Université de Saida– Dr. Moulay Tahar

Faculté des Sciences

Thèse

Présentée pour obtenir le diplôme de

Doctorat 3ème Cycle

Spécialité : Systèmes Dynamiques

Filière : Mathématiques

Par :

KRIM SALIM

Thème :

Etude de quelques systèmes d’équations différentielles implicites d’ordre fractionnaire

Thèse soutenue le 09/05/2022 devant le jury composé de :

N°	Nom et prénom	Grade	Etablissement	Qualité
01	KANDOUCI Abd El Djebbar	Prof.	Université de Saida – Dr. Moulay Tahar	Président
02	ABBAS Saïd	Prof.	Université de Saida – Dr. Moulay Tahar	Rapporteur
03	BENCHOHRA Mouffak	Prof.	Université de Sidi Bel-Abbès – Djilali liabes	Examinateur
04	LAZREG Jamel Eddine	Prof.	Université de Sidi Bel-Abbès – Djilali liabes	Examinateur
05	GUENDOUZI Toufik	Prof..	Université de Saida – Dr. Moulay Tahar	Examinateur

Remerciements

Je tiens á remercier en tout premier le *Prof. Saïd Abbas* pour m'avoir fait confiance en 2018 quand il m'a accepté en thèse. Il m'a transmis ses connaissances et ses idées avec générosité. Je le remercie vivement pour m'avoir encouragée et guidée pendant ces trois années.

Adresser mes remerciements au *Prof. Abdeldjebbar Kandouci*, vous me faites l'honneur et la la gentillesse de participer à mon jury de thèse. Merci pour le le temps que vous m'avez accordé pour m'aider.

J'adresse également mes remerciements au *Prof. Moufouk Benchohra*, Je suis trés reconnaissant d'avoir accepté de participer dans le jury de soutenance de ma thése. je me souviendrai de ton gentillesse, tu as toujours été disponible ainsi que l'écriture ma thése. La pertinence de vos commentaires et l'exactitude de vos corrections. Merci pour le le temps que vous m'avez accordé pour m'aider dans mes recherches.

J'adresse également mes remerciements au *Prof. Jamal Eddine Lazreg*, Je suis trés reconnaissant d'avoir accepté de participer dans le jury de soutenance de ma thése. je me souviendrai de ton gentillesse, tu as toujours été disponible ainsi que l'écriture ma thése. La pertinence de vos commentaires et l'exactitude de vos corrections.

J'adresse également mes remerciements au *Prof. Toufik Guendouzi*, Je suis trés reconnaissant d'avoir accepté de participer dans le jury de soutenance de ma thése. je me souviendrai de ton gentillesse, tu as toujours été disponible ainsi que l'écriture ma thése. La pertinence de vos commentaires et l'exactitude de vos corrections.

Cette thèse n'aurait pas vu le jour sans la confiance, les encouragements et le soutien indéfectible de mes parents. Je remercie mon père qui a tout fait pour que je puisse continuer mes études. Maman, je te remercie de tout mon coeur pour ton affection, pour le temps que tu m'as consacré durant ces années. Un grand merci à mes frères.

Krim Salim

الملخص :
في هذه الرسالة ، ندرس وجود وتفرد الحلول وثبات نوع أولام لبعض أصناف المعادلات التفاضلية الضمنية الكسرية مع مشتقات كابيتوا و هدمارد و كابيتو-فابريزيو و كاتيكمبولة ونذكر جميع المشتقات. المشاكل التي تمت دراستها هي مع الشروط الأولية والحدية .تستند النتائج التي تم الحصول عليها إلى بعض نظريات النقطة الثابتة وقياس عدم الاكتناز في فضاءات باناك وفريشي و بميتريك.
الكلمات المفتاحية: معادلة تفاضلية ، ترتيب كسري ، حل ، استقرار ، ضمني ، ثابت ، تأخير محدود ، تأخير لانهائي ، تأخير يعتمد على الحالة ، قياس عدم الاكتناز ، فضاء فريشت ، مساحة باناك ، مساحة ب-متري.

Résumé :

Dans cette thèse, nous étudions l'existence et l'unicité de solutions et la stabilité de type Ulam de certaines classes d'équations différentielles implicites fractionnaires avec les dérivées de Caputo, Hadamard, Caputo-Fabrizio, Katugampola, et mentionnons toutes les dérivées. Les problèmes étudiés sont à conditions initiales et aux limites. Les résultats obtenus sont basés sur quelques théorèmes de point fixe et la mesure de non-compacité dans les espaces de Banach, fréchet et b-Métrique.
Mots clés : Equation différentielle, ordre fractionnaire, solution, stabilité, implicite, fixe, retard fini, retard infini, retard dépendant de l'état, mesure de non compacité, espace de Fréechet, espace de Banach, espace de b-métrique.

Abstract :

In this thesis, we study the existence and uniqueness of solutions and the Ulam-type stability of some classes of fractional implicit differential equations with the derivatives of Caputo, Hadamard, Caputo-Fabrizio, Katugampola, and mention all the derivatives. The problems studied are with initial and boundary conditions. The results obtained are based on some fixed point theorems and the measure of non-compactness in the spaces of Banach, fréchet and b-Metric.
Key words: Differential equation, fractional order, solution, stability, implicit, fixed, finite delay, infinite delay, state-dependent delay, measure of non compactness, Fréechet space, Banach space, b-Metric space.

Publications

1. **S. Krim**, S. Abbas, M. Benchohra and M.A. Darwish, Boundary value problem for implicit Caputo–Fabrizio fractional differential equations, Int. J. Difference Equ. ISSN 0973-6069, 15 (2) (2020), 493-510.

2. **S. Krim**, S. Abbas, and M. Benchohra, Caputo-Hadamard implicit fractional differential equations with delay, Pãulo Journal of Mathematical Sciences, 15 (1) (2021), 463-484.

3. **S. Krim**, S. Abbas, M. Benchohra and E. Karapinar, Terminal value problem for implicit Katugampola fractional differential equations in b-metric spaces, *Journal of Function Spaces 2021*, Article ID 5535178, 7 pages.

4. **S. Krim**, S. Abbas, and M. Benchohra, Initial value problems for Caputo–Fabrizio implicit fractional differential equations in b-metric spaces, Bulletin of the Transilvania University of Braşov, 1 (63) No. 1 - 2021. SeriesIII: Mathematics, Informatics, Physics,1-13.

5. **S. Krim**, S. Abbas and M. Benchohra , Caputo-Hadamard implicit fractional differential equations with two boundary conditions (submitted).

6. **S. Krim**, S. Abbas, M. Benchohra and J.J.Nieto, Implicit Caputo-Fabrizio fractional differential equations with delay (submitted).

7. **S. Krim**, S. Abbas and M. Benchohra , Functional Katugampola fractional differential equations in b-metric spaces (submitted).

8. **S. Krim**, S. Abbas and M. Benchohra , Initial value problem for implicit Caputo-Katugampola fractional differential equations in b-metric spaces (submitted).

9. **S. Krim**, S. Abbas and M. Benchohra, Initial value problems for Caputo-Katugampola implicit fractional differential equations with delay (submitted).

10. **S. Krim**, A. Salim, S. Abbas and M. Benchohra , Functional k-generalised ψ-Hilfer fractional differential equations in b-metric spaces (submitted).

11. A. Salim, **S. Krim**, S. Abbas and M. Benchohra , Random solutions for improved conformable fractional differential equations with retarded and advanced arguments (submitted).

12. **S. Krim**, A. Salim, S. Abbas and M. Benchohra , Conformable fractional differential equations with delay in b-metric spaces (submitted).

13. **S. Krim**, A. Salim, S. Abbas and M. Benchohra , Implicit impulsive Caputo-Katugampola fractional differential equations with infinite delay (submitted).

14. A. Salim, M. Benchohra, **S. Krim** and S. Abbas , Initial value problem for hybrid improved conformable fractional differential equations with retardation and anticipation (submitted).

Contents

Introduction

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals, briefly differintegrals). In particular, this discipline involves the notion and methods of solving of differential equations involving fractional derivatives of the unknown function (called fractional differential equations).

The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The first known reference can be found in the correspondence of G. W. Leibniz and Marquis de l'Hopital in 1695 where the question of meaning of the semi-derivative has been raised. In 1730 the subject of fractional calculus did not escape Eulers attention. J. L. Lagrange in 1772 contributed to fractional calculus indirectly, when he developed the law of exponents for differential operators. In 1812, P. S. Laplace defined the fractional derivative by means of integral and in 1819 S. F. Lacroix mentioned a derivative of arbitrary order in his 700-page long text, followed by J. B. J. Fourier in 1822, who mentioned the derivative of arbitrary order. The first use of fractional operations was made by N. H. Abel in 1823 in the solution of tautochrome problem. J. Liouville made the first major study of fractional calculus in 1832, where he applied his definitions to problems in theory. In 1867, A. K. Grunwald worked on the fractional operations. G. F. B. Riemann developed the theory of fractional integration during his school days and published his paper in 1892. A. V. Letnikov wrote several papers on this topic from 1868 to 1872. Oliver Heaviside published a collection of papers in 1892, where he showed the so-called Heaviside operational calculus concerned with linear generalized operators. In the period of 1900 to 1970 the principal contributors to the subject of fractional calculus were, for example, H. H. Hardy, S. Samko, H. Weyl, M. Riesz, S. Blair, etc. From 1970 to the present, they are for instance J. Spanier, K. B. Oldham, B. Ross, K. Nishimoto, O. Marichev, A. Kilbas, H. M. Srivastava, R. Bagley, K. S. Miller, M. Caputo, I. Podlubny, and many others.

In recent years, there has been a significant development in the theory of fractional differential equations. It is caused by its applications in the modeling of many phenomena in various fields of science and engineering such as acoustic, control theory, chaos and fractals, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, optics, economics, astrophysics, chaotic dynamics, statistical physics, thermodynamics, proteins, biosciences, bioengineering, etc. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. See for example [22, 23, 60, 64, 86, 92, 99, 106, 107, 108].

There are two measures which are the most important ones. The Kuratowski measure of noncompactness $\alpha(B)$ of a bounded set B in a metric space is defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of sets of diameter smaller than r. The Hausdorf measure of noncompactness $\chi(B)$ defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of balls of radii smaller than r. Several authors have studied the measures of noncompactness in Banach spaces. See, for example, the books such as [13, 24] and the articles [15, 25, 26, 32, 33, 35, 67, 88, 100, 101, 102], and references therein.

In the theory of ordinary differential equations, of partial differential equations, and in the theory of ordinary differential equations in a Banach space there are several types of data dependence . On the other hand, in the theory of functional equations there are some special kind of data dependence: Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers- Bourgin, Aoki-Rassias [98].

The stability of problems with functional equations originated from a question of Ulam [110, 111] concerning the stability of group homomorphisms: "*Under what conditions does there exist an additive mapping near an approximately additive mapping ?*" Hyers [68] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers Theorem was generalized by Aoki [20] for additive mappings and by Th.M. Rassias [94] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Gavruta [51]. After, many interesting results of the generalized Hyers-Ulam stability to a number of functional equations have been investigated by a number of mathematicians; see [6, 17, 31, 69, 70, 72, 73, 74, 77, 103, 104, 105, 114, 115, 116] and the books [42, 95, 96] and references therein.

We have organized this thesis as follows:

Chapter 1: This chapter consists of six sections. In the first section, we present some "*notations and definitions*", in Section 1.2, we present some "*notations and definitions of fractional calculus theory*", in Section 1.3, we present some " *notations and definitions of b-Metric Spaces*" and in Section 1.4, we present some "*definitions and proprieties of noncompactness measure*",.
Finally, in the last Section, we recall some preliminary : some **fixed point theorems**) which are used throughout this thesis.

Chapter 2: This chapter consists of two sections. In the first section, we investigate the existence of solutions for the following class of Caputo-Hadamard fractional differential equation

$$({}^{Hc}D_1^r u)(t) = f(t, u(t), ({}^{Hc}D_1^r u)(t)),\ t \in I := [1, T], \tag{1}$$

with the boundary conditions

$$u(1) = u_1,\ u'(T) = u_T, \tag{2}$$

where $T > 1,\ r \in (1, 2],\ u_1, u_T \in E,\ f : I \times E \times E \to E$ is a given continuous function, E is a real (or complex) Banach space with a norm $\|\cdot\|$, ${}^{Hc}D_1^r$ is the Caputo-Hadamard fractional derivative of order r.
In the second Section , first we investigate the existence and uniqueness of solutions for the following class of boundary value problems of Caputo-Hadamard fractional differential equations with finite delay:

$$\begin{cases} u(t) = \varphi(t);\ t \in [1-h, 1], \\ ({}^{Hc}D_1^r u)(t) = f(t, u_t, ({}^{Hc}D_1^r u)(t));\ t \in I := [1, T], \\ u'(T) = u_T, \end{cases} \tag{3}$$

where $h > 0,\ T > 1,\ r \in (1, 2],\ u_T \in \mathbb{R},\ \varphi \in \mathcal{C}, f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{Hc}D_1^r$ is the Caputo-Hadamard fractional derivative of order r, and $\mathcal{C} := C([1-h, 1], \mathbb{R})$ is the space of continuous functions on $[1-h, 1]$.
For any $t \in I$, we define u_t by

$$u_t(s) = u(t + s - 1);\ \text{for } s \in [1-h, 1].$$

Next, we investigate the following class of Caputo-Hadamard fractional differential equations with infinite delay:

$$\begin{cases} u(t) = \varphi(t);\ t \in (-\infty, 1], \\ ({}^{Hc}D_1^r u)(t) = f(t, u_t, ({}^{Hc}D_1^r u)(t));\ t \in I, \\ u'(T) = u_T, \end{cases} \tag{4}$$

where $\varphi : [-\infty, 0] \to \mathbb{R}, f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\mathcal{B}$ is called a phase space that will be specified later.
For any $t \in I$, we define $u_t \in \mathcal{B}$ by

$$u_t(s) = u(t + s - 1);\ \text{for } s \in (-\infty, 1].$$

In the third subsection, we investigate the following class of Caputo-Hadamard fractional differential equations with state dependent finite delay:

$$\begin{cases} u(t) = \varphi(t);\ t \in [1-h, 1], \\ ({}^{Hc}D_1^r u)(t) = f(t, u_{\rho(t, u_t)}, ({}^{Hc}D_1^r u)(t));\ t \in I, \\ u'(T) = u_T, \end{cases} \tag{5}$$

where $\varphi \in \mathcal{C},\ \rho : I \times \mathcal{C} \to \mathbb{R},\ f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Finally, we consider the following class of Caputo-Hadamard fractional differential equations with state dependent infinite delay:

$$\begin{cases} u(t) = \varphi(t);\ t \in [-\infty, 1], \\ ({}^{Hc}D_1^r u)(t) = f(t, u_{\rho(t,u_t)}, ({}^{Hc}D_1^r u)(t));\ t \in I, \\ u'(T) = u_T, \end{cases} \tag{6}$$

where $\varphi : (-\infty, 0] \to \mathbb{R},\ f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Chapter 3: This chapter consists of two sections. In the first section, we investigate the existence of solutions and some Ulam stability results for the following class of Caputo-Fabrizio fractional differential equation

$$({}^{CF}D_0^r u)(t) = f(t, u(t), ({}^{CF}D_0^r u)(t));\ t \in I := [0, T], \tag{7}$$

with the boundary conditions

$$au(0) + bu(T) = c, \tag{8}$$

where $T > 0,\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, a, b, c are real constants with $a + b \neq 0,\ {}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r \in (0, 1)$.

Next, we discuss the existence of solutions for problem (7)-(8), when $f : I \times E \times E \to E$ is a given continuous function, $c \in E$, and E is a real (or complex) Banach space with a norm $\| \cdot \|$,
In the second Section, first we investigate the the following class of boundary value problems of Caputo-Fabrizio fractional differential equations with finite delay:

$$\begin{cases} \wp(t) = \zeta(t);\ t \in [-h, 0], \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_t, ({}^{CF}D_0^r \wp)(t));\ t \in I := [0, T], \end{cases} \tag{9}$$

where $h > 0,\ T > 0,\ \zeta \in \mathcal{C}, f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r \in (0, 1]$, and $\mathcal{C} := C([-h, 0], \mathbb{R})$ is the space of continuous functions on $[-h, 0]$.
For any $t \in I$, we define u_t by

$$\wp_t(s) = \wp(t + s);\ \text{for } s \in [-h, 0].$$

Next, we investigate the following class of Caputo-Fabrizio fractional differential equations with infinite delay:

$$\begin{cases} \wp(t) = \zeta(t);\ t \in (-\infty, 0], \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_t, ({}^{CF}D_0^r \wp)(t));\ t \in I, \end{cases} \tag{10}$$

where $\zeta : [-\infty, 0] \to \mathbb{R}, f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\mathcal{B}$ is called a phase space that will be specified later.
For any $t \in I$, we define $\wp_t \in \mathcal{B}$ by

$$\wp_t(s) = \wp(t+s);\ \text{for } s \in (-\infty, 0].$$

In the third subsection, we investigate the following class of Caputo-Fabrizio fractional differential equations with state dependent finite delay:

$$\begin{cases} \wp(t) = \zeta(t);\ t \in [-h, 0], \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_{\rho(t,\wp_t)}, (({}^{CF}D_0^r \wp)(t));\ t \in I, \end{cases} \tag{11}$$

where $\zeta \in \mathcal{C},\ \rho : I \times \mathcal{C} \to \mathbb{R},\ f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Finally, we consider the following class of Caputo-Fabrizio fractional differential equations with state dependent infinite delay:

$$\begin{cases} \wp(t) = \zeta(t);\ t \in [-\infty, 0], \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_{\rho(t,\wp_t)}, (({}^{CF}D_0^r \wp)(t));\ t \in I, \end{cases} \tag{12}$$

where $\zeta : (-\infty, 0] \to \mathbb{R},\ f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Chapter 4: This chapter consists of three sections. In the first section, we discuss the existence of solutions for the following class of Katugampola implicit fractional differential equations:

$$\begin{cases} ({}^{\rho}D_{0+}^r \wp)(t) = f(t, \wp(t), ({}^{\rho}D_{0+}^r \wp)(t));\ t \in I := [0, T], \\ ({}^{\rho}I_{0+}^{1-r} \wp)(0) = u_0 \in \mathbb{R}, \end{cases} \tag{13}$$

where $T, \rho > 0,\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{\rho}I_{0+}^r$ is the Katugampola fractional integral of order $r \in (0, 1]$, ${}^{\rho}D_{0+}^r$ is the Katugampola fractional derivative of order r.

In the second Section, we investigate the existence and uniqueness of solutions for the following class of initial value problems of Caputo-Fabrizio fractional differential equations:

$$\begin{cases} ({}^{CF}D_0^r u)(t) = f(t, u(t), ({}^{CF}D_0^r u)(t));\ t \in I := [0, T], \\ u(0) = u_0, \end{cases} \tag{14}$$

where $T > 0,\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r \in (0, 1)$, and $u_0 \in \mathbb{R}$.

In the third section, we discuss the existence of solutions for the following class of Katugampola implicit fractional differential equations:

$$\begin{cases} ({}^\rho D^r_{0+}\wp)(t) = f(t, \wp(t), ({}^\rho D^r_{0+}\wp)(t)); \ t \in I := [0, T], \\ ({}^\rho I^{1-r}_{0+}\wp)(0) = u_0 \in \mathbb{R}, \end{cases} \tag{15}$$

where $T, \rho > 0$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^\rho I^r_{0+}$ is the Katugampola fractional integral of order $r \in (0, 1]$, ${}^\rho D^r_{0+}$ is the Katugampola fractional derivative of order r.

Chapter 1

Basic Ingredients

In this chapter, we discuss the necessary mathematical tools, notations and concepts we need in the succeeding chapters. We look at some essential properties of fractional differential operators. We also review some of the basic properties of measures of noncompactness and fixed point theorems which are crucial in our results regarding fractional differential equations.

1.1 Notations and Definitions

Consider the Banach space $C(I) := C(I, E)$ of continuous functions from I into E equipped with the usual norm

$$\|u\|_\infty := \sup_{t \in I} \|u(t)\|.$$

In the case $E = \mathbb{R}$, we have

$$\|u\|_C = \sup_{t \in I} |u(t)|.$$

By $L^1(I, E)$ we denote the Banach space of measurable function $u : I \to E$ which are Bochner integrable, equipped with the norm

$$\|u\|_{L^1} = \int_0^T \|u(t)\| dt.$$

In the case $E = \mathbb{R}$, we have

$$\|u\|_{L^1} = \int_0^T |u(t)| dt.$$

1.2 Fractional Calculus Theory

In this section, we recall some definitions of fractional integral and fractional differential operators that include all we use throughout this thesis. We conclude it by some necessary lemmata, theorems and properties.

Definition 1.2.1 (Gamma Function [93]) *The gamma function* $\Gamma(z)$ *is defined by the integral :*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$$

which converges in the right half of the complex plane $Re(z) > 0$.
One of the basic properties of the gamma function is that it satisfies the following functional equation:

$$\Gamma(z+1) = z\Gamma(z)$$

so, for positive integer values n, the Gamma function becomes $\Gamma(n) = (n-1)!$ *and thus can be seen as an extension of the factorial function to real values.*
A useful particular value of the function: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, *is used throughout many examples in this thesis.*

Definition 1.2.2 (Hadamard fractional integral [55, 81]) *The Hadamard fractional integral of order* r, *of a function* $h : [1, \infty) \to X$ *is defined as*

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} h(s) \frac{ds}{s}, \quad r > 0,$$

provided that the integral exists.

Definition 1.2.3 (Caputo-Hadamard fractional derivative [50, 81]) *For at least n-times differentiable function* $h : [1, \infty) \to X$, *the Caputo-type Hadamard derivative of fractional order* r *is defined as*

$$D^r h(t) = \frac{1}{\Gamma(n-r)} \int_1^t \left(\log \frac{t}{s}\right)^{n-r-1} \delta^n h(s) \frac{ds}{s};$$

$$n - 1 < r < n,$$

where $\delta = t\left(\frac{d}{dt}\right)$, $\log(.) = \log_e(.)$, *and* $[r]$ *denotes the integer part of the real number* r.

Lemma 1.2.4 ([50]) *Let* $u \in AC^n_\delta[a,b]$ *or* $C^r_\delta[a,b]$ *and* $r \in \mathbb{R}$, *where*

$$X^n_\delta[a,b] = \{h : [a,b] \to C : \delta^{n-1} h(t) \in X[a,b]\}.$$

Then, one has

$$I^r(D^r)u(t) = u(t) - \sum_{k=0}^{n-1} C_k (\log t)^k,$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$, $(n = [r] + 1)$.

Definition 1.2.5 (Caputo-Fabrizio fractional integral[40, 83, 28]) *The Caputo-Fabrizio fractional integral of order* $0 < r < 1$ *for a function* $h \in L^1(I)$ *is defined by*

$$^{CF}I^r h(\tau) = \frac{2(1-r)}{M(r)(2-r)}h(\tau) + \frac{2r}{M(r)(2-r)}\int_0^\tau h(x)dx, \quad \tau \geq 0,$$

where $M(r)$ is normalization constant depending on r.

Definition 1.2.6 (Caputo-Fabrizio fractional derivative [40, 83, 28]) *The Caputo-Fabrizio fractional derivative for a function* $h \in C^1(I)$ *of order* $0 < r < 1$, *is defined by*

$$^{CF}D^r h(\tau) = \frac{(2-r)M(r)}{2(1-r)}\int_0^\tau \exp(-\frac{r}{1-r}(\tau - x))h'(x)dx; \ \tau \in I.$$

Note that $(^{CF}D^r)(h) = 0$ *if and only if* h *is a constant function.*

Definition 1.2.7 (Katugampola fractional integral [75, 82, 29]) *The Katugampola fractional integrals of order* $r > 0$ *and* $\rho > 0$ *of a function* $y \in X_c^p[0, T]$ *is defined by*

$$^{\rho}T^r_{0^+}y(t) = \frac{\rho^{1-r}}{\Gamma(r)}\int_0^t \frac{s^{\rho-1}y(s)}{(t^\rho - s^\rho)^{1-r}}ds, \ t \in I. \tag{1.1}$$

Definition 1.2.8 (Katugampola fractional derivative [75, 76, 29]) *The generalized fractional derivatives of order* $r > 0$ *and* $\rho > 0$ *corresponding to the Katugampola fractional integrals (1.1) defined for any* $t \in I$ *by*

$$\begin{aligned} ^{\rho}D^r_{0^+}y(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n (^{\rho}T^{n-r}_{0^+}y)(t) \\ &= \frac{\rho^{r-n+1}}{\Gamma(n-r)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_0^t \frac{s^{\rho-1}y(s)}{(t^\rho - s^\rho)^{r-n+1}}ds, \end{aligned} \tag{1.2}$$

where $n = [r] + 1$; *if the integrals exist.*

Remark 1.2.9 *([75, 76]) As a basic example, we quote for* $r, \rho > 0$ *and* $\theta > -\rho$,

$$^{\rho}D^r_{0^+}t^\theta = \frac{\rho^{r-1}\Gamma\left(1 + \frac{\theta}{\rho}\right)}{\Gamma\left(1 - r + \frac{\theta}{\rho}\right)}t^{\theta - r\rho}.$$

Giving in particular:

$$^{\rho}D^r_{0^+}t^{\rho(r-i)} = 0, \ for \ each \ i = 1, 2, \ldots, n.$$

In fact, for $r, \rho > 0$ *and* $\theta > -\rho$, *we have:*

$$\begin{aligned} ^{\rho}D^r_{0^+}t^\theta &= \frac{\rho^{r-n+1}}{\Gamma(n-r)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_0^t s^{\rho+\theta-1}(t^\rho - s^\rho)^{n-r-1}ds \\ &= \frac{\rho^{r-1}\Gamma\left(1+\frac{\theta}{\rho}\right)}{\Gamma\left(1+n-r+\frac{\theta}{\rho}\right)}\left[n - r + \frac{\theta}{\rho}\right]\ldots\left[1 - r + \frac{\theta}{\rho}\right]t^{\theta - r\rho} \\ &= \frac{\rho^{r-1}\Gamma\left(1+\frac{\theta}{\rho}\right)}{\Gamma\left(1-r+\frac{\theta}{\rho}\right)}t^{\theta - r\rho}. \end{aligned} \tag{1.3}$$

If we put $i = r - \frac{\theta}{\rho}$, *we obtain from* (1.3)*:*

$$^{\rho}D^{r}_{0+}t^{\theta(r-i)} = \rho^{r-1}\frac{\Gamma\left(r-i+1\right)}{\Gamma\left(n-i+1\right)}(n-i)(n-i-1)\dots(1-m)t^{-\rho i}.$$

So, for each $i = 1, 2, \dots, n$, *we have* $^{\rho}D^{r}_{0+}t^{\rho(r-i)} = 0,\ \forall r, \rho > 0.$

Theorem 1.2.10 *([76]) Let* $r, \rho, c \in \mathbb{R}$, *be such that* $r, \rho > 0$. *Then for any* $f, g \in X^{p}_{c}(I)$, *where* $1 \leq p \leq \infty$, *we have:*
Inverse property:

$$^{\rho}D^{r}_{0+}{}^{\rho}I^{r}_{0+}f(t) = f(t),\ \textit{for all } r \in (0,1]. \tag{1.4}$$

Linearity property: for all $r \in (0,1)$, *we have:*

$$\begin{cases} {}^{\rho}D^{r}_{0+}(f+g)(t) = {}^{\rho} D^{r}_{0+}f(t) + {}^{\rho} D^{r}_{0+}g(t). \\ {}^{\rho}I^{r}_{0+}(f+g)(t) = {}^{\rho} I^{r}_{0+}f(t) + {}^{\rho} I^{r}_{0+}g(t). \end{cases} \tag{1.5}$$

We state the following generalization of ***Gronwall***'s lemma for singular kernels.

Lemma 1.2.11 *([118]) Let* $v : [0, T] \to [0, +\infty)$ *be a real function and* $w(\cdot)$ *is a nonnegative, locally integrable function on* $[0, T]$. *Assume that there are constants* $a > 0$ *and* $0 < \alpha < 1$ *such that*

$$v(t) \leq w(t) + a\int_{0}^{t}(t-s)^{-\alpha}v(s)ds,$$

Then, there exists a constant $K = K(\alpha)$ *such that*

$$v(t) \leq w(t) + Ka\int_{0}^{t}(t-s)^{-\alpha}w(s)ds,\ \textit{for every } t \in [0, T].$$

Bainov and Hristova [21] introduced the following integral inequality of Gronwall type for piecewise continuous functions which can be used in the sequel.

Lemma 1.2.12 *Let for* $t \geq t_0 \geq 0$ *the following inequality hold*

$$x(t) \leq a(t) + \int_{t_0}^{t}g(t,s)x(s)ds + \sum_{t_0<t_k<t}\beta_k(t)x(t_k),$$

where $\beta_k(t)(k \in \mathbb{N})$ *are nondecreasing functions for* $t \geq t_0$, $a \in PC([t_0, \infty), \mathbb{R}_+)$, a *is nondecreasing and* $g(t, s)$ *is a continuous nonnegative function for* $t, s \geq t_0$ *and nondecreasing with respect to* t *for any fixed* $s \geq t_0$. *Then, for* $t \geq t_0$, *the following inequality is valid:*

$$x(t) \leq a(t)\prod_{t_0<t_k<t}(1+\beta_k(t))exp\left(\int_{t_0}^{t}g(t,s)ds\right).$$

Theorem 1.2.13 *[57](theorem of Ascoli-Arzela). Let $A \subset C(J, \mathbb{R})$, A is relatively compact (i.e $\overline{A}$ is compact) if:*

1. *A is uniformly bounded i.e, there exists $M > 0$ such that*

$$|f(x)| < M \text{ for every } f \in A \text{ and } x \in J.$$

2. *A is equicontinuous i.e, for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, \overline{x} \in J$, $|x - \overline{x}| \leq \delta$ implies $|f(x) - f(\overline{x})| \leq \epsilon$, for every $f \in A$.*

1.3 b-Metric Spaces

The notion of b-metric was proposed by Czerwik [44, 45]. Following these initial papers, the existence fixed point for the various classes of operators in the setting of b-metric spaces have been investigated extensively; see [37, 43, 46, 91], and related references therein.

Definition 1.3.1 *[8, 9] Let $c \geq 1$ and M be a set. A distance function $d : M \times M \to \mathbb{R}_+^*$ is called b-metric if for all $\mu, \nu, \xi \in M$, the following are fulfilled:*

- *(bM1) $d(\mu, \nu) = 0$ if and only if $\mu = \nu$;*
- *(bM2) $d(\mu, \nu) = d(\nu, \mu)$;*
- *(bM3) $d(\mu, \xi) \leq c[d(\mu, \nu) + d(\nu, \xi)]$.*

The tripled (M, d, c) is called a b-metric space.

Example 1.3.2 *[8, 9] Let $d : C(I) \times C(I) \to \mathbb{R}_+^*$ be defined by*

$$d(u, v) = \|(u - v)^2\|_\infty := \sup_{t \in I} \|u(t) - v(t)\|^2; \;\; for \;\, all \;\, u, v \in C(I).$$

It is clear that d is a b-metric with $c = 2$.

Example 1.3.3 *[8, 9] Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}_+^*$ be defined by*

$$d(x, y) = |x^2 - y^2|; \;\; for \;\, all \;\, x, y \in X.$$

It is clear that d is not a metric, but it is easy to see that d is a b-metric space with $r \geq 2$.

Let Φ be the set of all increasing and continuous function $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ satisfying the property: $\phi(c\mu) \leq c\phi(\mu) \leq c\mu$, for $c > 1$ and $\phi(0) = 0$. We denote by $\mathcal{F}$ the family of all nondecreasing functions $\lambda : \mathbb{R}_+^* \to [0, \frac{1}{c^2})$ for some $c \geq 1$.

Definition 1.3.4 *[8, 9] For a b-metric space (M, d, c), an operator $T : M \to M$ is called a generalized $\alpha - \phi-$Geraghty contraction type mapping whenever there exists $\alpha : M \times M \to \mathbb{R}_+^*$, and some $L \geq 0$ such that for*

$$D(x,y) = \max\left\{d(x,y), d(x,T(x)), d(y,T(y)), \frac{d(x,T(y)) + d(y,T(x))}{2s}\right\},$$

and

$$N(x,y) = \min\{d(x,y), d(x,T(x)), d(y,T(y))\},$$

we have

$$\alpha(\mu,\nu)\phi(c^3 d(T(\mu),T(\nu)) \leq \lambda(\phi(D(\mu,\nu))\phi(D(\mu,\nu)) + L\psi(N(\mu,\nu); \tag{1.6}$$

for all $\mu, \nu \in M$, where $\lambda \in \mathcal{F}$, $\phi\,\psi \in \Phi$.

Remark 1.3.5 *In the case when $L = 0$ in Definition 1.3.4, and the fact that*

$$d(x,y) \leq D(x,y);\ \textit{for all } x,y \in M,$$

the inequality (1.6) becomes

$$\alpha(\mu,\nu)\phi(c^3 d(T(\mu),T(\nu)) \leq \lambda(\phi(d(\mu,\nu))\phi(d(\mu,\nu)). \tag{1.7}$$

Definition 1.3.6 *[8, 9] Let M be a non empty set, $T : M \to M$, and $\alpha : M \times M \to \mathbb{R}_+^*$ be a given mappings. We say that T is $\alpha-$admissible if for all $\mu, \nu \in M$, we have*

$$\alpha(\mu,\nu) \geq 1 \Rightarrow \alpha(T(\mu),T(\nu)) \geq 1.$$

Definition 1.3.7 *[8, 9] Let (M, d) be a b-metric space and let $\alpha : M \times M \to \mathbb{R}_+^*$ be a function. M is said to be $\alpha-$regular if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_n$ with $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

1.4 Measure of Noncompactness

We define in this Section the Kuratowski (1896-1980) and Hausdorf measures of noncompactness (**MNC** for short) and give their basic properties.

Definition 1.4.1 *([79]) Let (X, d) be a complete metric space and $\mathcal{B}$ the family of bounded subsets of X. For every $B \in \mathcal{B}$ we define the Kuratowski measure of noncompactness $\alpha(B)$ of the set B as the infimum of the numbers d such that B admits a finite covering by sets of diameter smaller than d.*

Remark 1.4.2 *The diameter of a set B is the number* $\sup\{d(x,y) : x \in B, y \in B\}$ *denoted by* $diam(B)$, *with* $diam(\emptyset) = 0$.
It is clear that $0 \leq \alpha(B) \leq diam(B) < +\infty$ *for each nonempty bounded subset B of X and that* $diam(B) = 0$ *if and only if B is an empty set or consists of exactly one point.*

Definition 1.4.3 *([24]) Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map* $\alpha : \Omega_E \to [0, \infty]$ *defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E,$$

where

$$diam(B_i) = sup\{||x - y|| : x, y \in B_i\}.$$

The Kuratowski measure of noncompactness satisfies the following properties:

Lemma 1.4.4 *([13, 24, 25, 79]) Let A and B bounded sets.*

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is relatively compact), where $\overline{B}$ denotes the closure of B.*

(b) nonsingularity : α is equal to zero on every one element-set.

(c) If B is a finite set, then $\alpha(B) = 0$.

(d) $\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$, *where $convB$ is the convex hull of B.*

(e) monotonicity : $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(f) algebraic semi-additivity : $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, *where*

$$A + B = \{x + y : x \in A, \quad y \in B\}.$$

(g) semi-homogencity : $\alpha(\lambda B) = |\lambda|\alpha(B)$; $\lambda \in \mathbb{R}$, *where* $\lambda(B) = \{\lambda x : x \in B\}$.

(h) semi-additivity : $\alpha(A \bigcup B) = max\{\alpha(A), \alpha(B)\}$.

(i) $\alpha(A \bigcap B) = min\{\alpha(A), \alpha(B)\}$.

(j) invariance under translations : $\alpha(B + x_0) = \alpha(B)$ *for any* $x_0 \in E$.

Remark 1.4.5 *The a-measure of noncompactness was introduced by Kuratowski in order to generalize the Cantor intersection theorem*

Theorem 1.4.6 *([79]) Let (X, d) be a complete metric space and $\{B_n\}$ be a decreasing sequence of nonempty, closed and bounded subsets of X and* $\lim_{n\to\infty} \alpha(B_n) = 0$. *Then the intersection B_∞ of all B_n is nonempty and compact.*

In [67], Horvath has proved the following generalization of the Kuratowski theorem:

Theorem 1.4.7 *([79]) Let (X, d) be a complete metric space and $\{B_i\}_{i\in I}$ be a family of nonempty of closed and bounded subsets of X having the finite intersection property. If $\inf_{i\in I} \alpha(B_i) = 0$ then the intersection B_∞ of all B_i is nonempty and compact.*

Lemma 1.4.8 *([54]) If $V \subset C(J, E)$ is a bounded and equicontinuous set, then*

(i) the function $t \to \alpha(V(t))$ is continuous on J, and

$$\alpha_c(V) = \sup_{0\leq t\leq T} \alpha(V(t)).$$

(ii) $\alpha\left(\int_0^T x(s)ds : x \in V\right) \leq \int_0^T \alpha(V(s))ds,$

where

$$V(s) = \{x(s) : x \in V\}, \ s \in J.$$

In the definition of the Kuratowski measure we can consider balls instead of arbitrary sets. In this way we get the definition of the Hausdorff measure:

Definition 1.4.9 *([79]) The Hausdorff measure of noncompactness $\chi(B)$ of the set B is the infimum of the numbers r such that B admits a finite covering by balls of radius smaller than r.*

Theorem 1.4.10 *([79]) Let $B(0, 1)$ be the unit ball in a Banach space X. Then*

$$\alpha(B(0, 1)) = \chi(B(0, 1)) = 0$$

if X is finite dimensional, and $\alpha(B(0, 1)) = 2$, $\chi(B(0, 1)) = 1$ otherwise.

Theorem 1.4.11 *([79]) Let $S(0, 1)$ be the unit sphere in a Banach space X. Then $\alpha(S(0, 1)) = \chi(S(0, 1)) = 0$ if X is finite dimensional, and $\alpha(S(0, 1)) = 2$, $\chi(S(0, 1)) = 1$ otherwise.*

Theorem 1.4.12 *([79]) The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(B) \leq \alpha(B) \leq 2\chi(B).$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

Example 1.4.13 *Let l^∞ be the space of all real bounded sequences with the supremum norm, and let A be a bounded set in l^∞. Then $\alpha(A) = 2\chi(A)$.*

For further facts concerning measures of noncompactness and their properties we refer to [13, 24, 25, 79] and the references therein.

1.5 Some Fixed Point Theorems

In this section we present some fixed point theorems.

Theorem 1.5.1 (Banach's fixed point theorem (1922) [52]) *Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 1.5.2 (Schauder's fixed point theorem[52]) *Let X be a Banach space, D be a bounded closed convex subset of X and $T : D \to D$ be a compact and continuous map. Then T has at least one fixed point in D.*

Theorem 1.5.3 (Schaefer's fixed point theorem[52]) *Let X be a Banach space, and $N : X \longrightarrow X$ completely continuous operator.*
If the set $\mathcal{E} = \{y \in X : \ y = \lambda N y, \ for some \ \lambda \in (0, 1)\}$ is bounded, then N has fixed points.

For our purpose we will only need the following fixed point theorem, and the important Lemma.

Theorem 1.5.4 (Darbo's Fixed Point Theorem [52]) *Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $N : C \to C$ is such that for all closed subsets D of C,*

$$\alpha(T(D)) \leq k\alpha(D), \tag{1.8}$$

where $0 \leq k < 1$, and α is the Kuratowski measure of noncompactness. Then T has a fixed point in C.

Remark 1.5.5 *Mappings satisfying the Darbo-condition (1.8) have subsequently been called k-set contractions.*

Theorem 1.5.6 *[8, 9] Let (M, d) be a complete b-metric space and $T : M \to M$ be a generalized $\alpha - \phi -$Geraghty contraction type mapping such that*

- *(i) T is $\alpha-$admissible;*
- *(ii) there exists $\mu_0 \in M$ such that $\alpha(\mu_0, T(\mu_0)) \geq 1$;*
- *(iii) either T is continuous or M is $\alpha-$regular.*

Then T has a fixed point. Moreover, if

- *(iv) for all fixed points μ, ν of T, either $\alpha(\mu, \nu) \geq 1$ or $\alpha(\nu, \mu) \geq 1$,*

then T has a unique fixed point.

Theorem 1.5.7 (Mönch's Fixed Point Theorem [10, 88]) *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0 \tag{1.9}$$

holds for every subset V of D, then N has a fixed point.
Here α is the Kuratowski measure of noncompactness.

For more details see [10, 18, 79, 52, 119]

Chapter 2

Caputo-Hadamard Implicit Fractional Differential Equations

2.1 Introduction

The purpose of this chapter is the study of two results for a class of existence results for a class of Caputo-Hadamard implicit fractional differential equations with two boundary conditions and for classes of Caputo- Hadamard implicit fractional differential equations with two boundary conditions and delay. The results are based on some fixed point theorems and the concept of measure of noncompactness.

There are different definitions of fractional derivatives available in the literature. However, the most commonly used is the Hadamard fractional derivative given by Hadamard [55]. Butzer et al. [78, 38] studied various properties of Hadamard-type derivatives which aremore generalized than theHadamard fractional derivatives. In this context, the readers are also referred to [76] for a detailed study on generalized fractional derivatives and references therein. Caputo introduced another type of fractional derivative [39] which has an advantage over derivative in a differential equation since it does not require to define the fractional order initial conditions (see, for example, [16]). Recently the authors in [71] utilized Caputo-type modification on Hadamard factional derivatives. Moreover Trujillo et al. in [109] derived Taylor formula with Riemann-Liouville derivatives, and Odibat and Shawagfeh [90] derived the same based on Caputo fractional derivative. Gulsu et al. [53] extended the work of previous authors and proposed a numerical scheme to approximate solutions of relaxation oscillation equation by using the fractional Taylor series. Finally, Fernandez and Baleanu [49] developed mean value theorem and Taylor theorem for certain fractional differential operators.

2.2 Caputo-Hadamard Implicit Fractional Differential Equations with two Boundary Conditions

The outcome of our study in section is the continuation of the problem raised recently in [34], in it, Benchohra et al. investigated existence and Uniqueness Results for Nonlinear Implicit Fractional Differential Equations with Boundary Conditions:

$$\begin{cases} ({}^cD^r u)(t) = f(t, u(t), ({}^cD^r u)(t)), \ for \ each \ t \in I = [0, T], \ T > 0, \ 1 < r \leq 2, \\ u(0) = u_0, \ u'(T) = u_1, \end{cases}$$

where ${}^cD^r$ is the Caputo fractional derivative, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $u_0, u_1 \in \mathbb{R}$.

In this section we investigate the existence of solutions for the following class of Caputo-Hadamard fractional differential equation:

$$({}^{Hc}D_1^r u)(t) = f(t, u(t), ({}^{Hc}D_1^r u)(t)), \ t \in I := [1, T], \tag{2.1}$$

with the boundary conditions:

$$u(1) = u_1, \ u'(T) = u_T, \tag{2.2}$$

where $T > 1, \ r \in (1, 2], \ u_1, u_T \in E, \ f : I \times E \times E \to E$ is a given continuous function, E is a real (or complex) Banach space with a norm $\|\cdot\|$, ${}^{Hc}D_1^r$ is the Caputo-Hadamard fractional derivative of order r.

2.2.1 Existence of Solutions

For the existence of solution for the problem $(2.1) - (2.2)$, we need the following auxiliary lemmas:

Lemma 2.2.1 *Let $h \in C(I)$, and $\alpha \in (1, 2]$. Then the unique solution of problem*

$$\begin{cases} ({}^{Hc}D_1^r u)(t) = h(t); \ t \in I, \\ u(1) = u_1, \ u'(T) = u_T, \end{cases}$$

is given by:

$$u(t) = u_1 + T u_T \log t$$
$$+ \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{h(s)}{s} ds. \tag{2.3}$$

Proof. Solving the equation
$$({}^{Hc}D_1^r u)(t) = h(t),$$
we get
$$u(t) = {}^H I_1^r h(t) + c_0 + c_1 \log t.$$
Thus
$$u'(t) = {}^H I_1^{r-1} h(t) + \frac{c_1}{t}.$$
From the boundary conditions, we get
$$c_0 = u_1, \ and \ c_1 = T u_T - {}^H I_1^{r-1} h(T).$$
Hence, we obtain (2.3).
Conversely, if u satisfies the integral equation (2.3), then
$$\begin{cases} ({}^{Hc}D_1^r u)(t) = h(t); \ t \in I, \\ u(1) = u_1, \quad u'(T) = u_T. \end{cases}$$
In order to prove the main theorems, we list the following hypotheses:

- (H1) The function $f : I \times E \times E \to E$ is continuous.
- (H2)There exist constants $K > 0$ *and* $0 < L < 1$ such that
$$\|f(t,u,v) - f(t,u_1,v_1)\| \leq K\|u - u_1\| + L\|v - v_1\|,$$
for each $u, \ u_1, \ v, \ v_1 \in E$ and $t \in I$.
$$\alpha(f(t,M,N) \leq K\alpha(M) + L\alpha(N).$$

Remark 2.2.2 *[11] The hypothesis (H2) is equivalent to the following hypothesis:*

- *(H3) For each bounded sets $M, N \subset E$ and each $t \in I$,*
$$\alpha(f(t,M,N) \leq K\alpha(M) + L\alpha(N).$$

Theorem 2.2.3 *Assume that $(H1) - (H3)$ hold. If*
$$\ell := \frac{K(\log T)^r(1 + rT)}{(1-L)\Gamma(r+1)} < 1, \tag{2.4}$$
then the problem (2.1)-(2.2) has at least one solution defined on I.

Proof. In order to prove the existence solutions of the problem (2.1)-(2.2), we consider the operator $F:\ C(I,E)\to C(I,E)$ defined by:

$$(Fu)(t)=u_1+Tu_T\log t$$
$$+\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{h(s)}{s}ds-\frac{T\log t}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\frac{h(s)}{s}ds, \tag{2.5}$$

where $h\in C(I,E)$ be such that

$$h(t)=f(t,u(t),h(t)).$$

Let $R>0$ such that

$$R\geq\frac{f^*\left(\log T\right)^r\left(1+Tr\right)+[\|u_1\|+T\log T\|u_T\|]\Gamma(r+1)(1-L)}{\Gamma(r+1)(1-L)-K\left(\log T\right)^r\left(1+Tr\right)}, \tag{2.6}$$

where

$$f^*=\sup_{t\in I}\|f(t,0,0)\|.$$

Define the ball

$$B_R=\{x\in C(I,E),\ \|x\|_\infty\leq R\}.$$

The proof will be given in four steps.

Step 1. F is continuous on B_R.
Let u_n be a sequence such that $u_n\to u$ in B_R. For each $t\in I$, we have

$$\begin{array}{rcl}\|(Fu_n)(t)-(Fu)(t)\| & \leq & \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\|h_n(s)-h(s)\|\frac{ds}{s}\\ & & +\frac{T\log t}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\|h_n(s)-h(s)\|\frac{ds}{s},\end{array} \tag{2.7}$$

where $h_n,\ h\in C(I,E)$ such that

$$h_n(t)=f(t,u_n(t),h_n(t))\quad and\quad h(t)=f(t,u(t),h(t)).$$

By $(H2)$, we have

$$\begin{array}{rcl}\|h_n(t)-h(t)\| & = & \|f(t,u_n(t),h_n(t))-f(t,u(t),h(t))\|\\ & \leq & K\|u_n(t)-u(t)\|+L\|h_n(t)-h(t)\|.\end{array}$$

Then

$$\|h_n(t)-h(t)\|\leq\frac{K}{1-L}\|u_n(t)-u(t)\|.$$

Since $u_n\to u$ then get $h_n(t)\to h(t)$ for each $t\in I$. And let $b>0$ be such that, for each $t\in I$ we have $\|h_n(t)\|\leq b$ and $\|h(t)\|\leq b$. Then, we have

$$\begin{array}{rcl}\frac{(\log\frac{t}{s})^{r-1}}{s}\|h_n(s)-h(s)\| & \leq & \frac{(\log\frac{t}{s})^{r-1}}{s}\|h_n(s)-h(s)\|\\ & \leq & \frac{(\log\frac{t}{s})^{r-1}}{s}(\|h_n(s)\|+\|h(s)\|)\\ & \leq & \frac{2b(\log\frac{t}{s})^{r-1}}{s}.\end{array}$$

Also, we have

$$\begin{array}{rcl}\frac{(\log\frac{T}{s})^{r-2}}{s}\|h_n(s)-h(s)\| & \leq & \frac{(\log\frac{T}{s})^{r-2}}{s}\|h_n(s)-h(s)\|\\ & \leq & \frac{(\log\frac{T}{s})^{r-2}}{s}(\|h_n(s)\|+\|h(s)\|)\\ & \leq & \frac{2b(\log\frac{T}{s})^{r-2}}{s}.\end{array}$$

The function $s\to\frac{2b(\log\frac{T}{s})^{r-2}}{s}$ is integrable on $[1,T]$, and the function $s\to\frac{2b(\log\frac{t}{s})^{r-1}}{s}$ is integrable on $[1,t]$, then the Lebesgue domination convergence theorem and (2.17) imply that

$$\|F(u_n)(t)-F(u)(t)\|\to 0 \quad as \quad n\to\infty.$$

Hence

$$\|F(u_n)-F(u)\|_\infty\to 0 \quad as \quad n\to\infty.$$

hence, F is continuous operator on B_R.

Step 2. $F(B_R)\subset B_R$.
Let $u\in B_R$, From $(H2)$, for each $t\in I$ we have

$$\begin{array}{rcl}\|h(t)\| & \leq & \|f(t,u(t),h(t))-f(t,0,0)+f(t,0,0)\|\\ & \leq & K\|u(t)\|+L\|h(t)\|+f^*\\ & \leq & K\|u\|_\infty+L\|h\|_\infty+f^*\\ & \leq & KR+L\|h\|_\infty+f^*.\end{array}$$

Then

$$\|h\|_\infty\leq\frac{f^*+KR}{1-L}:=\lambda.$$

Thus,

$$\begin{array}{rcl}\|(Fu)(t)\| & \leq & \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\|h(t)\|\frac{ds}{s}\\ & + & \frac{T\log T}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\|h(t)\|\frac{ds}{s}+\|u_1\|+T\log T\|u_T\|\\ & \leq & \frac{\lambda}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{ds}{s}+\frac{\lambda T\log T}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\frac{ds}{s}+\|u_1\|+T\log T\|u_T\|\\ & \leq & \frac{\lambda(\log T)^r(1+Tr)}{\Gamma(r+1)}+\|u_1\|+T\log T\|u_T\|.\end{array}$$

Hence

$$\|F(u)\|_\infty\leq R.$$

Consequently, $F(B_R)\subset B_R$.

Step 3. $F(B_R)$ is equicontinuous
For $1\leq t_1\leq t_2\leq T$, and $u\in B_R$, we have

$$\|(Fu)(t_1)-(Fu)(t_2)\|$$

$$\begin{array}{rl} \leq & \left\| \frac{1}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] h(s) \frac{ds}{s} \right. \\ + & \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-2} h(s) \frac{ds}{s} \\ & \left. + \frac{T(\log t_2 - \log t_1)}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} h(s) \frac{ds}{s} + T u_T \left[\log t_2 - \log t_1\right] \right\| \\ \leq & \frac{\lambda}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] \frac{ds}{s} + \frac{\lambda}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-2} \frac{ds}{s} \\ & + \frac{\lambda T(\log t_2 - \log t_1)}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{ds}{s} + T\|u_T\| \left[\log t_2 - \log t_1\right]. \end{array}$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero, we conclude that $F(B_R)$ is equicontinuous.

Step 4. Let $A \in B_R$ and $t \in I$, we have

$$\begin{array}{rcl} \alpha((FA)(t)) = \alpha\{(Fu)(t),\ u \in A\} & \leq & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \{\alpha(h(s)),\ u \in A\} \frac{ds}{s} \\ & & + \frac{T \log T}{\Gamma(r)} \int_1^T \left(\log \frac{t}{s}\right)^{r-2} \{\alpha(h(s)),\ u \in A\} \frac{ds}{s}, \end{array}$$

where $h \in C(I, E)$ such that $h(t) = f(t, u(t), h(t))$.
Hypothesis $(H3)$ and Lemma 1.4.4 imply that, for each $s \in I$,

$$\begin{array}{rcl} \alpha(\{h(s),\ u \in A\}) & = & \alpha(\{f(s, u(s), h(s)),\ u \in A\}) \\ & \leq & K\alpha(\{u(s),\ u \in A\}) + L\alpha(\{h(s),\ u \in A\}). \end{array}$$

Thus

$$\alpha(\{h(s),\ u \in A\}) \leq \frac{K}{1-L} \alpha(\{V(s),\ u \in A\}).$$

Hence

$$\begin{array}{rcl} \alpha(F(A)(t)) & \leq & \frac{K}{(1-L)\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \{\alpha(u(s)),\ u \in A\} \frac{ds}{s} \\ & & + \frac{K}{1-L} \frac{T \log T}{\Gamma(r)} \int_1^T \left(\log \frac{t}{s}\right)^{r-2} \{\alpha(u(s),\ u \in A\} \frac{ds}{s} \\ & \leq & \frac{K(\log T)^r}{(1-L)\Gamma(r+1)} \alpha_c(A) + \frac{TK(\log T)^r}{(1-L)\Gamma(r)} \alpha_c(A) \\ & \leq & \frac{K(\log T)^r(1+rT)}{(1-L)\Gamma(r+1)} \alpha_c(A). \end{array}$$

This implies that

$$\alpha_c(F(A)) \leq \ell \alpha_c(A).$$

Therefore, the condition (2.4) implies that F is a contraction. From Darbo's fixed Point theorem, there exists a fixed point u of F in B_R, which is a solution of problem (2.1)-(2.2).

The following result is based on Mönch's fixed point theorem.

Theorem 2.2.4 *Assume taht $(H1) - (H3)$ and the condition (2.4) hold. Then the problem (2.1)-(2.2) has at leat one solution defined on I.*

Proof. Consider the operator $F : C(I, E) \to C(I, E)$ defined in (2.5). As in the proof of Theorem 2.2.3, $F : B_R \to B_R$ is continuous, and $F(B_R)$ is bounded and equicontinuous.

Next, For every subset V of B_R, and each $t \in I$, we have

$$\begin{aligned}
\alpha(F(V)(t)) &\leq \tfrac{K}{(1-L)}\left\{\tfrac{1}{\Gamma(r)}\int_1^t \left(\log\tfrac{t}{s}\right)^{r-1}\{\alpha(u(s))\}\tfrac{ds}{s},\ u \in A\right\} \\
&\quad + \tfrac{K}{(1-L)}\left\{\tfrac{T\log T}{\Gamma(r)}\int_1^T \left(\log\tfrac{t}{s}\right)^{r-2}\{\alpha(u(s))\}\tfrac{ds}{s},\ u \in V\right\} \\
&\leq \tfrac{K(\log T)^r}{(1-L)\Gamma(r+1)}\alpha_c(V) + \tfrac{TK(\log T)^r}{(1-L)\Gamma(r)}\alpha_c(A) \\
&\leq \tfrac{K(\log T)^r(1+rT)}{(1-L)\Gamma(r+1)}\alpha_c(V(t)).
\end{aligned}$$

That is

$$\alpha((FV)(t) \leq \ell\alpha_c(V(t)).$$

From Mönch condition, we have

$$\alpha(V(t)) \leq \alpha(conv((F(V)) \cup \{0\}) = \alpha(F(V)) \leq \ell\alpha_c(V(t)).$$

Thus

$$\alpha_c(V(t)) \leq \ell\alpha_c(V(t)),$$

which implies that

$$\alpha_c(V(t)) = 0,$$

and gives $\alpha(V(t)) = 0$. for each $t \in I$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_R. Hence, from Mönch's fixed point Theorem, there exists a fixed point u of operator F, which is a solution of the problem (2.1)-(2.2).

2.2.2 An Example

Let

$$l^1 = \left\{u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty\right\},$$

be the Banach space with the norm

$$\|u\|_{l^1} = \sum_{n=1}^{\infty} |u_n|.$$

consider the following problem:

$${}^{Hc}D_1^{\frac{3}{2}} u_n(t) = \frac{2^{-n}}{180(1+\|u\|_{l^1})}\sin(u_n(t))$$

$$+\frac{2^{-n}}{60(1+\|^{Hc}D_1^{\frac{3}{2}}u\|_{l^1})}\cos\left(^{Hc}D_1^{\frac{3}{2}}u_n(t)\right);\ t\in[1,e], \tag{2.8}$$

$$u_n(1)=u_n'(e)=0, \tag{2.9}$$

where $f=(f_1,f_2,\ldots,f_n,\ldots)$, $u=(u_1,u_2,\ldots,u_n,\ldots)$,
$^{Hc}D_1^{\frac{3}{2}}u=(^{Hc}D_1^{\frac{3}{2}}u_1,^{Hc}D_1^{\frac{3}{2}}u_2,\ldots,^{Hc}D_1^{\frac{3}{2}}u_n,\ldots)$, and

$$f_n(t,u,v)=\frac{2^{-n}}{180(1+\|u\|_{l^1})}\sin(u_n)+\frac{2^{-n}}{60(1+\|v\|_{l^1})}\cos(v_n);$$

for $t\in[1,e]$ and $u,v\in l^1$.
Clearly, the function f is continuous.
For any $u,\widetilde{u},v,\widetilde{v}\in l^1$ and $t\in[1,e]$, we have

$$\begin{aligned}\|f(t,u,v)-f(t,\widetilde{u},\widetilde{v})\|_{l^1} &= \sum_{n=1}^{\infty}|f_n(t,u,v)-f_n(t,\widetilde{u},\widetilde{v})|\\
&\le \tfrac{\|u-\widetilde{u}\|_{l^1}}{180}\sum_{n=1}^{\infty}2^{-n}|\sin(u_n)-\sin(\widetilde{u_n})|\\
&+ \tfrac{\|v-\widetilde{v}\|_{l^1}}{60}\sum_{n=1}^{\infty}2^{1-n}|\cos(v_n)-\cos(\widetilde{v}_n)|\\
&\le \tfrac{\|u-\widetilde{u}\|_{l^1}}{180}\sum_{n=1}^{\infty}2^{1-n}+\frac{\|v-\widetilde{v}\|_{l^1}}{60}\sum_{n=1}^{\infty}2^{1-n}\\
&\le \tfrac{1}{90}\|u-\widetilde{u}\|_{l^1}+\tfrac{1}{30}\|v-\widetilde{v}\|_{l^1}.\end{aligned}$$

Thus

$$\|f(t,u,v)-f(t,\widetilde{u},\widetilde{v})\|_{l^1}\le\frac{1}{90}\|u-\widetilde{u}\|_E+\frac{1}{30}\|v-\widetilde{v}\|_{l^1}.$$

Hence the hypothesis $(H2)$ is satisfied with

$$K=\frac{1}{90}\ \ and\ \ L=\frac{1}{30}.$$

Next, the condition (2.4) is satisfies with $T=e\ \ and\ \ r=\frac{3}{2}$. Indeed,

$$\frac{K(\log T)^r(1+rT)}{(1-L)\Gamma(r+1)}=\frac{\frac{1}{90}(1+\frac{3}{2}e)}{(1-\frac{1}{30})\Gamma(\frac{5}{2})}=\frac{1+\frac{3}{2}e}{29\sqrt{\pi}}<1.$$

Simple computations show that all conditions of Theorem 2.2.3 are satisfied. It follows that the problem (2.8)-(2.9) has at least one solution defined on $[1,e]$.

2.3 Caputo-Hadamard Implicit Fractional Differential Equations with Delay

Functional differential equations with delay have become an active area of research, and appear frequently in applications as model of equations. Several phenomena in engineering, physics and life sciences can be described by means of differential equations with delay [1, 56, 58, 59, 61, 62, 63, 65, 66, 112, 113]. We can cite some papers subject of fractional differential equations with finite delay [1, 31], infinite delay [3, 58]. The study of functional differential equations with state-dependent delay has received great attention in the last year; see for instance [3, 4, 5, 61, 62, 63]. The literature related to functional differential quations with state-dependent delay is limited, some papers by Hernández werw considered iin the space of Lipschitz functions. In [61, 62, 63], the authors studied some abstract differential equations with state-dependent delay.

Inspired by the above works, in this section, first we investigate the existence and uniqueness of solutions for the following class of boundary value problems of Caputo-Hadamard fractional differential equations with finite delay:

$$\begin{cases} u(t) = \varphi(t);\ t \in [1-h, 1], \\ ({}^{Hc}D_1^r u)(t) = f(t, u_t, ({}^{Hc}D_1^r u)(t));\ t \in I := [1, T], \\ u'(T) = u_T, \end{cases} \tag{2.10}$$

where $h > 0,\ T > 1,\ r \in (1, 2],\ u_T \in \mathbb{R},\ \varphi \in \mathcal{C}, f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{Hc}D_1^r$ is the Caputo-Hadamard fractional derivative of order r, and $\mathcal{C} := C([1-h, 1], \mathbb{R})$ is the space of continuous functions on $[1-h, 1]$.
For any $t \in I$, we define u_t by

$$u_t(s) = u(t + s - 1);\ \text{for } s \in [1-h, 1].$$

Next, we investigate the following class of Caputo-Hadamard fractional differential equations with infinite delay:

$$\begin{cases} u(t) = \varphi(t);\ t \in (-\infty, 1], \\ ({}^{Hc}D_1^r u)(t) = f(t, u_t, ({}^{Hc}D_1^r u)(t));\ t \in I, \\ u'(T) = u_T, \end{cases} \tag{2.11}$$

where $\varphi : [-\infty, 0] \to \mathbb{R}, f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\mathcal{B}$ is called a phase space that will be specified later.
For any $t \in I$, we define $u_t \in \mathcal{B}$ by

$$u_t(s) = u(t + s - 1);\ \text{for } s \in (-\infty, 1].$$

In the third subsection, we investigate the following class of Caputo-Hadamard fractional differential equations with state dependent finite delay:

$$\begin{cases} u(t) = \varphi(t); \ t \in [1-h, 1], \\ ({}^{Hc}D_1^r u)(t) = f(t, u_{\rho(t,u_t)}, ({}^{Hc}D_1^r u)(t)); \ t \in I, \\ u'(T) = u_T, \end{cases} \tag{2.12}$$

where $\varphi \in \mathcal{C}, \ \rho : I \times \mathcal{C} \to \mathbb{R}, \ f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Finally, we consider the following class of Caputo-Hadamard fractional differential equations with state dependent infinite delay:

$$\begin{cases} u(t) = \varphi(t); \ t \in [-\infty, 1], \\ ({}^{Hc}D_1^r u)(t) = f(t, u_{\rho(t,u_t)}, ({}^{Hc}D_1^r u)(t)); \ t \in I, \\ u'(T) = u_T, \end{cases} \tag{2.13}$$

where $\varphi : (-\infty, 0] \to \mathbb{R}, \ f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In the last section, we present some examples illustrating the presented results.

2.3.1 Existence of Solutions

Consider the Banach space $C(I) := C(I, \mathbb{R})$ of continuous real functions on $I := [1, T]$ equipped with the usual norm

$$\|u\|_\infty := \sup_{t \in I} |u(t)|.$$

Also, $C := C([1-h, T])$ is a Banach space with the norm

$$\|u\|_C := \sup_{t \in [1-h, T]} |u(t)|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}$, and by $L^1(I)$ we denote the space of measurable real functions $v : I \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|v\|_1 = \int_I |v(t)| dt.$$

Lemma 2.3.1 *Let $f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then problem (2.10) is equivalent to the problem of obtaining the solutions of the integral equation:*

$$\begin{cases} u(t) = \varphi(t); \ t \in [1-h, 1], \\ g(t) = f(t, u_t, g(t)); \ t \in I, \end{cases}$$

with

$$u_t(\tau) = \varphi(1) + Tu_T \log(t+\tau-1) + \frac{1}{\Gamma(r)}\int_1^{t+\tau-1}\left(\log\frac{t+\tau-1}{s}\right)^{r-1}\frac{g(s)}{s}ds$$

$$-\frac{T\log(t+\tau-1)}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\frac{g(s)}{s}ds;\ t\in I,\ \tau\in[1-h,1],$$

and if $g(\cdot)\in C(I)$, is the solution of this equation, then

$$\begin{cases} u(t)=\varphi(t);\ t\in[1-h,1],\\ u(t)=\varphi(1)+Tu_T\log t+\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{g(s)}{s}ds-\frac{T\log t}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\frac{g(s)}{s}ds;\ t\in I. \end{cases}$$

Existence of Solutions with Finite Delay

In this subsubsection, we establish the existence results for problem (2.10).

Definition 2.3.2 *By a solution of problem (2.10), we mean a function $u\in C$ such that*

$$u(t)=\begin{cases} \varphi(t);\ t\in[1-h,1],\\ \varphi(1)+Tu_T\log t+\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\frac{g(s)}{s}ds-\frac{T\log t}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\frac{g(s)}{s}ds;\ t\in I, \end{cases}$$

where $g\in C(I)$ with $g(t)=f(t,u_t,g(t))$.

The following hypotheses will be used in the sequel.

- (H_1) There exist constantes $\omega_1>0,\ 0<\omega_2<1$ such that:

  $$|f(t,u_1,v_1)-f(t,u_2,v_2)|\le\omega_1\|u_1-u_2\|_{[1-h,1]}+\omega_2|v_1-v_2|,$$

  for any $u_1,u_2\in\mathcal{C},\ v_1,v_2\in\mathbb{R}$, and each $t\in I$.

- (H_2) There exist constants $K>0$ *and* $0<L<1$ such that

  $$|f(t,u,v)|\le K\|u\|_{[1-h,1]}+L|v|,$$

  for any $u\in\mathcal{C},\ v\in\mathbb{R}$, and each $t\in I$.

Theorem 2.3.3 *Assume that the hypothesis (H_1) holds. If*

$$\phi:=\frac{\omega_1(1+rT)(\log T)^r}{(1-\omega_2)\Gamma(1+r)}<1, \tag{2.14}$$

then problem (2.10) has a unique solution on $[1-h,T]$.

Proof. Consider the operator $N: \ C \to C$ defined by:

$$(Nu)(t) = \begin{cases} \varphi(t); \ t \in [1-h, 1], \\ \varphi(1) + Tu_T \log t + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds \\ -\frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds; \ t \in I, \end{cases} \tag{2.15}$$

where $g \in C(I)$ such that $g(t) = f(t, u_t, g(t))$.

Let $u, v \in C(I)$. Then, for each $t \in [1-h, 1]$, we have

$$|(Nu)(t) - (Nv)(t)| = 0,$$

and for each $t \in I$, we have

$$\begin{aligned} |(Nu)(t) - (Nv)(t)| \ &\leq \ \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |g(s) - h(s)| \frac{ds}{s} \\ &\quad + \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} |g(s) - h(s)| \frac{ds}{s}, \end{aligned}$$

where $g, \ h \in C(I)$ such that

$$g(t) = f(t, u_t, g(t)) \quad and \quad h(t) = f(t, v_t, h(t)).$$

From (H_1), we have

$$\begin{aligned} |g(t) - h(t)| \ &= \ |f(t, u_t, g(t)) - f(t, v_t, h(t))| \\ &\leq \ \omega_1 \|u_t - v_t\|_{[1-h,1]} + \omega_2 |g(t) - h(t)|. \end{aligned}$$

This gives,

$$|g(t) - h(t)| \leq \frac{\omega_1}{1 - \omega_2} \|u_t - v_t\|_{[1-h,1]}.$$

Thus, for each $t \in I$, we get

$$\begin{aligned} |(Nu)(t) - (Nv)(t)| \ &\leq \ \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{\omega_1}{1-\omega_2} \|u_s - v_s\|_{[1-h,1]} \frac{ds}{s} \\ &\quad + \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{\omega_1}{1-\omega_2} \|u_s - v_s\|_{[1-h,1]} \frac{ds}{s} \\ &\leq \ \phi \left(\|u_t - v_t\|_{[1-h,1]}\right) \\ &\leq \ \frac{\omega_1}{1-\omega_2} \left(\frac{(\log T)^r}{\Gamma(1+r)} + \frac{T \log T (\log T)^{r-1}}{\Gamma(r)} \right) \|u - v\|_C \\ &\leq \ \frac{\omega_1 (1 + rT)(\log T)^r}{(1-\omega_2)\Gamma(1+r)} \|u - v\|_C \\ &\leq \ \phi \|u - v\|_C. \end{aligned}$$

Hence, we get

$$\|N(u) - N(v)\|_C \leq \phi \|u - v\|_C.$$

Consequently, from Theorem 1.5.1, the operator N has a unique fixed point, which is the unique solution of our problem (2.10) on $[1-h, T]$.

Theorem 2.3.4 *Assume that the hypothesis* (H_2) *holds. If*

$$\frac{K(1+Tr)(\log T)^r}{(1-L)\Gamma(1+r)} < 1,$$

Then problem (2.10) *has at least one solution on* $[1-h, T]$.

Proof. Consider the operator $N : C \to C$ be the operator defined in (2.15).
Let $R > 0$ such that

$$R \geq \max\left\{ \|\varphi\|_{C([1-h,0],\mathbb{R})} \frac{|\varphi(1)| + T|u_T| \log T}{1 - \frac{K(1+Tr)(\log T)^r}{(1-L)\Gamma(1+r)}} \right\}. \tag{2.16}$$

Define the ball

$$B_R = \{x \in C(I, \mathbb{R}),\ \|x\|_C \leq R\}.$$

We prove that the operator $N : B_R \to B_R$ satisfies all conditions of Theorem 1.5.2. The proof will be given in three steps.

Step 1. N is continuous .
Let u_n be a sequence such that $u_n \to u$ in B_R. For each $t \in [1-h, 1]$, we have

$$|(Nu_n)(t) - (Nu)(t)| = 0,$$

and for each $t \in I$, we have

$$\begin{aligned} |(Nu_n)(t) - (Nu)(t)| \;\leq\; & \tfrac{1}{\Gamma(r)} \textstyle\int_1^t \left(\log \tfrac{t}{s}\right)^{r-1} |g_n(s) - g(s)| \tfrac{ds}{s} \\ & + \tfrac{T \log t}{\Gamma(r-1)} \textstyle\int_1^T \left(\log \tfrac{T}{s}\right)^{r-2} |g_n(s) - g(s)| \tfrac{ds}{s}, \end{aligned} \tag{2.17}$$

where $g_n,\ g \in C(I, \mathbb{R})$ such that

$$g_n(t) = f(t, u_{nt}, g_n(t)) \quad \textit{and} \quad g(t) = f(t, u_t, g(t)).$$

Since $\|u_n - u\|_C \to 0$ as $n \to \infty$ and f, g and g_n are continuous, then the Lebesgue dominated convergence theorem, implies that

$$\|N(u_n) - N(u)\|_C \to 0 \quad \text{as } n \to \infty.$$

Hence, N is continuous.

Step 2. $N(B_R) \subset B_R$.
Let $u \in B_R$, If $t \in [1-h, 1]$ then $\|(Nu)(t)\| \leq \|\varphi\|_C \leq R$, and from (H_2), for each $t \in I$, we have

$$\begin{aligned} |g(t)| &\leq |f(t, u_t, g(t))| \\ &\leq K\|u_t\|_{[1-h,1]} + L|g(t)| \\ &\leq K\|u\|_C + L\|g\|_\infty \\ &\leq KR + L\|g\|_\infty. \end{aligned}$$

Then
$$\|g\|_\infty \leq \frac{RK}{1-L}.$$
Thus,
$$\begin{array}{rcl}
|(Nu)(t)| & \leq & \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}|g(t)|\frac{ds}{s} \\
& + & \frac{T\log T}{\Gamma(r-1)}\int_1^T \left(\log\frac{T}{s}\right)^{r-2}|g(t)|\frac{ds}{s} + |\varphi(1)| + T\log T|u_T| \\
& \leq & \frac{\|h\|_\infty}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{ds}{s} \\
& & +\frac{\|h\|_\infty T\log T}{\Gamma(r-1)}\int_1^T \left(\log\frac{T}{s}\right)^{r-2}\frac{ds}{s} + |\varphi(1)| + T|u_T|\log T \\
& \leq & \frac{RK(1+Tr)(\log T)^r}{(1-L)\Gamma(1+r)} + |\varphi(1)| + T|u_T|\log T \\
& \leq & R.
\end{array}$$
Hence
$$\|N(u)\|_C \leq R.$$
Consequently, $N(B_R) \subset B_R$.

Step 3. $N(B_R)$ is equicontinuous
For $1 \leq t_1 \leq t_2 \leq T$, and $u \in B_R$, we have
$$\begin{array}{rcl}
|N(u)(t_1) - N(u)(t_2)| & \leq & \Big|\frac{1}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1} - \left(\log\frac{t_1}{s}\right)^{r-1}\right]g(s)\frac{ds}{s} \\
& & +\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-2}g(s)\frac{ds}{s} \\
& & +\frac{T(\log t_2-\log t_1)}{\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}g(s)\frac{ds}{s} \\
& & +Tu_T\left[\log t_2 - \log t_1\right]\Big| \\
& \leq & \frac{RK}{(1-L)\Gamma(r)}\int_1^{t_1}[\left(\log\frac{t_2}{s}\right)^{r-1} - \left(\log\frac{t_1}{s}\right)^{r-1}]\frac{ds}{s} \\
& & +\frac{RK}{(1-L)\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-2}\frac{ds}{s} \\
& & +\frac{RKT(\log t_2-\log t_1)}{(1-L)\Gamma(r-1)}\int_1^T\left(\log\frac{T}{s}\right)^{r-2}\frac{ds}{s} \\
& & +T|u_T|(\log t_2 - \log t_1).
\end{array}$$
As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero, we conclude that $N(B_R)$ is equicontinuous.

As a consequence of the above three steps with the Arzelá-Ascoli theorem, we can conclude that N is continuous and compact. From an application of Theorem 1.5.2, we deduce that N has at least a fixed point which is a solution of problem (2.10).

Existence of Solutions with Infinite Delay

In this subsubsection, we establish some existence results for problem (2.11). Let the space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 1]$ into $\mathbb{R}$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato [58] for ordinary differential functional equations:

(A_1) If $u : (-\infty, T] \to \mathbb{R}$, and $u_0 \in \mathcal{B}$, then there are constants $L, M, H > 0$, such that for any $t \in I$ the following conditions hold:

- (i) *u_t is in $\mathcal{B}$,*
- (ii) $\|u_t\|_{\mathcal{B}} \le K\|u_1\|_{\mathcal{B}} + M \sup_{s\in[1,t]} |u(s)|$,
- (iii) $\|u(t)\| \le H\|u_t\|_{\mathcal{B}}$.

(A_2) For the function $u(\cdot)$ *in* $(A1)$, u_t is a $\mathcal{B}-$ valued continuous function on I.

(A_3) The space $\mathcal{B}$ is complete.

Consider the space

$$\Omega = \{u : (-\infty, T] \to \mathbb{R},\ u|_{(-\infty,1]} \in \mathcal{B},\ u|_I \in C(I)\}.$$

Definition 2.3.5 *By a solution of problem (2.11), we mean a continuous function $u \in \Omega$*

$$u(t) = \begin{cases} \varphi(t);\ t \in (-\infty, 1], \\ \varphi(1) + Tu_T \log t \\ +\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds;\ t \in I, \end{cases} \tag{2.18}$$

where $g \in C(I, E)$ such that $g(t) = f(t, u_t, g(t))$.

The following hypotheses will be used in the sequel.

- (H_{01}) The function f satisfies the Lipschitz condition:

  $$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le b_1\|u_1 - u_2\|_{\mathcal{B}} + b_2|v_1 - v_2|,$$

  for any $u_1, v_1 \in \mathcal{B}$, $u_2, v_2 \in \mathbb{R}$, and each $t \in I$, where $b_1 > 0$ and $0 < b_2 < 1$.

- (H_{02}) There exist constants $B_1 > 0$ *and* $0 < B_2 < 1$ such that

  $$|f(t, u, v)| \le B_1\|u\|_{\mathcal{B}} + B_2|v|,$$

  for any $u \in \mathcal{B},\ v \in \mathbb{R},$, and each $t \in I$.

First, we prove an existence and uniqueness result by using the Banach's fixed point theorem.

Theorem 2.3.6 *Assume that the hypothesis (H_{01}) holds. If*

$$\lambda := \frac{Mb_1(1 + rT)(\log T)^r}{(1 - b_2)\Gamma(1 + r)} < 1, \tag{2.19}$$

then problem (2.11) has a unique solution on $(-\infty, T]$.

Proof. Consider the operator $N_1 : \ \Omega \to \Omega$ defined by:

$$(N_1 u)(t) = \begin{cases} \varphi(t); \ t \in (-\infty, 1], \\ \varphi(1) + T u_T \log t \\ + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds; \ t \in I, \end{cases} \tag{2.20}$$

where $g \in C(I)$ such that $g(t) = f(t, u_t, g(t))$.
Let $x(\cdot) : (-\infty, T] \to \mathbb{R}$ be a function defined by

$$x(t) = \begin{cases} \varphi(t); & t \in (-\infty, 1], \\ \varphi(1) + T u_T \log t; & t \in I. \end{cases}$$

Then $x_0 = \varphi$, For each $z \in C(I)$, with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z} = \begin{cases} 0; & t \in t \in (-\infty, 1], \\ z(t), & t \in I. \end{cases}$$

If $u(\cdot)$ satisfies the integral equation

$$\begin{array}{rcl} u(t) & = & \varphi(1) + T u_T \log t \\ & + & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds. \end{array}$$

We can decompose $u(\cdot)$ as $u(t) = \overline{z}(t) + x(t)$; for $t \in I$, which implies that $u_t = \overline{z}_t + x_t$ for every $t \in I$, and the function $z(\cdot)$ satisfies

$$z(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds,$$

where

$$g(t) = f(t, \overline{z}_t + x_t, g(t)); \ t \in I.$$

Set

$$C_0 = \{z \in C(I); \ z_0 = 0\},$$

and let $\| \cdot \|_T$ be the seminorm in C_0 defined by

$$\|z\|_T = \|z_0\|_{\mathcal{B}} + \sup_{t \in I} |z(t)| = \sup_{t \in I} |z(t)|; \ z \in C_0.$$

C_0 is a Banach space with norm $\| \cdot \|_T$. Define the operator $P : C_0 \to C_0$; by

$$(Pz)(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds, \tag{2.21}$$

where

$$g(t) = f(t, \overline{z}_t + x_t, g(t)); \ t \in I.$$

Thus, the operator N has a fixed point is equivalent to P has a fixed point. We turn to proving that P has a fixed point. We shall show that $P : C_0 \to C_0$ is a contraction map. Let $z, z' \in C_0$, then we have for each $t \in I$

$$\begin{aligned} |P(z)(t) - P(z')(t)| \leq\ & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |g(s) - h(s)| \frac{ds}{s} \\ & + \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} |g(s) - h(s)| \frac{ds}{s}, \end{aligned} \tag{2.22}$$

where $g,\ h \in C(I)$ such that

$$g(t) = f(t, \overline{z}_t + x_t, g(t)) \quad and \quad h(t) = f(t, \overline{z'}_t + x_t, h(t)).$$

Since, for each $t \in I$, we have

$$|g(t) - h(t)| \leq \frac{b_1}{1 - b_2} \|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}},$$

then, for each $t \in I$; we get

$$\begin{aligned} |P(z)(t) - P(z')(t)| \leq\ & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{b_1}{1-b_2} \|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}} \frac{ds}{s} \\ & + \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{b_1}{1-b_2} \|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}} \frac{ds}{s} \\ \leq\ & \frac{b_1}{1-b_2} \left(\frac{(\log T)^r}{\Gamma(1+r)} + \frac{T \log T (\log T)^{r-1}}{\Gamma(r)} \right) \|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}} \\ \leq\ & \frac{b_1}{1-b_2} \left(\frac{(\log T)^r}{\Gamma(1+r)} + \frac{T \log T (\log T)^{r-1}}{\Gamma(r)} \right) M \sup_{t\in I}\ |\overline{z}(t) - \overline{z'}(t)\|_{\mathcal{B}} \\ \leq\ & \frac{M b_1 (1+rT)(\log T)^r}{(1-b_2)\Gamma(1+r)} \|\overline{z} - \overline{z'}\|_T \\ \leq\ & \lambda \|\overline{z} - \overline{z'}\|_T. \end{aligned}$$

Thus, we get

$$\|P(z)(t) - P(z')(t)\|_T \leq \lambda \|\overline{z} - \overline{z'}\|_T.$$

Hence, from Theorem 1.5.1, the operator P has a unique fixed point. Consequently, N has a unique fixed point which is the unique solution of problem (2.11).

Now, we prove an existence result by using the Scheafer's fixed point theorem.

Theorem 2.3.7 *Assume that the hypothesis* (H_{02}) *holds. Then problem (2.11) has at least one solution on* $(-\infty, T]$.

Proof. Let $P : C_0 \to C_0$ defined as in (2.21), For each given $R > 0$, we define the ball

$$B_R = \{x \in C_0,\ \|x\|_T \leq R\}.$$

We prove that the operator $P : C_0 \to C_0$ satisfies all conditions of Theorem 1.5.3. The proof will be given in four steps.

Step 1. P is continuous .
Let z_n be a sequence such that $z_n \to z$ in C_0. For each $t \in I$, we have

$$\begin{aligned} |(Pz_n)(t) - (Pz)(t)| \leq & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |g_n(s) - g(s)| \frac{ds}{s} \\ & + \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} |g_n(s) - g(s)| \frac{ds}{s}, \end{aligned} \tag{2.23}$$

where $g_n,\ g \in C(I)$ such that

$$g_n(t) = f(t, \overline{z}_{nt} + x_t, g_n(t)) \quad and \quad g(t) = f(t, \overline{z}_t + x_t, g(t)).$$

Since $\|z_n - z\|_T \to 0$ as $n \to \infty$ and f, g and g_n are continuous, then the Lebesgue dominated convergence theorem, implies that

$$\|P(u_n) - P(u)\|_T \to 0 \quad \text{as } n \to \infty.$$

Hence, P is continuous.

Step 2. $P(B_R)$ is bounded.
Let $z \in B_R$, for each $t \in I$, we have

$$\begin{aligned} |g(t)| & \leq |f(t, \overline{z}_t + x_t, g(t))| \\ & \leq B_1 \|\overline{z}_t + x_t\|_{\mathcal{B}} + B_2 |g(t)| \\ & \leq B_1 \left[\|\overline{z}_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}}\right] + B_2 \|g\|_\infty \\ & \leq B_1 M R + B_1 K \|\varphi\|_{\mathcal{B}} + B_2 \|g\|_\infty. \end{aligned}$$

Then

$$\|g\|_\infty \leq \frac{B_1 M R + B_1 K \|\varphi\|_{\mathcal{B}}}{1 - B_2}.$$

Thus,

$$\begin{aligned} |(Pz)(t)| & \leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |g(t)| \frac{ds}{s} + \frac{T \log T}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} |g(t)| \frac{ds}{s} \\ & \leq \frac{\|g\|_\infty}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{ds}{s} + \frac{\|g\|_\infty T \log T}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{ds}{s} \\ & \leq \frac{[B_1 M R + B_1 K \|\varphi\|_{\mathcal{B}}](1 + Tr)(\log T)^r}{(1 - B_2)\Gamma(1 + r)} \\ & := \ell. \end{aligned}$$

Hence

$$\|P(z)\|_T \leq \ell.$$

Consequently, P maps bounded sets into bounded sets in C_0.

Step 3. $P(B_R)$ is equicontinuous.

For $1 \leq t_1 \leq t_2 \leq T$, and $z \in B_R$, we have

$$\begin{array}{rcl} |P(z)(t1) & - & P(z)(t2)| \leq \Big| \frac{1}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] g(s) \frac{ds}{s} \\ & & + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-2} g(s) \frac{ds}{s} \\ & & + \frac{T(\log t_2 - \log t_1)}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} g(s) \frac{ds}{s} \Big| \\ & \leq & \frac{[B_1 MR + B_1 K \|\varphi\|_{\mathcal{B}}](1+Tr)(\log T)^r}{(1-B_2)\Gamma(r)} \int_1^{t_1} [\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1}] \frac{ds}{s} \\ & & + \frac{[B_1 MR + B_1 K \|\varphi\|_{\mathcal{B}}](1+Tr)(\log T)^r}{((1-B_2)\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-2} \frac{ds}{s} \\ & & + \frac{[B_1 MR + B_1 K \|\varphi\|_{\mathcal{B}}](1+Tr)(\log T)^r T(\log t_2 - \log t_1)}{(1-B_2)\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{ds}{s}. \end{array}$$

As $t_2 \to t_1$ the rigth-hand side of the above inequality tends to zero, we conclude that P maps bounded sets into equicontinuous sets in C_0.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $P : C_0 \to C_0$ is completely continuous.

Step 4. A priori bounds.
We prove that the set

$$\mathcal{E} = \{u \in C_0 : u = \lambda P(u);\ for\ some\ \lambda \in (0,1)\}$$

is bounded. Let $z \in C_0$. Let $u \in C_0$, such that $z = \lambda P(z);\ for\ some\ \lambda \in (0,1)$. Then for each $t \in I$, we have

$$z(t) = \lambda(Pz)(t) = \frac{\lambda}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds - \frac{\lambda T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds.$$

From (H_{02}) we have

$$\begin{array}{rcl} |g(t)| & \leq & |f(t, \overline{z}_t + x_t, g(t))| \\ & \leq & B_1 \|\overline{z}_t + x_t\|_{\mathcal{B}} + B_2 |g(t)| \\ & \leq & B_1 [\|\overline{z}_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}}] + B_2 \|g\|_\infty \\ & \leq & B_1 M \|z\|_T + B_1 K \|\varphi\|_{\mathcal{B}} + B_2 \|g\|_\infty. \end{array}$$

This gives,

$$\|g\|_\infty \leq \frac{B_1 M \|z\|_T + B_1 K \|\varphi\|_{\mathcal{B}}}{1 - B_2} := \beta.$$

Thus, for each $t \in I$, we obtain

$$\begin{array}{rcl} |z(t)| & \leq & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |g(t)| \frac{ds}{s} + \frac{T \log T}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} |g(t)| \frac{ds}{s} \\ & \leq & \frac{\beta}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{ds}{s} + \frac{\beta T \log T}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{ds}{s} \\ & \leq & \frac{\beta (\log T)^r (1+Tr)}{\Gamma(r+1)} \\ & \leq & \ell'. \end{array}$$

Hence

$$\|z\|_T \leq \ell'.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 1.5.3, the operator N has a fixed point which is a solution of problem (2.11).

Existence Results with State-Dependent Delay (The Finite Delay Case)

In this subsubsection, we establish the existence results for problem (2.12).

Definition 2.3.8 *By a solution of problem (2.12), we mean a continuous function $u \in C$ such that*

$$u(t) = \begin{cases} \varphi(t); \ t \in [1-h, 1], \\ \varphi(1) + Tu_T \log t + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds; \ t \in I, \end{cases}$$

where $g \in C(I)$ with $g(t) = f(t, u_{\rho(t,u_t)}, g(t))$.

The following hypotheses will be used in the sequel.

- (H_3) The function f satisfies the Lipschitz condition:

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \omega_3 \|u_1 - u_2\|_{[1-h,1]} + \omega_4 |v_1 - v_2|,$$

  for any $u_1, v_1 \in C([1-h, 1], \mathbb{R}), \ u_2, v_2 \in \mathbb{R}$, and each $t \in I$,
  where $\omega_3 > 0, \ 0 < \omega_4 < 1$.

- (H_4) There exist constants $A_1 > 0$ *and* $0 < A_2 < 1$ such that

$$|f(t, u, v)| \leq A_1 \|u\|_{[1-h,1]} + A_2 |v|,$$

  for any $u \in C([1-h, 1], \mathbb{R}), \ v \in \mathbb{R}$, and each $t \in I$.

As in Theorems 2.3.3 and 2.3.4, we give without proof, the following results:

Theorem 2.3.9 *Assume that the hypothesis (H_3) holds. If*

$$\frac{\omega_3(1 + rT)(\log T)^r}{(1 - \omega_4)\Gamma(1 + r)} < 1,$$

then problem (2.12) has a unique solution on $[1-h, T]$.

Theorem 2.3.10 *Assume that the hypothesis (H_4) holds. If*

$$\frac{A_1(1 + Tr)(\log T)^r}{(1 - A2)\Gamma(1 + r)} < 1,$$

then problem (2.12) has at least one solution on $[1-h, T]$.

Existence Results with State-Dependent Delay (The Infinite Delay Case)

Now, we establish the last problem (2.13).

Definition 2.3.11 *By a solution of problem (2.13), we mean a continuous $u \in \Omega$*

$$u(t) = \begin{cases} \varphi(t);\ t \in (-\infty, 1], \\ \varphi(1) + Tu_T \log t \\ +\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{g(s)}{s} ds - \frac{T \log t}{\Gamma(r-1)} \int_1^T \left(\log \frac{T}{s}\right)^{r-2} \frac{g(s)}{s} ds;\ t \in I, \end{cases} \tag{2.24}$$

where $g \in C(I)$ such that $g(t) = f(t, u_{\rho(t,u_t)}, g(t))$.

Set

$$R' := R'_{\rho^-} = \{\rho(t,u) : t \in I,\ u \in \mathcal{B}\ \rho(t,u) < 0\}$$

We always assume that $\rho : I \times \mathcal{B} \to \mathbb{R}$ is continuous and the function $t \to u_t$ is continuous from R' into $\mathcal{B}$. We will need the following hypothesis:

(H_φ) There exists a continuous bounded function $L : R'_{\rho^-} \to (0, \infty)$ such that

$$\|\varphi_t\|_{\mathcal{B}} \leq L(t)\|\varphi\|_{\mathcal{B}},\ for\ any\ t \in R'.$$

In the sequel we will make use of the following generalization of a consequence of the phase space axioms.

Lemma 2.3.12 *If $u \in \Omega$ then*

$$\|u_t\|_{\mathcal{B}} = (M + L')\|\varphi\|_{\mathcal{B}} + K \sup_{\theta \in [0, \max\{0,t\}]} \|u(\theta)\|,$$

where

$$L' = \sup_{t \in R'} L(t).$$

The following hypotheses will be used in the sequel.

- (H_{03}) The function f satisfies the Lipschitz condition:

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq b_3\|u_1 - u_2\|_{\mathcal{B}} + b_4|v_1 - v_2|,$$

  for any $u_1, v_1 \in \mathcal{B}$, $u_2, v_2 \in \mathbb{R}$, and each $t \in I$, where $b_3 > 0$ and $0 < b_4 < 1$.

- (H_{04}) There exist constants $B_3 > 0$ *and* $0 < B_4 < 1$ such that

$$|f(t,u,v)| \leq B_3\|u\|_{\mathcal{B}} + B_4|v|,$$

  for any $u \in \mathcal{B},\ v \in \mathbb{R}$, and each $t \in I$.

As in Theorems 2.3.6 and 2.3.7, we give without proof, the following results:

Theorem 2.3.13 *Assume that the hypothesis* (H_{03}) *holds. If*

$$\frac{Mb_3(1+rT)(\log T)^r}{(1-b_4)\Gamma(1+r)} < 1,$$

then problem (2.13) has a unique solution on $(-\infty, T]$.

Theorem 2.3.14 *Assume that the hypotheses* (H_φ) *and* (H_{04}) *hold. Then problem (2.13) has at least one solution on* $(-\infty, T]$.

2.3.2 Some Examples

Example 1. Consider the following problem

$$\begin{cases} u(t) = 1 + t^2 : \ t \in [-1, 1], \\ ({}^{Hc}D_1^{3/2}u)(t) = \frac{1}{90(1+\|u_t\|)} + \frac{1}{30\left(1+|({}^{Hc}D_1^{3/2}u(t))|\right)}; \ t \in [1, e], \\ u'(e) = 1. \end{cases} \tag{2.25}$$

Set

$$f(t, u, v) = \frac{1}{90\,(1 + \|u_t\|)} + \frac{1}{30\,(1 + |v|)}; \ t \in [1, e], \ u \in \mathcal{C}, \ v \in \mathbb{R}.$$

Clearly, the function f is continuous. For any $u, \widetilde{u} \in \mathcal{C}, \ v, \widetilde{v} \in \mathbb{R}$, and $t \in [1, e]$, we have

$$|f(t, u, v) - f(t, \widetilde{u}, \widetilde{v})| \leq \frac{1}{90}\|u - \widetilde{u}\|_{[1-h,1]} + \frac{1}{30}|v - \widetilde{v}|.$$

Hence hypothesis (H_1) is satisfied with

$$\omega_1 = \frac{1}{90} \quad and \quad \omega_2 = \frac{1}{30}.$$

Next, condition (2.14) is satisfied with $T = e$ and $r = \frac{3}{2}$. Indeed,

$$\begin{aligned} \frac{\omega_1(1+rT)(\log T)^r}{(1-\omega_2)\Gamma(1+r)} &\leq \frac{\frac{1}{90}(1+\frac{3}{2}e)}{(1-\frac{1}{30})\Gamma(\frac{5}{3})} \\ &= \frac{1+\frac{3}{2}e}{29\sqrt{\pi}} \\ &< 1. \end{aligned}$$

Simple computations show that all conditions of Theorem 2.3.3 are satisfied. It follows that problem (2.25) has a unique solution defined on $[-1, e]$.

Example 2. Consider now the following problem

$$\begin{cases} u(t) = t; \ t \in [-\infty, 1], \\ ({}^{Hc}D_1^{3/2}u)(t) = \frac{u_t e^{-\gamma t+t}}{180(e^t - e^{-t})(1+\|u_t\|)} + \frac{u(t)e^{-\gamma t+t}}{60(e^t-e^{-t})\left(1+|({}^{Hc}D_1^{3/2}u(t))|\right)}; \ t \in [1, e], \\ u'(e) = 1. \end{cases} \tag{2.26}$$

Let γ be a positive real constant and

$$B_\gamma = \{u \in C((-\infty, 1], \mathbb{R},) : \lim_{\theta \to -\infty} e^{\gamma\theta\theta} u(\theta) \ exists \ in \ \mathbb{R}\}. \tag{2.27}$$

The norm of B_γ is given by

$$\|u\|_\gamma = \sup_{\theta \in (-\infty, 1]} e^{\gamma\theta} |u(\theta)|.$$

Let $u : (-\infty, 1] \to \mathbb{R}$ be such that $u_0 \in B_\gamma$. Then

$$\begin{aligned} \lim_{\theta\to-\infty} e^{\gamma\theta} u_t(\theta) &= \lim_{\theta\to-\infty} e^{\gamma\theta} u(t+\theta-1) = \lim_{\theta\to-\infty} e^{\gamma(\theta-t+1)} u(\theta) \\ &= e^{\gamma(-t+1)} \lim_{\theta\to-\infty} e^{\gamma(\theta)} u_1(\theta) < \infty. \end{aligned}$$

Hence $u_t \in B_\gamma$. Finally we prove that

$$\|u_t\|_\gamma \leq K\|u_1\|_\gamma + M \sup_{s\in[1,t]} |u(s)|,$$

where $K = M = 1$ and $H = 1$. We have

$$\|u_t(\theta)\| = |u(t+\theta-1|.$$

If $t + \theta \leq 1$, we get

$$\|u_t(\beta)\| \leq \sup_{s\in(-\infty,1]} |u(s)|.$$

For $t + \theta \geq 1$, then we have

$$\|u_t(\beta)\| \leq \sup_{s\in[1,t]} |u(s)|.$$

Thus for all $t + \theta \in I$, we get

$$\|u_t(\beta)\| \leq \sup_{s\in(-\infty,1]} |u(s)| + \sup_{s\in[1,t]} |u(s)|.$$

Then

$$\|u_t\|_\gamma \leq \|u_1\|_\gamma + \sup_{s\in[1,t]} |u(s)|.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ a phase space. Set

$$f(t,u,v) = \frac{e^{-\gamma t+t}}{180\left(e^t - e^{-t}\right)\left(1+\|u\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\left(e^t - e^{-t}\right)\left(1+|v|\right)};\ t\in[1,e],\ u\in B_\gamma, v\in\mathbb{R}.$$

For any $u,\in B_\gamma,\ v\in\mathbb{R}$ and $t\in[1,e]$, we have

$$|f(t,u,v)| \le \frac{1}{180}\|u\|_{B_\gamma} + \frac{1}{60}|v|.$$

Hence hypothesis (H_{02}) is satisfied with

$$B_1 = \frac{1}{180} \quad and \quad B_2 = \frac{1}{60}.$$

Simple computations show that all conditions of Theorem 2.3.7 are satisfied. It follows that problem (2.26) has at least one solution defined on $(-\infty, e]$.

Example 3. We consider the following problem

$$\begin{cases} u(t) = 1+t^2;\ t\in[-1,1], \\ ({}^{Hc}D_1^{3/2}u)(t) = \frac{1}{90(1+|u(t-\sigma(u(t)))|)} + \frac{1}{30\left(1+|({}^{Hc}D_1^{3/2}u(t))|\right)};\ t\in[1,e], \\ u'(e) = 1, \end{cases} \tag{2.28}$$

where $\sigma\in C(\mathbb{R},[1,e])$. Set

$$\rho(t,\varphi) = t - \sigma(\varphi(0)), \quad (t,\varphi)\in[1,e]\times C([-1,1],\mathbb{R}),$$

$$f(t,u,v) = \frac{1}{90(1+|u(t-\sigma(u(t)))|)} + \frac{1}{30\left(1+|v(t)|\right)};\ t\in[1,e],\ u\in\mathcal{C},\ v\in\mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $u,\widetilde{u}\in\mathcal{C},\ v,\widetilde{v}\in\mathbb{R}$ and $t\in[1,e]$, we have

$$|f(t,u,v) - f(t,\widetilde{u},\widetilde{v})| \le \frac{1}{90}\|u-\widetilde{u}\|_{[1-h,1]} + \frac{1}{30}|v-\widetilde{v}|.$$

Hence hypothesis (H_3) is satisfied with

$$\omega_3 = \frac{1}{90} \quad and \quad \omega_4 = \frac{1}{30}$$

Next, we can see that the condition

$$\frac{\omega_3(1+rT)(\log T)^r}{(1-\omega_4)\Gamma(1+r)} < 1,$$

is satisfied with $T=e$ and $r=\frac{3}{2}$.

Simple computations show that all conditions of Theorem 2.3.9 are satisfied. It follows that problem (2.28) has a unique solution defined on $[-1,e]$.

Example 4. Finally, we consider now the following problem

$$\begin{cases} u(t) = t;\ t \in [-\infty, 1], \\ ({}^{Hc}D_1^{3/2}u)(t) = \frac{u(t-\lambda(u(t))e^{-\gamma t+t}}{180(e^t-e^{-t})(1+|u(t-\sigma(u(t))|)} + \frac{u(t)e^{-\gamma t+t}}{60(e^t-e^{-t})\left(1+|({}^{Hc}D_1^{3/2}u(t))|\right)};\ t \in [1, e], \\ u'(e) = 1. \end{cases} \tag{2.29}$$

Let γ be a positive real constant and the phase space B_γ defined in Example 2.

Define

$$\rho(t, \varphi) = t - \lambda(\varphi(0)), \quad (t, \varphi) \in [1, e] \times B_\gamma,$$

and set

$$f(t, u, v) = \frac{e^{-\gamma t+t}}{180\left(e^t - e^{-t}\right)\left(1 + \|u\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\left(e^t - e^{-t}\right)\left(1 + |v|\right)};\ t \in [1, e],\ u \in B_\gamma,\ v \in \mathbb{R}.$$

Simple computations show that all conditions of Theorem 2.3.7 are satisfied. It follows that problem (2.29) has at least one solution defined on $(-\infty, e]$.

Chapter 3

Implicit Caputo-Fabrizio Fractional Differential Equations

3.1 Introduction

The purpose of this chapter is the study of two results for a class of existence and Ulam stability results for a class of Boundary Value Problem for Implicit Caputo-Fabrizio Fractional Differential Equations and for Caputo- Fabrizio implicit fractional differential equations with two boundary conditions and delay. The results are based on some fixed point theorems and the concept of measure of noncompactness.

In recent times, a new fractional differential operator having a kernel with exponential decay has been introduced by Caputo and Fabrizio [40]. This approach of fractional derivative is known as the Caputo-Fabrizio operator which has attracted many research scholars due to the fact that it has a non-singular kernel. Several mathematicians were recently busy in development of Caputo-Fabrizio fractional differential equations, see; [27, 85, 117, 80], and the references therein.

3.2 Implicit Caputo-Fabrizio Fractional Differential Equations

The outcome of our study in section is the continuation of the problem raised recently in [7], in it, Abbas et al. discuss the existence, uniqueness and Ulam-Hyers-Rassias stability of solutions for the following implicit fractional $q-$difference equation:

$$\begin{cases} ({}^cD_q^r u)(t) = f(t, u(t), ({}^cD_q^r u)(t)); \ t \in I = [0, T], \\ u(0) = u_0, \end{cases}$$

where $q \in (0,1)$, $r \in (0,1]$, ${}^cD_q^r$ is the Caputo fractional $q-$difference derivative of order r and $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $u_0, u_1 \in \mathbb{R}$.

In this section we investigate the existence of solutions and some Ulam stability results for the following class of Caputo-Fabrizio fractional differential equation:

$$({}^{CF}D_0^r u)(t) = f(t, u(t), ({}^{CF}D_0^r u)(t));\ t \in I := [0,T], \tag{3.1}$$

with the boundary conditions:

$$au(0) + bu(T) = c, \tag{3.2}$$

where $T > 0$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, a, b, c are real constants with $a + b \neq 0$, ${}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r \in (0,1)$.

Next, we discuss the existence of solutions for problem (3.1)-(3.2), when $f : I \times E \times E \to E$ is a given continuous function, $c \in E$, and E is a real (or complex) Banach space with a norm $\|\cdot\|$.

3.2.1 Existence of Solutions and Ulam Stability Results

Let $\mathcal{M}_X$ denote the class of all bounded subsets of a metric space X.

Lemma 3.2.1 *Let $h \in L^1(I, E)$. A function $u \in \mathcal{C}$ is a solution of problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) = h(t); \quad t \in I := [0,T] \\ au(0) + bu(T) = c, \end{cases} \tag{3.3}$$

if and only if u satisfies the following integral equation

$$u(t) = C_0 + a_r h(t) + b_r \int_0^t h(s)ds + \frac{bb_r}{a+b}\int_0^T h(s)ds. \tag{3.4}$$

$$a_r = \frac{2(1-r)}{(2-r)M(r)},\ b_r = \frac{2r}{(2-r)M(r)},$$
$$C_0 = \frac{1}{a+b}[c - ba_r(h(T) - h(0))] - a_r h(0).$$

proof. Suppose that u satisfies (3.3). From Proposition 1 in [85]; the equation

$$({}^{CF}D_0^r u)(t) = h(t),$$

implies that

$$u(t) - u(0) = a_r(h(t) - h(0)) + b_r \int_0^t h(s)ds.$$

Thus,

$$u(T) = u(0) + a_r(h(T) - h(0)) + b_r \int_0^T h(s)ds.$$

From the mixed boundary conditions $au(0) + bu(T) = c$, we get

$$au(0) + b(u(0) + a_r(h(T) - h(0)) + b_r \int_0^T h(s)ds) = c.$$

Hence,

$$u(0) = \frac{c - b(a_r(h(T) - h(0)) - b_r \int_0^T h(s)ds)}{a+b}.$$

So; we get (3.4).
Conversely, if u satisfies (3.4), then $({}^{CF}D_0^r u)(t) = h(t)$; for $t \in I$,
and $au(0) + bu(T) = c$.

Lemma 3.2.2 *A function u is a solution of problem (3.1)-(3.2), if and only if u satisfies the following integral equation*

$$u(t) = c_0 + a_r g(t) + b_r \int_0^t g(s)ds + \frac{bb_r}{a+b} \int_0^T g(s)ds, \tag{3.5}$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$ and

$$c_0 = \frac{1}{a+b}[c - ba_r(g(T) - g(0))] - a_r g(0).$$

Let $\epsilon > 0$ and $\Phi : I \to \mathbb{R}_+$ be a continuous function. We consider the following inequalities

$$\|({}^{HF}D_0^r u)(t) - f(t, u(t), ({}^{HF}D_0^r u)(t))\| \leq \epsilon, \ t \in I. \tag{3.6}$$

$$\|({}^{HF}D_0^r u)(t) - f(t, u(t), ({}^{HF}D_0^r u)(t))\| \leq \Phi(t), \ t \in I. \tag{3.7}$$

$$\|({}^{HF}D_0^r u)(t) - f(t, u(t), ({}^{HF}D_0^r u)(t))\| \leq \epsilon\Phi(t), \ t \in I. \tag{3.8}$$

Definition 3.2.3 *[2] The problem(3.1)-(3.2) is Ulam–Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (3.6), there exists a solution $v \in \mathcal{C}$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq \epsilon c_f, \ t \in I.$$

Definition 3.2.4 *[2] The problem (3.1)-(3.2) is generalized Ulam–Hyers stable if there exists $c_f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $c_f(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (3.6), there exists a solution $v \in \mathcal{C}$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq c_f(\epsilon), \ t \in I.$$

Definition 3.2.5 *[2] The problem (3.1)-(3.2) is Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (3.8), there exists a solution $v \in \mathcal{C}$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq \epsilon c_{f,\Phi}\Phi(t), \ t \in I.$$

Definition 3.2.6 *[2] The problem (3.1)-(3.2) is generalized Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each solution $u \in \mathcal{C}$ of the inequality (3.7), there exists a solution $v \in \mathcal{C}$ of (3.1)-(3.2) with*

$$\|u(t) - v(t)\| \leq c_{f,\Phi}\Phi(t), \ t \in I.$$

Remark 3.2.7 *A function $u \in \mathcal{C}$ is a solution of the inequality (3.7) if and only if there exist a function $h \in \mathcal{C}$ (which depend on u) such that*

$$\|h(t)\| \leq \Phi(t),$$
$$({}^{HF}D_0^r u)(t) = f(t, u(t), ({}^{HF}D_0^r u)(t)) + h(t), \ for \ t \in I.$$

Lemma 3.2.8 *If $u \in$ is a solution of the inequality (3.7) then u is a solution of the following integral inequality*

$$\begin{aligned} &\left\| u(t) - c_0 - a_r g(t) - b_r \int_0^t g(s)ds - \frac{bb_r}{a+b}\int_0^T g(s)ds \right\| \\ &\quad \leq \left(a_r + Tb_r + T\frac{bb_r}{a+b}\right)\Phi(t), \ if \ t \in I, \end{aligned} \tag{3.9}$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$ and

$$c_0 = \frac{1}{a+b}[c - ba_r(g(T) - g(0))] - a_r g(0).$$

Proof. By remark 3.2.7, for $t \in I$ we have

$$u(t) = C_0 + a_r[g(t) + h(t)] + b_r \int_0^t [g(s) + h(s)]ds + \frac{bb_r}{a+b}\int_0^T [g(s) + h(s)]ds.$$

Thus, we obtain

$$\begin{aligned} \|u(t) \ &- \ C_0 - a_r g(t) - b_r \textstyle\int_0^t g(s)ds - \frac{bb_r}{a+b}\int_0^T g(s)ds\| \\ &\leq \ a_r\|h(t)\| + b_r \textstyle\int_0^t \|h(s)\|ds + \frac{bb_r}{a+b}\int_0^T \|h(s)\|ds \\ &\leq \ \left(a_r + Tb_r + T\frac{bb_r}{a+b}\right)\Phi(t). \end{aligned}$$

Hence, we get (3.9).

Definition 3.2.9 *By a solution of problem (3.1)-(3.2), we mean a function $u \in \mathcal{C}$ such that*

$$u(t) = c_0 + a_r g(t) + b_r \int_0^t g(s)ds + \frac{bb_r}{a+b}\int_0^T g(s)ds,$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$ and

$$c_0 = \frac{1}{a+b}[c - ba_r(g(T) - g(0))] - a_r g(0).$$

3.2.2 The Scalar Case

The following hypotheses will be used in the sequel:

(H_1) There exist a nondecreasing continuous function $\psi : \mathbb{R}_+ \to (0, \infty)$ and continuous functions $p, q : I \to \mathbb{R}_+$ such that

$$|f(t, u, v)| \leq p(t)\psi(|u|) + q(t)|v|, \text{ for each } t \in I \ u, v \in \mathbb{R}.$$

(H_2) There exists a constant $R > 0$, such that

$$R \geq |c_0| + \left[a_r + Tb_r + T\frac{bb_r}{a+b}\right]\frac{p^*\psi(R)}{1-q^*}, \tag{3.10}$$

where $p^* = \sup_{t\in I} p(t)$, and $q^* = \sup_{t\in I} q(t)$, with $0 < q^* < 1$.

(H_3) There exist constants $d_1 > 0,\ 0 < d_2 < 1$, such that

$$(1 + |u_1 - u_2|)|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq d_1\Phi(t)|u_1 - u_2| + d_2|v_1 - v_2|,$$

for each $t \in I$ and $u_i, v_i \in \mathbb{R};\ i = 1, 2$.

(H_4) There exists a constant $\lambda_\Phi > 0$, such that for each $t \in I$ we have

$$\int_0^T \Phi(t)dt \leq \lambda_\Phi \Phi(t).$$

Remark 3.2.10 *From (H_3), for a each $t \in I$, and $u \in \mathbb{R}$, we have that*

$$|f(t, u, v)| \leq |f(t, 0, 0)| + d_1\Phi(t)|u| + d_2|v|.$$

So, (H_3) implies (H_1) with

$$\psi(x) = 1 + x,\ p(t) = \max\{d_1\Phi(t), |f(t, 0, 0)|\},\ q(t) = d_2.$$

Now, we prove an existence result for the problem (3.1)-(3.2) based on Schauder's fixed point theorem.

Theorem 3.2.11 *Assume that the hypotheses (H_1) and (H_2) hold. Then the problem (3.1)-(3.2) has a least one solution defined on I.*

Proof. Consider the operator $N : \mathcal{C} \to \mathcal{C}$ such that,

$$(Nu)(t) = c_0 + a_r g(t) + b_r\int_0^t g(s)ds + \frac{bb_r}{a+b}\int_0^T g(s)ds, \tag{3.11}$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$ and

$$c_0 = \frac{1}{a+b}[c - ba_r(g(T) - g(0))] - a_r g(0).$$

Consider the ball $B_R := \{u \in \mathcal{C} : \|u\|_C \leq R\}$. Let $u \in B_R$ From (H_1), for each $t \in I$, we have

$$\begin{array}{rcl} |g(t)| & = & |f(t, u(t), g(t))| \\ & \leq & p(t)\psi(\|u\|_C) + q(t)|g(t)| \\ & \leq & p^*\psi(R) + q^*\|g\|_C. \end{array}$$

Thus, from (H_2) we get

$$\|g\|_C \leq \frac{p^*\psi(R)}{1 - q^*}. \tag{3.12}$$

Next, we have

$$\begin{array}{rcl} |(Nu)(t)| & \leq & |c_0| + |a_r g(t)| + |b_r \int_0^t g(s)ds| + |\frac{bb_r}{a+b} \int_0^T g(s)ds| \\ & \leq & |c_0| + a_r|g(t)| + b_r \int_0^t |g(s)|ds + \frac{bb_r}{a+b} \int_0^T |g(s)|ds \\ & \leq & |c_0| + \left[a_r + Tb_r + T\frac{bb_r}{a+b}\right] \frac{p^*\psi(R)}{1-q^*} \\ & \leq & R. \end{array}$$

Hence

$$\|N(u)\|_C \leq R.$$

This proves that N transforms the ball B_R into itself.
We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 1.5.2. The proof will be given in two steps.

Step 1. *$N : B_R \to B_R$ is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I$, we have

$$\begin{array}{rcl} |(Nu_n)(t) - (Nu)(t)| & \leq & |a_r(g_n(t) - g(t))| \\ & & +|b_r \int_0^t (g_n(s) - g(s)ds| \\ & & + \left|\frac{bb_r}{a+b} \int_0^T (g_n(s) - g(s))ds\right|, \end{array} \tag{3.13}$$

where $g_n,\ g \in \mathcal{C}$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)) \quad \textit{and} \quad g(t) = f(t, u(t), g(t)).$$

Since$\|u_n - u\|_C \to 0$ as $n \to \infty$ and f, g and g_n are continuous, then the Lebesgue dominated convergence theorem, implies that

$$\|N(u_n) - N(u)\|_C \to 0 \quad \text{as } n \to \infty.$$

Hence, the operator N is continuous.

Step 2. *$N(B_R)$ is bounded and equicontinuous.*
Since $N(B_R) \subset B_R$ and B_R is bounded, then $N(B_R)$ is bounded.

Next, let $t_1, t_2 \in I$, with $0 \le t_1 \le t_2 \le T$, and let $u \in B_R$. Then we have

$$\begin{aligned}
|(Nu)(t_2) - (Nu)(t_1)| &\le |a_r g(t_2) + b_r \textstyle\int_0^{t_2} g(s)ds + \frac{bb_r}{a+b} \int_0^T g(s)ds - a_r g(t_1) \\
&\quad - b_r \textstyle\int_0^{t_1} g(s)ds - \frac{bb_r}{a+b} \int_0^T g(s)ds| \\
&\le |a_r g(t_2) + b_r \textstyle\int_0^{t_2} g(s)ds - a_r g(t_1) + b_r \int_{t_1}^0 g(s)ds| \\
&\le a_r |g(t_2) - g(t_1)| + b_r \textstyle\int_{t_1}^{t_2} |g(s)|ds \\
&\le a_r |g(t_2) - g(t_1)| + b_r (t_2 - t_1) \|g\|_C.
\end{aligned}$$

Since $\|g\|_C \le \frac{p^*\psi(R)}{1-q^*}$, in view to (3.12), we obtain

$$|(Nu)(t_2) - (Nu)(t_1)| \le a_r |g(t_2) - g(t_1)| + b_r (t_2 - t_1) \frac{p^*\psi(R)}{1 - q^*}.$$

As $t_2 \to t_1$ the continuity of g implies that the right-hand side of the above inequality tends to zero.

As a consequence of the above two steps, together with the Ascoli–Arzelá theorem, we can conclude that $N : B_R \to B_R$ is continuous and compact. From an application of Theorem 1.5.2, we deduce that N has a fixed point u which is a solution of problem (3.1)- (3.2).

Now, we are concerned with the generalized Ulam–Hyers–Rassias stability of problem (3.1)-(3.2).

Theorem 3.2.12 *Assume that the hypotheses $(H_2) - (H_4)$ hold. Then the problem (3.1)-(3.2) has at least one solution defined on I and it is generalized Ulam–Hyers–Rassias stable.*

Proof. From Remark 3.2.10, there exists a solution v of the problem (3.1)-(3.2). That is

$$v(t) = c_h + a_r g(t) + b_r \int_0^t h(s)ds + \frac{bb_r}{a+b} \int_0^T h(s)ds,$$

where $h \in \mathcal{C}$, with $h(t) = f(t, v(t), h(t))$ and

$$c_h = \frac{1}{a+b}[c - ba_r(h(T) - h(0))] - a_r h(0).$$

Let u be a solution of the inequality (3.7), then from Lemma 3.2.8, u is a solution of the integral inequality (3.9), that is

$$\left| u(t) - c_g - a_r g(t) - b_r \int_0^t g(s)ds - \frac{bb_r}{a+b} \int_0^T g(s)ds \right|$$

$$\leq \left(a_r + Tb_r + T\frac{bb_r}{a+b}\right)\Phi(t),$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$ and

$$c_g = \frac{1}{a+b}[c - ba_r(g(T) - g(0))] - a_r g(0).$$

Thus, for each $t \in I$, we obtain

$$\begin{aligned}
|u(t,w) - v(t,w)| &\leq \left|u(t) - c_g - a_r g(t) - b_r \int_0^t g(s)ds - \frac{bb_r}{a+b}\int_0^T g(s)ds\right| \\
&+ \Bigg|c_g + a_r g(t) + b_r \int_0^t g(s)ds + \frac{bb_r}{a+b}\int_0^T g(s)ds \\
&- c_h - a_r g(t) + b_r \int_0^t h(s)ds - \frac{bb_r}{a+b}\int_0^T h(s)ds\Bigg|.
\end{aligned}$$

This implies that,

$$\begin{aligned}
|u(t,w) - v(t,w)| &\leq \left(a_r + Tb_r + T\frac{bb_r}{a+b}\right)\Phi(t) \\
&+ |c_g - c_h| + a_r|g(t) - h(t)| + b_r \int_0^t |g(s) - h(s)|ds \\
&+ \frac{bb_r}{a+b}\int_0^T |g(s) - h(s)|ds.
\end{aligned}$$

On the other hand, from (H_3), for each $t \in I$, we have

$$\begin{aligned}
|g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, v(t), h(t))| \\
&\leq d_1\Phi(t) + d_2|g(t) - h(t)|,
\end{aligned}$$

which gives

$$|g(t) - h(t)| \leq \frac{d_1}{1-d_2}\Phi(t). \tag{3.14}$$

Again,

$$\begin{aligned}
|c_g - c_h| &\leq \tfrac{ba_r}{a+b}(|g(T) - h(T)| + |g(0) - h(0)|) + a_r|g(0) - h(0)| \\
&\leq \left(\tfrac{2ba_r d_1}{(a+b)(1-d_2)} + \tfrac{a_r d_1}{1-d_2}\right)\Phi(t).
\end{aligned}$$

Thus, we obtain

$$|u(t,w) - v(t,w)| \leq \left(a_r + Tb_r + T\frac{bb_r}{a+b}\right)\Phi(t)$$

$$
\begin{aligned}
&+ \left[\left(\frac{2ba_rd_1}{(a+b)(1-d_2)} + \frac{a_rd_1}{1-d_2}\right) + \frac{a_rd_1}{1-d_2}\right]\Phi(t)\\
&+ \frac{b_rd_1}{1-d_2}\int_0^t \Phi(s)ds\\
&+ \frac{bb_rd_1}{(a+b)(1-d_2)}\int_0^T \Phi(s)ds.
\end{aligned}
$$

Hence, from (H_4), we get

$$
\begin{aligned}
|u(t,w)-v(t,w)| &\leq \left(a_r + Tb_r + T\frac{bb_r}{a+b}\right.\\
&+ \frac{2ba_rd_1}{(a+b)(1-d_2)} + \frac{a_rd_1}{1-d_2} + \frac{a_rd_1}{1-d_2}\\
&+ \left.\frac{\lambda_\Phi b_rd_1}{1-d_2} + \frac{\lambda_\Phi bb_rd_1}{(a+b)(1-d_2)}\right)\Phi(t)\\
&= c_{f,\Phi}\Phi(t).
\end{aligned}
$$

3.2.3 Results in Banach Spaces

The following hypotheses will be used in the sequel:

(H_{01}) There exist a nondecreasing continuous function $\Psi : \mathbb{R}_+ \to (0,\infty)$ and continuous functions $\overline{p}, \overline{q} : I \to \mathbb{R}_+$ such that

$$\|f(t,u,v)\| \leq \overline{p}(t)\Psi(\|u\|) + \overline{q}(t)\|v\|, \text{ for each } t \in I\ u, v \in E.$$

(H_{02}) There exists a constant $M > 0$, such that

$$M \geq \|c_0\| + \left[a_r + Tb_r + T\frac{bb_r}{a+b}\right]\frac{\overline{p}^*\Psi(M)}{1-\overline{q}^*}, \tag{3.15}$$

where $\overline{p}^* = \sup_{t\in I}\overline{p}(t)$, and $\overline{q}^* = \sup_{t\in I}\overline{q}(t)$, with $0 < \overline{q}^* < 1$.

(H_{03}) For each bounded sets $\widetilde{K}, \widetilde{L} \subset E$ and each $t \in I$,

$$\mu(f(t,\widetilde{K},\widetilde{L})) \leq \overline{p}(t)\mu(\widetilde{K}) + \overline{q}(t)\mu(\widetilde{L}),$$

where μ is the Kuratowski measure of noncompactness on the space E.

Now, we prove an existence result for the problem (3.1)-(3.2) based on Monch's fixed point theorem.

Theorem 3.2.13 *Assume that the hypothesis* $(H_{01})-(H_{03})$ *hold. If*

$$\rho := \frac{\overline{p}^*}{1-\overline{q}^*}\left(a_r + Tb_r + \frac{Tbb_r}{a+b}\right) < 1, \tag{3.16}$$

then the problem (3.1)-(3.2) has a least one solution defined on I.

Proof. Consider the operator $N : \mathcal{C} \to \mathcal{C}$ be the operator defined in (3.11).
Define the ball

$$B_M = \{x \in \mathcal{C},\ \|x\|_C \leq M\}.$$

Let $u \in B_M$, from (H_{01}), for each $t \in I$, we have

$$\begin{array}{rcl} \|g(t)\| & \leq & \|f(t,u(t),g(t))\| \\ & \leq & \overline{p}(t)\Psi(\|u\|) + \overline{q}(t)\|g(t)\| \\ & \leq & \overline{p}^*\Psi(\|u\|_C) + \overline{q}^*\|g\|_C \\ & \leq & \overline{p}^*\Psi(\|u\|_C) + \overline{q}^*\|g\|_C. \end{array}$$

This gives

$$\|g\|_C \leq \frac{\overline{p}^*\Psi(M)}{1-\overline{q}^*}.$$

Thus, from (H_{02}), we obtain

$$\begin{array}{rcl} \|(Nu)(t)\| & \leq & \|c_0\| + \left[a_r + Tb_r + T\frac{bb_r}{a+b}\right]\frac{\overline{p}^*\Psi(M)}{1-\overline{q}^*}. \\ & \leq & M. \end{array}$$

Hence

$$\|N(u)\|_C \leq M.$$

This proves that N transforms the ball B_M into itself.

We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 1.5.7. We have $N(B_R) \subset B_R$, and as in the proof of Theorem 3.2.11, we can easily show that $N : B_R \to B_R$ is continuous, and $N(B_R)$ is equicontinuous.

Next, we prove that Mönch's condition (1.9)is satisfied.
Let V be a subset of B_M such that $V \subset \overline{N(V)} \cup \{0\}$, V is bounded and equicontinuous and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on I. From (H_{03}) and the properties of the measure μ, for each $t \in I$, we have

$$\begin{array}{rcl} v(t) & \leq & \mu((NV)(t) \cup \{0\}) \\ & \leq & \mu((NV)(t)) \\ & \leq & a_r\{\mu(g(t)) : u \in V\} + b_r\displaystyle\int_0^t \{\mu(g(s)) : u \in V\}ds \end{array}$$

$$+\frac{bb_r}{a+b}\int_0^T \{\mu(g(s)) : u \in V\}ds,$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$.
However, hypothesis (H_{01}) implies that for each $t \in I$,

$$\begin{array}{rcl} \mu(\{g(t) : u \in V\}) & = & \mu(\{f(t, u(t), g(t)) : u \in V\}) \\ & \leq & \overline{p}^*\mu(\{u(t) : u \in V\})) + \overline{q}^*\mu(\{g(t) : u \in V\}), \end{array}$$

which gives

$$\begin{array}{rcl} \mu(\{g(t) : u \in V\}) & \leq & \frac{\overline{p}^*}{1-\overline{q}^*}\mu\{u(s) : u \in V\} \\ & = & \frac{\overline{p}^*}{1-\overline{q}^*}\mu(V(t). \end{array}$$

Thus, we get

$$\begin{array}{rcl} v(t) & \leq & \frac{\overline{p}^*}{1-\overline{q}^*}\left[a_r\mu(V(t) + b_r\int_0^t \mu(V(s)ds + \frac{bb_r}{a+b}\int_0^T \mu(V(s)ds\right] \\ & \leq & \frac{\overline{p}^*}{1-\overline{q}^*}\left[a_r\|v\|_C + b_r\int_0^t \|v\|_C ds + \frac{bb_r}{a+b}\int_0^T \|v\|_C ds\right] \\ & \leq & \frac{\overline{p}^*}{1-\overline{q}^*}\left(a_r + Tb_r + \frac{Tbb_r}{a+b}\right)\|v\|_C. \end{array}$$

Hence

$$\|v\|_C \leq \rho\|v\|_C.$$

From (3.16), we get $\|v\|_C = 0$, that is $v(t) = \mu(V(t)) = 0$, for each $t \in I$, and then $V(t)$ is relatively compact in E. In view of the Ascoli–Arzelà theorem, V is relatively compact in B_M. From Mönch's fixed point Theorem (Theorem 1.5.7), we conclude that N has a fixed point which is a solution of the problem (3.1)-(3.2).

As in the proof of Theorem 3.2.11, we present (without proof) a result about the generalized Ulam–Hyers–Rassias stability.

Theorem 3.2.14 *Assume that the hypotheses* (H_{02}), (H_{03}), (H_4) *and the following hypothesis holds.*

(H_{04}) *There exist constants* $\overline{d}_1 > 0,\ 0 < \overline{d}_2 < 1,$ *such that*

$$(1 + \|u_1 - u_2\|)\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \overline{d}_1\Phi(t)\|u_1 - u_2\| + \overline{d}_2\|v_1 - v_2\|,$$

for each $t \in I$ *and* $u_i, v_i \in E;\ i = 1, 2.$

Then the problem (3.1)-(3.2) has at least one solution defined on I *and it is generalized Ulam–Hyers–Rassias stable.*

3.2.4 An Examples

Example 1. Consider the Caputo–Fabrizio implicit fractional differential equation

$$({}^{CF}D_0^{\frac{1}{4}}u)(t) = \frac{1+\ln(1+t^2)}{10(1+|u(t)|+|({}^{CF}D_0^{\frac{1}{4}}u)(t)|)},\ t \in [0,1], \tag{3.17}$$

with the boundary conditions

$$u(0) + 2u(1) = 1. \tag{3.18}$$

Set

$$f(t,u(t),v(t)) = \frac{1+\ln(1+t^2)}{10(1+|u(t)|+|v(t)|)},\ t \in [0,1].$$

The hypothesis (H_3) is satisfied with

$$d_1 = d_2 = \frac{1+\ln(2)}{10}.$$

Simple computations show that all conditions of Theorems 3.2.11 and 3.2.12 are satisfied. Hence problem (3.17)-(3.18) has a solution, and it is generalized Ulam–Hyers–Rassias stable.

Example 2. Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\},$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Consider the Caputo–Fabrizio fractional differential equation

$$({}^{CF}D_0^{\alpha}u)(t) = \frac{c(2^{-n}+u_n(t))}{\exp(t+3)(1+|u(t)|+|({}^{CF}D_0^{\alpha}u)(t)|)},\ t \in [0,1], \tag{3.19}$$

with the boundary conditions

$$u(0) + u(1) = (2^{-1}, 2^{-2}, \ldots, 2^{-n}, \ldots). \tag{3.20}$$

Set $f = (f_1, f_2, \ldots, f_n, \ldots)$,

$$f_n(t,u(t),v(t)) = \frac{c(2^{-n}+u_n(t))}{\exp(t+3)(1+|u(t)|+|v(t)|)},\ t \in [0,1].$$

Simple computations with a good choice of the constant c, show that all conditions of Theorem 3.2.13 are satisfied. Consequently, Theorem 3.2.13 implies that the problem (3.19)-(3.20) has at least one solution defined on $[0,1]$.

Also, hypothesis (H_4) is satisfied with $\lambda_\Phi = e-1$. Indeed

$$\int_0^T \Phi(t,w)dt = \int_0^T e^{-t}dt = 1-e^{-1} \leq \lambda_\Phi e^{-t} = \lambda_\Phi \Phi(t,w),\ t\in[0,1].$$

Consequently, Theorem 3.2.14 implies that problem (3.19)-(3.20) is generalized-Ulam–Hyers–Rassias stable.

3.3 Implicit Caputo-Fabrizio Fractional Differential Equations with Delay

Motivated by the works mentioned in the Introduction of the section 2.3, in this section, first we investigate the the followig class of boundary value problems of Caputo-Fabrizio fractional differential equations with finite delay:

$$\begin{cases} \wp(t) = \zeta(t);\ t\in[-h,0], \\ ({}^{CF}D_0^r\wp)(t) = f(t,\wp_t,({}^{CF}D_0^r\wp)(t));\ t\in I := [0,T], \end{cases} \tag{3.21}$$

where $h>0,\ T>0,\ \zeta\in\mathcal{C}, f: I\times\mathcal{C}\times\mathbb{R}\to\mathbb{R}$ is a given continuous function, ${}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r\in(0,1]$, and $\mathcal{C} := C([-h,0],\mathbb{R})$ is the space of continuous functions on $[-h,0]$.
For any $t\in I$, we define u_t by

$$\wp_t(s) = \wp(t+s);\ \text{for } s\in[-h,0].$$

Next, we investigate the following class of Caputo-Fabrizio fractional differential equations with infinite delay:

$$\begin{cases} \wp(t) = \zeta(t);\ t\in(-\infty,0], \\ ({}^{CF}D_0^r\wp)(t) = f(t,\wp_t,({}^{CF}D_0^r\wp)(t));\ t\in I, \end{cases} \tag{3.22}$$

where $\zeta: [-\infty,0]\to\mathbb{R}, f: I\times\mathcal{B}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions, and $\mathcal{B}$ is called a phase space that will be specified later.
For any $t\in I$, we define $\wp_t\in\mathcal{B}$ by

$$\wp_t(s) = \wp(t+s);\ \text{for } s\in(-\infty,0].$$

In the third subsection, we investigate the following class of Caputo-Fabrizio fractional differential equations with state dependent finite delay:

$$\begin{cases} \wp(t) = \zeta(t); \ t \in [-h, 0], \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_{\rho(t,\wp_t)}, (({}^{CF}D_0^r \wp)(t)); \ t \in I, \end{cases} \tag{3.23}$$

where $\zeta \in \mathcal{C}, \ \rho : I \times \mathcal{C} \to \mathbb{R}, \ f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Finally, we consider the following class of Caputo-Fabrizio fractional differential equations with state dependent infinite delay:

$$\begin{cases} \wp(t) = \zeta(t); \ t \in [-\infty, 0], \\ ({}^{CF}D_0^r \wp)(t) = f(t, \wp_{\rho(t,\wp_t)}, (({}^{CF}D_0^r \wp)(t)); \ t \in I, \end{cases} \tag{3.24}$$

where $\zeta : (-\infty, 0] \to \mathbb{R}, \ f : I \times \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

3.3.1 Existence of Solutions

Let $AC(I)$ denotes the space of absolutely continuous real functions on I, and by $L^1(I)$ we denote the space of measurable real functions on I which are Lebesgue integrable with the norm

$$\|\xi\|_1 = \int_I |\xi(t)| dt.$$

Lemma 3.3.1 *[80] Let $h \in L^1(I)$. Then the linear problem*

$$\begin{cases} ({}^{CF}D_0^r \wp)(t) = h(t); \ t \in I := [0, T] \\ \wp(0) = \wp_0, \end{cases} \tag{3.25}$$

has a unique solution given by

$$\wp(t) = \wp_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s) ds, \tag{3.26}$$

where

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)}.$$

Existence of Solutions with Finite Delay

In this subsubsection, we establish the existence results for problem (3.21).

Definition 3.3.2 *By a solution of problem (3.21), we mean a function $u \in C$ such that*

$$\wp(t) = \begin{cases} \zeta(t); \ t \in [-h, 0], \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases}$$

where $g \in C(I)$ with $g(t) = f(t, \wp_t, g(t))$.

The following hypotheses will be used in the sequel.

- (H_1) There exist constantes $\omega_1 > 0, \ 0 < \omega_2 < 1$ such that:

$$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \leq \omega_1 \|\wp_1 - \wp_2\|_{[-h,0]} + \omega_2 |\Im_1 - \Im_2|,$$

  for any $\wp_1, \wp_2 \in \mathcal{C}, \ \Im_1, \Im_2 \in \mathbb{R}$, and each $t \in I$.

- (H_2) For any bounded set $B \subset C$, the set:

$$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r \wp)(t)) : \wp \in B\};$$

  is equicontinuous in C.

Theorem 3.3.3 *If (H_1) holds, and*

$$\ell := \frac{\omega_1(2a_r + Tb_r)}{1 - \omega_2} < 1, \tag{3.27}$$

then problem (3.21) has a unique solution on $[-h, T]$.

Proof. Consider the operator $N : C \to C$ defined by:

$$(N\wp)(t) = \begin{cases} \zeta(t); \ t \in [-h, 0], \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases} \tag{3.28}$$

where $g \in C(I)$ such that $g(t) = f(t, \wp_t, g(t))$.

Let $u, v \in C(I)$. Then, for each $t \in [-h, 0]$, we have

$$|(N\wp)(t) - (N\Im)(t)| = 0,$$

and for each $t \in I$, we have

$$|(N\wp)(t) - (N\Im)(t)| \leq a_r|g(0) - h(0)| + a_r|g(t) - h(t)| + b_r \int_0^t |g(s) - h(s)|ds$$

where $g, \ h \in C(I)$ such that

$$g(t) = f(t, \wp_t, g(t)) \quad and \quad h(t) = f(t, \Im_t, h(t)).$$

From (H_1), we have

$$\begin{array}{rcl} |g(t)-h(t)| & = & |f(t,\wp_t,g(t))-f(t,\Im_t,h(t))| \\ & \leq & \omega_1\|\wp_t-\Im_t\|_{[-h,0]}+\omega_2|g(t)-h(t)|. \end{array}$$

This gives,

$$|g(t)-h(t)| \leq \frac{\omega_1}{1-\omega_2}\|\wp_t-\Im_t\|_{[-h,0]}.$$

Thus, for each $t\in I$, we get

$$\begin{array}{rcl} |(N\wp)(t)-(N\Im)(t)| & \leq & a_r\frac{\omega_1}{1-\omega_2}\|\wp_t-\Im_t\|_{[-h,0]}+a_r\frac{\omega_1}{1-\omega_2}\|\wp_t-\Im_t\|_{[-h,0]} \\ & & +b_r\int_0^t\frac{\omega_1}{1-\omega_2}\|\wp_s-\Im_s\|_{[-h,0]}ds \\ & \leq & 2a_r\frac{\omega_1}{1-\omega_2}\|\wp-\Im\|_C+Tb_r\frac{\omega_1}{1-\omega_2}\|\wp-\Im\|_C \\ & \leq & \frac{\omega_1(2a_r+Tb_r)}{1-\omega_2}\|\wp-\Im\|_C \\ & \leq & \ell\|\wp-\Im\|_C. \end{array}$$

Hence, we get

$$\|N(\wp)-N(\Im)\|_C \leq \ell\|\wp-\Im\|_C.$$

Since $\ell<1$, the Banach contraction principle implies that problem (3.21) has a unique solution.

Theorem 3.3.4 *If* (H_1) *and* (H_2) *hold, and*

$$\frac{\omega_1}{1-\omega_2}(2a_r+Tb_r)<1,$$

then problem (3.21) *has at least one solution on* $[-h,T]$.

Proof. Consider the operator $N:\ C\to C$ defined in (3.28).
Let $R>0$ such that

$$R \geq \max\left\{\|\zeta\|_{C([-h,0])},\frac{|\zeta(0)|+\frac{f^*}{1-\omega_2}(2a_r+Tb_r)}{1-\frac{\omega_1}{1-\omega_2}(2a_r+Tb_r)}\right\}, \tag{3.29}$$

where $f^*:=\sup_{t\in I}|f(t,0,0)|$.
Define the ball

$$B_R=\{x\in C(I,\mathbb{R}),\ \|x\|_C\leq R\}.$$

Step 1. N is continuous .
Let $\{\wp_n\}_n$ be a sequence such that $\wp_n\to\wp$ on B_R. For each $t\in[-h,0]$, we have

$$|(N\wp_n)(t)-(N\wp)(t)|=0,$$

and for each $t\in I$, we have

$$\begin{array}{rcl} |(N\wp_n)(t)-(N\wp)(t)| & \leq & a_r|g_n(0)-g(0)|+a_r|g_n(t)-g(t)|) \\ & & +b_r\int_0^t|g_n(s)-g(s)|ds, \end{array} \tag{3.30}$$

where $g_n,\ g \in C(I,\mathbb{R})$ such that

$$g_n(t) = f(t, \wp_{nt}, g_n(t)) \quad and \quad g(t) = f(t, \wp_t, g(t)).$$

Since $\|\wp_n - \wp\|_C \to 0$ as $n \to \infty$ and f, g and g_n are continuous, then the Lebesgue dominated convergence theorem, implies that

$$\|N(\wp_n) - N(\wp)\|_C \to 0 \quad \text{as } n \to \infty.$$

Hence, N is continuous.

Step 2. $N(B_R) \subset B_R$.

Let $\wp \in B_R$, If $t \in [-h, 0]$ then $\|(N\wp)(t)\| \leq \|\zeta\|_C \leq R$. From (H_1), for each $t \in I$, we have

$$\begin{array}{rcl} |g(t)| & = & |f(t, \wp_t, g(t))| \\ & \leq & |f(t,0,0)| + \omega_1\|\wp_t\|_{[-h,0]} + \omega_2|g(t)| \\ & \leq & f^* + \omega_1\|\wp\|_C + \omega_2\|g\|_\infty \\ & \leq & f^* + \omega_1 R + \omega_2\|g\|_\infty. \end{array}$$

Then

$$\|g\|_\infty \leq \frac{f^* + \omega_1 R}{1 - \omega_2}.$$

Thus,

$$\begin{array}{rcl} |(N\wp)(t)| & \leq & |\zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds| \\ & \leq & |\zeta(0)| + a_r|g(0)| + a_r|g(t)| + b_r \int_0^t |g(s)|ds \\ & \leq & |\zeta(0)| + \frac{f^*+\omega_1 R}{1-\omega_2}(2a_r + b_r \int_0^t ds) \\ & \leq & |\zeta(0)| + \frac{f^*+\omega_1 R}{1-\omega_2}(2a_r + Tb_r) \\ & \leq & R. \end{array}$$

Hence

$$\|N(\wp)\|_C \leq R.$$

Consequently, $N(B_R)$
$anglesubset B_R$.

Step 3. $N(B_R)$ is equicontinuous
For $1 \leq t_1 \leq t_2 \leq T$, and $u \in B_R$, we have

$$\begin{array}{rcl} |N(\wp)(t_1) - N(\wp)(t_2)| & \leq & a_r|g(t_2) - g(t_1)| + b_r \int_{t_1}^{t_2} |g(s)|ds| \\ & \leq & a_r|g(t_2) - g(t_1)| + \frac{RKb_r}{1-L}(t_2 - t_1). \end{array}$$

Thus, from (H_2), $a_r|g(t_2) - g(t_1)| + \frac{RKb_r}{1-L}(t_2 - t_1) \to 0$; as $t_2 \to t_1$. This gives the equicontinuity of $N(B_R)$.

From the above steps and the Arzelá-Ascoli theorem, we conclude that N is continuous and compact. Consequently, from Schauder's theorem we deduce that problem (3.21) has at least one solution.

Existence of Solutions with Infinite Delay

In this subsubsection, we establish some existence results for problem (3.22). Let the space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 1]$ into $\mathbb{R}$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato [58] for ordinary differential functional equations:

(A_1) If $u : (-\infty, T] \to \mathbb{R}$, and $u_0 \in \mathcal{B}$, then there exist constants $L, M, H > 0$, such that for each $t \in I$; we have:

- (i) *$\wp_t$ is in $\mathcal{B}$,*
- (ii) $\|\wp_t\|_{\mathcal{B}} \leq K\|\wp_1\|_{\mathcal{B}} + M\sup_{s\in[0,t]}|\wp(s)|$,
- (iii) $\|\wp(t)\| \leq H\|\wp_t\|_{\mathcal{B}}$.

(A_2) For the function $\wp(\cdot)$ *in* $(A1)$, u_t is a $\mathcal{B}-$ valued continuous function on I.

(A_3) The space $\mathcal{B}$ is complete.

Consider the space

$$\Omega = \{\wp : (-\infty, T] \to \mathbb{R},\ \wp|_{\mathbb{R}_-} \in \mathcal{B},\ \wp|_I \in C(I)\}.$$

Definition 3.3.5 *By a solution of problem (3.22), we mean a continuous function $u \in \Omega$*

$$\wp(t) = \begin{cases} \zeta(t);\ t \in \mathbb{R}_-, \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds;\ t \in I, \end{cases} \tag{3.31}$$

where $g \in C$ such that $g(t) = f(t, \wp_t, g(t))$.

Let us introduce the following hypotheses:

- (H_{01}) The function f satisfies the Lipschitz condition:

$$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \leq b_1\|\wp_1 - \wp_2\|_{\mathcal{B}} + b_2|\Im_1 - \wp_2|,$$

  for any $\wp_1, \Im_1 \in \mathcal{B}$, $\wp_2, \Im_2 \in \mathbb{R}$, and each $t \in I$, where $b_1 > 0$ and $0 < b_2 < 1$.

- (H_{02}) For any bounded set $B_1 \subset \Omega$, the set:

$$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r\wp)(t)) : \wp \in B_1\};$$

  is equicontinuous in Ω.

First, we prove an existence and uniqueness result by using the Banach's fixed point theorem.

Theorem 3.3.6 *Assume that the hypothesis* (H_{01}) *holds. If*

$$\lambda := (2a_r + Tb_r)\frac{b_1}{1-b_2} < 1, \tag{3.32}$$

then problem (3.22) has a unique solution on $(-\infty, T]$.

Proof. Consider the operator $N_1 : \Omega \to \Omega$ defined by:

$$(N_1\wp)(t) = \begin{cases} \zeta(t);\ t \in \mathbb{R}_-, \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds;\ t \in I, \end{cases} \tag{3.33}$$

where $g \in C(I)$ such that $g(t) = f(t, \wp_t, g(t))$.
Let $x(\cdot) : (-\infty, T] \to \mathbb{R}$ be a function defined by

$$x(t) = \begin{cases} \zeta(t); & t \in \mathbb{R}_-, \\ \zeta(0)- & t \in I. \end{cases}$$

Then $x_0 = \zeta$, For each $z \in C(I)$, with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z} = \begin{cases} 0; & t \in t \in \mathbb{R}_-, \\ z(t), & t \in I. \end{cases}$$

If $\wp(\cdot)$ satisfies the integral equation

$$\wp(t) \ = \ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds.$$

We can decompose $\wp(\cdot)$ as $\wp(t) = \overline{z}(t) + x(t)$; for $t \in I$, which implies that $\wp_t = \overline{z}_t + x_t$ for every $t \in I$, and the function $z(\cdot)$ satisfies

$$z(t) = -a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds,$$

where

$$g(t) = f(t, \overline{z}_t + x_t, g(t));\ t \in I.$$

Set

$$C_0 = \{z \in C(I);\ z_0 = 0\},$$

and let $\|\cdot\|_T$ be the norm in C_0 defined by

$$\|z\|_T = \|z_0\|_{\mathcal{B}} + \sup_{t\in I}|z(t)| = \sup_{t\in I}|z(t)|;\ z \in C_0.$$

C_0 is a Banach space with norm $\|\cdot\|_T$. Define the operator $P : C_0 \to C_0$; by

$$(Pz)(t) = -a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds, \tag{3.34}$$

where

$$g(t) = f(t, \overline{z}_t + x_t, g(t));\ t \in I.$$

We shall show that $P : C_0 \to C_0$ is a contraction map. Let $z, z' \in C_0$, then we have for each $t \in I$

$$|P(z)(t) - P(z')(t)| \leq a_r|g(0) - h(0)| + a_r|g(t) - h(t)| + b_r \int_0^t |g(s) - h(s)|ds, \tag{3.35}$$

where $g,\ h \in C(I)$ such that

$$g(t) = f(t, \overline{z}_t + x_t, g(t)) \quad and \quad h(t) = f(t, \overline{z'}_t + x_t, h(t)).$$

Since, for each $t \in I$, we have

$$|g(t) - h(t)| \leq \frac{b_1}{1 - b_2}\|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}}.$$

Then, for each $t \in I$; we get

$$\begin{aligned} |P(z)(t) - P(z')(t)| &\leq (2a_r + b_r \textstyle\int_0^t ds)\frac{b_1}{1-b_2}\|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}} \\ &\leq (2a_r + Tb_r)\frac{b_1}{1-b_2}\|\overline{z}_t - \overline{z'}_t\|_{\mathcal{B}} \\ &= \lambda\|\overline{z} - \overline{z'}\|_T. \end{aligned}$$

Thus, we get

$$\|P(z)(t) - P(z')(t)\|_T \leq \lambda\|\overline{z} - \overline{z'}\|_T.$$

Hence, from the Banach contraction principle, the operator P has a unique fixed point. Consequently, N has a unique fixed point which is the unique solution of problem (3.22).

Now, we prove an existence result by using Schaefer's fixed point theorem.

Theorem 3.3.7 *Assume that the hypotheses* (H_{01}) *and* H_{02} *hold. Then problem (3.22) has at least one solution on* $(-\infty, T]$.

Proof. Let $P : C_0 \to C_0$ defined as in (3.34), For each given $R > 0$, we define the ball

$$B_R = \{x \in C_0,\ \|x\|_T \leq R\}.$$

Step 1. N is continuous .
Let z_n be a sequence such that $z_n \to z$ in C_0. For each $t \in I$, we have

$$|(Pz_n)(t) - (Pz)(t)| \leq a_r|g_n(0) - g(0)| + a_r|g_n(t) - g(t)| + b_r \int_0^t |g_n(s) - g(s)|ds, \tag{3.36}$$

where $g_n,\ g \in C(I)$ such that

$$g_n(t) = f(t, \overline{z}_{nt} + x_t, g_n(t)) \quad and \quad g(t) = f(t, \overline{z}_t + x_t, g(t)).$$

Since $\|z_n - z\|_T \to 0$ as $n \to \infty$ and f, g and g_n are continuous, then

$$\|P(\wp_n) - P(\wp)\|_T \to 0 \quad \text{as } n \to \infty.$$

Hence, P is continuous.

Step 2. $P(B_R)$ is bounded.
Let $z \in B_R$, for each $t \in I$, we have

$$\begin{array}{rcl} |g(t)| & \leq & |f(t, \overline{z}_t + x_t, g(t))| \\ & \leq & |f(t,0,0)| + b_1\|\overline{z}_t + x_t\|_{\mathcal{B}} + b_2|g(t)| \\ & \leq & f^* + b_1\left[\|\overline{z}_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}}\right] + b_2\|g\|_\infty \\ & \leq & f^* + b_1 MR + b_1 K\|\zeta\|_{\mathcal{B}} + b_2\|g\|_\infty. \end{array}$$

Then

$$\|g\|_\infty \leq \frac{f^* + b_1 MR + b_1 K\|\zeta\|_{\mathcal{B}}}{1 - b_2}.$$

Thus,

$$\begin{array}{rcl} |(Pz)(t)| & \leq & a_r|g(0)| + a_r|g(t)| + b_r \int_0^t |g(s)|ds \\ & \leq & (2a_r + b_r \int_0^t ds)\frac{f^* + b_1 MR + b_1 K\|\zeta\|_{\mathcal{B}}}{1-b_2} \\ & \leq & (2a_r + Tb_r)\frac{f^* + b_1 MR + b_1 K\|\zeta\|_{\mathcal{B}}}{1-b_2} \\ & := & \ell. \end{array}$$

Hence

$$\|P(z)\|_T \leq \ell.$$

Consequently, P maps bounded sets into bounded sets in C_0.

Step 3. $P(B_R)$ is equicontinuous.
For $1 \leq t_1 \leq t_2 \leq T$, and $z \in B_R$, we have

$$\begin{array}{rcl} |P(z)(t1) - P(z)(t2)| & \leq & a_r|g(t_2) - g(t_1)| + b_r \int_{t_1}^{t_2} |g(s)|ds \\ & \leq & a_r|g(t_2) - g(t_1)| + b_r(t_2 - t_1)\frac{f^* + b_1 MR + b_1 K\|\zeta\|_{\mathcal{B}}}{1-b_2}. \end{array}$$

By (H_{02}), as $t_2 \to t_1$ the right-hand side of the above inequality tends to zero, we conclude that P maps bounded sets into equicontinuous sets in C_0.

Step 4. The priori bounds.
We prove that the set

$$\mathcal{E} = \{\wp \in C_0 : \Im = \lambda P(\wp);\ for\ some\ \lambda \in (0,1)\}$$

is bounded. Let $z \in C_0$. Let $u \in C_0$, such that $z = \lambda P(z);\ for\ some\ \lambda \in (0,1)$. Then for each $t \in I$, we have

$$z(t) = \lambda(Pz)(t) = \lambda\zeta(0) + \lambda a_r(g(t) - g(0)) + \lambda b_r \int_0^t g(s)ds.$$

From (H_{01}) we have

$$\begin{aligned} |g(t)| &\leq |f(t, \overline{z}_t + x_t, g(t))| \\ &\leq f^* + b_1\|\overline{z}_t + x_t\|_{\mathcal{B}} + b_2|g(t)| \\ &\leq f^* + b_1 \left[\|\overline{z}_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}}\right] + b_2\|g\|_\infty \\ &\leq f^* + b_1 M\|z\|_T + b_1 K\|\zeta\|_{\mathcal{B}} + b_2\|g\|_\infty. \end{aligned}$$

This gives,

$$\|g\|_\infty \leq \frac{f^* + b_1 M\|z\|_T + b_1 K\|\zeta\|_{\mathcal{B}}}{1 - b_2} := \eta.$$

Thus, for each $t \in I$, we obtain

$$\begin{aligned} |z(t)| &\leq |\zeta(0)| + a_r|g(0)| + a_r g(t) + b_r \int_0^t |g(s)|ds \\ &\leq |\zeta(0)| + \eta(2a_r + Tb_r) \\ &:= \eta'. \end{aligned}$$

Hence

$$\|z\|_T \leq \eta'.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's theorem [**?**], the operator N has a fixed point which is a solution of problem (3.22).

Existence Results with State-Dependent Delay (The Finite Delay Case)

In this subsubsection, we establish the existence results for problem (3.23).

Definition 3.3.8 *By a solution of problem (3.23), we mean a continuous function $u \in C$ such that*

$$\wp(t) = \begin{cases} \zeta(t);\ t \in [-h, 0], \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds;\ t \in I, \end{cases}$$

where $g \in C(I)$ with $g(t) = f(t, \wp_{\rho(t,\wp_t)}, g(t))$.

- (H_4) The function f satisfies the Lipschitz condition:

  $$|f(t, \wp_1, \Im_1) - f(t, \wp_2, \Im_2)| \leq \omega_3\|\wp_1 - \wp_2\|_{[-h,0]} + \omega_4|\Im_1 - \Im_2|,$$

  for any $\wp_1, \Im_1 \in C([-h, 0], \mathbb{R})$, $\wp_2, \Im_2 \in \mathbb{R}$, and each $t \in I$,
  where $\omega_3 > 0$, $0 < \omega_4 < 1$.

- (H_5) For any bounded set $B_2 \subset C$, the set:

  $$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r\wp)(t)) : \wp \in B_2\};$$

  is equicontinuous in C.

As in Theorems 3.3.3 and 3.3.4, we give without proof, the following results:

Theorem 3.3.9 *Assume that the hypothesis (H_4) holds. If*

$$(2a_r + Tb_r)\frac{\omega_3}{1-\omega_4} < 1,$$

then problem (3.22) has a unique solution on $[-h, T]$.

Existence Results with State-Dependent Delay (The Infinite Delay Case)

Now, we establish the last problem (3.24).

Definition 3.3.10 *By a solution of problem (3.24), we mean a continuous $u \in \Omega$*

$$\wp(t) = \begin{cases} \zeta(t); \ t \in \mathbb{R}_-, \\ \zeta(0) - a_r g(0) + a_r g(t) + b_r \int_0^t g(s)ds; \ t \in I, \end{cases} \tag{3.37}$$

where $g \in C(I)$ such that $g(t) = f(t, \wp_{\rho(t,\wp_t)}, g(t))$.

Set

$$R' := R'_{\rho^-} = \{\rho(t,\wp) : t \in I, \ \wp \in \mathcal{B} \ \rho(t,\wp) < 0\}$$

We always assume that $\rho : I \times \mathcal{B} \to \mathbb{R}$ is continuous and the function $t \to \wp_t$ is continuous from R' into $\mathcal{B}$. We will need the following hypothesis:

(H_ζ) There exists a continuous bounded function $L : R'_{\rho^-} \to (0,\infty)$ such that

$$\|\zeta_t\|_{\mathcal{B}} \le L(t)\|\zeta\|_{\mathcal{B}}, \ for \ any \ t \in R'.$$

Lemma 3.3.11 *If $\wp \in \Omega$ then*

$$\|\wp_t\|_{\mathcal{B}} = (M + L')\|\zeta\|_{\mathcal{B}} + K \sup_{\theta\in[0,\max\{0,t\}]} \|\wp(\theta)\|,$$

where

$$L' = \sup_{t\in R'} L(t).$$

- (H_{04}) The function f satisfies the Lipschitz condition:

  $$|f(t,\wp_1,\Im_1) - f(t,\wp_2,\Im_2)| \le b_3\|\wp_1 - \wp_2\|_{\mathcal{B}} + b_4|\Im_1 - \Im_2|,$$

  for any $\wp_1, \Im_1 \in \mathcal{B}$, $\wp_2, \Im_2 \in \mathbb{R}$, and each $t \in I$, where $b_3 > 0$ and $0 < b_4 < 1$.

- (H_{05}) For any bounded set $B_2 \subset \Omega$, the set:

  $$\{t \mapsto f(t, \wp_t, ({}^{CF}D_0^r\wp)(t)) : u \in B_2\};$$

  is equicontinuous in Ω.

As in Theorems 3.3.6 and 3.3.7, we give without proof, the following results:

Theorem 3.3.12 *Assume that the hypothesis* (H_{04}) *holds. If*

$$(2a_r + Tb_r)\frac{b_3}{1-b_4} < 1,$$

then problem (3.24) has a unique solution on $(-\infty, T]$.

Theorem 3.3.13 *Assume that the hypotheses* (H_ς), (H_{04}) *and* (H_{05}) *hold. Then problem (3.24) has at least one solution on* $(-\infty, T]$.

3.3.2 Some Examples

Example 1. Consider the following problem

$$\begin{cases} \wp(t) = 1 + t^2;\ t \in [-1, 0], \\ ({}^{CF}D_0^{1/2}\wp)(t) = \frac{\varsigma}{90(1+\|\wp_t\|)} + \frac{1}{30\left(1+|({}^{CF}D_0^{1/2}\wp(t))|\right)};\ t \in [0, 2], \end{cases} \tag{3.38}$$

where $\varsigma < \frac{87}{2a_{\frac{1}{2}}+2b_{\frac{1}{2}}}$.

Set

$$f(t, \wp, \Im) = \frac{\varsigma}{90\,(1+\|\wp\|)} + \frac{1}{30\,(1+|\Im|)};\ t \in [1, e],\ \wp \in \mathcal{C},\ \Im \in \mathbb{R}.$$

Clearly, the function f is continuous. For any $\wp, \widetilde{\wp} \in \mathcal{C}$, $\wp, \widetilde{\wp} \in \mathbb{R}$, and $t \in [0, 2]$, we have

$$|f(t, \wp, \Im) - f(t, \widetilde{\wp}, \widetilde{\Im})| \leq \frac{\varsigma}{90}\|\wp - \widetilde{\wp}\|_{[-1,0]} + \frac{1}{30}|\Im - \widetilde{\Im}|.$$

Hence hypothesis (H_1) is satisfied with

$$\omega_1 = \frac{\varsigma}{90} \quad and \quad \omega_2 = \frac{1}{30}.$$

Next, condition (3.27) is satisfied with $T = 2$ and $r = \frac{1}{2}$. Indeed,

$$\begin{aligned} \tfrac{\omega_1(2a_r+Tb_r)}{1-\omega_2} &= \tfrac{\varsigma\left(2a_{\frac{1}{2}}+2b_{\frac{1}{2}}\right)}{87} \\ &< 1. \end{aligned}$$

Theorem 3.3.3 implies that problem (3.38) has a unique solution defined on $[-1, 2]$.

Example 2. Consider now the following problem

$$\begin{cases} \wp(t) = t;\ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^{2/3}\wp)(t) = \frac{\wp_t e^{-\gamma t+t}}{180(e^t-e^{-t})(1+\|\wp_t\|)} + \frac{\wp(t)e^{-\gamma t+t}}{60(e^t-e^{-t})\left(1+|({}^{CF}D_0^{2/3}\wp(t))|\right)};\ t \in [0, 1]. \end{cases} \tag{3.39}$$

Let γ be a positive real constant and

$$B_\gamma = \{\wp \in C((-\infty,1],\mathbb{R},) : \lim_{\theta\to-\infty} e^{\gamma\theta\theta}\wp(\theta) \text{ } exists \text{ } in \text{ } \mathbb{R}\}. \tag{3.40}$$

The norm of B_γ is given by

$$\|\wp\|_\gamma = \sup_{\theta\in(-\infty,1]} e^{\gamma\theta}|\wp(\theta)|.$$

Let $\wp : \mathbb{R}_- \to \mathbb{R}$ be such that $\wp_0 \in B_\gamma$. Then

$$\begin{aligned}\lim_{\theta\to-\infty} e^{\gamma\theta}\wp_t(\theta) &= \lim_{\theta\to-\infty} e^{\gamma\theta}\wp(t+\theta-1) = \lim_{\theta\to-\infty} e^{\gamma(\theta-t+1)}\wp(\theta)\\ &= e^{\gamma(-t+1)}\lim_{\theta\to-\infty} e^{\gamma(\theta)}\wp_1(\theta) < \infty.\end{aligned}$$

Hence $\wp_t \in B_\gamma$. Finally we prove that

$$\|\wp_t\|_\gamma \leq K\|\wp_1\|_\gamma + M\sup_{s\in[0,t]}|\wp(s)|,$$

where $K = M = 1$ and $H = 1$. We have

$$\|\wp_t(\theta)\| = |\wp(t+\theta)|.$$

If $t+\theta \leq 1$, we get

$$\|\wp_t(\beta)\| \leq \sup_{s\in\mathbb{R}_-}|\wp(s)|.$$

For $t+\theta \geq 0$, then we have

$$\|\wp_t(\beta)\| \leq \sup_{s\in[0,t]}|\wp(s)|.$$

Thus for all $t+\theta \in I$, we get

$$\|\wp_t(\beta)\| \leq \sup_{s\in\mathbb{R}_-}|\wp(s)| + \sup_{s\in[0,t]}|\wp(s)|.$$

Then

$$\|\wp_t\|_\gamma \leq \|\wp_0\|_\gamma + \sup_{s\in[0,t]}|\wp(s)|.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ a phase space. Set

$$f(t,\wp,\Im) = \frac{e^{-\gamma t+t}}{180\,(e^t-e^{-t})\left(1+\|\wp\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\,(e^t-e^{-t})\,(1+|\Im|)};\ t\in[0,1],\ \wp\in B_\gamma, \Im\in\mathbb{R}.$$

We can verify that the hypothesis (H_{01}) is satisfied with

$$B_1 = \frac{1}{180} \quad and \quad B_2 = \frac{1}{60}.$$

Theorem 3.3.7 ensures that problem (3.39) has a solution defined on $(-\infty, 1]$.

Example 3. We consider the following problem

$$\begin{cases} \wp(t) = 1 + t^2;\ t \in [-1, 0], \\ ({}^{CF}D_0^{1/2}\wp)(t) = \frac{1}{90(1+|\wp(t-\sigma(\wp(t)))|)} + \frac{1}{30\left(1+|({}^{CF}D_0^{1/2}\wp(t))|\right)};\ t \in [0, 1], \end{cases} \tag{3.41}$$

where $\sigma \in C(\mathbb{R}, [0, 1])$. Set

$$\rho(t, \zeta) = t - \sigma(\zeta(0)), \quad (t, \zeta) \in [0, e] \times C([-1, 0], \mathbb{R}),$$

$$f(t, \wp, \Im) = \frac{1}{90(1 + |\wp(t - \sigma(\wp(t)))|)} + \frac{1}{30\,(1 + |\Im(t)|)};\ t \in [1, e],\ \wp \in \mathcal{C},\ \Im \in \mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $\wp, \widetilde{\wp} \in \mathcal{C},\ \Im, \widetilde{\Im} \in \mathbb{R}$ and $t \in [0, 1]$, we have

$$|f(t, \wp, \Im) - f(t, \widetilde{\wp}, \widetilde{\Im})| \leq \frac{1}{90}\|\wp - \widetilde{\wp}\|_{[-1,0]} + \frac{1}{30}|\Im - \widetilde{\Im}|.$$

Hence hypothesis (H_{04}) is satisfied with

$$\omega_3 = \frac{1}{90} \quad and \quad \omega_4 = \frac{1}{30}.$$

From Theorem 3.3.9, problem (3.41) has a unique solution on $[-1, 1]$.

Example 4. Consider now the problem

$$\begin{cases} \wp(t) = t^2;\ t \in \mathbb{R}_-, \\ ({}^{CF}D_0^{1/4}\wp)(t) = \frac{\wp(t-\lambda(\wp(t)))e^{-\gamma t+t}}{180(e^t-e^{-t})(1+|\wp(t-\sigma(\wp(t))|)} + \frac{\wp(t)e^{-\gamma t+t}}{60(e^t-e^{-t})\left(1+|({}^{CF}D_0^{1/4}\wp(t))|\right)};\ t \in [0, 3]. \end{cases} \tag{3.42}$$

Let γ be a positive real constant and the phase space B_γ defined in Example 2.

Define

$$\rho(t, \zeta) = t - \lambda(\zeta(0)), \quad (t, \zeta) \in [0, 3] \times B_\gamma,$$

and set

$$f(t, \wp, \Im) = \frac{e^{-\gamma t+t}}{180\,(e^t - e^{-t})\left(1 + \|\wp\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\,(e^t - e^{-t})\,(1 + |\Im|)};\ t \in [0, 3],\ \wp \in B_\gamma,\ \Im \in \mathbb{R}.$$

By Theorem 3.3.7, problem (3.42) has a solution defined on $(-\infty, 3]$.

Chapter 4

Implicit Fractional Differential Equations in b-Metric Spaces

4.1 Introduction

The purpose of this chapter is the study of three results for a class of existence results for a class of Terminal Value Problem for Katugampola implicit fractional differential equations in b-metric spaces, Caputo-Fabrizio implicit fractional differential equations in b-Metric Spaces with initial conditions, implicit Katugampola fractional differential equations in b-metric spaces and for Caputo-Katugampola implicit fractional differential equations in b-metric spaces. The results are based on the $\alpha - \phi$-Geraghty type contraction and the fixed point theory.

The Katugampola fractional differential operator has been introduced in [75, 76]. It is a generalization of the Caputo and the Caputo-Hadamard fractional derivatives. Some fundamental properties of this operator are presented in [14, 19], and the references therein.

4.2 Terminal Value Problem for Implicit Katugampola Fractional Differential Equations in b-Metric Spaces

The outcome of our study in section is the continuation of the problem raised recently in [19], in it, Arioua et al. study in a general manner, the existence and uniqueness of solutions of nonlinear fractional differential equations:

$$\begin{cases} ({}^{\rho}D^{r}_{0^+}u)(t) = f(t, u(t), ({}^{\rho}D^{r}_{0^+}u)(t)); \ t \in I := [0, T], \\ u(0) = 0, \end{cases}$$

where $0 < r \leq 1$, $\rho > 0$, $T \leq (pc)^{\frac{1}{pc}}$ for any $1 \leq p \leq \infty$, $c > 0$, is a finite positive constant, ${}^\rho D^r_{0^+}$ is the the Katugampola fractional derivative and $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function.

In this section, we discuss the existence of solutions for the following class of terminal value problems of Katugampola implicit fractional differential equations:

$$\begin{cases} ({}^\rho D^r_{0^+} u)(t) = f(t, u(t), ({}^\rho D^r_{0^+} u)(t)); \ t \in I := [0, T], \\ u(T) = u_T \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $T > 0$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^\rho D^r_{0^+}$ is the Katugampola fractional derivative of order $r \in (0, 1]$.

4.2.1 Existence of Solutions

Let $C_{r,\rho}(I)$ we denote the weighted space of continuous functions defined by

$$C_{r,\rho}(I) = \{u : (0, T] \to \mathbb{R} : t^{\rho(1-r)} u(t) \in C(I)\},$$

with the norm

$$\|u\|_C := \sup_{t \in I} \|t^{\rho(1-r)} u(t)\|.$$

Lemma 4.2.1 *([76]) Let $r, \rho > 0$. If $u \in C(I)$, then the fractional differential equation ${}^\rho D^r_{0^+} u(t) = 0$, has a unique solution*

$$u(t) = C_1 t^{\rho(r-1)} + C_2 t^{\rho(r-2)} + \ldots + C_n t^{\rho(r-n)},$$

where $C_i \in \mathbb{R}$ with $i = 1, 2, \ldots, n$.

Proof. Let $r, \rho > 0$. from Remark 1.2.9, we have

$${}^\rho D^r_{0^+} t^{\rho(r-i)} = 0, \ for \ each \ i = 1, 2, \ldots, n.$$

Then the fractional equation ${}^\rho D^r_{0^+} u(t) = 0$, has a particular solution as follows:

$$u(t) = C_i t^{\rho(r-i)}, \ C_i \in \mathbb{R}, \ for \ each \ i = 1, 2, \ldots, n. \tag{4.2}$$

Thus, the general solution of ${}^\rho D^r_{0^+} u(t) = 0$ is a sum of particular solutions (4.2) , i.e.

$$u(t) = C_1 t^{\rho(r-1)} + C_2 t^{\rho(r-2)} + \ldots + C_n t^{\rho(r-n)}, \ C_i \in \mathbb{R}; \ (i = 1, 2, \ldots, n).$$

Lemma 4.2.2 *Let $r, \rho > 0$. If $u, {}^\rho D^r_{0^+} u \in C(I)$ and $0 < r \leq 1$ then*

$${}^\rho I^r_{0^+} {}^\rho D^r_{0^+} u(t) = u(t) + c t^{\rho(r-1)}, \tag{4.3}$$

for some constant $c \in \mathbb{R}$.

Proof. Let ${}^{\rho}D^{r}_{0+}u \in C(I)$ be the fractional derivatives (1.2) of order $0 < r \leq 1$. If we apply the operator ${}^{\rho}D^{r}_{0+}$ to ${}^{\rho}I^{r}_{0+}{}^{\rho}D^{r}_{0+}u(t) - u(t)$, and use the properties (1.4) and (1.5), we get

$$\begin{aligned} {}^{\rho}D^{r}_{0+}[{}^{\rho}I^{r}_{0+}{}^{\rho}D^{r}_{0+}u(t) - u(t)] &= {}^{\rho}D^{r}_{0+}{}^{\rho}I^{r}_{0+}{}^{\rho}D^{r}_{0+}u(t) -{}^{\rho} D^{r}_{0+}u(t) \\ &= {}^{\rho}D^{r}_{0+}u(t) -{}^{\rho} D^{r}_{0+}u(t) = 0. \end{aligned}$$

From the proof of Lemma 4.2.1 we deduce that there exists $c \in \mathbb{R}$, such that:

$${}^{\rho}I^{r}_{0+}{}^{\rho}D^{r}_{0+}u(t) - u(t) = ct^{\rho(r-1)},$$

which implies (4.3).

Lemma 4.2.3 *Let $h \in L^1(I, \mathbb{R})$, and $0 < r \leq 1$ and $\rho > 0$. A function $u \in C(I)$*

$$\begin{cases} ({}^{\rho}D^{r}_{0+}u)(t) = h(t); \ t \in I, \\ u(T) = u_T, \end{cases} \tag{4.4}$$

if and only if u satisfies the following integral equation

$$u(t) = (u_T -{}^{\rho} I^{r}_{0+}h(T)) \left(\frac{t}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)} \int_0^t s^{\rho-1}(t^{\rho} - s^{\rho})^{r-1}h(s)ds. \tag{4.5}$$

Proof. Let $r, \rho > 0$. and $0 < r \leq 1$, Suppose that u satisfies (4.4). Applying the operator ${}^{\rho}I^{r}_{0+}$ to the both sides of the equation

$$({}^{\rho}D^{r}_{0+}u)(t) = h(t), \tag{4.6}$$

we obtain

$${}^{\rho}I^{r}_{0+}{}^{\rho}D^{r}_{0+}u(t) ={}^{\rho} I^{r}_{0+}h(t).$$

From Lemma 4.2.2, we get

$$u(t) + ct^{\rho(r-1)} ={}^{\rho} I^{r}_{0+}h(t). \tag{4.7}$$

for some $c \in \mathbb{R}$. If we use the terminal condition $u(T) = u_T$ in (4.7), we find

$$u(T) = u_T ={}^{\rho} I^{r}_{0+}h(T) - cT^{\rho(r-1)},$$

which shows,

$$c = ({}^{\rho}I^{r}_{0+}h(T) - u_T)T^{\rho(1-r)}.$$

So; we get (4.5).
Conversely, if u satisfies (4.5), then $({}^{\rho}D^{r}_{0+}u)(t) = h(t)$; for $t \in I$ and $u(t) = u_T$.

As in the proof of the above lemma, we can show the following one

Lemma 4.2.4 *A function u is a solution of problem (4.1), if and only if u satisfies the following integral equation:*

$$u(t) = (u_T -^\rho I^r_{0+}g(T))\left(\frac{t}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1}g(s)ds, \tag{4.8}$$

where $g \in C(I)$, and $g(t) = f(t, u(t), g(t))$.

Let $(C_{r,\rho}(I), d, 2)$ be the complete b-metric space with $c = 2$, such that $d : C_{r,\rho}(I) \times C_{r,\rho}(I) \to \mathbb{R}^*_+$ is given by:

$$d(u, v) = \|(u - v)^2\|_C := \sup_{t\in I} t^{\rho(1-r)}|u(t) - v(t)|^2.$$

Then $(C_{r,\rho}(I), d, 2)$ is a b-metric space.

Definition 4.2.5 *By a solution of the problem (4.1) we mean a function $u \in C_{r,\rho}(I)$ that satisfies*

$$u(t) = (u_T -^\rho I^r_{0+}g(T))\left(\frac{t}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1}g(s)ds,$$

where $g \in C(I)$, and $g(t) = f(t, u(t), g(t))$.

The following hypotheses will be used in the sequel.

(H_1) There exist $\phi \in \Phi$, $p : C(I) \times C(I) \to (0, \infty)$ and $q : I \to (0, 1)$ such that for each $u, v, u_1, v_1 \in C_{r,\rho}(I)$, and $t \in I$

$$|f(t, u, v) - f(t, u_1, v_1)| \le t^{\frac{\rho}{2}(1-r)}p(u, v)|u - u_1| + q(t)|v - v_1|,$$

with $\left\|\frac{\rho^{1-r}}{\Gamma(r)}\int_0^T s^{\rho-1}(T^\rho - s^\rho)^{r-1}\frac{p(u,v)}{1-q*}ds\right\|^2_C$

$$+\left\|\frac{\rho^{1-r}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1}\frac{p(u, v)}{1 - q*}ds\right\|^2_C \le \phi(\|(u - v)^2\|_C).$$

(H_2) There exist $\mu_0 \in C_{r,\rho}(I)$ and a function $\theta : C_{r,\rho}(I) \times C_{r,\rho}(I) \to \mathbb{R}$, such that

$$\theta\left(\mu_0(t), (u_T -^\rho I^r_{0+}g(T))\left(\frac{t}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1}g(s)ds\right) \ge 0,$$

where $g \in C(I)$, with $g(t) = f(t, \mu_0(t), g(t))$.

(H_3) For each $t \in I$, and $u, v \in C_{r,\rho}(I)$, we have:

$$\theta(u(t), v(t)) \geq 0$$

implies

$$\theta\left(\frac{\rho^{1-r}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1}g(s)ds, \frac{\rho^{1-r}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1}h(s)ds\right) \geq 0,$$

where $g, h \in C(I)$, with $g(t) = f(t, u(t), g(t))$ and $h(t) = f(t, v(t), h(t))$.

(H_4) If $u_{nn\in N} \subset C(I)$ with $u_n \to u$ and $\theta(u_n, u_{n+1}) \geq$, then

$$\theta(u_n, u) \geq 1.$$

Theorem 4.2.6 *Assume that hypotheses $(H_1) - (H_4)$ hold. Then the problem (4.1) has a least one solution defined on I.*

Proof. Consider the operator $N : C_{r,\rho}(I) \to C_{r,\rho}(I)$ defined by

$$(Nu)(t) = (u_T -^\rho I_{0+}^r g(T))\left(\frac{t}{T}\right)^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1}g(s)ds,$$

where $g \in C(I)$, with $g(t) = f(t, u(t), g(t))$.
By using Lemma 4.2.4, it is clear that the fixed points of the operator N are solutions of (4.1).

Let $\alpha : C_{r,\rho}(I) \times C_{r,\rho}(I) \to (0, \infty)$ be the function defined by:

$$\begin{cases} \alpha(u, v) = 1; & if\ \theta(u(t), v(t)) \geq 0,\ t \in I, \\ \alpha(u, v) = 0; & elese. \end{cases}$$

First, we prove that N is a generalized α-ϕ-Geraghty operator:
For any $u, v \in C(I)$ and each $t \in I$, we have

$$\begin{aligned} |t^{\rho(1-r)}(Nu)(t) - t^{\rho(1-r)}(Nv)(t)| &\leq t^{\rho(1-r)}|^\rho I_{0+}^r(g-h)(T))|\left(\tfrac{t}{T}\right)^{\rho(r-1)} \\ &\quad + \tfrac{\rho^{1-r}t^{\rho(1-r)}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1}|g(s) - h(s)|ds \end{aligned}$$

where $g, h \in C(I)$, with

$$g(t) = f(t, u(t), g(t)),$$

and

$$h(t) = f(t, v(t), h(t)).$$

From (H_1) we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, v(t), h(t))| \\ &\leq p(u, v)t^{\frac{\rho}{2}(1-r)}|u(t) - v(t)| + q(t)|g(t) - h(t)| \\ &\leq p(u, v)(t^{\rho(1-r)}|u(t) - v(t)|^2)^{\frac{1}{2}} + q(t)|g(t) - h(t)|. \end{aligned}$$

Thus,

$$|g(t)-h(t)| \leq \frac{p(u,v)}{1-q*}\|(u-v)^2\|_C^{\frac{1}{2}},$$

where $q* = \sup_{t\in I}|q(t)|$.
Next, we have

$$\begin{array}{rl} & |t^{\rho(1-r)}(Nu)(t) - t^{\rho(1-r)}(Nv)(t)| \\ \leq & t^{\rho(1-r)}|{}^\rho I_{0^+}^r(g-h)(T))| \left(\frac{t}{T}\right)^{\rho(r-1)} \\ + & \frac{\rho^{1-r}t^{\rho(1-r)}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{r-1}\frac{p(u,v)}{1-q*}\|(u-v)^2\|_C^{\frac{1}{2}}ds \\ \leq & \frac{\rho^{1-r}t^{\rho(1-r)}}{\Gamma(r)}\int_0^T s^{\rho-1}(T^\rho-s^\rho)^{r-1}\frac{p(u,v)}{1-q*}\|(u-v)^2\|_C^{\frac{1}{2}}ds \\ + & \frac{\rho^{1-r}t^{\rho(1-r)}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{r-1}\frac{p(u,v)}{1-q*}\|(u-v)^2\|_C^{\frac{1}{2}}ds. \end{array}$$

Thus

$$\begin{array}{rl} & \alpha(u,v)|t^{\rho(1-r)}(Nu)(t) - t^{\rho(1-r)}(Nv)(t)|^2 \\ \leq & \|(u-v)^2\|_C\alpha(u,v)\left\|\frac{\rho^{1-r}}{\Gamma(r)}\int_0^T s^{\rho-1}(T^\rho-s^\rho)^{r-1}\frac{p(u,v)}{1-q*}ds\right\|_C^2 \\ + & \|(u-v)^2\|_C\alpha(u,v)\left\|\frac{\rho^{1-r}}{\Gamma(r)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{r-1}\frac{p(u,v)}{1-q*}ds\right\|_C^2 \\ \leq & \|(u-v)^2\|_C\phi(\|(u-v)^2\|_C). \end{array}$$

Hence

$$\alpha(u,v)\phi(2^3 d(N(u),N(v)) \leq \lambda(\phi(d(u,v))\phi(d(u,v)),$$

where $\lambda \in \digamma$, $\phi \in \Phi$, with $\lambda(t) = \frac{1}{8}t$, and $\phi(t) = t$.
So, N is generalized α-ϕ-Geraghty operator.

Let $u, v \in C_{r,\rho}(I)$ such that

$$\alpha(u,v) \geq 1.$$

Thus, for each $t \in I$, we have

$$\theta(u(t), v(t)) \geq 0.$$

This implies from (H_3) that

$$\theta(Nu(t), Nv(t)) \geq 0,$$

which gives

$$\alpha(N(u), N(v)) \geq 1.$$

Hence, N is a α-admissible.

Now, from (H_2), there exists $\mu_0 \in C_{r,\rho}(I)$ such that

$$\alpha(\mu_0, N(\mu_0)) \geq 1.$$

Finally, from (H_4), If $\mu_{n n\in N} \subset M$ with $\mu_n \to \mu$ and $\alpha(\mu_n, \mu_{n+1}) \geq 1$, then

$$\alpha(\mu_n, \mu) \geq 1.$$

From an application of Theorem 1.5.6, we deduce that N has a fixed point u which is a solution of problem (4.1).

4.2.2 An Example

Let $(C_{r,\rho}([0,1]), d, 2)$ be the complete b-metric space, such that $d : C_{r,\rho}([0,1]) \times C_{r,\rho}([0,1]) \to \mathbb{R}_+^*$ is given by:

$$d(u,v) = \|(u-v)^2\|_C.$$

Consider the following fractional differential problem

$$\begin{cases} ({}^\rho D_{0^+}^r u)(t) = f(t, u(t,)({}^\rho D_{0^+}^r u)(t)); \ t \in [0,1], \\ u(1) = 2, \end{cases} \tag{4.9}$$

where

$$f(t,u(t),v(t)) = \frac{t^{\frac{\rho}{2}(1-r)}(1+\sin(|u(t)|))}{4(1+|u(t)|)} + \frac{e^{-t}}{2(1+|v(t))|}; \ t \in [0,1].$$

Let $t \in (0,1]$, and $u, v \in C_{r,\rho}([0,1])$. If $|u(t)| \le |v(t)|$, then

$$\begin{aligned}
& |f(t,u(t),u_1(t)) - f(t,v(t),v_1(t))| = t^{\frac{\rho}{2}(1-r)} \left| \frac{1+\sin(|u(t)|)}{4(1+|u(t)|)} - \frac{1+\sin(|v(t)|)}{4(1+|v(t)|)} \right| \\
+ \quad & \left| \frac{e^{-t}}{2(1+|u_1(t))|} - \frac{e^{-t}}{2(1+|v_1(t))|} \right| \\
\le \quad & \frac{t^{\frac{\rho}{2}(1-r)}}{4} ||u(t)| - |v(t)|| + \frac{t^{\frac{\rho}{2}(1-r)}}{4} |\sin(|u(t)|) - \sin(|v(t)|)| \\
+ \quad & \frac{t^{\frac{\rho}{2}(1-r)}}{4} \, ||u(t)|\sin(|v(t)|) - |v(t)|\sin(|u(t)|)| \\
+ \quad & \frac{e^{-t}}{2} |u_1(t) - v_1(t))| \\
\le \quad & \frac{t^{\frac{\rho}{2}(1-r)}}{4} |u(t) - v(t)| + \frac{t^{\frac{\rho}{2}(1-r)}}{4} |\sin(|u(t)|) - \sin(|v(t)|)| \\
+ \quad & \frac{t^{\frac{\rho}{2}(1-r)}}{4} \, ||v(t)|\sin(|v(t)|) - |v(t)|\sin(|u(t)|)| \\
+ \quad & \frac{e^{-t}}{2} |u_1(t) - v_1(t))| \\
= \quad & \frac{t^{\frac{\rho}{2}(1-r)}}{4} |u(t) - v(t)| + \frac{t^{\frac{\rho}{2}(1-r)}}{4} (1+|v(t)|) \, |\sin(|u(t)|) - \sin(|v(t)|)| \\
+ \quad & \frac{e^{-t}}{2} |u_1(t) - v_1(t))| \\
\le \quad & \frac{t^{\frac{\rho}{2}(1-r)}}{4} |u(t) - v(t)| + \frac{t^{\frac{\rho}{2}(1-r)}}{2} (1+|v(t)|) \\
\times \quad & \left| \sin\left(\frac{||u(t)| - |v(t)|}{2} \right) \right| \left| \cos\left(\frac{|u(t)| + |v(t)|}{2} \right) \right|
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{e^{-t}}{2}|u_1(t)-v_1(t)| \\
&\leq \quad \frac{t^{\frac{\rho}{2}(1-r)}}{4}(2+|v(t)|)|u(t)-v(t)| + \frac{e^{-t}}{2}|u_1(t)-v_1(t)|.
\end{aligned}
$$

The case when $|v(t)| \leq |u(t)|$, we get

$$|f(t,u(t))-f(t,v(t))| \leq \frac{t^{\frac{\rho}{2}(1-r)}}{4}(2+|u(t)||u(t)-v(t)| + \frac{e^{-t}}{2}|u_1(t)-v_1(t)|.$$

Hence

$$|f(t,u(t))-f(t,v(t))| \leq \frac{T^{\frac{\rho}{2}(1-r)}}{4}\min_{t\in I}\{2+|u(t)|,2+|v(t)|\}|u(t)-v(t)|+\frac{e^{-t}}{2}|u_1(t)-v_1(t)|.$$

Thus, hypothesis (H_1) is satisfied with

$$p(u,v) = \frac{T^{\frac{\rho}{2}(1-r)}}{4}\min_{t\in I}\{2+|u(t)|,2+|v(t)|\},$$

and

$$q(t) = \frac{1}{2}e^{-t}.$$

Define the functions $\lambda(t) = \frac{1}{8}t$, $\phi(t) = t$, $\alpha : C_{r,\rho}([0,1]) \times C_{r,\rho}([0,1]) \to \mathbb{R}_+^*$ with

$$
\begin{cases}
\alpha(u,v) = 1;\ if\ \delta(u(t),v(t)) \geq 0,\ t \in I, \\
\alpha(u,v) = 0;\ else,
\end{cases}
$$

and $\delta : C_{r,\rho}([0,1]) \times C_{r,\rho}([0,1]) \to \mathbb{R}$ with $\delta(u,v) = \|u-v\|_C$.
Hypothesis (H_2) is satisfied with $\mu_0(t) = u_0$. Also, (H_3) holds from the definition of the function δ. Hence by Theorem 4.2.6, problem (4.9) has at least one solution defined on $[0,1]$.

4.3 Functional Katugampola Fractional Differential Equations in b-Metric Spaces

Motivated by the works mentioned in the Introduction of the section 4.2 , in this section, we discuss the existence of solutions for the following class of Katugampola implicit fractional differential equations:

$$
\begin{cases}
({}^\rho D^r_{0^+}\wp)(t) = f(t,\wp(t),({}^\rho D^r_{0^+}\wp)(t));\ t \in I := [0,T], \\
({}^\rho I^{1-r}_{0^+}\wp)(0) = u_0 \in \mathbb{R},
\end{cases}
\tag{4.10}
$$

where $T,\rho > 0$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^\rho I^r_{0^+}$ is the Katugampola fractional integral of order $r \in (0,1]$, ${}^\rho D^r_{0^+}$ is the Katugampola fractional derivative of order r.

4.3.1 Existence of Solutions

Let $C_{r,\rho}(I)$ we denote the weighted space of continuous functions defined by

$$C_{r,\rho}(I) = \{\wp : (0,T] \to \mathbb{R} : t^{\rho(1-r)}\wp(t) \in C(I)\},$$

with the norm

$$\|\wp\|_C := \sup_{t\in I} \|t^{\rho(1-r)}\wp(t)\|.$$

Lemma 4.3.1 *Let $h \in L^1(I,\mathbb{R})$, and $0 < r \le 1$ and $\rho > 0$. A function $u \in C(I)$*

$$\begin{cases} ({}^\rho D^r_{0+}\wp)(t) = h(t);\ t \in I, \\ ({}^\rho I^{1-r}_0 \wp)(t) = \wp_0, \end{cases} \tag{4.11}$$

if and only if u satisfies the following integral equation

$$\wp(t) = \frac{\wp_0\rho^{1-r}}{\Gamma(r)} t^{\rho(r-1)} + ({}^\rho I^r_{0+}h)(t). \tag{4.12}$$

Proof. Let $r, \rho > 0$. and $0 < r \le 1$, Suppose that $\wp$ satisfies (4.11). Applying the operator ${}^\rho I^r_{0+}$ to the both sides of the equation

$$({}^\rho D^r_{0+}\wp)(t) = h(t), \tag{4.13}$$

we obtain

$${}^\rho I^r_{0+}\,{}^\rho D^r_{0+}\wp(t) = {}^\rho I^r_{0+}h(t).$$

From Lemma 4.2.2, we get

$$\wp(t) = Ct^{\rho(r-1)} + ({}^\rho I^r_{0+}h)(t), \tag{4.14}$$

for some $C \in \mathbb{R}$. If we use the condition $({}^\rho I^{1-r}_0\wp)(t) = \wp_0$ in (4.14), we find

$$\wp_0 = C\frac{\rho^{r-1}\Gamma(r)}{\Gamma(1)},$$

which shows,

$$C = \frac{\wp_0\rho^{1-r}}{\Gamma(r)}.$$

So; we get (4.12).
Conversely, if $\wp$ satisfies (4.12), then $({}^\rho D^r_{0+}\wp)(t) = h(t)$; for $t \in I$, and $({}^\rho I^{1-r}_0\wp)(t) = \wp_0$.

As in the proof of the above lemma, we can show the following one

Lemma 4.3.2 *A function u is a solution of problem (4.10), if and only if u satisfies the following integral equation*

$$\wp(t) = \frac{\wp_0 \rho^{1-r}}{\Gamma(r)} t^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1} g(s) ds, \tag{4.15}$$

where $g \in C(I)$, and $g(t) = f(t, \wp(t), g(t))$.

Let $(C_{r,\rho}(I), d, 2)$ be the complete b-metric space with $c = 2$, such that $d : C_{r,\rho}(I) \times C_{r,\rho}(I) \to \mathbb{R}_+^*$ is given by:

$$d(\wp, \Im) = \|(\wp - \Im)^2\|_C := \sup_{t \in I} t^{\rho(1-r)} |\wp(t) - \Im(t)|^2.$$

Then $(C_{r,\rho}(I), d, 2)$ is a b-metric space.

Definition 4.3.3 *By a solution of the problem (4.10) we mean a function $u \in C_{r,\rho}(I)$ that satisfies*

$$\wp(t) = \frac{\wp_0 \rho^{1-r}}{\Gamma(r)} t^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1} g(s) ds,$$

where $g \in C(I)$, and $g(t) = f(t, \wp(t), g(t))$.

The following hypotheses will be used in the sequel.

(H_1) There exist $\phi \in \Phi$, $p : C(I) \times C(I) \to (0, \infty)$ and $q : I \to (0, 1)$ such that for each $\wp, \Im, \wp_1, \Im_1 \in C_{r,\rho}(I)$, and $t \in I$

$$|f(t, \wp, \Im) - f(t, \wp_1, \Im_1)| \leq t^{\frac{\rho}{2}(1-r)} p(\wp, \Im) |\wp - \wp_1| + q(t) |\Im - \Im_1|,$$

with

$$\left\| \frac{\rho^{1-r}}{\Gamma(r)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1} \frac{p(\wp, \Im)}{1 - q*} ds \right\|_C^2 \leq \phi(\|(\wp - \Im)^2\|_C),$$

where $g, h \in C(I)$

(H_2) There exist $\mu_0 \in C_{r,\rho}(I)$ and a function $\theta : C_{r,\rho}(I) \times C_{r,\rho}(I) \to \mathbb{R}$, such that

$$\theta \left(\mu_0(t), \frac{\wp_0 \rho^{1-r}}{\Gamma(r)} t^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1} g(s) ds \right) \geq 0,$$

where $g \in C(I)$, with $g(t) = f(t, \mu_0(t), g(t))$.

(H_3) For each $t \in I$, and $u, v \in C_{r,\rho}(I)$, we have:

$$\theta(\wp(t), \Im(t)) \geq 0$$

implies

$$\theta\left(\frac{\wp_0 \rho^{1-r}}{\Gamma(r)} t^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1} g(s)ds, \right.$$
$$\left. \frac{\Im_0 \rho^{1-r}}{\Gamma(r)} t^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1} h(s)ds\right) \geq 0,$$

where $g, h \in C(I)$, with $g(t) = f(t, \wp(t), g(t))$ and $h(t) = f(t, \Im(t), h(t))$.

(H_4) If $\wp_{n n \in N} \subset C(I)$ with $\wp_n \to u$ and $\theta(\wp_n, \wp_{n+1}) \geq$, then

$$\theta(\wp_n, \wp) \geq 1.$$

Theorem 4.3.4 *Assume that hypotheses $(H_1) - (H_4)$ hold. Then the problem (4.10) has a least one solution defined on I.*

Proof. Consider the operator $N : C_{r,\rho}(I) \to C_{r,\rho}(I)$ defined by

$$(N\wp)(t) = \frac{u_0 \rho^{1-r}}{\Gamma(r)} t^{\rho(r-1)} + \frac{\rho^{1-r}}{\Gamma(r)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1} g(s)ds,$$

where $g \in C(I)$, with $g(t) = f(t, \wp(t), g(t))$.
By using Lemma 4.3.2, it is clear that the fixed points of the operator N are solutions of (4.10).

Let $\alpha : C_{r,\rho}(I) \times C_{r,\rho}(I) \to (0, \infty)$ be the function defined by:

$$\begin{cases} \alpha(\wp, \Im) = 1; & if\ \theta(\wp(t), \Im(t)) \geq 0,\ t \in I, \\ \alpha(\wp, \Im) = 0; & elese. \end{cases}$$

First, we prove that N is a generalized α-ϕ-Geraghty operator:
For any $\wp, \Im \in C(I)$ and each $t \in I$, we have

$$|t^{\rho(1-r)}(N\wp)(t) - t^{\rho(1-r)}(N\Im)(t)| \leq \frac{\rho^{1-r} t^{\rho(1-r)}}{\Gamma(r)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{r-1} |g(s) - h(s)|ds,$$

with

$$g(t) = f(t, \wp(t), g(t)),$$

and

$$h(t) = f(t, \Im(t), h(t)).$$

From (H_1) we have,

$$\begin{aligned} |g(t)-h(t)| &= |f(t,\wp(t),g(t))-f(t,\Im(t),h(t))| \\ &\leq p(\wp,\Im)t^{\frac{\rho}{2}(1-r)}|\wp(t)-\Im(t)|+q(t)|g(t)-h(t)| \\ &\leq p(\wp,\Im)(t^{\rho(1-r)}|\wp(t)-\Im(t)|^2)^{\frac{1}{2}}+q(t)|g(t)-h(t)|. \end{aligned}$$

Thus,

$$|g(t)-h(t)| \leq \frac{p(\wp,\Im)}{1-q*}\|(\wp-\Im)^2\|_C^{\frac{1}{2}},$$

where $q* = \sup_{t\in I}|q(t)|$.
Next, we have

$$\begin{aligned} &|t^{\rho(1-r)}(N\wp)(t)-t^{\rho(1-r)}(N\Im)(t)| \\ \leq\ & \tfrac{\rho^{1-r}t^{\rho(1-r)}}{\Gamma(r)}\textstyle\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{r-1}\frac{p(\wp,\Im)}{1-q*}\|(\wp-\Im)^2\|_C^{\frac{1}{2}}ds \\ \leq\ & \tfrac{\rho^{1-r}t^{\rho(1-r)}}{\Gamma(r)}\textstyle\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{r-1}\frac{p(\wp,\Im)}{1-q*}\|(\wp-\Im)^2\|_C^{\frac{1}{2}}ds. \end{aligned}$$

Thus

$$\begin{aligned} &\alpha(\wp,\Im)|t^{\rho(1-r)}(N\wp)(t)-t^{\rho(1-r)}(N\Im)(t)|^2 \\ \leq\ & \|(\wp-\wp)^2\|_C\alpha(\wp,\Im)\left\|\tfrac{\rho^{1-r}}{\Gamma(r)}\textstyle\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{r-1}\frac{p(\wp,\Im)}{1-q*}ds\right\|_C^2 \\ \leq\ & \|(\wp-\Im)^2\|_C\phi(\|(\wp-\Im)^2\|_C). \end{aligned}$$

Hence

$$\alpha(\wp,\Im)\phi(2^3d(N(\Im),N(\Im)) \leq \lambda(\phi(d(\wp,\Im))\phi(d(\wp,\Im)),$$

where $\lambda \in F$, $\phi \in \Phi$, with $\lambda(t)=\frac{1}{8}t$, and $\phi(t)=t$.
So, N is generalized α-ϕ-Geraghty operator.

Let $\wp,\Im \in C_{r,\rho}(I)$ such that

$$\alpha(\wp,\Im) \geq 1.$$

Thus, for each $t \in I$, we have

$$\theta(\wp(t),\Im(t)) \geq 0.$$

This implies from (H_3) that

$$\theta(N\wp(t),N\Im(t)) \geq 0,$$

which gives

$$\alpha(N(\wp),N(\Im)) \geq 1.$$

Hence, N is a α-admissible.

Now, from (H_2), there exists $\mu_0 \in C_{r,\rho}(I)$ such that

$$\alpha(\mu_0,N(\mu_0)) \geq 1.$$

Finally, from (H_4), If $\{\mu_n\}_{n\in N} \subset M$ with $\mu_n \to \mu$ and $\alpha(\mu_n,\mu_{n+1}) \geq 1$, then

$$\alpha(\mu_n,\mu) \geq 1.$$

From an application of Theorem 1.5.6, we deduce that N has a fixed point u which is a solution of problem (4.10).

4.3.2 An Example

Let $(C_{r,\rho}([0,1]), d, 2)$ be the complete b-metric space, such that
$d : C_{r,\rho}([0,1]) \times C_{r,\rho}([0,1]) \to \mathbb{R}_+^*$ is given by:

$$d(\wp, \Im) = \|(\wp - \Im)^2\|_C.$$

Consider the following fractional differential problem

$$\begin{cases} ({}^\rho D_{0^+}^r \wp)(t) = f(t, \wp(t), ({}^\rho D_{0^+}^r \wp)(t)); \; t \in [0,1], \\ ({}^\rho I_{0^+}^{1-r} \wp)(0) = 0, \end{cases} \tag{4.16}$$

where

$$f(t, \wp(t), \Im(t)) = \frac{t^{\frac{\varrho}{2}(1-r)}(1 + \sin(|\wp(t)|))}{4(1 + |\wp(t)|)} + \frac{e^{-t}}{2(1 + |\Im(t))|}; \; t \in [0,1].$$

Let $t \in (0,1]$, and $\wp, \Im \in C_{r,\rho}([0,1])$. If $|\wp(t)| \leq |v(t)|$, then

$$\begin{aligned}
&|f(t, \wp(t), \wp_1(t)) - f(t, \Im(t), \Im_1(t))| = t^{\frac{\varrho}{2}(1-r)} \left| \frac{1 + \sin(|\wp(t)|)}{4(1 + |\wp(t)|)} - \frac{1 + \sin(|\Im(t)|)}{4(1 + |\Im(t)|)} \right| \\
+ &\left| \frac{e^{-t}}{2(1 + |\wp_1(t))|} - \frac{e^{-t}}{2(1 + |\Im_1(t))|} \right| \\
\leq &\frac{t^{\frac{\varrho}{2}(1-r)}}{4} ||\wp(t)| - |\Im(t)|| + \frac{t^{\frac{\varrho}{2}(1-r)}}{4} |\sin(|\wp(t)|) - \sin(|\Im(t)|)| \\
+ &\frac{t^{\frac{\varrho}{2}(1-r)}}{4} \, ||\wp(t)| \sin(|\Im(t)|) - |\Im(t)| \sin(|\wp(t)|)| \\
+ &\frac{e^{-t}}{2} |\wp_1(t) - \Im_1(t))| \\
\leq &\frac{t^{\frac{\varrho}{2}(1-r)}}{4} |\wp(t) - \Im(t)| + \frac{t^{\frac{\varrho}{2}(1-r)}}{4} |\sin(|\wp(t)|) - \sin(|\Im(t)|)| \\
+ &\frac{t^{\frac{\varrho}{2}(1-r)}}{4} \, ||\Im(t)| \sin(|\Im(t)|) - |\Im(t)| \sin(|\wp(t)|)| \\
+ &\frac{e^{-t}}{2} |\wp_1(t) - \Im_1(t))| \\
= &\frac{t^{\frac{\varrho}{2}(1-r)}}{4} \wp(t) - \Im(t)| + \frac{t^{\frac{\varrho}{2}(1-r)}}{4} (1 + |\Im(t)|) \, |\sin(|\wp(t)|) - \sin(|\Im(t)|)| \\
+ &\frac{e^{-t}}{2} |\wp_1(t) - \Im_1(t))| \\
\leq &\frac{t^{\frac{\varrho}{2}(1-r)}}{4} |\wp(t) - \Im(t)| + \frac{t^{\frac{\varrho}{2}(1-r)}}{2} (1 + |\Im(t)|) \\
\times &\left| \sin\left(\frac{||\wp(t)| - |\Im(t)|}{2} \right) \right| \left| \cos\left(\frac{|\wp(t)| + |\Im(t)|}{2} \right) \right|
\end{aligned}$$

$$
\begin{aligned}
+ &\quad \frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t)| \\
\leq &\quad \frac{t^{\frac{\varrho}{2}(1-r)}}{4}(2+|\Im(t)|)|\wp(t)-\Im(t)| + \frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t)|.
\end{aligned}
$$

The case when $|\Im(t)| \leq |\wp(t)|$, we get

$$|f(t,\wp(t))-f(t,\Im(t))| \leq \frac{t^{\frac{\varrho}{2}(1-r)}}{4}(2+|\wp(t)||\wp(t)-\Im(t)| + \frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t)|.$$

Hence

$$|f(t,\wp(t))-f(t,\Im(t))| \leq \frac{T^{\frac{\varrho}{2}(1-r)}}{4}\min_{t\in I}\{2+|\wp(t)|,2+|\Im(t)|\}|\wp(t)-\Im(t)|+\frac{e^{-t}}{2}|\wp_1(t)-\Im_1(t)|.$$

Thus, hypothesis (H_1) is satisfied with

$$p(\wp,\Im) = \frac{T^{\frac{\varrho}{2}(1-r)}}{4}\min_{t\in I}\{2+|\wp(t)|,2+|\Im(t)|\},$$

and

$$q(t) = \frac{1}{2}e^{-t}.$$

Define the functions $\lambda(t) = \frac{1}{8}t$, $\phi(t) = t$, $\alpha : C_{r,\rho}([0,1]) \times C_{r,\rho}([0,1]) \to \mathbb{R}_+^*$ with

$$\begin{cases} \alpha(\wp,\Im) = 1; \ if \ \delta(\wp(t),\Im(t)) \geq 0, \ t \in I, \\ \alpha(\wp,\Im) = 0; \ else, \end{cases}$$

and $\delta : C_{r,\rho}([0,1]) \times C_{r,\rho}([0,1]) \to \mathbb{R}$ with $\delta(\wp,\Im) = \|\wp-\Im\|_C$.
Hypothesis (H_2) is satisfied with $\mu_0(t) = \wp_0$. Also, (H_3) holds from the definition of the function δ. Hence by Theorem 4.3.4, problem (4.16) has at least one solution defined on $[0,1]$.

4.4 Initial Value Problems for Caputo-Fabrizio Implicit Fractional Differential Equations in b-Metric Spaces

Motivated by the works mentioned in the Introduction of the section 3.2 , in this section, we investigate the existence and uniqueness of solutions for the following class of initial value problems of Caputo-Fabrizio fractional differential equations:

$$\begin{cases} ({}^{CF}D_0^r u)(t) = f(t,u(t),({}^{CF}D_0^r u)(t)); \ t \in I := [0,T], \\ u(0) = u_0, \end{cases} \tag{4.17}$$

where $T > 0$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r \in (0,1)$, and $u_0 \in \mathbb{R}$.

4.4.1 Existence of Solutions

Lemma 4.4.1 *Let $h \in L^1(I, \mathbb{R})$. A function $u \in C(I)$ is a solution of problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) = h(t); \quad t \in I := [0, T] \\ u(0) = u_0, \end{cases} \tag{4.18}$$

if and only if u satisfies the following integral equation

$$u(t) = C + a_r h(t) + b_r \int_0^t h(s)ds. \tag{4.19}$$

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ \ b_r = \frac{2r}{(2-r)M(r)},$$
$$C = u_0 - a_r h(0).$$

proof. Suppose that u satisfies (4.18). From Proposition 1 in [**?**]; the equation

$$({}^{CF}D_0^r u)(t) = h(t),$$

implies that

$$u(t) - u(0) = a_r(h(t) - h(0)) + b_r \int_0^t h(s)ds.$$

Thus from the initial condition $u(0) = u_0$, we get

$$u(t) = u(0) + a_r h(t) - a_r h(0) + b_r \int_0^t h(s)ds.$$

So; we get (4.19).
Conversely, if u satisfies (4.19), then $({}^{CF}D_0^r u)(t) = h(t)$; for $t \in I$, and $u(0) = u_0$.

We can conclude the following lemma:

Lemma 4.4.2 *A function u is a solution of problem (4.17), if and only if u satisfies the following integral equation*

$$u(t) = c + a_r g(t) + b_r \int_0^t g(s)ds,$$

where $g \in X$, with $g(t) = f(t, u(t), g(t))$ and

$$c = u_0 - a_r g(0).$$

Let $(C(I), d, 2)$ be the complete b-metric space with $c = 2$, such that $d : C(I) \times C(I) \to \mathbb{R}_+^*$ is given by:

$$d(u, v) = \|(u - v)^2\|_\infty := \sup_{t \in I} |u(t) - v(t)|^2.$$

Then $(C(I), d, 2)$ is a b-metric space.

In this section, we are concerned with the existence results of the problem (4.17).

Definition 4.4.3 *By a solution of the problem (4.17) we mean a function $u \in C(I)$ that satisfies*

$$u(t) = c + a_r g(t) + b_r \int_0^t g(s)ds, \tag{4.20}$$

where $g \in C(I)$, with $g(t) = f(t, u(t), g(t))$ and

$$c = u_0 - a_r g(0).$$

The following hypotheses will be used in the sequel.

(H_1) There exist $p : C(I) \times C(I) \to (0, \infty)$ and $q : I \to (0, 1)$ such that for each $u, v, u_1, v_1 \in C(I)$ and $t \in I$

$$|f(t, u, v) - f(t, u_1, v_1)| \leq p(u, v)|u - u_1| + q(t)|v - v_1|,$$

with

$$\left\| 1 + 2a_r \frac{p(u, v)}{1 - q*} + b_r \int_0^t \frac{p(u, v)}{1 - q*} ds \right\|_\infty^2 \leq \phi(\|(u - v)^2\|_\infty).$$

(H_2) There exist $\phi \in \Phi$ and $\mu_0 \in C(I)$ and a function $\theta : C(I) \times C(I) \to \mathbb{R}$, such that

$$\theta\left(\mu_0(t), c + a_r g(t) + b_r \int_0^t g(s)ds\right) \geq 0,$$

where $g \in C(I)$, with $g(t) = f(t, \mu_0(t), g(t))$,

(H_3) For each $t \in I$, and $u, v \in C(I)$, we have:

$$\theta(u(t), v(t)) \geq 0$$

implies

$$\theta\left(c + a_r g(t) + b_r \int_0^t g(s)ds, c + a_r h(t) + b_r \int_0^t h(s)ds\right) \geq 0,$$

where $g, h \in C(I)$, with

$$g(t) = f(t, u(t), g(t)) \text{ and } h(t) = f(t, v(t), h(t)).$$

(H_4) If $u_{nn\in N} \subset C(I)$ with $u_n \to u$ and $\theta(u_n, u_{n+1}) \geq 1$, then

$$\theta(u_n, u) \geq 1,$$

(H_5) For all fixed solutions x, y of problem (4.17), either

$$\theta(x(t), y(t)) \geq 0,$$

or

$$\theta(y(t), x(t)) \geq 0.$$

Theorem 4.4.4 *Assume that the hypotheses $(H_1) - (H_4)$ hold. Then the problem (4.17) has a least one solution defined on I. Moreover, if (H_5) holds, then we get a unique solution.*

Proof. Consider the operator $N : C(I) \to C(I)$ such that,

$$(Nu)(t) = c + a_r g(t) + b_r \int_0^t g(s)ds,$$

where $g \in C(I)$, with $g(t) = f(t, u(t), g(t))$ and

$$c = u_0 - a_r g(0).$$

Using Lemma 4.4.2, it is clear that the fixed points of the operator N are solutions of our problem (4.17).

Let $\alpha : C(I) \times C(I) \to]0, \infty)$ be the function defined by:

$$\begin{cases} \alpha(u, v) = 1; & if\ \theta(u(t), v(t)) \geq 0,\ t \in I, \\ \alpha(u, v) = 0; & eles. \end{cases}$$

First, we prove that N is a generalized α-ϕ-Geraghty operator:
For any $u, v \in C(I)$ and each $t \in I$, we have

$$|(Nu)(t) - (Nv)(t)| \leq |c_g - c_h| + a_r|g(t) - h(t)| + b_r \int_0^t |g(s) - h(s)|ds$$

where $g, h \in C(I)$, with $g(t) = f(t, u(t), g(t))$ and $h(t) = f(t, v(t), h(t))$.
From (H_1) we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, v(t), h(t))| \\ &\leq p(u, v)|u(t) - v(t)| + q(t)|g(t) - h(t)| \\ &\leq p(u, v)(|u(t) - v(t)|^2)^{\frac{1}{2}} + q(t)|g(t) - h(t)|. \end{aligned}$$

Thus,

$$\|g - h\|_\infty \leq \frac{p(u, v)}{1 - q*}\|(u - v)^2\|_\infty^{\frac{1}{2}},$$

where $q* = \sup_{t\in I}|q(t)|$.
Next, we have

$$\begin{aligned}|(Nu)(t)-(Nv)(t)| &\leq \|(u-v)^2\|_\infty^{\frac{1}{2}} + 2a_r\tfrac{p(u,v)}{1-q*}\|(u-v)^2\|_\infty^{\frac{1}{2}}\\ &+ b_r\int_0^t \tfrac{p(u,v)}{1-q*}\|(u-v)^2\|_\infty^{\frac{1}{2}}ds.\end{aligned}$$

Thus

$$\begin{aligned}\alpha(u,v)|(Nu)(t)-(Nv)(t)|^2 &\leq \|(u-v)^2\|_\infty\alpha(u,v)\\ &\quad\left\|1+2a_r\tfrac{p(u,v)}{1-q*}+b_r\int_0^t\tfrac{p(u,v)}{1-q*}ds\right\|_\infty^2.\\ &\leq \|(u-v)^2\|_\infty\phi(\|(u-v)^2\|_\infty).\end{aligned}$$

Hence

$$\alpha(u,v)\phi(2^3d(N(u),N(v))\leq\lambda(\phi(d(u,v))\phi(d(u,v)),$$

where $\lambda\in\digamma$, $\phi\in\Phi$, with $\lambda(t)=\frac{1}{8}t$, and $\phi(t)=t$.
So, N is generalized α-ϕ-Geraghty operator.

Let $u,v\in C(I)$ such that

$$\alpha(u,v)\geq 1.$$

Thus, for each $t\in I$, we have

$$\theta(u(t),v(t))\geq 0.$$

This implies from (H_3) that

$$\theta(Nu(t),Nv(t))\geq 0,$$

which gives

$$\alpha(N(u),N(v))\geq 1.$$

Hence, N is a α-admissible.

Now, from (H_2), there exists $\mu_0\in C(I)$ such that

$$\alpha(\mu_0,N(\mu_0))\geq 1.$$

Finally, From (H_4), If $\mu_{nn\in N}\subset M$ with $\mu_n\to\mu$ and $\alpha(\mu_n,\mu_{n+1})\geq 1$, then

$$\alpha(\mu_n,\mu)\geq 1.$$

From an application of Theorem 1.5.6, we deduce that N has a fixed point u which is a solution of problem (4.17).

Moreover, (H_5), implies that if x and y are fixed points of N, then either $\theta(x,y)\geq 0$ or $\theta(y,x)\geq 0$. This implies that either $\alpha(x,y)\geq 0$ or $\alpha(y,x)\geq 0$.
Hence; problem (4.17) has the uniqueness.

4.4.2 An Example

Consider the Caputo-Fabrizio fractional differential problem

$$\begin{cases} ({}^{CF}D_0^r u)(t) = f(t, u(t), ({}^{CF}D_0^r u)(t));\ t \in [0,1] \\ u(0) = 0, \end{cases} \tag{4.21}$$

where

$$f(t,u,v) = \frac{1+\sin(|u|)}{4(1+|u|)} + \frac{1}{4(1+|v|)};\ t \in [0,1].$$

Let $(C([0,1]), d, 2)$ be the complete b-metric space, such that $d : C([0,1]) \times C([0,1]) \to \mathbb{R}_+^*$ is given by:

$$d(u,v) = \|(u-v)^2\|_\infty := \sup_{t\in[0,1]} |u(t)-v(t)|^2.$$

For each $u, v \in C([0,1])$, we have Let $t \in (0,1]$, and $u, v, \bar{u}, \bar{v} \in C([0,1])$. If $|u(t)| \leq |v(t)|$, then

$$\begin{aligned}
|f(t,u(t),\bar{u}(t)) - f(t,v(t),\bar{v}(t))| &\leq \left| \frac{1+\sin(|u(t)|)}{4(1+|u(t)|+|\bar{u}(t)|)} - \frac{1+\sin(|v(t)|)}{4(1+|v(t)|+|\bar{v}(t)|)} \right| \\
&+ \frac{|\bar{u}(t)-\bar{v}(t)|}{4} \\
&\leq \frac{1}{4}||u(t)|-|v(t)|| + \frac{1}{4}|\sin(|u(t)|) - \sin(|v(t)|)| \\
&+ ||u(t)|\sin(|v(t)|) - |v(t)|\sin(|u(t)|)| \\
&+ \frac{|\bar{u}(t)-\bar{v}(t)|}{4} \\
&\leq |u(t)-v(t)| + \frac{1}{4}|\sin(|u(t)|) - \sin(|v(t)|)| \\
&+ \frac{|\bar{u}(t)-\bar{v}(t)|}{4} \\
&+ ||v(t)|\sin(|v(t)|) - |v(t)|\sin(|u(t)|)| \\
&= |u(t)-v(t)| + (1+|v(t)|)\,|\sin(|u(t)|) - \sin(|v(t)|)| \\
&+ \frac{|\bar{u}(t)-\bar{v}(t)|}{4} \\
&\leq |u(t)-v(t)| + \frac{1}{2}(1+|v(t)|) \\
&\times \left|\sin\left(\frac{||u(t)|-|v(t)|}{2}\right)\right| \left|\cos\left(\frac{|u(t)|+|v(t)||}{2}\right)\right| \\
&\leq (2+\|v\|_\infty)\|u-v\|_\infty + \frac{\|\bar{u}-\bar{v}\|_\infty}{4}.
\end{aligned}$$

The case when $|v(t)| \leq |u(t)|$, we get

$$|f(t,u(t),\bar{u}(t)) - f(t,v(t),\bar{v}(t))| \leq (2+\|u\|_\infty)\|u-v\|_\infty + \frac{\|\bar{u}-\bar{v}\|_\infty}{4}.$$

Hence

$$|f(t,u(t),\bar{u}(t)) - f(t,v(t),\bar{v}(t))| \leq \min\{2+\|u\|_\infty, 2+\|v\|_\infty\}\|u-v\|_\infty + \frac{\|\bar{u}-\bar{v}\|_\infty}{4}.$$

Thus, hypothesis (H_2) is satisfied with

$$p(u,v) = \min\{2+\|u\|_\infty, 2+\|v\|_\infty\}, \text{ and } q(t) = \frac{1}{4}.$$

Define the functions $\lambda(t) = \frac{1}{8}t$, $\phi(t) = t$, $\alpha : C([0,1]) \times C([0,1]) \to \mathbb{R}_+^*$ with

$$\begin{cases} \alpha(u,v) = 1; \ if \ \delta(u(t),v(t)) \geq 0, \ t \in I, \\ \alpha(u,v) = 0; \ else, \end{cases}$$

and $\delta : C([0,1]) \times C([0,1]) \to \mathbb{R}$ with $\delta(u,v) = \|u-v\|_\infty$.
Hypothesis (H_2) is satisfied with $\mu_0(t) = u_0$. Also, (H_3) holds from the definition of the function δ.

Simple computations show that all conditions of Theorem 4.4.4 are satisfied. Hence, we get the existence of solutions and the uniqueness for problem (4.21).

Conclusion and Perspectives

In this thesis, we have presented some results to the theory of the existence of solutions and uniqueness and the Ulam-type stability of some classes of fractional implicit differential equations with the derivatives of Caputo, Hadamard, Caputo–Fabrizio, Katugampola, and mention all the derivatives. The problems studied are with initial and boundary conditions. The results obtained are based on some fixed point theorems and the measure of non-compactness.

In future research, we plan to study some fracional differential and integral equations and inclusions with impulses (instantaneous and not instantaneous) in Banach and fréchet spaces.

Bibliography

[1] S. Abbas, M. Benchohra and J.J. Nieto, Functional implicit hyperbolic fractional order differential equations with delay, *Afr. Diaspora J. Math.*, **15** (1) (2013), 74-96.

[2] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[3] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.

[4] S. Abbas, M. Benchohra and S. Sivasundaram, Coupled Pettis Hadamard fractional differential systems with retarded and advanced arguments, *Journal of Mathematics and Statistics*, 2018, **14** (1) : 56-63.

[5] S. Abbas, M. Benchohra and J.J. Trujillo, Fractional order impulsive hyperbolic implicit differential equations with state-dependent delay, *Dedicated to the memory of A. Kilbas* (2012), 31-42.

[6] S. Abbas and M. Benchohra, On the generalized Ulam-Hyers-Rassias stability for Darboux problem for partial fractional implicit differential equations. *Applied Mathematics E-Notes*, **14** (2014), 20-28.

[7] S. Abbas, M. Benchohra, N. Laledj and Y. Zhou, Existence and Ulam stability for implicit fractional q-difference equations. *Advances in Difference Equations*, SpringerOpen Journal, 2019.

[8] H. Afshari, H. Aydi, E. Karapinar, Existence of fixed points of set-valued mappings in b-metric spaces, *East Asian Math. J.* **32** (3) (2016), 319-332.

[9] H. Afshari, H. Aydi, E. Karapinar, On generalized $\alpha - \psi -$Geraghty contractions on b-metric spaces, *Georgian Math. J.* **27**(1) (2020), 9-21.

[10] R.P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[11] J. Appell, Implicit Functions, Nonlinear Integral Equations, and the Measure of Noncompactness of the Superposition Operator, *Journal of Mathematical Analysis and Applications.* **83** (1981), 251-263.

[12] B. Ahmad, and S. Sivasundaram, Theory of fractional differential equations with three-point, boundary conditions, *Commun. Appl. Anal.* **12** (2008), 479-484.

[13] K.K. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, *Measures of noncompactness and condensing operators*, Birkhäuser Verlag, Basel, Bostn, Berlin, 1992.

[14] R. Almeida, A. B. Malinowska and T. Odzijewicz, Fractional differential equations with dependence on the Caputo-Katugampola derivative, *J. Comput Nonlinear Dynam.* **11** (6) (2016), 061017 (11 pages).

[15] J.C. Alvàrez, Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces, *Rev. Real. Acad. Cienc. Exact. Fis. Natur. Madrid* **79** (1985), 53-66.

[16] A. Alsaedi, B. Ahmad and M. Alghanmi, Extremal solutions for generalized Caputo fractional differential equations with Steiltjes-type fractional integro-initial conditions. Appl. Math. Lett. 91, 113-120 (2019).

[17] C. Alsina and R. Ger, On some inequalities and stability results related to the exponential function. *J. Inequal. Appl.* **2**, (1998), 373-380.

[18] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems,* Kluwer Academin Publishers, Dordrecht (2003).

[19] Y. Arioua, B. Basti, N. Benhamidouche, Initial value problem for nonlinear implicit fractional differential equations with Katugampola derivative, *Appl. Math. E-Notes* **19** (2019), 397-412.

[20] T. Aoki, On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Japan* **2**, (1950), 64-66.

[21] D.D. Bainov and S.G. Hristova, Integral inequalities of Gronwall type for piecewise continuous functions ,*J. Appl. Math. Stoc. Anal.*, 10(1997), 89-94.

[22] D. Baleanu, Z.B. Güvenç and J.A.T. Machado, *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.

[23] D. Baleanu, J.A.T. Machado and A.C.-J. Luo *Fractional Dynamics and Control*, Springer, 2012.

[24] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[25] J. Banaȁs and L. Olszowy, Measures of noncompactness related to monotonicity, *Comment. Math. (Prace Mat.)* **41** (2001), 13-23.

[26] J. Banaś, J, Zajac, T: A new approach to the theory of functional integral equations of fractional order. J. Math. Anal. Appl. **375**, 375-387 (2011).

[27] T. Bashiri, S.M. Vaezpour, J.J. Nieto, *Approximating solution of Fabrizio-Caputo Volterra's model for population growth in a closed system by homotopy analysis method*, J. Funct. Spaces **2018** (2018), Article ID 3152502.

[28] F. Bekada, S. Abbas, and M. Benchohra, Boundary value problem for Caputo–Fabrizio random fractional differential equations, *Moroccan J. Pure Appl. Anal. (MJPAA)* **6** (2) (2020), 218-230.

[29] F. Bekada, S. Abbas, M. Benchohra, and N.Juan, Dynamics and stability for Katugampola random fractional differential equations, *AIMS Mathematics* (2021), 8654-8666 .

[30] M. Benchohra, S. Bouriah and J. Henderson, Existence and stability results for nonlinear implicit neutral fractional differential equations with finite delay and impulses, *Comm. Appl. Nonlinear Anal.* **22** (1) (2015), 46-67.

[31] M. Benchohra, S. Bouriah and J.Henderson, Existence and stability results for nonlinear implicit neutral fractional differential equations with finite delay and impulses, *Communications on Applied Nonlinear Analysis* Vol. 22 No. 1(2015), 46-67.

[32] M. Benchohra, A. Cabada and D. Seba, An existence result for non-linear fractional differential equations on Banach spaces, *Boundary Value Problems.* Volume 2009 (2009), Article ID 628916, 11 pages.

[33] M. Benchohra and J. E. Lazreg, Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions. *Romanian J. Math. Comput. Sc.* **4** (1) (2014), 60-72.

[34] M. Benchohra and J. E. Lazreg, Existence results for nonlinear implicit fractional differential equations. *Surv. Math. Appl.* **9** (2014), 79-92.

[35] M. Benchohra and D. Seba, Impulsive fractional differential equations in Banach Spaces *Electron. J. Qual. Theory Differ. Equ.*. Spec. Ed. I (2009), No. 8, pp. 1-14.

[36] M. Benjemaa, Taylor's formula involving generalized fractional derivatives. Appl. Math. Comput. 335, 182-195 (2018).

[37] M.F. Bota, L. Guran, and A. Petrusel, New fixed point theorems on b-metric spaces with applications to coupled fixed point theory, *J. Fixed Point Theo. Appl.* **22** (3) (2020),74.

[38] T.A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii-Schaefer type. *Math. Nachr.* **189** (1989), 23-31.

[39] M. Caputo, Linear model of dissipation whose Q is almost frequency independent. II. Geophys. J. Int. 13(5), 529-539 (1967).

[40] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.* **1** (2) (2015), 73–85.

[41] J. Chen, F. Liu, I. Turner and V. Anh, The fundamental and numerical solutions of the Riesz space-fractional reaction-dispersion equation, *The ANZIAM Journal* **50** (1) (2008), 45-57.

[42] Y.J. Cho , T. M. Rassias and R. Saadati, *Stability of Functional Equations in Random Normed Spaces*, Science-Business Media, Springer 52, 2013.

[43] S. Cobzas and S. Czerwik. The completion of generalized b-metric spaces and fixed points, *Fixed Point Theory* **21** (1) (2020), 133-150.

[44] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Semin. Mat. Fis. Univ. Modena.* **46** (2) 1998, 263-276.

[45] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inf. Univ. Ostrav.* **1** (1993), 5-11.

[46] D. Derouiche, H. Ramoul. New fixed point results for F-contractions of Hardy-Rogers type in b-metric spaces with applications. *J. Fixed Point Theo. Appl.* **22** (4) (2020), 86.

[47] N. Engheta, Fractional Curl Operator in Electromagnetics. *Microwave Opt. Tech. Lett.* **17** (1) (1998), 86-91.

[48] M. Faryad and Q. A. Naqvi, Fractional Rectangular Waveguide. *Progress In Electromagnetics Research* PIER **75** (2007), 383-396.

[49] A. Fernandez and D. Baleanu, The mean value theorem and Taylor's theorem for fractional derivatives with Mittag-Leffler kernel. Adv. Differ. Equ. 2018, 86 (2018).

[50] Y.Y. Gambo, F. Jarad, D. Baleanu and T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, Adv. Difference Equ. 2014 (2014), No. 10, 12 pp.

[51] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431-436.

[52] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[53] M. Gulsu, Y. Ozturk and A. Anapali, Numerical approach for solving fractional relaxation oscillation equation. Appl. Math. Model. 37, 5927-5937 (2013).

[54] D.J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces,* Kluwer Academic Publishers, Dordrecht, 1996.

[55] J. Hadamard, Essai sur l'étude des fonctions données par leur d'eveloppment de Taylor, J. Pure Appl. Math. 4 (8) (1892), 101-186.

[56] J.K. Hale, *Functional Differential Equations*, Springer-Verlag, New York, 1971.

[57] J. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, , Applied Mathematicals Sciences, 99, Springer-Verlag, New York, 1993.

[58] J. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21** (1978), 11-41.

[59] J.K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[60] R. Hermann, *Fractional Calculus: An Introduction For Physicists*, World Scientific Publishing Co. Pte. Ltd. 2011.

[61] E. Hernández, On abstract differential equations with state dependent non-local conditions, *J. Math. Anal. Appl.* **1** (4)(2018), 408-425.

[62] E. Hernandez, K.A.G. Azevedo and V. Rolnik, Well-posed of abstract differential equations with state-dependent delay, em Mathematische Nachrichten bf 291 (13) (2018), 12045-2056.

[63] E. Hernández, D. Fernandes and J, Wu, Well-posedness of abstract integro-differential equations with state-dependent delay, *Proc. Amer. Math. Soc.* **148** (4)(2020), 1595-1609.

[64] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[65] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Math., **1473**, Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[66] Y. Hino, S. Murakami, T. Naito and N.V. Minh, A variation-of-constants formula for abstract functional differential equations in phase space, *J. Differential Equations* **179** (2002) 336-355.

[67] C. Horvath, Measure of Non-compactness and multivalued mappings in complete metric topological spaces, *J. Math. Anal. Appl.* **108** (1985), 403-408.

[68] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad.Sci. U.S.A. 27 (1941), 222-224.

[69] R.W. Ibrahim, Stability for Univalent Solutions of Complex Fractional Differential Equations, *Proceedings of the Pakistan Academy of Sciences*, Vol. 49 **3**, 2012, 227-232.

[70] K.W. Jun and H.M. Kim, On the stability of an n-dimensional quadratic and additive functional equation, *Math. Inequal. Appl.* **19** (9) (2006), 854-858.

[71] F. Jarad, T. Abdeljawad and D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives. Adv. Differ. Equ. 2012, 142 (2012).

[72] S.M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.* **222** (1998), 126-137.

[73] S.M. Jung, Hyers-Ulam stability of linear differential equations of first order, II *Appl. Math. Lett.* **19** (2006), 854-858.

[74] S.M. Jung, K.S. Lee, Hyers-Ulam stability of first order linear partial differential equations with constant coefficients *Math. Inequal. Appl.* **10** (2007), 261-266.

[75] U.N. Katugampola, A new approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.* **6** (2014), 15.

[76] U.N. Katugampola, New approach to a generalized fractional integral, *Appl. Math. Comput.* **218** (2011), 860-865.

[77] G.H. Kim, On the stability of functional equations with square-symmetric operation, *Math. Inequal. Appl* **17** (4) (2001), 257-266.

[78] A.A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.* **38** (6) (2001) 1191-1204.

[79] W.A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, Springer-Science + Business Media, B.V, Dordrecht, 2001.

[80] S. Krim, S. Abbas, M. Benchohra and M.A. Darwish, Boundary value problem for implicit Caputo–Fabrizio fractional differential equations, *Int. J. Difference Equ.* **15** (2) (2020), 493-510.

[81] S. Krim, S. Abbas, and M. Benchohra, Caputo-Hadamard implicit fractional differential equations with delay, Pãulo Journal of Mathematical Sciences, 15 (1) (2021), 463-484.

[82] S. Krim, S. Abbas, M. Benchohra and E. Karapinar, Terminal value problem for implicit Katugampola fractional differential equations in b-metric spaces, *Journal of Function Spaces 2021*, Article ID 5535178, 7 pages.

[83] S. Krim, S. Abbas, and M. Benchohra, Initial value problems for Caputo–Fabrizio implicit fractional differential equations in b-metric spaces, Bulletin of the Transilvania University of Braşov, 1 (63) No. 1 - 2021. SeriesIII: Mathematics, Informatics, Physics,1-13.

[84] Q. Liu, F. Liu, I. Turner and V. Anh, Numerical simulation for the 3D seep age flow with fractional derivatives in porous media,

[85] J. Losada and J.J. Nieto. , Properties of a New Fractional Derivative without Singular Kernel, *Progr. Fract. Differ. Appl* **1**(2) (2015), 87–92.

[86] A.J. Luo and V. Afraimovich, *Long-range Interactions, Stochasticity and Fractional Dynamics*, Springer, New York, Dordrecht, Heidelberg, London, 2010.

[87] C. Li, F. Zeng, and F. Liu, Spectral approximations to the fractional integral and derivative, *Fractional Calculus and Applied Analysis* 15 (3) (2012), 383-406.

[88] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.

[89] Q.A. Naqvi and M. Abbas, Complex and Higher Order Fractional Curl Operator in Electromagnetics, *Optics Communications* **241**, (2004), 349-355.

[90] Z.M Odibat and N.T. Shawagfeh, Generalized Taylor's formula. Appl. Math. Comput. 186, 286-293 (2007).

[91] S.K. Panda, E. Karapinar, and A. Atangana. A numerical schemes and comparisons for fixed point results with applications to the solutions of Volterra integral equations in dislocated extended b-metric space, *Alexandria Engineering J.* **59** (2) (2020), 815-827.

[92] I. Petras, *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation* Springer Heidelberg Dordrecht London New York, 2011.

[93] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[94] T.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** , (1978), 297-300.

[95] J.M. Rassias, *Functional Equations, Difference Inequalities and Ulam Stability Notions (F.U.N.)*, Nova Science Publishers, Inc. New York, 2010.

[96] T.M. Rassias and J. Brzdek, *Functional Equations in Mathematical Analysis*, Springer 86, New York Dordrecht Heidelberg London 2012.

[97] S.S. Ray, A new approach for the application of Adomian decomposition method for the solution of fractional space diffusion equation with insulated ends, *Applied Mathematics and Computation* **202** , (2) (2008), 544-549.

[98] I.A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.* **26** (2010), 103-107.

[99] P. Sahoo, T. Barman and J. P. Davim, *Fractal Analysis in Machining*, Springer, New York, Dordrecht, Heidelberg, London, 2011.

[100] A. Salim, M. Benchohra, J. R. Graef and J. E. Lazreg, Boundary value problem for fractional generalized Hilfer-type fractional derivative with non-instantaneous impulses. *Fractal Fract.* **5** (2021), 1-21.

[101] A. Salim, M. Benchohra, J. R. Graef and J. E. Lazreg, Initial value problem for hybrid ψ-Hilfer fractional implicit differential equations. *J. Fixed Point Theory Appl.* **24** (2022), 14 pp.

[102] A. Salim, M. Benchohra, E. Karapinar and J. E. Lazreg, Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations. *Adv. Differ. Equ.* **2020** (2020), 21 pp.

[103] A. Salim, M. Benchohra, J. E. Lazreg and J. Henderson, Nonlinear implicit generalized Hilfer-type fractional differential equations with non-instantaneous impulses in Banach spaces. *Adv. Theory Nonlinear Anal. Appl.* **4** (2020), 332-348.

[104] A. Salim, M. Benchohra, J. E. Lazreg and J. Henderson, On k-Generalized ψ-Hilfer Boundary Value Problems with Retardation and Anticipation. *Adv. Theory Nonlinear Anal. Appl.* **6** (2022), 173-190.

[105] A. Salim, M. Benchohra, J. E. Lazreg and E. Karapinar, On k-Generalized ψ-Hilfer Impulsive Boundary Value Problem with Retarded and Advanced Arguments. *J. Math. Ext.* **15** (2021), 39 pp.

[106] A. Salim, M. Benchohra, J. E. Lazreg and G. N'Guérékata, Boundary Value Problem for Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations with Impulses. *Abstract and Applied Analysis.* **2021** (2021), 17pp.

[107] A. Salim, M. Benchohra, J. E. Lazreg, J. J. Nieto and Y. Zhou, Nonlocal Initial Value Problem for Hybrid Generalized Hilfer-type Fractional Implicit Differential Equations. *Nonauton. Dyn. Syst.* **8** (2021), 87-100.

[108] V.E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[109] J.J. Trujillo, M. Rivero and B. Bonilla, On a Riemann-Liouville generalized Taylor's formula. J. Math. Anal. Appl. 231, 255-265 (1999).

[110] S.M. Ulam, *Problems in Modern Mathematics*, Chapter 6, JohnWiley and Sons, New York, USA,1940).

[111] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York,1960.

[112] J. Vanterler da C. Sousa and E.Capelas de Oliveira, On the $\psi-$Hilfer fractional derivative, *Communications in Nonlinear Science and Numerical Simulation,* **60** (2019), 72-91.

[113] J. Vanterler da C. Sousa and E.Capelas de Oliveira, Leibniz type rule: $\psi-$Hilfer fractional operator, *Communications in Nonlinear Science and Numerical Simulation,* **77** (2018), 305-311.

[114] J. Wang, M. Fečkan and Y. Zhou, Ulam's type stability of impulsive ordinary differential equations, *J. Math. Anal. Appl.* **395** (2012), 258-264.

[115] J. Wang, L. Lv and Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electron. J. Qual. Theory Differ. Equat.* **63** (2011), 1-10.

[116] J. Wang and Y. Zhang, Existence and stability of solutions to nonlinear impulsive differential equations in β-normed spaces, *Electron. J. Differential Equations*, (2014), No. 83, 1-10.

[117] X.J. Xiao-Jun, H.M. Srivastava, J.T. Machado, *A new fractional derivative without singular kernel*, Therm. Sci. **20** (2) (2016), 753-756.

[118] H. Ye, J. Gao, and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.* **328** (2007), 1075-1081.

[119] K. Yosida, *Fonctional Analysis,* 6th edn. Springer-Verlag, Berlin, 1980.

[120] S. Zhai, X. Feng, and Z. Weng, New high-order compact adi algorithms for 3D nonlinear time-fractional convection-diffusion equation, *Math. Prob. Eng* **2013** (2013), 11 pages.