Université de Saida– Dr. Moulay Tahar

Faculté des Sciences

Thèse

Présentée pour obtenir le diplôme de

Doctorat 3ème Cycle

Spécialité: Systèmes Dynamiques

Filière : Mathématiques

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Thème :

Contributions à l'étude de quelques classes d'équations différentielles aléatoires d'ordre fractionnaire

Thèse soutenue le 09/05/2022 devant le jury composé de :

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Acknowledgement

First of all, thanks to my God, for his rains of blessings throughout my work to complete the research.

I would like to express my special gratitude and thank you to my mentor, Professor SAID ABBAS, You do me the great honor of mentoring. Thank you for allowing me to do this work. you have been a wonderful mentor for me. I want to thank you for supporting my research. Your advice on both research and my career has been invaluable. I was extremely lucky to have a supervisor who cared so much about my job, and who answered my questions and requests if quickly.

address my thanks to Prof. MOUFFAK BENCHOHRA from the University of Sidi Bel Abbes, you do me the honor and the kindness to participate in my thesis jury.

I also address my thanks to the JAMAL EDDINE LAZREG from the University of Sidi Bel Abbes I am extremely grateful to you for having accepted to participate in the defense jury of my thesis. I will remember your kindness, you have always been available as well as writing my thesis. The relevance of your comments and the accuracy of your corrections

I also thank the members of the jury Prof. ABD EL DJEBBAR KANDOUCI and Prof. TOUFIK GUENDOUZI from the University of Saida for having accepted to examine the content of my thesis, as well as for their advice, remarks and orientations.

Loving thanks to my parents For her repeated trust throughout my work, and who always encouraged me during this period, to my dear sister and brothers and to all my family for believing in me and being there for practical support in all of these things in life.

To love thanks to my best friends who played such an important role throughout the work. together with their boundless friendship, I hope I will never disappoint you.

Publications

1. **F. Bekada**, S. Abbas, and M. Benchohra, Boundary value problem for Caputo-Fabrizio random fractional differential equations, Moroccan J. Pure Appl. Anal. (MJPAA) 6 (2) (2020), 218-230.

2. **F. Bekada**, S. Abbas, M. Benchohra, and J.J.Nieto, Dynamics and stability for Katugampola random fractional differential equations, AIMS Mathematics (2021), 8654-8666.

3. **F. Bekada**, S. Abbas, and M. Benchohra, Caputo−Hadamard random fractional differential equations in finite and infinite dimensional Banach spaces (Submitted).

4. **F. Bekada**, S. Abbas, and M. Benchohra, Existence and attractivity for Caputo-Fabrizio random fractional differential equations (Submitted).

5. **F. Bekada**, S. Abbas, and M. Benchohra, Random Caputo-Fabrizio fractional differential equations in Fréchet spaces (Submitted).

6. **F. Bekada**,S. Abbas, and M. Benchohra, Caputo-Fabrizio Fractional Differential Equations with Delay and Random Effects (Submitted).

7. **F. Bekada**,S. Abbas, and M. Benchohra, Katugampola random fractional differential equations with delay (Submitted).

8. **F. Bekada**,S. Abbas, and M. Benchohra, Attractivity for Caputo-Katugampola random fractional differential equations in Fréchet spaces (Submitted).

9. **F. Bekada**, S. Abbas, and M. Benchohra, Caputo-Katugampola random fractional differential inclusions (Submitted).

Résumé en Français

Contrubutions à l'étude de quelques classes d'équations différentielles aléatoires d'ordre fractionnaire

Resumé :

Dans cette thèse, nous considérons l'étude de l'existence des solutions alèatoires et la stabilité de type Ulam et l'attractivité de quelques classes d'équations différentielles avec les dérivées fractionnaires de Caputo, Hadamard, Fabrizio et Katugampola dans des espaces de Fréchet. Les méthodes utilisées sont basées sur la théorie de point fixe et la mesure de non compacité dans les espaces de Fréchet .Nous avons également montré l'existence de solutions aléatoires pour certaines classes d'equations différentielles fractionnaires alèatoires avec retard. De plus, pour la justification de nos résultats, nous donnons divers exemples ilustratifs.

Mots clés : **équation différentielle, équation intégrale, dérivée fractionnaire, solution aléatoire, espace de Banach, stabilité d'Ulam, point fixe, attractivité, problème non local, retard fini, retard infini, retard dépendant de l'état, mesure de non compacité, espace de Fréchet.**

Abstract in English

Contributions to the study of some classes of random differential equations of fractional order

Abstract :

In this thesis, we consider the study of the existence of random solutions and the Ulam stability and the attractivity of serveral classes of differential equations with fractional derivatives of Caputo, Hadamard, Fabrizio and Katugampola in Fréchet spaces. The used methods are the random fixed point and the technique of the measure non-compactness. We have also shown the existence of random solutions for certain classes of random fractional differential equations with delay. In addition, for the justification of our results, we give various examples in each chapter.

Keywords :**Differential equation, fractional integral, fractional derivative, random solution, Banach space, Ulam stability, fixed point, attractivity, non-local problem, finite delay, infinite delay, state-dependent delay, measure of non compactness, Fréchet space.**

ملخص

في هذه الرسالة، نأخذ في الاعتبار دراسة وجود الحلول العشوائية واستقرار أولام وجاذبية الفئات الخدمية للمعادلات التفاضلية مع المشتقات الكسرية لكابوتو، هادامارد، فابريزيو وكاتوجامبولا في فضاء فريشي. الطرق المستخدمة هي النقطة الثابتة العشوائية وتقنية قياس عدم التراص. لقد أظهرنا أيضًا وجود حلول عشوائية لفئات معينة من المعادلات التفاضلية الجزئية العشوائية مع تأخير. بالإضافة إلى ذلك، لتبرير نتائجنا، نقدم أمثلة مختلفة في كل فصل.

الكلمات مفتاحية: *معادلة تفاضلية، تكامل كسري، مشتق كسري، حل عشوائي، فضاء باناخ، استقرار أولام، نقطة ثابتة، جاذبية، مشكلة غير محلية، تأخير محدود، تأخير لانهائي، تأخير معتمد على الحالة، قياس عدم التراص، فضاء فريتشي.*

CONTENTS

INTRODUCTION

$\ll$ **This is an apparent paradox from which, one day, useful consequences will be drawn.**$\gg$. It is an apparent paradox which will one day have beneficial consequences Drawn These words are Leibniz's response to the letter from Hopital in which he was asked the next question $\ll$ **What if the order will be $\frac{1}{2}$?** $\gg$.

Several authors consider this letter dated September $30, 1695$, as time birth of fractional calculus. So fractional calculus is a mathematical subject dating back over 300 years.

The fractional calculus it its origin in the works by *Leibnitz*, *L'Hopital* (1695), *Bernoulli* (1697), *Euler* (1730), and *Lagrange* (1772). Some years later, *Laplace* (1812), *Fourier* (1822), *Abel* (1823), *Liouville* (1832), *Riemann* (1847), *Grünwald* (1867), *Letnikov* (1868), *Nekrasov* (1888), *Hadamard* (1892), *Heaviside* (1892), *Hardy* (1915), *Weyl* (1917), *Riesz* (1922), *P. Levy*(1923), *Davis* (1924), *Kober* (1940), *Zygmund* (1945), *Kuttner* (1953), *J. L. Lions* (1959), and *Liverman* (1964)... have developed the basic concept of fractional calculus.

However, fractional calculus can be considered a new topic because only a little over twenty years old, he was the subject of specialized conferences at his The first lecture is due to *B. Ross* who organized this lecture at New Haven University in June 1974 under the title *"Fractional Calculus and Its Applications"* and he published the procedure again, see [113]. For the first monograph, another merit is attributed to *K.B. Oldham and J. Spanier*, see [107], who have started a collaboration in 1968 ,published a work on fractional calculus in 1974.

At present, the list of procedures devoted exclusively or partially to fractional calculus and its applications contain several titles [52], of which the encyclopedia treated by *Samko, Kilbas* and *Marichev* is the most important; In addition, we recall the work of *Davis*, *Erdèlyi Gelfand* and *Shilov* , *Djrbashian* , *Caputo* , *Babenko* , *Gorenflo* and *Vessella* , who contain a detailed analysis of certain mathematical or physical aspects of applications of fractional calculus.

In recent years, there has been considerable interest in nominations fractional derivatives (of non-integer order) in several fields. in the field of interdisciplinary, many systems can be described by fractional differential equations.

for example :

- Fractional derivatives have been used widely in the mathematical model visco-elastic materials .
- Electromagnetic problems can be described using the equations fractional integrodifferentials .
- In physicochemistry, the current is proportional to the fractional derivatives of the voltage when the fractal interface is put between a metal and an ionic medium .
- In the theory of the fractional capacitor, if one of the electrodes of the capacitor has a rough surface, the current passing through it is proportional to the derivatives of order, not an integer of its voltage . Also, the existing memory in dioelectric used in capacitors is justified by the fractional derivative .
- Another example for an element with fractional order pattern is fractionance. Fractance is an electrical circuit with non-integer order impedance , this element has properties that lie between resistance and capacity; Citing the case of both well-known examples of fractances: the shaft fractance and the chain .
- The heating of the conductance as a dynamic process can be model both by fractional order models and by order models integer .
- In biology, it has been deduced that the membranes of cells of biological organism have fractional order electrical conductance and then is classified into a group of non-integer order models.

- In economics, some financial systems can display a dynamic fractional order , examples on fractional order dynamics.
- In addition, applications of fractional calculus have been reported in several areas such as:
  Signal processing , image processing , automatic control and robotics , these and many other similar samples clarify perfectly the importance of consideration and analysis of dynamic systems with the fractional order models...

The study of fractional problems is very topical and several methods are applied to solve these problems. However, the methods based on the principle of the fixed point play a big role.
Fixed point theorems are the basic mathematical tools, showing the existence of solutions in various kinds of equations. The fixed point theory is at the heart analysis nonlinear since it provides the necessary tools to have theorems existence in many different nonlinear problems. The development of the fixed point theory, which is the cardinal branch of analysis nonlinear gave great effects on the advancement of nonlinear analysis, considered as a stand-alone branch of mathematics, nonlinear analysis was developed in the 1950s by mathematicians like *Felix Browder* as a combination of functional analysis and variational analysis.

The method based inthe approximation is associated with the names of famous mathematicians such as *Cauchy, Liouville, Lipschitz* and above all, *Picard*. In fact, the precursors of the theory of the approximate fixed point are explicit in the works of *Picard*. However, it is the Polish mathematician *Stefan Banach*, who is credited with placing an abstract idea.
The principle of contracting application is one of the few constructive theorems of mathematical analysis. It constitutes a great tool fields of application a priori, in the study of nonlinear equations that play a crucial role in both mathematics and applied science. The principle is the theorem of the Banach fixed point or that of *Picard* which ensures the existence of a single fixed point for a contracting application of a complete metric space within itself.

The fixed point is the limit of an iterative process defined from an image repetition by this contracting mapping of an arbitrary starting point in this space. This concept has been proven in first, by *Banach* in 1922 then developed by several mathematicians including us let us cite *Brouwer* and *Schauder* in 1930 as well as *Krasnoselskii* in 1955.
The Schauder's fixed point theorem, which is by the way, an extension of Brouwer's in

infinite dimension is more topological than that of Banach and asserts that a continuous map on a convex compact admits a fixed point which is not necessarily unique. It is therefore not necessary to establish surcharges on the function but simply its continuity.

The measure of non-compactness which is one of the fundamental tools of the theory nonlinear analysis, was initiated by the pioneering articles of Alvàrez [30], Mönch [104] and was developed by Banas and Goebel [33] and many researchers in the literature. the measure of non-compactness has been applied in several works (see [33, 34, 42, 83] and references).

Byszewski is the first who proved the existence and the uniqueness of the mild solutions of the non-local Cauchy problems[44, 45, 46]. The non local condition may be more useful than the standard initial condition to describe some phenomena. Fractional differential equations with nonlocal conditions have been discussed in [26, 27, 28, 53, 57, 71, 135, 136] and the references therein.

Probabilistic functional analysis is an important mathematical research due to its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. Indeed,in many cases the mathematical models or equations used to describe phenomena in the biological, physical, engineering, and systems sciences contain certain parameters or coefficients which have specific interpretations, but whose values are unknown. Therefore, it is more realistic to consider such equations as random operator equations. These equations are much more difficult to handle mathematically than deterministic equations. Important contributions to the study of the mathematical aspects of such random equations have been undertaken in [40, 54, 81, 96, 122] among others.

The importance of random fixed point theory lies in its vast applicability in probabilistic Functional analysis and various probabilistic models. The introduction of randomness however leads to several new questions of measurability of solutions, probabilistic and statistical aspects of random solutions. It is well known that random fixed point theorems are stochastic generalization of classical fixed point theorems what we call as deterministic results. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by *$\tilde{S}$pa$\check{c}$ek* [124] and *Han$\tilde{s}$* (see [70]). The survey article by *Bharucha-Reid* [41] in 1976 attracted the attention of several mathematicians and gave wings to this theory. *Itoh* [81] extended *$\tilde{S}$pa$\check{c}$ek* and *Han$\tilde{s}$* theorems to multivalued contraction mappings. Random fixed point theorems with an application to Random differential equations in Banach spaces are obtained by *Itoh* [81]. *Sehgal* and *Waters*

[120] had obtained several random fixed point theorems including random analogue of the classical results due to *Rothe* [114]. In recent past, several fixed point theorems including *Kannan* type [88] *Chatterjeea* [49] and *Zamfirescu* type [138] have been generalized in stochastic version (see for detail in *Joshi* and *Bose* [84], *Saha* et al. ([117, 118]).

The rondom functional differential equations with delay have many important applications in mathematical models of real phenomena, and The study of this type of equations has received much attention in recent years.

On the other hand, the stability of the functional equations was raised by *Ulam* in 1940 in a talk given at the University of Wisconsin, (for details, see [131]). The first answer to the problem posed by *Ulam* was given by *Hyers* in 1941 in [76]. Subsequently, this type of stability is called stability in the sense of Ulam-Hyers. In 1978, Rassias [111] provided a remarkable generalization of stability in the sense of Ulam-Hyers. Considerable attention has been paid to the study of stability in the sense of Ulam-Hyers and in the sense of Ulam-Hyers-Rassias differential equations, one can see the monographs of [82].

In addition, there is little work on stability in the Ulam sense of fractional differential equations. First, stability in the sense of Ulam for the differential equations fractional with Caputo derivative is proposed by J. Wang et al. [131], while with the Riemann-Liouville derivative by R.Ibrahim [78]. More details of recent developments such stabilities are reported in [17, 20, 35, 36, 59, 79, 80, 97, 129].

Thesis overview

This thesis is divided into 6 chapters

Chapter 1: This chapter consists of three Sections. In Section one, we present "*Some notations and definitions of Fractional Calculus Theory*", and in Section two, we present some "*Some definitions and proprieties of noncompactness measure*".
Finally, in the last Section, we recall some preliminary : some basic concepts, and useful famous theorems and results (notations, definitions, lemmas and fixed point theorems) which are used throughout this thesis.

Chapter 2: In this chapter we investigate the existence of random solutions for the following class of Caputo-Hadamard fractional differential equation

$$({}^{Hc}D_1^r u)(t,w) = f(t,u(t,w),w);\ t \in I := [1,T],\ w \in \Omega, \tag{1}$$

with the boundary conditions

$$\begin{cases} u(1,w) = u_1(w) \\ u'(T,w) = u_T(w) \end{cases} ; \ w \in \Omega, \tag{2}$$

where $r \in (1,2]$, $T > 1$, $f : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, $u_1, u_T : \Omega \to \mathbb{R}$, ${}^{Hc}D_1^r$ is the Caputo-Hadamard fractional derivative of order r, and Ω is the sample space in a probability space (Ω, F).

In Section 2.3, we consider the problem (1)-(2), where $f : I \times E \times \Omega \to E$ is a given function, $u_1, u_T : \Omega \to E$, and E is a real (or complex) Banach space with a norm $\|\cdot\|$. Finally, some examples are given to illustrate the applicability of our main results.

Chapter 3: We establish the existence and the Ulam-Hyers stability results in a class of fractional random problems in Banach spaces.

Here two results are discussed, the first is based on the existence of random solutions and the stability of Ulam results for a class of Caputo-Fabrizio random fractional dierential equations in the form

$$({}^{CF}D_0^\alpha u)(t,w) = f(t, u(t,w), w); \ t \in I := [0,T], \ w \in \Omega,$$

with the boundary conditions

$$au(0,w) + bu(T,w) = c(w); \ w \in \Omega,$$

where $T > 0$, $f : I \times E \times \Omega \to E$ is a given function, $a, b \in \mathbb{R}$, $, c : \Omega \to E$, with $a + b \neq 0$, ${}^{CF}D_0^\alpha$ is the Caputo–Fabrizio fractional derivative of order $\alpha \in (0,1)$, and Ω is the sample space in a probability space (Ω, F), and E is a real (or complex) Banach space with a norm $\|\cdot\|$.

The second is based on the existence of random solutions and the stability Ulam for a class of random fractional differential equations of Katugampola

$$({}^\rho D_0^\varsigma x)(\xi, w) = f(\xi, x(\xi,w), w); \ \xi \in I = [0,T], \ w \in \Omega,$$

with the terminal condition

$$x(T,w) = x_T(w);\ w \in \Omega,$$

where $x_T : \Omega \to E$ is a measurable function, $\varsigma \in (0,1],\ T > 0,\ f : I \times E \times \Omega \to E,\ {}^{\rho}D_0^{\varsigma}$ is the Katugampola operator of order ς, and Ω is the sample space in a probability space.

Our results are based on the theory of the fixed point and random operators. Illustrative examples are presented in each section.

Chapter 4: we study the existence and attractivity for several classes of functional fractional differential equations.

$$({}^{CF}D_0^r u)(t,w) = f(t,u(t,w),w);\ t \in \mathbb{R}_+ = [0,\infty),\ w \in \Omega,$$

with the initial condition

$$u(0,w) = u_0(w);\ w \in \Omega,$$

where $T > 0,\ f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, $u_0 : \Omega \to \mathbb{R},\ {}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0,1)$, and Ω is the sample space in a probability space (Ω, F).

An illustrative example is presented in the last section.

Chapter 5: we prove the existence of random solutions and the Ulam stability for functional differential equations involving the Caputo-Fabrizio fractional derivative in Fréchet spaces of the from

$$({}^{CF}D_0^r u)(t,w) = f(t,u(t,w),w);\ t \in \mathbb{R}_+ = [0,\infty),\ w \in \Omega, \tag{3}$$

with the initial condition

$$u(0,w) = u_0(w);\ w \in \Omega, \tag{4}$$

where $u_0 : \Omega \to \mathbb{R}$, is a measurable function, $f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, ${}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0,1)$, and Ω is the sample space in a probability space (Ω, F).

Later, we consider the following nonlocal problem

$$\begin{cases} ({}^{CF}D_0^r u)(t,w) = f(t,u(t,w),w); \ t \in \mathbb{R}_+, \\ u(0,w) + Q(u(\cdot,w)) = u_0(w), \end{cases} \quad w \in \Omega,$$

where $u_0, \ f$ are as in problem (3)-(4), $Q : \Omega \times X \to \mathbb{R}$ is a given function, and X is the Fréchet space defined later.
At last, an example is included to show the applicability of our results.

Chapter 6: we prove the existence of random solutions for some classes of Caputo-Fabrizio random fractional differential equations delay. Our results are based on the random fixed point theory.
The second section, we investigate the following class of random Caputo-Fabrizio fractional differential equations with finite delay

$$\begin{cases} u(t,w) = \varphi(t,w); \ t \in [-h,0], \\ ({}^{CF}D_0^r u)(t,w) = f(t,u_t(\cdot,w),w); \ t \in I := [0,T], \end{cases} ; \ w \in \Omega,$$

where $h > 0, \ T > 0, \ \ \varphi \in \mathcal{C}, f : I \times \mathcal{C} \times \Omega \to \mathbb{R}$ is a given function, ${}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r \in (0,1]$, and $\mathcal{C} := C([-h,0],\mathbb{R})$ is the space of continuous functions on $[-h,0]$.

For any $t \in I$, we define $u_t(\cdot,w)$ by

$$u_t(s,w) = u(t+s,w); \ \text{for } s \in [-h,0], \text{and } w \in \Omega.$$

In the third section, we investigate the following class of random Caputo-Fabrizio fractional differential equations with infinite delay

$$\begin{cases} u(t,w) = \varphi(t,w); \ t \in \mathbb{R}_- := (-\infty,0], \\ ({}^{CF}D_0^r u)(t,w) = f(t,u_t(\cdot,w),w); \ t \in I, \end{cases} ; \ w \in \Omega,$$

where $\varphi : [-\infty,0] \to \mathbb{R}, f : I \times \mathcal{B} \times \Omega \to \mathbb{R}$ are given functions, and $\mathcal{B}$ is called a phase space that will be specified later.

For any $t \in I$, we define $u_t \in \mathcal{B}$ by

$$u_t(s,w) = u(t+s,w); \ \text{for } s \in \mathbb{R}_-, \text{and } w \in \Omega.$$

In the section 6.4 , we investigate the following class of random Caputo-Fabrizio fractional differential equations with state dependent finite delay

$$\begin{cases} u(t,w) = \varphi(t,w);\ t \in [-h,0], \\ ({}^{CF}D_0^r u)(t,w) = f(t, u_{\rho(t,u_t(\cdot,w))}(\cdot,w), w);\ t \in I, \end{cases}$$

where $\varphi \in \mathcal{C},\ \rho : I \times \mathcal{C} \times \Omega \to \mathbb{R},\ f : I \times \mathcal{C} \times \Omega \to \mathbb{R}$ are given functions.

Finally, we consider the following class of Caputo-Fabrizio fractional differential equations with state dependent infinite delay

$$\begin{cases} u(t,w) = \varphi(t,w);\ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^r u)(t,w) = f(t, u_{\rho(t,u_t(\cdot,w))(\cdot,w)}, w);\ t \in I, \end{cases} \quad ;\ w \in \Omega,$$

where $\varphi : \mathbb{R}_- \to \mathbb{R},\ f : I \times \mathcal{B} \times \Omega \to \mathbb{R}$ are given functions.

Finally,an example for each section.

CHAPTER 1

BASIC INGREDIENTS

The main purpose of this chapter is to provided the necessary background material to the reader. Here we shall introduce definitions, notations and theoretical results that will be used along this thesis

1.1 Some notations and definitions of fractional calculus theory

Let $C(I, E)$ be the Banach space of all continuous functions from $I = [0, T], T > 0$ into E with the norm

$$\|u\|_\infty = \sup\{\|u(t)\| : t \in I\}.$$

and $L^1(I, E)$ we denote the Banach space of measurable function $u : I \to E$ with are Bochner integrable, equipped with the norm

$$\|u\|_{L^1} = \int_0^T \|u(t)\|dt.$$

1.1.1 Random Operators

Let β_E be the $\sigma-$algebra of Borel subsets of E. A mapping $v : \Omega \to E$ is said to be measurable if for any $B \in \beta_E$, one has

$$v^{-1}(B) = \{w \subset \Omega : v(w) \subset B\} \subset A.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 1.1.1 *A mapping $T : \Omega \times E \to E$ is called jointly measurable if for any $B \subset \beta_E$, one has*

$$T^{-1}(B) = \{(w, v) \subset \Omega \times E : T(w, v) \subset B\} \subset A \times \beta_E$$

where $A \times \beta_E$ is the direct product of the $\sigma-$algebras A and β_E those defined in Ω and E respectively.

Lemma 1.1.2 *[54] Let $T : \Omega \times E \to E$ be a mapping such that $T(\cdot, v)$ is measurable for all $v \subset E$, and $T(w, \cdot)$ is continuous for all $w \subset \Omega$. Then the map $(w, v) \to T(w, v)$ is jointly measurable.*

Definition 1.1.3 *[66] A function $f : I \times E \times \Omega \to E$ is called random Carathéodory if the following conditions are satisfied:*

- *(i) The map $(t, w) \to f(t, u, w)$ is jointly measurable for all $u \subset E$, and*
- *(ii) The map $u \to f(t, u, w)$ is continuous for almost all $t \in I$ and $w \subset \Omega$.*

Definition 1.1.4 *$T : \Omega \times E \to E$ be a mapping. then T is called a random operator if $T(w, u)$ is measurable in w for all $u \subset E$ and it is expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is random operator on E. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, u)$ is continuous (resp. compact, totally bounded and completely continuous)in u for all $w \subset \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh.*

Definition 1.1.5 *[58] Let $P(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $P(Y)$. A mapping $T : \{(w, u) : w \subset \Omega, y \subset C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e for all closed $A \subset Y$, $\{w \subset \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $u \subset Y, \{w \subset \Omega : u \subset C(w), T(w, u) \subset D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T , a mapping $u : \Omega \to Y$ is called random (stochastic) fixed point of T if for P-almost all $w \subset \Omega, u(w) \subset C(w)$ and $T(w)u(w) = u(w)$ and for all open $D \subset Y, \{w \subset \Omega : u(w) \subset D\}$ is measurable.*

1.1.2 Fractional calculus

Definition 1.1.6 *([93, 110]). The fractional (arbitrary) order integral of the function $f \in L^1([0,T],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where Γ is the gamma function.

Theorem 1.1.7 *[93]. For any $f \in C([0,T],\mathbb{R})$ the Riemann-Liouville fractional integral satisfies*

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t),$$

for $\alpha, \beta > 0$.

Definition 1.1.8 *([92]). For a function f given on the interval $[0,T]$, the Caputo fractional-order derivative of order α of h, is defined by*

$$({}^cD^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 1.1.9 *([103]) Let $\alpha \geq 0$ and $n = [\alpha]+1$. Then*

$$I^\alpha({}^cD^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

Remark 1.1.10 *([7, 25, 103])The Caputo derivative of a constant is equal to zero.*

We need the following auxiliary lemmas.

Lemma 1.1.11 *([23, 139]) Let $\alpha > 0$. Then the differential equation*

$${}^cD^\alpha f(t) = 0$$

has solutions $f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\alpha]+1$.

Lemma 1.1.12 *([139]) Let $\alpha > 0$. Then*

$$I^{\alpha c}D^\alpha f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\alpha]+1$.

Lemma 1.1.13 *([54], Lemma 3.11) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $m = [\alpha]$. Moreover assume that $f \in C^m[a,b]$. Then*

$$^{c}D_a^{\alpha} f \in C[a,b]$$

and

$$^{c}D_a^{\alpha} f(a) = 0.$$

Definition 1.1.14 *(Hadamard fractional integral)[15]. The Hadamard fractional integral of order r is defined as*

$$I_0^{\varsigma} f(\xi) = \frac{1}{\Gamma(\varsigma)} \int_1^{\xi} \left(\log \frac{\xi}{s}\right)^{\varsigma-1} f(s) \frac{ds}{s}, \quad \varsigma > 0.$$

Definition 1.1.15 *(Hadamard fractional derivative)[15]. The Hadamard fractional derivative of order r is defined as*

$$D_0^{\varsigma} h(\xi) = \frac{1}{\Gamma(n-\varsigma)} \left(\xi \frac{d}{d\xi}\right)^n \int_1^{\xi} \left(\log \frac{\xi}{s}\right)^{n-\varsigma-1} h(s) \frac{ds}{s}, \quad \varsigma > 0.$$

We denote by $AC_{\delta}^n(I)$ the space defined by

$$AC_{\delta}^n([1,T],E) = \{h : [1,T] \to E : \delta^{n-1} h(t) \in AC(I,E)\},$$

where $\delta = t\frac{d}{dt}$ is the Hadamard derivative and $AC(I,E)$ is the space of absolutely continuous functions on I.

Definition 1.1.16 *(Caputo-Hadamard fractional derivative)[13] The Caputo-Hadamard fractional derivative of order $q > 0$ applied to the function $u \in AC_{\delta}^n$ is defined as*

$$({}^{Hc}D_1^q u(x)) = ({}^{H}I_1^{n-q} \delta^n u)(x).$$

Definition 1.1.17 *[13] The Caputo-type Hadamard derivative of fractional order q is defined as*

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \int_1^t \left(\log \frac{t}{s}\right)^{n-q-1} \delta^n f(s) \frac{ds}{s},$$

where $n-1 < q < n, \quad n = [q]+1$, and Γ is the Gamma function.

Lemma 1.1.18 *Let $u \in AC_{\delta}^n[1,T]$ or $C_{\delta}^r[1,T]$ and $q > 0$, then one has*

$$I^q(D^r)u(t) = u(t) - \sum_{k=0}^{n-1} C_k (\ln t)^k,$$

where $c_k \in \mathbb{R}$, $k = 0, 1, \ldots, n-1$, $(n = [q] + 1)$.

Definition 1.1.19 *[47, 95, 100] The Caputo-Fabrizio fractional integral of order* $0 < r < 1$ *for a function* $h \in L^1(I)$ *is defined by*

$$^{CF}I_0^r h(\tau) = \frac{2(1-r)}{M(r)(2-r)} h(\tau) + \frac{2r}{M(r)(2-r)} \int_0^\tau h(x)dx; \ \tau \geq 0$$

where $M(r)$ is normalization constant depending on r.

Definition 1.1.20 *[47, 100] The Caputo-Fabrizio fractional derivative for a function* $h \in C^1(I)$ *of order* $0 < r < 1$, *is defined by*

$$^{CF}D^r h(\tau) = \frac{(2-r)M(r)}{2(1-r)} \int_0^\tau \exp(-\frac{r}{1-r}(\tau - x))h'(x)dx; \ \tau \in I.$$

Note that $(^{CF}D^r)(h) = 0$ *if and only if* h *is a constant function.*

Remark 1.1.21 *[47, 60, 100] Note that,according to the previous definition,the fractional integral of Caputo-Fabrizio type of a function of order* $0 < r < 1$ *is an average between function* f *and its integral of order one.*
Imposing

$$\frac{2(1-r)}{(2-r)M(r)} + \frac{2r}{(2-r)M(r)} = 1$$

we obtain an explicit formula for $M(r)$

$$M(r) = \frac{2}{2-r}$$

Example 1.1.22 *[47]*
1- For $h(t) = t$ *and* $0 < r \leq 1$, *we have*

$$(^{CF}D^r h)(t) = \frac{M(r)}{r} \left(1 - \exp\left(-\frac{r}{1-r}t\right)\right).$$

2- For $g(t) = e^{\lambda t}$, $\lambda \geq 0$ *and* $0 < r \leq 1$, *we have*

$$(^{CF}D^r g)(t) = \frac{\lambda M(r)}{r + \lambda(1-r)} e^{\lambda t} \left(1 - \exp\left(-\lambda - \frac{r}{1-r}t\right)\right).$$

Definition 1.1.23 *(Katugampola fractional integral)[38, 89]. The Katugampola frac-*

tional integrals of order ($\varsigma > 0$) *is defined by*

$$ {}^{\rho}I_0^{\varsigma}x(t) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_0^{\xi}\frac{s^{\rho-1}}{(\xi^{\rho}-s^{\rho})^{1-\varsigma}}x(s)ds \tag{1.1}$$

for $\rho > 0$ *and* $\xi \in I$.

Definition 1.1.24 *(Katugampola fractional derivative)[38, 89]. The Katugampola fractional derivative of order* $\varsigma > 0$ *is defined by:*

$$\begin{aligned} {}^{\rho}D_0^{r}u(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n ({}^{\rho}I_0^{n-r}u)(t) \\ &= \frac{\rho^{r-n+1}}{\Gamma(n-r)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_0^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{r-n+1}}u(s)ds. \end{aligned}$$

We present in the following theorem some properties of Katugampola fractional integrals and derivatives.

Theorem 1.1.25 *[89] Let* $0 < Re(\varsigma)1$ *and* $0 < Re(\eta) < 1$ *and* $\rho > 0$, *for* $a > 0$:

- *Index property:*

$$\begin{aligned} ({}^{\rho}D_a^{\varsigma})({}^{\rho}D_a^{\eta}h)(t) &= {}^{\rho}D_a^{\varsigma+\eta}h(t) \\ ({}^{\rho}I_a^{r})({}^{\rho}I_a^{\eta}h)(t) &= {}^{\rho}I_a^{r+\eta}h(t) \end{aligned}$$

- *Inverse property:*

$$({}^{\rho}D_a^{r})({}^{\rho}I_a^{r}h)(t) = h(t)$$

- *Linearity property:*

$$\begin{aligned} {}^{\rho}D_a^{r}(h+g) &= {}^{\rho}D_a^{r}h(t) +^{\rho} D_a^{r}g(t) \\ {}^{\rho}I_a^{r}(h+g) &= {}^{\rho}I_a^{r}h(t) +^{\rho} I_a^{r}g(t) \end{aligned}$$

and we have

$$(t^{1-\rho}\frac{d}{dt})I_0^{r}(I_0^{1-r})u(s)ds.$$

Theorem 1.1.26 *[89] Let* r *be a complex number,* $Re(r) \geq 0$, $n = [Re(r)]$ *and* $\rho > 0$. *Then, for* $t > a$;

1. $\lim_{\rho\to 1}({}^{\rho}I_a^{r}h)(t) = \frac{1}{\Gamma(r)}\int_a^t (t-\tau)^{r-1}h(\tau)d\tau$.

2. $\lim_{\rho\to 0^+}({}^\rho I_a^r h)(t) = \frac{1}{\Gamma(r)}\int_a^t (\log\frac{t}{\tau})^{r-1} h(\tau)\frac{d\tau}{\tau}$.

3. $\lim_{\rho\to 1}({}^\rho D_a^r h)(t) = (\frac{d}{dt})^n \frac{1}{\Gamma(n-r)}\int_a^t \frac{h(\tau)}{(t-\tau)^{r-n+1}} d\tau$.

4. $\lim_{\rho\to 0^+}({}^\rho D_a^r h)(t) = \frac{1}{\Gamma(n-r)}(t\frac{d}{dt})^n \int_a^t (\log\frac{t}{\tau})^{n-r-1} h(\tau)\frac{d\tau}{\tau}$.

Remark 1.1.27

1. $\lim_{\rho\to 1}({}^\rho I_a^r h)(t) = ({}^{RL} I_a^r h)(t)$.

2. $\lim_{\rho\to 0^+}({}^\rho I_a^r h)(t) = ({}^{H} I_a^r h)(t)$.

3. $\lim_{\rho\to 1}({}^\rho D_a^r h)(t) = ({}^{RL} D_a^r h)(t)$.

4. $\lim_{\rho\to 0^+}({}^\rho D_a^r h)(t) = ({}^{H} D_a^r h)(t)$.

Lemma 1.1.28 *Let $0 < r < 1$. The fractional equation $({}^\rho D_0^r v)(t) = 0$, has as solution*

$$v(t) = ct^{\rho(r-1)}, \tag{1.2}$$

with $c \in \mathbb{R}$.

Lemma 1.1.29 *Let $0 < r < 1$. Then*

$${}^\rho I^r({}^\rho D_0^r u)(t) = u(t) + ct^{\rho(r-1)}.$$

Proof. We have

$$\begin{aligned}
I_0^r D_0^r u(t) &= \left(t^{1-p}\frac{d}{dt}\right) I_0^{r+1} D_0^r u(t) \\
&= (t^{1-\rho}\frac{d}{dt})\left(\frac{\rho^{-r}}{\Gamma(r+1)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{-r}}({}^\rho D_0^r u(s))ds\right) \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left(\frac{\rho^{-r}}{\Gamma(r+1)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{-r}}\left[\left(s^{1-\rho}\frac{d}{ds}\right)(I_0^{1-r}u)(s)\right]ds\right) \\
&= \left(t^{1-\rho}\frac{d}{dt}\right)\left(\frac{\rho^{-r}}{\Gamma(r+1)}\int_0^t (t^\rho - s^\rho)^r\left[\frac{d}{ds}(I_0^{1-r}u)(s)\right]ds\right).
\end{aligned}$$

Thus, $I_0^r D_0^r u(t) = I_1 + I_2$, with

$$I_1 = \left(t^{1-\rho}\frac{d}{dt}\right)\frac{\rho^{-r}}{\Gamma(r+1)}\left(\left[(t^\rho - s^\rho)^r I_0^{1-r}u(s)\right]_0^t\right),$$

and

$$I_2 = \left(t^{1-\rho}\frac{d}{dt}\right)\frac{\rho^{-r}}{\Gamma(r+1)}\int_0^t r\rho s^{\rho-1}(t^\rho - s^\rho)^{r-1} I_0^{1-r}u(s)ds.$$

Hence, we get

$$I_1 = ct^{\rho(r-1)}$$

and

$$\begin{aligned} I_2 &= \left(t^{1-\rho}\frac{d}{dt}\right)\frac{\rho^{1-r}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1} I_0^{1-r} u(s)ds \\ &= \left(t^{1-\rho}\frac{d}{dt}\right) I_0^r (I_0^{1-r}) u(s)ds \\ &= u(t). \end{aligned}$$

Finally we obtain

$$(I_0^r)(D_0^r u)(t) = u(t) + ct^{\rho(r-1)}.$$

1.2 Some definitions and proprieties measure of noncompactness

Now, we give the definition of the concept of a measure of noncompactness.

Definition 1.2.1 *([33]) Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty)$ defined by*

$$\alpha(B) = inf\{\xi > 0 : B \subseteq \bigcup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \xi\}; hereB \in \Omega_E,$$

where $diam(B_i) = sup\{\|x_y\| : x, y \in B_i\}$

Proposition 1.2.2 *([29, 33, 34, 94])*

1. *$\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact), where $\overline{B}$ denotes the closure of B*
2. *nonsingularity: α is equal to 0 on every one element-set.*
3. *$\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$, where $convB$ is the convex hull of B*
4. *monotonocity $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$*
5. *algebraic semi-additivity:*

$$\alpha(A + B) \leq \alpha(A) + \alpha(B)$$

where $A + B = \{x + y : x \in A, y \in B\}$

6. *semi-homogencity*

$$\alpha(\lambda B) = |\lambda|\alpha(B), \ \lambda \in \mathbb{R}$$

where $\lambda B = \{\lambda x : x \in B\}$

7. *semi-additivity:* $\alpha(A \cup B) = max\{\alpha(A), \alpha(B)\}$.

8. *invariance under translations:*

$$\alpha(B + x_0) = \alpha(B)$$

for any $x_0 \in E$.

Definition 1.2.3 *Let* $T : X \to X$ *be a continuous mapping of Banach space* X*, them* T *is called a* $k-$*set contraction if for all* $A \subset X$ *with* A *bounded, for* $0 < K < 1$,, $T(A)$ *is bounded and*

$$\alpha(TA) \leq K\alpha(A).$$

If $\alpha(TA) \leq \alpha(A)$, T *called condensing mapping .*

Lemma 1.2.4 *If* $\{u_k\}_{k=1}^{\infty} \subset L^1(I)$ *is uniformly integrable, then* $\alpha(\{u_k\}_{k=1}^{\infty})$ *is measurable and for each* $t \in I$

$$\alpha(\left\{\int_0^t u_k(s)ds\right\}_k^{\infty}) \leq 2\int_0^t \alpha(\{u_k(s)\}_k^{\infty})ds. \tag{1.3}$$

Lemma 1.2.5 *If* Y *is bounded subset of a Banach space* X, *then for each* $\xi > 0$, *there is a sequence* $\{u_k\}_k^{\infty} \subset Y$ *such that*

$$\alpha(Y) \leq 2\alpha(\{u_k\}_k^{\infty}) + \xi. \tag{1.4}$$

For further facts concerning measures of noncompactness and their properties we refer to [29, 31, 33, 34, 94] and the references therein.

1.2.1 Auxiliary Lemmas

We state the following generalization of Gronwall's lemma for singular kernels.

Lemma 1.2.6 *([137]) Let* $v : [0, T] \to [0, +\infty)$ *be a real function and* $w(\cdot)$ *is a non-negative, locally integrable function on* $[0, T]$. *Assume that there are constants* $a > 0$ *and* $0 < \alpha < 1$ *such that*

$$v(t) \leq w(t) + a\int_0^t (t - s)^{-\alpha} v(s)ds,$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq w(t) + Ka \int_0^t (t-s)^{-\alpha} w(s)ds, \text{ for every } t \in [0, T].$$

Theorem 1.2.7 *[68](**theorem of Ascoli-Arzela**). Let $A \subset C(J, \mathbb{R})$, A is relatively compact (i.e $\overline{A}$ is compact) if:*

1. *A is uniformly bounded i.e, there exists $M > 0$ such that*

$$|f(x)| < M \text{ for every } f \in A \text{ and } x \in J.$$

2. *A is equicontinuous i.e, for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, \overline{x} \in J$, $|x - \overline{x}| \leq \delta$ implies $|f(x) - f(\overline{x})| \leq \epsilon$, for every $f \in A$.*

1.3 Some fixed point theorems

Theorem 1.3.1 *[81] Let X be a nonempty, closed convex bounded subset of the separable Banach space E and let $N : \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $N(w)u = u$ has a random solution.*

Theorem 1.3.2 *[81] Let X be a separable closed convex subset of Banach space, $f : \Omega \times X \to X$ a condensing random operator. Suppose that for any $w \in \Omega, f(w, X)$ is bounded. then there exists a random fixed Point $\xi : \Omega \to X$ of f .*

Theorem 1.3.3 *[65] Let K be a compact convex subset of a Fréchet space X and $T : \Omega \times K \to K$ be a continuous affine random operator. Then T has a random fixed point.*

CHAPTER 2

CAPUTO-HADAMARD RANDOM FRACTIONAL DIFFERENTIAL EQUATIONS IN FINITE AND INFINITE DIMENSIONAL BANACH SPACES

2.1 Introduction and Motivations

The theory of fractional differential equations is a good tool for modeling such phenomena. When our knowledge about the parameters of a dynamic system are of statistical nature [126], that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations [48, 54, 58, 108].

The problem of fixed points for random mappings was initialed by the Prague school of probability. The first results were obtained in 1955-1956 by Špacek and Hanš in the context of Fredholm integral equations with random kernels. In a separable metric space, random fixed point theorems for contraction mappings were proved by Hanš [70], Hanš and Špacek [69], Mukherjea [105, 106].

Recently, several researchers obtained other results by application of the technique of measure of noncompactness; see [30, 31, 33, 127], and the references therein.

This chapter deals with some existence of random solutions for a class of Caputo-Hadamard random fractional differential equations with two boundary conditions

$$({}^{Hc}D_1^r u)(t,w) = f(t,u(t,w),w);\ t \in I := [1,T],\ w \in \Omega, \tag{2.1}$$

with the boundary conditions

$$\begin{cases} u(1,w) = u_1(w) \\ u'(T,w) = u_T(w) \end{cases} ;\ w \in \Omega, \tag{2.2}$$

where $r \in (1,2],\ T > 1,\ f : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, $u_1, u_T : \Omega \to \mathbb{R}$, ${}^{Hc}D_1^r$ is the Caputo-Hadamard fractional derivative of order r, and Ω is the sample space in a probability space (Ω, F).

Next, we consider the problem (2.1)-(2.2), where $f : I \times E \times \Omega \to E$ is a given function, $u_1, u_T : \Omega \to E$, and E is a real (or complex) Banach space with a norm $\|\cdot\|$. Our results are based on some random fixed point theorems and the measure of noncompactness.

2.2 Random Caputo-Hadamard fractional differential equations Results

Let $C := C(I,\mathbb{R})$ is assumed to be endowed with the standard norm

$$\|u\|_\infty = \sup\{|u(t)| : t \in I\}.$$

Lemma 2.2.1 *A function $u \in C$ is a solution of problem*

$$\begin{cases} ({}^{Hc}D_1^r u)(t) = h(t); \quad t \in I := [1,T] \\ u(1) = u_1 \\ u'(T) = u_T \end{cases} \tag{2.3}$$

if and only if u satisfies the following integral equation

$$u(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\ln \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds - \frac{T \ln t}{\Gamma(r-1)} \int_1^T \left(\ln \frac{T}{s}\right)^{r-2} \frac{h(s)}{s} ds + u_1 + T u_T \ln t. \tag{2.4}$$

Proof. Solving the equation

$$({}^{Hc}D_1^r u)(t) = h(t),$$

we get

$$u(t) =^H I_1^r h(t) + c_0 + c_1 \ln t,$$

and then

$$u'(t) =^H I_1^{r-1} h(t) + \frac{c_1}{t}.$$

From the boundary conditions, we get

$$\begin{aligned} c_0 &= u_1 \\ c_1 &= T(u_T -^H I_1^{r-1} h(T)) \end{aligned}$$

hence, we obtain (2.4).
Conversely, if u satisfies the integral equation (2.4), then

$$\begin{cases} ({}^{Hc}D_1^r u)(t) = h(t); \ t \in I, \\ u(1) = u_1, \quad u'(T) = u_T. \end{cases}$$

From the above Lemma, we conclude with the following lemma

Lemma 2.2.2 *A function u is a random solution of problem (2.1)-(2.2), if and only if u satisfies the following integral equation*

$$\begin{aligned} u(t,w) &= u_1(w) + T u_T(w) \ln t + \frac{1}{\Gamma(r)} \int_1^t \left(\ln \frac{t}{s} \right)^{r-1} f(s, u(s,w), w) \frac{ds}{s} \\ &\quad - \frac{T \ln t}{\Gamma(r-1)} \int_1^T \left(\ln \frac{T}{s} \right)^{r-2} f(s, u(s,w), w) \frac{ds}{s}. \end{aligned}$$

2.2.1 Existence of solutions in the Scalar Case

The following hypotheses will be used in the sequel:

(H_1) The function f is random Carathéodory.

(H_2) There exist measurable and bounded functions $p_i : \Omega \to C(I, \mathbb{R}_+); \ i = 1, 2$ such that

$$|f(t,u,w)| \le p_1(t,w) + p_2(t,w)|u|, \text{ for all } u \in \mathbb{R} \text{ and } t \in I,$$

with

$$p_i^*(w) = \sup_{t \in I} p_i(t,w); \ i = 1, 2, \ w \in \Omega.$$

Theorem 2.2.3 *Assume that the hypotheses* (H_1) *and* (H_2) *hold. If*

$$p_2^*(w)\left(\frac{(\ln T)^r}{\Gamma(r+1)}+T\frac{(\ln T)^r}{\Gamma(r)}\right)<1, \tag{2.5}$$

then the problem (2.1)-(2.2) has a random solution defined on $I\times\Omega$.

Proof. From Lemma 2.2.2 for any $w\in\Omega$ and each $t\in I$, the problem (2.1)-(2.2) is equivalent to the operator equation $N(w)u=u$, where $N:\Omega\times C\to C$ be the operator defined by

$$\begin{aligned}(Nu)(t,w)&=u_1(w)+Tu_T(w)\ln t+\frac{1}{\Gamma(r)}\int_1^t\left(\ln\frac{t}{s}\right)^{r-1}f(s,u(s,w),w)\frac{ds}{s}\\ &-\frac{T\ln t}{\Gamma(r-1)}\int_1^T\left(\ln\frac{T}{s}\right)^{r-2}f(s,u(s,w),w)\frac{ds}{s}.\end{aligned} \tag{2.6}$$

Since the function f is absolutely continuous for all $w\in\Omega$ and $t\in I$, then u is a solution for the problem (2.1)-(2.2) if and only if $u=N(u)(t,w)$. Let

$$R(w)>\frac{|u_1(w)|+T\ln T|u_T(w)|+p_1^*(w)\left(\frac{(\ln T)^r}{\Gamma(r+1)}+T\frac{(\ln T)^r}{\Gamma(r)}\right)}{1-p_2^*(w)\left(\frac{(\ln T)^r}{\Gamma(r+1)}+T\frac{(\ln T)^r}{\Gamma(r)}\right)}\quad w\in\Omega. \tag{2.7}$$

Define the ball

$$B_R=B(0,R(w))=\{u\in C:\| u\|\leq R(w)\}.$$

For any $w\in\Omega$ and each $t\in I$, we have

$$\begin{aligned}|(Nu)(t,w)|&\leq|u_1(w)+Tu_T(w)\ln T|\\ &+\left|\frac{1}{\Gamma(r)}\int_1^t(\ln\frac{t}{s})^{r-1}f(s,u(s,w),w)\frac{ds}{s}\right|\\ &+\left|\frac{T\ln t}{\Gamma(r-1)}\int_1^T(\ln\frac{T}{s})^{r-2}f(s,u(s,w),w)\frac{ds}{s}\right|\\ &\leq|u_1(w)|+T\ln T|u_T(w)|\\ &+\frac{(\ln T)^r}{\Gamma(r+1)}|f(s,u(s,w),w)|+T\ln T\frac{(\ln T)^{r-1}}{\Gamma(r)}|f(s,u(s,w),w)|\\ &\leq|u_1(w)|+T\ln T|u_T(w)|+\frac{(\ln T)^r}{\Gamma(r+1)}\left(p_1^*(w)+p_2^*(w)R(w)\right)\\ &+T\frac{(\ln T)^r}{\Gamma(r)}\left(p_1^*(w)+p_2^*(w)R(w)\right)\\ &\leq R(w).\end{aligned}$$

This proves that $N(w)$ transforms the ball B_R into itself. We shall prove in several steps that the operator $N : \Omega \times B_R \to B_R$ satisfies assumptions of Theorem 1.3.1.

Step 1. $N(w)$ is a random operator.

Since $f(t,u,w)$ is random Carathéodory, the map $w \longrightarrow f(t,u,w)$ is measurable in view Definition 1.1.5 and further the integral is a limit of a finite sum of measurable functions therefore the map

$$\begin{aligned} w &\mapsto u_1(w) + Tu_T(w)\ln t + \frac{1}{\Gamma(r)}\int_1^t \left(\ln\frac{t}{s}\right)^{r-1} f(s,u(s,w),w)\frac{ds}{s} \\ &- \frac{T\ln t}{\Gamma(r-1)}\int_1^T \left(\ln\frac{T}{s}\right)^{r-2} f(s,u(s,w),w)\frac{ds}{s} \end{aligned}$$

is measurable. As a result, $N(w)$ is a random operator.

Step 2. $N(w)$ is continuous.

Let u_n be a sequence such that $u_n \to U$ in C . Then, for each $t \in I$ we have

$$\begin{aligned} &|(Nu_n)(t,w) - (Nu)(t,w)| \\ &\leq \frac{1}{\Gamma(r)}\int_1^t (\ln\frac{t}{s})^{r-1} |f(s,u_n(s,w),w) - f(s,u(s,w),w)| \frac{ds}{s} \\ &+ \frac{T\ln t}{\Gamma(r-1)}\int_1^T (\ln\frac{T}{s})^{r-2} |f(s,u_n(s,w),w) - f(s,u(s,w),w)| \frac{ds}{s} \\ &\leq \left(\tfrac{(\ln T)^r}{\Gamma(r+1)} + \tfrac{T(\ln T)^r}{\Gamma(r)}\right) \|f(\cdot,u_n(\cdot,w),w) - f(\cdot,u(\cdot,w),w)\|_\infty. \end{aligned}$$

Since f is of Carathéodory type, then by the Lebesgue dominated convergence theorem, we get

$$\|(Nu_n)(\cdot,w) - (Nu)(\cdot,w)\|_\infty \to 0 \ as \ n \to \infty.$$

Since $N(w)$ is a continuous random operator with stochastic domain. We can conclude that $N(w)B_R \subset B_R$ is bounded.

Step 3. $N(w)B_R$ is equicontinuous.

For $1 \leq t_1 \leq t_2 \leq T$, and $u \in B_R$, we have

$$\begin{aligned}
& |(Nu)(t_1,w) - (Nu)(t_2,w)| \\
\leq\ & |Tu_T(w)\ln t_1 + \frac{1}{\Gamma(r)}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{r-1} f(s,u(s,w),w)\frac{ds}{s} \\
-\ & \tfrac{T\ln t_1}{\Gamma(r-1)}\int_1^{T}\left(\ln\frac{T}{s}\right)^{r-2} f(s,u(s,w),w)\frac{ds}{s} \\
-\ & Tu_T(w)\ln t_2 - \tfrac{1}{\Gamma(r)}\int_1^{t_2}\left(\ln\frac{t_2}{s}\right)^{r-1} f(s,u(s,w),w)\frac{ds}{s} \\
+\ & \tfrac{T\ln t_2}{\Gamma(r-1)}\int_1^{T}\left(\ln\frac{T}{s}\right)^{r-2} f(s,u(s,w),w)\frac{ds}{s}| \\
\leq\ & Tu_T(w)\left[\ln t_2 - \ln t_1\right] \\
+\ & \tfrac{1}{\Gamma(r)}\int_1^{t_1}\left[\left(\ln\frac{t_2}{s}\right)^{r-1} - \left(\ln\frac{t_1}{s}\right)^{r-1}\right]|f(s,u(s,w),w)|\,\frac{ds}{s} \\
+\ & \int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{r-2}|f(s,u(s,w),w)|\,\frac{ds}{s} \\
+\ & \tfrac{T(\ln t_2-\ln t_1)}{\Gamma(r-1)}\int_1^{T}\left(\ln\frac{T}{s}\right)^{r-2}|f(s,u(s,w),w)|\,\frac{ds}{s} \\
\leq\ & Tu_T(w)\left[\ln t_2 - \ln t_1\right] \\
+\ & \tfrac{1}{\Gamma(r)}\int_1^{t_1}\left[\left(\ln\frac{t_2}{s}\right)^{r-1} - \left(\ln\frac{t_1}{s}\right)^{r-1}\right](P_1^*(w) + P_2^*(w)\,|u|)\,\frac{ds}{s} \\
+\ & \int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{r-2}(P_1^*(w) + P_2^*(w)|u|)\,\frac{ds}{s} \\
+\ & \tfrac{T(\ln t_2-\ln t_1)}{\Gamma(r-1)}\int_1^{T}\left(\ln\frac{T}{s}\right)^{r-2}(P_1^*(w) + P_2^*(w)|u|)\,\frac{ds}{s} \\
\leq\ & Tu_T(w)\left[\ln\left(\tfrac{t_2}{t_1}\right)\right] \\
+\ & \tfrac{1}{\Gamma(r)}\int_1^{t_1}\left[\left(\ln\frac{t_2}{t_1}\right)^{r-1}(P_1^*(w) + P_2^*(w)R(w))\right]\frac{ds}{s} \\
+\ & \int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{r-2}(P_1^*(w) + P_2^*(w)R(w))\,\frac{ds}{s} \\
+\ & \tfrac{T(\ln t_2-\ln t_1)}{\Gamma(r-1)}\int_1^{T}\left(\ln\frac{T}{s}\right)^{r-2}(P_1^*(w) + P_2^*(w)R(w))\,\frac{ds}{s}.
\end{aligned}$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \times B_R \to B_R$ is continuous and compact. From an application of Theorem 1.3.1, we deduce that the operator equation $Nu(w) = u$ has a random solution.

2.2.2 Existence Results in Banach Space

In this section we prove the existence of random solutions for our problem in the Banach space E by using the measure of noncompactness.

Let us introduce the following hypotheses:

(H_1') The function f is random Carathéodory.

(H_2') There exist measurable and bounded functions $l_i : \Omega \to C(I, E)$ and $i = 1, 2$ such that

$$(1 + \|u\|)\|f(t, u, w)\| \leq l_1(t, w) + l_2(t, w)\|u\|$$

for each $u \subset E$ and $t \in I$. with

$$l_i^*(w) = \sup_{t \in I} \|l_i(t, w)\|; \; i = 1, 2, \; w \in \Omega.$$

(H_3') For any bounded $B \in E$ and $t \in I$

$$\alpha(f(t, B, w)) \leq l_2(t, w)\alpha(B).$$

Theorem 2.2.4 *Assume (H_1')-(H_3') hold. If*

$$M := 4\left[\frac{(\ln T)^r}{\Gamma(r+1)} l_2^*(w) + \frac{T(\ln T)^r}{\Gamma(r)} l_2^*(w)\right] \leq 1,$$

then the problem (2.1)-(2.2) has at least one solution defined on I.

Proof. From hypotheses (H_1') and (H_2'), for each $w \subset \Omega$ and $t \in I$ the problem (2.1)-(2.2) is equivalent to the Operator $N : \Omega \times C(I, E) \to C(I, E)$ defined in (2.6).
Since the function f is absolutely continuous for all $w \subset \Omega$ and $t \in I$. Hence u is a solution for the problem (2.1)-(2.2) if and only if $u = N(u)(t, w)$, we shall show that the operator N satisfied all conditions of Theorem 1.3.2. The proof will be given in several steps.
Step 1.($N(w)$is a random operator)
Since $f(t, u, w)$ is a random Carathéodory, the maps $w \longrightarrow f(t, u, w)$ is measurable in view definition 1.1.5 and further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \;\; \mapsto \;\; u_1(w) + Tu_T(w)\ln t + \frac{1}{\Gamma(r)} \int_1^t \left(\ln \frac{t}{s}\right)^{r-1} f(s, u(s, w), w)\frac{ds}{s}$$

$$- \quad \frac{T\ln t}{\Gamma(r-1)}\int_1^T \left(\ln\frac{T}{s}\right)^{r-2} f(s,u(s,w),w)\frac{ds}{s}$$

is measurable. As a result, $N(w)$ is a random operator.

Step 2. ($N(w)$ is bounded)

$$\begin{aligned}
\|(Nu)(t,w)\| &\leq \|u_1(w)+Tu_T(w)\ln T\| \\
&+ \left\|\frac{1}{\Gamma(r)}\int_1^t \left(\ln\frac{t}{s}\right)^{r-1} f(s,u(s,w),w)\frac{ds}{s}\right\| \\
&+ \left\|\frac{T\ln t}{\Gamma(r-1)}\int_1^T \left(\ln\frac{T}{s}\right)^{r-2} f(s,u(s,w),w)\frac{ds}{s}\right\| \\
&\leq \|u_1(w)\| + T\ln T\|u_T(w)\| \\
&+ \frac{(\ln T)^r}{\Gamma(r+1)}\|f(s,u(s,w),w)\| + T\ln T\frac{(\ln T)^{r-1}}{\Gamma(r)}\|f(s,u(s,w),w)\| \\
&\leq \|u_1(w)\| + T\ln T\|u_T(w)\| + \frac{(\ln T)^r}{\Gamma(r+1)}\left(l_1^*(w)+l_2^*(w)\right) \\
&+ T\frac{(\ln T)^r}{\Gamma(r)}\left(l_1^*(w)+l_2^*(w)\right) \\
&\leq \|u_1(w)\| + T\ln T\|u_T(w)\| + (l_1^*(w) \\
&+ l_2^*(w))\left(\frac{(\ln T)^r}{\Gamma(r+1)} + T\frac{(\ln T)^r}{\Gamma(r)}\right) \\
&:= \ell.
\end{aligned}$$

Hence, we conclude that $N(w)$ is bounded.

Step 3.($N(w)$ is a condensing operator)

For each bounded B of $C(I,E)$, we have

$$\begin{aligned}
\alpha((N(w)B)(t,w)) &= \alpha(\{u_1(w)+Tu_T(w)\ln t \\
&\quad + \frac{1}{\Gamma(r)}\int_1^t \left(\ln\frac{t}{s}\right)^{r-1} f(s,u(s,w),w)\frac{ds}{s} \\
&\quad - \frac{T\ln t}{\Gamma(r-1)}\int_1^T \left(\ln\frac{T}{s}\right)^{r-2} f(s,u(s,w),w)\frac{ds}{s},\ u\in B\}) \\
&\leq 2\alpha(\{\frac{1}{\Gamma(r)}\int_1^t \left(\ln\frac{t}{s}\right)^{r-1} f(s,u(s,w),w)\frac{ds}{s} \\
&\quad - \frac{T\ln t}{\Gamma(r-1)}\int_1^T \left(\ln\frac{T}{s}\right)^{r-2} f(s,u(s,w),w)\frac{ds}{s}\}) + \xi
\end{aligned}$$

$$
\begin{aligned}
&\leq 4\Big[\frac{1}{\Gamma(r)}\int_1^t\left(\ln\frac{t}{s}\right)^{r-1}\alpha(\{f(s,u(s,w),w)\})\frac{ds}{s} \\
&\quad+\frac{T\ln t}{\Gamma(r-1)}\int_1^T\left(\ln\frac{T}{s}\right)^{r-2}\alpha(\{f(s,u(s,w),w)\})\frac{ds}{s}\Big]+\xi \\
&\leq 4\left[\frac{(\ln T)^r}{\Gamma(r+1)}l_2^*(w)+\frac{T(\ln T)^r}{\Gamma(r)}l_2^*(w)\right]\alpha(B)+\xi \\
&\leq M\alpha(B)+\xi.
\end{aligned}
$$

Since $\xi > 0$ is arbitrary and $M \leq 1$, then

$$\alpha(N(B)) \leq \alpha(B).$$

Hence N is a condensing random operator. Consequently, from the above three steps; the problem (2.1)-(2.2) has a random solution.

2.3 Examples

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$.

Example 1. Consider the random equation of Caputo-Hadamard fractional differential equations of the from

$$({}^{Hc}D_1^r u)(t,w) = \frac{cw^2}{\exp(t+3)(1+w^2+|u(t,w)|)};\ t\in[1,e],\ w\in\Omega, \tag{2.8}$$

with the boundary conditions

$$\begin{cases} u(1,w) = \sin w \\ u'(T,w) = \cos w \end{cases};\ w\in\Omega, \tag{2.9}$$

where $0 < c < e^3\left(\frac{1}{\Gamma(r+1)}+\frac{e}{\Gamma(r)}\right)^{-1}$. Set

$$f(t,u(t,w),w) = \frac{cw^2}{\exp(t+3)(1+w^2+|u(t,w))|};\ t\in[1,e],\ w\in\Omega,$$

and

$$\begin{cases} u_0(w) = \sin w \\ u_T(w) = \cos w. \end{cases} \tag{2.10}$$

The condition (H_2) is satisfied with $p_1(t,w) = 0$ and $p_2(t,w) = ce^{-3-t}$. The condition (2.5) is satisfies, indeed;

$$p_2^*(w)\left(\frac{(\ln T)^r}{\Gamma(r+1)} + T\frac{(\ln T)^r}{\Gamma(r)}\right) = ce^{-3}\left(\frac{1}{\Gamma(r+1)} + \frac{e}{\Gamma(r)}\right) < 1.$$

Consequently, Theorem 2.2.3 implies that the problem (2.8)-(2.9) has at least one random solution.

Example 2. Let

$$E = l^1 = \left\{u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty\right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Consider the random Caputo-Hadamard fractional differential equation

$$^{Hc}D_1^r(t,u_n) = \frac{c(2^{-n} + u_n)}{(1+w^2)(1+|u(t,w)|)};\ t \in [1,e],\ w \in \Omega, \tag{2.11}$$

with the boundary conditions

$$u_n(1,w) = u_n'(\exp 1, w) = 0, \tag{2.12}$$

with

$$0 < c \leq \left(\frac{4}{\Gamma(r+1)} + \frac{4e}{\Gamma(r)}\right)^{-1},\ u = (u_1, u_2, \cdots),\ f = (f_1, f_2, \cdots).$$

Set

$$f_n(t,u,w) = \frac{cw^2(2^{-n} + u_n)}{1+|u(t,w)|}.$$

The condition (H_2') is satisfied with $l_1(t,w) = l_2(t,w) = c$. Also, the condition $M \leq 1$ is satisfied and we have

$$M := 4\left(\frac{(\ln T)^r}{\Gamma(r+1)} l_2^*(w) + \frac{T(\ln T)^r}{\Gamma(r)} l_2^*(w)\right) = 4c\left(\frac{1}{\Gamma(r+1)} + \frac{e}{\Gamma(r)}\right) \leq 1.$$

Simple computations show that all conditions of Theorem 2.2.4 are satisfied. Consequently, the problem (2.11)-(2.12) has at least one random solution.

CHAPTER 3

ULAM STABILITIES FOR RONDOM FRACTIONAL DIFFERENTIAL EQUATIONS

3.1 Introduction and Motivations

There are different definitions of fractional derivatives. The popular derivatives of fractional order we mention Riemann-Liouville, Caputo, Hadamard, and Hilfer.

Caputo and Fabrizio developed and proposed a new version of fractional derivative by changing the Kernel $(t-s)^{-\alpha}$ by the function $(t,s) \mapsto \exp(\frac{(-\alpha(t-s))}{(1-\alpha)})$ and $\frac{1}{\Gamma(1-\alpha)}$ by $\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}$. For more details; see [98]. Katugampola introduced a derivative that is a generalization of the Riemann-Liouville fractional operators and the fractional integral of Hadamard in a single
form [89, 90].

The question of stability for functional differential equations was introduced by Ulam and Hyers. Thereafter; this type of stability is called the Ulam-Hyers stability [82, 116]. In 1978, Rassias provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables of stability for a functional equation arises when we replace the functional equation by an inequality. For more details; see the monographs [16, 77, 83, 85], the papers [21, 86, 112, 115, 116, 130, 131, 132], and the references therein

In one section we investigate the following class of Caputo–Fabrizio fractional differential equation

$$({}^{CF}D_0^{\alpha}u)(t,w) = f(t,u(t,w),w);\ t \in I := [0,T],\ w \in \Omega, \tag{3.1}$$

with the boundary conditions

$$au(0,w) + bu(T,w) = c(w);\ w \in \Omega, \tag{3.2}$$

where $T > 0,\ f : I \times E \times \Omega \to E$ is a given function, $a, b \in \mathbb{R},\ ,c : \Omega \to E$, with $a + b \neq 0,\ {}^{CF}D_0^{\alpha}$ is the Caputo–Fabrizio fractional derivative of order $\alpha \in (0,1)$, and Ω is the sample space in a probability space (Ω, F), and E is a real (or complex) Banach space with a norm $\|\cdot\|$. Next we investigate the following class of Katugampola random fractional differential equation

$$({}^{\rho}D_0^{\varsigma}x)(\xi,w) = f(\xi,x(\xi,w),w);\ \xi \in I = [0,T],\ w \in \Omega, \tag{3.3}$$

with the terminal condition

$$x(T,w) = x_T(w);\ w \in \Omega, \tag{3.4}$$

where $x_T : \Omega \to E$ is a measurable function, $\varsigma \in (0,1],\ T > 0,\ f : I \times E \times \Omega \to E,\ {}^{\rho}D_0^{\varsigma}$ is the Katugampola operator of order ς, and Ω is the sample space in a probability space, and $(E, \|\cdot\|)$ is a Banach space.

3.2 Boundary Value Problem for Caputo–Fabrizio Random Fractional Differential Equations

Let $\mathcal{C} := C(I,E)$ be the Banach space of all continuous functions from I into E with the norm

$$\|u\|_{\infty} = \sup\{\|u(t)\| : t \in I\}.$$

Lemma 3.2.1 *Let $h \in L^1(I,E)$. A function $u \in \mathcal{C}$ is a solution of problem*

$$\begin{cases} ({}^{CF}D_0^{\alpha}u)(t) = h(t); \quad t \in I := [0,T] \\ au(0) + bu(T) = c, \end{cases} \tag{3.5}$$

where $a, b \in \mathbb{R},\ c \in E$ with $a + b \neq 0$, if and only if u satisfies the following integral *equation*

$$u(t) = C_0 + a_\alpha h(t) + b_\alpha \int_0^t h(s)ds + \frac{bb_\alpha}{a+b} \int_0^T h(s)ds, \tag{3.6}$$

$$a_\alpha = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)},\ b_\alpha = \frac{2\alpha}{(2-\alpha)M(\alpha)},$$

$$C_0 = \frac{1}{a+b}[c - ba_\alpha(h(T) - h(0))] - a_\alpha h(0).$$

Proof. Suppose that u satisfies (3.5). From Proposition 1 in [98], the equation $({}^{CF}D_0^\alpha u)(t) = h(t)$ implies that

$$u(t) - u(0) = a_\alpha(h(t) - h(0)) + b_\alpha \int_0^t h(s)ds.$$

Thus,

$$u(T) = u(0) + a_\alpha(h(T) - h(0)) + b_\alpha \int_0^T h(s)ds.$$

From the mixed boundary conditions $au(0) + bu(T) = c$, we get

$$au(0) + b(u(0) + a_\alpha(h(T) - h(0)) + b_\alpha \int_0^T h(s)ds) = c.$$

Hence,

$$u(0) = \frac{c - b(a_\alpha(h(T) - h(0)) - b_\alpha \int_0^T h(s)ds)}{a+b}.$$

So; we get (3.6).
Conversely, if u satisfies (3.6), then $({}^{CF}D_0^\alpha u)(t) = h(t)$; for $t \in I := [0, T]$, and $au(0) + bu(T) = c$.

From the above Lemma, we can conclude the following Lemma:

Lemma 3.2.2 *A function u is a random solution of problem (3.1)-(3.2), if and only if u satisfies the following integral equation:*

$$u(t,w) = C_0(w) + a_\alpha f(t, u(t,w), w)$$

$$+b_\alpha \int_0^t f(s, u(s,w), w)ds + \frac{bb_\alpha}{a+b} \int_0^T f(s, u(s,w), w)ds,$$

where

$$C_0(w) = \frac{1}{a+b}[c(w) - ba_\alpha(f(T, u(T,w), w) - f(0, u(0,w), w))] - a_\alpha f(0, u(0,w), w).$$

3.2.1 Existence of solutions

Definition 3.2.3 *By a random solution of problem (3.1)-(3.2), we mean a function* $u \in C$ *that satisfies the equation*

$$u(t,w) = C_0(w) + a_\alpha f(t,u(t,w),w)$$

$$+b_\alpha \int_0^t f(s,u(s,w),w)ds + \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds,$$

where

$$C_0(w) = \frac{1}{a+b}[c(w) - ba_\alpha(f(T,u(T,w),w) - f(0,u(0,w),w))] - a_\alpha f(0,u(0,w),w).$$

The following hypotheses will be used in the sequel:

(H_1) The function f is random Carathéodory.

(H_2) There exist measurable and bounded functions $p_i : \Omega \to C(I,[0,\infty));\ i = 1,2$ such that

$$\|f(t,u,w)\| \le p_1(t,w) + p_2(t,w)\|u\|;$$

for all $u \subset E$ and $t \in I$ with

$$p_i^*(w) = \sup_{t\in I} p_i(t,w);\ i = 1,2,\ w \in \Omega.$$

Now, we prove an existence result for the problem (3.1)-(3.2) based on Itoh's fixed point theorem.

Theorem 3.2.4 *Assume that the hypotheses* $(H_1) - (H_2)$ *hold. If*

$$\left(a_\alpha + Tb_\alpha + T\frac{bb_\alpha}{a+b}\right)p_2^*(w) < 1, \tag{3.7}$$

then the problem (3.1)-(3.2) has at least one random solution defined on I.

Proof. From Lemma 3.2.2 for any $w \in \Omega$ and each $t \in I$, the problem (3.1)-(3.2) is equivalent to the operator equation $(Nw)u = u$, where $N : \Omega \times \mathcal{C} \to \mathcal{C}$ be the operator defined by

$$(Nu)(t,w) = C_0(w) + a_\alpha f(t,u(t,w),w)$$

$$+b_\alpha \int_0^t f(s,u(s,w),w)ds + \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds. \tag{3.8}$$

Since the function f is absolutely continuous for all $w \in \Omega$ and $t \in I$, then u is a random solution for the problem (3.1)-(3.2) if and only if $u = (Nu)(t,w)$. Set

$$R(w) > \frac{\|C_0(w)\| + \left[a_\alpha + Tb_\alpha + T\frac{bb_\alpha}{a+b}\right] p_1^*(w)}{1 - \left[a_\alpha + Tb_\alpha + T\frac{bb_\alpha}{a+b}\right] p_2^*(w)} \quad w \in \Omega. \tag{3.9}$$

Define the ball

$$B_R = B(0,R(w)) := \{u \in \mathcal{C} : \|u\| \leq R(w)\}.$$

For any $w \in \Omega$ and each $t \in I$, we have

$$\begin{aligned}
\|(Nu)(t,w)\| &\leq \|C_0(w)\| + \|a_\alpha f(t,u(t,w),w)\| \\
&+ \left\|b_\alpha \int_0^t f(s,u(s,w),w)ds\right\| + \left\|\frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds\right\| \\
&\leq \|C_0(w)\| + a_\alpha \|f(t,u(t,w),w)\| \\
&+ b_\alpha \int_0^t \|f(s,u(s,w),w)\|ds + \frac{bb_\alpha}{a+b}\int_0^T \|f(s,u(s,w),w)\|ds \\
&\leq \|C_0(w)\| + \left[a_\alpha + Tb_\alpha + T\frac{bb_\alpha}{a+b}\right](p_1^*(w) + p_2^*(w)R(w)) \\
&\leq R(w).
\end{aligned}$$

This proves that $N(w)$ transforms the ball B_R into itself. We shall prove in three steps that the operator $N : \Omega \times B_R \to B_R$ satisfies all the assumptions of Theorem 1.3.1.

Step 1. $N(w)$ is a random operator.
Since $f(t,u,w)$ is random Carathéodory, the map $w \longrightarrow f(t,u,w)$ is measurable in view Definition 1.1.5 and further the integral is a limit of a finite sum of measurable functions therefore the map

$$\begin{aligned}
w &\mapsto C_0(w) + a_\alpha f(t,u(t,w),w) \\
&+ b_\alpha \int_0^t f(s,u(s,w),w)ds + \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds,
\end{aligned}$$

is measurable. As a result, $N(w)$ is a random operator.

Step 2. $N(w)$ is continuous and bounded.
Let u_n be a sequence such that $u_n \to U$ in $\mathcal{C}$. Then, for each $t \in I$ we have

$$\|(Nu_n)(t,w) - (Nu)(t,w)\| \leq \|a_\alpha(f(t,u(t,w),w) - f(t,u_n(t,w),w))\|$$

$$\begin{aligned}
&+ \left\| b_\alpha \int_0^t (f(t,u(t,w),w) - f(t,u_n(t,w),w))ds \right\| \\
&+ \left\| \frac{bb_\alpha}{a+b} \int_0^T (f(t,u(t,w),w) - f(t,u_n(t,w),w)) \right\| \\
&\leq a_\alpha \|f(t,u(t,w),w) - f(t,u_n(t,w),w)\| \\
&+ b_\alpha \int_0^t \|f(t,u(t,w),w) - f(t,u_n(t,w),w)\| ds \\
&+ \frac{bb_\alpha}{a+b} \int_0^T \|f(t,u(t,w),w) - f(t,u_n(t,w),w)\| ds.
\end{aligned}$$

Since f is Carathéodory, then by the Lebesgue dominated convergence theorem, we get

$$\|(Nu_n)(\cdot,w)) - (Nu)(\cdot,w)\|_\infty \to 0 \ as \ n \to \infty.$$

Since $N(w)$ is a continuous random operator with stochastic domain. We can conclude that $N(w)B_R \subset B_R$ is bounded.

Step 3. $N(w)B_R$ is equicontinuous.
For $1 \leq t_1 \leq t_2 \leq T$, and $u \in B_R$, we have

$$\begin{aligned}
\|(Nu)(t_2,w) &- (Nu)(t_1,w)\| \leq \Big\| a_\alpha f(t_2,u(t_2,w),w) + b_\alpha \textstyle\int_0^{t_2} f(s,u(s,w),w)ds \\
&+ \frac{bb_\alpha}{a+b} \int_0^T f(s,u(s,w),w)ds - a_\alpha f(t_1,u(t_1,w),w) \\
&- b_\alpha \int_0^{t_1} f(s,u(s,w),w)ds - \frac{bb_\alpha}{a+b} \int_0^T f(s,u(s,w),w)ds \Big\| \\
&\leq a_\alpha \|f(t_2,u(t_2,w),w) - f(t_1,u(t_1,w),w)\| \\
&+ b_\alpha \int_{t_1}^{t_2} \|f(s,u(s,w),w)ds\| \\
&\leq a_\alpha \|f(t_2,u(t_2,w),w) - f(t_1,u(t_1,w),w)\| \\
&+ b_\alpha (t_2 - t_1)(p_1^*(w) + p_2^*(w)R(w)) \\
&\to 0 \ as \ t_2 \to t_1.
\end{aligned}$$

As a consequence of the above steps and the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \times B_R \to B_R$ is continuous and compact. From an application of Theorem 1.3.1, the operator equation $Nu(w) = u$ has a random solution.

3.2.2 Ulam-Hyers Rassias stability

Now, we are concerned with the generalized Ulam-Hyers-Rassias stability of our problem (3.1)-(3.2).

Let $\epsilon > 0$ and $\Phi : I \times \Omega \to \mathbb{R}_+$ be a measurable function. We consider the following inequalities

$$\|({}^{CF}D_0^{\alpha}u)(t,w) - f(t,u(t,w),w)\| \leq \epsilon;\ t \in I,\ w \in \Omega. \tag{3.10}$$

$$\|({}^{CF}D_0^{\alpha}u)(t,w) - f(t,u(t,w),w)\| \leq \Phi(t,w);\ t \in I,\ w \in \Omega. \tag{3.11}$$

$$\|({}^{CF}D_0^{\alpha}u)(t,w) - f(t,u(t,w),w)\| \leq \epsilon\Phi(t,w);\ t \in I,\ w \in \Omega. \tag{3.12}$$

Definition 3.2.5 *[16] The problem(3.1)-(3.2) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $u(\cdot,w) \in C(I)$ of the inequality (3.10), there exists a solution $v() \in C(I)$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq \epsilon c_f;\ t \in I.$$

Definition 3.2.6 *[16] The problem (3.1)-(3.2) is generalized Ulam-Hyers stable if there exists $c_f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $c_f(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u(w) \in C(I)$ of the inequality (3.10), there exists a solution $v \in C(I)$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq c_f(\epsilon);\ t \in I.$$

Definition 3.2.7 *[16] The problem (3.1)-(3.2) is Ulam-Hyers-Rassias stable with respect to ϕ if there exists a real number $c_{f,\phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u(w) \in C(I)$ of the inequality (3.12), there exists a solution $v \in C(I)$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq \epsilon c_{f,\phi}\phi(t,w);\ t \in I.$$

Definition 3.2.8 *[16] The problem (3.1)-(3.2) is generalized Ulam-Hyers-Rassias stable with respect to ϕ if there exists a real number $c_{f,\phi} > 0$ such that for each solution $u \in C(I)$ of the inequality (3.11), there exists a solution $v(w) \in C(I)$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq c_{f,\phi}\phi(t,w);\ t \in I.$$

Remark 3.2.9 *A function $u(\cdot,w) \in \mathcal{C}$ is a solution of the inequality (3.11) if and only if* *there exist a function $g(\cdot, w) \in \mathcal{C}$ (which depend on u) such that*

$$\|g(t,w)\| \leq \Phi(t,w),$$

and

$$({}^{CF}D_0^{\alpha}u)(t,w) = f(t,u(t,w)) + g(t,w);\ for\ t \in I,\ and\ w \in \Omega.$$

The following hypotheses will be used in the sequel.

(H_3) $\Phi(\cdot, w) \in L^1(\mathbb{R}_+)$, and there exists a measurable and bounded function $q : \Omega \to C(I, [0,\infty))$; such that

$$(1 + \|u - v\|)\|f(t,u(t,w),w) - f(t,v(t,w),w)\| \leq q(t,w)\Phi(t,w)\|u - v\|;$$

for all $u, v \in E$ and each $t \in I$, with

$$q^*(w) = \sup_{t\in I} q(t,w);\ w \in \Omega.$$

(H_4) There exists a constant $\lambda_\Phi > 0$, such that for any $w \in \Omega$, and each $t \in I$ we have

$$\int_0^T \Phi(t,w)dt \leq \lambda_\Phi \Phi(t,w).$$

Remark 3.2.10 *From (H_3), for any $w \in \Omega$, and each $t \in I$, and $u \in E$, we have that*

$$\|f(t,u,w)\| \leq \|f(t,0,w)\| + q(t,w)\Phi(t,w)\|u\|.$$

So, (H_3) implies (H_2), with $p_1(t,w) = \|f(t,0,w)\|$, and $p_2(t,w) = q(t,w)\Phi(t,w)$,

Lemma 3.2.11 *If $u \in \mathcal{C}$ is a solution of the inequality (3.11) then u is a solution of the following integral inequality*

$$\|u(t,w) - C_0(w) - a_\alpha f(s,u(s,w),w) - b_\alpha \int_0^t f(s,u(s,w),w)ds$$

$$-\frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds\| \leq \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi \frac{bb_\alpha}{a+b}\right)\Phi(t,w);\ t \in I;\ w \in \Omega. \quad (3.13)$$

Proof. By Remark 3.2.9; for any $w \in \Omega$ and each $t \in I$, we have

$$u(t,w) \;=\; C_0(w) + a_\alpha[f(s,u(s,w),w) + g(s,w)]$$

$$
\begin{aligned}
&+ \quad b_\alpha \int_0^t [f(s,u(s,w),w) + g(s,w)]ds \\
&+ \quad \frac{bb_\alpha}{a+b} \int_0^T [f(s,u(s,w),w) + g(s,w)]ds.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\|u(t,w) \quad &- \quad C_0(w) - a_\alpha f(s,u(s,w),w) - b_\alpha \int_0^t f(s,u(s,w),w)ds \\
&- \quad \frac{bb_\alpha}{a+b} \int_0^T f(s,u(s,w),w)ds\| \\
&\leq \quad a_\alpha \|g(s,w)\| + b_\alpha \int_0^t \|g(s,w)\|ds + \frac{bb_\alpha}{a+b} \int_0^T \|g(s,w)\|ds \\
&\leq \quad \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi \frac{bb_\alpha}{a+b} \right) \Phi(t,w).
\end{aligned}
$$

Theorem 3.2.12 *Assume that the hypotheses* $(H_1), (H_3), (H_4)$ *and the condition (3.7) hold. Then the problem (3.1)-(3.2) has at least one solution on* I *and it is generalized Ulam-Hyers-Rassias stable.*

Proof. From Remark 3.2.10, there exists a random solution v of the random problem (3.1)-(3.2). That is

$$
\begin{aligned}
v(t,w) \quad &= \quad C_0(w) + a_\alpha f(t,v(t,w),w) \\
&+ \quad b_\alpha \int_0^t f(s,v(s,w),w)ds + \frac{bb_\alpha}{a+b} \int_0^T f(s,v(s,w),w)ds.
\end{aligned}
$$

Let u be a solution of the inequality (3.11), then from Lemma 3.2.11, for any $w \in \Omega$, and each $t \in I$, we have

$$
\begin{aligned}
\|u(t,w) \quad &- \quad C_0(w) + a_\alpha f(t,u(t,w),w) - b_\alpha \int_0^t f(s,u(s,w),w)ds \\
&- \quad \frac{bb_\alpha}{a+b} \int_0^T f(s,u(s,w),w)ds \Bigg\| \leq \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi \frac{bb_\alpha}{a+b} \right) \Phi(t,w).
\end{aligned}
$$

Then, for any $w \in \Omega$, and each $t \in I$, we obtain

$$
\begin{aligned}
\|u(t,w) - v(t,w)\| \quad &\leq \quad \|u(t,w) - C_0(w) - a_\alpha f(t,u(t,w),w) - b_\alpha \int_0^t f(s,u(s,w),w)ds \\
&- \quad \frac{bb_\alpha}{a+b} \int_0^T f(s,u(s,w),w)ds + a_\alpha f(t,u(t,w),w)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad b_\alpha \int_0^t f(s,u(s,w),w)ds + \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds \\
&- \quad a_\alpha f(t,v(t,w),w) - b_\alpha \int_0^t f(s,v(s,w),w)ds \\
&- \quad \frac{bb_\alpha}{a+b}\int_0^T f(s,v(s,w),w)ds\|.
\end{aligned}
$$

This implies that,

$$
\begin{aligned}
\|u(t,w)-v(t,w)\| \quad &\leq \quad \|u(t,w) - C_0(w) - a_\alpha f(t,u(t,w),w) - b_\alpha \int_0^t f(s,u(s,w),w)ds \\
&- \quad \frac{bb_\alpha}{a+b}\int_0^T f(s,u(s,w),w)ds\| \\
&+ \quad \|a_\alpha f(t,u(t,w),w) + b_\alpha \int_0^t f(s,u(s,w),w)ds - a_\alpha f(t,v(t,w),w) \\
&- \quad b_\alpha \int_0^t f(s,v(s,w),w)ds - \frac{bb_\alpha}{a+b}\int_0^T f(s,v(s,w),w)ds\| \\
&\leq \quad \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi \frac{bb_\alpha}{a+b}\right)\Phi(t,w) \\
&+ \quad a_\alpha \|f(t,u(t,w),w) - f(t,v(t,w),w)\| \\
&+ \quad b_\alpha \int_0^t \|f(s,u(s,w),w) - f(s,v(s,w),w)\|ds \\
&+ \quad \frac{bb_\alpha}{a+b}\int_0^T \|f(s,u(s,w),w) - f(s,v(s,w),w)\|ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|u(t,w)-v(t,w)\| \quad &\leq \quad \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi \frac{bb_\alpha}{a+b}\right)\Phi(t,w) \\
&+ \quad a_\alpha q^*(w)\Phi(t,w)\frac{\|u(t,w)-v(t,w)\|}{1+\|u(t,w)-v(t,w)\|} \\
&+ \quad b_\alpha \int_0^t q^*(w)\Phi(t,w)\frac{\|u(s,w)-v(s,w)\|}{1+\|u(s,w)-v(s,w)\|}ds \\
&+ \quad \frac{bb_\alpha}{a+b}\int_0^T q^*(w)\Phi(t,w)\frac{\|u(s,w)-v(s,w)\|}{1+\|u(s,w)-v(s,w)\|}ds \\
&\leq \quad \left(a_\alpha + \lambda_\Phi b_\alpha + \lambda_\Phi \frac{bb_\alpha}{a+b}\right)\Phi(t,w) \\
&+ \quad a_\alpha q^*(w)\Phi(t,w) + b_\alpha q^*(w)\int_0^t \Phi(t,w)ds \\
&+ \quad \frac{bb_\alpha q^{*(w)}}{a+b}\int_0^T \Phi(t,w)ds.
\end{aligned}
$$

Hence, from (H_4), we get

$$\begin{aligned}
\|u(t,w)-v(t,w)\| &\leq \left(a_\alpha+\lambda_\Phi b_\alpha+\lambda_\Phi\frac{bb_\alpha}{a+b}\right)\Phi(t,w)+a_\alpha q^*(w)\Phi(t,w)\\
&+ \left(b_\alpha q^*(w)+\frac{bb_\alpha q^*(w)}{a+b}\right)\int_0^T\Phi(s,w)ds\\
&\leq \left(a_\alpha+\lambda_\Phi b_\alpha+\lambda_\Phi\frac{bb_\alpha}{a+b}\right)(1+q^*(w))\Phi(t,w)\\
&:= c_{f,\Phi}\Phi(t,w).
\end{aligned}$$

This conclude that our problem (3.1)-(3.2) is generalized Ulam-Hyers-Rassias stable.

3.3 Dynamics and Stability for Katugampola Random Fractional Differential Equations

3.3.1 Existence of solutions

By $C(I):=C(I,E)$ we denote the Banach space of all continuous functions $x:I\to E$ with the norm

$$\|x\|_\infty=\sup_{t\in I}\|x(\xi)\|,$$

Let $C_{\varsigma,\rho}(I)$ be the weighted space of continuous functions defined by

$$C_{\varsigma,\rho}(I)=\{x:(0,T]\to\mathbb{R}:\xi^{\rho(1-\varsigma)}x(\xi)\in C(I)\},$$

with the norm

$$\|x\|_C:=\sup_{\xi\in I}\|\xi^{\rho(1-\varsigma)}x(\xi)\|.$$

Lemma 3.3.1 *The problem*

$$\begin{cases}({}^\rho D_0^r x)(t)=h(t); \quad t\in I:=[0,T]\\ x(T)=x_T\end{cases} \tag{3.14}$$

has the following solution

$$u(t)=\frac{\rho^{1-r}}{\Gamma(r)}\int_0^t\frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-r}}h(t)ds-Ct^{\rho(r-1)} \tag{3.15}$$

where

$$C=\frac{1}{T^{\rho(r-1)}}\left(\frac{\rho^{1-r}}{\Gamma(r)}\int_0^T\frac{s^{\rho-1}}{(T^\rho-s^\rho)^{1-r}}h(T)ds-u_T\right).$$

Proof. Solving the equation

$$({}^{\rho}D_0^r u)(t) = h(t),$$

we get

$$u(t) =^{\rho} I_0^r h(t) - ct^{\rho(r-1)}.$$

From the condition, we get

$$C = \frac{{}^{\rho}I_0^r h(T) - u_T}{T^{\rho(r-1)}}$$

hence, we obtain (3.15).

Lemma 3.3.2 *u is a random solution of (3.3)-(3.4), if and only if it satisfies*

$$x(\xi, w) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^{\xi} \frac{s^{\rho-1}}{(\xi^{\rho} - s^{\rho})^{1-\varsigma}} f(\xi, x, w) ds - C(w)\xi^{\rho(\varsigma-1)} \tag{3.16}$$

where

$$C(w) = \frac{1}{T^{\rho(\varsigma-1)}} \left(\frac{\rho^{1-\varsigma}}{\Gamma(r)} \int_0^T \frac{s^{\rho-1}}{(T^{\rho} - s^{\rho})^{1-\varsigma}} f(T, x, w) ds - x_T(w) \right).$$

Definition 3.3.3 *By a random solution of problem (3.3)-(3.4), we mean a measurable function $x(w, \cdot) \in C_{\varsigma,\rho}(I)$ such that*

$$x(t, w) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{1-\varsigma}} f(t, x, w) ds - C(w)t^{\rho(\varsigma-1)}, \tag{3.17}$$

where

$$C(w) = \frac{1}{T^{\rho(r-1)}} \left(\frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^T \frac{s^{\rho-1}}{(T^{\rho} - s^{\rho})^{1-\varsigma}} f(T, u, w) ds - u_T(w) \right).$$

We shall make use of the following hypotheses:

(H_1) f is a random Carathéodory function.

(H_2) There exist measurable and essentially bounded functions $l_i : \Omega \to C(I);\ i = 1, 2$ such that

$$\|f(t, x, w)\| \leq l_1(t, w) + l_2(t, w)t^{\rho(1-r)}\|x\|,$$

for all $x \in E$ and $t \in I$ with

$$l_i^*(w) = \sup_{t \in I} l_i(t, w);\ i = 1, 2,\ w \in \Omega.$$

Theorem 3.3.4 *If (H_1) and (H_2) hold, and*

$$\frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_2^*(w) < 1, \tag{3.18}$$

then there exists a random solution for (3.3)-(3.4).

Proof. Let $N : \Omega \times C_{\varsigma,\rho}(I) \to C_{\varsigma,\rho}(I)$ be the operator defined by

$$(Nx)(t,w) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w)ds - C(w)t^{\rho(\varsigma-1)}, \tag{3.19}$$

and set

$$R(w) > \frac{\|C(w)\| + \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_1^*(w)}{1 - \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_2^*(w)}; \quad w \in \Omega, \tag{3.20}$$

and define the ball

$$B_R = B(0, R(w)) := \{x \in C_{\varsigma,\rho}(I) : \|x\|_C \leq R(w)\}.$$

For any $w \in \Omega$ and each $t \in I$, we have

$$\begin{aligned}
\|t^{\rho(1-\varsigma)}(Nx)(t,w)\| &\leq \|C(w)\| + \|\frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} f(s,x(s,w),w)ds\| \\
&\leq \|C(w)\| + \frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}}\|l_1(s,w)\|ds \\
&+ \frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}}\|s^{\rho(1-\varsigma)}l_2(s,w)x(s,w)\|ds \\
&\leq \|C(w)\| + \frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\frac{T^{\varsigma\rho}}{\varsigma\rho}l_1^*(w) \\
&+ \frac{l_2^*(w)\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(r)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}}\|s^{\rho(1-\varsigma)}x(s,w)\|ds \\
&\leq \|C(w)\| + \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_1^*(w) + \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_2^*(w)\|x\|_C \\
&\leq \|C(w)\| + \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_1^*(w) + \frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}l_2^*(w)R(w) \\
&\leq R(w).
\end{aligned}$$

Thus

$$\|N(w)(u\|_C \leq R(w).$$

Hence $N(w)(B_R) \subset B_R$. We shall prove that $N : \Omega \times B_R \to B_R$ satisfies the assumptions of Theorem 1.3.1.

Step 1. $N(w)$ is a random operator.
From (H_1), the map $w \longrightarrow f(t,x,w)$ is measurable and further the integral is a limit of a finite sum of measurable functions therefore the map

$$w \mapsto \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} f(s,x(s,w),w)ds - C(w)t^{\rho(r-1)},$$

is measurable.
Step 2. $N(w)$ is continuous.
Consider the sequence $(x_n)_n$ such that $x_n \to u$ in $C_{\varsigma,\rho}$.
Set

$$v_n(t,w) = t^{\rho(1-\varsigma)}(Nx_n)(t,w), \quad and \quad v(t,w) = t^{\rho(1-\varsigma)}(Nx)(t,w).$$

Then

$$\|v_n(t,w) - v(t,w)\|$$

$$\begin{aligned}
&\leq \left\| \frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}}(f(s,x_n(s,w),w) - f(s,x(s,w),w))ds \right\| \\
&\leq \frac{\rho^{1-\varsigma}T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}}\|f(s,x_n(s,w),w) - f(s,x(s,w),w))\|ds.
\end{aligned}$$

By (H_1) we obtain

$$\|v_n(\cdot,w) - v(\cdot,w)\|_C \to 0 \ as \ n \to \infty,$$

Consequently, $N(w) : B_R \subset B_R$ is continuous.

Step 3. $N(w)B_R$ is equicontinuous.
For $1 \leq t_1 \leq t_2 \leq T$, and $x \in B_R$, we have

$$\|t_2^{\rho(1-\varsigma)}(Nx)(t_2,w) - t_1^{\rho(1-\varsigma)}(Nx)(t_1,w)\|$$

$$
\begin{aligned}
&\leq \left\| \frac{\rho^{1-\varsigma} t_2^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_2} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds \right. \\
&- \left. \frac{\rho^{1-\varsigma} t_1^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_1} \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds \right\| \\
&\leq \left\| \frac{\rho^{1-\varsigma} t_2^{\rho(1-\varsigma)}}{\Gamma(r)} \int_{t_1}^{t_2} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds \right. \\
&- \frac{\rho^{1-\varsigma} t_1^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_1} \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds \\
&+ \left. \frac{\rho^{1-\varsigma} t_2^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_1} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds \right\| \\
&\leq \frac{\rho^{1-\varsigma} T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_{t_1}^{t_2} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\varsigma}} \|f(s, x(s,w), w)\| ds \\
&+ \frac{\rho^{1-\varsigma} T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_1} \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\varsigma}} \|f(s, x(s,w), w)\| ds \\
&+ \frac{\rho^{1-\varsigma} T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^{t_1} \frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\varsigma}} \|f(s, x(s,w), w)\| ds \\
&\leq \frac{t_2^{\varsigma\rho} + t_1^{\varsigma\rho} + 2(t_2^\rho - t_1^\rho)^\varsigma}{\rho^\varsigma \Gamma(1+\varsigma)} T^{\rho(1-\varsigma)} (l_1^*(w) + l_2^*(w) R(w)) \\
&\to 0; \quad as \quad t_2 \to t_1.
\end{aligned}
$$

Arzelá-Ascoli theorem implies that $N : \Omega \times B_R \to B_R$ is continuous and compact. Hence; from Theorem 1.3.1, we deduce the existence of random solution to problem (3.3)-(3.4).

3.3.2 Ulam-Hyers Rassias stability

we prove a result concerning the generalized Ulam-Hyers-Rassias stability of (3.3)-(3.4).

Let $\epsilon > 0$ and $\Phi : \Omega \times I \to \mathbb{R}_+$ be a jointly measurable function. We consider the following inequality

$$\|({}^\rho D_0^r x)(\xi, w) - f(\xi, u(\xi, w), w)\| \leq \Phi(\xi, w); \; for \; \xi \in I, \quad and \; w \in \Omega. \tag{3.21}$$

Definition 3.3.5 *[16] The problem (3.3)-(3.4) is generalized Ulam-Hyers-Rassias stable with respect to* Φ *if there exists* $c_{f,\phi} > 0$ *such that for each solution* $x(\cdot, w) \in C_{\varsigma,\rho}(I)$ *of the inequality (3.21), there exists* $y(\cdot, w) \in C_{\varsigma,\rho}(I)$ *satisfies (3.3)-(3.4) with*

$$\|\xi^{\rho(1-\varsigma)} x(\xi, w) - \xi^{\rho(1-\varsigma)} y(\xi, w)\| \leq c_{f,\phi} \phi(\xi, w); \; \xi \in I; \; w \in \Omega.$$

We introduce the following additional hypotheses:

(H_3) For any $w \in \Omega$, $\Phi(t, \cdot) \subset L^1[0, \infty)$, and there exists a measurable and essentially bounded function $q : \Omega \to C(I, [0, \infty))$; such that

$$(1 + \|x - y\|)\|f(t, x(t, w), w) - f(t, y(t, w), w)\| \leq q(t, w)\Phi(t, w)t^{\rho(1-\varsigma)}\|x - y\|.$$

(H_4) There exists $\lambda_\Phi > 0$ such that

$$^{\rho}I_0^{\varsigma}\Phi(t, w) \leq \lambda_\Phi \Phi(t, w).$$

Remark 3.3.6 *Hypothesis (H_3) implies (H_2) with*

$$l_1(w, t) = f(t, 0, w),\ \ l_2(w) = q(t, w)\Phi(t, w).$$

Set

$$\Phi^*(w) = \sup_{t\in I} \Phi(t, w),\ \ q^*(w) = \sup_{t\in I} q(t, w);\ \ w \in \Omega.$$

Theorem 3.3.7 *If (H_1), (H_3), (H_4) and*

$$\frac{\rho^{-\varsigma}T^{\rho}}{\Gamma(1+\varsigma)}\Phi^*(w)q^*(w) < 1, \tag{3.22}$$

hold. Then the problem (3.3)-(3.4) has random solutions defined on I, and it is generalized Ulam-Hyers-Rassias stable.

Proof. From (H_1), (H_3) and Remark 3.3.6; the problem (3.3)-(3.4) has at least one random solution y. Then, we have

$$y(t, w) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} f(s, y(s, w), w)ds - C(w)t^{\rho(\varsigma-1)}.$$

Assume x be a random solution of (3.21). We obtain

$$\begin{aligned} \|t^{\rho(1-\varsigma)}x(t, w) &- \frac{\rho^{1-\varsigma}t^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} f(s, v(s, w), w)ds + C(w)\| \\ &\leq T^{\rho(1-\varsigma)}(^{\rho}I_0^{\varsigma}\Phi)(t, w). \end{aligned}$$

From hypotheses (H_3) and (H_4), we have

$$\|t^{\rho(1-\varsigma)}x(t, w) - t^{\rho(1-\varsigma)}y(t, w)\|$$

$$\leq \|t^{\rho(1-\varsigma)}x(t, w) - \frac{\rho^{1-\varsigma}t^{\rho(1-\varsigma)}}{\Gamma(\varsigma)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} f(s, x(s, w), w)ds + C(w)\|$$

$$
\begin{aligned}
&+ \quad \| \frac{\rho^{1-\varsigma} t^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} f(s, x(s,w), w) ds - C(w) \\
&- \quad \frac{\rho^{1-\varsigma} t^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} f(s, y(s,w), w) ds + C(w) \| \\
&\leq \quad T^{\rho(1-\varsigma)} ({}^\rho I_0^\varsigma \Phi)(t,w) \\
&+ \quad \frac{\rho^{1-\varsigma} T^{\rho(1-\varsigma)}}{\Gamma(\varsigma)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} \| f(s, x(s,w), w) - f(s, y(s,w), w) \| ds \\
&\leq \quad T^{\rho(1-\varsigma)} ({}^\rho I_0^\varsigma \Phi)(t,w) \\
&+ \quad \frac{\rho^{1-\varsigma} T^{\rho(1-r)}}{\Gamma(\varsigma)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\varsigma}} q^*(w) \Phi(s,w) s^{\rho(1-\varsigma)} \frac{\|x-y\|}{1+\|x-y\|} ds \\
&\leq \quad T^{\rho(1-\varsigma)} \lambda_\Phi \Phi(t,w) + T^{2\rho(1-\varsigma)} \lambda_\Phi \Phi(t,w) q^*(w).
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\| t^{\rho(1-\varsigma)} x(t,w) - t^{\rho(1-\varsigma)} y(t,w) \| &\leq (1 + T^{\rho(1-\varsigma)} q^*(w)) T^{\rho(1-\varsigma)} \lambda_\Phi \Phi(t,w) \\
&:= c_{f,\Phi} \Phi(t,w).
\end{aligned}
$$

Hence, problem (3.3)-(3.4) is generalized Ulam-Hyers-Rassias stable.

3.4 Examples

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$, and

$$
E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}
$$

be the Banach space with the norm

$$
\|u\|_E = \sum_{n=1}^{\infty} |u_n|.
$$

Example 1. Consider the Caputo-Fabrizio fractional differential equation

$$
({}^{CF}D_0^\alpha u_n)(t,w) = \frac{cw^2(2^{-n} + u_n(t,w))}{\exp(t+3)(1+w^2+|u_n(t,w)|)}; \ t \in [0,1], \ w \in \Omega, \tag{3.23}
$$

with the boundary conditions

$$u_n(0,w)+u_n(1,w)=\frac{1}{1+w^2};\ w\in\Omega. \tag{3.24}$$

Set $0<c<\frac{2}{2a_\alpha+3b_\alpha}$, and

$$f(t,u(t,w),w)=\frac{cw^2(2^{-n}+u_n(t,w)))}{\exp(t+3)(1+w^2+|u(t,w)|)};\ t\in[0,1],\ w\in\Omega.$$

The hypothesis (H_2) is satisfied with $p_1(t,w)=p_2(t,w)=\frac{cw^2}{1+w^2}e^{-t}$,
and then $p_1^*(w)=p_2^*(w)=c$. The condition (3.7) is satisfied. Indeed;

$$\left(a_\alpha+Tb_\alpha+T\frac{bb_\alpha}{a+b}\right)p_2^*(w)=c\left(a_\alpha+\frac{3b_\alpha}{2}\right)<1,$$

Consequently, Theorem 6.2.2 implies that the problem (3.23)-(3.24) has at least one random solution defined on $[0,1]$.

Example 2. Consider now the Caputo-Fabrizio fractional differential equation

$$({}^{CF}D_0^\alpha u_n)(t,w)=\frac{cw^2 2^{-n}}{\exp(t+3)(1+w^2+|u_n(t,w)|)};\ t\in[0,1],\ w\in\Omega, \tag{3.25}$$

with the boundary conditions

$$u_n(0,w)+u_n(1,w)=\frac{w}{1+w^2};\ w\in\Omega. \tag{3.26}$$

Set

$$f(t,u(t,w),w)=\frac{cw^2 2^{-n}}{\exp(t+3)(1+w^2+|u(t,w)|)};\ t\in[0,1],\ w\in\Omega.$$

The hypothesis (H_3) is satisfied with $q(t,w)=\frac{cw^2}{1+w^2}$ and $\Phi(t)=e^{-t}$. The condition (3.7) is satisfied with a good choice of the constant c.
Also; the hypotheses (H_4) is satisfied with $\lambda_\Phi=e-1$. Indeed;

$$\int_0^T\Phi(t,w)dt=\int_0^T e^{-t}dt=1-e^{-1}\le\lambda_\Phi e^{-t}=\lambda_\Phi\Phi(t,w);\ t\in[0,1].$$

Consequently, Theorem 3.2.12 implies that the problem (3.25)-(3.26) has at least one random solution and it is generalized-Ulam-Hyers-Rassias stable.

Example 3. Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$, and let

$$l^1 = \left\{ x = (x_1, x_2, \ldots, x_n, \ldots), \sum_{n=1}^{\infty} |x_n| < \infty \right\}$$

be the Banach space with the norm

$$\|x\| = \sum_{n=1}^{\infty} |x_n|.$$

Consider the Katugampola random fractional differential equation

$$({}^{\rho}D_{0+}^{r} x_n)(t, w) = f_n(t, x(t, w), w); \ t \in [0, 1], \ w \in \Omega, \tag{3.27}$$

with the terminal condition

$$x(T, w) = ((1 + w^2)^{-1}, 0, 0, \cdots); \ w \in \Omega, \tag{3.28}$$

with $x = (x_1, x_2, \ldots, x_n, \ldots), \ f = (f_1, f_2, \ldots, f_n, \ldots)$,

$${}^{\rho}D_{0+}^{r} x = ({}^{\rho}D_{0+}^{r} x_1, \ldots, {}^{\rho}D_{0+}^{r} x_n, \ldots),$$

and

$$f_n(t, x(t, w), w) = \frac{w^2 t^{\rho(1-r)}(2^{-n} + x_n(t, w))}{2(1 + w^2)(1 + \|x\|)} \left(e^{-7-w^2} + \frac{1}{e^{t+5}} \right); \ t \in [0, 1], \ w \in \Omega.$$

We have

$$\|f(t, x, w) - f(t, y, w)\| \leq (e^{-7-w^2} + e^{-t-5}) \frac{w^2 t^{\rho(1-r)} \|x - y\|}{1 + \|x - y\|}.$$

Hence, hypotheses (H_3) and (H_4) are satisfied with

$$q(t, w) = e^{-7-w^2} + e^{-t-5}, \quad \Phi(t, w) = w^2.$$

Hence by theorems 3.3.4 and 3.3.7, problem (3.27)-(3.28) admits a random solution, and is generalized Ulam-Hyers-Rassias stable.

CHAPTER 4

EXISTENCE AND ATTRACTIVITY FOR CAPUTO–FABRIZIO RANDOM FRACTIONAL DIFFERENTIAL EQUATIONS

4.1 Introductions and Motivations

Fractional differential equations have recently been applied in various areas of scientific disciplines, see; [9, 126]. In recent years, several works and development of fractional differential equation and inclusions are cited to the monographs [9, 15, 16, 18, 91, 98, 119, 140], the papers [11, 104] and the reference therein.

The physical constants and parameters in formulating differential equations; may be considered to be random variables whose values are determined by some probability distribution or law. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. We refer the reader to the monographs [40, 121, 123, 128], the papers [99, 100, 125], and the references therein. The initial value problems of ordinary random differential equations have been studied in the literature on bounded as well as unbounded intervals [43, 141]. Recently, fractional random differential equations is largely studied by many authors, see for example [2, 3, 24, 141].

In [1, 9, 12, 14, 18], Abbas *et al.* studied the existence and attractivity for several classes of functional fractional differential equations. In this paper we investigate the following class of Caputo–Fabrizio fractional differential equation

$$({}^{CF}D_0^r u)(t,w) = f(t,u(t,w),w);\ t \in \mathbb{R}_+ = [0,\infty),\ w \in \Omega, \tag{4.1}$$

with the initial condition

$$u(0,w) = u_0(w);\ w \in \Omega, \tag{4.2}$$

where $T > 0$, $f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, $u_0 : \Omega \to \mathbb{R}$, ${}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0,1)$, and Ω is the sample space in a probability space (Ω, F).

4.2 Existence and attractivity of solutions

Let $I := [0,T];\ T > 0$. Denote by $\mathcal{C} := C(I,\mathbb{R})$ the Banach space of all continuous functions from I into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup_{t\in I} |u(t)|.$$

Let $\mathcal{BC} := BC(\mathbb{R}_+,\mathbb{R})$ be the Banach space of all real continuous and bounded functions on $\mathbb{R}_+$ with the norm

$$\|u\|_{BC} = \sup_{t\in\mathbb{R}_+} |u(t)|.$$

Let $\emptyset \neq \Lambda \subset \mathcal{BC}$, and let $G : \Lambda \to \Lambda$ and consider the solution of the random equation

$$G(w)u(t) = u(t,w). \tag{4.3}$$

Inspired by the definition of the attractivity of solutions of integral equations, we introduce the following concept of attractivity of solutions for the random equation (4.3).

Definition 4.2.1 *Solutions of equation (4.3) are locally attractive if there exists a ball $B(u_0,\eta)$ in the space $\mathcal{BC}$ such that, for arbitrary solutions $v = v(t,w)$ and $z = z(t,w)$ of equation (4.3) belonging to $B(u_0,\eta) \cap \Lambda$, we have*

$$\lim_{t\to\infty} (v(t,w) - z(t,w)) = 0. \tag{4.4}$$

When the limit (4.4) is uniform with respect to $B(u_0,\eta) \cap \Lambda$, solutions of equation (4.3) *are said to be uniformly locally attractive (or equivalently that solutions of (4.3) are locally asymptotically stable).*

Definition 4.2.2 *[32] The solution $v = v(t, w)$ of equation (4.3) is said to be globally attractive if (4.4) holds for each solution $z = z(t, w)$ of (4.3). If condition (4.4) is satisfied uniformly with respect to the set Λ, solutions of equation (4.3) are said to be globally asymptotically stable (or uniformly globally attractive).*

Lemma 4.2.3 *[50] Let $D \subset \mathcal{BC}$. Then D is relatively compact in $\mathcal{BC}$ if the following condition hold:*

1. *D is uniformly bounded in $\mathcal{BC}$;*
2. *The functions beloning to D are almost equicontinuous on $\mathbb{R}_+$ i.e., equicontinuous on every compact subset of $\mathbb{R}_+$;*
3. *The functions from D are equiconvergent, that is, given $\epsilon > 0$ there corresponds $T(\epsilon, w) > 0$ such that*
$$|u(t, w) - \lim_{t\to\infty} u(t, w)| < \epsilon;$$
for any $t \geq T(\epsilon, w)$ and $u \in D$.

Lemma 4.2.4 *Let $h \in L^1(I)$. A function $u \in \mathcal{C}$ is a solution of problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) = h(t); \quad t \in I := [0, T] \\ u(0) = u_0, \end{cases} \tag{4.5}$$

if and only if u satisfies the following integral equation

$$u(t) = C + a_r h(t) + b_r \int_0^t h(s)ds. \tag{4.6}$$

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)},$$
$$C = u_0 - a_r h(0).$$

From the above Lemma, we can conclude the following Lemma:

Lemma 4.2.5 *A function u is a random solution of problem (4.1)-(4.2), if and only if u satisfies the following integral equation*

$$u(t, w) = C(w) + a_r f(t, u(t, w), w) + b_r \int_0^t f(s, u(s, w), w)ds \tag{4.7}$$

where

$$C(w) = u_0(w) - a_r f(0, u(0, w), w).$$

Definition 4.2.6 *By a random solution of problem (4.1)-(4.2), we mean a function* $u : \Omega \to \mathcal{BC}$ *that satisfies the integral equation*

$$u(t, w) = C(w) + a_r f(t, u(t, w), w) + b_r \int_0^t f(s, u(s, w), w)ds$$

where

$$C(w) = u_0(w) - a_r f(0, u(0, w), w).$$

The following hypotheses will be used in the sequel:

(H_1) The function f is random Carathéodory.

(H_2) There exist measurable, positive and bounded functions $p_i : \Omega \to \mathcal{BC};\ i = 1, 2$; such that

$$|f(t, u, w)| \leq p_1(t, w) + p_2(t, w)|u|;$$

for any $w \in \Omega$, and for each $t \in \mathbb{R}_+$ and $u \in \mathbb{R}$, with

$$\lim_{t\to\infty} p_i(t, w) = 0, \quad \textit{and} \quad \lim_{t\to\infty} \int_0^t p_i(s, w)ds = 0.$$

Set

$$p_i^*(w) = \sup_{t\in\mathbb{R}_+} p_i(t, w);\ w \in \Omega, \quad \textit{and} \quad \tilde{p}_i(w) = \sup_{t\in\mathbb{R}_+} \int_0^t p_i(s, w)ds;\ i = 1, 2,\ w \in \Omega.$$

Now,we prove an existence result for the problem (4.1)-(4.2) based on the Itoh's fixed point theorem.

Theorem 4.2.7 *Assume that the hypotheses Assume that the hypotheses* (H_1) *and* (H_2) *hold. Then the problem (4.1)-(4.2) has at least one random solution defined on* $\mathbb{R}_+$.

Proof. From Lemma 4.2.5 for any $w \in \Omega$ and each $t \in \mathbb{R}_+$, the problem (4.1)-(4.2) is equivalent to the operator equation $(Nw)u = u(w)$, where $N : \Omega \times \mathcal{BC} \to \mathcal{BC}$ be the operator defined by

$$(Nu)(t, w) = C(w) + a_r f(t, u(t, w), w) + b_r \int_0^t f(s, u(s, w), w)ds.$$

Since the function f is continuous for all $w \in \Omega$, and the indefinite integral is continuous on $\mathbb{R}_+$, then u is a random solution for the problem (4.1)-(4.2) if and only if $u = (Nu)(t,w)$. We shall show that $N : \Omega \times \mathcal{BC} \to \mathcal{BC}$ satisfies the conditions of Theorem 1.3.3. The proof will be given in serval steps.

Step 1. $N(w)$ is a random operator.
Since $f(t,u,w)$ is random Carathéodory, the maps $w \longrightarrow f(t,u,w)$ and $w \longrightarrow \int_0^t f(s,u,w)ds$ are measurable in view Definition 1.1.5. Therefore the map $w \mapsto (Nu)(t,w)$ is measurable. As a result, $N(w)$ is a random operator on $\Omega \times \mathcal{BC} \to \mathcal{BC}$.

Step 2. $N(w)$ is continuous.
Let u_n be a sequence such that $u_n \to u$ in $\mathcal{BC}$. Them, for each $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
|(Nu_n)(t,w) - (Nu)(t,w)| &\leq |a_r(f(t,u(t,w),w) - f(t,u_n(t,w),w))| \\
&+ |b_r \int_0^t (f(t,u(t,w),w) - f(t,u_n(t,w),w))ds| \\
&\leq a_r|(f(t,u(t,w),w) - f(t,u_n(t,w),w))| \\
&+ b_r \int_0^t |(f(s,u(s,w),w) - f(s,u_n(s,w),w))|ds \\
&\leq a_r p_1(t,w)\|u_n - u\|_{BC} \\
&+ b_r \int_0^t p_2(s,w)\|u_n - u\|_{BC} ds.
\end{aligned}$$

Thus

$$\|(Nu_n)(t,w) - (Nu)(t,w)\| \leq \left(a_r p_1(t,w) + b_r \int_0^t p_2(s,wds \right) \|u_n - u\|_{BC}. \tag{4.8}$$

Claim 1. If $t \in [0,T], T > 0$, then since $u_n \to u$ as $n \to \infty$, (4.8) implies that

$$\|(Nu_n)(\cdot,w)) - (Nu)(\cdot,w)\|_{BC} \to 0 \ as \ n \to \infty.$$

Claim 2. If $t \in [T,\infty);\ T > 0$, then, since $u_n \to u$ as $n \to \infty$ and $t \to \infty$, then from (H_2), (4.8) implies that

$$\|(Nu_n)(\cdot,w)) - (Nu)(\cdot,w)\|_{BC} \to 0.$$

Step 3. $N(w)$ is uniformly bounded for each bounded set.
For any $w \in \Omega$, and each $t \in \mathbb{R}_+$ and $u \in \mathcal{BC}$, there exists $R(w) > 0$, such that $\|u\|_{BC} \leq$

$R(w)$, and

$$\begin{aligned}
|(Nu)(t,w)| &\leq \left|C(w) + a_r f(t,u(t,w),w) + b_r \int_0^t f(s,u(s,w),w)ds\right| \\
&\leq \|C(w)\| + a_r|f(t,u(t,w),w)| + b_r \int_0^t |f(s,u(s,w),w)|ds \\
&\leq |C(w)| + a_r(p_1(t,w) + p_2(t,w)\|u\|_{BC}) \\
&+ b_r \int_0^t (p_1(s,w) + p_2(s,w)\|u\|_{BC})ds \\
&\leq |C(w)| + a_r(p_1^*(w) + p_2^*(w)R(w) + b_r(\tilde{p}_1(w) + \tilde{p}_2(w)R(w)) \\
&:= \ell(w).
\end{aligned}$$

Hence, $(Nu)(w) \in \mathcal{BC}$ for any $w \in \Omega$, and each $u \in \mathcal{BC}$.

Step 4. $N(w)$ maps bounded sets into equicontinuous sets on every compact subset $[0,T] \subset \mathbb{R}_+;\ T > 0$.
Consider the bounded set $B \subset \mathcal{BC}$. For any $w \in \Omega$, and each $0 \leq t_1 \leq t \leq t_2 \leq T$, and $u \in B$, then there exists $R(w) > 0$, such that $\|u\|_{BC} \leq R(w)$, and

$$\begin{aligned}
|(Nu)(t_2,w) - (Nu)(t_1,w)| &\leq |a_r f(t_2,u(t_2,w),w) + b_r \textstyle\int_0^{t_2} f(s,u(s,w),w)ds \\
&- a_r f(t_1,u(t_1,w),w) - b_r \textstyle\int_0^{t_1} f(s,u(s,w),w)ds| \\
&\leq a_r|f(t_2,u(t_2,w),w) - f(t_1,u(t_1,w),w)| \\
&+ b_r \textstyle\int_{t_1}^{t_2} |f(s,u(s,w),w)ds| \\
&\leq a_r|f(t_2,u(t_2,w),w) - f(t_1,u(t_1,w),w)| \\
&+ b_r(t_2-t_1)(p_1^*(w) + p_2^*(w)\|u\|_{BC}) \\
&\leq a_r|f(t_2,u(t_2,w),w) - f(t_1,u(t_1,w),w)| \\
&+ b_r(t_2-t_1)(p_1^*(w) + p_2^*(w)R(w).
\end{aligned}$$

Since f is Carathéodory, then as $t_2 \to t_1$ the right-hand side of the above inequality tends to zero.

Step 5. $N(w)B$ is equiconvergent for each bounded set $B \subset \mathcal{BC}$.
Let $u \in B$, then for any $w \in \Omega$, and each $t \in \mathbb{R}_+$ there exists $R(w) > 0$, such that $\|u\|_{BC} \leq R(w)$, and

$$\begin{aligned}
|(Nu)(t,w)| &\leq \left|C(w) + a_r f(t,u(t,w),w) + b_r \int_0^t f(s,u(s,w),w)ds\right| \\
&\leq |C(w)| + a_r|f(t,u(t,w),w)| + b_r \int_0^t |f(s,u(s,w),w)|ds \\
&\leq |C(w)| + a_r(p_1(t,w) + p_2(t,w)\|u\|_{BC})
\end{aligned}$$

$$\begin{aligned}
&+ \quad b_r\|u\|_{BC}\int_0^t (p_1(s,w)+p_2(s,w))ds \\
&\leq \quad |C(w)| + a_r(p_1(t,w)+p_2(t,w)R(w) \\
&+ \quad b_rR(w)\int_0^t (p_1(s,w)+p_2(s,w))ds.
\end{aligned}$$

Then, from (H_2) we deduce that, for any $w \in \Omega$ and each $t \in \mathbb{R}_+$, we get

$$|(Nu)(t,w)| \to \|C(w)\| \ as \ t \to \infty.$$

Hence

$$|(Nu)(t,w) - (Nu)(\infty,w)| \to 0 \ as \ t \to \infty.$$

As a consequence of steps 1 to 5 together with the lemma 4.2.3, we can conclude that N is continuous and compact random operator. Theorem 1.3.3 implies that the operator equation $(Nw)u = u$ has a random solution.

Now, we are concerned with the attractivity of problem (4.1)-(4.2). The following hypothesis will be used in the sequel:

(H_3) There exists a measurable, positive and bounded function $q : \Omega \to BC;\ i = 1,2;$ such that

$$(1+|u-v|)|f(t,u,w) - f(t,v,w)| \leq q(t,w)|u-v|;$$

for any $w \in \Omega$, and for each $t \in \mathbb{R}_+$ and $u, v \in \mathbb{R}$, with

$$\lim_{t\to\infty} q(t,w) = 0, \quad and \quad \lim_{t\to\infty}\int_0^t q(s,w)ds = 0.$$

Moreover, we assume that for any $w \in \Omega$, the function $t \mapsto f(t,0,w)$ is bounded on $\mathbb{R}_+$, with $\lim_{t\to\infty}|f(t,0,w)| = 0$.

Set

$$q^*(w) = \sup_{t\in\mathbb{R}_+} q(t,w), \quad and \quad \tilde{q}(w) = \sup_{t\in\mathbb{R}_+}\int_0^t p_i(s,w)ds;\ w \in \Omega.$$

Remark 4.2.8 *We can easily verify that (H_3) implies (H_2) with $p_1(t,w) = |f(t,0,w)|$ and $p_2(t,w) = q(t,w)$.*

Theorem 4.2.9 *Assume that the hypotheses (H_1) and (H_3) hold. Then all solution of the problem (4.1)-(4.2) are globally asymptotically stable.*

Proof. From Remark 4.2.8 and Theorem 4.2.7, our problem (4.1)-(4.2) has at least one random solution v defined on $\mathbb{R}_+$. Thus for any $w \in \Omega$, and each $t \in \mathbb{R}_+$ we have

$$v(t,w) = C(w) + a_r f(t, v(t,w), w) + b_r \int_0^t f(s, v(s,w), w) ds.$$

Consider the ball $B(v, R_\eta(w)) := \{u \in \mathcal{BC} : \|u - v\|_{BC} \leq \eta(w)\}$. Take $u \in B(v, R_\eta(w))$, then for any $w \in \Omega$, and each $t \in \mathbb{R}_+$ we have

$$\begin{aligned}
|(Nu)(t,w) - v(t,w)| &= |(Nu)(t,w) - (Nv)(t,w)| \\
&= |a_r f(t, u(t,w), w) + b_r \int_0^t f(t, u(t,w), w) ds \\
&- a_r f(t, v(t,w), w) - b_r \int_0^t f(t, v(t,w), w) ds| \\
&\leq a_r |f(t, u(t,w), w) - f(t, v(t,w), w)| \\
&+ b_r \int_0^t |f(s, u(s,w), w) - f(s, v(s,w), w)| ds \\
&\leq a_r q(t,w) |u(t,w) - v(t,w)| \\
&+ b_r \int_0^t q(s,w) |u(s,w) - v(s,w)| ds \\
&\leq (a_r q^*(w) + b_r \tilde{q}(w)) R_\eta(w).
\end{aligned}$$

Thus $N(w)$ is a continuous operator such that $N(w)(B(v, R_\eta(w))) \subset B(v, R_\eta(w))$. Moreover, if u is a solution of problem (4.1)-(4.2), then from (H_3) for any $w \in \Omega$, and each $t \in \mathbb{R}_+$ we have

$$\begin{aligned}
|u(t,w) - v(t,w)| &= |(Nu)(t,w) - (Nv)(t,w)| \\
&\leq a_r |f(t, u(t,w), w) - f(t, v(t,w), w)| \\
&+ b_r \int_0^t |f(s, u(s,w), w) - f(s, v(s,w), w)| ds \\
&\leq a_r q(t,w) \frac{|u(t,w) - v(t,w)|}{1 + |u(t,w) - v(t,w)|} \\
&+ b_r \int_0^t q(s,w) \frac{|u(s,w) - v(s,w)|}{1 + |u(s,w) - v(s,w)|} ds \\
&\leq a_r q(t,w) + b_r \int_0^t q(s,w) ds.
\end{aligned}$$

Hence, we deduce that

$$|u(t,w) - v(t,w)| \to 0 \; as \; t \to \infty.$$

Consequently, all solutions of problem (4.1)-(4.2) are globally asymptotically stable.

4.3 An Example

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Consider the Caputo-Fabrizio fractional differential equation

$$({}^{CF}D_0^{\frac{1}{4}}u)(t,w) = \frac{w^2(1-tw^2)e^{-tw^2}(1+\sin(u(t,w)))}{(1+t)(1+w^2+|u(t,w)|)};\ t \in \mathbb{R}_+,\ w \in \Omega, \tag{4.9}$$

with the initial condition

$$u(0,w) = \frac{1}{1+w^2};\ w \in \Omega. \tag{4.10}$$

Set

$$f(t,u(t,w),w) = \frac{w^2(1-tw^2)e^{-tw^2}(1+\sin(u(t,w)))}{(1+t)(1+w^2+|u(t,w)|)};\ t \in \mathbb{R}_+,\ w \in \Omega.$$

For any $w \in \Omega$, and for each $t \in \mathbb{R}_+$ and $u, v \in \mathbb{R}$, we have

$$|f(t,u,w) - f(t,v,w)| \le |1-tw^2|e^{-tw^2t}\frac{|u-v|}{1+|u-v|};$$

Hence, the hypothesis (H_3) is satisfied with

$$q(t,w) = |1-tw^2|e^{-tw^2}.$$

So; we have

$$\lim_{t\to\infty} q(t,w) = 0, \quad and \quad \lim_{t\to\infty}\int_0^t q(s,w)ds = \lim_{t\to\infty} te^{-tw^2} = 0.$$

Moreover, for any $w \in \Omega$, the function

$$t \mapsto f(t,0,w) = \frac{w^2(1-tw^2)e^{-tw^2}}{(1+t)(1+w^2)}$$

is bounded on $\mathbb{R}_+$, with $\lim_{t\to\infty} |f(t,0,w)| = 0$.

Simple computations show that all conditions of Theorem 4.2.9 are satisfied. Hence problem (4.9)-(4.10) has random solutions, and all solutions are globally asymptotically stable.

CHAPTER 5

RANDOM CAPUTO-FABRIZIO FRACTIONAL DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

5.1 Introductions and Motivations

In recent years, Caputo and Fabrizio [47] introduced a new approach of fractional derivative having a kernel with exponential decay known as the Caputo-Fabrizio operator. Several rechearchers were recently busy in development of Caputo-Fabrizio fractional differential equations, see; [51, 62, 63, 64, 100, 133], and the references therein.

The initial value problems of ordinary random differential equations have been studied in the literature on bounded as well as unbounded intervals [43, 141]. Recently, fractional randon differential equations is largely studied by many authors, see for example [2, 3, 24, 141].

Considerable attention has been given to the study of the Ulam-Hyers-Rassias stability of all kinds of functional equations, see; the monographs [16, 87], and the papers [4, 5, 6, 19, 22]. More details from historical point of view, and developments of such stabilities are reported in [82, 86, 109, 112, 115].

Fractional differential equations in Fréchet spaces have studied by many mathematicians; see [4, 6, 10, 55, 56]. In this article we investigate the following class of Caputo–

Fabrizio random fractional differential equation

$$({}^{CF}D_0^r u)(t,w) = f(t,u(t,w),w);\ t \in \mathbb{R}_+ = [0,\infty),\ w \in \Omega, \tag{5.1}$$

with the initial condition

$$u(0,w) = u_0(w);\ w \in \Omega, \tag{5.2}$$

where $u_0 : \Omega \to \mathbb{R}$, is a measurable function, $f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, ${}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0,1)$, and Ω is the sample space in a probability space (Ω, F).

Nonlocal problems are used to represent mathematical models for evolution of various phenomena, such as nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion, see for example [44, 53, 102, 134, 135]. In our next results, we discuss the existence of random solutions the Ulam stability for the nonlocal problem of fractional differential equations

$$\begin{cases} ({}^{CF}D_0^r u)(t,w) = f(t,u(t,w),w);\ t \in \mathbb{R}_+, \\ u(0,w) + Q(u(\cdot,w)) = u_0(w), \end{cases} \quad w \in \Omega, \tag{5.3}$$

where $u_0,\ f$ are as in problem (5.1)-(5.2), $Q : \Omega \times X \to \mathbb{R}$ is a given function, and X is the Fréchet space defined later.

5.2 Existence of Random Solutions and Ulam stability

Let $I := [0,T];\ T > 0$. Denote by $C(I) := C(I,\mathbb{R})$ the Banach space of all real continuous functions on I with the norm

$$\|u\|_\infty = \sup_{t \in I} |u(t)|.$$

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}^*}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le ... \quad for\ every\ x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \le \overline{M}_n \quad for\ all\ y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows : For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [61].

For each $p \in \mathbb{N}\backslash\{0\}$, we set $I_p := [0,p]$, we consider following set, $C_p = C([0,p])$, and we define in $X := C(\mathbb{R}_+)$ the semi-norms by

$$\|u\|_p = \sup_{t\in[0,p]} |u(t)|.$$

Then X is a Fréchet space with the family of semi-norms $\{\|u\|_p\}$.

Lemma 5.2.1 *Let $h \in L^1(I)$. A function $u \in \mathcal{C}$ is a solution of problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) = h(t); \quad t \in I := [0,T] \\ u(0) = u_0, \end{cases} \tag{5.4}$$

if and only if u satisfies the following integral equation

$$u(t) = C + a_r h(t) + b_r \int_0^t h(s)ds. \tag{5.5}$$

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)},$$
$$C = u_0 - a_r h(0).$$

Now, we consider the Ulam stability for the problem (5.1)-(5.2). Let $\epsilon > 0$ and $\Phi : \Omega \times I_p \to \mathbb{R}_+;\ p \in \mathbb{N}$ be a measurable and continuous function. We consider the following inequalities

$$|({}^{CF}D_0^r u)(t,w) - f(t,u(t,w),w)| \le \epsilon;\ t \in I_p,\ w \in \Omega. \tag{5.6}$$

$$|({}^{CF}D_0^r u)(t,w) - f(t,u(t,w),w)| \le \Phi(t,w);\ t \in I_p,\ w \in \Omega. \tag{5.7}$$

$$|({}^{CF}D_0^r u)(t,w) - f(t,u(t,w),w)| \le \epsilon\Phi(t,w);\ t \in I_p,\ w \in \Omega. \tag{5.8}$$

Definition 5.2.2 *[16] The problem(5.1)-(5.2) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $u(\cdot,w) \in X$ of the inequality*

(5.6), there exists a solution $v(\cdot,w)\in X$ of (5.1)-(5.2) with

$$|u(t,w)-v(t,w)|\leq \epsilon c_f;\ t\in I_p,\ w\in\Omega.$$

Definition 5.2.3 *[16] The problem (5.1)-(5.2) is generalized Ulam-Hyers stable if there exists $c_f\in C(\mathbb{R}_+,\mathbb{R}_+)$ with $c_f(0)=0$ such that for each $\epsilon>0$ and for each solution $u(\cdot,w)\in X$ of the inequality (5.6), there exists a solution $v(\cdot,w)\in X$ of (5.1)-(5.2) with*

$$|u(t,w)-v(t,w)|\leq c_f(\epsilon);\ t\in I_p,\ w\in\Omega.$$

Definition 5.2.4 *[16] The problem (5.1)-(5.2) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi}>0$ such that for each $\epsilon>0$ and for each solution $u(\cdot,w)\in X$ of the inequality (5.8), there exists a solution $v(\cdot,w)\in x$ of (5.1)-(5.2) with*

$$|u(t,w)-v(t,w)|\leq \epsilon c_{f,\Phi}\Phi(t,w);\ t\in I_p,\ w\in\Omega.$$

Definition 5.2.5 *[16] The problem (5.1)-(5.2) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi}>0$ such that for each solution $u(\cdot,w)\in X$ of the inequality (5.7), there exists a solution $v(\cdot,w)\in X$ of (5.1)-(5.2) with*

$$|u(t,w)-v(t,w)|\leq c_{f,\Phi}\Phi(t,w);\ t\in I_p,\ w\in\Omega.$$

Remark 5.2.6 *A function $u(\cdot,w)\in X$ is a solution of the inequality* (5.7) *if and only if there exist a function $g(\cdot,w)\in C(I_p)$ (wich depend on u) such that*

$$|g(t,w)|\leq\Phi(t,w),$$

$$({}^{CF}D_0^r u)(t,w)=f(t,u(t,w),w)+g(t,w);\ for\ t\in I_p,\ and\ w\in\Omega.$$

Let us introduce the following hypotheses.

(H_1) The function $f:I_p\times\mathbb{R}\times\Omega\mapsto f(t,u,w)\in\mathbb{R}$ is random Carathéodory on $I_p\times\mathbb{R}\times\Omega$, and affine with respect to u,

(H_2) There exists a measurable and bounded function $\ell:\Omega\to C(I_p,\mathbb{R}_+)$, such that

$$|f(t,u,w)-f(t,v,w)|\leq\ell(t,w)|u-v|;\ for\ a.e.\ t\in I_p,\ and\ each\ u,v\in\mathbb{R},\ w\in\Omega,$$

(H_3) There exists $\lambda_\Phi > 0$ such that for each $t \in I_p$, and $w \in \Omega$, we have

$$({}^{CF}I_0^r\Phi)(t,w) \leq \lambda_\Phi \Phi(t,w),$$

(H_4) The function $Q : \Omega \times X \to \mathbb{R}$ is jointly measurable, affine with respect to u, and there exists a measurable function $\nu : \Omega \to \mathbb{R}_+$, such that

$$(1 + \|u(\cdot,w) - v(\cdot,w)\|_p)|Q(u(\cdot,w)) - Q(v(\cdot,w))|$$

$$\leq \Phi(t,w)\nu(w)\|u(\cdot,w) - v(\cdot,w)\|_p; \ for \ each \ u(\cdot,w), v(\cdot,w) \in X.$$

For any $p \in \mathbb{N}$, we set $\ell_p^*(w) = \sup_{t\in I_p} \ell(t,w)$, $\Phi^*(w) = \sup_{t\in I_p} \Phi(t.w)$,
and $f_p^*(w) = \sup_{t\in I_p} |f(t,0,w)|$.

5.2.1 The Initial Value Problem

In this section, we are concerned with the existence and Ulam stability results of the problem (5.1)-(5.2).

Definition 5.2.7 *By a random solution of problem (5.1)-(5.2), we mean a measurable function $u(\cdot,w) \in X;\ w \in \Omega$ that satisfies the integral equation*

$$u(t,w) = c(w) + a_r f(t, u(t,w), w) + b_r \int_0^t f(s, u(s,w), w)ds,$$

where

$$c(w) = u_0(w) - a_r f(0, u(0,w), w).$$

Now, we shall prove the following theorem concerning the existence of random solutions and the generalized Ulam-Hyers-Rassias stability of problem (5.1)-(5.2).

Theorem 5.2.8 *Assume that the hypotheses (H_1) and (H_2) hold. If*

$$\ell_p^*(w)(a_r + pb_r) < 1, \tag{5.9}$$

for any $w \in \Omega$, then problem (5.1)-(5.2) has at least one random solution in the space X. Furthermore, if the hypothesis (H_3) holds, then problem (5.1)-(5.2) is generalized Ulam-Hyers-Rassias stable.

Proof. Define a mapping $N : \Omega \times X \to X$ by:

$$(N(w)u)(t) = c(w) + a_r f(t, u(t,w), w) + b_r \int_0^t f(s, u(s,w), w)ds. \tag{5.10}$$

The map $w \to c(w)$ is measurable for all $w \in \Omega$. Again, as the function f is a random Carathéodory, the map $w \to f(t, u, w)$ is measurable in view of Definition **??**. Similarly, the integral is measurable, then $N(w)$ is a random operator on X, and defines a mapping $N : \Omega \times X \to X$. Thus the random solutions of problem (5.1)-(5.2) are random fixed points of the random operator N.

Next, for each $p \in \mathbb{N}\backslash\{0\}$ and any $w \in \Omega$, we can show that $N(w)$ transforms the ball $B_R = \{u \in X : \|u\|_p \leq R_p(w)\}$ into itself, where

$$R_p(w) \geq \frac{|c(w)| + (a_r + pb_r)f_p^*(w)}{1 - \ell_p^*(w)(a_r + pb_r)}.$$

Indeed, for any $w \in \Omega$, and each $u \in B_R$ and $t \in \tilde{I}_p$, we have

$$\begin{aligned}
|(Nu)(t,w)| &\leq \left| c(w) + a_r f(t, u(t,w), w) + b_r \int_0^t f(s, u(s,w), w)ds \right| \\
&\leq |c(w)| + a_r |f(t, u(t,w), w)| + b_r \int_0^t |f(s, u(s,w), w)|ds \\
&\leq |c(w)| + a_r(|f(t, 0, w)| + \ell(t,w)|u(t)|) \\
&\quad + b_r \int_0^t (|f(s, 0, w)| + \ell(s,w)|u(s)|)ds \\
&\leq |c(w)| + a_r(f_p^*(w) + \ell_p^*(w)|u(t)|) \\
&\quad + b_r \int_0^t (f_p^*(w) + \ell_p^*(w)|u(s)|)ds \\
&\leq |c(w)| + (a_r + pb_r)(f_p^*(w) + \ell_p^*(w)R_p(w)) \\
&\leq R_p(w).
\end{aligned}$$

Thus

$$\|N(w)u\|_p \leq R_p(w). \tag{5.11}$$

The proof of Theorem 5.2.8 be given in two steps.

Step 1. *The operator $N : \Omega \times B_R \to B_R$ has a random fixed point.*

We shall show that the operator $N : \Omega \times B_R \to B_R$ satisfies all the assumptions of Theo-

rem 1.3.3.
Since $N(w)$ is a random operator on $\Omega \times B_R$ into B_R, it remains for us to demonstrate that $N(w)$ is continuous and affine. The proof will be given in two claims.

Claim 1. *$N(w)$ is continuous.*
Let u_n be a sequence such that $u_n \to u$ in B_R Them, for each $t \in I_p$, and $w \in \Omega$, we have

$$\begin{aligned} |(Nu_n)(t,w) - (Nu)(t,w)| &\leq a_r|f(t,u(t,w),w) - f(t,u_n(t,w),w)| \\ &+ b_r \int_0^t |f(t,u(t,w),w) - f(t,u_n(t,w),w)|ds \\ &\leq a_r \ell_p(w)(t,w)\|u_n - u\|_p + pb_r\ell_p(w)\|u_n - u\|_p. \end{aligned}$$

Hence

$$\|(Nu_n)(\cdot,w)) - (Nu)(\cdot,w)\|_p \to 0 \; as \; n \to \infty.$$

Claim 2. *$N(w)$ is affine.*
We use the fact that $f(t,u,w)$ is affine with respect to u, for each $u,v \in B_R,\ t \in I_p$, and any $\lambda \in (0,1)$ and $w \in \Omega$, we have

$$\begin{aligned} N(w)(\lambda u + (1-\lambda)v) &= c(w) + a_r f(t,(\lambda u(t,w)) + (1-\lambda)v)(t,w),w) \\ &\quad + b_r \textstyle\int_0^t f(s,(\lambda u(s,w) + (1-\lambda)v)(s,w),w)ds \\ &= \lambda c(w) + \lambda a_r f(t,u(t,w),w) + \lambda b_r \textstyle\int_0^t f(s,u(s,w),w)ds \\ &\quad + (1-\lambda)c(w) + (1-\lambda)a_r f(t,v(t,w),w) \\ &\quad + (1-\lambda)b_r \textstyle\int_0^t f(s,v(s,w),w)ds \\ &= \lambda N(w)(u) + (1-\lambda)N(w)(v). \end{aligned}$$

Hence $N(w)$ is affine.

As a consequence of the above claims, together with the Theorem 1.3.3, we deduce that N has a random fixed point v which is a random solution of the problem (5.1)-(5.2).

Step 2. *The generalized Ulam-Hyers-Rassias stability.*
Let u be a random solution of the inequality (5.7), and let us assume that v is a random

solution of problem (5.1)-(5.2). Thus, we have

$$v(t,w) = c(w) + a_r f(t, v(t,w), w) + b_r \int_0^t f(s, v(s,w), w)ds.$$

From the inequality (5.7) for each $t \in I_p$, and $w \in \Omega$, we have

$$|u(t,w) - c(w) - a_r f(t, u(t,w), w) - b_r \int_0^t f(s, u(s,w), w)ds| \leq ({}^{CF}I_0^r \Phi)(t).$$

From hypotheses (H_2) and (H_3), for each $t \in I_p$, and $w \in \Omega$, we get

$$\begin{aligned}
|u(t,w) - v(t,w)| &\leq \| u(t,w) - c(w) - a_r f(t, u(t,w), w) \\
&- b_r \int_0^t f(s, u(s,w), w)ds| \\
&+ a_r |f(s, u(s,w), w) - f(s, v(s,w), w)| \\
&+ b_r \int_0^t |f(s, u(s,w), w) - f(s, v(s,w), w)|ds \\
&\leq ({}^{CF}I_0^r \Phi)(t,w) \\
&+ a_r \ell_p^*(w)|u(s,w) - v(s,w)| + b_r \ell_p^*(w) \int_0^t |u(s,w) - v(s,w)|ds \\
&\leq \lambda_\Phi \Phi(t,w) \\
&+ a_r \ell_p^*(w)|u(s,w) - v(s,w)| + b_r \ell_p^*(w) \int_0^t |u(s,w) - v(s,w)|ds.
\end{aligned}$$

Thus, we get

$$|u(t,w) - v(t,w)| \leq \frac{\lambda_\Phi}{1 - a_r \ell_p^*(w)} \Phi(t,w) + \frac{b_r \ell_p^*(w)}{1 - a_r \ell_p^*(w)} \int_0^t |u(s,w) - v(s,w)|ds.$$

By applying the classical Gronwall lemma, we obtain

$$\begin{aligned}
|u(t,w) - v(t,w)| &\leq \frac{\lambda_\Phi}{1 - a_r \ell_p^*(w)} \Phi(t,w) \exp\left(\frac{b_r \ell_p^*(w)}{1 - a_r \ell_p^*(w)} \int_0^t ds \right) \\
&\leq \frac{\lambda_\Phi}{1 - a_r \ell_p^*(w)} \exp\left(\frac{p b_r \ell_p^*(w)}{1 - a_r \ell_p^*(w)} \right) \Phi(t,w) \\
&:= c_{f,\Phi} \Phi(t,w).
\end{aligned}$$

Hence, our problem (5.1)-(5.2) is generalized Ulam-Hyers-Rassias stable.

5.2.2 The Nonlocal Problem

Now, we are concerned with the nonlocal problem (5.3).

Definition 5.2.9 *By a random solution of problem (5.3), we mean a measurable function* $u(\cdot,w) \in X;\ w \in \Omega$ *that satisfies the integral equation*

$$u(t,w) = c(w) - Q(u(\cdot,w)) + a_r f(t,u(t,w),w) + b_r \int_0^t f(s,u(s,w),w)ds.$$

Theorem 5.2.10 *Assume that the hypotheses* $(H_1),\ (H_2),\ (H_4)$*, and the condition (5.9) hold. Then problem (5.3) has at least one random solution in the space* X*. Furthermore, if the hypothesis* (H_3) *holds, then problem (5.3) is generalized Ulam-Hyers-Rassias stable.*

Proof. Define a mapping $G : \Omega \times X \to X$ by:

$$(G(w)u)(t) = c(w) - Q(u(\cdot,w)) + a_r f(t,u(t,w),w) + b_r \int_0^t f(s,u(s,w),w)ds. \tag{5.12}$$

The map $w \to c(w)$ is measurable for all $w \in \Omega$, and the map $w \to Q(u(\cdot,w))$ is measurable. Again, as the function f is a random Carathéodory, the map $w \to f(t,u,w)$ is measurable in view of Definition 1.1.5. Similarly, the integral is measurable, then $G(w)$ is a random operator on X, and defines a mapping $G : \Omega \times X \to X$. Thus the random solutions of the nonlocal problem (5.3) are random fixed points of the random operator G.

Next, for each $p \in \mathbb{N}\backslash\{0\}$ and any $w \in \Omega$, we can show that $G(w)$ transforms the ball $B_\rho = \{u \in X : \|u\|_p \le \rho_p(w)\}$ into itself, where

$$\rho_p(w) \ge \frac{|c(w)| + |Q(0)| + \Phi^*(w)\nu(w) + (a_r + pb_r)f_p^*(w)}{1 - \ell_p^*(w)(a_r + pb_r)}.$$

Indeed, for any $w \in \Omega$, and each $u \in B_\rho$ and $t \in \tilde{I}_p$, we have

$$\begin{aligned}
|(Gu)(t,w)| &\le \left| c(w) - Q(u(\cdot,w)) + a_r f(t,u(t,w),w) + b_r \int_0^t f(s,u(s,w),w)ds \right| \\
&\le |c(w)| + |Q(u(\cdot,w))| + a_r|f(t,u(t,w),w)| + b_r \int_0^t |f(s,u(s,w),w)|ds \\
&\le |c(w)| + |Q(0)| + \frac{\Phi^*(w)\nu(w)\|u\|_p}{1+\|u\|_p} + a_r(|f(t,0,w)| + \ell(t,w)|u(t)|) \\
&\quad + b_r \int_0^t (|f(s,0,w)| + \ell(s,w)|u(s)|)ds
\end{aligned}$$

$$\begin{aligned}
&\leq |c(w)| + |Q(0)| + \Phi^*(w)\nu(w) + a_r(f_p^*(w) + \ell_p^*(w)|u(t)|) \\
&\quad + b_r \int_0^t (f_p^*(w) + \ell_p^*(w)|u(s)|)ds \\
&\leq |c(w)| + |Q(0)| + \Phi^*(w)\nu(w) + (a_r + pb_r)(f_p^*(w) + \ell_p^*(w)\rho_p(w)) \\
&\leq \rho_p(w).
\end{aligned}$$

Thus

$$\|G(w)u\|_p \leq \rho_p(w). \tag{5.13}$$

The proof of Theorem 5.2.10 be given in two steps.

Step 1. *The operator $N : \Omega \times B_R \to B_R$ has a random fixed point.*
Since $G(w)$ is a random operator on $\Omega \times B_R$ into B_R, and Q is jointly measurable and affine, then as in the proof of Theorem 6.2.2, we can demonstrate that $G(w)$ is continuous and affine. Hence the operator $G : \Omega \times B_R \to B_R$ satisfies all the assumptions of Theorem 1.3.3, and then we deduce that G has a random fixed point v which is a random solution of the problem (5.3).

Step 2. *The generalized Ulam-Hyers-Rassias stability.*
Let u be a random solution of the inequality (5.7), and let us assume that v is a random solution of problem (5.3). Thus, we have

$$v(t,w) = c(w) - Q(v(\cdot,w)) + a_r f(t, v(t,w), w) + b_r \int_0^t f(s, v(s,w), w)ds.$$

From the inequality (5.7) for each $t \in I_p$, and $w \in \Omega$, we have

$$|u(t,w) - c(w) + Q(u(\cdot,w)) - a_r f(t, u(t,w), w) - b_r \int_0^t f(s, u(s,w), w)ds| \leq ({}^{CF}I_0^r\Phi)(t).$$

From hypotheses $(H_2) - (H_4)$, for each $t \in I_p$, and $w \in \Omega$, we have

$$\begin{aligned}
|u(t,w) - v(t,w)| &\leq |u(t,w) - c(w) + Q(u(\cdot,w)) - a_r f(t, u(t,w), w) \\
&- b_r \int_0^t f(s, u(s,w), w)ds| \\
&+ |Q(u(\cdot,w)) - Q(v(\cdot,w))| + a_r|f(s, u(s,w), w) - f(s, v(s,w), w)| \\
&+ b_r \int_0^t |f(s, u(s,w), w) - f(s, v(s,w), w)|ds.
\end{aligned}$$

Then, we obtain

$$\begin{aligned} |u(t,w)-v(t,w)| &\leq ({}^{CF}I_0^r\Phi)(t,w)+\nu(w)\Phi(t,w) \\ &+ a_r\ell_p^*(w)|u(s,w)-v(s,w)|+b_r\ell_p^*(w)\int_0^t|u(s,w)-v(s,w)|ds \\ &\leq \lambda_\Phi\Phi(t,w)+\nu(w)\Phi(t.w) \\ &+ a_r\ell_p^*(w)|u(s,w)-v(s,w)|+b_r\ell_p^*(w)\int_0^t|u(s,w)-v(s,w)|ds. \end{aligned}$$

Thus, we get

$$|u(t,w)-v(t,w)|\leq\frac{\lambda_\Phi+\nu(w)}{1-a_r\ell_p^*(w)}\Phi(t,w)+\frac{b_r\ell_p^*(w)}{1-a_r\ell_p^*(w)}\int_0^t|u(s,w)-v(s,w)|ds.$$

By applying the classical Gronwall lemma, we obtain

$$\begin{aligned} |u(t,w)-v(t,w)| &\leq \frac{\lambda_\Phi\nu(w)}{1-a_r\ell_p^*(w)}\Phi(t,w)\exp\left(\frac{b_r\ell_p^*(w)}{1-a_r\ell_p^*(w)}\int_0^t ds\right) \\ &\leq \frac{\lambda_\Phi\nu(w)}{1-a_r\ell_p^*(w)}\exp\left(\frac{pb_r\ell_p^*(w)}{1-a_r\ell_p^*(w)}\right)\Phi(t,w) \\ &:= c_{f,\Phi}^*\Phi(t,w). \end{aligned}$$

Hence, our problem (5.3) is generalized Ulam-Hyers-Rassias stable.

5.3 An Example

Let $\Omega=(-\infty,0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$. As an application of our results, we consider the following problem

$$\begin{cases} ({}^{CF}D_0^{\frac{1}{4}}u)(t,w)=f(t,u(t,w),w);\ t\in[0,\infty), \\ u(t)|_{t=0}=1, \end{cases} \quad w\in\Omega, \tag{5.14}$$

where

$$\begin{cases} f(t,u,w)=\dfrac{c_p(1+u)\sin t}{(1+\sqrt{t})(1+w^2)(1+w^2+|u|)};\ t\in(0,\infty),\ u\in\mathbb{R}, \\ f(0,u,w)=0; \qquad u\in\mathbb{R}, \end{cases} \quad w\in\Omega,$$

for each $t \in [0, p]$, where $0 < c_p < \frac{1}{a_{\frac{1}{2}} + p b_{\frac{1}{2}}}$; $p \in \mathbb{N} - \{0\}$.
The hypothesis (H_2) is satisfied with

$$\begin{cases} \ell_p(t,w) = \dfrac{c_p|\sin t|}{(1+\sqrt{t})(1+w^2)}; \ t \in (0,p], \\ \ell_p(0,w) = 0, \end{cases} \quad w \in \Omega.$$

The hypothesis (H_2) is satisfied with $\ell_p^*(w) = c_p$. Also, we can easily verify the condition (5.9). Indeed; for any $p \in \mathbb{N} - \{0\}$, we have $\ell_p^*(w)(a_r + p b_r) < \frac{1}{a_{\frac{1}{2}} + p b_{\frac{1}{2}}}(a_r + p b_r) = 1$.

Again; the hypotheses (H_3) is satisfied with $\Phi(t,w) = w^2 e^t$, and $\lambda_\Phi = \frac{8}{7M(\frac{1}{4})}$. Indeed; for each $t \in [0, p]$,

$$\begin{aligned} {}^{CF}I_0^{\frac{1}{4}}\Phi(t,w) &= \frac{6}{7M(\frac{1}{4})} w^2 e^t + \frac{2w^2}{7M(\frac{1}{4})}\int_0^t e^s ds \\ &\leq \frac{8}{7M(\frac{1}{4})} w^2 e^t \\ &:= \lambda_\Phi \Phi(t,w). \end{aligned}$$

Simple computations show that conditions of Theorem 5.2.8 are satisfied. Hence, problem (5.14) has at least one random solution defined on $\mathbb{R}_+$. Moreover, problem (5.14) is generalized Ulam-Hyers-Rassias stable.

CHAPTER 6

CAPUTO-FABRIZIO FRACTIONAL DIFFERENTIAL EQUATIONS WITH DELAY AND RANDOM EFFECTS

6.1 Introduction and Motivations

The functional differential equations with finite delay, infinite delay , and state-dependent delay have received a lot of attention in recent years, the study of this type of equations were carried out by Abbas *et al.* [8, 14, 15, 16],and the papers [37, 67, 71, 72, 73, 74].

The functional differential equations with random effects are differential equations with a stochastic process, they play a very important fundamental role in the theory of random dynamic systems, in addition they are used in various branches of science and engineering. We refer the reader to the monographs [54, 58, 101, 106] and their references.

In this chapter, first we investigate the the following class of random Caputo-Fabrizio fractional differential equations with finite delay

$$\begin{cases} u(t,w) = \varphi(t,w);\ t \in [-h,0], \\ ({}^{CF}D_0^r u)(t,w) = f(t,u_t(\cdot,w),w);\ t \in I := [0,T], \end{cases} ;\ w \in \Omega, \tag{6.1}$$

where $h > 0,\ T > 0,\ \ \varphi \in \mathcal{C}, f : I \times \mathcal{C} \times \Omega \to \mathbb{R}$ is a given function, ${}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r \in (0,1]$, and $\mathcal{C} := C([-h,0],\mathbb{R})$ is the space of continuous functions on $[-h, 0]$.

For any $t \in I$, we define $u_t(\cdot, w)$ by

$$u_t(s, w) = u(t+s, w);\ \text{for } s \in [-h, 0], \text{and } w \in \Omega.$$

Next, we investigate the following class of random Caputo-Fabrizio fractional differential equations with infinite delay

$$\begin{cases} u(t, w) = \varphi(t, w);\ t \in \mathbb{R}_- := (-\infty, 0], \\ ({}^{CF}D_0^r u)(t, w) = f(t, u_t(\cdot, w), w);\ t \in I, \end{cases} ;\ w \in \Omega, \tag{6.2}$$

where $\varphi : (-\infty, 0] \to \mathbb{R}, f : I \times \mathcal{B} \times \Omega \to \mathbb{R}$ are given functions, and $\mathcal{B}$ is called a phase space.

For any $t \in I$, we define $u_t \in \mathcal{B}$ by

$$u_t(s, w) = u(t+s, w);\ \text{for } s \in \mathbb{R}_-, \text{and } w \in \Omega.$$

In the third section, we investigate the following class of random Caputo-Fabrizio fractional differential equations with state dependent finite delay

$$\begin{cases} u(t, w) = \varphi(t, w);\ t \in [-h, 0], \\ ({}^{CF}D_0^r u)(t, w) = f(t, u_{\rho(t, u_t(\cdot, w))}(\cdot, w), w);\ t \in I, \end{cases} \tag{6.3}$$

where $\varphi \in \mathcal{C},\ \rho : I \times \mathcal{C} \times \Omega \to \mathbb{R},\ f : I \times \mathcal{C} \times \Omega \to \mathbb{R}$ are given functions.

and fractional differential equations with state dependent infinite delay

$$\begin{cases} u(t, w) = \varphi(t, w);\ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^r u)(t, w) = f(t, u_{\rho(t, u_t(\cdot, w))(\cdot, w)}, w);\ t \in I, \end{cases} ;\ w \in \Omega, \tag{6.4}$$

where $\varphi : \mathbb{R}_- \to \mathbb{R},\ f : I \times \mathcal{B} \times \Omega \to \mathbb{R}$ are given functions.

6.2 Existence of Random Solutions with Finite Delay

In this section, we establish some existence results for problem (6.1).

Definition 6.2.1 *By a solution of problem (6.1), we mean a function $u \in C$ such that*

$$u(t,w) = \begin{cases} \varphi(t,w); \ t \in [-h,0], \\ \varphi(0,w) - a_r f(0,u_0,w) + a_r f(t,u_t,w) + b_r \int_0^t f(s,u_s,w)ds; \ t \in I. \end{cases}$$

We shall make use of the following hypotheses:

(H_1) f is a random Carathéodory function.

(H_2) The function $t \mapsto \varphi(t,w)$ is continuous on $[-h,0]$.

(H_3) There exist measurable and essentially bounded functions $l, \tilde{l} : \Omega \to C(I)$ such that

$$|f(t,u,w)| \leq l(t,w) + \tilde{l}(t,w)\|u\|_{[-h,0]}, \text{ for all } u \in \mathcal{C}, t \in I.$$

(H_4) For any bounded set $B \subset C$, the set:

$$\{t \mapsto f(t,u_t,w) : u \in B\};$$

is equicontinuous in C.

Set

$$\begin{aligned} l^*(w) &= \sup_{t\in I} l(t,w); \quad w \in \Omega. \\ \tilde{l}^*(w) &= \sup_{t\in I} \tilde{l}(t,w). \end{aligned}$$

Theorem 6.2.2 *Assume that hypotheses $(H_1) - (H_4)$ hold. If*

$$(2a_r + Tb_r)\tilde{l}^*(w) < 1, \tag{6.5}$$

then the problem (6.1) has at least one solution on $[-h,T]$.

Proof. Let $N : \Omega \times C \to C$ be the operator defined by

$$N(w)u(t) = \begin{cases} \varphi(t,w); \ t \in [-h,0], \quad w \in \Omega \\ \varphi(0,w) - a_r f(0,u_0,w) + a_r f(t,u_t,w) + b_r \int_0^t f(s,u_s,w)ds; \ t \in I, \end{cases} \tag{6.6}$$

and set

$$R(w) \geq \max\left\{\|\varphi\|_{C([-h,0],\mathbb{R})}, \frac{|\varphi(0,w)| + (2a_r + Tb_r)l^*(w)}{1-(2a_r+Tb_r)\tilde{l}^*(w)}\right\}. \tag{6.7}$$

Define the ball

$$B_R := \{x \in C(I,\mathbb{R}) : \|x\|_C \leq R(w)\}.$$

Let $u \in B_R$ and $t \in [-h,0]$, then

$$\|N(w)u(t)\| \leq \|\varphi\|_C \leq R(w).$$

For any $w \in \Omega$ and each $t \in I$, we have

$$\begin{aligned}
|N(w)u(t)| &\leq |\varphi(0,w)| + a_r|f(0,u_0,w)| + a_r|f(t,u_t,w)| + b_r \textstyle\int_0^t |f(s,u_s,w)|ds \\
&\leq |\varphi(0,w)| + 2a_r(l(t,w) + \tilde{l}(t,w)\|u_t\|_{[-h,0]}) \\
&\quad + b_r(l(t,w) + \tilde{l}(t,w)\|u_t\|_{[-h,0]}) \textstyle\int_0^t ds \\
&\leq |\varphi(0,w)| + 2a_r(l^*(w) + \tilde{l}^*(w)\|u\|_C) + b_r(l^*(w) + \tilde{l}^*(w)\|u\|_C) \textstyle\int_0^t ds \\
&\leq |\varphi(0,w)| + 2a_r(l^*(w) + \tilde{l}^*(w)R(w)) + b_r(l^*(w) + \tilde{l}^*(w)R(w)) \textstyle\int_0^t ds \\
&\leq |\varphi(0,w)| + (2a_r + Tb_r)(l^*(w) + \tilde{l}^*(w)R(w)) \\
&\leq R(w).
\end{aligned}$$

Thus

$$\|N(w)u\|_C \leq R(w).$$

Hence $N(w)(B_R) \subset B_R$. We shall prove that $N : \Omega \times B_R \to B_R$ satisfies the assumptions of Theorem 1.3.1.

Step 1. $N(w)$ is a random operator.

The map $w \longrightarrow f(t,u_t,w)$ is measurable, and then the map

$$w \quad \mapsto \quad \varphi(0,w) - a_r f(0,u_0,w) + a_r f(t,u_t,w) + b_r \int_0^t f(s,u_s,w)ds,$$

is measurable. Hence N is a random operator on $\Omega \times C$ into C.

Step 2. $N(w)$ is continuous.

Let u_n be a sequence such that $u_n \to u$ in B_R. For each $t \in [-h,0]$, we have

$$|N(w)u_n(t) - N(w)u(t)| = 0,$$

and for each $t \in I$, we have

$$\begin{aligned} |N(w)u_n(t) - N(w)u(t)| \leq\ & a_r|f_n(0,u_0,w) - f(0,u_0,w)| \\ & +a_r|f_n(t,u_t,w) - f(t,u_t,w)| \\ & +b_r \int_0^t |f_n(s,u_s,w) - f(s,u_s,w)|ds, \end{aligned} \tag{6.8}$$

we obtain

$$\|u_n(\cdot,w) - u(\cdot,w)\|_C \to 0 \; as \; n \to \infty,$$

then the Lebesgue dominated convergence theorem implies that

$$\|N(u_n) - N(u)\|_C \to 0 \quad \text{as } n \to \infty.$$

Consequently, $N(w)$ is continuous.
Step 3. $N(w)B_R$ is equicontinuous.
For $1 \leq t_1 \leq t_2 \leq T$, and $u \in B_R$, we have

$$\begin{aligned} |(Nu)(t_1,w) - (Nu)(t_2,w)| &\leq a_r|f(t_2,u_{t_2},w) - f(t_1,u_{t_1},w)| + b_r \int_{t_1}^{t_2} |f(s,u_s,w)|ds \\ &\leq a_r|f(t_2,u_{t_2},w) - f(t_1,u_{t_1},w)| + b_r(l(t,w) \\ &\quad +\tilde{l}(t,w)\|u_t\|_{[-h,0]}) \int_{t_1}^{t_2} ds \\ &\leq a_r|f(t_2,u_{t_2},w) - f(t_1,u_{t_1},w)| + b_r(l^*(w) \\ &\quad +\tilde{l}^*(w)R(w))(t_2 - t_1). \end{aligned}$$

Thus, from (H_4), as $t_2 \to t_1$ the right-hand side of the above inequality tends to zero. Arzelá-Ascoli theorem implies that $N : \Omega \times B_R \to B_R$ is continuous and compact. Hence; from Theorem 1.3.1, we deduce the problem (6.1) has at least one solution on $[-h, T]$.

6.3 Existence of Random Solutions with Infinite Delay

In this section, we establish some existence results for problem (6.2). Let the space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ as a seminormed linear space of functions mapping $\mathbb{R}_-$ into $\mathbb{R}$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato [67] for ordinary differential functional equations.

(A_1) If $u : (-\infty, T] \to \mathbb{R}$, and $u_0 \in \mathcal{B}$, then there are constants $L, M, H > 0$, such that for any $t \in I$ the following conditions hold:

(i) u_t *is in* $\mathcal{B}$,

(*ii*) $\|u_t\|_{\mathcal{B}} \leq K\|u_1\|_{\mathcal{B}} + M \sup_{s\in[0,t]} |u(s)|$,

(*iii*) $\|u(t)\| \leq H\|u_t\|_{\mathcal{B}}$.

(A_2) For the function $u(\cdot)$ *in* (A_1), u_t is a $\mathcal{B}-$ valued continuous function on I.

(A_3) The space $\mathcal{B}$ is complete.

Consider the space

$$\Delta = \{u : (-\infty, T] \to \mathbb{R} : u|_{\mathbb{R}_-} \in \mathcal{B},\ u|_I \in C(I)\}.$$

Definition 6.3.1 *The problem (6.2) is equivalent to the integral equation*

$$u(t,w) = \begin{cases} \varphi(t,w);\ t \in \mathbb{R}_-, \\ \varphi(0,w) - a_r f(0,u_0,w) + a_r f(t,u_t,w) + b_r \int_0^t f(s,u_s,w)ds;\ t \in I. \end{cases} \tag{6.9}$$

The following hypotheses will be used in the sequel.

(H_{01}) f is a random Carathéodory function.

(H_{02}) The function $t \mapsto \varphi(t,w)$ is continuous and bounded on $\mathbb{R}_-$.

(H_{03}) There exist measurable and essentially bounded functions $m, \tilde{m} : \Omega \to C(I)$; such that

$$|f(t,u,w)| \leq m(t,w) + \tilde{m}(t,w)\|u\|_{\mathcal{B}},\ \text{for all } u \in \mathcal{B},\ t \in I.$$

(H_{04}) For any bounded set $B_1 \subset \Delta$, the set:

$$\{t \mapsto f(t,u_t,w) : u \in B_1\};$$

is equicontinuous in Δ.

Set

$$\begin{aligned} m^*(w) &= \sup_{t\in I} m(t,w);\quad w \in \Omega, \\ \tilde{m}^*(w) &= \sup_{t\in I} \tilde{m}(t,w). \end{aligned}$$

Theorem 6.3.2 *Assume that the hypotheses* $(H_{01}) - (H_{04})$ *hold. If*

$$(2a_r + Tb_r)M\tilde{m}^*(w) < 1, \tag{6.10}$$

then problem (6.2) has at least one random solution on $(-\infty, T]$.

Consider the operator $N_1 : \ \Omega \times \Delta \to \Delta$ defined by:

$$(N_1 u)(t,w) = \begin{cases} \varphi(t,w); \ t \in \mathbb{R}_-, \\ \varphi(0,w) - a_r f(0,u_0,w) + a_r f(t,u_t,w) + b_r \int_0^t f(s,u_s,w)ds; \ t \in I. \end{cases} \tag{6.11}$$

Let $x(\cdot,w) : (-\infty,T] \times \Omega \to \mathbb{R}$ be a function defined by

$$x(t,w) = \begin{cases} \varphi(t,w); & t \in \mathbb{R}_-, \\ \varphi(0,w) & t \in I. \end{cases}$$

Then $v_0 = \varphi$, For each z continuous on I with $z(0,w) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t,w) = \begin{cases} 0; & t \in \mathbb{R}_-, w \in \Omega, \\ z(t,w), & t \in I. \end{cases}$$

If $u(\cdot,w)$ satisfies the integral equation

$$u(t,w) \ = \ \varphi(0,w) - a_r f(0,u_0,w) + a_r f(t,u_t,w) + b_r \int_0^t f(s,u_s,w)ds.$$

We can decompose $u(\cdot,w)$ as $u(t,w) = \overline{z}(t,w) + x(t,w)$; for $t \in I$, which implies that $u_t = \overline{z}_t + x_t$ for every $t \in I$, and $w \in \Omega$ and the function $z(\cdot,w)$ satisfies

$$z(t,w) = -a_r f(0,\overline{z}_0 + x_0,w) + a_r f(t,\overline{z}_t + x_t,w) + b_r \int_0^t f(s,\overline{z}_s + x_s,w)ds.$$

Set

$$C_0 = \{z \in C(I); \ z_0 = 0\},$$

and let $\|\cdot\|_T$ be the norm in C_0 defined by

$$\|z\|_T = \|z_0\|_{\mathcal{B}} + \sup_{t\in I} |z(t)| = \sup_{t\in I} |z(t)|; \ z \in C_0.$$

C_0 is a Banach space with norm $\|\cdot\|_T$.
Let the operator $P : C_0 \to C_0$; defined by

$$(Pz)(t,w) = -a_r f(0,\overline{z}_0 + x_0,w) + a_r f(t,\overline{z}_t + x_t,w) + b_r \int_0^t f(s,\overline{z}_s + x_s,w)ds. \tag{6.12}$$

For each given $R(w) > 0$, we define the ball

$$B_R = \{x \in C_0,\ \|x\|_T \leq R(w)\}.$$

Let $z \in B_R$, for each $t \in I$, and $w \in \Omega$ we have

$$\begin{array}{rcl} |(Pz)(t,w)| & \leq & a_r|f(0,\overline{z}_0 + x_0, w)| + a_r|f(t,\overline{z}_t + x_t, w)| + b_r\int_0^t |f(s,\overline{z}_s + x_s, w)|ds \\ & \leq & 2a_r(m(t,w) + \tilde{m}(t,w)\|\overline{z}_t + x_t\|_{\mathcal{B}}) \\ & + & b_r(m(t,w) + \tilde{m}(t,w)\|\overline{z}_t + x_t\|_{\mathcal{B}})\int_0^t ds \\ & \leq & (2a_r + Tb_r)(m^*(w) + \tilde{m}^*(w)[\|\overline{z}_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}}]) \\ & \leq & (2a_r + Tb_r)(m^*(w) + \tilde{m}^*(w)[MR(w) + K\|\varphi\|_{\mathcal{B}}]) \\ & := & \ell(w). \end{array}$$

Hence

$$\|P(z)\|_T \leq \ell(w).$$

We prove that the operator $P : C_0 \to C_0$ satisfies all conditions of Theorem 1.3.3. The proof will be given in several steps.

Step 1. $P(w)$ is a random operator.

Since the map $w \longrightarrow f(t,u_t,w)$ is measurable, we obtain that the map

$$w \mapsto -a_rf(0,\overline{z}_t + x_t, w) + a_rf(t,\overline{z}_0 + x_0, w) + b_r\int_0^t f(s,\overline{z}_s + x_s, w)ds,$$

is measurable, and hence $P(w)$ is a random operator on $\Omega \times C_0$ into C_0.

Step 2. $P(w)$ is continuous .

Let z_n be a sequence such that $z_n \to z$ in C_0. For each $t \in I$, we have

$$\begin{array}{rcl} |(Pz_n)(t,w) - (Pz)(t,w)| & \leq & a_r|f(0,\overline{z}_0 + x_0, w) - f(0,\overline{z}_0 + x_0, w)| \\ & + & a_r|f(t,\overline{z}_{nt} + x_t, w) - f(t,\overline{z}_t + x_t, w)| \\ & + & b_r\int_0^t |f(s,\overline{z}_{ns} + x_s, w) - f(s,\overline{z}_s + x_s, w)|ds. \end{array} \tag{6.13}$$

Since $\|z_n - z\|_T \to 0$ as $n \to \infty$ and f is Carathéodory, the Lebesgue dominated convergence theorem, implies that

$$\|P(u_n) - P(u)\|_T \to 0 \quad \text{as } n \to \infty.$$

Hence, $P(w)$ is continuous.

Step 3. $P(B_R)$ is equicontinuous.
For $1 \le t_1 \le t_2 \le T$, and $z \in B_R$, we have

$$\begin{aligned} & |(Pz)(t_2,w)-(Pz)(t_1,w)| \\ \le\ & a_r|f(t_2,\overline{z}_{t_2}+x_{t_2},w)-f(t_1,\overline{z}_{t_1}+x_{t_1},w)| \\ & +b_r\int_{t_1}^{t_2}|f(s,\overline{z}_s+x_s,w)|ds \\ \le\ & a_r|f(t_2,\overline{z}_{t_2}+x_{t_2},w)-f(t_2,\overline{z}_{t_2}+x_{t_2},w)-f(t_1,\overline{z}_{t_1}+x_{t_1},w)| \\ +\ & b_r(t_2-t_1)(m^*(w)+\tilde{m}^*(w)[MR(w)+K\|\varphi\|_{\mathcal{B}}]). \end{aligned}$$

By (H_{04}), as $t_2 \to t_1$ the right-hand side of the above inequality tends to zero, we conclude that P maps bounded sets into equicontinuous sets in C_0. Hence problem (6.2) has at least one solution.

6.4 Existence Results with State-Dependent Delay

In this section, we establish some existence of random solutions for problems (6.3) and (6.4).

6.4.1 Existence of Soluions In Finite Delay Case

Definition 6.4.1 *By a solution of problem (6.3), we mean a function $u \in C$ such that*

$$u(t,w)=\begin{cases}\varphi(t,w);\ \ t\in[-h,0],\\ \varphi(0,w)-a_rf(0,u_{\rho(0,u_0)},w)+a_rf(t,u_{\rho(t,u_t(t,w))},w)\\ +b_r\int_0^t f(t,u_{\rho(s,u_s(s,w))}(s,w),w)ds;\ \ t\in I.\end{cases}$$

We shall make use of the following hypotheses:

(H_5) There exist measurable and essentially bounded functions $l_1,\tilde{l}_1:\Omega\to C(I)$; such that
$$|f(t,u,w)|\le l_1(t,w)+\tilde{l}_1(t,w)\|u\|_{[-h,0]},\ \text{for all}\ u\in\mathcal{C},\ t\in I.$$

(H_6) For any bounded set $B_2\subset C$, the set:

$$\{t\mapsto f(t,u_t,w):u\in B_2\};$$

is equicontinuous in C.

Set

$$\begin{aligned} l_1^*(w) &= \sup_{t\in I} l_1(t,w); \quad w\in\Omega. \\ \tilde{l}_1^*(w) &= \sup_{t\in I} \tilde{l}_1(t,w). \end{aligned}$$

Theorem 6.4.2 *Assume that the hypothesis* $(H_1),(H_2),(H_5)$ *and* (H_6) *hold. If*

$$(2a_r+Tb_r)\tilde{l}_1^*(w)<1, \tag{6.14}$$

then problem (6.3) has at least one solution on $[-h,T]$.

6.4.2 Existence of Solutions In Infinite Delay Case

Definition 6.4.3 *The problem (6.4) is equivalent to the integral equation*

$$u(t,w)=\begin{cases} \varphi(t,w);\ t\in\mathbb{R}_-, \\ \varphi(0,w)-a_rf(0,u_{\rho(0,u_0)},w)+a_rf(t,u_{\rho(t,u_t(t,w))}) \\ +b_r\int_0^t f(t,u_{\rho(s,u_s(s,w))}(s,w),w)ds;\ t\in I. \end{cases} \tag{6.15}$$

Set

$$R':=R'_{\rho^-}(w)\{\rho(t,u,w):t\in I,\ u\in\mathcal{B},\ w\in\Omega,\ \rho(t,u,w)<0\}.$$

We always assume that $\rho: I\times\mathcal{B}\times\Omega\to\mathbb{R}$ is continuous and the function $t\to u_t$ is continuous from R' into $\mathcal{B}$. We will need the following hypothesis:

(H_φ) There exists a continuous bounded function $L: R'_{\rho^-}\to(0,\infty)$ such that

$$\|\varphi_t\|_{\mathcal{B}}\le L(t)\|\varphi\|_{\mathcal{B}},\ for\ any\ t\in R'.$$

In the sequel we will make use of the following generalization of a consequence of the phase space axioms.

Lemma 6.4.4 *If* $u\in\Delta$ *then*

$$\|u_t\|_{\mathcal{B}}=(M+L')\|\varphi\|_{\mathcal{B}}+K\sup_{\theta\in[0,\max\{0,t\}]}\|u(\theta)\|,$$

where

$$L' = \sup_{t\in R'} L(t).$$

The following hypotheses will be used in the sequel.

(H_{05}) There exist measurable and essentially bounded functions $m_1, \tilde{m}_1 : \Omega \to C(I)$; such that

$$|f(t,u,w)| \leq m_1(t,w) + \tilde{m}_1(t,w)\|u\|_{\mathcal{B}}, \text{ for all } u \in \mathcal{B},\ t \in I.$$

(H_{06}) For any bounded set $B_2 \subset \Delta$, the set:

$$\{t \mapsto f(t,u_t,w) : u \in B_2\}$$

is equicontinuous in Δ.

Set

$$\begin{aligned} m_1^*(w) &= \sup_{t\in I} m_1(t,w); \quad w \in \Omega. \\ \tilde{m}_1^*(w) &= \sup_{t\in I} \tilde{m}_1(t,w). \end{aligned}$$

Theorem 6.4.5 *Assume that the hypotheses (H_φ), $(H_{01}), (H_{02}), (H_{05})$ and (H_{06}) hold. Then problem (6.4) has at least one solution on $(-\infty, T]$.*

6.5 Some Examples

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$.

Example 1. Consider now the following random problem

$$\begin{cases} u(t,w) = \frac{2w^2}{t^2+1};\ t \in [-1,0], \\ ({}^{CF}D_0^r u)(t,w) = \frac{w^2 d(w)}{e^{2t+1}(1+\|u_t\|)};\ t \in [0,1], \end{cases} \tag{6.16}$$

where $d(w) < \frac{e^3}{(2a_r+b_r)w^2}$.

Set

$$f(t,u,w) = \frac{w^2 d(w)}{e^{2t+1}\left(1+\|u_t\|\right)};\ t \in [0,1],\ u \in \mathcal{C}.$$

Clearly, the function f is continuous. For any $u \in \mathcal{C}$ and $t \in [0,1]$, we have

$$|f(t,u,w)| \leq \frac{w^2 d(w)}{e^3}\|u\|_{[-1,0]}.$$

Hence hypothesis (H_3) is satisfied with

$$l^*(w) = 0 \quad and \quad \tilde{l}^*(w) = \frac{w^2 d(w)}{e^3}.$$

Next, condition (6.5) is satisfied with $T = 1$.Indeed,

$$\begin{array}{rcl}(2a_r + Tb_r)\tilde{l}^*(w) & = & \frac{w^2 d(w)}{e^3}(2a_r + b_r) \\ & < & 1.\end{array}$$

Simple computations show that all conditions of Theorem 6.2.2 are satisfied. It follows that problem (6.16) has a solution defined on $[-1,1]$.

Example 2. Consider now the following problem

$$\begin{cases} u(t,w) = t\sin w;\ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^r u)(t,w) = \frac{c(w)w^2 u_t e^{-\gamma t + t}}{(e^t - e^{-t})(1+w^2)(1+\|u_t\|)};\ t \in [0,1], \end{cases} \tag{6.17}$$

where $c(w) < \frac{1}{2a_r + b_r}$. Let γ be a positive real constant and

$$B_\gamma = \{u \in C((-\infty,1],\mathbb{R},) : \lim_{\theta \to -\infty} e^{\gamma\theta}u(\theta)\ exists\ in\ \mathbb{R}\}. \tag{6.18}$$

The norm of B_γ is given by

$$\|u\|_\gamma = \sup_{\theta \in (-\infty,1]} e^{\gamma\theta}|u(\theta)|.$$

Let $u : \mathbb{R}_- \to \mathbb{R}$ be such that $u_0 \in B_\gamma$. Then

$$\begin{array}{rl}\lim_{\theta \to -\infty} e^{\gamma\theta}u_t(\theta) & = \lim_{\theta \to -\infty} e^{\gamma\theta}u(t+\theta-1) = \lim_{\theta \to -\infty} e^{\gamma(\theta-t+1)}u(\theta) \\ & = e^{\gamma(-t+1)}\lim_{\theta \to -\infty} e^{\gamma(\theta)}u_1(\theta) < \infty.\end{array}$$

Hence $u_t \in B_\gamma$. Finally we prove that

$$\|u_t\|_\gamma \leq K\|u_1\|_\gamma + M\sup_{s \in [0,t]}|u(s)|,$$

where $K = M = 1$ and $H = 1$. We have

$$\|u_t(\theta)\| = |u(t+\theta|.$$

If $t+\theta \leq 1$, we get

$$\|u_t(\beta)\| \leq \sup_{s\in\mathbb{R}_-} |u(s)|.$$

For $t+\theta \geq 0$, then we have

$$\|u_t(\beta)\| \leq \sup_{s\in[0,t]} |u(s)|.$$

Thus for all $t+\theta \in I$, we get

$$\|u_t(\beta)\| \leq \sup_{s\in\mathbb{R}_-} |u(s)| + \sup_{s\in[0,t]} |u(s)|.$$

Then

$$\|u_t\|_\gamma \leq \|u_0\|_\gamma + \sup_{s\in[0,t]} |u(s)|.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ a phase space. Set

$$f(t,u,w) = \frac{c(w)w^2 e^{-\gamma t+t}}{(e^t - e^{-t})(1+w^2)\left(1+\|u\|_{B_\gamma}\right)};\ t\in[0,1],\ u\in B_\gamma.$$

For any $u, \in B_\gamma$ and $t\in[0,1]$, we have

$$|f(t,u,w)| \leq \frac{c(w)w^2}{1+w^2}\|u\|_{B_\gamma}.$$

Hence hypotheses $(H_{01}) - (H_{03})$ are satisfied with

$$\tilde{m}^*(w) = \frac{c(w)w^2}{1+w^2} \quad and \quad m^*(w) = 0.$$

Next we obtain

$$\begin{array}{rcl}(2a_r + Tb_r)M\tilde{m}^*(w) & = & \frac{c(w)w^2}{1+w^2}(2a_r+b_r) \\ & < & 1.\end{array}$$

It follows from Theorem 6.3.2 that problem (6.17) has at least one solution defined on $(-\infty, 1]$.

Example 3. We consider the following problem

$$\begin{cases} u(t,w) = \frac{2w^2}{t^2+1};\ t \in [-1,0], \\ ({}^{CF}D_0^r u)(t,w) = \frac{w^2}{e^{2t+1}(1+|u(t-\sigma(u(t)))|)};\ t \in [0,1], \end{cases} \tag{6.19}$$

where $\sigma \in C(\mathbb{R},[0,1])$. Set

$$\rho(t,\varphi,w) = t - \sigma(\varphi(0,w)), \quad (t,\varphi,w) \in [0,e] \times C([-1,0],\mathbb{R}) \times \Omega.$$

$$f(t,u,w) = \frac{w^2}{e^{2t+1}(1+|u(t-\sigma(u(t)))|)};\ t \in [0,1],\ u \in \mathcal{C}$$

Clearly, the function f is jointly continuous. For any $u \in \mathcal{C}$ and $t \in [0,1]$, we have

$$|f(t,u,w)| \leq \frac{w^2}{e^3}\|u\|_{[-1,0]}.$$

Hence hypothesis (H_{05}) is satisfied with

$$\tilde{m}^*(w) = \frac{w^2}{1+w^2} \quad and \quad m^*(w) = 0.$$

It follows from Theorem 6.4.2 that problem (6.19) has a solution defined on $[-1,1]$.

Example 4. Consider now the problem

$$\begin{cases} u(t,w) = \frac{t^2}{w^2+2};\ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^r u)(t,w) = \frac{u(t-\lambda(u(t)))e^{-\gamma t+t}}{w^2(e^t-e^{-t})(1+|u(t-\sigma(u(t,w),w)|)};\ t \in [0,2]. \end{cases} \tag{6.20}$$

Let γ be a positive real constant and the phase space B_γ defined in Example 2.

Define

$$\rho(t,\varphi,w) = t - \lambda(\varphi(0,w)), \quad (t,\varphi) \in [0,2] \times B_\gamma \times \Omega,$$

and set

$$f(t,u,w) = \frac{e^{-\gamma t+t}}{w^2\left(e^t-e^{-t}\right)\left(1+\|u\|_{B_\gamma}\right)};\ t \in [0,2],\ u \in B_\gamma.$$

Simple computations show that all conditions of Theorem 6.4.5 are satisfied. It follows that problem (6.20) has at least one solution defined on $(-\infty,2]$.

CONCLUSION AND PERSPECTIVES

In this thesis, we have presented some results.

The fisrt result is based on the existence of random solutions for the following class of Caputo-Hadamard fractional differential equation

$$({}^{Hc}D_1^r u)(t,w) = f(t,u(t,w),w);\ t \in I := [1,T],\ w \in \Omega,$$

with the boundary conditions

$$\begin{cases} u(1,w) = u_1(w) \\ u'(T,w) = u_T(w) \end{cases} ;\ w \in \Omega,$$

The second is based on the existence of random solutions and the stability of Ulam results for a class of Caputo-Fabrizio random fractional dierential equations in the form

$$({}^{CF}D_0^\alpha u)(t,w) = f(t,u(t,w),w);\ t \in I := [0,T],\ w \in \Omega,$$

with the boundary conditions

$$au(0,w) + bu(T,w) = c(w);\ w \in \Omega,$$

and the existence of random solutions and the stability Ulam for a class of random fractional differential equations of Katugampola

$$({}^{\rho}D_0^{\varsigma}x)(\xi, w) = f(\xi, x(\xi, w), w);\ \xi \in I = [0, T],\ w \in \Omega,$$

with the terminal condition

$$x(T, w) = x_T(w);\ w \in \Omega,$$

In chapter 4 we study the existence and attractivity for several classes of functional fractional differential equations.

$$({}^{CF}D_0^{r}u)(t, w) = f(t, u(t, w), w);\ t \in \mathbb{R}_+ = [0, \infty),\ w \in \Omega, \tag{6.21}$$

with the initial condition

$$u(0, w) = u_0(w);\ w \in \Omega, \tag{6.22}$$

and in chapter 5 we are proved the existence and the Ulam stability results in Fréchet spaces of the problem (6.21)-(6.22).
We have presented the following nonlocal problem

$$\begin{cases} ({}^{CF}D_0^{r}u)(t, w) = f(t, u(t, w), w);\ t \in \mathbb{R}_+, \\ u(0, w) + Q(u(\cdot, w)) = u_0(w), \end{cases} \quad w \in \Omega,$$

We have also proved the existence of random solutions for some classes of Caputo-Fabrizio random fractional differential equations delay.

Our results are based on the random fixed point theory.

In future research we plan to investigate a some problems for random implicit fractional differential equations, thus problems with and without impulses (instantaneous and not instantaneous).

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Abstract :

In this thesis, we consider the study of the existence of random solutions and the Ulam stability and the attractivity of serveral classes of differential equations with fractional derivatives of Caputo, Hadamard, Fabrizio and Katugampola in Fréchet spaces. The used methods are the random fixed point and the technique of the measure non-compactness. We have also shown the existence of random solutions for certain classes of random fractional differential equations with delay. In addition, for the justification of our results, we give various examples in each chapter.

Keywords :Differential equation, fractional integral, fractional derivative, random solution, Banach space, Ulam stability, fixed point, attractivity, nonlocal problem, finite delay, infinite delay, state-dependent delay, measure of non compactness, Fréchet space.

Resumé :

Dans cette thèse, nous considérons l'étude de l'existence des solutions aléatoires et la stabilité de type Ulam et l'attractivité de quelques classes d'équations différentielles avec les dérivées fractionnaires de Caputo, Hadamard, Fabrizio et Katugampola dans des espaces de Fréchet. Les méthodes utilisées sont basées sur la théorie de point fixe et la mesure de non compacité dans les espaces de Fréchet .Nous avons également montré l'existence de solutions aléatoires pour certaines classes d'équations différentielles fractionnaires aléatoires avec retard. De plus, pour la justification de nos résultats, nous donnons divers exemples illustratifs.

Mots clés : équation différentielle, équation intégrale, dérivée fractionnaire, solution aléatoire, espace de Banach, stabilité d'Ulam, point fixe, attractivité, problème non local, retard fini, retard infini, retard dépendant de l'état, mesure de non compacité, espace de Fréchet.

الملخص

في هذه الرسالة، نأخذ في الاعتبار دراسة وجود الحلول العشوائية واستقرار أولام وجاذبية الفئات الخدمية للمعادلات التفاضلية مع المشتقات الكسرية لكابوتو، هادامارد، فابريزيو وكاتوجامبولا في فضاء فريشي. الطرق المستخدمة هي النقطة الثابتة العشوائية وتقنية قياس عدم التراص. لقد أظهرنا أيضًا وجود حلول عشوائية لفئات معينة من المعادلات التفاضلية الجزئية العشوائية مع تأخير. بالإضافة إلى ذلك، لتبرير نتائجنا، نقدم أمثلة مختلفة في كل فصل

الكلمات مفتاحية: *معادلة تفاضلية، تكامل كسري، مشتق كسري، حل عشوائي، فضاء باناخ، استقرار أولام، نقطة ثابتة، جاذبية، مشكلة غير محلية، تأخير محدود، تأخير لانهائي، تأخير معتمد على الحالة، قياس عدم التراص، فضاء فريتشي*