H is neither a semimartingale nor a Markov process when $H \neq \frac{1}{2}$.

1.1.2 Long and short range dependence

Definition 1.1.2. *A stationary sequence $\{X_n, n \in \mathbb{N}\}$ exhibits long-range dependence or short range dependence if $\rho(n) = Cov(X_1, X_n)$ satisfies:*

$$\sum_{n=1}^{\infty} \rho(n) = \infty \quad or \quad \sum_{n=1}^{\infty} \rho(n) < \infty,$$

respectively.

Remark 1.1.2. *If a stationary sequence $\{X_n, n \in \mathbb{N}\}$ is long-range dependent, then the covariance function slowly decays as a power law when n tends to infinity in the sense that*

$$\lim_{n \to \infty} \frac{\rho(n)}{cn^{-\alpha}} = 1,$$

where c is a constant and $\alpha \in (0, 1)$.

Proposition 1.1.1. *(Tudor [102]). A fractional Brownian motion is long range dependence if $H > \frac{1}{2}$ and short range dependence if $H < \frac{1}{2}$*

Proof. Let $X_1 = B_1^H - B_0^H$ and $X_n = B_n^H - B_{n-1}^H$. Then, we have

$$\begin{aligned}\rho(n) &= Cov(X_1, X_n)\\ &= \mathbb{E}(B_1(B_n^H - B_{n-1}^H))\\ &= \frac{1}{2}\big[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\big]\\ &= 2H(2H-1)n^{2H-2} + o(n^{2H-2}).\end{aligned}$$

We see that $\rho(n)$ is a general term of divergent series if and only if $2H - 2 > 0$, i.e $H > \frac{1}{2}$

It is clear that if $H = \frac{1}{2}$, $\rho(n) = 0$ for all $n \in \mathbb{N}^*$ and then, the increments are independents. In other hand we see that $\rho(n) < 0$ for $H < \frac{1}{2}$ and $\rho(n) > 0$ for $H > \frac{1}{2}$. Hence the increments of fractional Brownian motion are:

- Positively correlated if $H > \frac{1}{2}$,
- Negatively correlated if $H < \frac{1}{2}$,

The following graph present $\rho(n)$ in function of n and H.

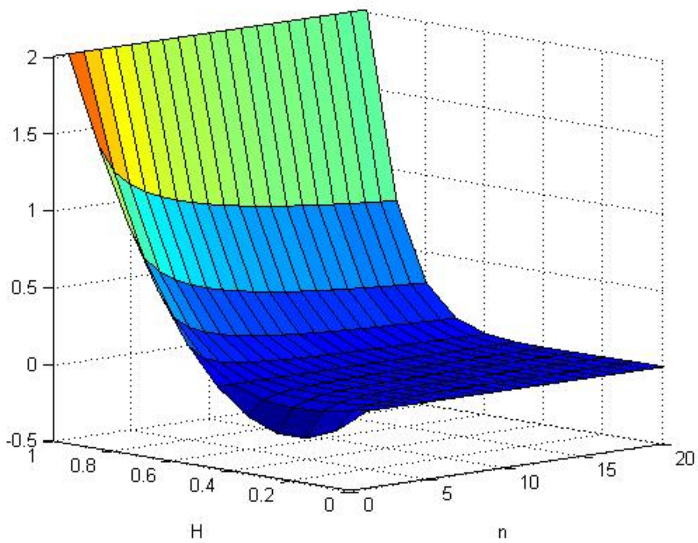

Figure 1.1: Graphical representation of $(H, n) \rightarrow \rho(n)$.

1.1.3 Markov property

Lemma 1.1.1. *(Revuz and Yor [90]). If X is a Gaussian centered Markovian process, then for all $s < t < u$*

$$\mathbb{E}(X_t X_s)\mathbb{E}(X_t X_u) = \mathbb{E}(X_t X_t)\mathbb{E}(X_u X_s).$$

Theorem 1.1.1. *(Tudor [102]). The fractional Brownian motion B^H of Hurst index $H \in (0,1)/\frac{1}{2}$ is not a Markov process.*

Proof. Assume that B^H is a Markov process. Since it is a Gaussian process we have, for $s = 1 < t = 2 < u = 3$

$$\mathbb{E}(B_1^H B_2^H)\mathbb{E}(B_2^H B_3^H) = \mathbb{E}(B_2^H B_2^H)\mathbb{E}(B_1^H B_3^H), \tag{1.1}$$

we have:

$$\mathbb{E}(B_1^H B_2^H)\mathbb{E}(B_2^H B_3^H) = 2^{2H}(2^{2H} + 3^{2H} - 1),$$

and

$$\mathbb{E}(B_2^H B_2^H)\mathbb{E}(B_1^H B_3^H) = 2^{2H+1}.$$

Then, for all $H \in (0,1)\ \frac{1}{2}$, Equation (1.1) is not verified which leads to a contradiction.

1.1.4 Semimartingale property

Proposition 1.1.2. *(Tudor [102]). Fractional Brownian motion is not a semimartingale expect when $H = \frac{1}{2}$.*

Proof. For the case when $H = \frac{1}{2}$, the fBm restricts to a standard Brownian motion $B^{\frac{1}{2}}$ which is a martingale since $E[B_t^{\frac{1}{2}}/\mathcal{F}_s] = B_s^{\frac{1}{2}}$ for $s < t$, where $\mathcal{F}_s$ is the σ-algebra generated by the Brownian motion up to time s. However, when $H \neq \frac{1}{2}$, fBm is not semimartingale. This can be shown by the following argument provided in Rogers [92]:

Let $p > 0$, the two processes $\{M_{n,p}, n \geq 1\}$ and $\{M^*_{n,p}, n \geq 1\}$ where

$$M_{n,p} = n^{pH-1} \sum_{i=1}^{n} \mid B^H_{i/n} - B^H_{(i-1)/n} \mid^p,$$

$$M^*_{n,p} = n^{-1} \sum_{i=1}^{n} \mid B^H_i - B^H_{i-1} \mid^p,$$

have the same distribution because of the self-similar property of fBm. By the Ergodic Theorem $M^*_{n,p}$ converges to $E[\mid B^H_1 \mid^p]$ in L^1 almost surely as n goes to infinity. This implies $M_{n,p}$ converges to $E[\mid B^H_1 \mid^p]$ in probability as n goes to infinity. Hence the process

$$N_{n,p} = \sum_{i=1}^{n} \mid B^H_{i/n} - B^H_{(i-1)/n} \mid^p,$$

converges to zero in probability as n goes to infinity when $pH > 1$, and to infinity when $pH < 1$. The p-variation of fBm is defined as $\lim_{n\to\infty} N_{n,p}$. Consider the following two cases:

- When $H < \frac{1}{2}$, choosing $p > 2$, then $pH < 1$. Then, the quadratic variation variation when $p = 2$ is infinite.
- When $H > \frac{1}{2}$, if $\frac{1}{H} < p < 2$, then $pH > 1$ and the quadratic variation is zero. However if we choose $1 < p < \frac{1}{H}$, then $pH < 1$ and the quadratic variation is infinite.

Any semimartingale can be decomposed into a process of limit variation i.e. with vanishing quadratic variation and a local martingale having locally finite quadratic variation according to Doob-Meyer Theorem for semimartingale. Then, we conclude that it cannot be a semimartingale.

□

1.1.5 Integral representations of fBm

Moving Average Representation

Mandelbrot and Van Ness [73] considered the moving average representation of B^H, via the Wiener process $\{B_t, t \geq 0\}$ over an infinite interval

$$B^H(t)\frac{1}{C_1(H)}\int_{\mathbb{R}}\left((t-u)_+^{H-\frac{1}{2}}-(-u)^{H-\frac{1}{2}}\right)dB_u, \quad t\in\mathbb{R},$$

where

$$C_1(H)=\left(\int_{\mathbb{R}}\left((1+s)^{H-\frac{1}{2}}-s^{H-\frac{1}{2}}\right)^2 ds+\frac{1}{2H}\right)^{\frac{1}{2}}.$$

Levy-Hida Representation

Note that the fractional Brownian motion is a particular case of Volterra processes. Following Decreusfond and Üstünel[30], we have this kernel representation

$$B^H(H)=\int_0^t K_H(t,s)dB_s, \qquad 0<s<t<\infty,$$

where

$$K_H(t,s)=C_2(H)s^{\frac{1}{2}-H}\int_s^t |u-s|^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du, \qquad t>s,$$

and

$$C_2(H)=\left(\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}\right)^{\frac{1}{2}}.$$

1.1.6 Sample paths of fBm

The simulation of sample paths of fractional Brownian motion gives a better understanding of its characteristic. Various method have been introduced in order to give an exact numerical approximation based on Cholesky decomposition of covariance matrix, we cite

Cholesky method [5], Davies and Harte mothod [29] which was later simultaneously generalized by Dietrich and Newsam [32] and Wood and Chan [105]. Although this methods are known, approximative methods that simulate this process have been proposed to reduce the computation time. A natural idea is to approximate this integral by Riemann-type sums to simulate the process. As we saw in Mandelbrot and van Ness [73] defined fractional Brownian motion by a stochastic integral with respect to standard Brownian motion. A natural idea is to approximate this integral by Riemann-type sums to simulate the process. For $i = 1, ..., N$, the approximations given by

$$B^H(i) = C_1(H) \left[\sum_{j=-b}^{0} \left((i-j)^{H-1/2} - (-j)^{H-1/2} \right) \xi_j + \sum_{j=0}^{i} (i-j)^{H-1/2} \eta_j \right], \tag{1.2}$$

where ξ_j resp. η_j are vectors of $b+1$ resp. $N+1$ i.i.d. standard normal variables. We can generate sample paths of fBm directly on Matab using the the function *wfbm* or in **R** via the function *fbm* using the package *somebm*.

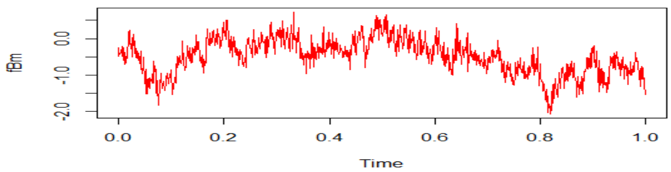

Figure 1.2: Sample paths of fBm when H = 0.2

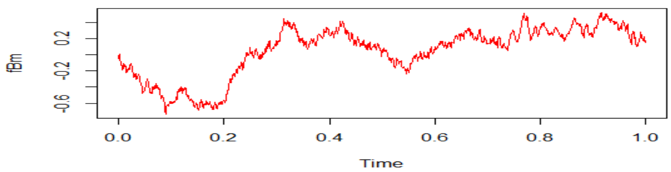

Figure 1.3: Sample paths of fBm when H = 0.5

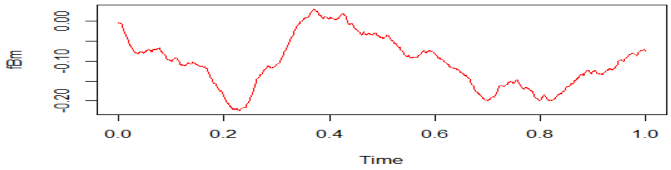

Figure 1.4: Sample paths of fBm when H = 0.9

1.2 Sub-fractional Brownian motion

Definition 1.2.1. *Sub-fractional Brownian motion* $S^H = (S_t^H; \forall t \geq 0)$ *with a Hurst parameter* $H \in (0,1)$ *is defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *as a centered gaussian process with continuous simple paths such that and for all* $t \geq 0$ *,*

$$C^H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}[(t+s)^{2H} - |t-s|^{2H}]$$

and verify

- $S_0^H = 0$,
- $S_t^H = \dfrac{B_t^H + B_{-t}^H}{\sqrt{2}}$,
- $\mathbb{E}\left[|S_t^H|^2\right] = (2 - 2^{2H-1})t^{2H}$.

Particulary, if $H = \frac{1}{2}$, $C^{\frac{1}{2}}(s,t) = s + t - \frac{1}{2}[(t+s) - |t-s|] = t \wedge s$, then sub-fBm reduces to the standard Brownian motion.

The following properties of a sub-fBm are established in Bojdecki el al. [14], Tudor [99, 100].

1.2.1 Main properties

1. The process S^H is self similar, for each $a > 0$

$$\{S_{at}^H, t \geq 0\} \overset{law}{=} \{a^H S_t^H, t \geq 0\}.$$

2. The increments of S^H are not stationary. Indeed, for all $s > 0, t > 0$, the second moment of increments is given by:

   - $\mathbb{E}\left[|S_t^H - S_s^H|^2\right] = -2^{2H-1}(t^{2H} + s^{2H}) + (t+s)^{2H} + (t-s)^{2H}$
   - $\mathbb{E}\left[|S_t^H|^2\right] = (2 - 2^{2H-1})t^{2H}$

   Hence, the increments of S^H are not stationary and admit the following estimates:

$$(t-s)^{2H} \leq \mathbb{E}\left[|S_t^H - S_s^H|^2\right] \leq (2 - 2^{2H-1})(t-s)^{2H} \text{ if } H < \frac{1}{2},$$

   and

$$(2 - 2^{2H-1})(t-s)^{2H} \leq \mathbb{E}\left[|S_t^H - S_s^H|^2\right] \leq (t-s)^{2H} \text{ if } H > \frac{1}{2}.$$

3. S^H has Hölder paths, by Komogorov's continuity criterion, for each $\gamma < H$ and each $T > 0$, $\exists$ a random variable $K_{\gamma,T}$ such that

$$|S_t^H - S_s^H| \leq K_T |t-s|^{H-\gamma}, \quad s,t \geq 0, \ a.s.$$

4. S^H is neither a semimartingale nor a Markov process when $H \neq \frac{1}{2}$.

5. The process S^H has a short memory for all $H \in (0,1)$. Indeed, for each $n \geq 1$

$$\begin{aligned} r(n) &= Cov(S_1^H, S_{n+1}^H - S_n^H) \\ &= 1 + (n+1)^{2H} - \frac{1}{2}\left[(n+2)^{2H} + n^{2H}\right] - 1 - n^{2H} + \frac{1}{2}\left[(n+1)^{2H} + (n-1)^{2H}\right] \\ &= 2H(1-H)(2H-1)n^{2H-3} + o(n^{2H-3}). \end{aligned}$$

   Then, $r(n) < \infty$ and $\sum_n |r(n)| < \infty$, for all $H \in (0,1)$.

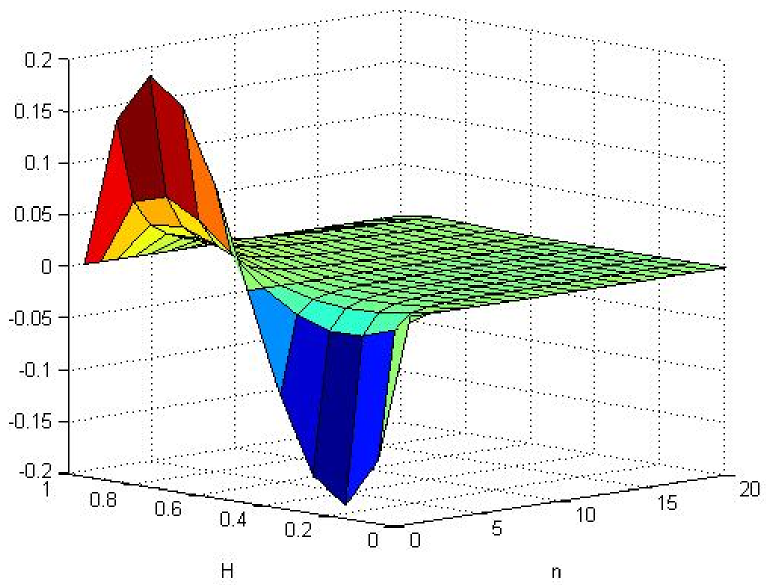

Figure 1.5: Graphical representation of $(H, n) \to r(n)$.

1.2.2 Integral representation of sub-fBm

Moving average representation

Bojedecki el al.[14] considered the following integral representation of S^H Based on the moving average representation of fBm: for any $t \geq 0$

$$S^H(t) = \frac{1}{C_3(H)} \int_{\mathbb{R}} \left[(t-s)_+^{H-\frac{1}{2}} + (t+s)_-^{H-\frac{1}{2}} - 2(-s)_+^{H-\frac{1}{2}} \right] dB(s), \tag{1.3}$$

where B is the Brownian process on $\mathbb{R}$ and

$$C_3(H) = \left[2 \int_0^{+\infty} \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds + \frac{1}{2H} \right]^{\frac{1}{2}}. \tag{1.4}$$

Integral representation on a finite time interval

According Dzapharize and Van Zaten [36], the sub-fractional Brownian motion is the even part of the fractional Brownian motion with the following representation over a finite time interval:

$$S^H(t) = C_4(H)\int_0^t n_H(t,s)dB(s) \tag{1.5}$$

where

$$n_H(t,s) = \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-\frac{1}{2})} s^{\frac{3}{2}-H}\left(\int_s^t (x^2-s^2)^{H-\frac{3}{2}}dx\right)\mathbb{I}_{[0,t]}(s) \tag{1.6}$$

and

$$C_4(H) = \left[\frac{\Gamma(1+2H)\sin(\pi H)}{\pi}\right]\frac{1}{2}.$$

1.2.3 Sample paths of sub-fBm

Morozewics and Filatova [80] suggested a simulation algorithm by discretize the integral representation of sub-fBm on $\mathbb{R}$ given by (1.3) and the integral $C_3(H)$ given by (1.4).

Kuang and Xie [59] proposed a simulation algorithm of the trajectories of sub-fractional Brownian motion using the integral representation on a finite interval, the formula is developed as follows:

$$S^H_{t_{i+1}} \approx S^H_{t_i} + \frac{C_4(H)}{\sqrt{n}}\sum_{j=1}^{i}\left[n(t_{i+1},\frac{t_j+t_{j-1}}{2}) - n(t_i,\frac{t_j+t_{j-1}}{2})\right]\zeta_j + \frac{C_4(H)}{\sqrt{n}}\left[n(t_{i+1},\frac{t_i+t_{i+1}}{2})\right]\eta_i,$$

where $\zeta_j, j=\overline{1,i}$ and $\eta_i, i=\overline{0,n-1}$ are gaussian standard variables and $S^H_{t_0} = S^H_0 = 0$. The kernel 1.6 is approximated by

$$n_H(t,s) \approx \frac{\sqrt{\pi}s^{\frac{3}{2}-H}}{2^H\Gamma(H+\frac{1}{2})}\left[\frac{(t^2-s^2)^{H-\frac{1}{2}}}{t} + \frac{\left[\left(\frac{t+s}{2}\right)^2 - s^2\right]^{H-\frac{1}{2}}(t-s)}{\left(\frac{t+s}{2}\right)^2}\right],\ 0<s<t.$$

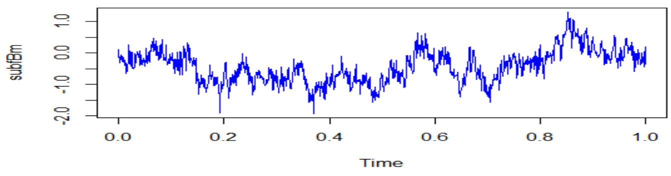

Figure 1.6: Sample paths of sub-fBm, H = 0.2

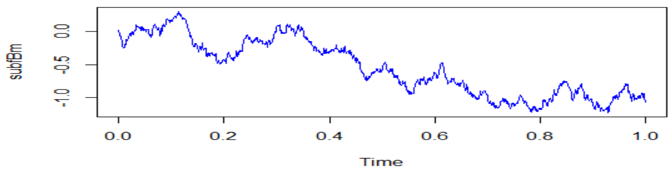

Figure 1.7: Sample paths of sub-fBm, H = 0.5

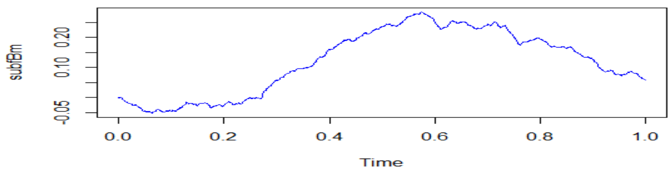

Figure 1.8: Sample paths of sub-fBm, H = 0.9

1.2.4 Some comparisons between fBm and sub-fBm

1. The increment of fBm are self-similar in the sense that for each a>0,

$$B^H(t+as) - B^H(t) \stackrel{law}{=} a^H(B^H(t+s) - B^H(t)),$$

   but sub-fBm does not have this property.

2. Covariance: for all $s, t > 0$, the covariance function $C_H(s,t) > 0$. Moreover

$$C_H(s,t) > R_H(s,t) \text{ if } H < \frac{1}{2}$$

   and

$$C_H(s,t) < R_H(s,t) \text{ if } H > \frac{1}{2}.$$

FBm and sub-fBm become similar for large t in the sense that for each $\tau > 0$

$$\lim_{t\to+\infty} \frac{C_H(t,t+\tau)}{R_H(t,t+\tau)} = 2 - 2^{2H-1}.$$

3. Covariance between increments for $0 \leq u < v \leq s < t$, let

$$R_H(u,v,s,t) = \mathbb{E}[(B_v^H - B_u^H)(B_t^H - B_s^H)]$$

and

$$C_H(u,v,s,t) = \mathbb{E}[(S_v^H - S_u^H)(S_t^H - S_s^H)]$$

Then,

$$R_H(u,v,s,t) < C_H(u,v,s,t) < 0 \ \text{ if } \ H < \frac{1}{2},$$

$$0 < C_H(u,v,s,t) < R_H(u,v,s,t) \ \text{ if } \ H > \frac{1}{2},$$

where

$$\begin{aligned} C_h =& \frac{1}{2}\big((t+u)^{2H} + (t-u)^{2H} + (s+v)^{2H} + (s-v)^{2H} \\ & - (t+v)^{2H} - (t-v)^{2H} - (s+u)^{2H} + (s-u)^{2H}\big), \end{aligned}$$

and

$$R_h = \frac{1}{2}\left((s-v)^{2H} + (t-u)^{2H} - (s-u)^{2H} - (t-u)^{2H}\right).$$

Therefore the covariance of increments of sfBm over non-overlapping intervals have the same sign but are smaller in absolute value than those of fBm $\lim_{s,t\to\infty} C_H(u,v,s,t) = 0$ for all $H \in (0,1)$ but $\lim_{s,t\to\infty} R_H(u,v,s,t) = 0$ for $H \in (0,\frac{1}{2})$.

4. Correlation between increments: for $u \geq 0$, $r > 0$ let $\rho_{B^H}(u,r)$ and $\rho_{S^H}(u,r)$ denote the correlation coefficient of the increment $B_{u+r}^H - B_u^H$,$B_{u+2r}^H - B_{u+r}^H$ and $S_{u+r}^H - S_u^H$,$S_{u+2r}^H - S_{u+r}^H$ We have:

$$|\rho_{S^H}(u,r)| \leq |\rho_{B^H}|.$$

Then, the increments of sub-fBm on the intervals $[u, u+r], [u+r, u+2r]$ are more weakly correlated than those of fBm.

5. Long-range dependence:

   For $0 \leq u < v \leq s < t$, we have:

$$R_H(u, v, s+\tau, t+\tau) \sim 2H(H-\frac{1}{2})(t-s)(v-u)\tau^{2H-2} \text{ as } \tau \to \infty,$$

   and

$$C_H(u, v, s+\tau, t+\tau) \sim 2H(H-\frac{1}{2})(2H-2)(v^2-u^2)\tau^{2(H-\frac{3}{2})} \text{ as } \tau \to \infty.$$

   The long-range dependence decays at a height rate for sub-fBm than for fBm.

1.3 Mixed fractional Brownian motion

An extension of the fractional Brownian motion has been introduced by Cheridito [20] to present an interesting stochastic model of the discount stock price in some financial markets under the form $M = B + \alpha B^H$, $\alpha > 0$, this process is called mixed fraction Brownian motion (mfBm in short). Zili [109] considered a more general version of mixed fBm as a linear combination of standard Bm B and independent fractional Bm B^H with Hurst parameter $H \in (0, 1)$

$$\forall t \geq 0, \quad M^H = M_t^H = M_t^H(a, b) = aB_t + bB_t^H,$$

where a and b are two real constants. Cheridito [20] claimed that the process M^H is equivalent to aB if and only if $H > \frac{3}{4}$. In the case when $H < \frac{1}{4}$, the mixed fBm is equivalent to bB^H (see Van Zanten [108]). The Lemma 1.3.1 gives the main properties of mixed fractional Brownian motion, the proofs are detailed in Zili [109]

Lemma 1.3.1. *(Zili [109]). The mfBm satisfies the following properties:*

- *M^H is a centered gaussian process;*

- *Second moment: for all $t \in \mathbb{R}_+$, $\mathbb{E}((M_t^H(a,b))^2) = a^2 t + b^2 t^{2H}$.*

- *Covariance function: for all $t, s \geq 0$;*

$$Cov(M_t^H(a,b), M_s^H(a,b)) = a^2 \min(t,s) + \frac{b^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- *The increments of the mfBm are stationary.*

- *Mixed self similarity: for all $t \in \mathbb{R}_+$ $\left(M_{\alpha t}^H(a,b)\right)$ and $\left(M_t^H(a\alpha^{\frac{1}{2}}, b\alpha)\right)$ have the same distribution.*

- *For all $H \in (0,1)\backslash\{\frac{1}{2}\}, a \in \mathbb{R}, b \in \mathbb{R}$, $(M_t^H(a,b))_{t\geq 0}$ is not a markovian process.*

- *Long and short memory: the mfBm is long memory when $H > \frac{1}{2}$ and short memory when $H < \frac{1}{2}$, indeed, for $n \in \mathbb{N}^*$,*

$$\begin{aligned} r(n) &= \mathbb{E}\left((M_{n+1}^H - M_n^H)M_1^H\right) = \frac{b^2}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right) \\ &= b^2 H(H-1) n^{2H-2}\epsilon(n), \end{aligned}$$

  where $\lim\limits_{n\to+\infty} \epsilon(n) = 0$.

  we see that $\sum_{n\in\mathbb{N}^*} r(n) = \infty$ if and only if $H > \frac{1}{2}$.

1.3.1 Correlation between the increments

We denote the correlation coefficient between the increments $M_{t+h}^H - M_t^H$, $M_{s+h}^H - M_s^H$ by $\forall s \in \mathbb{R}_+, \forall t \in \mathbb{R}_+, \forall h \in \mathbb{R}_+, 0 \leq h \leq t-s$

$$\rho(M_{t+h}^H - M_t^H, M_{s+h}^H - M_s^H) \;=\; \frac{b^2}{2(a^{2h+b^2h^{2H}})}\left((t-s+h)^{2H} - 2(t-s)^{2H} + (t-s-h)^{2H})\right)$$

Corollary 1.3.1. *(Zili [109]). for all* $a \in \mathbb{R}$ *et* $b \in \mathbb{R}\backslash\{0\}$*, the increments of* $(M_t^H(a,b))_{t\in\mathbb{R}_+}$ *are:*

1. *positively correlated if* $\frac{1}{2} < H < 1$.
2. *negatively correlated if* $0 < H < \frac{1}{2}$.
3. *uncorrelated if* $H = \frac{1}{2}$.

1.3.2 Hölder continuity

Lemma 1.3.2. *(Zili [109]). For all* $T > 0$ *and* $\beta < \frac{1}{2} \wedge H$*, the mfBm has a modification which sample paths having a Hölder continuity, with order* β*, on the interval* $[0,T]$ *such that:* $\forall q > 0, \Delta > 0$,

$$\mathbb{E}\left(\left|M_{t+\Delta}^H - M_t^H\right|^q\right) \leq C_q \Delta^{q(H\wedge\frac{1}{2})}, \tag{1.7}$$

where

$$C_q = C_1|a|^q T^{q((1/2)-H)}\mathbb{E}(|B_1|^q) + C_2|b|^q\mathbb{E}(|B_1^H|^q), \text{ when } H < \frac{1}{2},$$

and

$$C_q = C_1'|a|^q\mathbb{E}(|B_1|^q) + C_2'|b|^q T^{q((1/2)-H)}E(|B_1^H|^q), \text{ when } H < \frac{1}{2},$$

where C_1, C_2, C_1', C_2' *are positive constants depending on* q.

1.3.3 Non-differentiability

Definition 1.3.1. *Let* f *be a continuous function on* $[a,b]$*, and let* $\alpha \in]0,1[$*. We call a local fractional* $\alpha-$*derivative of* f *at* $t_0 \in [a,b]$*, the following quantity*

$$d_\sigma^\alpha f(t_0) = \Gamma(1+\alpha)\lim_{t\to t_0^\sigma}\frac{\sigma(f(t)-f(t_0))}{|t-t_0|^\alpha},$$

for $\sigma = +(resp, \sigma = -)$*, where* Γ *is the Euler function.*

Definition 1.3.2. *Let f be a continuous function on $[a,b]$, and let $\alpha \in]0,1[$. The function f is $\alpha-$ differentiable at $t_0 \in [a,b]$, if and only if $d^{\alpha}_{-}f(t_0) = d^{\alpha}_{+}f(t_0)$ exist and are equal. In this case we call $d^{\alpha}f(t_0)$ the $\alpha-$derivative of f at t_0*

Theorem 1.3.1. *(Zili [109]). For all $\alpha \in]0, \frac{1}{2} \wedge H[$, the sample paths of the mfbm are almost surely $\alpha-$differentiable at evrey $t_0 \geq 0$.*

Theorem 1.3.2. *(Zili [109]). For all $\alpha \in]\frac{1}{2} \wedge H, 1[$, the sample paths of mfbm are nowhere $\alpha-$ differentiable almost surely.*

1.3.4 Semimartingale property

Definition 1.3.3. *(Cheridito [20]). Let $\{\mathcal{F}_t,\, t \in [0,1]\}$ be the natural filtration. A process X_t is a $\mathcal{F}_t$-weak semimartingale if it is $\mathcal{F}_t$-adapted and satisfying the following property:*

$$I_X(\beta(\mathcal{F}_t)) \text{ is bounded in } L^0, \tag{1.8}$$

where

$$\beta(\mathcal{F}_t) = \left\{ \sum_{j=0}^{n-1} f_j \mathbf{1}_{(t_j,t_{j-1})}, n \in \mathbb{N}, 0 \leq t_0 \leq \ldots \leq t_n \leq 1, \right.$$

$$\left. \forall j, f_j \text{ is } \mathcal{F}_t - \text{measurable and } |f_j| \leq 1 \quad p.s. \right\}$$

and

$$I_X(\vartheta) = \sum_{j=0}^{n-1} f_j(X_{t_{j+1}} - X_{t_j}) \text{ for } \vartheta = \sum_{j=0}^{n-1} f_j \mathbf{1}_{(t_j,t_{j+1}]} \in \beta(\mathcal{F}_t)$$

an a.s. right-continuous, $\mathcal{F}_t$-adapted stochastic process $(X_t)_{t\in[0,1]}$ is a semimartingale if and only if X satisfying (1.8).

Theorem 1.3.1. *(Cheridito [20]). M^H is not a weak semimartingale if $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}]$, it is equivalent to $\sqrt{1+\alpha^2}B_t$ if $H = \frac{1}{2}$ and equivalent in distribution to Brownian motion if $H \in (\frac{3}{4}, 1]$.*

1.4 Fractional Ornstein-Uhlenbeck process

1.4.1 An overview on Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck (O.U.) process can be defined as the strong solution of the Langevin equation with Brownian noise. Langevin described in his pioneer paper [65] published in 1908 the random movement of a free particle in a fluid, he described the velocity of the particle via the following differential equation

$$\frac{dv(t)}{dt} = \frac{f}{m}v(t) + \frac{F(t)}{m}, \tag{1.9}$$

where $m > 0$ is the mass of the particle, $f > 0$ is the friction coefficient and $F(t)$ is the force on the particle by the impact of molecules in the fluid. In 1930, Ornstein and Uhlenbeck [103] imposed a probability hypothesis on $F(t)$ and then derived that for $v(0) = x \in R$, $v(t)$ is normally distributed with mean $xe^{-\lambda t}$ and variance $\frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda t})$, for $\lambda = \frac{f}{m}$ and $\sigma = \sqrt{\frac{2fkt}{m^2}}$, where k is the Boltzmann constant and T is the temperature. Then, the differential equation (1.9) is written as:

$$dv(t) = -\lambda(v(t) - v(0))dt + \sigma\eta(t), \quad t \geq 0,$$

where $\eta(t)$ is a white Gaussian process.

In application fields as physics, engineering and finance, the common representation of O.U process is a solution of a stochastic differential equation with a Brownian noise under the form:

$$dX_t = -\lambda X_t dt + \sigma dB_t, \quad t \geq 0,$$

where $\lambda > 0$, $\sigma > 0$ and B_t denotes the Brownian motion. Another version of O.U. process by adding an additional drift term is as:

$$dX_t = \lambda(m - X_t)dt + \sigma dB_t, \quad t \geq 0.$$

The classical Ornstein-Uhlenbeck process with parameters $\lambda > 0$ and $\sigma > 0$ starting at $x \in \mathbb{R}$, is the unique strong solution of the Langevin equation (1.4.1).

$$X_t = x - \lambda \int_0^t X_s ds + \sigma B_t, \quad t \geq 0,$$

It is given by the almost surely continuous Gaussian Markov process

$$X_t = e^{-\lambda t}\left(x + \sigma \int_0^t e^{\lambda u} dB_u\right) \quad t \geq 0.$$

Considering that B is a two-sided Brownian motion through 0, the unique strong solution of (1.4.1) with random initial condition ξ

$$\xi = \sigma \int_{-\infty}^0 e^{\lambda u} dB_u$$

is the stationary, almost surely continuous, centered Gaussian Markov process

$$X_t = \sigma \int_{-\infty}^0 e^{-\lambda(t-u)} dB_u.$$

It can easily be checked that

$$Cov(X_t, X_{t+s}) = \frac{\sigma^2}{2\lambda} e^{-\lambda|s|}.$$

1.4.2 Definition and properties of fractional Ornstein-Uhlenbeck process

The fractional Ornstein-Uhlenbeck process (fOU for short) was introduced in Cheridito [21] as a solution of a Langevin-type equation driven by a fractional Brownian motion B^H given as follows:

$$\begin{cases} dX_t = -\lambda X_t dt + \sigma dB_t^H, \quad t \geq 0, \\ X_0 = \xi. \end{cases}$$

The solution can be expressed in a pathwise sense as:

$$X_t = e^{-\lambda t}\left(\xi + \sigma \int_0^t e^{\lambda u} dB_u^H\right), \quad t \geq 0.$$

In particular, if we consider $\xi = \int_{-\infty}^0 e^{\lambda u} dB_u^H$, we write the solution of (1.4.2) in the following form:

$$X_t^{\xi} = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dB_u^H, \quad t \in \mathbb{R}.$$

The results shown below are derived from the following proposition:

Proposition 1.4.1. *(Cheridito et al. [21]). Let $(B_t^H \in \mathbb{R})$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ and $\xi \in L^0(\Omega)$. Let $-\infty \leq a < +\infty$ and $\lambda, \sigma > 0$. Then for almost all $\omega \in \Omega$, we have the following:*

1. *For all $t > a$,*

$$\int_0^a e^{\lambda u} dB_u^H(\omega)$$

exists as a Riemann-Stieltjes integral and is equal to

$$e^{\lambda t} B_t^H(\omega) - e^{\lambda a} B_a^H(\omega) - \lambda \int_a^t B_u^H(\omega) e^{\lambda u} du.$$

2. *The function*

$$\int_0^a e^{\lambda u} dB_u^H(\omega), \quad t > a,$$

is continuous in t

3. *The unique continuous function y that solves the equation*

$$y(t) = \xi(\omega) - \lambda \int_0^t y(s) ds + \sigma B_t^H(\omega), \quad t \geq 0,$$

is given by

$$y(t) = e^{\lambda t}\left(\xi(\omega) + \sigma \int_0^t e^{\lambda u} dB_u^H(\omega)\right), \quad t \geq 0.$$

In particular, the unique continuous solution of the equation,

$$y(t) = \sigma \int_{-\infty}^0 e^{\lambda u} dB_u^H(\omega) - \lambda \int_0^t y(s)ds + \sigma B_t^H(\omega), \quad t \geq 0.$$

is given by

$$y(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dB_u^H(\omega), \quad t \geq 0.$$

The covariance of fractional O.U. process decays as a power function and is very similar to the decays of fBm's increments.

Theorem 1.4.1. *(Cheridito et al. [21]). Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and $N \in \mathbb{N}$. Then for a fixed $t \in \mathbb{R}$ and $s \to \infty$,*

$$Cov(X_t^\xi, X_{t+s}^\xi) = \frac{1}{2}\sigma^2 \sum_{n=1}^{N} a^{-2n}\left(\prod_{k=0}^{2n-1}(2H-k)\right)s^{2H-2n} + O(s^{2H-2N-2}).$$

Proposition 1.4.2. *(Kaarakka and Salminen [55]). The stationary sequence process X^ξ is long range dependent when $H > \frac{1}{2}$, and short range dependent when $H < \frac{1}{2}$.*

Proof. The leading term of the sum in (1.4.1) is of the order t^{2H-2}. Consequently,

$$\sum_{n=0}^{\infty}\left|Cov(X_i^\xi, X_{i+n}^\xi)\right| \simeq \sum_{n=0}^{\infty} n^{2H-2}.$$

□

In particular, for $\xi = x \in \mathbb{R}$, Cheridito [21] claimed that

$$Cov(X_t^x, X_{t+s}^x) = Cov(X_t^\xi, X_{t+s}^\xi) - e^{\lambda t}Cov(X_0^\xi, X_{t+s}^\xi) + O(e^{\lambda s}).$$

The next corollary shows that the solution X_t^x decays also like a power function of the order $2H-2$.

Corollary 1.4.1. *(Cheridito et al. [21]). Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and $N \in \mathbb{N}$. Then for a fixed $t \in \mathbb{R}$ and $s \to \infty$,*

$$Cov(X_t^x, X_{t+s}^x) = \frac{1}{2}\sigma^2 \sum_{n=1}^{N} a^{-2n} \left(\prod_{k=0}^{2n-1} (2H-k) \right) \left[s^{2H-2n} - e^{-at}(t+s)^{2H-2n} \right] + O(s^{2H-2N-2}).$$

Chapter 2

Stochastic integration of non-adapted processes related to fractional Gaussian processes

The issue of stochastic integration of anticipating integrand has received an interest since a considerable attention since the past fifty years (Hitsuda [46] in 1972, Skorohod [97] in 1975, etc).

In 1976, Itô [52] highlighted the problem of anticipating integral in the international symposium on stochastic differential equations. He gave a solution for

$$\int_0^t B(1)dB(t), \quad 0 \leq t \leq 1, \tag{2.1}$$

which is not an Itô integral as the integrand $B(1)$ is not adapted to $\mathcal{F}_t$, the filtration generated by $B(t), 0 \leq t \leq 1$. His idea is to enlarge filtration by letting $\mathcal{G}_t$ the σ-field generated by $\mathcal{F}_t$ and $B(1)$. Then, the integrand $B(1)$ is evidently adapted to $\mathcal{G}_t$.
However, the stochastic process $B(t)$ is no more a Brownian motion with respect to $\mathcal{G}_t$, but it is a $\mathcal{G}_t$-quasimartingale under the decomposition:

$$B(t) = \left(B(t) - \int_0^t \frac{B(1)-B(u)}{1-u}du\right) + \int_0^t \frac{B(1)-B(u)}{1-u}du.$$

Therefore, the stochastic integral (2.1) can be defined with respect to a quasimartingale. Then:

$$\int_0^t B(1)dB(t) = B(1)B(t), \quad 0 \le t \le 1.$$

In 2008, Ayed and Kuo [6] introduced a new approach to define stochastic integrals of anticipating integrands using Itô's idea. As we seen above, Itô [52] kept the integrand and decomposed the integrator (with respect to an enlarged filtration), while Ayed and Kuo [6] kept the integrator (and the filtration) and decomposed the integrand into a linear combination of an adapted part and an instantly independent part. Then, the integral is defined as a Riemann-like sum, the evaluation points for the instantly independent part are the right endpoints of subintervals, while those for the adapted part are the left endpoints.

In our work, we extend the above study for the class of fractional type process, namely, the sub-fractional and mixed fractional Brownian motions by exploiting the fact that the stochastic integral with respect to such processes can be defined as a Riemann sum when $H > \frac{1}{2}$.

2.1 Anticipating stochastic integral: Ayed and Kuo approach

In this section, we describe the idea of Ayed and Kuo on to define the new stochastic integral, their idea came from a very simple observation of integrand's decomposition. The following are some simple examples that demonstrate the decompositions.

Example 2.1.1. The anticipating stochastic process $B(1)$ can be decomposed as

$$B(1) = (B(1) - B(t)) + B(t).$$

Example 2.1.2. Consider another anticipating stochastic process $B(1)^2$. This anticipating integrand $B(1)^2$ can be decomposed as

$$B(1)^2 = (B(1) - B(t))^2 + 2B(t)(B(1) - B(t)) + B(t)^2.$$

Example 2.1.3. For $n \in \mathbb{N}$, it follows from the binomial theorem that $B(1)^n$ can be decomposed as

$$B(1)^n = (B(1) - B(t) + B(t))^n = \sum_{k=1}^{n} \binom{n}{k} (B(1) - B(t))^k B(t)^{n-k}.$$

Example 2.1.4. The stochastic process $e^{B(1)}$ can be written as

$$e^{B(1)} = e^{B(1)-B(t)} e^{B(t)}.$$

from above examples, we note that every decomposition is a linear combination of products of an adapted part (e.g., $B(t), B(t)^k, e^{B(t)}$) and an anticipating part with a special property (e.g., $B(1) - B(t), (B(1) - B(t))^k, e^{B(1)-B(t)}$) given in the following definition.

Definition 2.1.1. *A stochastic process $\{X_t, 0 \le t \le T\}$, is said to be instantly independent with respect to the filtration $\mathcal{F}_t$ if for each t, the random variable X_t is independent of the σ-field $\mathcal{F}_t$.*

According to above definition, the following stochastic processes are all instantly independent with respect to $\mathcal{F}_t :$ $(B(1) - B(t)), [B(1) - B(t)]^n; n \in \mathbb{N}, e^{B(1)-B(t)}$ for $0 \le t \le 1$.

Example 2.1.5. Let $\mathcal{F}_t$ be the underlying filtration of Brownian Motion $B(t)$:

- $(B(1) - B(t))$ is instantly independent of $\mathcal{F}_t$ for $t \in [0, 1]$.
- $(B(1) - B(t))$ is adapted to $\mathcal{F}_t$ for $t \ge 1$.

Lemma 2.1.1. *(Ayed and Kuo [6]). If a stochastic process X_t is both adapted and instantly independent with respect to the filtration $\mathcal{F}_t$, then X_t must be a deterministic function.*

Proof. Let us examine the conditional expectation of X_t with respect to $\mathcal{F}_t$. On one hand, X_t is adapted to $\mathcal{F}_t$. So, by the property of the conditional expectation, we have

$$\mathbb{E}(X_t/\mathcal{F}_t) = X_t,$$

but at the same time it is independent of $\mathcal{F}_t$. Hence,

$$\mathbb{E}(X_t/\mathcal{F}_t) = \mathbb{E}(X_t).$$

Therfore, $X_t = \mathbb{E}(X_t)$ and so X_t is a deterministic function. $\square$

From this Lemma, we conclude that instantly independent processes are independent of the past and present contrarily to the adapted processes. Thus, we can consider the class of instantly independent processes as a counterpart of Itô's theory. In addition, we can deduce that many anticipating stochastic processes can be decomposed into sums of the products of Itô parts (adapted processes) and counterparts (instantly independent processes). This turns out to be a key idea of Ayed and Kuo approach to define the new stochastic integral. To use their idea to evaluate an anticipating stochastic integral, one needs to:

1. keep the filtration $\mathcal{F}_t$ and the integrator $B(t)$.
2. decompose the integrand into a sum of the products of adapted stochastic processes and instantly independent stochastic processes.
3. evaluate each stochastic integral of a product of an adapted stochastic process and an instantly independent stochastic process.

Hence the next question we need to answer is how one can define a stochastic integral of a product of an adapted process and an instantly independent process. Recall that Itô integral measures the integrand using left endpoint for each subinterval. For instantly independent part, if we also use the left endpoint to approximate, we lose its important properties as it has been seen in Example 2.1.5. However, if we measure the instantly independent part using right endpoint, its properties will be conserved. This lead to Ayed and Kuo's definition of the new integral.

Definition 2.1.2. *(Ayed and Kuo [6]). Let $B(t)$ be a Brownian motion, for an adapted stochastic process $f(t)$ with respect to the filtration $\mathcal{F}_t$ and an instantly independent stochastic process $g(t)$ with respect to the same filtration, we define the stochastic integral of $f(t)g(t)$ to be the limit:*

$$\int_0^T f(t)g(t)dB(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(t_{i-1})g(t_i)(B(t_i) - B(t_{i-1}))$$

provided that the limit in probability exists, where $\Delta_n = \{0 = t_0 < t_1 < .... < t_n = T\}$ is the partition of interval $[0, T]$.

Remark 2.1.1. *Note that, by the above definition, if $f(t)$ is continuous and $g(t) = 1$,*

$$\int_0^T f(t)dB(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(t_{i-1})(B(t_i) - B(t_{i-1}))$$

We see that this new integral reduces to the Itô integral. This is why it can be seen as an extension of Itô integral.

Next, let us provide some examples to show how this idea works. More examples can be found in Ayed and Kuo [6, 7]. First, we begin with the simplest case when $f(t) = 1$.

Example 2.1.6. We have to find $\displaystyle\int_0^1 (B(1) - B(t))dB(t)$.

Let $\Delta_n = \{0 = t_0 < t_1 < t_2 < ... < t_n = 1\}$ be a partition of the interval $[0, 1]$. Here, $g(t) = B(1) - B(t)$ is instantly independent. Thus, on each subinterval $[t_{i-1}, t_i]$, we take the right endpoint t_i as the evaluation point to form a Riemann like sum. So, we have:

$$\begin{aligned}
\int_0^1 (B(1)-B(t))dB(t) &= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} [B(1)-B(s_i)](B(s_i)-B(s_{i-1})) \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} B(1)(B(s_i)-B(s_{i-1})) - \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} B(s_i)(B(s_i)-B(s_{i-1})) \\
&= B(1)^2 - \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} [[B(s_i)-B(s_{i-1})]+B(s_{i-1})](B(s_i)-B(s_{i-1})) \\
&= B(1)^2 - \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} (B(s_i)-B(s_{i-1}))^2 - \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} B(s_{i-1})(B(s_i)-B(s_{i-1})) \\
&= B(1)^2 - 1 - \int_0^1 B(t)dB(t).
\end{aligned}$$

The last integral $\int_0^1 B(t)dB(t)$ is an Itô integral since $B(t)$ is adapted. Hence, by the decomposition (2.1.1), we have:

$$\begin{aligned}
\int_0^1 B(t)dB(t) &= \int_0^1 (B(1)-B(t))dB(t) + \int_0^1 B(t)dB(t) \\
&= \left(B(1)^2 - 1 - \int_0^1 B(t)dB(t)\right) + \int_0^1 B(t)dB(t) \\
&= B(1)^2 - 1.
\end{aligned}$$

Example 2.1.7. We evaluate the stochastic integral

$$\int_0^t B(s)(B(1)-B(s))dB(s), \qquad 0 \le t \le 1.$$

For $0 \le t \le 1$, $f(t) = B(t)$ is adapted and $g(t) = B(1) - B(t)$ is instantly independent. Let $\Delta_n = \{0 = s_0 < s_1 < s_2 < ... < s_n = t\}$ be a partition of the interval $[0, t]$. By definition:

$$\begin{aligned}
&\int_0^t B(s)(B(1)-B(s))dB(s) \\
&\qquad = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} B(s_{i-1})(B(1)-B(s_i))(B(s_i)-B(s_{i-1})) \\
&\qquad = B(1) \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} B(s_{i-1})(B(s_i)-B(s_{i-1})) \\
&\qquad - \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} B(s_{i-1})\left(B(s_i)-B(s_{i-1})+B(s_{i-1})\right)\left(B(s_i)-B(s_{i-1})\right) \\
&\qquad = B(1) \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} B(s_{i-1})(B(s_i)-B(s_{i-1})) \\
&\qquad - \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} B(s_{i-1})(B(s_i)-B(s_{i-1}))^2 - \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} B(s_{i-1})^2(B(s_i)-B(s_{i-1})).
\end{aligned}$$

Note that, as $\|\Delta_n\| \to 0$, the above three summations converge in probability to $\int_0^t B(s)dB(s), \int_0^t B(s)ds$ and $\int_0^t B(s)^2dB(s)$, respectively. Therefore,

$$\int_0^t B(s)(B(1)-B(s))dB(s) = B(1)\int_0^t B(s)dB(s) - \int_0^t B(s)ds - \int_0^t B(s)^2dB(s).$$

Since both $\int_0^t B(s)dB(s)$ and $\int_0^t B(s)^2dB(s)$are Itô integrals, we can apply the regular Itô's formula to evaluate these two integrals. So, for $0 \le t \le 1$, we have

$$\begin{aligned}
&\int_0^t B(s)(B(1)-B(s))dB(s) \\
&\qquad = B(1)\left[\frac{1}{2}(B(t)^2-t)\right] - \int_0^t B(s)ds - \left[\frac{1}{3}B(t)^3 - \int_0^t B(s)ds\right] \\
&\qquad = \frac{B(1)}{2}\left(B(t)^2-t\right) - \frac{B(t)^3}{3}.
\end{aligned}$$

2.1.1 Near martingale property and anticipating integral

In Itô integration theory, one of the important property of a stochastic process is the "martingale property". Recall that if $f(t)$ is an adapted and square integrable process,

$B(t)$ is a Brownian motion and $\mathcal{F}_t$ is the natural Brownian filtration, then the stochastic process

$$X_t = \int_0^t f(s)dB(s), \quad t \in [0,T] \tag{2.2}$$

is a martingale with respect to $\mathcal{F}_t$.

In the anticipating integration theory, we use an instantly independent process (or generally a product of adapted and instantly independent process) which is non-adapted. Hence, the process X_t defined by (2.2) is not $\mathcal{F}_t$-adapted which involves that it is not a $\mathcal{F}_t$-martingale.

Recall that a martingale with respect to the filtration $\mathcal{F}_t$ is an $\mathcal{F}_t$-adapted stochastic process with $\mathbb{E}(|X_t|) < \infty$ and $\mathbb{E}(X_t/\mathcal{F}_s) = X_s$ for any $s \leq t$. Thus, this property makes sense only for adapted stochastic processes. Therefore, it is natural to ask if we will have a similar property to that martingale property in this new theory which still makes sense for non-adapted stochastic processes. Obviously, in order to obtain such similar property, the first assumption that X_t is $\mathcal{F}_t$-adapted must be removed and consider only the assumption $\mathbb{E}(X_t/\mathcal{F}_s) = X_s$, for any $s \leq t$. However, this assumption can be rewritten as

$$\mathbb{E}(X_t - X_s/\mathcal{F}_s) = 0, \qquad \forall s \leq t. \tag{2.3}$$

Observe that, the expression in Equation (2.3) still makes sense for non-adapted stochastic processes including instantly independent stochastic processes in the new theory. Therefore, this property motivates a definition of new concept called "near-martingale". We consider backward and forward filtration (see the work of Protter and Pardoux [85]).

Definition 2.1.3. *(Protter and Pardoux [85])*

1. *We say that a family $\mathcal{F}_t$ of sub-σ-fields is a forward filtration if $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $0 \leq s \leq t$.*

2. *We say that a family $\mathcal{F}^{(t)}$ of sub-σ-fields is a backward filtration if $\mathcal{F}^{(s)} \supseteq \mathcal{F}^{(t)}$, for $s \leq t$.*

Definition 2.1.4. *(Kuo et al.[63]). Let X_t be a stochastic process with $\mathbb{E}(|X_t|) < \infty$ for all t. We say that X_t is a near-martingale with respect to a forward filtration $\mathcal{F}_t$ if*

$$\mathbb{E}(X_t - X_s/\mathcal{F}_s) = 0, \qquad \forall s \leq t.$$

On the other hand, we say that X_t is a near-martingale with respect to a backward filtration $\mathcal{F}^{(t)}$ if

$$\mathbb{E}(X_t - X_s/\mathcal{F}^{(t)}) = 0, \qquad \forall s \leq t.$$

Remark 2.1.2. *If X_t is a near-martingale with respect to the filtration $\mathcal{F}_t$ and is adapted to $\mathcal{F}_t$, then it is a martingale with respect to $\mathcal{F}_t$. This is the reason why we call this class of stochastic processes near-martingales.*

Instantly independent processes take a central place on the anticipating integral theory. The next theorems state the cases when such processes are near-martingales. Next Theorem establishes the conditions under which this product of stochastic processes is a near-martingale.

Theorem 2.1.1. *(Kuo et al.[63]). Suppose $g(t)$ is instantly independent with respect to a forward filtration $\mathcal{F}_t$ and $\mathbb{E}(|X_t|) < \infty$ for all t. Then, $g(t)$ is a near-martingale with respect to $\mathcal{F}_t$ if and only if $\mathbb{E}[g(t)] = \mathbb{E}[g(s)]$ for all s and t, i.e., $\mathbb{E}[g(t)]$ is constant.*

Next, consider the case of a product of an adapted process and an instantly independent process. Next Theorem establishes the conditions under which this product of stochastic processes is a near-martingale.

Theorem 2.1.2. *(Kuo et al.[63]). Let $\mathcal{F}_t$ be a forward filtration. Assume that $f(t)$ and $g(t)$ are stochastic processes such that*

1. *$f(t)$ is a martingale with respect to $\mathcal{F}_t$;*
2. *$g(t)$ is instantly independent with respect to $\mathcal{F}_t$ and $\mathbb{E}[g(t)]$ is constant;*
3. *$\mathbb{E}|f(t)g(t)| < \infty$ for all t.*

Then, $\theta(t) = f(t)g(t)$ is a near-martingale with respect to $\mathcal{F}_t$.

Remark 2.1.3. *Observe that*

1. *By Theorem 2.1.1, condition (2) implies that $g(t)$ is a near-martingale.*

2. *Theorem 2.1.2 is false if we assume that $g(t)$ is a near-martingale without the instantaneous independence of $g(t)$.*

Remark 2.1.4. *The same theorems holds when dealing with the backward filtration $\mathcal{F}^{(t)}$.*

In the new theory, we can also define these stochastic processes associated with the new integrals. That is, for continuous functions f and g, we define X_t to be the stochastic process

$$X_t = \int_0^t f(B(s))g(B(T) - B(s))dB(s); \qquad 0 \leq t \leq T,$$

and we prove, under some conditions, that X_t is a near-martingale with respect to the forward filtration $\mathcal{F}_t$, where $\mathcal{F}_t = \sigma\{B(s), 0 \leq s \leq t\}$. Moreover, we introduce another associated stochastic process defined by

$$Y_t = \int_t^T f(B(s))g(B(T) - B(s))dB(s); \qquad 0 \leq t \leq T,$$

then, under the same conditions, we can also show that the stochastic process Y_t is a near-martingale with respect to the same filtration

Theorem 2.1.3. *(Kuo et al.[63]). Let $\mathcal{F}_t$ be a forward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $\mathbb{E}\big[X_t\big] < +\infty$,

2. $\mathbb{E}\big[Y_t\big] < +\infty$.

Then,

$$X_t = \int_0^t f(B(s))g(B(T) - B(s))dB(s); \qquad 0 \leq t \leq T,$$

and

$$Y_t = \int_t^T f(B(s))g(B(T) - B(s))dB(s); \qquad 0 \leq t \leq T$$

exist and are near-martingales with respect to the forward filtration $\mathcal{F}_t$.

Now, we turn our attention to the backward filtration $\mathcal{F}^{(t)}$ where

$$\mathcal{F}^{(t)} = \sigma\{B(T) - B(s), 0 \leq s \leq t\}$$

It can also be shown that the stochastic process X_t and $\mathbf{Y}_t$ are both near-martingales with respect to this backward filtration $\mathcal{F}^{(t)}$.

Theorem 2.1.4. *(Kuo et al.[63]). Let* $\mathcal{F}^{(t)}$ *be a backward filtration and let* $f(x)$ *and* $g(x)$ *be continuous functions such that:*

1. $\mathbb{E}\big[X_t\big] < +\infty$,

2. $\mathbb{E}\big[Y_t\big] < +\infty$.

Then

$$X_t = \int_0^t f(B(s))g(B(T) - B(s))dB(s); \qquad 0 \leq t \leq T, \tag{2.4}$$

and

$$Y_t = \int_t^T f(B(s))g(B(T) - B(s))dB(s); \qquad 0 \leq t \leq T \tag{2.5}$$

exist and are near-martingales with respect to the backward filtration $\mathcal{F}^{(t)}$.

The proofs of Theorems can be found in Kuo et al [63].

2.2 Anticipating stochastic integral related to fractional Gaussian processes

We treat in this section the problem of integration of non-adapted processes with respect to fractional Gaussian processes. Particulary, we consider the mixed fractional Brownian motion M^H and sub fractional Brownian motion S^H with $H \in (0, 1)$ defined in chapter 1. Nevertheless, when $H \neq \frac{1}{2}$ this processes are not semimartingales. Therefore, we cannot apply directly Itô's theory in this case. Moreover, the Riemann-Stieltjes integration cannot be used since the paths of the fBm (then those of mfBm) and sub-fBm have unbounded variations. Many technics have been introduced in order to study the integral with respect to fBm and sub-fBm. In the case where $H > \frac{1}{2}$, the processes have a p-bounded variation, which allows to use the pathwise approach that allows us to write the integral as a limit of Riemann sum (Young [106], Zähle [107], and Feyel and Pradelle [38] and the references therein). In our study, we use this approach in order to give a definition of the anticipating integral with respect to a mixed fractional Brownian motion M^H and sub fractional Brownian motion S^H, then study the near-martingale property.

2.2.1 Fractional stochastic integral and Riemann sum

In order to define the stochastic integral with respect to sub-fBm and mixed fBm we start by defining the Riemann-Stieltjes integral.

Definition 2.2.1. *(Riemann-Stieltjes Integrals). Let $f : [0, T] \to \mathbb{R}$ be continuous and $g : [0, T] \to \mathbb{R}$ be a function of bounded variation. We define the Riemann-Stieltjes integral as follows:*

$$\int_0^T f(t)dg(t) = \lim_{||\Delta_n|| \to 0} \sum_{i=1}^{n} f(\tau_i)(g(t_i) - g(t_{i-1}))$$

where $0 = t_0 < .... < t_n = t$, $\Delta_n = t_i - t_{i-1}$, $i = \overline{1, n}$ and τ_i is an evaluation point in the interval $[t_{i-1}; t_i]$.

Recall that in the Brownian case, it is impossible to define

$$\int_0^T f(t)dB(t) \tag{2.6}$$

via Riemann-Stieltjes approach for all continuous process f since B has unbounded variation but it has bounded quadratic variation. Then, Itô defined the integral (2.6) as a limit of Riemann-Stieltjes for adapted and square integrable processes.

Definition 2.2.2. *We denote by $L^2_{ad}(\Omega, [0,T])$ the space of all adapted stochastic processes $H(t)$ such that*

$$\mathbb{E}\Big(\int_0^T H^2(s)ds < \infty\Big).$$

Definition 2.2.3. *(Itô integral). Suppose that $H \in L^2_{ad}(\Omega \times [0,T])$. We define the Itô integral of f with respect to Brownian motion as*

$$I(H) = \int_0^T H(t)dB(t) = \lim_{||\Delta_n||\to 0} \sum_{i=1}^{n} H(t_{i-1})(B(t_i) - B(t_{i-1}))$$

whenever the limit in probability exists.

Proposition 2.2.1. *(Kuo[60]). The process $M(t)$, $t \in [0,T]$, defined by*

$$M(t) = \int_0^t H(s)dB(s) \tag{2.7}$$

is a-martingale with respect to $\mathcal{F}_t = \sigma\{B(s), s \le t\}$.

Now, we extend the Riemann-Stieltjes integral to functions of unbounded variation. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing, continuous, unbounded, convex function and $\Phi(0) = 0$. For each partition $\Delta_n = \{0 = t_0 < t_1 < t_2 < ... < t_n = T\}$ of the interval $[0,T]$. For a function f: $[0,T] \to \mathbb{R}$, set

$$v_\Phi(f, \Delta_n) = \sum_{i=1}^{n} \Phi|f(t_i) - f(t_{i-1})|.$$

Definition 2.2.4. *We define Φ-variation of the function f over the interval $[0, T]$ by*

$$v_\Phi(f) = \sup_{\Delta_n} v_\Phi(f, \Delta_n), \tag{2.8}$$

where the spermium is taken over all partitions Δ_n of the interval $[0, T]$. If $v_\Phi(f) < \infty$, we say that f has the bounded Φ-variation property and we denote by $\mathcal{W}_\Phi$ the class of all functions f with bounded Φ-variation.

The case $p = 1$ corresponds to the classical case of bounded variation. For the function $\Phi(x) = x^p$, let $v_\Phi(f) = v_p(f)$ and $\mathcal{W}_\Phi = \mathcal{W}_p$. Moreover, we define the index of the function f by

$$v(f) := \inf\{p \geq 1,\ v_p(f) < \infty\}.$$

The family of functions with bounded p-variation is denoted by $\mathcal{W}_p$ and it becomes a Banach space under the norm

$$\|f\|_{[p]} = \max\left(v_p(f)^{\frac{1}{p}}, \|f\|_\infty\right).$$

We denote by $\mathcal{H}_{[0,T],\alpha}$ the class of all α-Hölder functions $f : [0, T] \to \mathbb{R}$ with $f(0) = 0$ and define

$$\|f\|_{[0,T],\alpha} = \sup_{0 \leq u < v \leq T} \frac{|f(u) - f(v)|}{|v - u|^\alpha}.$$

The next proposition shows the link between Hölder continuous and bounded p-variation functions.

Proposition 2.2.2. *(Dudley and Norvaisa [34]). Let $1 \leq p < \infty$. Then the function $f : [0, T] \to \mathbb{R}$ belongs to $\mathcal{W}_p$ if and only if $f = g \circ h$, where h is a bounded, non-negative and increasing function on $[0, T]$ and g is a $\frac{1}{p}$-Hölder continuous function on $[h(0), h(T)]$.*

Theorem 2.2.1. *(Young [106]). If $f(t)$ and $g(t)$ are continuous paths of finite p, q variation, respectively, where $\frac{1}{p} + \frac{1}{q} > 1$, then the integral $\int_0^t f(s)dg(s)$ may be defined as the corresponding Riemann-Stieltjes sum.*

Proposition 2.2.3. *(Feyel and Pradelle [38]). If f is α-Hölder, g is β-Hölder with $\alpha+\beta>1$, then the Steiltjes integral $\int_0^T f(s)dg(s)$ exists as limit of the corresponding Riemann-Stieltjes sums and is β-Hölder. Moreover for every $0<\varepsilon<\alpha+\beta-1$,*

$$\left|\int_0^T f(s)dg(s)\right| \le C(\alpha,\beta) \parallel f \parallel_{[0,T],\alpha} \parallel g \parallel_{[0,T],\beta} T^{1+\varepsilon}.$$

Concerning the variation of fBm and sub-fBm we have the following results:

Proposition 2.2.4. *(Mishura[79]). For every $p>0$, we have:*

- $v_p(B^H)<\infty$ *if* $p>\frac{1}{H}$.
- $v_p(B^H)=\mathbb{E}(|\mathcal{N}(0,1)|^p)$ *if* $p=\frac{1}{H}$.
- $v_p(B^H)=\infty$ *if* $p<\frac{1}{H}$,

Proposition 2.2.5. *(Tudor[100]). For every $p>0$, we have:*

- $v_p(S^H)<\infty$ *if* $p>\frac{1}{H}$.
- $v_p(S^H)=\mathbb{E}(|\mathcal{N}(0,1)|^p)$ *if* $p=\frac{1}{H}$.
- $v_p(S^H)=\infty$ *if* $p<\frac{1}{H}$,

where $\mathcal{N}(0,1)$ is standard Gaussian random variable.

Thus, B^H and S^H in $\mathcal{W}_p$ and if and only if $p\ge\frac{1}{H}$. Then, for every process $f(t), t\in[0,T]$ in $\mathcal{W}_q$, the Riemann-Stieltjes integrals $\int_0^t f(u)dB^H(u)$ and $\int_0^t f(u)dS^H(u)$ are well defined, i.e.

$$\int_0^T f(t)dB^H(t) = \lim_{||\Delta_n||\to 0}\sum_{i=1}^n f(t_{i-1})(B^H(t_i)-B^H(t_{i-1})), \tag{2.9}$$

and

$$\int_0^T f(t)dS^H(t) = \lim_{||\Delta_n||\to 0} \sum_{i=1}^{n} f(t_{i-1})(S^H(t_i) - S^H(t_{i-1})), \tag{2.10}$$

where $0 = t_0 < .... < t_n = t$, $\Delta_n = t_i - t_{i-1}$, $i = \overline{1,n}$.

In particular, as B^H and S^H are γ-Hölder for all $\gamma < H$, then for every α-Hölder process $f(t)$ for some $\alpha > 1 - H$,the Riemann-Stieltjes integrals (2.9) and (2.10) are well defined and have γ-Hölder paths, for every $\gamma < H$.

Now, we can define the stochastic integrals (2.9) and (2.10) for adapted integrands for $H > \frac{1}{2}$ in a pathwise sense.

Remark 2.2.1. *Recall that*

$$M^H(t) = M_t^H(a,b) = aB(t) + bB^H(t),\ t \in \mathbb{R}_+, \qquad a, b \in \mathbb{R}^*,$$

where B and B^H are independent standard and fractional Brownian motions, respectively. It follows that

$$\int_0^t f(u)dM^H(u) = a\int_0^t f(u)dB^H(u) + b\int_0^t f(u)dB(u). \tag{2.11}$$

then, the integral (2.11) is defined in a similar way as $\int_0^t f(u)dB^H(u)$ in pathwise sense.

2.2.2 New anticipating integral with respect to sub-fractional Brownian motion

In this section, we give a definition of stochastic integral of the product $f(t)g(t)$ as in Definition 2.1.2 by taking the sub-fBm S^H as integrator. Formally, we have:

Definition 2.2.5. *Let S^H, $H > \frac{1}{2}$ be a sub-fractional Brownian motion and let $\mathcal{F}_t$ be the σ-field generated by $\{S^H(t), t \geq 0\}$, for an adapted stochastic process $f(t)$ with respect to* *the filtration $\mathcal{F}_t$ and an instantly independent stochastic process $g(t)$ with respect to the same filtration. We define the stochastic integral of $f(t)g(t)$ to be the limit:*

$$\int_0^T f(t)g(t)dS^H(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(t_{i-1})g(t_i)(S^H(t_i) - S^H(t_{i-1})) \tag{2.12}$$

provided that the limit in probability exists.

It is clear that the anticipating integral (2.12) is not a $\mathcal{F}_t$-martingale since the integrand is non-adapted to $\mathcal{F}_t$. Moreover the integrator S^H is not a $\mathcal{F}_t$-semimartingale what brings us to verify if it satisfy the near-martingale property with respect to forward and backward filtrations given respectively by $\mathcal{F}_t = \sigma\{B(s), S^H(s); 0 \le s \le t\}$ and $\mathcal{F}^{(t)} = \sigma\{B(T) - B(s), S^H(T) - S^H(s); 0 \le s \le t\}$.

Theorem 2.2.2. *Let $\mathcal{F}_t$ be a forward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $\mathbb{E}\big[\int_0^T f(B(t))g(B(T) - B(t))dS^H(t)\big] < +\infty$,
2. $\mathbb{E}\big[g(B(T) - B(t))\big] = 0$.

Then,

$$X_t = \int_0^t f(B(s))g(B(T) - B(s))dS^H(s); \qquad 0 \le t \le T \tag{2.13}$$

exists and is a near-martingale with respect to the forward filtration $\mathcal{F}_t$.

Proof. We need to verify that $\mathbb{E}[X_t - X_s/\mathcal{F}_s] = 0$ for $0 \le s \le t$. Notice that

$$X_t - X_s = \int_s^t f(B(u))g(B(T) - B(u))dS^H_u.$$

Let $\Delta_n = \{s = t_0 < t_1 < ... < t_{n-1} < t_n = t\}$ a partition of interval $[s,t]$ and $\Delta S_i^H = S^H(t_i) - S^H(t_{i-1})$, we have:

$$\mathbb{E}[X_t - X_s/\mathcal{F}_s] = \mathbb{E}\left[\int_s^t f(B(u))g(B(T) - B(u))dS^H(u)/\mathcal{F}_s\right].$$

From Definition 2.2.6

$$
\begin{aligned}
\mathbb{E}[X_t - X_s/\mathcal{F}_s] &= \mathbb{E}\Bigg[\lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n} f(B(t_{i-1}))g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}_s\Bigg] \\
&= \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\mathbb{E}\Bigg[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}_s\Bigg].
\end{aligned}
$$

It is sufficient to verify that every component of the last sum is zero. Recall that $f(B(t_{i-1}))$ is $\mathcal{F}_{t_{i-1}}$-measurable and $g(B(T) - B(t_i))$ is independent of $\mathcal{F}_{t_{i-1}}$, using the properties of conditional expectation, we obtain:

$$
\begin{aligned}
&E\Bigg[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}_s\Bigg] \\
&\qquad = E\Bigg[E\big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}_{t_i}\big]/\mathcal{F}_s\Bigg] \\
&\qquad = E\Bigg[f(B(t_{i-1}))\Delta S_i^H E\big[g(B(T)-B(t_i))/\mathcal{F}_{t_i}\big]/\mathcal{F}_s\Bigg] \\
&\qquad = E\Bigg[f(B(t_{i-1}))\Delta S_i^H E\big[g(B(T)-B(t_i))\big]/\mathcal{F}_s\Bigg].
\end{aligned}
$$

From the independence of Brownian increments and the zero expectation of $g(B(T) - B(t_i))$, we have:

$$
\begin{aligned}
&\mathbb{E}\Bigg[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}_s\Bigg] \\
&\qquad = \mathbb{E}\big[g(B(T)-B(t_i))\big]\mathbb{E}\Bigg[f(B(t_{i-1}))\Delta S_i^H/\mathcal{F}_s\Bigg] \\
&\qquad = 0.
\end{aligned}
$$

Thus X_t is a near-martingale with respect to $\mathcal{F}_t$. □

Theorem 2.2.3. *Let $\mathcal{F}_t$ be a forward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $\mathbb{E}\big[\int_0^T f(B(t))g(B(T)-B(t))dS^H(t)\big] < +\infty$,

2. $\mathbb{E}\big[g(B(T)-B(t))\big]=0.$

Then

$$Y_t=\int_t^T f(B(s))g(B(T)-B(s))dS^H(s); \qquad 0\leq t\leq T \tag{2.14}$$

exists and is a near-martingale with respect to the forward filtration $\mathcal{F}_t$.

Proof. Notice that for $0\leq s<t\leq T$, we have

$$Y_t-Y_s=-\int_s^t f(B(u))g(B(T)-B(u))dS^H(u)=-(X_t-X_s), \tag{2.15}$$

where X_t is given in (2.13). Thus Y_t is a near-martingale with respect to $\mathcal{F}_t$.

□

Theorem 2.2.4. *Let $\mathcal{F}^{(t)}$ be a backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $\mathbb{E}\big[\int_0^T f(B(t))g(B(T)-B(t))dS^H(t)\big]<+\infty,$

2. $\mathbb{E}\big[f(B(t))\big]=0.$

Then

$$X_t=\int_0^t f(B(s))g(B(T)-B(s))dS^H(s);0\leq t\leq T$$

exists and is a near-martingale with respect to the backward filtration $\mathcal{F}^{(t)}$.

Proof. According to the proof of Theorem 2.2.2, we just have to prove that

$$\mathbb{E}\Bigg[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}^{(t)}\Bigg]=0,$$

where $0\leq s<t\leq T$ and $s=t_0<t_1<....<t_{n-1}<t_n=t$. Notice that

$$\Delta S_i^H=(S_T^H-S_{t_{i-1}}^H)-(S_T^H-S_{t_i}^H)\in\mathcal{F}^{(t_{i-1})}.$$

Next, by the $\mathcal{F}^{(t_{i-1})}$- measurability of ΔS_i^H and the conditional expectation properties, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}^{(t)}\Big] \\
&\qquad = \mathbb{E}\Big[\mathbb{E}\big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}^{(t_{i-1})}\big]/\mathcal{F}^{(t)}\Big] \\
&\qquad = \mathbb{E}\Big[g(B(T)-B(t_i))\Delta S_i^H\mathbb{E}\big[f(B(t_{i-1}))/\mathcal{F}^{(t_{i-1})}\big]/\mathcal{F}^{(t)}\Big].
\end{aligned}$$

Notice that for each $s > t_{i-1}$, $B(T) - B(s)$ is independent of $\mathcal{F}_{t_{i-1}}$. This implies the independence of the σ-fields $\mathcal{F}^{(t_{i-1})}$ and $\mathcal{F}_{t_{i-1}}$. Since $f(B(t_{i-1}))$ is $\mathcal{F}_{t_{i-1}}$ measurable, it follows that $f(B(t_{i-1}))$ is independent of $\mathcal{F}^{(t_{i-1})}$. Thus

$$\begin{aligned}
&\mathbb{E}\Big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}^{(t)}\Big] \\
&\qquad = \mathbb{E}\Big[g(B(T)-B(t_i))\Delta S_i^H\mathbb{E}\big[f(B(t_{i-1}))\big]/\mathcal{F}^{(t)}\Big].
\end{aligned}$$

Since $\mathbb{E}\big[f(B(t_{i-1}))\big]$ is $\mathcal{F}^{(t)}$- measurable, then

$$\begin{aligned}
&\mathbb{E}\Big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}^{(t)}\Big] \\
&\qquad = \mathbb{E}\big[f(B(t_{i-1}))\big]\mathbb{E}\Big[g(B(T)-B(t_i))\Delta S_i^H/\mathcal{F}^{(t)}\Big] \\
&\qquad = 0.
\end{aligned}$$

□

Theorem 2.2.5. *Let $\mathcal{F}^{(t)}$ be a backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $\mathbb{E}\big[\int_0^T f(B(t))g(B(T)-B(t))dS^H(t)\big] < +\infty$,

2. $\mathbb{E}\big[f(B(t))\big] = 0.$

Then,

$$Y_t = \int_t^T f(B(s))g(B(T) - B(s))dS^H(s); \qquad 0 \le t \le T \tag{2.16}$$

exists and is a near-martingale with respect to the backward filtration $\mathcal{F}^{(t)}$.

Proof. From Theorem 2.2.4, we have $Y_t - Y_s = -(X_t - X_s)$. Consequently, the proof is completed. $\square$

2.2.3 New anticipating integral with respect to mixed fractional Brownian motion

Similarly to sub-fractional case when $H > \frac{1}{2}$, we give a definition of the stochastic integral of the product $f(t)g(t)$ by taking the mfBm M^H as an integrator. Formally, we have

Definition 2.2.6. *Let* $M^H(t)$, $H > \frac{1}{2}$ *be a mixed fractional Brownian motion and let* $\mathcal{F}_t$ *denote the* σ*-field generated by* $\{M^H(t), t \ge 0\}$. *For an adapted stochastic process* $f(t)$ *with respect to the filtration* $\mathcal{F}_t$, *and an instantly independent stochastic process* $g(t)$ *with respect to the same filtration. We define the stochastic integral of* $f(t)g(t)$ *as:*

$$\int_0^T f(t)g(t)dM^H(t) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^n f(t_{i-1})g(t_i)(M^H(t_i) - M^H(t_{i-1})) \tag{2.17}$$

provided that the convergence in probability exists.

It is quite clear that the anticipating integral (2.17) is not a $\mathcal{F}_t$-martingale. Thus, we have to check if this latter satisfies the near-martingale property. We show that this latter satisfy near martingale property with respect to the forward filtration

$$\mathcal{F}_t = \sigma\{B(s), M^H(s); 0 \le s \le t\}$$

and with respect to the backward filtration

$$\mathcal{F}^{(t)} = \sigma\{B(T) - B(s), M^H(T) - M^H(s), \ 0 \le s \le t\}.$$

Theorem 2.2.1. *Let $\mathcal{F}_t$ be a forward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $\mathbb{E}\big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\big] < +\infty$,

2. $\mathbb{E}\big[g(B(T)-B(t))\big] = 0.$

Then,

$$X_t = \int_0^t f(B(s))g(B(T)-B(s))dM^H(s); \qquad 0 \leq t \leq T \tag{2.18}$$

exists and is a near-martingale with respect to the forward filtration $\mathcal{F}_t$.

Theorem 2.2.2. *Let $\mathcal{F}_t$ be a forward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $\mathbb{E}\big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\big] < +\infty$

2. $\mathbb{E}\big[g(B(T)-B(t))\big] = 0.$

Then,

$$Y_t = \int_t^T f(B(s))g(B(T)-B(s))dM^H(s), \ \ 0 \leq t \leq T \tag{2.19}$$

exists and is a near-martingale with respect to the forward filtration $\mathcal{F}_t$.

Theorem 2.2.3. *Let $\mathcal{F}^{(t)}$ be a backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $\mathbb{E}\big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\big] < +\infty$,

2. $\mathbb{E}\big[f(B(t))\big] = 0.$

Then,

$$X_t = \int_0^t f(B(s))g(B(T) - B(s))dM^H(s), \ \ 0 \le t \le T \tag{2.20}$$

exists and is a near-martingale with respect to the backward filtration $\mathcal{F}^{(t)}$.

Theorem 2.2.4. *Let $\mathcal{F}^{(t)}$ be a backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $\mathbb{E}\big[\int_0^T f(B(t))g(B(T) - B(t))dM^H(t)\big] < +\infty$,

2. $\mathbb{E}\big[f(B(t))\big] = 0$.

Then,

$$Y_t = \int_t^T f(B(s))g(B(T) - B(s))dM^H(s), \ \ 0 \le t \le T \tag{2.21}$$

exists and is a near-martingale with respect to the backward filtration $\mathcal{F}^{(t)}$.

Proof. The proofs of Theorems 2.2.1-2.2.4 are similar to Theorems 2.2.2-2.2.5 (when dealing with S^H). □

Some results in the case where $H \in (\frac{3}{4}, 1)$

This section presents some results establishing the relationship between standard Bm and mixed-fBm in the case where $H > \frac{3}{4}$. We show that our anticipating integral with respect to M^H can be written as a Riemann sum depending on standard Bm satisfying the near martingale property.

Proposition 2.2.6. *Let $M^H(t); H > \frac{3}{4}$ be a mixed fractional Brownian motion and $\mathcal{F}_t = \sigma\{M^H(t), t \ge 0\}$. For an $\mathcal{F}_t$-adapted stochastic process $f(t)$ and an $\mathcal{F}_t$-instantly independent stochastic process $g(t)$, we have*

$$\int_0^T f(t)g(t)dM^H(t) = a \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^n f(t_{i-1})g(t_i)(B(t_i) - B(t_{i-1})) \tag{2.22}$$

provided that the convergence in probability exists.

Proof. The proof is a direct result of Theorem 1.7 of Cheridito [20]. □

Proposition 2.2.7. *Let $\mathcal{F}_t$ be a forward filtration, $\mathcal{F}^{(t)}$ denotes the backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

$$\mathbb{E}\big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\big] < +\infty.$$

Then,

$$X_t = \int_0^t f(B(s))g(B(T)-B(s))dM^H(s); \qquad 0 \le t \le T,$$

and

$$Y_t = \int_t^T f(B(s))g(B(T)-B(s))dM^H(s); \qquad 0 \le t \le T$$

exist and are near-martingales with respect to $\mathcal{F}_t$ and $\mathcal{F}^{(t)}$ respectively.

Proof. The proof of this proposition is based on Theorem 1.7 in Cheridito [20] and Theorems 3.5-3.8 given in Kuo et al. [63]. □

In what follows, we give some examples where we evaluate some anticipating stochastic integrals with respect to mixed fractional Brownian motion when $H > \frac{3}{4}$, using the result obtained in the Proposition 2.2.6.

Example 2.2.1. Consider the following integral

$$\int_0^t B(T)^2 dM^H(s), \qquad 0 \le t \le T.$$

The integrand $B(T)^2$ is decomposed as

$$B(1)^2 = [(B(T)-B(s))]^2 + 2B(s)[B(T)-B(s)] + B(s)^2.$$

In addition, the integral converges in probability to

$$\sum_{i=1}^{n}\big([(B(T)-B(s_i))]^2+2B(s_{i-1})[B(T)-B(s_i)]+B(s_{i-1})^2\big)(M^H(s_i)-M^H(s_{i-1})).$$

As M^H and aB are equivalent (in law), then the above sum can be expressed as

$$a\sum_{i=1}^{n}\big([(B(T)-B(s_i))]^2+2B(s_{i-1})[B(T)-B(s_i)]+B(s_{i-1})^2\big)(B(s_i)-B(s_{i-1})).$$

Therefore, we have

$$\int_0^t B(T)^2 dM^H(s)=aB(T)^2B(t)-2aB(T)t, \qquad 0\le t\le T.$$

In general, for any $n\in\mathbb{N}^*$, it is easy to check that

$$\int_0^t B(T)^n dM^H(s)=aB(T)^nB(t)-anB(T)^{n-1}t, \qquad 0\le t\le T.$$

Example 2.2.2. Consider the integrand $B(s)B(T)$, equivalently,

$$B(s)(B(T)-B(s))+B(s)^2.$$

Then,

$$\begin{aligned}\int_0^t & B(s)B(T)dM^H(s)\\ &= a\lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\big(B(s_{i-1})(B(T)-B(s_i))+B(s_{i-1})^2\big)(B(s_i)-B(s_{i-1}))\\ &=\frac{a}{2}B(T)(B(t)^2-t)-a\int_0^t B(s)ds, \qquad 0\le t\le T.\end{aligned}$$

In the same manner, an integrand of the form $\phi(B(s))B(T)$ can be decomposed as

$$\phi(B(s))(B(T)-B(s))+\phi(B(s))B(s),$$

for any continuous function $\phi(x)$. Therefore, the integral

$$\int_0^t \phi(B(s))B(T)dM^H(s), \qquad 0 \leq t \leq T$$

converges in probability to

$$aB(T)\sum_{i=1}^{n}(\phi(B(s_{i-1}))(B(s_i)-B(s_{i-1}))-a\sum_{i=1}^{n}\phi(B(s_{i-1}))(B(s_i)-B(s_{i-1}))^2,$$

which is equivalent to

$$aB(T)\int_0^t \phi(B(s))dB(s) - a\int_0^t \phi(B(s))ds.$$

Example 2.2.3. The integral

$$\int_0^t e^{B(T)}dM^H(s), \qquad 0 \leq t \leq T$$

is the limit of the sum

$$e^{B(T)}\sum_{i=1}^{n} e^{(B(s_i)-B(s_{i-1}))}(M(s_i)-M(s_{i-1})).$$

Using Taylor series expansions of exponential function, Equation (2.2.3) converges in probability to

$$ae^{B(T)}\sum_{i=1}^{n}\big(1-(B(s_i)-B(s_{i-1}))-\frac{1}{2}(B(s_i)-B(s_{i-1}))^2+o((B(s_i)-B(s_{i-1}))^2)(B(s_i)-B(s_{i-1})).$$

Consequently,

$$\int_0^t e^{B(T)}dM^H(s) = ae^{B(T)}(B(t)-t), \qquad 0 \leq t \leq T.$$

2.2.4 Practical application of of the anticipating integral

Our study has a notable application in finance and economy. For instance, we consider a financial stock market where the process $f(t)$ is a quantity of the stock at time t, adapted to $\mathcal{F}_t$, the σ-field represents information available by time t, and $B(t)$ (the standard Brownian motion) characterizes the stock price at time t. The integral $\int_0^T f(t)dB(t)$ describes the change of the stock market wealth over the trading period $[0,T]$. By dividing the time integral into the subintervals $[t_{i-1},t_i]$, $\int_0^T f(t)dB(t)$ can be computed as a limit of Riemann-like sums of $f(t_{i-1})(B(t_i)-B(t_{i-1}))$. The use of the left endpoint of subintervals comes from the fact that $f(t)$ depends on the past and present but not on the future. If one comes across the case where the quantity of stock $f(t)$ is independent of past and present, i.e for each $t \in [0,T])$, $f(t)$ is $\mathcal{F}_t$-independent, then the future change in stocks can be known and one can use the right endpoint t_i as an evaluation point for the above stochastic integral. On the other hand, it has been interesting, in recent years, to divide the noise of stock price into two parts: the first describes the stochastic behavior of stock markets which is considered as a white noise, the other one represents the random state of the stock price which has a long memory, this motivates researchers to take such a situation into consideration and to provide a mixture of processes in accordance with the requirements of the phenomena.

Furthermore, over the past, there has been an extensive studies on option pricing. It has been shown that the distributions of logarithmic returns of financial assets generally exhibit properties of self-similarity and long-term dependence, and since the fractional Brownian motion has these two important properties, it has the ability to capture the behavior of the underlying asset price. The Black-Scholes model supposed that the volatility of the underlying security is constant, while stochastic volatility models classified the price of the underlying security as a random variable or, more generally, a stochastic process. In turn, the dynamics of this stochastic process can be driven by another process (usually by Brownian motion). In a stochastic volatility model, the volatility randomly changes according to stochastic processes. In our work, the process used is the mixture between fBm (fractional Brownian motion) and Bm(Brownian motion). The current study helps to solve the stochastic differential equations (SDEs) driven by a mixed fractional Brownian motion in the case of non adapted integrands which contributes to the resolution of the phenomena linked to volatility in the above situations.

Chapter 3

Stochastic volatility modeling via fractional Gaussian processes

The concept of volatility is probably one of the most researched topics in the field of financial mathematics. The interest is motivated by two important reasons: the increasing number of companies using risk management tools and the large number of derivatives traded in the world's financial markets.

The fair price of an option is determined by a number of factors including the volatility of the underlying asset. All factors are directly observable in the market except for volatility. The most well-known study in the world of mathematical finance is certainly the Black & Scholes option pricing model, which has been an immediate success with researchers and financial practitioners . The model allows the price of an option to be assessed in the non-arbitrage case based on the assumption that the valuation of a security follows a geometric Brownian motion and that the volatility is constant. However, the assumption of constant volatility in the Black & Scholes formula was rejected from the beginning as demonstrated in various works, mainly Black [12] in 1976 and especially after the 1987 Krash which highlighted the smile effect which should not appear under the Black & Scholes hypothesis. Accordingly, alternative models have been proposed since eighties to capture the empirical behavior of the volatility. We give in this chapter an overview volatility dynamic in financial markets and its different classes: constant, local and stochastic volatility, then we present the fractional stochastic volatility models: Comte & Renault model with $H > \frac{1}{2}$ and the rough stochastic volatility model with $H < \frac{1}{2}$. Finally we give our mixed stochastic volatility model

3.1 A review of volatility dynamics

3.1.1 Types of volatility in markets

In the context of option pricing theory, volatility takes two main and distinct forms:

Realized volatility (historical volatility)

The concept of realized volatility, introduced by Andersen and Bollerslev [3], is an alternative measure of daily volatility in financial markets. The realized volatility modeling is based on the idea of using the sum of the squared intra-day returns to generate more accurate measures of daily volatility. Merton [76] was the first to use high frequency data to measure volatility. Realized volatility is a statistical measure of the dispersion of returns of a given price of a financial instrument over a given period of time. In general, this measure is calculated by determining the average deviation from the average price over the given period. Realized volatility is calculated from underlying price changes over a certain period. If this period is in the past, we call it historical volatility. Different sources may use slightly different historical volatility formulas, the following is the most common approach: calculating historical volatility as standard deviation of logarithmic returns, based on daily closing prices. Given the (discrete) stock price time series $S(t_0), ..., S(t_n)$, the realized volatility σ in discrete time is obtained by calculating the standard deviation of the stock's continuously compound return R_i for each period (sampling interval) $\Delta t = t_i - t_{i-1}$, such that :

$$R_i = \ln\left(\frac{S(t_i)}{S(t_{i-1})}\right).$$

The realized variance of the log-returns of the stock is given by:

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \overline{R})^2,$$

where the sample mean $\overline{R}$ is given by:

$$\overline{R} = \frac{1}{n}\sum_{i=1}^{n} R_i.$$

The realized volatility converges in probability to the integrated volatility as $n \to +\infty$ during a certain time period $[0, T]$:

$$\sigma_t^2 = \frac{1}{T}\int_0^T \sigma^2(u)du.$$

Implied volatility

Implied volatility consists of using observed market prices to extract a volatility; it is related to the present value of the market. The Black & Scholes implied volatility is given by the volatility that equals to the option price given by the Black & Scholes formula and the option price observed on the market.

$$C_t(S_t, ..., T, K, \hat{\sigma}) = \hat{C}_t,$$

where C_t is the theoretical premium calculated through the model and $\hat{C}_t$ is the premium of call observed on the market. Its calculation requires the inversion of the Black & Scholes valuation formula. Among the methods of calculating implied volatility, we present two common numerical methods:

- The Newton-Raphson algorithm that requires a priori knowledge of the option premium derivative in relation to the volatility

$$\hat{\sigma}_{i+1} = \hat{\sigma}_i + a(C_t(\hat{\sigma}_i) - \hat{C}_t)\left(\frac{\partial C_t(\hat{\sigma}_i)}{\partial \hat{\sigma}_i}\right)^{-1},$$

where $0 < a \leq 1$ (for more details see Manaster and Koehler [71]).

- Method of bisection which is based on the choice of two volatilities, one low σ_l and other high σ_h, corresponding respectively to premiums C_t^l and C_t^h such as $C_t^l < \hat{C}_t < C_t^h$ and

$$\hat{\sigma} = \sigma_l + \frac{(\hat{C}_t - C_t^l)(\sigma_h - \sigma_l)}{(C_t^h - C_t^l)}.$$

(See Chapters 3 and 4 of Brent [17] for more details).

3.1.2 Stylized facts of volatility

Volatility clustering

In 1963, Mandelbrot [72] observed an important characteristic of volatility which he summarizes with the famous remark : "large changes tend to be followed by large changes - of either sign- and small changes tend to be followed by small changes". This phenomena is called "volatility clustering". it can be explained also by a positive autocorrelation of the absolute log-returns over few days.

Leverage effect

The leverage effect is another empirical characteristic of volatility, it determined a negative relation between stock returns and volatilities This effect is first observed by Black [12] in 1976 and describes the negative correlation with stock prices and volatility. Other empirical studies on leverage have been carried out by Chirtie [22], Nelson[81], Schwert [94], Bollerslev [16] and other references.

Long memory

The long memory in volatility is considered a stylized fact and has been notably reported in Ding and Granger [33] in 1993 and Andersen et al.[4] in 2001. The long memory is referred to the slow decay of auto-covariance function of volatility of scale $n > 0$

$$Cov(\sigma_{t+n}, \sigma_n) \sim \frac{C}{t^\gamma}, \quad n \to +\infty$$

with $\gamma < 1$. The auto-covariance function is not integrable, i.e.

$$\sum_{t=1}^{+\infty} Cov(\sigma_{t+n}, \sigma_n) = +\infty.$$

3.1.3 Volatility in Black & Scholes world

Black & Scholes model

In 1973, Black and Scholes [13] proposed to model stock price as a geometric Brownian motion with a constant volatility assumption. Although the constant volatility is obviously inconsistent with the stylized facts presented above. This helps us to understand how advanced mathematics is applied in finance. Moreover, Black & Scholes model is very practical in the sense that it provides a closed form expressions for the price of basic options like calls and puts.

The Black and Scholes model assumed that the dynamics of the price of the underlying asset is given by :

$$\forall t \in [0, T], \quad dS_t = S_t(\mu_t dt + \sigma_t dB_t), \quad S_0 > 0,$$

under the historical probability $\mathbb{P}$, where B is a standard Brownian motion, μ is the drift, σ is the volatility and μ, σ are assumed to be constant. In other words, the value at time t of the risky asset is given as the exponential of Brownian motion, it is a geometric Brownian motion as follows:

$$\forall t \in [0, T], \quad S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right).$$

and the risk-less asset Z_t with interest rate r under the structure

$$dZ_t = Z_t r dt.$$

Proposition 3.1.1. *(Jeanblanc et al. [53]). In the Black and Scholes model, there exists a unique equivalent risk-neutral measure $\mathbb{Q}$, precisely $\mathbb{Q}|_{\mathcal{F}_t} = \exp(-\theta B_t - \frac{\theta^2}{2}t)\mathbb{P}|_{\mathcal{F}_t}$, where $\theta = \frac{\mu - r}{\sigma}$ is the risk premium. The risk-neutral dynamic of the asset is given as:*

$$dS_t = S_t(rdt + \sigma dW_t), \tag{3.1}$$

where W is a $\mathbb{Q}$-Brownian motion. In a closed form, we have

$$\forall t \in [0, T], \quad S_t = S_0 e^{rt} \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right). \tag{3.2}$$

In the case of a European call, we have:

$$C(S_t, t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} \max(S_T - K, 0)]. \tag{3.3}$$

with $\mathbb{E}^{\mathbb{Q}}$ is the expectation under the risk-neutral probability. This expectation is explicitly calculated to obtain the value of the European options in the Black and Scholes model which is expressed by the Black and Scholes formula:

Theorem 3.1.1. *(Black and Scholes [13]). Let $dS_t = S_t(\mu_t dt + \sigma_t dW_t)$ be the dynamic of the price of a risky asset and assume that the interest rate is a constant r. The value at time t of an European call with maturity T and a strike K is C^{BS} where*

$$C^{BS}(t, S_t, \sigma, r, K, T) = S_t \mathcal{N}(d_1) - K e^{-r(T-t)} \mathcal{N}(d_2), \tag{3.4}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \tau\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}},$$

$$d_2 = d_1 - \sigma\sqrt{\tau},$$

$$\mathcal{N}(d) = \frac{1}{2\pi}\int_{-\infty}^{d} e^{-u^2} du$$

and $\tau = T - t$ is the time to maturity.

Implied volatility and volatility surface

In Black & Scholes world, the volatility is assumed to be constant. However, empirical data shows that options written on the same underlying asset with different strike price K and exercise time T have different implied volatilities. The plot of implied volatility as a function of strike price and expiry time produces the volatility surface, which is discussed in detail in Gatheral [40]. Such a Volatility Surface generated from a stochastic volatility inspired (for more details see Gatheral and Jaquier [41]) fit to closing prices of S&P options as of June 20, 2013 is is illustrated in Figure 3.1 in Gatheral et al. [42]. The plot of implied volatility against strike price K for a fixed maturity T is often symmetric and "U-shape" looking, hence called volatility smile. As we mentioned, the Black & Scholes model assumes a constant volatility, thus will have a volatility surface which is completely flat as illustrated in Figure 3.2. But in reality this is not the case. This is also a motivation of various models which are designed later on.

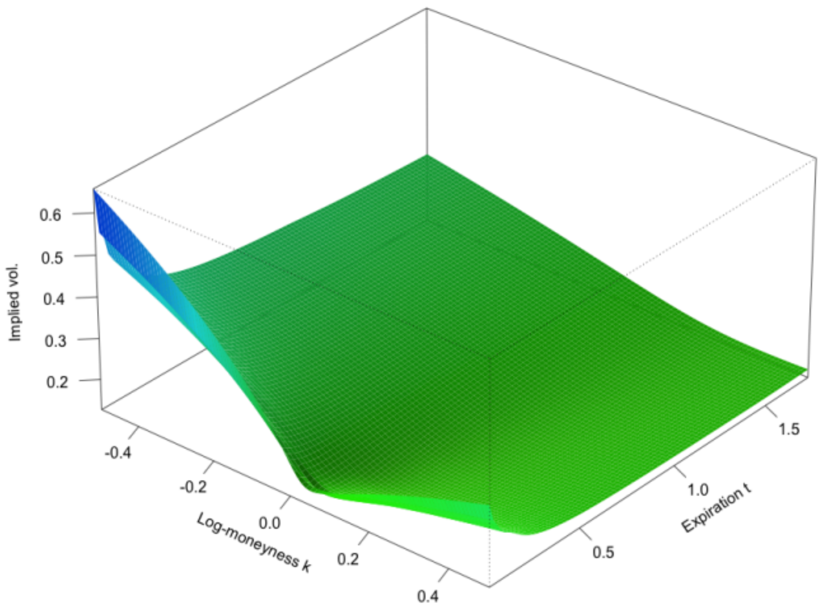

Figure 3.1: The S&P implied volatility surface as of June 20, 2013.

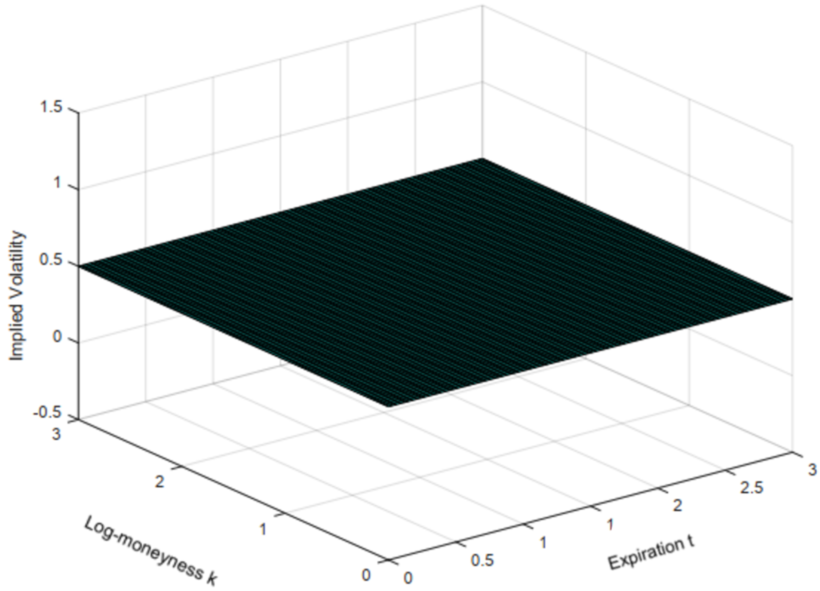

Figure 3.2: Black & Scholes implied volatility surface.

3.1.4 Local volatility model

The inability of generating empirical features such as leverage effect and the implied volatility surface via the constant volatility motivates the next generation model: the local volatility models where volatility is a deterministic function of time and stock price. The structure of local volatility models is given as follows:

$$dS_t = S_t(\mu dt + \sigma dB_t),$$

$$\sigma = f(t, S_t).$$

Dupire [35], Derman and Kani [31] give a relation between the volatility and the price of European calls. They have showed that for local volatility there exists a unique such risk-neutral process that is consistent with option data such that he stock price modeled by a risk natural diffusion process

$$dS_t = S_t(rdt + \sigma(t, S_t)dW_t),$$

and the price of the European call option by risk neutral valuation is given by

$$\begin{aligned} C(K,T) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}(S_T - K)^+ \\ &= e^{-rT}\int_K^{\infty}(S_T - K)^+ p(S_T) dS_T \end{aligned}$$

where $(S_T - K)^+ = \max((S_T - K), 0)$, $p(S_T)$ is the risk neutral probability density function for S_T, and K is the strike. Dupire used Fokker-Planck equation (see Risken [91]) to obtain local volatility formula presented in the next proposition:

Proposition 3.1.2. *(Dupire [35]). Assume that the European call prices*

$$C(K,T) = \mathbb{E}(e^{-rT}(S_T - K)^+))$$

for any maturity T and any strike K are known. If, under the risk-neutral probability, the stock price dynamics are given by

$$dS_t = S_t(rdt + \sigma(t, S_t)dW_t)$$

where σ is a deterministic function, then

$$\sigma^2(T,K) = \frac{\partial_T C(K,T) + rK\partial_K C(K,T)}{\frac{1}{2}K^2\partial^2_{KK}C(K,T)}$$

where ∂_T (resp. ∂_K) is the partial derivative operator with respect to the maturity (resp. the strike).

3.1.5 Stochastic volatility models

The local volatility models enable volatility to vary with strike K and maturity T, but they capture only some features of empirical volatility and still have some limitations. For instance, because volatility is a deterministic function of time and stock price, it is perfectly correlated with stock price, i.e. the absolute value of the correlation between volatility and stock price is 1. However, there is no empirically observed perfect correlation. Furthermore, volatility clustering is not captured in local volatility models. These failures motivated the stochastic volatility models which are described under the dynamic

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dB_t, \\ d\sigma_t = f(t, \sigma_t)dt + g(t, \sigma_t)dW_t, \\ corr(B_t, W_t) = \rho. \end{cases}$$

Note that ρ can take any value in $(-1, 1)$ but in order to capture leverage, ρ must be negative.

Stochastic volatility models have many more advantages over previous classes of volatility models, capturing more empirical features such as volatility clustering and leverage by choosing a negative value for ρ. In addition, they have greater variability and are able to capture extreme events such as the dramatic changes in volatility during the financial crisis. The disadvantages of stochastic volatility are mainly due to the additional randomness of the second stochastic process. The inconveniences of stochastic volatility are mainly due to the additional randomness of the second stochastic process W. Stochastic volatility has less practical use because the market is no longer complete, therefore not all contingent claims can be perfectly hedged and the price cannot be uniquely determined.

Hull and White

We refer to Hull and White [47] as the first authors who proposed the use of stochastic volatility models, this model describes the dynamic of asset price $S_t, t \geq 0$ as follows:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t,$$

$$d\sigma_t^2 = \theta \sigma_t^2 dt + \gamma \sigma_t^2 dB_t,$$

where σ_t denotes the stochastic volatility of price S_t, W_t and B_t, $t \in [0, T]$ are two standards Bm with correlation coefficient $\rho \in (-1, 1)$ and μ, θ, γ are constants.

Hull & White model provides a closed form solution to European option prices when $\rho = 0$. The solution is obtained by using Black & Scholes pricing formula with

$$\sigma_t = \sqrt{\frac{1}{T-t}\int_t^T \sigma_s^2 ds}. \tag{3.5}$$

Stein and Stein [98] adopted the same model but with a correlation between the asset price and the volatility.

SABR model

Hagan et al. [43] introduced SABR model which is a stochastic version of elasticity of variance (CEV) model. Under risk natural measure, the price follows the following system of stochastic differential equations

$$dS_t = \sigma_t S_t^\beta dB_t, \quad t > 0,$$

$$d\sigma_t = v\sigma_t dW_t, \quad t > 0.$$

Here, σ models the stochastic volatility of asset price, β is the elasticity parameter, $v > 0$ is the volatility of volatility. Note that in this model the volatility increases exponentially for long term options.

Heston model

Heston [44] suggested that the variance follows a Cox-Ingersoll-Ross (CIR) interest process.

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \tag{3.6}$$

$$d\sigma_t^2 = \alpha(m - \sigma_t^2)dt + \gamma\sigma_t^2 dB_t, \tag{3.7}$$

where m is the long-term average of $\sigma^2, t > 0, \alpha > 0$ is the rate of mean-reversion, γ is the volatility of volatility coefficient. The quantities μ, α, m, γ are real constants, $\alpha > 0$, and $W_t, B_t, t > 0$, are standard Brownian motions with correlation coefficient $\rho \in (-1, 1)$.

This model enjoys success since it is possible to deduce analytical formula for the price of an European call (put) options whose underlying asset price and its stochastic volatility satisfy Equations (3.6) and (3.7) respectively. The formula for the option prices are expressed as integrals of explicitly known integrands. These integrals must be numerically evaluated but due to the incompleteness of the market, the risk neutral measure has to be specified first.

3.2 Fractional stochastic volatility models

Stochastic volatility models can reproduce some important features of implied volatility such as the variation from the strike price, represented graphically as a smile or a skew. The effect of stochastic volatility are not easily explained, however, because of its dependence on the time of maturity. For example, the effects of stochastic volatility remain important for long maturities (see Bollosev and Mikkelsen [15]). In practice, the slope of the observed short-term implied volatility skewness tends to infinity when the time to maturity tends to zero, while for classical stochastic volatility models this limit is a constant (see Lee [66]). On the other side, The empirical results show that the autocorrelation function of squared hight frequency returns decays slowly towards zero, then it has been proposed that squared returns may be modeled as a long memory process whose autocorrelation function decays slowly. (see Andersen and Bollerslev [2], Andersen et al.[4] and Ding et al.[33]).

3.2.1 Comte and Renault model

Comte and Renault [25] introduced for a first time a fractional stochastic volatility model (FSV) which is a long memory volatility process, using a model based on a fractional Brownian motion with Hurst $H > \frac{1}{2}$. This model is a generalization of Hull & White model by replacing the Wiener process by a fBm. More precisely, the model considered that that the risky asset dynamic's price follows (3.6) and the log-volatility is a fractional

Ornstein-Uhlenbeck process where $H > \frac{1}{2}$, to ensure the long-range dependence:

$$dX_t = -\kappa X_t dt + \gamma dB_t^H, X_0 = 0,\ \kappa > 0,\ H > \frac{1}{2}. \tag{3.8}$$

The solution can be written as

$$X_t = \gamma \int_0^t e^{-\kappa(t-s)} dB_t^H.$$

Integration with respect to B^H is defined only in the Wiener L^2 sense and for the integration of deterministic functions. We thus obtain families of Gaussian processes. The process X_t can also be written as $\int_0^t a(t-s)dW_s$, such that

$$\begin{aligned} a(x) &= \frac{\gamma}{\Gamma(H-\frac{1}{2})} \frac{d}{dx} \int_0^x e^{-\kappa u}(x-u)^{H-\frac{1}{2}} du \\ &= \frac{\gamma}{\Gamma(H-\frac{1}{2})} \left(xH - \frac{1}{2} - \kappa e^{-\kappa x} \int_0^x e^{\kappa u} u^{H-\frac{1}{2}} \right). \end{aligned}$$

We denote by Y_t the stationary version of X_t, $Y_t = \int_{-\infty}^t a(t-s)dW_s$. Therefore, the solution X of the fractional SDE (3.8) is given by

$$X_t = \int_0^t \frac{(t-s)^{\frac{1}{2}-H}}{\Gamma(H-\frac{1}{2})} dX_s^H,$$

where

$$X_t^H = \frac{d}{dt} \int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(\frac{3}{2}-H)} = \gamma \int_0^t e^{-\kappa(t-s)} dW_s.$$

It seems interesting to note that long-memory fractional processes as considered in Comte and Renault [25] and solutions of (3.8), in particular, have the following properties proved in Comte [23]

- The covariance function $\rho = \rho_Y$ associated with X satisfies for $h \to 0$ and C constant:

$$\rho(h) = \rho(0) + \frac{1}{2}C|h|^{2H} + o(|h|^{2H}).$$

- X is ergodic in L^2 sense.

- There is a process Z_t equivalent to X_t, such that the sample function of Z satisfies a Lipschitz condition of order β, $\forall \beta \in (0, H)$, a.s.

Consider a truncated version $\widetilde{B}_t^H$ of the fBm such that:

$$\widetilde{B}_t^H = \int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} dW_t, \; H > \frac{1}{2}.$$

The volatility process σ_t is asymptotically equivalent (in quadratic variation) to the stationary process:

$$\widetilde{\sigma}_t = exp\left\{\gamma \int_{-\infty}^t e^{-\kappa(t-s)} d\widetilde{B}_t^H\right\}. \tag{3.9}$$

As in usual diffusion models of stochastic volatility, the volatility process is assumed to be asymptotically stationary and nowhere differentiable. The fractional exponent H provides some degree of regularity. Indeed, it is possible to show for σ_t, the same type of regularity property as for the fractional process $X_t = ln(\sigma_t)$

Proposition 3.2.1. *(Comte and Renault [25]).*
Let $r_\sigma(h) = Cov(\widetilde{\sigma}_{t+h}, \widetilde{\sigma}_t)$, where $\widetilde{\sigma}$ is given by (3.9). Then, for $h \to 0$,

$$r_\sigma(h) = r_\sigma(0) + \eta|h|^{2H} + o(|h|^{2H}),$$

where η is a given constant.

In fact, we can see that when $H > \frac{1}{2}$:

$$\frac{r_\sigma(h) - r_\sigma(0)}{h} \to 0 \quad as \quad h \to 0.$$

In the main, the autocorrelation function of the stationary process satisfies the regularity condition that ensures the Lipschitz characteristic of the sample paths. As the H is larger, the smoothness of the path of the volatility process is higher.

3.2.2 Rough fractional volatility model

Although correlated fBm volatility models (with Hurst parameter $H > \frac{1}{2}$) can explain the observed long range properties of implied volatility, it has been shown in Alòs et al. [1] that these models cannot describe its empirical behavior in the short range. In particular, if $H > \frac{1}{2}$, the slope of the skewness of implied volatility at money tends to zero with time to maturity when the time to maturity tends to zero, whereas this slope tends to infinity in the case of non-correlation $H < \frac{1}{2}$. These properties were also investigated by Fukasawa [39].

According to the above results, Gatheral et al. [42] recently suggested a model of fractional volatility with $H < \frac{1}{2}$. This model was found to be very efficient in describing real market data. Gatheral et al. [42] suggested to use a more rough version of the previous model, they proved that the log-volatility behaves like a fractional Brownian motion with Hurst index H towards 0.1 and showed that this model is remarkably consistent with empirical financial time series data, and gives better forecasts of realized volatility. The model has the following structure:

$$dX_t = \gamma dB_t^H + \kappa(\theta - X_t)dt, \tag{3.10}$$

with $\theta \in \mathbb{R}$, κ and γ are positive parameters and $H < \frac{1}{2}$.

Using daily realized variance estimates as proxies for daily spot volatilities, authors found that the empirical quantity

$$m(\Delta, q) = \frac{1}{N}\sum_{k=1}^{N} |\log(\sigma_{k\Delta}) - \log(\sigma_{(k-1)\Delta})|^q$$

with $k \in \{[t/\Delta]\}, N = [T/\Delta]$ and $q \geq 0$, enjoy the scaling property of the fBm it the sense that:

$$\mathbb{E}\big[|\log(\sigma_{t+\Delta}) - \log(\sigma_t)|^q\big] = K_q v^q \Delta^{qH}, \tag{3.11}$$

where $0,06 < H < 0,2$ and it can be seen as a measure of smoothness characteristic of the underlying volatility process. In addition, the distributions of increments of log-volatility is close to normal distribution. Then, naturally the log-volatility may be modeled using fractional Brownian motion under the form:

$$\log(\sigma_{t+\Delta}) - \log(\sigma_t) = v(B^H_{t+\Delta} - B^H_t). \tag{3.12}$$

Clearly, (3.12) can be written under the form

$$\sigma_t = \alpha \exp\{vB^H_t\}, \qquad \alpha \in \mathbb{R},$$

which is non-stationary model. The solution of the stochastic differential equation (3.10) is the stationary version of RFSV model, then the final specification of the RFSV model for the volatility on the time interval is as follows:

$$\sigma_t = \exp\{X_t\}.$$

3.3 Stochastic volatility modeling via mixed fractional Brownian motion

We propose a generalization of rough stochastic volatility model by dealing with the mixed fractional Brownian motion with $H < \frac{1}{2}$, we show that the realized volatility enjoy the scaling property of mixed fBm with $0.09 < H < 0.2$.

3.3.1 Empirical results related to the increments of log-volatility

The scaling property and gaussianity

In this section, we are interesting in the dynamic of log-volatility process, since it is not directly observed. We use non parametric daily data from Oxford-Man Institute of Quantitative Finance Realized Library during 5311 days (from January 3, 2000 to November 2,

2020) which include a collection of daily non parametric estimates of integrated variance of all indices in the Oxford-Man dataset[1], we use the realized kernel estimates (Tukey-Hanning(2)) denoted by rk_th2 in order to proxy the spot volatility. We consider in our study the S&P and FTSE indices. Following the same steps as Gatheral et al.[42], we observe the behavior of the empirical quantity defined by:

$$m(\Delta, q) = \frac{1}{N} \sum_{k=1}^{N} |\log(\sigma_{k\Delta}) - \log(\sigma_{(k-1)\Delta})|^q, \tag{3.13}$$

where $\{\sigma_{k\Delta}\}_k$ is a volatility process on time interval $[0, T]$, $k \in \{0, N\}, N = [T/\Delta]$ and $q > 0$. We recall that $m(\Delta, q)$ is the empirical counterpart of $\mathbb{E}(|\log \sigma_{t+\Delta} - \log \sigma_t|^q)$.

Our aim is to show that the log-volatility can be approximated by mixed fractional Brownian motion, i.e, $m(\Delta, q)$ satisfies the scaling property of mfBm (1.7) in the sense that:

$$m(\Delta, q) \leq \widetilde{C}_q \Delta^{f_q}, \tag{3.14}$$

where $\widetilde{C}_q$ is a constant depending on q. The parameter f_q can be viewed as the smoothness parameter which controls the regularity of trajectories of the log-volatility process. To give the best estimation of the parameter H for which

$$f_q \sim q\left(\frac{1}{2} \wedge H\right),$$

we first display the $\log m(\Delta, q)$ against $\log \Delta$ for different values of q.

[1]Data obtained from https://realized.oxford-man.ox.ac.uk/data/download

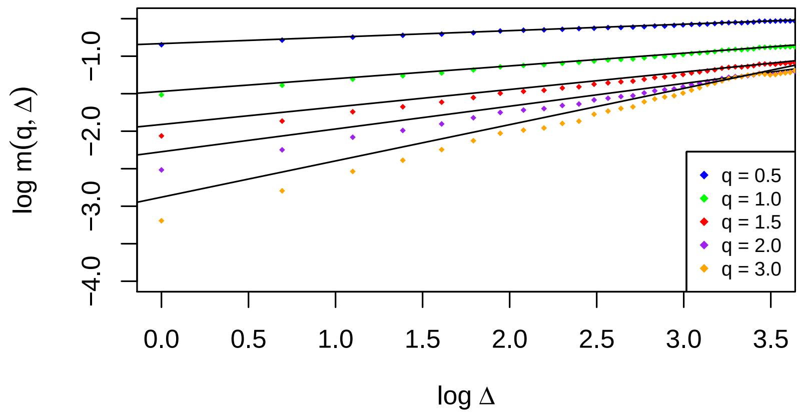

Figure 3.3: $\log m(\Delta, q)$ vs. $\log \Delta$ of S&P index.

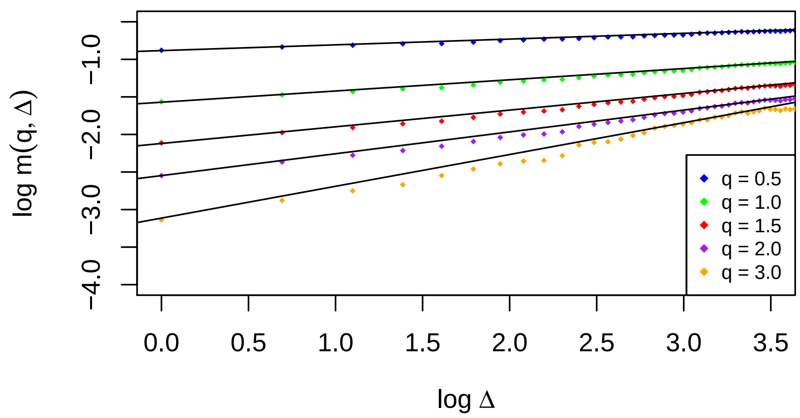

Figure 3.4: $\log m(\Delta, q)$ vs. $\log \Delta$ of FTSE index.

Then, in order to give a graphic illustration of (3.14), we plot a straight line of the form

$$y : \log \Delta \mapsto f_q \log(\Delta) + v_q,$$

where f_q is the slope of line associated with q and v_q is a positive constant, such that for all q:

$$\log m(\Delta, q) \leq f_q \log(\Delta) + v_q.$$

We can see easily that for some f, the inequality (3.14) holds. Now, we give the estimation of the smoothness parameter by regressing f_q against q. We get $f_q \sim q\left(\frac{1}{2} \wedge H\right)$ with $H = 0.156$ for S&P and with $H = 0.147$ for FTSE.

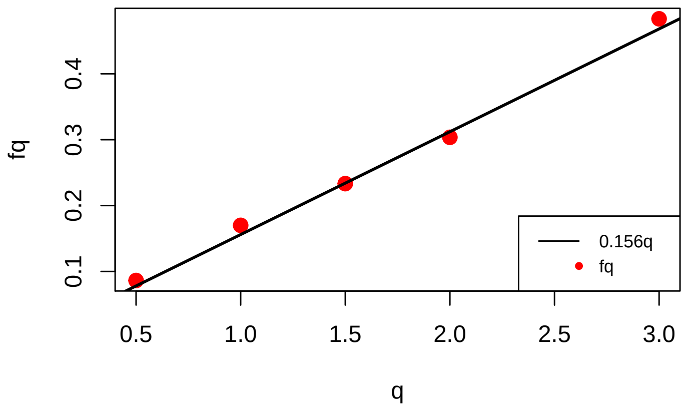

Figure 3.5: f_q against q of S&P index.

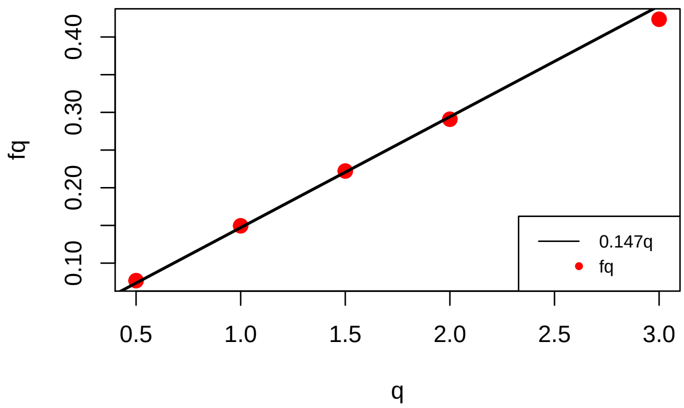

Figure 3.6: f_q against q of FTSE index.

Furthermore, the comparison between the increments of empirical log-volatility over different time scales ($\Delta = 1$, $\Delta = 10$, $\Delta = 100$) and the normal fit shows that the distribution of log-volatility increments tends to Normal distribution. We display histograms of log-volatility increments and the normal fit of mfBm for S&P index, the same result is obtained for FTSE index.

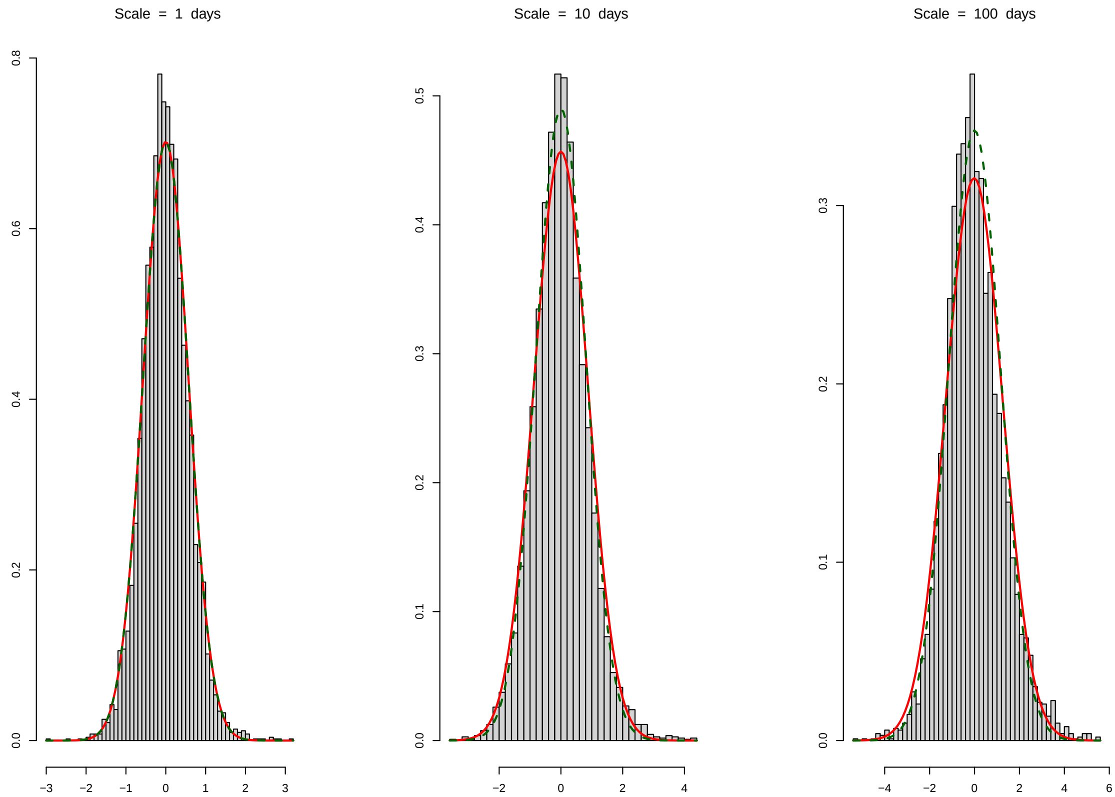

Figure 3.7: The distribution of the increment of log-volatility, standard normal distribution in red and distribution of the increments of mfBm with H=0.156 in green, S&P index.

Smoothness parameter of other indices under the MFSV model and comparison with rough model

To obtain the best estimation of the smoothness parameter, we repeat the statistical study applied on S&P and FTSE indices for all other indices of the data from Oxford-Man Institute, we find that the estimated parameter H varies between 0.09 and 0.2. Moreover, comparing the results obtained using our mixed fractional volatility (MFSV) model and those obtained from the rough volatility (RFSV) model, we remark a slight difference between (see Table 3.1). We conclude that the choice of a stochastic volatility model generated by the mixed fractional Brownian motion with Hurst index $H < \frac{1}{2}$ can be seen as an alternative of rough model.

Index	MFSV	RFSV	Index	MFSV	RFSV
AEX.rk_th2	0.161	0.162	KSE.rk_th2	0.0795	0.117
AORD.rk_th2	0.109	0.115	MXX.rk_th2	0.0961	0.0974
BFX.rk_th2	0.154	0.152	N225.rk_th2	0.144	0.139
BVSP.rk_th2	0.14	0.14	OMXC20.rk_th2	0.123	0.124
DJI.rk_th2	0.161	0.174	OMXHPI.rk_th2	0.131	0.132
FCHI.rk_th2	0.148	0.149	OSEAX.rk_th2	0.144	0.152
FTMIB.rk_th2	0.171	0.154	RUT.rk_th2	0.133	0.134
FTSE.rk_th2	0.147	0.15	SMSI.rk_th2	0.124	0.124
GDAXI.rk_th2	0.163	0.164	SPX.rk_th2	0.156	0.17
GSPTSE.rk_th2	0.149	0.151	SSEC.rk_th2	0.141	0.137
HSI.rk_th2	0.111	0.119	SSMI.rk_th2	0.202	0.198
IBEX.rk_th2	0.145	0.141	STI.rk_th2	0.0793	0.0789
IXIC.rk_th2	0.153	0.156	STOXX50E.rk_th2	0.127	0.128
KS11.rk_th2	0.108	0.13			

Table 3.1: Estimated values of H under mixed fractional model and rough model.

3.3.2 Mixed fractional stochastic volatility model

The empirical results given in Section 3.3.1 showed that the increments of log-volatility behave as those of mfBm and their probability distribution tends to Normal distribution. Thus, in the light of rough model and taking into account the properties of $(M^H_{t+\Delta} - M^H_t)$, let:

$$\log(\sigma_{t+\Delta}) - \log(\sigma_t) = \gamma(M^H_{t+\Delta} - M^H_t), \tag{3.15}$$

where γ is a positive constant and H the empirical measure of smoothness of log-volatility. Equivalently, we have :

$$X_t = \gamma M^H_t + \kappa, \qquad \kappa \in \mathbb{R}^+, \;\; t \in [0, T], \tag{3.16}$$

where $X_t = \log(\sigma_t)$. X_t is clearly non-stationary model because of the non-stationarity of the process M^H. The first natural way to get a stationary model is to define an Ornstein Uhlenbeck process as the solution to the following Langevin equation:

$$dX_t = -\lambda(X_t - \mu)dt + \gamma dM_t^H, \;\; t \in [0, T], \tag{3.17}$$

where λ, μ and γ are positive constants. The explicit solution of (3.17) is given by Riemann Stieltjes integral as :

$$X_t^\lambda = \gamma \int_{-\infty}^{t} e^{-\lambda(t-s)} dM_s^H + \mu.$$

If λ tends to zero, the mixed fractional Ornstein Uhlenbeck process tends to the non-stationary model (3.16), i.e. the behavior of the stationary log-volatility model is approximatively similar to that of mfBm. Moreover the scaling property of its trajectories tends to that of M^H. These facts are established.

Proposition 3.3.1. *Let M^H be a mixed fractional Brownian motion and X^λ defined by (3.3.2) for a given $\lambda > 0$. As $\lambda > 0$ tends to zero, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \mid X_t^\lambda - X_0^\lambda - \gamma M_t^H \mid\right] \to 0. \tag{3.18}$$

Proof. We have

$$X_t^\lambda = \gamma \int_{-\infty}^{t} e^{-\lambda(t-s)} dM_s^H + \mu.$$

Applying integration by parts, we obtain

$$X_t^\lambda = \gamma M_t^H - \gamma\lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} M_s^H ds + \mu.$$

Then,

$$\begin{aligned} X_t^\lambda - X_0^\lambda - \gamma M_t^H &= \gamma\lambda \left(\int_{-\infty}^{t} e^{-\lambda(t-s)} M_s^H ds + \int_{-\infty}^{0} e^{\lambda s} M_s^H ds \right) \\ &= \gamma\lambda \left(\int_{-\infty}^{0} (e^{\lambda s} - e^{-\lambda(t-s)}) M_s^H ds - \int_{0}^{t} e^{-\lambda(t-s)} M_s^H ds \right). \end{aligned} \tag{3.19}$$

Using the fact that $e^x \geq 1 + x$, we have

$$\sup_{t\in[0,T]} \mid X_t^\lambda - X_0^\lambda - \gamma M_t^H \mid \leq \gamma\lambda \int_{-\infty}^{0} (T\lambda e^{\lambda s}) \hat{M}_s^H ds - \gamma\lambda T \hat{M}_T^H, \tag{3.20}$$

where $\hat{M}_T^H = \sup_{t\in[0,T]} |M_s^H|$. Then,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \mid X_t^\lambda - X_0^\lambda - \gamma M_t^H \mid\right] \leq c\gamma\lambda \int_{-\infty}^{0} (T\lambda e^{\lambda s}) \mathbb{E}\left[\hat{M}_s^H\right] ds - \gamma\lambda T \mathbb{E}\left[\hat{M}_T^H\right], \tag{3.21}$$

where c is a constant. Using the inequality of Novikov [83] for the fBm, classical Burkholder inequalities for the standard Brownian motion [19] and taking into account that $\hat{M}_T^H \leq \hat{B}_T^H + \hat{W}_T$, we get

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]} \mid X_t^\lambda - X_0^\lambda - \gamma M_t^H \mid\right] \\ &\qquad \leq c\gamma\lambda \int_{-\infty}^{0} (T\lambda e^{\lambda s}) \left(|s|^H + |s|^{\frac{1}{2}}\right) ds - \gamma\lambda T \left(|T|^H + |T|^{\frac{1}{2}}\right). \end{aligned} \tag{3.22}$$

Clearly,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \mid X_t^\lambda - X_0^\lambda - \gamma M_t^H \mid\right] \xrightarrow[\lambda\to 0]{} 0. \tag{3.23}$$

□

Corollary 3.3.1. *Let $q > 0$, $t > 0$ and $\Delta > 0$. As $\lambda \to 0$, we have*

$$\mathbb{E}\left[|X_{t+\Delta}^\lambda - X_t^\lambda|^q\right] \to \gamma^q K_q \Delta^{Hq}. \tag{3.24}$$

Proof. We have

$$\begin{aligned}
X_t^\lambda &= \gamma \int_{-\infty}^{t} e^{-\lambda(t-s)} dM_s^H + \mu \\
&= \gamma \int_{-\infty}^{t} e^{-\lambda(t-s)} dB_s^H + \gamma \int_{-\infty}^{t} e^{-\lambda(t-s)} dB_s + \mu \\
&= Z_t^\lambda + Y_t^\lambda + \mu,
\end{aligned}$$

where Z_t^λ is a fractional Ornstein Uhlenbeck process and Y_t^λ is the classical Ornstein Uhlenbeck process. Due to the independence between B^H and B, Z_t^λ and Y_t^λ are independents. Therefore we have

$$Cov(X_{t+\Delta}^\lambda, X_t^\lambda) = Cov(Z_{t+\Delta}^\lambda, Z_t^\lambda) + Cov(Y_{t+\Delta}^\lambda, Y_t^\lambda).$$

According to Cheridito [21],

$$Cov(Z_{t+\Delta}^\lambda, Z_t^\lambda) = K \int_{\mathbb{R}} e^{i\Delta x} \frac{|x|^{1-2H}}{\lambda^2 + x^2} dx,$$

with $K = \frac{\gamma^2 \Gamma(2H+1) sin(\pi H)}{2\pi}$, then, it follows

$$Cov(Y_{t+\Delta}^\lambda, Y_t^\lambda) = K \int_{\mathbb{R}} e^{i\Delta x} \frac{1}{\lambda^2 + x^2} dx.$$

The second moment $\mathbb{E}\left[|X_{t+\Delta}^\lambda - X_t^\lambda|^2\right]$ is equal to

$$2V\left[X_t^\lambda\right] - 2Cov\left[X_{t+\Delta}^\lambda, X_t^\lambda\right].$$

Then,

$$\mathbb{E}\left[|X_{t+\Delta}^\lambda - X_t^\lambda|^2\right] = 2K \int_{\mathbb{R}} \left(1 - e^{i\Delta x}\right) \frac{1 + |x|^{1-2H}}{\lambda^2 + x^2} dx.$$

The right hand side is uniformly bounded by

$$2K \int_{\mathbb{R}} \left(1 - e^{i\Delta x}\right) \frac{1 + |x|^{1-2H}}{x^2} dx,$$

for some fixed Δ.

Here, $X^{\lambda}_{t+\Delta} - X^{\lambda}_{t}$ is a Gaussian random variable, according to the properties of Normal distribution, the family $|X^{\lambda}_{t+\Delta} - X^{\lambda}_{t}|^q$ is uniformly integrable. Proposition 3.3.1 claimed that $|X^{\lambda}_{t} - X^{\lambda}_{0}| \to \gamma M^{H}_{t}$ in distribution as λ tends toward zero. Thus by stationarity, we get

$$\mathbb{E}\left[|X^{\lambda}_{t+\Delta} - X^{\lambda}_{t}|^q\right] \underset{\lambda\to 0}{\longrightarrow} \gamma^q \mathbb{E}\left[|M^{H}_{t+\Delta} - M^{H}_{t}|^q\right].$$

□

Conclusion and Perspectives

Conclusion

In this thesis, we introduced an anticipating stochastic integral with respect to a sub-fractional Brownian motion and a mixed fractional Brownian motion (mfBm) in the case where $H > \frac{1}{2}$. This gives a new concept of stochastic integration of non-adapted processes. Under some conditions, we showed that our anticipating integral turns out to be a near-martingale. In addition, few specific cases when dealing with M^H when $H > \frac{3}{4}$ have been treated.

Moreover, we shed light on an important issue on modern financial modeling which is the description of volatility process. Due to the important characteristics of the mixed fractional Brownian motion and their adequacy with observed time series data, we give a description of volatility process under our mixed fractional stochastic volatility model as:

$$\sigma_t = exp\left(X_t\right), \qquad t \in [0, T],$$

where X_t is the mixed fractional Ornstein-Uhlenbeck process with index $H < \frac{1}{2}$.

We gave in the first chapter a general overview on fractional, sub-fractional and mixed fractional Brownian motions as well as fractional Ornstein-Uhlenbeck process.

In the second chapter, we defined integrals of non-adapted process which is a product of adapted and instantly independent processes with respect to sub-fractional and mixed fractional Brownian motions as a Riemann sum. Then and we showed that our anticipating integrals are near-martingales under some assumptions.

In the third chapter, we proposed a stochastic volatility model which is represented as an Ornstein-Uhlenbeck process driven by a mixed fractional Brownian motion M^H when $H < \frac{1}{2}$.

Perspectives

The study carried out in this thesis can be used to develop new research on anticipating stochastic integration theory as well as stochastic volatility modeling. for further work , there is many interesting issues to address such as :

- Develop an anticipating Itô formula of the anticipating integral with respect to sub-fBm and mixed fBm.
- Define anticipating stochastic integrals with respect to fractional, sub-fractional and mixed fractional Brownian motion when $H < \frac{1}{2}$ using other technics of stochastic integration.
- Extend the study dealing with a more general fractional Gaussian processes.
- Introduce new stationary stochastic volatility models via Gaussian processes with long/short range dependence .
- Introduce an European pricing option formula using MFSV model.

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"التكامل العشوائي المتعلق بعمليات غوسيان والتطبيق في نماذج التقلب العشوائي الكسرية المختلطة"

الملخص: الهدف من هذه الأطروحة هو نمذجة التقلب العشوائي من خلال عمليات غوسيان الكسرية ذات الذاكرة الطويلة والمسارات غير المنتظمة. نستخدم بيانات عالية التردد لتقدير انتظام مسارات لوغاريتم التقلب العشوائي. نقترح الحركة البراونية الكسرية المختلطة كنموذج تقلب عشوائي ونقدم عملية أورنشتاين أولنبك لما $H < 1/2$ كنموذج ثابت للوغاريتم التقلب العشوائي.
علاوة على ذلك، أظهرنا نتيجة أساسية لتكامل العمليات غير المكيفة بالنسبة لعمليات غوسيان من النوع الكسري (الحركة البراونية الكسرية المختلطة، الحركة البراونية الكسرية الجزئية) لمّا يكون $H > 1/2$ كمجموع ريمان مع الاختيار المناسب لقيم النقاط في المجالات الفرعية وهذا من خلال تفكيك عملية الدمج إلى مجموعة خطية من العمليات المتكيفة والمستقلة على الفور. تعتبر هذه الدراسة بمثابة تعميم لتلك التي تم وضعها في الإطار البراوني. بالإضافة إلى ذلك، في ظل شروط معينة، أثبتنا أن هذه التكاملات قريبة من مارتينغال.

الكلمات المفتاحية: عمليات غوسيان، الحركة البراونية الكسرية المختلطة، الحركة البراونية الكسرية الفرعية، العملية غير المكيفة، القرب من مارتينغال، التقلب العشوائي.

« Intégration stochastique par rapport à des processus Gaussiens et application aux modèles de volatilité stochastique fractionnaire mixte »

Résumé : L'objectif de cette thèse est de modéliser la volatilité par des processus Gaussiens fractionnaires à mémoire longue et à trajectoires irrégulières. Nous utilisons des données à haute fréquence pour estimer la régularité des trajectoires de la log-volatilité. Nous proposons le mouvement Brownien fractionnaire mixte avec H< ½ comme un modèle de volatilité stochastique et nous construisons un processus d'Ornstein-Uhlenbeck fractionnaire mixte stationnaire comme modèle stationnaire de la log-volatilité
Par ailleurs, nous avons démontrer un résultat fondamental sur l'intégration des processus non adaptés par rapport aux processus de type fractionnaire (mouvement Brownien sous fractionnaire, mouvement Brownien fractionnaire mixte) lorsque H> ½ comme une somme de Riemann avec un choix approprié de points d'évaluation de sous-intervalles tout en décomposant le processus intégrant à une combinaison linéaire des processus adaptés et instantanément indépendants. Cette étude est considérée comme une généralisation du celle établie dans le cadre Brownien. De plus, sous certaines conditions, nous prouvons que nos intégrales anticipées sont des près-martingales.

Mots clés : Processus gaussiens, mouvement brownien fractionnaire mixte, mouvement brownien sous-fractionnaire, processus non-adapté, près-martingale, volatilité stochastique.

« Stochastic Integration with respect to Gaussian Processes and application to mixed fractional volatility models »

Abstract: The objective of this thesis is to model volatility by fractional Gaussian processes with long memory and irregular trajectories. We use high-frequency data to estimate the regularity of the log-volatility paths. We propose the mixed fractional Brownian motion with H< ½ as a stochastic volatility model and construct a stationary mixed fractional Ornstein-Uhlenbeck process as a stationary model of log-volatility.
Furthermore, we demonstrate a fundamental result to the integration of non-adapted processes with respect to fractional type processes (sub-fractional Brownian motion, mixed fractional Brownian motion) when H> ½ as a Riemann sum with an appropriate choice of sub-interval evaluation points by decomposing the integrand processes to a linear combination of adapted and instantly independent processes. This study is considered as a generalization of the one established in the Brownian framework. We prove that our anticipating integrals are near-martingales under some conditions.

Keywords: Gaussian processes, mixed fractional Brownian motion, sub-fractional Brownian motion, non-adapted process, near-martingale, stochastic volatility.