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Dedication

This senior thesis is dedicated to my parents:

My Mother, in the first place, is the one who sincerely raised me with her caring and offered me unconditional love, a very special thank you for the myriad of ways in which, throughout my life, you have actively supported me in my determination to find and realise my potential.

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Contents

Introduction

Fractional Calculus is the feilds of mathematical analysis which deals with the investigation and application of the integrals and derivative of arbitrery order. The term fractional is misnomer but it is retained following the prevailing use.

The fractional calculus may be considered an old and yet novel topic. It is an old topic since, starting from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed up to nowadays. In fact the idea of generalizing the notion of derivative to non integer order, in particular to the order 1/2, is contained in the correspondence of Leibniz with Bernoulli, L'Hôpital and Wallis. Euler took the first step by observing that the resuit of the evaluation of the derivative of the power function has a meaning for non-integer order thanks to his Gamma function.

A list of mathematicians, who have provided important contributions up to the middle of the 20-th century, includes P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823-1826), J. Liouville (1832-1837), B. Riemann (1847), A.K. Grûnwald (1867-1872), P.A. Nekrassov (1888), J. Hadamard (1892), O. Heaviside (1892-1912), G.H. Hardy and J.E. Littlewood (1917-1928), H. Weyl (1917), P. Levy (1923), A. Marchaud (1927), H.T. Davis (1924-1936), A. Erdélyi (1939-1965), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949), W. Feller (1952).

However, it may be considered a novel topic as well, since only from a little more than thirty years it has been object of specialized conferences and treatises. B. Ross organized the First Conference on Fractional Calculus and its Applications 1974.

Nowadays, to our knowledge, the list of texts in book form devoted to fractional calculus includes less than 20 titles. In recent years considerable interest in fractional calculus has been stimulated by the applications that it finds in different fields of science, including numerical analysis, physics, biology, economics and finance.

This senior thesis is orgnized as follows. In chapter 1 we develop the Wiener integration w.r.t fractional Brownian motion. In this chapter we will give definitions and properties of the needed theory. We briefly recall some basic notions of the Fractional calculus, then we skim through the Fractional Brownian Motion we review rapidly the basic concepts, then we discuss Wiener integration with respect to fBm and various relations between different "integrable spaces" related to fBm. Finally, we provide new and rather simple proofs of some basic properties not only for the fractional Brownian motion. But for Wiener integration w.r.t fractional Brownian motion.

Next, Chapter 2 is devoted to stochastic integration w.r.t. fractional Brownian motion and other aspects of stochastic calculus of fBm. There exist several approaches to stochastic integration w.r.t. fractional Brownian motion: pathwise integration in Sobolev-type spaces, Wick integration, Skorohod integration and some others that are not mentioned here.

Chapter 1

Wiener Integration with Respect to Fractional Brownian Motion

In this chapter, we have two linked aims. Define the Wiener integral, and give some properties of fractional Brownian motion and of integral with respect to this process. The main references for this chapter are [21], [26], [25].

1.1 The Elements of Fractional Calculus

Definition 1.1.1. *Let f be a deterministic real valued function that belongs to $L_1(a,b)$, where (a,b) is a finite interval of $\mathbb{R}$. Define the Riemann Liouville left-right sided fractional integration on (a,b) of order $\alpha > 0$ by*

$$(I_{a+}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt,$$

and

$$(I_{b-}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt,$$

respectively.

Definition 1.1.2. *The Riemann-Liouville fractional integrals on $\mathbb{R}$ are defined respectively by*

$$(I_{+}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(t)(x-t)^{\alpha-1} dt,$$

and

$$(I_{-}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^{\infty} f(t)(t-x)^{\alpha-1} dt,$$

The fonction $f \in \mathcal{D}(I^{\alpha}_{a+(b-)})$ (respectively $\mathcal{D}(I^{\alpha}_{\pm})$) if the respective integrals converge for almost all $x \in (a,b)$(respectively $x \in \mathbb{R}$).

According to [26], we have inclusion $L_p(\mathbb{R}) \subset \mathcal{D}(I^{\alpha}_{\pm})$, $1 \le p \le \frac{1}{\alpha}$. Moreover, the following theorem holds.

Theorem 1.1.1. ([26].) *Let $1 \le p, q < \infty, 0 < \alpha < 1$. Then the operators $I^{\alpha}_{\pm}$ are bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ if and only if $1 < p < \frac{1}{\alpha}$ and $q = p(1-\alpha p)^{-1}$. This means, in particular, that for any $1 < p < \frac{1}{\alpha}$ and $q = \frac{p}{1-\alpha p}$, there exists a constant $C_{p,q,\alpha}$ such that*

$$\left(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(u)||x-u|^{\alpha-1} du\Big)^q dx\right)^{\frac{1}{q}} \le C_{p,q,\alpha} \|f\|_{L_p(\mathbb{R})}. \tag{1.1.1}$$

Fractional integration admits the following composition formulas

$$I^{\alpha}_{\pm} I^{\beta}_{\pm} f = I^{\alpha+\beta}_{\pm} f$$

for $f \in L_p(\mathbb{R}), \alpha, \beta > 0$ and $\alpha + \beta < \frac{1}{p}$.

Integration-by-parts formula for fractional integrals

Let $f \in L_p(\mathbb{R}), g \in L_q(\mathbb{R}), p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. Then

$$\int_{\mathbb{R}} g(x)(I^{\alpha}_{+} f)(x) dx = \int_{\mathbb{R}} f(x)(I^{\alpha}_{-} g)(x) dx. \tag{1.1.2}$$

Let $C^{\lambda}(T)$ be the set of Hölder continuous functions $f : T \to \mathbb{R}$ of order λ, If $\alpha > 0$ and $\alpha p > 1$. Then $I^{\alpha}_{\pm}(L_p(\mathbb{R})) \subset C^{\lambda}[a,b]$ for any $-\infty < a < b < \infty$ and $0 < \lambda \le \alpha - \frac{1}{p}$.

Definition 1.1.3. *For $p \ge 1$, denote by $I^{\alpha}_{\pm}(L_p[a,b])$ the class of functions f, that can be presented as Riemann −Liouville integrals. For $0 < \alpha < 1$ it coincides for a.a. $x \in [a,b]$ with the left-(right-) sided Riemann-Liouville fractional derivative of f of order α. These derivatives are denoted by*

$$(I^{-\alpha}_{a+} f)(x) = (D^{\alpha}_{a+} f)(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x f(t)(x-t)^{-\alpha} dt,$$

and

$$(I^{-\alpha}_{a+} f)(x) = (D^{\alpha}_{a+} f)(x) := \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b f(t)(t-x)^{-\alpha} dt,$$

respectively.

Weyl representation of fractional derivatives
Let $f \in L_p[a,b]$, the Weyl representation of fractional derivatives holds:

$$(D^\alpha_{a+}f)(x) = \frac{1}{\Gamma(1-\alpha)}(f(x)(x-a)^{-\alpha}+\alpha\int_a^x (f(x)-f(t))(x-t)^{-\alpha-1}dt).\mathbb{1}_{(a,b)}(x),$$

and

$$(D^\alpha_{b-}f)(x) = \frac{1}{\Gamma(1-\alpha)}(f(x)(b-x)^{-\alpha}+\alpha\int_x^b (f(x)-f(t))(t-x)^{-\alpha-1}dt).\mathbb{1}_{(a,b)}(x),$$

respectively.

Let $f \in I^\alpha_\pm(L_p(\mathbb{R})), 0<\alpha<1$ and $p \geq 1$. Then

$$I^\alpha_\pm I^{-\alpha}_\pm f = f; \tag{1.1.3}$$

moreover, for $f \in L_1(\mathbb{R})$ we have that

$$I^{-\alpha}_\pm I^\alpha_\pm f = f. \tag{1.1.4}$$

We set $I^0_\pm f := f$.
The composition formula for fractional derivatives has the form

$$D^\alpha_{a+}D^\beta_{a+}f = D^{\alpha+\beta}_{a+}f, \tag{1.1.5}$$

where $\alpha \geq 0, \beta \geq 0$ and $f \in I^{\alpha+\beta}_{a+}(L_1(\mathbb{R}))$.
Also, under the assumptions $0<\alpha<1$, $f \in I^\alpha_{a+}(L_p[a,b])$ and $g \in I^\alpha_{b-}(L_q[a,b])$, $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ we have the integration-by-parts formula for fractional derivatives

$$\int_a^b (D^\alpha_{a+}f)(x)g(x)dx = \int_a^b f(x)(D^\alpha_{b-}g)(x)dx. \tag{1.1.6}$$

Lemma 1.1.1. *Let $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$ and $\alpha = H-\frac{1}{2}$. Then, for all $t \in \mathbb{R}$, we have the equality*

$$(I^\alpha_-\mathbb{1}_{(0,t)})(x) = \frac{1}{\Gamma(1+\alpha)}((t-x)^\alpha_+ - (-x)^\alpha_+).$$

Proof. Let $H \in (\frac{1}{2},1)$ and, for example, $x<0<t$. Then,

$$(I^\alpha_-\mathbb{1}_{(0,t)})(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty \mathbb{1}_{(0,t)}(u)(u-x)^{\alpha-1}du$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (u-x)^{\alpha-1} du = \frac{1}{\Gamma(\alpha+1)}((t-x)^\alpha - (-x)^\alpha). \qquad (1.1.7)$$

Let $H \in (0, \frac{1}{2})$. According to the definition of the fractional derivative and (1.1.3), we must prove that

$$\int_x^\infty ((t-u)_+^\alpha - (-u)_+^\alpha)(u-x)^{-\alpha-1} du = \Gamma(-\alpha)\Gamma(\alpha+1)\mathbb{1}_{(0,t)}(x). \qquad (1.1.8)$$

Let, for example, $0 < x < t$. Then the left-hand side of (1.1.8) equals

$$\int_x^t (t-u)^\alpha (u-x)^{-\alpha-1} du \mathbb{1}_{(0,t)}(x)$$

$$= B(\alpha+1, -\alpha)\mathbb{1}_{(0,t)}(x) = \Gamma(-\alpha)\Gamma(\alpha+1)\mathbb{1}_{(0,t)}(x).$$

The other cases can be considered similarly.

$\square$

Definition 1.1.4. *The Fourier transform of f is defined as*

$$(\mathcal{F}f)(x) = \widehat{f}(x) = \int_{\mathbb{R}} e^{ixt} f(t) dt.$$

Theorem 1.1.2. *([26]) $(\imath)$ For any $0 < \alpha < 1$ and $f \in L_1(\mathbb{R})$ it holds that*

$$\mathcal{F}(I_\pm^\alpha f)(x) = \widehat{f}(x).(\mp ix)^{-\alpha}$$

where $(\mp ix)^{-\alpha} = |x|^\alpha exp\left\{ \mp \frac{\alpha\pi i}{2} sign\ x \right\}$.

$(\imath\imath)$ For any $0 < \alpha < 1$ and $f \in S(\mathbb{R})$ it holds that

$$\mathcal{F}(I_\pm^{-\alpha} f) = \widehat{f}(x).(\mp ix)^\alpha$$

Definition 1.1.5. *f is step function, or elementary function, if there exist a finite number of points $t_k \in \mathbb{R}, 0 \le k \le n-1$, and $a_k \in \mathbb{R}, 1 \le k \le n$, such that*

$$f(t) = \sum_{k=1}^n a_k \mathbb{1}_{[t_{k-1}, t_k)}(t).$$

1.2 Fractional Brownian Motion

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition 1.2.1. *The (two−sided, normalized) fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, P)$, having the properties*
$(i) B_0^H = 0,$
$(ii) EB_t^H = 0, \quad t \in \mathbb{R}$
$(iii) EB_t^H B_s^H = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad t, s \in \mathbb{R}$

Remark 1.2.1. *Since $E(B_t^H - B_s^H)^2 = |t-s|^{2H}$ and $B^H$is a Gaussian process, it has a continuous modification, according to the Kolmogorov theorem(see,[21]).*

The characteristic function has the form

$$\varphi_\lambda(t) = Eexp\{i \sum_{k=1}^{n} \lambda_k B_{t_k}^H\} = exp\{-\frac{1}{2}(C_t\lambda, \lambda)\},$$

where $C_t = E(B_{t_k}^H B_{t_i}^H)_{1 \le i,k \le n}$. Therefore, it follows from item (iii) of Definition 1.2.1, that for any $\beta > 0$

$$\varphi_\lambda(\beta t) = exp\{-\frac{1}{2}\beta^{2H}(C_t\lambda, \lambda)\}. \tag{1.2.1}$$

Definition 1.2.2. *A stochastic process $X = X_t, t \in \mathbb{R}$ is called b-self-similar if*

$$\{X_{at}, t \in \mathbb{R}\} \stackrel{d}{=} \{a^b X_t, t \in \mathbb{R}\}$$

in the sense of finite-dimensional distributions.

From Definition 1.2.2 and (1.2.1) it follows that B^H is H-self-similar. Note that

$$E(B_t^H - B_s^H)(B_u^H - B_v^H) = \frac{1}{2}(|s-u|^{2H} + |t-v|^{2H} - |t-u|^{2H} - |s-v|^{2H}). \tag{1.2.2}$$

It follows from (1.2.2) that the process B_H has stationary increments . Let $H = \frac{1}{2}$. Then the increments of B^H are non-correlated, and consequently independent. So B^H is a Wiener process which we denote further by B or W. For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $t_1 < t_2 < t_3 < t_4$, it follows from (1.2.2) for $\alpha = H - \frac{1}{2}$ that

$$E(B_{t_4}^H - B_{t_3}^H)(B_{t_2}^H - B_{t_1}^H) = 2\alpha H \int_{t_1}^{t_2} \int_{t_3}^{t_4} (u-v)^{2\alpha-1} du dv.$$

Furthermore, for any $n \in \mathbb{Z}/\{0\}$ the autocovariance function is given by

$$r(n) := EB_1^H(B_{n+1} - B_n^H) = 2\alpha H \int_0^1 \int_n^{n+1} (u-v)^{2\alpha-1} dudv.$$

$$\sim 2\alpha H \mid n \mid^{2\alpha-1}, \quad \mid n \mid \to \infty.$$

If $H \in (0, \frac{1}{2})$, then $\sum_{n\in\mathbb{Z}} |r(n)| \sim \sum_{n\in\mathbb{Z}/\{0\}} \mid n \mid^{2\alpha-1} < \infty$.

If $H \in (\frac{1}{2}, 1)$, then $\sum_{n=1}^{\infty} \mid r(n) \mid \sim \sum_{n\in\mathbb{Z}/\{0\}} \mid n \mid^{2\alpha-1} = \infty$. In this case we say that fBm B^H has the property of long-range dependence.

1.3 Mandelbrot-van Ness Representation of fBm

Let $W = \{W_t, t \in \mathbb{R}\}$ be the two-sided Wiener process, *i.e.* the Gaussian process with independent increments satisfying $EW_t = 0$ and $EW_tW_s = s \wedge t,\ s, t \in \mathbb{R}$. Evidently, $W = B^{\frac{1}{2}}$. Denote $k_H(t,u) := (t-u)_+^\alpha - (-u)_+^\alpha$ where $\alpha = H - \frac{1}{2}$. The following representation is due to Mandelbrot and van Ness ([19]).

Theorem 1.3.1. *The process $\overline{B}^H = \{\overline{B}_t^H, t \in \mathbb{R}\}$ define by*

$$\overline{B}_t^H = C_H^{(2)} \int_{\mathbb{R}} k_H(t,u) dW_u, \quad H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$$

where

$$C_H^{(2)} = \left(\int_{\mathbb{R}_+} ((1+s)^\alpha - s^\alpha)^2 ds + \frac{1}{2H} \right)^{-\frac{1}{2}} = \frac{(2H \sin \pi H \Gamma(2H))^{1/2}}{\Gamma(H+1/2)}$$

has a continuous modification which is a normalized two-sided fBm.

Proof. Evidently, $\overline{B}^H$ is a Gaussian process with $\overline{B}_0^H = 0$ and $E\overline{B}_t^H = 0$. Furthermore, it holds that for $t > 0$,

$$E(\overline{B}_t^H)^2 = \left(C_H^{(2)}\right)^2 \left(\int_{-\infty}^0 k_H^2(t,u) du + \int_0^t (t-u)^{2\alpha} du \right) = t^{2H}.$$

For $t < 0$ we have that

$$E(\overline{B}_t^H)^2 = \left(C_H^{(2)}\right)^2 \left(\int_{-\infty}^t k_H^2(t,u) du + \int_t^0 (-u)^{2\alpha} du \right) = (-t)^{2H}.$$

Furthermore, for $h > 0$, it holds that

$$\overline{B}^H_{s+h} - \overline{B}^H_s = C_H^{(2)} \int_{-\infty}^{s} (k_H(s+h,u) - k_H(s,u))dW_u$$

$$+ \int_s^{s+h} (k_H(s+h,u))dW_u =: I_1 + I_2. \tag{1.3.1}$$

Note that I_1 and I_2 are independent, and W has stationary increments. Therefore,

$$I_1 \stackrel{d}{=} C_H^{(2)} \int_{-\infty}^{0} (k_H(h,u) - k_H(0,u))dW_u, \quad I_2 \stackrel{d}{=} \int_0^h (k_H(h,u))dW_u.$$

and $E(\overline{B}^H_{s+h} - \overline{B}^H_s)^2 = E(\overline{B}^H_h)^2 = h^{2H}$. By combining these results, we obtain that

$$\begin{aligned} E\overline{B}^H_s \overline{B}^H_t &= \frac{1}{2}(E(\overline{B}^H_s)^2 + E(\overline{B}^H_t)^2 - E(\overline{B}^H_t - \overline{B}^H_s)^2) \\ &= \frac{1}{2}(\mid t \mid^{2H} + \mid s \mid^{2H} - \mid t - s \mid^{2H}). \end{aligned} \tag{1.3.2}$$

The proof follows immediately from Definition 1.2.1 and Remark 1.2.1.

□

Definition 1.3.1. *Define the operator*

$$M_{\pm}^H f := \begin{cases} C_H^{(3)} I_{\pm}^{\alpha} f, & H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\ f, & H = \frac{1}{2}, \end{cases} \tag{1.3.3}$$

where $C_H^{(3)} = C_H^{(2)} \Gamma(H + \frac{1}{2})$.

Corollary 1.3.1. *It follows from Lemma* 1.1.1 *and Theorem* 1.3.1, *that for any* $H \in (0,1)$ *the process*

$$B_t^H = \int_{\mathbb{R}} (M_-^H \mathbb{1}_{(0,t)})(s) dW_s, \tag{1.3.4}$$

is a normalized fractional Brownian motion.

1.4 Fractional Brownian Motion with $H \in (\frac{1}{2}, 1)$ on the White Noise Space

Definition 1.4.1. *Let* $S(\mathbb{R})$ *denotes the Schwartz space of rapidly decreasing smooth functions on* $\mathbb{R}$, *and let* $\Omega = S'(\mathbb{R})$ *be its dual space ,usually called* *the space of tempered distributions . Let P be the probability measure on the $\sigma-$algebra of Borel sets $\mathcal{F}(S'(\mathbb{R}))$ definde by the property that*

$$Eexp(i\langle f, \omega\rangle) = exp\{\frac{1}{2} \parallel f \parallel^2_{L_2(\mathbb{R})}\}, \quad f \in S(\mathbb{R}) \tag{1.4.1}$$

where $\langle ., .\rangle$ denotes the dual operation.

Using (1.4.1) one can prove that

$$E(\langle f, \omega\rangle) = 0, \quad E(\langle f, \omega\rangle)^2 = \parallel f \parallel^2_{L_2(\mathbb{R})} \quad \text{for all} \quad f \in S(\mathbb{R}), \tag{1.4.2}$$

from (1.4.1), (1.4.2), it follows that the process $W_t = \langle \mathbb{1}_{[0,t]}, \omega\rangle$, is a standard Brownian motion.

Define two stochastic processes

$$B^H_{\pm}(t)(\omega) = \langle M^H_{\pm}\mathbb{1}_{(0,t)}, \omega\rangle, \quad t \in \mathbb{R}.$$

Then the processes $B^H_{\pm}(t)$ are Gaussian, $EB^H_+(t) = EB^H_-(t) = 0$. For the covariance function, it holds that

$$EB^H_{\pm}(t)B^H_{\pm}(s) = \int_{\mathbb{R}} (M^H_{\pm}\mathbb{1}_{(0,t)})(x)(M^H_{\pm}\mathbb{1}_{(0,s)})(x)dx. \tag{1.4.3}$$

By considering the sign $''-''$, we obtain from (1.3.4) that the right-hand side of (1.4.3) coincides with

$$\begin{aligned} EB^H_t B^H_s &= \int_{\mathbb{R}}(M^H_-\mathbb{1}_{(0,t)})(x)(M^H_-\mathbb{1}_{(0,s)})(x)dx \\ &= \tfrac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \end{aligned}$$

One obtains the same result if one considers the sign $'' +''$.
Therefore, each of the processes $B^H_{\pm}$ has a modification that is a normalized fBm. The process $B^H_-(t) = \int_{\mathbb{R}}(M^H_-\mathbb{1}_{(0,t)})(s)dW_s$, is called a $''backward''$ fBm, depends only on the past, $i.e.$ on $\{W_s, s \in (-\infty, t)\}$. where $W_t(\omega) = \langle \mathbb{1}_{(0,t)}, \omega\rangle$. The process $B^H_+(t)$ is called a $''forward''$ fBm; it admits the representation $B^H_+ = \int_{\mathbb{R}}(M_+\mathbb{1}_{(0,t)})(s)dW_s$, and depends on future values of W, i.e. on $\{W_s, s \in (t, +\infty)\}$.

Consider the linear combinations of the operators $M^{H_k}_{\pm}$ and of fractional Brownian motions with different Hurst indices

$$M_{\pm}f(x) := \sum_{k=1}^{m} \sigma_k M^{H_k}_{\pm} f(x), \quad \sigma_k > 0$$

and

$$B_{\pm}^{M}(t) := \sum_{k=1}^{m} \sigma_k B_{\pm}^{H_k}(t) = \langle M_{\pm}\mathbb{1}_{(0,t)}, \omega\rangle. \tag{1.4.4}$$

Clearly, the operators $M_{\pm}$ are mutually adjoint in the same way as $M_{\pm}^{H}$.

1.5 Fractional Noise on White Noise Space

Let $\mathcal{J}$ be the set of all finite multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{N}_0$. Denote $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! := \alpha_1! \dots \alpha_n!$. Define the Hermite polynomials by

$$h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

and Hermite functions

$$\widetilde{h}_n(x) = \pi^{-1/4}(n!)^{-1/2} 2^{-n/2} h_n(x) e^{-x^2/2}, \quad n \geq 0.$$

Define

$$\mathcal{H}_\alpha(\omega) := \prod_{i=1}^{n} h_{\alpha_i}(\langle \widetilde{h}_i, \omega\rangle),$$

the product of Hermite polynomials and consider a random variable

$$F = F(\omega) \in L_2(\Omega) := L_2(S'(\mathbb{R}), \mathcal{F}, P).$$

Then, according to ([12], *Theorem* 2.2.4), $F(\omega)$ admits the representation

$$F(\omega) = \textstyle\sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{H}_\alpha(\omega), \tag{1.5.1}$$

and

$$\| f \|_{L_2(\Omega)}^2 = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2 < \infty.$$

Next, we introduce the following dual spaces.

(i) $F \in S$ if the coeffcients from expansion (1.5.1) satisfy

$$\| f \|_k^2 = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty.$$

for any $k \geq 1$, where $(2\mathbb{N})^\gamma = \prod_{j=1}^{m} (2j)^{\gamma_j}, \gamma = (\gamma_1 \dots \gamma_m \in \mathcal{J})$.

(ii) $F \in S^*$ if F admits the formal expansion (1.5.1) with finite negative norm

$$\| f \|_{-q}^2 = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty.$$

for at least one $q \in \mathbb{N}$(in this case we say that $F \in S_{-q}$). For $F = \sum_\alpha c_\alpha \mathcal{H}_\alpha \in S$, $G = \sum_\alpha d_\alpha \mathcal{H}_\alpha \in S^*$, we define

$$\langle\!\langle F, G \rangle\!\rangle = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha d_\alpha.$$

Now we want to present the linear combination $B_\pm^M(t)$ of fBms in terms of $\widetilde{h}_k, k \geq 1$.

Lemma 1.5.1. *It holds that*

$$B_\pm^M(t) = \sum_{k=1}^{\infty} \int_0^t M_\mp \widetilde{h}_k(x) dx \langle \widetilde{h}_k, \omega \rangle, \ \ t \in \mathbb{R}, \ \ \omega \in S'(\mathbb{R}), \tag{1.5.2}$$

and the series converges in $L_2(\Omega)$.

Now, we introduce the fractional noise $\dot{B}^H$ as the formal expansion

$$\dot{B}_x^H(\omega) = \sum_{k=1}^{\infty} M_+^H \widetilde{h}_k(x) \langle \widetilde{h}_k, \omega \rangle,$$

and the linear combination of fractional noises as

$$\dot{B}_x^M(\omega) = \sum_{k=1}^{\infty} M_+ \widetilde{h}_k(x) \langle \widetilde{h}_k, \omega \rangle.$$

Recall, that here we consider only $H \in [1/2, 1)$ and that

$$\dot{B}_x(\omega) = \sum_{k=1}^{\infty} \widetilde{h}_k(x) \langle \widetilde{h}_k, \omega \rangle$$

is white noise.

Lemma 1.5.2. *The fractional noise $\dot{B}_x^H$ and the linear combination $\dot{B}_x^M$ of such noises belong to S^* for any $x \in \mathbb{R}$.*

Proof. (See[21])

1.6 Wiener Integration with Respect to fBm

Let $(\Omega, \mathcal{F}, P)$, an arbitrary complete probability space, and consider $L_2^H(\mathbb{R}) = \{f : M_-^H f \in L_2(\mathbb{R})\}$ equipped with the norm $\|f\|_{L_2^H(\mathbb{R})} = \|M_-^H f\|_{L_2(\mathbb{R})}$.

Definition 1.6.1. *Let $f \in L_2^H(\mathbb{R})$. Then the Wiener integral w.r.t. fBm is defined as*

$$I_H(f) := \int_{\mathbb{R}} f(s) dB_s^H := \int_{\mathbb{R}} (M_-^H f)(s) dW_s. \tag{1.6.1}$$

Here, B_s^H and W_s are connected as in (1.3.4).

As a particular case, consider the step function f defined as in definition 1.1.5. Then, from the linearity of the operator M_-^H, we have that

$$I_H(f) := \sum_{k=1}^{n} a_k \int_{\mathbb{R}} M_-^H \mathbb{1}_{[t_{k-1},t_k)}(s) dW_s = \sum_{k=1}^{n} a_k (B_{t_k}^H - B_{t_{k-1}}^H). \tag{1.6.2}$$

A question arises: in which sense can we consider formula (1.6.1) as the extension of the sum (1.6.2)?

Note, that for a step function, it holds that

$$\begin{aligned} \|I_H(f)\|_{L_2(\Omega)}^2 &= \textstyle\sum_{i,k=1}^{n} a_i a_k \int_{\mathbb{R}} M_-^H \mathbb{1}_{[t_{k-1},t_k)}(x) M_-^H \mathbb{1}_{[t_{i-1},t_i)}(x) dx \\ &= \| M_-^H f \|_{L_2(\mathbb{R})}^2 = 2\alpha H \textstyle\int_{\mathbb{R}^2} f(u) f(v) \mid u - v \mid^{2\alpha - 1} dudv, \end{aligned} \tag{1.6.3}$$

where the last equality holds for $H \in (1/2, 1)$ but not for $H \in (0, 1/2)$. Nevertheless, for any $0 < H < 1$ we have the following:

Lemma 1.6.1. *([4]) For $0 < H < 1$, it holds that the linear span of the set $\{M_-^H \mathbb{1}_{(u,v)}, u, v \in \mathbb{R}\}$ is dense in $L_2(\mathbb{R})$.*

Proof. We invite the reader to commet ([21], p.16) for more information about the proof of this result.

Theorem 1.6.1. *The space L_2^H is incomplete for $H \in (1/2, 1)$.*

Proof. The operator $M_-^H : L_2^H(\mathbb{R}) \to L_2(\mathbb{R})$ is isometric. So, $L_2^H(\mathbb{R})$ can be identified with its image in $L_2(\mathbb{R})$. According to Lemma 1.6.1, $L_2^H(\mathbb{R})$ is dense in $L_2(\mathbb{R})$, but in ([21], remark 1.6.1) it was demonstrate that $L_2^H(\mathbb{R}) \neq L_2(\mathbb{R})$. Therefore, the image $M_-^H(L_2^H(\mathbb{R}))$ and hence $L_2^H(\mathbb{R})$ it self, is incomplete. $\square$

In spite of the incompleteness of $L_2^H(\mathbb{R})$ for $H \in (1/2, 1)$, due to Lemma 1.6.1, we can approximate any $f \in L_2^H(\mathbb{R})$ by step functions $f_n \in L_2^H(\mathbb{R})$. Then $M_-^H f_n \to M_-^H f$ in $L_2(\mathbb{R})$, and we have that

$$\begin{aligned} I_H(f) := \int_{\mathbb{R}} f(x) dB_s^H &= \int_{\mathbb{R}} (M_-^H f)(s) dW_s \\ &= \lim_{n\to\infty} \textstyle\int_{\mathbb{R}} (M_-^H f_n)(s) dW_s = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(s) dB_s^H, \end{aligned}$$

where the convergence is in $L_2(\Omega)$. Furthermore, for $H \in (1/2, 1)$, we have that

$$E \mid I_H(f) \mid^2 = \int_{\mathbb{R}} \mid (M_-^H f)(x) \mid^2 dx$$

for $f \in L_2^H(\mathbb{R})$; however, in general, it does not hold (compare with (1.6.3)) that

$$E \mid I_H(f) \mid^2 = 2\alpha H \int_{\mathbb{R}^2} f(u) f(v) \mid u - v \mid^{2\alpha - 1} dudv, \tag{1.6.4}$$

even if the last integral is finite. This equality can be obtained only if we can apply the Fubini theorem or if we can prove that the integral $\int_{\mathbb{R}^2} f_n(u) f_n(v) |u - v|^{2\alpha-1} dudv$ with step functions f_n converges to $\int_{\mathbb{R}^2} f(u) f(v) |u - v|^{2\alpha-1} dudv$. Both things need some additional assumptions.

For $H \in (1/2, 1)$, define the space of measurable functions by

$$\mid R_H \mid := \left\{ f : \mathbb{R} \to \mathbb{R} \middle| \int_{\mathbb{R}_+^2} \mid f(u) \mid\mid f(v) \mid\mid u - v \mid^{2\alpha-1} dudv < \infty \right\},$$

with the norms

$$\parallel f \parallel_{|R_H|,1}^2 = 2\alpha H \int_{\mathbb{R}_+^2} f(u) f(v) \mid u - v \mid^{2\alpha-1} dudv \tag{1.6.5}$$

and

$$\parallel f \parallel_{|R_H|,2}^2 = 2\alpha H \int_{\mathbb{R}_+^2} \mid f(u) \mid\mid f(v) \mid\mid u - v \mid^{2\alpha-1} dudv. \tag{1.6.6}$$

For $H \in (0, 1)$, we introduce one more space,

$$\mathfrak{F}_H := \left\{ f : \mathbb{R} \to \mathbb{R} \middle| f \in L_2(\mathbb{R}) \int_{\mathbb{R}} \mid \widehat{f}(x) \mid^2 \mid x \mid^{-2\alpha} dx < \infty \right\},$$

with the norm

$$\|f\|_{\mathfrak{F}_H}^2 = \int_{\mathbb{R}} |f(x)|^2 |x|^{-2\alpha} dx. \tag{1.6.7}$$

Moreover, consider $L_H^2(\mathbb{R})$ with the norm

$$\|f\|^2_{L_2^H(\mathbb{R})} = \int_{\mathbb{R}} |(M_-^H f)(x)|^2 dx. \tag{1.6.8}$$

Below we study the most important features of these spaces.

Note, at first, that the norms defined in (1.6.5) – (1.6.8) are all generated by corresponding inner products. Namely,

$$(f,g)_{|R_H|,1} = 2\alpha H \int_{\mathbb{R}_+^2} f(u)g(v)|u-v|^{2\alpha-1}dudv, \tag{1.6.9}$$

$$(f,g)_{|R_H|,2} = 2\alpha H \int_{\mathbb{R}_+^2} |f(u)||g(v)||u-v|^{2\alpha-1}dudv, \tag{1.6.10}$$

$$(f,g)_{\mathcal{F}_H} = \int_{\mathbb{R}} \widehat{f}(x)\widehat{g}(x)|x|^{1-2H}dx \tag{1.6.11}$$

and

$$(f,g)_{L_2^H(\mathbb{R})} = \int_{\mathbb{R}} (M_-^H f)(x)(M_-^H g)(x)dx. \tag{1.6.12}$$

Thus, all these spaces are spaces with inner products. Furthermore, (1.6.5) is indeed a norm on $|\ R_H\ |$. Indeed, we can apply the Fubini theorem, use the following relation from ([11]):

$$\int_{-\infty}^{s\wedge t} (s-u)^{\alpha-1}(t-u)^{\alpha-1}du = C_H^{(4)}|t-s|^{2\alpha-1},$$

where $C_H^{(4)} = \frac{\Gamma(H-1/2)\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}$, and rewrite (1.6.5) as

$$2\alpha H \int_{\mathbb{R}} f(u)f(v)|u-v|^{2\alpha-1}dudv$$

$$= (C_H^{(4)})^{-1} 2\alpha H \int_{\mathbb{R}_+^2} f(u)f(v) \int_{-\infty}^{u\wedge v} (u-z)^{\alpha-1}(v-z)^{\alpha-1}dzdudv$$

$$= (C_H^{(4)})^{-1} 2\alpha H \int_{\mathbb{R}} \int_z^{\infty} f(u)(u-z)^{\alpha-1}du \int_z^{\infty} f(v)(v-z)^{\alpha-1}dvdz$$

$$= (C_H^{(4)})^{-1} 2H\alpha (C_H^{(3)})^{-2} \parallel M_-^H f \parallel^2_{L_2(\mathbb{R})} = 2\alpha H (C_H^{(4)})^{-1}(C_H^{(3)})^{-2} \|f\|^2_{L_2^H(\mathbb{R})}. \tag{1.6.13}$$

Lemma 1.6.2. *We have that the space* $L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \subset L_{1/H}(\mathbb{R}) \subset |\ R_H\ |$ *for any* $H \in (1/2, 1)$.

Proof. It is enough to prove that for any $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ the iterated integral is finite,

$$I := \int_{\mathbb{R}} \mid f(u) \mid \left(\int_{\mathbb{R}} \mid f(v) \mid\mid u - v \mid^{2\alpha - 1} dv \right) du < \infty.$$

From Theorem 1.1.1 with $\alpha = 2H - 1$,$p = \frac{1}{H}$ and $q = \frac{p}{1 - 2\alpha p} = \frac{1}{1-H}$ we obtain that

$$I \leq \left(\int_{\mathbb{R}} \mid f(u) \mid^{\frac{1}{H}} du \right)^H \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mid f(v) \mid\mid u - v \mid^{2H-1} dv \right)^{\frac{1}{1-H}} du \right)^{1-H}$$

$$\leq \| f \|_{L_{1/H}(\mathbb{R})} C_{1/H, 1/1-H, 2H-1} \| f \|_{L_{1/H}(\mathbb{R})} = C_H \| f \|^2_{L_{1/H}(\mathbb{R})} .$$

Obviously, $L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \subset L_{1/H}(\mathbb{R})$ for $H \in (1/2, 1)$, whence the claim follows.

□

Lemma 1.6.3. *The inclusion $L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \subset \mathfrak{F}_H$ is valid if and only if $H \in (1/2, 1)$.*

Proof. Assume that $H \in (1/2, 1)$. Since $|\widehat{f}(x)| \leq \|f\|_{L_1(\mathbb{R})}$ for any $x \in \mathbb{R}$, we have that

$$\int_{\mathbb{R}} |\widehat{f}(x)|^2 |x|^{-2\alpha} dx = \int_{|x| \geq 1} |\widehat{f}(x)|^2 |x|^{-2\alpha} dx + \int_{|x|<1} |\widehat{f}(x)|^2 |x|^{-2\alpha} dx$$

$$\leq \int_{\mathbb{R}} |\widehat{f}(x)|^2 dx + \|f\|_{L_1(\mathbb{R})} \int_{|x|<1} |x|^{-2\alpha} dx \leq \|f\|^2_{L_2(\mathbb{R})} + (1 - H)^{-1} \|f\|^2_{L_1(\mathbb{R})}.$$

Let $H \in (0, \frac{1}{2})$. According to ([24]), take the function $f(u) = sign\ u \frac{\varepsilon^{-|u|}}{|u|^p}$ with $p \in (H, \frac{1}{2})$. Evidently, $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. Nevertheless, due to ([10], p.491),

$$\widehat{f}(\lambda) = 2\Gamma(1 - p)(\lambda^2 + 1)^{\frac{p-1}{2}} \sin((1 - p) \arctan \lambda) \sim |\lambda|^{p-1}$$

as $|\lambda| \to \infty$, and $2p - 2 > 2\alpha - 1 > -1$, which means that $\|f\|_{\mathfrak{F}_H} = +\infty$.

□

Lemma 1.6.4. *For any $H \in (0, 1)$, we have that $\mathfrak{F}_H \subset L_2^H(\mathbb{R})$.*

Proof. For $H = \frac{1}{2}$, the statement is evident and $\mathcal{F}_{\frac{1}{2}} = L^2_{\frac{1}{2}}(\mathbb{R}) = L_2(\mathbb{R})$. Let $H \in (\frac{1}{2}, 1)$ and $f \in \mathcal{F}_H$. Then, in particular, $f \in L_2(\mathbb{R})$, and, therefore, according to Theorem 1.1.1, the operator $I^{\alpha}_{-}f$ is well defined and bounded from $L_2(\mathbb{R})$ to $L_{\frac{1}{1-H}}(\mathbb{R})$. Moreover, according to Theorem 1.1.2 and since

$\int_{\mathbb{R}} |\widehat{f(x)}|^2|x|^{-2\alpha}dx < \infty$, it follows that $I^{\alpha}_{-}f \in L_2(\mathbb{R})$. Therefore, $f \in L^H_2(\mathbb{R})$. Let $H \in (0, \frac{1}{2})$. We must prove, that for any $f \in L_2(\mathbb{R})$ with $\int_{\mathbb{R}} |\widehat{f(x)}|^2|x|^{-2\alpha}dx < \infty$, there exists $\widetilde{\varphi} \in L_2(\mathbb{R})$, such that

$$\widetilde{\varphi} = M^H_{-}f = C^{(3)}_H D^{-\alpha}_{-}f. \tag{1.6.14}$$

Consider the function $\psi(x) = \widehat{f}(x) \mid x \mid^{-\alpha} C_H(x)$. Since $\mid C_H(x) \mid= 1, \psi \in L_2(\mathbb{R})$ and $\overline{\psi(x)} = \psi(-x)$, we conclude that $\psi(x) = \widehat{\varphi}(x)$ for some function $\varphi \in L_2(\mathbb{R})$. Now we prove that $C^{(3)}_H\varphi$ satisfies (1.6.14). Indeed,

$$\widehat{f}(x) = \widehat{\varphi}(x) \mid x \mid^{\alpha} C_H(-x), \tag{1.6.15}$$

whence $|\widehat{f}(x)|^2 = |\widehat{\varphi}(x)|^2|x|^{2\alpha}$. Since $\widehat{f} \in L_2(\mathbb{R})$, we have that $\varphi \in \mathcal{F}_{1-H}$, and from Theorem 1.1.2 and (1.6.15), it follows that

$$f = I^{-\alpha}_{-}\varphi.$$

Therefore, $\widetilde{\varphi}(x) = C^{(3)}_H\varphi(x)$ satisfies (1.6.14), whence the claim follows. $\square$

Lemma 1.6.5. *Let* $0 < H < 1$. *Then* $M^{1-H}_{-}\mathbb{1}_{(0,t)} \in L^H_2(\mathbb{R})$ *for all* $t \in \mathbb{R}$, *and the underlying Wiener process* W *admits the representation*

$$W_t = \widetilde{C_H} \int_{\mathbb{R}} M^{1-H}\mathbb{1}_{(0,t)}(s)dB^H_s,$$

where $\widetilde{C_H} = (C^{(3)}_H C^{(3)}_{1-H})^{-1}$.

Proof. We must check that $M^{1-H}_{-}\mathbb{1}_{(0,t)} \in L^H_2(\mathbb{R})$. Indeed,

$$M^H_{-}.M^{1-H}_{-}\mathbb{1}_{(0,t)} = C^{(3)}_H C^{(3)}_{1-H} I^{H-\frac{1}{2}}_{-}(I^{\frac{1}{2}-H}_{-}\mathbb{1}_{(0,t)}) = (\widetilde{C_H})^{-1}\mathbb{1}_{(0,t)} \in L_2(\mathbb{R}).$$

Furthermore, according to Definition 1.6.1, it holds that

$$\begin{aligned}\widetilde{C_H} \int_{\mathbb{R}} (M^{1-H}_{-}\mathbb{1}_{(0,t)})(s)dB^H_s &= \widetilde{C_H} \int_{\mathbb{R}} (M^H_{-}M^{1-H}_{-}\mathbb{1}_{(0,t)})(s)dW_s \\ &= \int_{\mathbb{R}} \mathbb{1}_{(0,t)}(s)dW_s = W_t. \end{aligned} \tag{1.6.16}$$

$\square$

Corollary 1.6.1. *Any fBm* B^H *admits a Mandelbrot van Ness representation with respect to the Wiener process* W *from representation* (1.6.16).

1.7 The Space of Gaussian Variables Generated by fBm.

Denote
$$\mathcal{B}_H = \overline{span}\{B_t^H, t \in \mathbb{R}\},$$
where the closure is taken in $L_2(\Omega)$.

Theorem 1.7.1. *Let $\mathcal{I}$ be some class of integrands and let $\mathcal{I}_s \subset \mathcal{I}$ be the class of step functions. Under the assumptions*

(i) *$\mathcal{I}$ is a space with inner product $(f,g)_I$, $f,g \in \mathcal{I}$,*

(ii) *for $f,g \in \mathcal{I}_s$ $(f,g)_I = EI(f)I(g)$,*

(iii) *the set $\mathcal{I}_s$ is dense in $\mathcal{I}$,*

we have the following:

(a) *there is an isometry between the space $\mathcal{I}$ and a linear subspace of $\mathcal{B}_H$ which is an extension of the map $f \to I(f)$ for $f \in \mathcal{I}_s$*

(b) *$\mathcal{I}$ is isometric to $\mathcal{B}_H$ if and only if $\mathcal{I}$ is complete.*

Proof. (a) Let $f \in \mathcal{I}$. By (iii), there exists $f_n \in \mathcal{I}_s$, such that $\{f_n, n \geq 1\}$ is a Cauchy sequence in $\mathcal{I}$ with norm $\|\cdot\|_{\mathcal{I}} = (\cdot,\cdot)_I$. According to (ii), $I(f_n)$ is a Cauchy sequence in $L_2(\Omega)$, hence it converges to some r.v. $\xi \in L_2(\Omega)$. We set $I(f) := \xi$. Since $I(f_n) \in \mathcal{B}_H$ and $\mathcal{B}_H$ is a closed subspace of $L_2(\Omega)$, we obtain that $I(f) \in \mathcal{B}_H$. So, we can define the map $I : \mathcal{I} \to \mathcal{B}_H$. For any $f, g \in \mathcal{I}$ it holds that
$$(f,g)_{\mathcal{I}} = \lim_{n\to\infty} (f_n, g_n)_{\mathcal{I}} = \lim_{n\to\infty} EI(f_n)I(g_n) = EI(f)I(g).$$
Moreover, ξ does not depend on the choice of the sequence $f_n \to f$ in $\mathcal{I}$. Since the map I is linear, we get an isometry between $\mathcal{I}$ and some subspace of $\mathcal{B}_H$.

(b) Since $\mathcal{B}_H$ is complete as a closed subspace of the complete space $L_2(\Omega)$, it follows that $\mathcal{I}$ is complete if I is an isometry between $\mathcal{I}$ and $\mathcal{B}_H$. Conversely, let $\mathcal{I}$ be complete. Then, for any $\eta \in \mathcal{B}_H$, it holds that $\eta = \lim \eta_n, \eta_n = I(f_n) \in span\{B_t^H, t \in \mathbb{R}\}, f_n \in \mathcal{I}_s$. So, $I(f_n) \to \eta$ in $L_2(\Omega)$. Therefore, from (ii) it follows that f_n is a Cauchy sequence in $\mathcal{I}$, and from completeness, $f_n \to f$ in $\mathcal{I}, \eta = I(f)$. □

Corollary 1.7.1. *From Lemma* 1.6.1, *Theorem* 1.6.1, *and according to ([21], Remark* 1.6.3*), we obtain the following: the space $\mathcal{I} = L_2^H(\mathbb{R})$ is complete for $H \in (0, \frac{1}{2})$ and incomplete for $H \in (\frac{1}{2}, 1)$. Step functions are dense in $L_2^H(\mathbb{R})$ for any $H \in (0,1)$. Therefore, $L_2^H(\mathbb{R})$ is isometric to $\mathcal{B}_H$for $H \in (0, \frac{1}{2})$ and isometric to a subspace of $\mathcal{B}_H$ for $H \in (\frac{1}{2}, 1)$.*

1.8 Representation of fBm via the Wiener Process on a Finite Interval

Sometimes it is convenient to consider a "$one-sided$" fBm $B^H = \{B_t^H,\ t \geq 0\}$ and to represent it as a functional of the form $B_t^H = \varphi(B_s,\ 0 \leq s \leq t)$, of some Wiener process $B = \{B_t,\ t \geq 0\}$, For this purpose consider the kernel

$$l_H(t,s) = C_H^{(5)} s^{-\alpha}(t-s)^{-\alpha}\mathbb{1}_{\{0<s<t\}},$$

and

$$m_H(t,s) = C_H^{(6)}\left((\frac{t}{s})^{\alpha}(t-s)^{\alpha} - \alpha s^{-\alpha}\int_s^t u^{\alpha-1}(u-s)^{\alpha}du\right),$$

where

$$C_H^{(5)} = \left(\frac{\Gamma(2-2\alpha)}{2H\Gamma(1-\alpha)^3\Gamma(1+\alpha)}\right)^{\frac{1}{2}}, \quad C_H^{(6)} = \left(\frac{2H\Gamma(1-\alpha)}{\Gamma(1-2\alpha)\Gamma(\alpha+1)}\right)^{\frac{1}{2}},$$

and $\alpha = H - \frac{1}{2}, H \in (0,1)$. By using the equality

$$\int_0^1 t^{-\mu}(1-t)^{-\mu} \mid x-t \mid^{2\mu-1} dt = B(\mu, 1-\mu), \tag{1.8.1}$$

that was established in ([22], $Lemma$ 2.2) for any $\mu \in (0,1), x \in (0,1)$, we obtain that for any $t > 0$

$$\begin{array}{rl} \| l_H(t,\cdot) \|_{|R_H|,2} & \\ = & (C_H^{(5)})^2 2H\alpha \int_0^t\int_0^t (t-u)^{-\alpha}(t-s)^{-\alpha}u^{-\alpha}s^{-\alpha} \mid u-s \mid^{2\alpha-1} duds \\ = & t^{1-2\alpha}(C_H^{(5)})^2 2H\alpha \int_0^1 u^{-\alpha}(1-u)^{-\alpha}(\int_0^1 s^{-\alpha}(1-s)^{-\alpha}|u-s|^{2\alpha-1}ds)du \\ = & t^{1-2\alpha}(C_H^{(5)})^2 2H\alpha B(\alpha, 1-\alpha)B(1-\alpha, 1-\alpha) \\ = & t^{1-2\alpha}\frac{\Gamma(2-2\alpha)\Gamma(\alpha)\Gamma(1-\alpha)^3}{\Gamma(1-\alpha)^3\Gamma(\alpha)\Gamma(2-2\alpha)} = t^{1-2\alpha} < \infty. \end{array} \tag{1.8.2}$$

Therefore, we can consider the integral

$$\begin{aligned} I_t^H(l_H) &= \int_0^t l_H(t,s)dB_s^H := \int_{\mathbb{R}} l_H(t,s)dB_s^H \\ &= \int_{\mathbb{R}} (M_-^H l_H)(t,\cdot)(x)dW_x, \end{aligned} \tag{1.8.3}$$

where $W = \{W_x, x \in \mathbb{R}\}$ is the underlying Wiener process. Similarly to (1.8.2), for any $0 < t < t'$, we obtain that

$$EI_t^H(l_H)I_{t'}^H(l_H) = (l_H(t,\cdot), l_H(t',\cdot))_{|R_H|,2}$$

$$= (C_H^{(5)})^2 2H\alpha \int_0^t (t-u)^{-\alpha} u^{-\alpha} (\int_0^{t'} (t'-s)^{-\alpha} s^{-\alpha} \mid u-s \mid^{2\alpha-1} ds) du$$

$$= (C_H^{(5)})^2 2H\alpha t^{1-2\alpha} B(\alpha, 1-\alpha) B(1-\alpha, 1-\alpha) = t^{1-2\alpha}. \tag{1.8.4}$$

From (1.8.3), it follows that $\{I_t^H, t \geq 0\}$ is a centered Gaussian process. Moreover, from (1.8.4), we obtain for any $0 < s < t \leq s' < t'$ that

$$E(I_{t'}^H(l_H) - I_{s'}^H(l_H))(I_t^H(l_H) - I_s^H(l_H)) = 0.$$

Thus, the increments of $I_t^H(l_H)$ are uncorrelated, and hence independent. It follows that $I_t^H(l_H)$ is a martingale w.r.t. its natural filtration

$$\mathcal{F}_t^H = \sigma\{I_s^H(l_H), 0 \leq s \leq t\},$$

having angle bracket $\langle I_t^H(l_H)\rangle = t^{1-2\alpha}$ and $I_0^H(l_H) = 0$. By the L'evy theorem, there exists some Wiener process $B = \{B_t, t \geq 0\}$ such that

$$M_t^H := I_t^H(l_H) = \widetilde{\alpha} \int_0^t s^{-\alpha} dB_s. \tag{1.8.5}$$

where $\widetilde{\alpha} = (1-\alpha)^{1/2}$. The process M^H is called the Molchan martingale, or the fundamental martingale.

Theorem 1.8.1. *Let B^H be an fBm with $H \in (0,1)$, and let*

$$M_t^H = I_t^H(l_H) = \int_0^t l_H(t,s) dB_s^H. \tag{1.8.6}$$

Then there exists a Wiener process B such that (1.8.5) *holds. Moreover,* $\sigma\{B_s^H, 0 \leq s \leq t\} = \sigma\{B_s, 0 \leq s \leq t\}$.

The inverse relation can be obtained. For $H \in (0,1)$, and for any $t > 0$, the random variable $Y_t := \int_0^t s^{-\alpha} dB_s^H$ is well defined. Therefore, it holds that

$$Y_t = t^{-\alpha} B_t^H + \alpha \int_0^t B_s^H s^{-\alpha-1} ds,$$

is an integral equation with respect to $\{B_s^H, 0 \leq s \leq t\}$ and its solution has the form

$$B_t^H = t^{\alpha} Y_t - \alpha \int_0^t s^{\alpha-1} Y_s ds = \int_0^t s^{\alpha} dY_s.$$

Let $M_t^H = I_t^H(l_H)$ be the Molchan martingale. Then, for $H \in (0, \frac{1}{2})$, integration by parts leads to the equality

$$M_t^H = C_H^{(5)} \int_0^t (t-s)^{-\alpha} s^{-\alpha} dB_s^H = -\alpha C_H^{(5)} \int_0^t (t-s)^{-\alpha-1} Y_s ds,$$

whence

$$\int_0^t (t-u)^\alpha M_u^H du = -\alpha C_H^{(5)} \int_0^t Y_s \left(\int_s^t (t-u)^\alpha (u-s)^{-1-\alpha} du \right) ds$$

$$= -\alpha C_H^{(5)} B(\alpha+1, -\alpha) \int_0^t Y_s ds,$$

and

$$Y_t = C_H^{(6)} \hat{\alpha} \int_0^t (t-u)^\alpha dM_u^H, \qquad (1.8.7)$$

where $\hat{\alpha} = (1-\alpha)^{-1/2}$. Therefore,

$$B_t^H = \hat{\alpha} C_H^{(6)} \Big(t^\alpha \textstyle\int_0^t (t-u)^\alpha dM_u^H$$
$$-\alpha \textstyle\int_0^t s^{\alpha-1} \big(\int_0^s (s-u)^\alpha dM_u^H \big) ds \Big) = \int_0^t m_H(t,s) dB_s. \qquad (1.8.8)$$

Let $H \in (\frac{1}{2}, 1)$. Then, by using Theorem 1.8.1, we obtain that

$$\begin{aligned} &\textstyle\int_0^t (t-u)^\alpha dM_u^H = \alpha \int_0^t (t-u)^{\alpha-1} M_u^H du \\ &= C_H^{(5)} \alpha \textstyle\int_0^t (t-u)^{\alpha-1} \int_0^u (u-s)^{-\alpha} s^{-\alpha} dB_s^H du \\ &= C_H^{(5)} \alpha \textstyle\int_0^t \left(\int_s^t (t-u)^{\alpha-1} (u-s)^{-\alpha} du \right) s^{-\alpha} dB_s^H \\ &= C_H^{(5)} \alpha B(\alpha, 1-\alpha) Y_t = (C_H^{(6)})^{-1} \tilde{\alpha} Y_t, \end{aligned} \qquad (1.8.9)$$

i.e. we have (1.8.7) and obtain (1.8.8). In this case the kernel $m_H(t,s)$ can be simplified to $m_H(t,s) = \alpha C_H^{(6)} s^{-\alpha} \int_s^t u^\alpha (u-s)^{\alpha-1} du$.

1.9 The Inequalities for the Moments of the Wiener Integrals with Respect to fBm

In this section we introduce the estimates for the moments of the Wiener integrals with respect to fBm. For details one can refer to ([20]).

Theorem 1.9.1. *(i) Let $H \in (0, \frac{1}{2})$. Then $L_2^H(\mathbb{R}) \subset L_{\frac{1}{H}}(\mathbb{R})$ and there exists a constant $C_H > 0$ such that for any $f \in L_2^H(\mathbb{R})$, it holds that*

$$\| f \|_{L_{\frac{1}{H}}(\mathbb{R})} \leq C_H \| f \|_{L_2^H(\mathbb{R})} . \tag{1.9.1}$$

(ii) Let $H \in (\frac{1}{2}, 1)$. Then $L_{\frac{1}{H}}(\mathbb{R}) \subset L_2^H(\mathbb{R})$ and there exists a constant $C_H > 0$ such that for any $f \in L_{\frac{1}{H}}(\mathbb{R})$.

$$\| f \|_{L_2^H(\mathbb{R})} \leq C_H \| f \|_{L_{\frac{1}{H}}(\mathbb{R})} . \tag{1.9.2}$$

Proof. (i) Let $f \in L_2^H(\mathbb{R})$, this means that $M_-^H(\mathbb{R}) = C_H^{(3)} D_-^{-\alpha} f \in L_2(\mathbb{R})$. Evidently, $f = I_-^{\alpha} D_-^{-\alpha} f$ and from the Hardy-Littlewood theorem (Theorem 1.1.1 with $q = \frac{1}{H}, p = 2$ and $\alpha = \frac{1}{2} - H$), it follows that

$$\| f \|_{L_{\frac{1}{H}}(\mathbb{R})} = \| I_-^{\alpha} D_-^{-\alpha} f \|_{L_{\frac{1}{H}}(\mathbb{R})} \leq C_{2,\frac{1}{H},-\alpha} \| D_-^{-\alpha} f \|_{L_2(\mathbb{R})} = C_H \| f \|_{L_2^H(\mathbb{R})} .$$

(ii) We directly apply the Hardy-Littlewood theorem with $p = \frac{1}{2}, \alpha = H - \frac{1}{2}$ and $q = 2$:

$$\| f \|_{L_2^H(\mathbb{R})} = \| M_-^H f \|_{L_2(\mathbb{R})} \leq C_H \| f \|_{L_{\frac{1}{H}}(\mathbb{R})} .$$

□

Corollary 1.9.1. *Let $f \in L_2^H(\mathbb{R})$. Then there exists $I(f) = \int_{(\mathbb{R})} f(s) dB_s^H$ and $E|I(f)|^2 = \| f \|^2_{L_2^H(\mathbb{R})}$. Therefore, we have for $H \in (0, \frac{1}{2})$ that $E|I(f)|^2 \geq C_H^{-2} \| f \|^2_{L_{\frac{1}{H}}(\mathbb{R})}$ and, for $H \in (\frac{1}{2}, 1)$, it holds that $E|I(f)|^2 \leq C_H^2 \| f \|^2_{L_{\frac{1}{H}}(\mathbb{R})}$. Since $I(f)$ is a Gaussian random variable, we obtain the following inequalities for the moments of the Wiener integrals with respect to fBm: for any $r > 0$, there exists a constant $C(H, r)$, such that for $H \in (\frac{1}{2}, 1)$*

$$E|I(f)|^r \leq C(H, r) \| f \|^r_{L_{\frac{1}{H}}(\mathbb{R})}$$

and such that for $H \in (0, \frac{1}{2})$, we have that

$$\| f \|^r_{L_{\frac{1}{H}}(\mathbb{R})} \leq C(H, r) E|I(f)|^r .$$

Corollary 1.9.2. *Let $H \in (\frac{1}{2}, 1)$ and $f \in L_{\frac{1}{H}}(\mathbb{R})$. Then it follows from Theorem* 1.9.1, *(ii)*, (1.6.7) *and* (1.6.13), *that*

$$\| f \|_{|R_H|,2} \leq C \| f \|_{L_{\frac{1}{H}}(\mathbb{R})} .$$

Corollary 1.9.3. *Let $f \in L_{\frac{1}{H}}[a,b]$ and $f = 0$ outside (a,b). Then we obtain the following estimates: for any $r > 0$, there exists a constant $C(H,r)$, such that for $H \in (\frac{1}{2}, 1)$, it holds that*

$$E\left|\int_a^b f(s)dB_s^H\right|^r \le C(H,r) \parallel f \parallel^r_{L_{\frac{1}{H}}[a,b]}$$

and

$$E\left|\int_a^b f(s)dB_s^H \int_a^b g(s)dB_s^H\right|^r \le C(H,r) \parallel f \parallel^r_{L_{\frac{1}{H}}[a,b]} \parallel g \parallel^r_{L_{\frac{1}{H}}[a,b]} .$$

Furthermore, for $H \in (0, \frac{1}{2})$ the opposite inequality holds:

$$\parallel f \parallel^r_{L_{\frac{1}{H}}[a,b]} \le C(H,r) E\left|\int_a^b f(s)dB_s^H\right|^r .$$

Remark 1.9.1. *Let $H \in (\frac{1}{2}, 1)$ and $f \in |R_H|$. Then, from Hölder inequality, we obtain the estimate*

$$\|f\|^2_{|R_H|,2} = \int_{\mathbb{R}} |f(s)| \left(\int_{\mathbb{R}} |f(u)| \left|(s-u)\right|^{2\alpha-1} du\right) ds$$

$$\le \left(\int_{\mathbb{R}} |f(s)|^{\frac{1}{H}} ds\right)^H \left(\int_{\mathbb{R}} ds \left(\int_{\mathbb{R}} |f(u)| \left|(s-u)\right|^{2\alpha-1} du\right)^{\frac{1}{1-H}}\right)^{1-H} .$$

Further, from theorem (1.1.1) with $\alpha = 2H-1, q = \frac{1}{1-H}$ and $p = \frac{1}{H}$, we obtain that

$$\left(\int_{\mathbb{R}} ds \left(\int_{\mathbb{R}} |f(u)| \left|(s-u)\right|^{2\alpha-1} du\right)^{\frac{1}{1-H}}\right)^{1-H} \le C_H \|f\|_{L_{\frac{1}{H}}(\mathbb{R})}.$$

Therefore,

$$\|f\|_{|R_H|,2} \le C_H \|f\|_{L_{\frac{1}{H}}(\mathbb{R})}.$$

1.10 The Conditions of Continuity of Wiener Integrals with Respect to fBm

Let $H \in (\frac{1}{2}, 1)$. As mentioned in $([21], p.41)$, let $f \in L_{\frac{1}{H}}[0,T]$. and consider on $[0,T]$ the semi-metric ρ_I generated by the process I, i.e.

$$\rho_I^2(s,t) = E(I_t - I_s)^2 = E\left|\int_s^t f(u)dB_u^H\right|^2.$$

Where $I_t(f) = \int_0^t f(s)dB_s^H$. For any $\varepsilon > 0$ denote by $\mathcal{N}([0,T],\varepsilon)$ the metric ε-capacity of $([0,T],\rho)$. Also, let $\mathcal{H}([0,T],\varepsilon) := log\mathcal{N}([0,T],\varepsilon)$ be the metric ε-entropy of this interval in ρ_I, and let $D(T,\varepsilon) = \int_0^\varepsilon \mathcal{H}([0,T],u)^{1/2}du$ be the Dudley integral.

According to ([18]), a suficient condition for the continuity of separable modification of $I_t(f)$ on $[0,T]$ is the finiteness of the Dudley integral $\int_0^\varepsilon \mathcal{H}([0,T],u)^{\frac{1}{2}}du$. But in our case, from ([21], *Theorem* 1.10.3) with ε instead of $\frac{\sigma}{2}$ it follows that

$$\int_0^\varepsilon \mathcal{H}([0,T],u)^{1/2}du \le \int_0^\varepsilon \big(\log(1 + u^{-\frac{1}{H}}\widetilde{C}_H \int_0^T |f(u)|^{\frac{1}{H}}du)\big)^{\frac{1}{2}}du$$

$$\le \int_0^\varepsilon u^{-\frac{1}{2H}}du \cdot \big(\widetilde{C}_H \int_0^T |f(u)|^{\frac{1}{H}}du\big)^{\frac{1}{2}} < \infty.$$

This means that the separable modification of the Wiener integral w.r.t. fBm with $H \in (\frac{1}{2},1)$ is continuous if $f \in L_{\frac{1}{H}}[0,T]$.

Now, let $H \in (0,\frac{1}{2})$. Then, according to ([21], *Theorem* 1.10.4) with ε instead of $\frac{\sigma}{2}$, we have that $\int_0^\varepsilon \mathcal{H}([0,T],u)^{1/2}du.$ is finite for any $f \in L_p[0,T] \cap D_p^H[0,T]$, $p > \frac{1}{H}$. So, for such f a separable modification of $I_t(f)$ is continuous on $[0,T]$.

1.11 Stochastic Fubini Theorem for the Wiener Integrals w.r.t fBm

Consider only the case $H \in (1/2,1)$. Let $\mathcal{P}_T = [0,T]^2$.

Theorem 1.11.1. *Let the measurable function $f = f(t,s) : \mathcal{P}_T \to \mathbb{R}$ satisfy the conditions*

$$\int_{[0,T]^3} |f(t,u)||f(t,s)||s-u|^{2\alpha-1}ds\ du\ dt < \infty \tag{1.11.1}$$

and

$$\int_{[0,T]^4} |f(t_1,u)||f(t_2,s)||s-u|^{2\alpha-1}ds\ du\ dt_1\ dt_2 < \infty. \tag{1.11.2}$$

Then both the repeated integrals $I_1 := \int_0^T(\int_0^T f(t,s)dt)dB_s^H$ and $I_2 := \int_0^T(\int_0^T f(t,s)dB_s^H)dt$ exist and $I_1 = I_2$ with probability 1.

Proof. The existence of the integral I_1 is evident, due to (1.11.2). As to I_2, $\int_0^T f(t,s)dB_s^H$ exists , and according to (1.11.1), it holds that

$$E\int_0^T\left|\int_0^T f(t,s)dB_s^H\right|dt \leq T^{1/2}\left(E\int_0^T\left|\int_0^T f(t,s)dB_s^H\right|^2 dt\right)^{1/2}$$

$$\leq (T2\alpha H\int_{[0,T]^3}|f(t,s)||f(t,u)||s-u|^{2\alpha-1}du\ \ ds\ \ dt)^{1/2} < \infty.$$

We consider at first only the measurable and bounded functions. Let $f^* := \sup_{(t,s)\in[0,T]^2}|f(t,s)| < \infty$. Then there exists the sequence of simple and totally bounded functions $f_n = f_n(t,s)$, such that $f_n \to f$ uniformly on $\mathcal{P}_T$. The statement of the theorem is evident for f_n. Further, denote $g_n(t,s) := f(t,s) - f_n(t,s)$ and obtain the estimate

$$|I_1 - I_2| \leq \left|\int_0^T(\int_0^T g_n(t,s)dt)dB_s^H\right| + \left|\int_0^T(\int_0^T g_n(t,s)dB_s^H)dt\right|$$

$$=: I_{1n} + I_{2n}.$$

Furthermore,

$$\begin{aligned}E|I_{1n}|^2 &= 2\alpha H\int_{\mathcal{P}_T}\left(\int_0^T g_n(t_1,s)dt_1\right)\left(\int_0^T g_n(t_2,s)dt_2\right)|s-u|^{2\alpha-1}dsdu\\ &\leq 2\alpha HT^2\sup_{(t,s)\in[0,T]^2}|g_n(t,s)|^2\int_{\mathcal{P}_T}|s-u|^{2\alpha-1}dsdu\\ &= T^{2H+2}\sup_{(t,s)\in\mathcal{P}_T}|g_n(t,s)|^2 \to 0,\end{aligned}$$

and

$$E|I_{2n}|^2 \leq T\int_0^T E\left|\int_0^T g_n(t,s)dB_s^H\right|^2 dt \leq \sup_{(t,s)\in\mathcal{P}_T}|g_n(t,s)|^2 T^{2H+2} \to 0,$$

as $n \longrightarrow \infty$, and we obtain the proof for bounded f. Now, let f satisfy (1.11.1) and (1.11.2). For $f_n(t,s) = f(t,s)\mathbb{1}_{\{|f(t,s)|\leq n\}}$, $n \geq 1$ the theorem is already proved. Define

$$C_n = \{(t,s,u) \in [0,T]^3/|f(t,s)| \geq n\}, \quad \overline{f}_n = f - f_n.$$

Then for any $n \geq 1$ we have that

$$|I_1 - I_2| \leq \left| \int_0^T \Big(\int_0^T f(t,s) \mathbb{1}_{\{|f(t,s)|>n\}} dt \Big) dB_s^H \right|$$

$$+ \left| \int_0^T \Big(\int_0^T f(t,s) \mathbb{1}_{\{|f(t,s)|>n\}} dB_s^H \Big) dt \right| =: I'_{1n} + I'_{2n}.$$

Furthermore, we have that

$$E|I'_{1n}|^2 = 2\alpha H \int_{[0,T]^2} \left(\int_0^T \overline{f}_n(t_1,s) dt_1 \right) \left(\int_0^T \overline{f}_n(t_2,s) dt_2 \right) |s-u|^{2\alpha-1} ds \ du$$

$$\leq 2\alpha H \int_{[0,T]^4} |\overline{f}_n(t_1,s)| |\overline{f}_n(t_2,s)| |s-u|^{2\alpha-1} ds \ du \ dt_1 \ dt_2 \to 0,$$

as $n \longrightarrow \infty$, according to (1.11.2), and

$$E|I'_{2n}|^2 \leq T 2\alpha H \int_{[0,T]^3} |\overline{f}_n(t,s)| |\overline{f}_n(t,u)| |s-u|^{2\alpha-1} ds \ du \ dt \to 0,$$

as $n \longrightarrow \infty$, according to (1.11.1). □

1.12 Martingale Transforms and Girsanov Theorem for Long-memory Gaussian Processes

In this section we consider long-memory Gaussian processes that can be presented as integrals $V_t = \int_0^t h(t-s)\varphi(s) dW_s$ with some Wiener process W_t and establish the conditions allowing us to transform these processes, into square-integrable martingales, similarly to

$$M_t^H := C_H^{(5)} \int_0^t s^{-\alpha} (t-s)^{-\alpha} dB_s^H$$

Where $B_t := \ \widehat{\alpha} \int_0^t \ s^{\alpha} \ dM_s^H$, is a Wiener process. In turn $B_t^H = C_H^{(6)} \int_0^t m_H (t-s)^{-\alpha} dB_s$ Moreover, the process

$$Y_t = C_H^{(6)} \int_0^t (t-s)^{\alpha} s^{-\alpha} dB_s \tag{1.12.1}$$

has the property that $M_t^H = C_H^{(5)} \int_0^t (t-s)^{-\alpha} dY_s$ is square-integrable martingale. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with $\mathcal{F} = \mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t^W$.

Define the convolution of two measurable integrable functions φ_1 and $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}$ by $(\varphi_1 * \varphi_2)(t) = \int_0^t \varphi_1(t-s)\varphi_2(s)ds, t \in \mathbb{R}_+$. Let h and φ satisfy the assumption

$$\varphi \in L_2(0,t), \;\; (h^2 * \varphi^2)_t < \infty, \;\; t > 0. \tag{1.12.2}$$

Let $\mathcal{F}_t^X = \sigma\{X_s, 0 \leq s \leq t\}$ and $\mathcal{H}_t^X = \mathcal{H}\{X_s, 0 \leq s \leq t\}$ be, correspondingly, σ-fields and Gaussian subspaces, generated by the process X on the interval $(0,t], X = W, V$. It follows from ([7], Proposition 15) that $\mathcal{F}_t^V = \mathcal{F}_t^W, t \in \mathbb{R}_+$ if and only if $\mathcal{H}_t^V = \mathcal{H}_t^W$. A necessary and suficient condition for this coincidence can be formulated as

the only function f such that $\forall t \in \mathbb{R}_+$
$f \in L_2(0,t)$ and $((f \cdot \varphi) * h)_t = 0$ is the zero function. (1.12.3)

Denote by $L_2(V) = L_2(W) = L_2(\Omega, \mathcal{F}_\infty, P)$ the space of $\mathcal{F}_\infty$-measurable ξ with $E\xi^2 < \infty$. Let $\mathcal{H}(V)$ be the closed subspace of $L_2(V)$ consisting of linear functionals of V. Suppose that the function $R : \mathbb{R}_+^2 \to \mathbb{R}$ has a bounded variation $|R|_t := var_{\mathcal{P}_t} R$ on any rectangle $\mathcal{P}_t, t \to \mathbb{R}_+^2$, and consider the measurable function $g : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\int_{\mathcal{P}(s,t)} |g(s-u)||g(t-v)|d|R|_{uv} < \infty, \;\; s,t \in \mathbb{R}_+. \tag{1.12.4}$$

As stated by ([13]), let $I(f) = \int_{\mathbb{R}} f dV \in \mathcal{H}(V)$, and let

$$M_t := \int_0^t g(t-u)dV_u := I(\tilde{g}),$$

where $\tilde{g}(s) = g(t-s)\mathbb{1}_{\{s\leq t\}}, t \geq 0$. Then $\{M_t, \mathcal{F}_t^W, t \geq 0\}$ is a Gaussian process and

$$EM_sM_t = \int_{\mathcal{P}(s,t)} g(s-u)g(t-v)dR_{uv}.$$

Moreover, under the condition:
the double Riemann integral $\int_{\mathcal{P}(s,t)} g(s-u)g(t-v)dR_{uv}$ exists, (1.12.5)

the process M_t can be considered for any $t \geq 0$ as a limit of Riemann sums in the mean-square sense. Note that the following condition is suficient for (1.12.5): the derivative $h'(s), s > 0$, exists, $h(0) = 0$, and R_{uv} admits a representation

$$R_{uv} = \int_{\mathcal{P}(u,v)} \left[\int_0^{u_1 \wedge v_1} h'(u_1 - z) h'(v_1 - z) \varphi^2(z) dz \right] du_1 dv_1 \tag{1.12.6}$$

and

$$\int_{\mathcal{P}(s,t)} |g(s-u)||g(t-v)| \left[\int_0^{u \wedge v} h'(u-z) h'(v-z) \varphi^2(z) dz \right] du dv < \infty.$$

Now we are in a position to study conditions on φ, h and g supplying martingale properties of M_t.

Definition 1.12.1. *Gaussian process V is called (g)-transformable if the process*

$$M_t := \int_0^t g(t-s) dV_s$$

is a martingale.

Denote $U = \{f : \mathbb{R}_+ \to \mathbb{R} \big| (f * q)_t = 0,\ t \in \mathbb{R}_+,$ for such $q : \mathbb{R}_+ \to \mathbb{R}$ that $(|f| * |q|)_t < \infty,\ t \geq 0$ if and only if $q = 0\}$
$AC[0,t] = \{f : \mathbb{R}_+ \to \mathbb{R} \big| f(s) = \int_0^s f'(u) du;\ 0 \leq s \leq t$ with $\int_0^t |f'(u)| du < \infty\}$.

Theorem 1.12.1. 1) *Let φ, h, g satisfy conditions* (1.12.2), (1.12.3), (1.12.6) *and*

$$(|g| * |h'|)_t < \infty, \quad t > 0, \tag{1.12.7}$$
$$(g * h')_t = C_0,\ t > 0 \ for some\ C_0 \in \mathbb{R}. \tag{1.12.8}$$

Then V_t is (g)-transformable and $\langle M_t \rangle = C_0^2 \int_0^t \varphi^2(s) ds$.
2) *Let φ, h, g satisfy conditions* (1.12.2), (1.12.3), (1.12.6) *and* (1.12.7), $h \in U$, $\varphi \neq 0 (mod \lambda)$ *(λ is the Lebesgue measure), $(g * h')_t \in C(0, \infty), V_t$ be (g)-transformable. Then $(g * h')_t = C_0, t > 0$, for some $C_0 \in \mathbb{R}$.*

Theorem 1.12.2. 3) *Let φ and h satisfy* (1.12.2) *and* (1.12.3), $\varphi \neq 0 (mod \lambda)$, *$g$ satisfies* (1.12.5) *and*

$$g \in AC[0,t],\ t \geq 0,\ g(0) = 0, \tag{1.12.9}$$
$$(|g'| * (h^2 * \varphi^2)^{1/2})_t < \infty,\ t > 0, \tag{1.12.10}$$
$$(g' * h) t = C_0, t > 0\ for some\ C_0 \in \mathbb{R}. \tag{1.12.11}$$

Then V_t is (g)-transformable and $\langle M_t\rangle = C_0^2 \int_0^t \varphi^2(s)ds$.

4) *Let φ and h satisfy* (1.12.2), (1.12.3), $\varphi \neq 0$ *a.e.* $(mod\lambda)$, *the process V_t is (g)-transformable with g satisfying* (1.12.9), (1.12.10), $(g' * h)_t \in C(0,\infty)$. *Then $(g' * h)_t = C_0, t > 0$ for some $C_0 \in \mathbb{R}$.*

Proof . 3) Under condition (1.12.5) the integral M_t is a mean-square limit of Riemann sums, and condition (1.12.9) permits us to transform the sum:

$$\begin{aligned} M_t &= \lim_{|\lambda_N|\to 0} \sum_{i=0}^{N-1} g(t-s_i)(V_{s_{i+1}} - V_{s_i}) \\ &= \lim_{|\lambda_N|\to 0} \sum_{i=0}^{N-1} V(s_{i+1})(g(s_{i+1}) - g(s_i)) \\ &= \int_0^t g'(t-s)V_s ds = \int_0^t g'(t-s)\big(\int_0^s h(s-z)\varphi(z)dWz\big)ds, \end{aligned}$$

where $|\lambda_N| = \max_{0\le i\le N-1} |g(s_{i+1}) - g(s_i)|$, and the last integral is the limit of Riemann sums in the mean-square sense. Further, condition (1.12.10), according to ([25], p. 160) or ([17]), permits to apply to M_t the stochastic Fubini theorem, and we obtain from (1.12.11) that

$$M_t = \int_0^t \varphi(z)\big(\int_z^t g'(t-u)h(u-z)du\big)dW_s = C_0 \int_0^t \varphi(z)dW_z. \qquad (1.12.12)$$

4) If the process M_t is a square-integrable martingale, then from (1.12.12) it follows that for any $0 \le s \le t$

$$0 = E(M_t - M_s/\mathcal{F}_s^W) = \int_0^s \varphi(z)\eta(z)dW_z,$$

where

$$\eta(z) = (g' * h)_{t-z} - (g' * h)_{s-z}.$$

Hence $\int_0^s \varphi^2(z)\eta^2(z)dz = 0$, and, arguing similarly to the completion of the proof of Theorem 1.12.4, part 2), (see[21] ,p. 64), we obtain that

$$(g' * h)_t = C_0$$

for some $C_0 \in \mathbb{R}$. □

Now, let V_t be equal to Y_t from (1.12.1). Recall that $B_t^H = \int_0^t s^\alpha dV_s$ is an fBm with Hurst index H, and in this case B_t^H can be presented as $B_t^H = \int_0^t m_H(t,s)dB_s$, where B is a Wiener process and the kernel $m_H(t,s)$ is defined in Section 1.8. Consider general conditions on function $\psi : \mathbb{R}_+ \to \mathbb{R}$ for the process $N_t := \int_0^t \psi_s dV_s$ to be presented in a similar way.

Theorem 1.12.3. *Let conditions* (1.12.2), (1.12.3) *hold and also*

$$\lim_{\varepsilon\downarrow 0}\psi^2(\varepsilon)\int_0^\varepsilon h^2(\varepsilon-u)\varphi^2(u)du = 0; \tag{1.12.13}$$

the Riemann integral $\int_{[0,(s,t)]}\psi(u)\psi(v)dR_{uv}$ *exists,* $s,t>0;$ (1.12.14)
there exists a derivative $\psi'(s), s>0$ *and*

$$(h^2{*}\varphi^2)^{1/2}\psi'\in L_1(0,t), \qquad (|h|{*}|\psi'|)_t<\infty, \qquad t>0. \tag{1.12.15}$$

Then

$$\int_0^t\psi(s)dV_s=\int_0^t m(t,s)\varphi(s)dW_s,\ \ t>0,\ \ a.s.,$$

where

$$m(t,s)=\psi(t)h(t-s)-\int_s^t h(u-s)\psi'(u)du,$$

W *is a Wiener process. If* (1.12.15) *is strengthened to*

$$(h^2*\varphi^2)^{1/2}\psi'\in L_2(0,t), t>0, \tag{1.12.16}$$

then $E(\int_0^t\psi(s)dV_s)^2<\infty$.

Proof. Under (1.12.13)−(1.12.15), we can consider the integral $\int_0^t\psi(u)dV_u$ as a mean-square limit of Riemann sums, and integrating by parts, we obtain the following limits in the mean-square sense

$$\begin{aligned}\int_0^t\psi(u)dV_u &= \lim_{\varepsilon\downarrow 0}\int_\varepsilon^t\psi(u)dV_u\\ &= \psi(t)V(t)-\lim_{\varepsilon\downarrow 0}\psi(\varepsilon)V(\varepsilon)-\int_0^t\psi'(u)V(u)du\\ &= \psi(t)V(t)-\int_0^t\psi'(u)\big(\int_0^u h(u-s)\varphi(s)dW_s\big)du.\end{aligned}$$

Due to (1.12.15), the stochastic Fubini theorem can be applied to the last integral, and we obtain

$$\int_0^t\psi(u)dV_u=\int_0^t\psi(t)h(t-s)\varphi(s)ds-\int_0^t\varphi(s)\big(\int_0^u h(u-s)\psi'(u)du\big)dW_s$$

$$=\int_0^t m(t,s)\varphi(s)dW_s.$$

The second statement is evident. □

Now let P and $\widehat{P}$ be two probability measures on $(\Omega,\mathfrak{F})$. Denote by $P_t(\widehat{P}_t)$ the restriction of $P(\widehat{P})$ on $\mathfrak{F}_t$ and suppose that $\widehat{P}\overset{loc}{\ll}P$. Consider the density process $Z_t = \mathcal{E}(X_t) := exp\{X_t - 1/2\langle X^c\rangle_t\}\prod_{0\le s\le t}(1+\triangle X_s)e^{-\triangle X_s}$, X is a local martingale.

As before, we consider the Gaussian process $V_t = \int_0^t h(t-s)\varphi(s)dW_s$ and suppose that V_t is (g)-transformable by the function g; moreover, the conditions $(1.12.7)-(1.12.8)$ or $(1.12.9)-(1.12.11)$ hold. Let $M_t = C_0\int_0^t \varphi(s)dW_s$ with C_0 depending on g. Since M_t has continuous modification, the process $[M,X]$ has P-locally bounded variation (see ([14], Lemma 3.14)). Denote by $A_t := \langle M,X\rangle_t$ the P-compensator of $[M,X]$. Suppose further that the function ψ satisfies conditions $(1.12.13)-(1.12.15)$ of Theorem 1.12.3.

Lemma 1.12.1. *The integral* $\int_0^t m(t,s)dA_s$ *exists for any* $t>0$ $P-$ *and* $\widehat{P}-a.s.$

Proof. Since $m(t,s) = \psi(t)h(t-s) - \int_s^t h(u-s)\psi'(u)du$, we consider $\int_0^t h(t-s)dA_s$ and $\int_0^t\big(\int_s^t h(u-s)\psi'(u)du\big)dA_s$ individually. From Kunita's inequality and (1.12.2),

$$\int_0^t |h(t-s)|d|A|_s \le \left(\int_0^t |h(t-s)|^2 d\langle M\rangle_s\cdot\langle X\rangle_t\right)^{\frac{1}{2}}$$
$$= C_0\left(\int_0^t |h(t-s)|^2\varphi^2(s)ds\langle X\rangle_t\right)^{\frac{1}{2}} < \infty$$

P and $\widehat{P}-a.s.$
Similarly,

$$\int_0^t\left|\int_s^t \psi'(u)h(u-s)du\right|d|A|_s$$
$$\le C_0\left(\int_0^t\left|\int_s^t \psi'(u)h(u-s)du\right|^2\varphi^2(s)ds\cdot\langle X\rangle_t\right)^{\frac{1}{2}}$$
$$\le C_0\left(\int_0^t (h^2*\varphi^2)_u|\psi'(u)|^2du\cdot\langle X\rangle_t\right)^{\frac{1}{2}} < \infty,$$

P and $\widehat{P}-a.s.$ □

Theorem 1.12.4. *Let* V_t *be* (g)*-transformable with* g *satisfying* $(1.12.7)-(1.12.8)$ *or* $(1.12.9)-(1.12.11), \psi$ *satisfying* $(1.12.13)-(1.12.15), \varphi\neq 0$ *a.e. Then* $\widehat{N}_t := N_t - C_0^{-1}\int_0^t m(t,s)dA_s$ *is a Gaussian process w.r.t.* $\widehat{P}$ *and admits the representation* $\widehat{N}_t = \int_0^t m(t,s)\varphi(s)d\widehat{W}_s$*, where* $\widehat{W}_t$ *is a Wiener process w.r.t.* $\widehat{P}$.

Proof. According to the classical Girsanov theorem, $\widehat{M}_t := M_t - \langle M, X\rangle_t$ is a $\widehat{P}$-local martingale with the angle bracket $\langle \widehat{M}\rangle_t = \langle M\rangle_t = C_0^2 \int_0^t \varphi^2(s)ds$. Therefore, $\widehat{M}_t$ is a continuous square-integrable $\widehat{P}-$ martingale. Since $\varphi \neq 0 \quad a.e.(mod\lambda)$, we obtain from the Lévy theorem that $\widehat{M}_t = C_0 \int_0^t \varphi_s d\widehat{W}_s, \widehat{W}$ is $\widehat{P}$-Wiener process. According to Theorem 1.12.2, $\widehat{B}_t = C_0^{-1} \int_0^t z(t,s)d(M_s - \langle M, X\rangle_s) = C_0^{-1} \int_0^t m(t,s)d\widehat{M}_s = \int_0 m(t,s)\varphi(s)d\widehat{W}_s$. □

According to the Theorem 1.12.4, we obtain that the drift has the form $D_t := C_0^{-1} \int_0^t m(t,s)dA_s$ in the case when the density process Z_t is known. Consider also the question: what "drifts" are admissible?

Theorem 1.12.5. *Let* (1.12.13) – (1.12.15) *and one of the following sets of conditions hold:*

1) *conditions* $(1.12.2), (1.12.3), (1.12.6) - (1.12.8)$ *and* $\varphi \neq 0$ *a.e.* $(mod\lambda)$;

2) $\int_s^t |h'(v-s)||\psi'(v)|dv < \infty, \quad 0 \le s \le t \quad a.s$;

3) *a process* $\{D_t, \mathcal{F}_t^W, t = 0\}$ *has a.s. bounded variation* $|D|_t = var_{[0,t]}D, t > 0, D_0 = 0$;

4)$\psi \neq 0$, *the integral* $\int_0^t |g(t-s)||\psi^{-1}(s)|d|D|_s < \infty \quad a.s., \ t > 0$, *and we have a representation*

$$\int_0^t g(t-s)\psi^{-1}(s)dD_s = \int_0^t \delta_s ds, \quad where \quad \int_0^t |\delta_s|ds < \infty \quad a.s.$$

$$E\int_0^t \varphi_s^{-2}\delta_s^2 ds < \infty, \quad t > 0;$$

5)$E\mathcal{E}(X_t) = 1$ *where*

$$X_t = C_0^{-1}\int_0^t \varphi_s^{-1}\delta_s dW_s, \quad \mathcal{E}(X_t) = \exp\{X_t - \frac{1}{2}\langle X\rangle_t\};$$

or

6) *conditions* $(1.12.2), (1.12.3), (1.12.5), (1.12.9) - (1.12.11)$;

7) *conditions* 3) – 5);

8) *a process* $E_t = \int_0^t m(t,s)\delta_s ds$ *has bounded variation and*

$$\int_0^t |g(t-s)||\psi^{-1}(s)|d|E|_s < \infty, \quad a.s., \quad t > 0;$$

9) $g' \in U$.

Then the process $\widehat{B}_t = B_t - D_t$ *is Gaussian and admits the representation* $\widehat{B}_t = \int_0^t m(t,s)\varphi(s)d\widehat{W}_s$ *under the measure* $\widehat{P} \overset{loc}{\ll} P$ *such that* $\frac{d\widehat{P}}{dP}\big|_{\mathcal{F}_t^W} = \mathcal{E}(X_t)$.

1.13 Nonsemimartingale Properties of fBm; How to Approximate Them by Semimartingales

A process $\{X_t, \mathcal{F}_t, t \geq 0\}$ is called semimartingale, if it admits the representation $X_t = X_0 + M_t + A_t$, where M is an $\mathcal{F}_t$-local martingale with $M_0 = 0$, A is a process of locally bounded variation, X_0 is $\mathcal{F}_0$-measurable. Evidently, any semimartingale has locally bounded quadratic variation; if X is continuous, then M and A are continuous. Let $X_t = B_t^H$ with $H \in (0, 1/2)$. Then its quadratic variation is infinite, therefore, it is not a semimartingale. If $H \in (1/2, 1)$ then the quadratic variation of X is zero, and if we suppose that X is semimartingale, then the quadratic variation of $M_t = X_t - X_0 - A_t$ is zero, and M is zero. But $X_t \neq A_t$ since X has unbounded variation. Therefore, $X_t = B_t^H$ is not a semimartingale for any $H \neq 1/2$. Nevertheless, there are many approaches to how to approximate fBm by a sequence of semimartingales.

1.13.1 Approximation of fBm by Continuous Processes of Bounded Variation

We follow here the approach of ([1],[2]). According to (1.8.5) and (1.8.9), we can represent $\{B_t^H, t \geq 0\}$ with $, H \in (1/2, 1)$ as

$$B_t^H = \int_0^t s^\alpha dY_s,$$

where

$$Y_t = C_H^{(8)} \int_0^t (t-s)^\alpha s^{-\alpha} dB_s,$$

and $C_H^{(8)} = C_H^{(6)} \widetilde{\alpha}$. We can rewrite Y_t as

$$Y_t = C_H^{(8)} \alpha \int_0^t \left(\int_s^t (u-s)^{\alpha-1} du \right) s^{-\alpha} dB_s. \tag{1.13.1}$$

If we formally apply the stochastic Fubini theorem to the right-hand side of (1.13.1), we obtain that

$$Y_t = C_H^{(8)} \alpha \int_0^t \left(\int_0^u (u-s)^{\alpha-1} s^{-\alpha} dB_s \right) du. \tag{1.13.2}$$

But the right-hand side of (1.13.2) does not exist, since the variance of interior integral is infinite,

$$\int_0^u (u-s)^{2\alpha-2} s^{-2\alpha} ds = \infty.$$

Thereupon, we introduce the "truncated" process for $\beta \in (0,1)$,

$$Y_t^{\beta} = C_H^{(8)} \alpha \int_0^t \left(\int_0^{\beta s} (s-u)^{\alpha-1} u^{-\alpha} dB_u \right) ds,$$

and

$$B_t^{H,\beta} = \int_0^t s^{\alpha} dY_s^{\beta} = C_H^{(8)} \alpha \int_0^t s^{\alpha} \left(\int_0^{\beta s} (s-u)^{\alpha-1} u^{-2\alpha} dB_u \right) ds \qquad (1.13.3)$$

is a process of bounded variation which will serve as an approximation of B_t^H.

Theorem 1.13.1. *We have that*

$$E(B_t^H - B_t^{H,\beta})^2 \le c_1 t^{2H} (1-\beta)^{2\alpha},$$

where $c_1 = c_1(H)$ is some constant, independent of t and β.

Proof. First, we want to change the limits of the integration in (1.13.3) and consider the process

$$\begin{aligned} Z_t^{\beta} &:= \alpha C_H^{(8)} \int_0^{\beta t} \left(\int_{u/\beta}^t (s-u)^{\alpha-1} ds \right) u^{-\alpha} dB_u \\ &= C_H^{(8)} \left(\int_0^{\beta t} (t-u)^{\alpha} u^{-\alpha} dB_u \left(\frac{1-\beta}{\beta} \right)^{\alpha} B_{\beta t} \right). \end{aligned} \qquad (1.13.4)$$

We cannot apply here the stochastic Fubini theorem ([25], Theorem IV.4.5), because it is valid if the integral $\int_0^{\beta t} \int_{u/\beta}^t (s-u)^{2\alpha-2} u^{-2\alpha} ds du$ is finite but it is infinite. Therefore, we must go an indirect way. We consider the integral $Y_t^{\beta,\varepsilon} = D \int_{\varepsilon}^t \left(\int_{\beta\varepsilon}^{\beta s} (s-u)^{\alpha-1} u^{-\alpha} dB_u \right) ds$, where $D = \alpha C_H^{(8)}$, and the Fubini theorem ensures the equality

$$Y_t^{\beta,\varepsilon} = Z_t^{\beta,\varepsilon} := D \int_{\beta\varepsilon}^{\beta t} (\int_{u/\beta}^t (s-u)^{\alpha-1} ds) u^{-\alpha} dB_u.$$

Furthermore,

$$E|Y_t^{\beta,\varepsilon} - Y_t^{\beta}| \le D \left(\int_0^{\varepsilon} \left(\int_0^{\beta s} (s-u)^{2\alpha-2} u^{-2\alpha} du \right)^{1/2} ds \right.$$

$$+\int_\varepsilon^t\left(\left(\int_0^{\beta\varepsilon}(s-u)^{2\alpha-2}u^{-2\alpha}du\right)^{1/2}ds\right)\le D\left(\int_0^\varepsilon u^{-1/2}du\left(\int_0^\beta(1-u)^{2\alpha-2}\right.\right.$$

$$\left.\left.\times u^{-2\alpha}du\right)+\widehat{\alpha}(\beta\varepsilon)^{1/2-\alpha}\int_\varepsilon^t(s-\beta\varepsilon)^{\alpha-1}ds\right)\to 0$$

and

$$E|Z_t^{\beta,\varepsilon}-Z_t^\beta|^2\le D^2\int_0^{\beta\varepsilon}\left(\int_{u/\beta}^t(s-u)^{\alpha-1}ds\right)^2u^{-2\alpha}du\le CD^2\beta\varepsilon^{1-2\alpha}\to 0$$

as $\varepsilon\to 0$, where $C>0$ is some constant. This means that $Y_t^\beta=Z_t^\beta \quad a.s.$ for any $t\in[0,T]$. Therefore, for $1/2<\beta<1$

$$E(Y_t-Y_t^\beta)^2=(C_H^{(8)})^2E\left(\int_{\beta t}^t(t-u)^\alpha u^{-\alpha}dB_u+\left(\frac{1-\beta}{\beta}\right)^\alpha B_{\beta t}\right)^2$$

$$\le 2(C_H^{(8)})^2\int_{\beta t}^t(t-u)^{2\alpha}u^{-2\alpha}du+2(C_H^{(8)})^2\left(\frac{1-\beta}{\beta}\right)^{2\alpha}\beta t$$

$$\le H^{-1}(C_H^{(8)})^2(\beta t)^{-2\alpha}t^{2H}(1-\beta)^{2H}+2(C_H^{(8)})^2\left(\frac{1-\beta}{\beta}\right)^{2\alpha}\beta t$$

$$\le c_2t(1-\beta)^{2\alpha}\ \ with\ \ c_2=(C_H^{(8)})^2\cdot 2^{2\alpha-1}(H^{-1}+2). \tag{1.13.5}$$

Integration by parts gives us

$$B_t^H-B_t^{H,\beta}=t^\alpha(Y_t-Y_t^\beta)-\alpha\int_0^t(Y_s-Y_s^\beta)s^{\alpha-1}ds$$

whence we obtain from (1.13.5) that

$$E(B_t^H-B_t^{H,\beta})^2\le 2t^{2\alpha}E(Y_t-Y_t^\beta)^2+2\alpha^2t\int_0^tE(Y_s-Y_s^\beta)^2s^{2\alpha-2}ds$$

$$\le 2c_2t^{2H}(1-\beta)^{2\alpha}+2\alpha^2t\int_0^ts^{2\alpha-1}ds\cdot c_2(1-\beta)^{2\alpha},$$

and we can put $c_1=2c_2(\alpha+1)$. $\square$

1.13.2 Convergence $B^{H,\beta} \to B^H$ in Besov space $W^\lambda[a,b]$

For $\lambda \in (0, 1/2)$ define the Besov space $W^\lambda[a,b]$ as the space of measurable functions $f : [a,b] \to \mathbb{R}$ such that

$$\|f\|_{a,b,\lambda} := \int_a^b \frac{|f(s)|}{(s-a)^\lambda} ds + \int_a^b \int_a^s \frac{|f(s) - f(y)|}{(s-y)^{\lambda+1}} dyds < \infty.$$

Theorem 1.13.2. *For any $\lambda \in (0, 1/2), H \in (1/2, 1)$ and any $[a,b] \subset [0,T]$*

$$E\|B^H - B^{H,\beta}\|_{a,b,\lambda} \le c_1(H, \lambda, T)(1-\beta)^\alpha.$$

Proof. Denote $\overline{B}_t^{H,\beta} := B_t^H - B_t^{H,\beta}$. We have

$$E\|\overline{B}^{H,\beta}\|_\lambda = E\int_a^b \frac{|\overline{B}_s^{H,\beta}|}{(s-a)^\lambda} ds + E\int_a^b \int_a^s \frac{|\overline{B}_s^{H,\beta} - \overline{B}_y^{H,\beta}|}{(s-y)^{\lambda+1}} dyds. \tag{1.13.6}$$

From Theorem 1.13.1,

$$E\int_a^b \frac{|\overline{B}_s^{H,\beta}|}{(s-a)^\lambda} ds \le \int_a^b \frac{(E(\overline{B}_s^{H,\beta})^2)^{1/2}}{(s-a)^\lambda} ds \le c_1^{1/2}(1-\beta)^\alpha \int_a^b \frac{s^H}{(s-a)^\lambda} ds$$
$$\le c_1(H, \lambda, T)(1-\beta)^\alpha, \tag{1.13.7}$$

with $c_1(H, \lambda, T) = c_1^{1/2} \cdot T^{H-\lambda+1} \cdot (H - \lambda + 1)^{-1}$. Consider the second term in the right-hand side of (1.13.6). Rewrite the difierence in the numerator as

$$\overline{B}_s^{H,\beta} - \overline{B}_y^{H,\beta} = (B_s^H - B_s^{H,\beta}) - (B_y^H - B_y^{H,\beta})$$
$$= \int_y^s u^\alpha d(Y_u - Y_u^\beta) = \int_y^s u^\alpha d\overline{Y}_u^\beta, \tag{1.13.8}$$

where $\overline{Y}_u^\beta = Y_u - Y_u^\beta$. Equality (1.13.8) and integration by parts give us the estimates

$$\int_a^b \int_a^s \frac{|\overline{B}_s^{H,\beta} - \overline{B}_y^{H,\beta}|}{(s-y)^{\lambda+1}} dyds$$
$$= \int_a^b \int_a^s (s-y)^{-\lambda-1} \Big| s^\alpha \overline{Y}_s^\beta - y^\alpha \overline{Y}_y^\beta + \alpha \int_y^s \overline{Y}_u^\beta u^\alpha du \Big| dyds$$
$$\le \int_a^b \int_a^s (s-y)^{-\lambda-1} s^\alpha |\overline{Y}_s^\beta - \overline{Y}_y^\beta| dyds$$

$$
\begin{aligned}
&+\int_a^b\int_a^s (s-y)^{-\lambda-1}(s^\alpha - y^\alpha)|\overline{Y}^\beta_y|dyds \\
&+\alpha\int_a^b\int_a^s (s-y)^{-\lambda-1}\Big(\int_y^s |\overline{Y}^\beta_u|u^{\alpha-1}du\Big)dyds \\
=: {}& I_1(\beta)+I_2(\beta)+\alpha I_3(\beta).
\end{aligned}
$$

Now we estimate $I_2(\beta)$:

$$
\begin{aligned}
EI_2(\beta) &\le \alpha\int_a^b\int_a^s y^{\alpha-1}(s-y)^{-\lambda}(E(\overline{Y}^\beta_y)^2)^{1/2}dyds \\
&\le c_2^{1/2}\alpha\int_a^b\int_a^s y^{\alpha-1}(s-y)^{-\lambda}y^{1/2}dyds\cdot(1-\beta)^\alpha \\
&\le c_2(H,\lambda,T)(1-\beta)^\alpha, \qquad (1.13.9)
\end{aligned}
$$

where $c_2(H,\lambda,T) = c_2^{1/2}\alpha T^{1-\lambda}$.

$$
\begin{aligned}
EI_3(\beta) &\le \textstyle\int_a^b\int_a^s (s-y)^{-\lambda-1}\Big(\int_y^s (E(\overline{Y}^\beta_u)^2)^{1/2}u^{\alpha-1}du\Big)dyds \\
&\le c_2^{1/2}\textstyle\int_a^b\int_a^s (s-y)^{-\lambda-1}\Big(\int_y^s u^{\alpha-1/2}du\Big)dyds\cdot(1-\beta)^\alpha \qquad (1.13.10) \\
&\le c_3(H,\lambda,T)(1-\beta)^\alpha,
\end{aligned}
$$

where $c_3(H,\lambda,T) = c_2^{1/2}\frac{T^{H-\lambda+1}}{H(H-\lambda)(H-\lambda+1)}$. Now we use the representation (1.13.4) to estimate $I_1(\beta)$:

$$
\begin{aligned}
|\overline{Y}^\beta_s - \overline{Y}^\beta_y| &\le C_H^{(8)}\left|\int_{\beta s}^s (s-u)^\alpha u^{-\alpha}dB_u - \int_{\beta y}^y (s-u)^\alpha u^{-\alpha}dB_u\right| \\
&+ C_H^{(8)}\left(\frac{1-\beta}{\beta}\right)^\alpha |B_{\beta s} - B_{\beta y}|,
\end{aligned}
$$

therefore

$$
\begin{aligned}
I_1(\beta) &\le C_H^{(8)}\textstyle\int_a^b\int_a^s (s-y)^{-\lambda-1}s^\alpha \\
&\times\left|\textstyle\int_{\beta s}^s (s-u)^\alpha u^{-\alpha}dB_u - \int_{\beta y}^y (s-u)^\alpha u^{-\alpha}dB_u\right|dyds
\end{aligned}
\qquad (1.13.11)
$$

$$+C_H^{(8)}\left(\tfrac{1-\beta}{\beta}\right)^\alpha \int_a^b\int_a^s s^\alpha(s-y)^{-\lambda-1}|B_{\beta s}-B_{\beta y}|dyds$$

$$=: \mathcal{I}_1(\beta)+\mathcal{I}_2(\beta).$$

Further,

$$\begin{aligned} E\mathcal{I}_2(\beta) &\le C_H^{(8)}\left(\tfrac{1-\beta}{\beta}\right)^\alpha \int_a^b\int_a^s s^\alpha(s-y)^{-\lambda-1/2}dyds\beta^{1/2} \\ &= c_4(H,\lambda,T)(1-\beta)^\alpha, \end{aligned} \tag{1.13.12}$$

where $c_4(H,\lambda,T) = C_H^{(8)}2^\alpha \cdot \frac{T^{H-\lambda+1}}{1/2-\lambda}$.(Here we see that indeed λ must be less than 1/2.) Next, we decompose $\mathcal{I}_1(\beta)$ into two integrals

$$\mathcal{I}_1(\beta) = C_H^{(8)}\int_a^b\int_a^{(\beta s)\vee a} + C_H^{(8)}\int_a^b\int_{(\beta s)\vee a}^s =: \mathcal{I}_3(\beta)+\mathcal{I}_4(\beta).$$

$$\begin{aligned} E\mathcal{I}_3(\beta) &\le C_H^{(8)}\int_a^b\int_a^{(\beta s)\vee a}(s-y)^{-\lambda-1}s^\alpha \\ &\times\Big(E\Big(\int_{\beta s}^s(s-u)^\alpha u^{-\alpha}dB_u-\int_{\beta y}^y(y-u)^\alpha u^{-\alpha}dB_u\Big)^2\Big)^{1/2}dyds \\ &\le \sqrt{2}C_H^{(8)}\int_a^b\int_a^{(\beta s)\vee a}(s-y)^{-\lambda-1}s^\alpha \\ &\times\big(\int_{\beta s}^s(s-u)^{2\alpha}u^{-2\alpha}du+\int_{\beta y}^y(y-u)^{2\alpha}u^{-2\alpha}du\big)^{1/2}dyds \\ &\le 2^\alpha H^{-1/2}C_H^{(8)}\int_a^b\int_a^{(\beta s)\vee a}(s-y)^{-\lambda-1}(s+y)^{1/2}s^\alpha dyds\cdot(1-\beta)^H \\ &\le c(H,\lambda,T)(1-\beta)^{H-\lambda} \end{aligned} \tag{1.13.13}$$

with $c(H,\lambda,T) = \frac{2^HT^{1+H-\lambda}}{\lambda(1-\lambda)H^{1/2}}$. Finally,

$$\begin{aligned} E\mathcal{I}_4(\beta) &\le C_H^{(8)}\int_a^b\int_{(\beta s)\vee a}^s(s-y)^{-\lambda-1}s^\alpha\times\big(E\big|\int_0^s((s-u)^\alpha u^{-\alpha}\mathbb{1}_{(\beta s,s)}(u) \\ &-(y-u)^\alpha u^{-\alpha}\mathbb{1}_{(\beta y,y)}(u))dB_u\big|^2\big)^{1/2}dyds \end{aligned}$$

$$= C_H^{(8)} \int_a^b \int_{(\beta s)\vee a}^s (s-y)^{-\lambda-1} s^\alpha \Big(\int_0^s ((s-u)^\alpha \mathbb{1}_{(\beta s,s)}(u)$$

$$-(y-u)^\alpha \mathbb{1}_{(\beta y,y)}(u))^2 u^{-2\alpha} du \Big)^{1/2} dyds$$

The interior integral equals

$$\int_{\beta s}^y ((s-u)^\alpha - (y-u)^\alpha)^2 u^{-2\alpha} du + \int_y^s (s-u)^{2\alpha} u^{-2\alpha} du$$

$$+ \int_{\beta y}^{\beta s} (y-u)^{2\alpha} u^{-2\alpha} du =: \mathcal{I}_5(\beta),$$

and via some routine calculations can be estimated as

$$\mathcal{I}_5(\beta) \leq C_H (1-\beta)^{2\alpha}(s-y),$$

where $C_H = 1 + 2^{2\alpha} + \frac{\alpha}{1-2\alpha}$. Therefore

$$E\mathcal{I}_4(\beta) \leq C_H^{(8)} (C_H)^{1/2} (1-\beta)^\alpha \int_a^b s^\alpha \int_{(\beta s)\vee a}^s (s-y)^{-\lambda-1/2} dyds$$

$$\leq C_H^{(8)} (C_H)^{1/2} (1-\beta)^\alpha \int_a^b s^{H-\lambda} ds \int_\beta^1 (1-y)^{-\lambda-1/2} dy$$

$$\leq C(H,\lambda,T)(1-\beta)^{H-\lambda} \tag{1.13.14}$$

with $C(H,\lambda,T) = C_H^{(8)} (C_H)^{1/2} \frac{T^{H-\lambda+1}}{(H-\lambda+1)(1/2-\lambda)}$. Summarizing (1.13.9), and (1.13.10), (1.13.12) – (1.13.14), we obtain the proof. □

We obtain another approximation, considering the "truncated" process of the form

$$Y_t^\beta := C_H^{(8)} \alpha \int_0^t \Big(\int_0^{(s-\beta)_+} (s-u)^{\alpha-1} u^{-\alpha} dB_u \Big) ds$$

and

$$B_t^{H,\beta} = \int_0^t s^\alpha dY_s^\beta, \quad t \geq 0, \quad H \in (1/2, 1). \tag{1.13.15}$$

Evidently, we intend to obtain the approximation while $\beta \to 0$.

1.14 Hölder Properties of the Trajectories of fBm and of Wiener Integrals w.r.t. fBm

Let $\{\xi_t, t \in [0,T]\}$ be a separable modification of Gaussian process, $\rho_\xi^2(s,t) = E(\xi_s - \xi_t)^2, G = G(x) : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous increasing function, $G(0) = 0, D(T,\varepsilon) = \int_0^\varepsilon \mathcal{H}(T,u)^{1/2} du$ be the Dudley integral, $\rho(s,t)$ be some semi-metric in $[0,T]$.

Definition 1.14.1. *A function $\Theta = \Theta(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is called a modulus of continuity if $\Theta(0) = 0$ and for any $x_1, x_2 \geq 0$*

$$\Theta(x_1) \leq \Theta(x_1 + x_2) \leq \Theta(x_1) + \Theta(x_2).$$

Definition 1.14.2. *Let $g : [0,T] \to \mathbb{R}$ be some function. The function*

$$\Delta_\rho(g,\varepsilon) := \sup_{\substack{\rho(s,t)\leq\varepsilon \\ s,t\in[0,T]}} |g(s) - g(t)|$$

is called a modulus of uniform continuity of the function g with respect to the semi-metric ρ.

Definition 1.14.3. *A modulus $\Theta(\cdot)$ is called a uniform modulus of a Gaussian process ξ with respect to the semi-metric ρ if for a.a. $\omega \in \Omega$*

$$\limsup_{\varepsilon\to 0} \Delta_\rho(\xi(\omega),\varepsilon)/\Theta(\varepsilon) < \infty.$$

Theorem 1.14.1. *([18])* 1. *Let for any $s,t \in [0,T]$*

$$\rho_\xi(s,t) \leq G(\rho(s,t)) \cdot \tag{1.14.1}$$

Then the function $\Theta(\varepsilon) := D(T, G(\varepsilon))$ is a uniform modulus of the Gaussian process ξ with respect to the semi-metric ρ.

2. *Under assumption (1.14.1) with $\rho(s,t) = |s-t|$, the function*

$$\Theta(\varepsilon) = \int_0^\varepsilon |\log r|^{1/2} dG(r)$$

is a uniform modulus of the Gaussian process ξ with respect to ρ.

Definition 1.14.4. *We say that the function $f : [0,T] \to \mathbb{R}$ belongs to the space $C^{\beta-}[0,T]$ if $f \in C^\gamma[0,T]$ for any $\gamma < \beta$.*

Let $\xi_t = B_t^H$ be an fBm with Hurst index $H \in (0,1)$. Then, evidently, we can take $G(x) = x^H$, so from the second statement of previous theorem, the function $\Theta(\varepsilon) \sim \varepsilon^H |\log \varepsilon|^{1/2}$ will be a uniform modulus of B^H on any $[0,T]$. In particular, $|B_t^H - B_s^H| \le c(\omega)|t-s|^{H-\beta}$ for any $0 < \beta < H$, i.e. $B^H \in C^{H-}[0,T]$ for a.a. ω and any $T > 0$.

Now, let $\xi_t = I_t(f) = \int_0^t f(s) dB_s^H$ with $f \in L_2^H[0,t]$ for any $0 \le t \le T, H \in (1/2,1)$. We can take $\rho(s,t) = \int_s^t |f(u)|^{\frac{1}{H}} du, G(x) = C_H x^H$,

$$\Delta_\rho(I,\varepsilon) = \sup_{\substack{0 \le s < t \le T: \\ \int_s^t |f(u)|^{\frac{1}{H}} du < \varepsilon}} |\xi_t - \xi_s|,$$

$D(T, G(\varepsilon)) = \int_0^{C_H \varepsilon^H} \mathcal{H}(T,u)^{1/2} du$. Then, according to the first statement of Theorem 1.14.1 and from ([21], Theorem 1.10.3)

$$\limsup_{\varepsilon \to 0} \Delta_\rho(I,\varepsilon)/D(T, G(\varepsilon)) < \infty.$$

Now we simplify the situation supposing that f is essentially bounded on $[0,T]$, $f_T^* := ess \sup_{0 \le t \le T} |f(t)| < \infty$. Then we can take $\rho(s,t) = |s-t|, G(x) = C^H f_T^* \cdot x^H$,and $\Theta(\varepsilon) \sim C^H f_T^* \varepsilon^H |\log \varepsilon|^{1/2}$ will be a uniform modulus of $I(f)$ on $[0,T]$.

1.15 Estimates for Fractional Derivatives of fBm via the Garsia-Rodemich-Rumsey Inequality

Consider for any $T > 0$ the random variable that is the right-sided Riemann-Liouville fractional derivative of order β of fBm B^H, where $1 - H < \beta < 1/2$ and $H \in (1/2,1)$

$$G_t := \frac{1}{\Gamma(\beta)} \sup_{0 \le s < z \le t} |D_{z-}^{1-\beta} B_{z-}^H(s)|, \quad t \in [0,T].$$

Lemma 1.15.1. *For any $1 - H < \beta < 1/2$ and any $p > 0$*

$$EG_t^p < \infty.$$

Proof. By the Garsia-Rodemich-Rumsey inequality ([9]), for any $p \ge 1$ and $\rho > p^{-1}$ there exists a constant $C_{\rho,p} > 0$ such that for any continuous function f on $[0,T]$ and for all $s < z \le t \in [0,T]$

$$|f(z) - f(s)|^p \le C_{\rho,p}|z-s|^{\rho p-1}\int_0^z\int_0^z \frac{|f(x)-f(y)|^p}{|x-y|^{\rho p+1}}dxdy.$$

Choose $\varepsilon < \beta - (1-H)$ and put $\rho = H - \frac{\varepsilon}{2}, p = \frac{2}{\varepsilon}$ and $f(t) = B_t^H$.

$$|B_z^H - B_s^H| \le C_{H,\varepsilon}|z-s|^{H-\varepsilon}\xi_{t,\varepsilon},$$

where

$$\xi_{t,\varepsilon} = \left(\int_0^t\int_0^t \frac{|B_x^H - B_y^H|^{\frac{2}{\varepsilon}}}{|x-y|^{\frac{2H}{\varepsilon}}}dxdy\right)^{\frac{\varepsilon}{2}}, \ \ 0<\varepsilon<H. \tag{1.15.1}$$

Since $B_x^H - B_y^H$ is a Gaussian random variable, and $E|B_x^H - B_y^H|^2 = |x-y|^{2H}$, we have that for the random variable $\xi_{t,\varepsilon}$ for any $q > 1$

$$E|\xi_{t,\varepsilon}|^q = E\left(\int_0^t\int_0^t \frac{|B_x^H - B_y^H|^{\frac{2}{\varepsilon}}}{|x-y|^{\frac{2H}{\varepsilon}}}dxdy\right)^{q\frac{\varepsilon}{2}}$$

$$\le C_{q,H,T}\int_0^T\int_0^T \frac{E|B_x^H - B_y^H|^q}{|x-y|^{Hq}}dxdy \le C_{q,H,T}$$

which means that all moments of $\xi_{t,\varepsilon}$ are finite. Further, for $\varepsilon < \beta - (1-H)$

$$G_t \le C_\beta \sup_{0\le s<z\le t}\left(\frac{|B_z^H - B_s^H|}{|z-s|^{1-\beta}} + \int_s^z \frac{|B_s^H - B_y^H|}{|s-y|^{2-\beta}}dy\right)$$

$$\le C_{\beta,H,\varepsilon}\sup_{0\le s<t}(t-s)^{H-\varepsilon-1+\beta}\xi_{t,\varepsilon} \le C_{\beta,H,\varepsilon}\xi_{t,\varepsilon},$$

so, $EG_t^p < \infty$ for any $p > 0$. □

Remark 1.15.1. 1) *It is easy to see that the random process* $\{G_t, t\in[0,T]\}$ *is dominated, up to a constant, by* $\xi_{t,\varepsilon}$.
2) *Evidently, all moments of the random variable* G_T *are finite.*
3) *It follows immediately from Corollary* 1.9.3 *that the same conclusions hold for a Wiener integral w.r.t. fBm with a bounded integrand and* $H \in (1/2, 1)$.

1.16 Power Variations of fBm and of Wiener Integrals w.r.t. fBm

Consider for fBm $\{B_t^H, t \geq 0\}$ with $H \in (0,1)$ and for $p > 0$ the sums

$$S_{n,p}(t) = \sum_{j=1}^{2^n} |B^H_{\frac{jt}{2^n}} - B^H_{\frac{(j-1)t}{2^n}}|^p \cdot 2^{n(pH-1)}, \tag{1.16.1}$$

and

$$\widetilde{S}_{n,p}(t) = 2^{-n} \sum_{j=1}^{2^n} |B^H_{jt} - B^H_{(j-1)t}|^p.$$

Then $Law(S_{n,p}(t)) = Law(\widetilde{S}_{n,p}(t))$, due to the self-similarity property of

$$B^H : (Law(B^H_{ct}, t > 0) = Law(c^H B^H_t, t > 0)).$$

The sequence $(B^H_k - B^H_{k-1})_{k \in \mathbb{N}}$ is stationary. Therefore, from the ergodic theorem

$$\widetilde{S}_{n,p}(t) \to E|B^H_t|^p =: C_p t^{pH} \ \ as \ \ n \to \infty$$

with probability 1 and in $L_1(P)$, whence

$$S_{n,p}(t) \xrightarrow{d} C_p t^{pH}, n \to \infty, \tag{1.16.2}$$

so $S_{n,p}(t) \xrightarrow{P} C_p t^{pH}, n \to \infty$.

From (1.16.1)-(1.16.2)

$$\sum_{j=1}^{2^n} |B^H_{\frac{jt}{2^n}} - B^H_{\frac{(j-1)t}{2^n}}|^p \xrightarrow{P} \begin{cases} 0, & p > \frac{1}{H}, \\ +\infty, & p < \frac{1}{H}, \\ E|B^H_t|^{1/H}, & p = \frac{1}{H}. \end{cases} \tag{1.16.3}$$

Now, consider the interval $[0,1]$; let $\{\pi_k, k \geq 1\}$ be a sequence of refining partitions and $\Pi(\delta)$ be the set of all partitions π of $[0,1]$ with $|\pi| < \delta$. Evidently, from (1.16.3) we obtain that

$$\lim_{\delta \to 0} \sup_{\pi \in \Pi(\delta)} S(|x|^p, \pi, B^H) = +\infty$$

with probability 1, where $p < \frac{1}{H}$ and

$$S(\psi(x), \pi, X) := \sum_{t_j \in \pi} \psi(X_{t_j} - X_{t_{j-1}}).$$

Theorem 1.16.1. *Let $X_t, 0 \le t \le 1$ be a centered Gaussian process with continuous trajectories such that*

$$E|X_t - X_s|^2 \le \sigma^2(|t-s|),$$

where $\{\sigma(t), 0 \le t \le 1\}$ is a continuous function with $\sigma(0) = 0$. Let $\{\psi(t), 0 \le t \le 1\}$ be a non-decreasing regular varying function with exponent $\alpha > 0$ satisfying

$$\psi(\sigma(t)) = t\gamma(t) \quad for \quad 0 \le t \le 1 \qquad and \quad \lim_{t \downarrow 0} \gamma(t) = 0.$$

Then $\lim_{\delta \to 0} \sup_{\pi \in \Pi(\delta)} S(\psi(x), \pi, X) =$ constant (including ∞) holds with probability 1.

Put $X_t = B_t^H, \sigma^2(t) = t^{2\alpha+1}, \psi(t) = t^{\frac{1}{H}+\varepsilon}$ for some $\varepsilon > 0$ (recall that a function is regularly varying if $\frac{\psi(xt)}{\psi(t)} \to \rho(x)$ as $t \to \infty$ and in this case $\rho(x) = x^\beta$ for some $\beta \ge 0$). Then $\psi(\sigma(t)) = t^{1+H\varepsilon}$ and all the assumptions of Theorem 1.16.1 are satisfied. So, $\lim_{\delta \to 0} \sup_{p \to \Pi(\delta)} S(|x|^p, \pi, B^H) = const$ for any $p > \frac{1}{H}$. Evidently, this constant is zero since for any $p' > p > \frac{1}{H}$

$$S(x^{p'}, \pi, B^H) \le \sup_{0 \le t < t' \le t+\delta \le 1} |B_t^H - B_{t'}^H|^{p'-p} \cdot S(x^p, \pi, B^H),$$

and the first factor tends to zero a.s. as $\delta \to 0$.

Now, let $H \in (0, \frac{1}{2})$. In this case we can use the following theorem for the case $p = \frac{1}{H}$.

Theorem 1.16.2. *([15])*

1) *Let the following assumptions hold:*

(*a*) $E|X_s - X_t|^2 \le \sigma^2(|t-s|)$;

(*b*) $\sigma(t)$ *is a non-decreasing regular varying function;*

(*c*) *the function* $\sigma(t)\sqrt{2 \log \log \frac{1}{t}}$ *is strictly increasing near the origin.*

Let $\tilde{\Pi(k)}$ be the set of all partitions such that $\min |t_j - t_{j-1}| \ge \frac{1}{k}$. Then

$$\limsup_{k \to \infty} \sup_{\pi \in \tilde{\Pi}(k)} \frac{S(\sigma^{-1}(x), \pi, X)}{\Phi(\frac{1}{k})} \le 1,$$

with probability 1, *where*

$$\Phi(t) = \sup_{s \ge t} \frac{\sigma^{-1}\big(\sigma(s)\sqrt{2 \log \log \frac{1}{s}}\big)}{s}$$

2) *Let the assumption* (b) *hold and also*
$(d) E|X_s - X_t|^2 \le \sigma^2(|t-s|)$;
(e) $\sigma^2(t) - \sigma^2(t-h) \le C\sigma^2(h)$ *for some* $C > 0$, *any small* t *and* $0 \le h \le t$.
Then $\liminf_{k\to\infty} \sup_{\pi\in\tilde{\Pi}(k)} \frac{S(\sigma^{-1}(x),\pi,X)}{\Phi(\frac{1}{k})} \ge 1$, *with probability* 1.

Put $\sigma(t) = t^H, X_t = B_t^H$. Then conditions $(a), (b), (c)$ and (d) hold. Moreover, for $H \in (0, \frac{1}{2}), \sigma^2(t) - \sigma^2(t-h) = t^{2\alpha+1} - (t-h)^{2\alpha+1} \le h^{2\alpha+1}$ for all $0 \le h \le t \le 1$. The function $\Phi(t)$ now has the form $\Phi(t) = (2\log\log\frac{1}{t})^{\frac{1}{H}}$, whence $\lim_{k\to\infty} \sup_{\pi\in\tilde{\Pi}(k)} \frac{S(|x|^{\frac{1}{H}},\pi,B^H)}{(2\log\log k)^{\frac{1}{2H}}} = 1$ or, in other words,

$\lim_{k\to\infty} \sup_{\pi\in\tilde{\Pi}(k)} \frac{\sum_{t_j\in\pi} |B_{t_j}^H - B_{t_{j-1}}^H|^{\frac{1}{H}}}{(2\log\log k)^{\frac{1}{2H}}} = 1$

For $(H \in \frac{1}{2}, 1)$ we have no assumption (e), so, give only upper bounds. Namely, from the first statement of Theorem 1.16.2, we can deduce that

$$\lim_{k\to\infty} \sup_{\pi\in\tilde{\Pi}(k)} \frac{\sum_{t_j\in\pi} |B_{t_j}^H - B_{t_{j-1}}^H|^{\frac{1}{H}}}{(2\log\log k)^{\frac{1}{2H}}} \le 1$$

Moreover, the following result holds.

Theorem 1.16.3. *Under assumptions* $(a) - (c)$

$$\lim_{\delta\to 0} \sup_{\pi\in\Pi(\delta)} S(\psi(x), \pi, X) \le 1,$$

with probability 1, *where* $\psi(x)$ *is the inverse function to* $\sigma(t)\sqrt{2\log\log\frac{1}{t}}$ *near the origin.*

In our case it means that

$$\lim_{\delta\to 0} \sup_{\pi\in\Pi(\delta)} \sum_{t_j\in\pi} \psi(|B_{t_j}^H - B_{t_{j-1}}^H|) \le 1,$$

where $\psi(t)$ is the inverse function to $t^H\sqrt{2\log\log\frac{1}{t}}$.

Definition 1.16.1. *For any* $p > 0$ *define* $p-$*variation of the function* f *on the interval* $[a, b]$ *as*

$$v_p(f) = \sup_{\pi\in\Pi} S(|x|^p, \pi, f).$$

Also, let p-variation index of the function f be $v(f) := \inf(p : v_p(f) < \infty)$. The last relations mean that $v(B_H) = \frac{1}{H}$ with probability 1, and, moreover,

$$v_p(B^H) < \infty \quad \text{for} \quad p > \frac{1}{H} \qquad \text{and} = \infty \quad \text{for} \quad p < \frac{1}{H}.$$

Now consider the Gaussian process $X_t = I_t(f) = \int_0^t f(s)dB_s^H$. Let $H \in (\frac{1}{2}, 1)$ and the function f is essentially bounded on $[0, 1]$, let $f^* = ess\sup_{0\le t\le 1} |f(t)|$. Then, according to $([21], Theorem\ 1.10.3)$, $E|X_t - X_s|^2 \le \sigma^2(|t-s|)$, where $\sigma^2(t) = C_H(f^*)^2 t^{2\alpha+1}$, therefore from Theorem 1.16.1 $\lim_{\delta\to 0}\sup_{\pi\in\Pi(\delta)} S(|x|^p, \pi, I) = 0$ for any $p > \frac{1}{H}$ and from Theorems 1.16.2 and 1.16.3

$$\limsup_{k\to\infty} \sup_{\pi\in\widetilde{\Pi}(k)} \frac{S(|x|^{\frac{1}{H}}, \pi, I)}{\Phi(\frac{1}{k})} \le 1 \;\; P - a.s., \tag{1.16.4}$$

$$\lim_{\delta\to\infty} \sup_{\pi\in\widetilde{\Pi}(\delta)} S(\psi(x), \pi, I) \le 1 \;\; P - a.s. \tag{1.16.5}$$

where $\psi(x)$is the inverse to $C_H^{1/2} f^* t^H \sqrt{2\log\log\frac{1}{t}}$ near the origin.

Let $f_* := ess\inf_{0\le t\le 1} f(t) > 0$. Then

$$E|I_t - I_s|^2 = C_H \int_s^t \int_s^t f(u)f(v)|u-v|^{2\alpha-1}dudv \ge C_H f_*^2 |t-s|^{2\alpha+1},$$

whence $S(|x|^p, \pi, I) \xrightarrow{P} \infty$ as $|\pi| \to 0$ and $p < \frac{1}{H}$, and together with Theorem 1.16.1 it means that

$$\lim_{\delta\to 0} \sup_{\pi\in\Pi(\delta)} S(|x|^p, \pi, I) = \infty \qquad P - a.s., \qquad p < \frac{1}{H}.$$

For $H \in (0, \frac{1}{2})$ and f with $f_* > 0$ we can immediately conclude from Theorem 1.9.1 that

$$E|I_t - I_s|^2 \ge C_H \|f\|^2_{L_{\frac{1}{H}}[s,t]} \ge C_H f_*^2 |t-s|^{2\alpha+1},$$

whence $S(|x|^p, \pi, I) \xrightarrow{P} \infty$ as $|\pi| \to 0$ and $p < \frac{1}{H}$. Let $f \in C^\beta[0, 1]$. Then we can deduce from $([21], Remark\ 1.10.7)$, that

$$E|I_t - I_s|^2 \le C_H \|f\|_{C^\beta([0,1])}((t-s)^{2\alpha+1} + (t-s)^{2H+2\beta}),$$

whence (1.16.4)-(1.16.5) follow for $H \in (0, \frac{1}{2})$.

Chapter 2

Stochastic Integration with Respect to fBm and Related Topics

The aim of this chapter is to provide a comprehensive overview of stochastic calculus with respect to fractional Brownian motion. For further details concerning the theory of stochastic integration with respect to fractional Brownian motion, we refer to [12], [16], [21], [23].

2.1 Pathwise Stochastic Integration

2.1.1 Pathwise Stochastic Integration in the Fractional Sobolev-type Spaces

In this subsection we consider pathwise integrals $\int_0^T f(t)dB_t^H$ for processes f from the fractional Sobolev type spaces $I_{a+}^{\alpha}(L^p)$ for some $p > 1$. This approach was developed by Zähle [30], [31].

Consider two deterministic functions $f, g : [a, b] \to \mathbb{R}$ such that the limits $f(u+) = \lim_{\delta\to 0} f(u + \delta)$ and $g(u-) = \lim_{\delta\to 0} g(u - \delta), a \le u \le b$, exist. Put

$f_{a+}(x) = (f(x) - f(a+))\mathbb{1}_{(a,b)}(x)$, $g_{b-}(x) = (g(b-) - g(x))\mathbb{1}_{(a,b)}(x)$. Suppose also that $f_{a+} \in I_{a+}^{\alpha}(L_p[a, b])$, and $g_{b-} \in I_{b-}^{1-\alpha}(L_p[a, b])$ for some $p, q \ge 1, 1/p + 1/q \le 1$, and $0 \le \alpha \le 1$.

Definition 2.1.1. *The generalized fractional Lebesgue-Stieltjes integral is defined as*

$$\int_a^b f(x)dg(x) := \int_a^b (D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx + f(a+)(g(b-) - g(a+)).$$

Remark 2.1.1. *The definition of generalized Lebesgue-Stieltjes integral does not depend on the possible choice of* α.

Let $\alpha p < 1$. Then $f_{a+} \in I^{\alpha}_{a+}(L_p[a,b])$ if and only if $f \in I^{\alpha}_{a+}(L_p[a,b])$ and in this case we can simplify the formula for the generalized integral:

$$\int_a^b f(x)dg(x) = \int_a^b \big((D^{\alpha}_{a+}f)(x) - \tfrac{1}{\Gamma(1-\alpha)} \cdot \tfrac{f(a+)}{(x-a)^{\alpha}}\big)(D^{1-\alpha}_{b-}g_{b-})(x)dx \tag{2.1.1}$$

$$+f(a+)(g(b-)-g(a+)) = \int_a^b (D^{\alpha}_{a+}f)(x)(D^{1-\alpha}_{b-}g_{b-})(x)dx$$

$$-f(a+)I^{1-\alpha}_{b-}(D^{1-\alpha}_{b-}g)(a) + f(a+)(g(b-)-g(a+))$$

$$= \int_a^b (D^{\alpha}_{a+}f)(x)(D^{1-\alpha}_{b-}g_{b-})(x)dx$$

Lemma 2.1.1. *Let* $g_{b-} \in I^{1-\alpha}_{b-}(L_q[a,b]) \cap C[a,b]$ *for some* $q > \frac{1}{1-\alpha}$ *and* $0 < \alpha < 1$. *Then for any* $a < c < d < b$

$$\int_a^b (D^{\alpha}_{a+}\mathbb{1}_{[c,d)})(x)(D^{1-\alpha}_{b-}g_{b-})(x)dx = g(d) - g(c). \tag{2.1.2}$$

Proof. We have that

$$(D^{\alpha}_{a+}\mathbb{1}_{[c,d)})(x) = \begin{cases} 0, & x \le c, \\ \frac{(x-c)^{-\alpha}}{\Gamma(1-\alpha)}, & c < x \le d, \\ \frac{(x-c)^{-\alpha}-(x-d)^{-\alpha}}{\Gamma(1-\alpha)}, & d \le x \le b. \end{cases}$$

Therefore, by using (2.1.1), we obtain for $\alpha p < 1$, or $q > \frac{1}{1-\alpha}$, that

$$\int_a^b (D^{\alpha}_{a+}\mathbb{1}_{[c,d)})(x)(D^{1-\alpha}_{b-}g_{b-})(x)dx = \tfrac{1}{\Gamma(1-\alpha)} \int_c^b (x-c)^{-\alpha}(D^{1-\alpha}_{b-}g_{b-})(x)dx$$

$$-\tfrac{1}{\Gamma(1-\alpha)} \int_d^b (x-d)^{-\alpha}(D^{1-\alpha}_{b-}g_{b-})(x)dx = I^{1-\alpha}_{b-}(D^{1-\alpha}_{b-}g_{b-})(c)$$

$$-I^{1-\alpha}_{b-}(D^{1-\alpha}_{b-}g_{b-})(d) = g(d) - g(c).$$

□

Corollary 2.1.1. *For any step function* $f_{\pi}(x) = \sum_{k=0}^{n-1} c_k \mathbb{1}_{[x_k,x_{k+1})}(x)$ *with* $a = x_0 < \ldots < x_n = b$ *and* g *satisfying the conditions of Lemma* 2.1.1, *we have that*

$$\int_a^b f(x)dg(x) = \sum_{k=0}^{n-1} c_k(g(x_{k+1}) - g(x_k)).$$

Denote by $BV[a,b]$ the class of functions of bounded variation on $[a,b]$, and suppose that $g(b-) = g(b)$ and $g(a+) = g(a)$.

Lemma 2.1.2. *Let the functions* $f_{a+} \in I^{\alpha}_{a+}(L_p[a,b])$, $g_{b-} \in I^{1-\alpha}_{b-}(L_q[a,b]) \cap BV[a,b]$ *with* $p \geq 1, q \geq 1, 1/p + 1/q \leq 1$ *and*

$$\int_a^b I^{\alpha}_{a+}(|(D^{\alpha}_{a+}f)|)(x)|g|(dx) < \infty. \tag{2.1.3}$$

Then

$$\int_a^b f(x)dg(x) = (L-S)\int_a^b f(x)dg(x)$$

Proof. We have that

$$(L-S)\int_a^b f(x)dg(x) = (L-S)\int_a^b I^{\alpha}_{a+}(D^{\alpha}_{a+}f)(x)dg(x) \tag{2.1.4}$$

$$= \frac{1}{\Gamma(1-\alpha)}(L-S)\int_a^b(\int_a^x (x-y)^{\alpha-1}(D^{\alpha}_{a+}f)(y)dy)dg(x).$$

Condition (2.1.3) together with Fubini theorem permits us to change the order of integration:

$$(L-S)\int_a^b(\int_a^x (x-y)^{\alpha-1}(D^{\alpha}_{a+}f)(y)dy)dg(x)$$

$$= \int_a^b (D^{\alpha}_{a+}f)(y)(\int_y^b (x-y)^{\alpha-1}dg(x))dy \tag{2.1.5}$$

$$= (\alpha-1)\int_a^b (D^{\alpha}_{a+}f)(y)(\int_y^b(\int_x^{\infty}(z-y)^{\alpha-2}dz)dg(x))dy.$$

Further, if $y \in (a,b)$ is the point of continuity of function g, then

$$\int_y^b(\int_x^{\infty}(z-y)^{\alpha-2}dz)dg(x) = \int_y^b(\int_y^z dg(x))(z-y)^{\alpha-2}dz$$

$$+\int_b^{\infty}(\int_y^b dg(x))(z-y)^{\alpha-2}dz = \int_y^b \frac{g(z)-g(y)}{(z-y)^{2-\alpha}}dz \tag{2.1.6}$$

$$+\frac{g(b)-g(y)}{(\alpha-1)(b-y)^{\alpha-1}} = \frac{\Gamma(\alpha)}{\alpha-1}(D^{1-\alpha}_{b-}g_{b-})(y).$$

Taking (2.1.4)−(2.1.6) together, we obtain the proof. □

Now we consider the case of Hölder functions f and g. The existence of $(R-S)\int_a^b f dg$ for $f \in C^{\lambda}[a,b]$, $g \in C^{\mu}[a,b]$ with $\lambda + \mu > 1$ was established by Kondurar [16].

Let $f \in C^{\lambda}[a,b]$ for some $0 < \lambda \leq 1$ and $|f(x)-f(y)| \leq c(\lambda)|x-y|^{\lambda}, x, y \in [a,b]$. Consider the following step function:

$$f_{\pi}(x) = \sum_{k=0}^{n-1} f(x_k)\mathbb{1}_{[x_k,x_{k+1})}(x)$$

where the partition $\pi = \{a = x_0 < x_1 < \ldots < x_n = b\}$.

Evidently, $\lim_{|\pi|\to 0} \sup_\pi \|f_\pi - f\|_{L_\infty[a,b]} = 0$.

Theorem 2.1.1. 1) *For any* $0 < \alpha < \lambda$

$$\lim_{|\pi|\to 0} \sup_\pi \|(D^\alpha_{a+} f_\pi) - (D^\alpha_{a+} f)\|_{L_1[a,b]} = 0.$$

2) *Let* $f \in C^\lambda([a,b]), g \in C^\mu[a,b]$ *with* $\lambda + \mu > 1$, *then* $(R-S)\int_a^b f dg$ *exists and*

$$\int_a^b f dg = (R-S)\int_a^b f dg.$$

Proof.

1) It is sufficient to prove that $\int_a^b \frac{|f_\pi(x) - f(x)|}{(x-a)^\alpha} dx \to 0$ and $\int_a^b \int_a^x (x-y)^{-\alpha-1} |f_\pi(x) - f(x) - f_\pi(y) + f(y)| dy dx \to 0$ as $|\pi| \to 0$. But $|f_\pi(x) - f(x)| \le |f(x_k) - f(x)| \le c(\lambda)|\pi|^\lambda$ for $x \in [x_k, x_{k+1})$.

$$\begin{array}{rcl} A(x) & = & \int_a^x (x-y)^{-\alpha-1} |f_\pi(x) - f(x) - f_\pi(y) + f(y)| dy \\ & \le & 2c(\lambda)|\pi|^\lambda \frac{(x-x_k)^{-\alpha}}{1-\alpha} + c(\lambda)\frac{(x-x_k)^{\lambda-\alpha}}{\lambda-\alpha} \\ & \le & 3c(\lambda)\frac{|\pi|^{\lambda-\alpha}}{\lambda-\alpha} \end{array}$$

which means that $\int_a^b A(x)dx \to 0$ as $|\pi| \to 0$.

2) We take $1 - \mu < \alpha < \lambda$, then the fractional derivatives $D^\alpha_{a+} f(x)$ and $(D^{1-\alpha}_{b-} g)_{b-}(x)$ exist, and, moreover,

$$\begin{array}{rcl} |(D^{1-\alpha}_{b-} g)_{b-}(x)| & \le & \frac{1}{\Gamma(1-\alpha)}\left(\frac{|g(b)-g(x)|}{(b-x)^{1-\alpha}} + (1-\alpha)\int_x^b \frac{|g(y)-g(x)|}{(y-x)^{2-\alpha}} dy\right) \\ & \le & \frac{1}{\Gamma(1-\alpha)} \cdot c(\lambda)(b-x)^{\mu+\alpha-1}\left(1 + \frac{1-\alpha}{\mu+\alpha-1}\right) \le C \end{array}$$

for some constant C. Therefore, according to part 1) of the proof,

$$\begin{aligned} \left|\int_a^b f_\pi dg - \int_a^b f dg\right| &\le \int_a^b |(D^\alpha_{a+} f_\pi)(x) - (D^\alpha_{a+} f)(x)||(D^{1-\alpha}_{b-} g)_{b-}(x)| dx \\ &\le C\int_a^b |(D^\alpha_{a+} f_\pi)(x) - (D^\alpha_{a+} f)(x)| dx \to 0, \end{aligned} \tag{2.1.7}$$

as $|\pi| \to 0$. Furthermore, according to Corollary 2.1.1,

$$\int_a^b f_\pi dg = \sum_{k=0}^{n-1} f(x_k)(g(x_{k+1}) - g(x_k)) \to (R-S)\int_a^b f dg, \tag{2.1.8}$$

and from (2.1.7) – (2.1.8) we obtain the desired equality. □

Proposition 2.1.1. *Some elementary properties of generalized Lebesgue-Stieltjes integrals are:*

(i) $\int_a^b \mathbb{1}_{(s,t)} f dg = \int_s^t f dg$, *if both integrals exist in the sense of generalized Lebesgue-Stieltjes integrals.*

(ii) $\int_s^t f dg + \int_t^u f dg = \int_s^u f dg$ *for* $a \leq s < t < u \leq b$, *if all the integrals exist as generalized Lebesgue-Stieltjes integrals.*

2.2 Wick Integration with Respect to fBm with $H \in [1/2, 1)$ as S^*-integration

2.2.1 Wick Products and S^*-integration

Recall (Sections 1.4 - 1.5), that the random variable F on the probability space $S'(\mathbb{R})$ belongs to S^* if F admits the formal expansion (1.5.1) with finite negative norm

$$\|F\|_{-q}^2 = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$$

for at least one $q \in \mathbb{N}$. Introduce the following notations:

(i) Let the function $Z : \mathbb{R} \to S^*$, and for any $F \in S$ we have that $\langle\langle Z(t), F \rangle\rangle \in L_1(\mathbb{R})$ as a function of $t \in \mathbb{R}$.

(ii) In this case, define $\int_{\mathbb{R}} Z(t)dt$ as the unique element of S^* such that

$$\left\langle\!\!\left\langle \int_{\mathbb{R}} Z(t)dt, F \right\rangle\!\!\right\rangle = \int_{\mathbb{R}} \langle\langle Z(t), F \rangle\rangle dt,$$

and say that Z is integrable in S^*.

Definition 2.2.1. *The Wick products of two fractional stockastic fonction* $F(\omega) = \sum_\alpha c_\alpha \mathcal{H}_\alpha(w), G(\omega) = \sum_\beta d_\beta \mathcal{H}_\beta(w)$, *is defined as*

$$(F \Diamond G)(\omega) = \sum_{\alpha,\beta} c_\alpha d_\beta \mathcal{H}_{\alpha+\beta}(w).$$

According to the ([12]), the wick product is commutative associative and distributive with respect to addition

Theorem 2.2.1. *Let the process* $Y(t) \in S^*$ *and admit an expansion* $Y(t) = \sum_\alpha c_\alpha(t) \mathcal{H}_\alpha(\omega), t \in \mathbb{R}$, *with the coeficients, satisfying the inequality*

$$K = \sup_\alpha \{\alpha! \|c_\alpha\|_{L_1(\mathbb{R})}^2 (2\mathbb{N})^{-q\alpha}\} < \infty$$

for some $q > 0$.
Then the Wick product $Y(t)\Diamond \dot{B}_t^M$ is S^*-integrable, and, moreover,

$$\int_{\mathbb{R}} Y(t)\Diamond \dot{B}_t^M dt = \sum_{\alpha,k} \int_{\mathbb{R}} c_\alpha(t) M_+ \tilde{h}_k(t) dt \cdot \mathcal{H}_{\alpha+\varepsilon_k}(\omega). \tag{2.2.1}$$

Proof. Consider only $\dot{B}_t^H$ and for $\dot{B}_t^M$ the proof is the same. Since $\langle \tilde{h}_k, \omega\rangle = \mathcal{H}_{\varepsilon_k}(\omega)$, we have that the Wick product $Y(t) \Diamond \dot{B}_t^H \in S^*$ and equals $\sum_{\alpha,k} c_\alpha(t) M_+^H \tilde{h}_k(t)\mathcal{H}_{\alpha+\varepsilon_k}(\omega)$. According to ([12], Lemmas 2.5.6 and 2.5.7), the S^*-integrability of $Y(t)\Diamond \dot{B}_t^H$ follows from the inequality

$$\sum_{\beta\in\mathcal{J}} \beta! \Big\| \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} c_\alpha(t) M_+^H \tilde{h}_k(t) \Big\|^2_{L_1(\mathbb{R})} (2\mathbb{N})^{-p\beta} < \infty$$

for some $p > 0$. According to ([21], lemma 1.5.2)

$$\int_{\mathbb{R}} |c_\alpha(t) M_+^H \tilde{h}_k(t)| dt \le C k^{5/12} \|c_\alpha\|_{L_1(\mathbb{R})}$$

for $k \ge 1$, $C > 0$, and

$$\Big\| \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} c_\alpha(t) M_+^H \tilde{h}_k(t) \Big\|^2_{L_1(\mathbb{R})} \le C \Big(\sum_{\alpha,k:\alpha+\varepsilon_k=\beta} k^{5/12} \|c_\alpha\|_{L_1(\mathbb{R})} \Big)^2 .$$

Consider

$$\begin{aligned} S &= \sum_{\beta\in\mathcal{J}} \beta! \Big(\sum_{\alpha,k:\alpha+\varepsilon_k=\beta} k^{5/12} \|c_\alpha\|_{L_1(\mathbb{R})} \Big)^2 (2\mathbb{N})^{-p\beta} \\ &\le \sum_{\beta\in\mathcal{J}} \beta! (l(\beta))^{5/6} \Big(\sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \|c_\alpha\|_{L_1(\mathbb{R})} \Big)^2 (2\mathbb{N})^{-p\beta}, \end{aligned}$$

where $l(\beta)$ is the length of the index β. Further, for any α, β there exists k, such that $\alpha + \varepsilon_k = \beta$. Therefore,

$$\Big(\sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \|c_\alpha\|_{L_1(\mathbb{R})} \Big)^2 \le l^2(\beta) \sum_{\alpha,k:\alpha+\varepsilon_k=\beta} \|c_\alpha\|^2_{L_1(\mathbb{R})}.$$

It means that

$$S \le \sum_{\alpha,k} (\alpha+\varepsilon_k)! (l(\alpha+\varepsilon_k))^{17/6} \|c_\alpha\|^2_{L_1(\mathbb{R})} (2\mathbb{N})^{-p\alpha-p\varepsilon_k}$$

$$\leq K \sum_{\alpha,k} \frac{(\alpha+\varepsilon_k)!}{\alpha!} (l(\alpha+\varepsilon_k))^3 (2\mathbb{N})^{-(p-q)\alpha - p\varepsilon_k}$$

$$\leq K \sum_{\alpha,k} (|\alpha|+1)^4 2^{-|\alpha|(p-q)} k^{-p} < \infty,$$

for $p > q+1$, we have established the S^*-integrability of $Y(t)\Diamond \dot{B}_t^H$. Now, for any $F = \sum_{\beta,k} d_{\beta,k} \mathcal{H}_{\beta+\varepsilon_k}(\omega) \in S$, we have from the definition of the S^*-integral and of Wick product, that

$$\left\langle\!\!\left\langle \int_{\mathbb{R}} Y(t)\Diamond \dot{B}_t^H dt, F \right\rangle\!\!\right\rangle = \int_{\mathbb{R}} \left\langle\!\!\left\langle \sum_{\alpha,k} c_\alpha(t) M_+^H \tilde{h}_k(t) \mathcal{H}_{\alpha+\varepsilon_k}(\omega), F \right\rangle\!\!\right\rangle dt$$

$$= \int_{\mathbb{R}} \sum_{\alpha,k} (\alpha+\varepsilon_k)! c_\alpha(t) d_{\alpha,k} M_+^H \tilde{h}_k(t)(\omega) dt. \qquad (2.2.2)$$

Note that

$$\sum_{\alpha,k} (\alpha+\varepsilon_k)! |d_{\alpha,k}| (2\mathbb{N})^{2q(\alpha+p\varepsilon_k)} = C_q < \infty$$

for any $q \in \mathbb{N}$. Therefore

$$\sum_{\alpha,k} \int_{\mathbb{R}} (\alpha+\varepsilon_k)! |c_\alpha(t)| |d_{\alpha,k}| |M_+^H \tilde{h}_k(t)| dt \leq \sum_{\alpha,k} \int_{\mathbb{R}} (\alpha+\varepsilon_k)! |d_{\alpha,k}| k^{5/6} \|c_\alpha\|_{L_1(\mathbb{R})}$$

$$\leq \left(C_q K \sum_{\alpha,k} k^{5/6} \frac{\beta_k!}{\alpha!} (2\mathbb{N})^{-q|\alpha|} k^{-2q} \right)^{1/2} < \infty$$

for $q > 11/12, \beta_k = \alpha + \varepsilon_k$, because $\sum_\alpha \frac{\beta_k!}{\alpha!} (2N)^{-q|\alpha|} \leq \sum_\alpha (|\alpha|+1) 2^{-q|\alpha|} < \infty$. So, we can change the signs of sum and integral in (2.2.2) and obtain

$$\begin{aligned} \left\langle\!\!\left\langle \textstyle\int_{\mathbb{R}} Y(t)\Diamond \dot{B}_t^H dt, F \right\rangle\!\!\right\rangle &= \textstyle\sum_{\alpha,k} (\alpha+\varepsilon_k)! d_{\alpha,k} + \int_{\mathbb{R}} c_\alpha(t) M_+^H \tilde{h}_k(t)(\omega) dt \\ &= \left\langle\!\!\left\langle \textstyle\sum_{\alpha,k} \int_{\mathbb{R}} c_\alpha(t) M_+^H \tilde{h}_k(t)(\omega) dt, F \right\rangle\!\!\right\rangle \end{aligned}$$

whence (2.2.1) follows. □

Corollary 2.2.1. *Let $Y(t) = \sum_\alpha c_\alpha(t) \mathcal{H}_\alpha(\omega) \in S^*$ be a process such that $\int_0^T EY^2(t) dt < \infty$ for some $T > 0$. Then $\sum_\alpha \alpha! \int_0^T c_\alpha^2(t) dt = \int_0^T EY^2(t) dt < \infty$ whence $K = \sup_\alpha \{\alpha! \|\bar{c}_\alpha\|^2_{L_1(\mathbb{R})} (2\mathbb{N})^{-q\alpha}\} < \infty$ for any $q > 0$, (hereafter we put $\bar{c}_\alpha = c_\alpha(t) \mathbb{1}_{[0,T]}(t)$).*

2.2.2 Comparison of Wick and Pathwise Integrals for "Markov" Integrands

In this subsection we consider the probability space $(\Omega, \mathcal{F}, P)$, the coordinate process $B : \Omega \to \mathbb{R}$ defined as,

$$B_t(\omega) = \omega(t), \quad \omega \in \Omega$$

is the Wiener process.

(i) Recall the notion of a stochastic derivative. Let F be a square-integrable random variable, and suppose that

$$\lim_{\beta \to 0} \frac{1}{\beta} \left(F(\omega. + \beta \int_0^{\cdot} h(s)ds) - F(\omega.) \right) \quad \text{exists in } L_2(P)$$

for any $h \in L_2(\mathbb{R})$. Then this limit is called the directional derivative $D_h F$.

(ii) If the directional derivative $D_h F, h \in L_2(\mathbb{R})$, is absolutely continuous w.r.t. the measure $h(x)dx$, i.e.

$$D_h F = \int_{\mathbb{R}} \frac{dD_h F}{dh}(x) \cdot h(x) dx,$$

and $(dD_h(F))/(dh)$ does not depend on h, then this derivative is called the stochastic derivative of F and is denoted by DxF.

(iii) Recall the notion of the class $\mathbb{D}_{1,2}$, obtained as a completion of the set $\mathcal{P}_0$ of smooth functionals $F = f(B_{t_1}, \ldots B_{t_i})$, w.r.t. the norm $\|F\|_{1,2} = \|F\|_{L_2(P)} + \|D_x \|F\|_{HS}\|_{L_1(P)}$, where $F \in \mathcal{P}_0$, and $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt norm.

Denote $L_2^M(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} |M_- f(x)|^2 dx < \infty\}$.

Lemma 2.2.1. *Let $F \in \mathbb{D}_{1,2}, f \in L_2^M(\mathbb{R})$. Suppose that the integrals*

$$\int_{\mathbb{R}} (M_- f)(s) \cdot D_s F ds \quad \text{and} \quad F \cdot \int_{\mathbb{R}} (M_- f)(s) dB_s = F \cdot \int_{\mathbb{R}} f(s) dB_s^M$$

belong to $L_2(P)$. Then $F \Diamond \int_{\mathbb{R}} f(s) dB_s^M$ exists and

$$F \Diamond \int_{\mathbb{R}} f(s) dB_s^M = \int_{\mathbb{R}} (F \cdot M_- f)(s) \delta B_s$$

$$= F \cdot \int_{\mathbb{R}} f(s) \; dB_s^M - \int_{\mathbb{R}} (M_- f)(s) \cdot D_s F ds \tag{2.2.3}$$

Proof. By using ([12], Corollary 2.5.12) and ([23], Theorem 3.2), we obtain for nonrandom f that

$$F\diamondsuit \int_{\mathbb{R}} f(s)dB_s^M = F\diamondsuit \int_{\mathbb{R}} (M_- f)(s)dB_s$$

$$= \int_{\mathbb{R}} (F\diamondsuit M_- f)(s)\delta B_s = \int_{\mathbb{R}} (F\cdot M_- f)(s)\delta B_s$$

$$= F\cdot \int_{\mathbb{R}} (M_- f)(s)\delta B_s - \int_{\mathbb{R}} (M_- f)(s)\cdot D_s F ds$$

$$= F\cdot \int_{\mathbb{R}} f(s)dB_s^M - \int_{\mathbb{R}} (M_- f)(s)\cdot D_s F ds.$$

Note that according to ([23], Theorem 3.2), the Skorohod integral $\int_{\mathbb{R}} F\cdot (M_- f)(s)\delta B_s$ exists if and only if the difference $F\cdot \int_{\mathbb{R}}(M_- f)(s)dB_s - \int_{\mathbb{R}}(M_- f)(s)\cdot D_s F ds$ belongs to $L_2(P)$. □

Lemma 2.2.2. *Let $\varphi \in C^1(\mathbb{R}), F_t = \varphi(B_t^H), f(s) = \mathbb{1}_{[t,t+h]}(s), t, h > 0$. If $\varphi'(B_t^H)$ and $F_t\cdot(B_{t+h}^H - B_t^H)$ belong to $L_2(P)$, then*

$$F_t\diamondsuit(B_{t+h}^H - B_t^H) = F\cdot(B_{t+h}^H - B_t^H)$$
$$-H\varphi'(B_t^H)t^{2\alpha}h + c(\omega)(t^{2\alpha-1}h^2 + h^{2H}),$$

where $c(\omega)$ is a.s. finite and independent of t and h.

Proof. According to equation (2.2.3), we can rewrite formally the left-hand side of the previous equality:

$$F_t\diamondsuit(B_{t+h}^H - B_t^H) = F_t\cdot(B_{t+h}^H - B_t^H)$$
$$- \int_{\mathbb{R}} (M_-^H \mathbb{1}_{[t,t+h]})(s) D_s\varphi(B_t^H)ds. \tag{2.2.4}$$

Further, according to ([21], lemma 2.3.5), it holds that

$$D_s\varphi(B_t^H) = \varphi'(B_t^H)D_s B_t^H,$$

and

$$D_s B_t^H = D_s \int_{\mathbb{R}} (M_-^H \mathbb{1}_{[0,t]})(u)dB_u = (M_-^H \mathbb{1}_{[0,t]})(s).$$

Therefore,

$$F_t \lozenge (B^H_{t+h} - B^H_t) = F_t \cdot (B^H_{t+h} - B^H_t)$$

$$-\varphi'(B^H_t) \int_{\mathbb{R}} (M^H_- \mathbb{1}_{[t,t+h]})(s) = (M^H_- \mathbb{1}_{[0,t]})(s) ds,$$

and under the conditions of the lemma the right-hand side of equation (2.2.4) is well-defined. Finally,

$$\int_{\mathbb{R}} (M^H_- \mathbb{1}_{[t,t+h]})(s)(M^H_- \mathbb{1}_{[0,t]})(s) ds = E(B^H_{t+h} - B^H_t) B^H_t$$

$$= \frac{1}{2}((t+h)^{2H} - t^{2H} - h^{2H}) = Ht^{2\alpha} h + 2H\alpha\theta^{2\alpha-1} h^2 - h^{2H}$$

where $\theta \in (t, t+h)$. The lemma is proved. $\square$

Now,fix some $T > 0$ and consider the sequence

$\pi_n = \{0 = t^n_0 < \ldots < t^n_n = T\}$ of partitions of $[0, T]$, such that $\pi_n \subset \pi_{n+1}$ and $|\pi_n| \to 0$ as $n \to \infty$. Suppose that

$$\varphi'(B^H_t) \in L_2(P), \ \ \varphi(B^H_t) \in L_{2+\varepsilon}(P), \ \ t \in [0, T] \tag{2.2.5}$$

for some $\varepsilon > 0$. According to Lemma 2.2.2, we can write

$$\sum_{i=1}^{n} \varphi(B^H_{t^n_{i-1}}) \lozenge \Delta B^H_{i,n} = \sum_{i=1}^{n} \varphi(B^H_{t^n_{i-1}}) \Delta B^H_{i,n}$$

$$-H \sum_{i=1}^{n} \varphi'(B^H_{t^n_{i-1}})(t^n_{i-1})^{2\alpha} \Delta t_{i,n} + R_n(T),$$

where $\Delta t_{i,n} = t^n_i - t^n_{i-1}$, $\Delta B^H_{i,n} = B^H_{t^n_i} - B^H_{t^n_{i-1}}$. Here $R_n(T)$, is a remainder term and $R_n(T) \to 0$ $a.s.$ as $n \to \infty$. Furthermore, the process $C_t := \varphi(B^H_t)$ is Hölder continuous up to order H. Also, by Theorem 2.1.1, part 2), the sum $\sum_{i=1}^{n} \varphi(B^H_{t^n_{i-1}}) \Delta B^H_{i,n}$ converges a.s. as $n \to \infty$ to the pathwise integral $\int_0^T \varphi(B^H_s) dB^H_s$. Clearly,

$$\sum_{i=1}^{n} \varphi'(B^H_{t^n_{i-1}})(t^n_{i-1})^{2\alpha} \Delta t_{i,n} \to \int_0^T \varphi'(B^H_s) s^{2\alpha} ds \quad a.s.$$

Therefore,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\varphi(B^H_{t^n_{i-1}})\Diamond\Delta B^H_{i,n}=\int_0^T\varphi(B^H_s)dB^H_s-H\int_0^T\varphi'(B^H_s)s^{2\alpha}ds\quad a.s.$$

Moreover, under assumption (2.2.5) and

$$E\int_0^T(\varphi(B^H_s))^2ds<\infty \tag{2.2.6}$$

there exists the Wick integral $\int_0^T\varphi(B^H_s)\Diamond dB^H_s$. And from ([21], Theoreme 2.3.7)

$$\int_0^T\varphi(B^H_s)\Diamond dB^H_s=\lim_{n\to\infty}\sum_{i=1}^{n}\varphi(B^H_{t^n_{i-1}})\Diamond\Delta B^H_{i,n}. \tag{2.2.7}$$

Theorem 2.2.2. *Under conditions* (2.2.5) *and*

$$E\sup_{s\le T}(\varphi(B^H_s))^2+E\sup_{s\le T}(\varphi'(B^H_s))^2<\infty \tag{2.2.8}$$

equality (2.2.6) *and* (2.2.7), *consequently, the equality*

$$\int_0^T\varphi(B^H_s)\Diamond dB^H_s=\int_0^T\varphi(B^H_s)dB^H_s-H\int_0^T\varphi'(B^H_s)s^{2\alpha}ds$$

holds a.s.

Proof. We invit the reader to commet ([21], p.149) for the proof of this theorem.

2.2.3 Reduction of Wick Integration w.r.t. Fractional Noise to the Integration w.r.t. White Noise

Recall that for nonrandom integrands $f\in L^H_2(\mathbb{R})$

$$\int_{\mathbb{R}}f(t)dB^H_t=\int_{\mathbb{R}}(M^H_-f)(t)dB_t.$$

In this subsection we reduce $\int_{\mathbb{R}}X_t\Diamond\dot{B}^H_tdt$ to the corresponding integral $\int_{\mathbb{R}}(M^H_-f)(t)\Diamond\dot{B}_tdt$ w.r.t. white noise.

Theorem 2.2.3. *Let the following conditions hold:*

$$E\int_{\mathbb{R}} |X_t|^2 dt < \infty \quad \textit{and} \quad E\int_{\mathbb{R}} ((M_-^H |X_t|(t))^2 dt < \infty.$$

Then

$$\int_{\mathbb{R}} X_t \Diamond \dot{B}_t^H dt = \int_{\mathbb{R}} (M_-^H X_t)(t) \Diamond \dot{B}_t dt \qquad a.s$$

Proof. According to Theorem 2.2.1 and Corollary 2.2.1, the condition $E\int_{\mathbb{R}} |X_t|^2 dt < \infty$ supplies the equality

$$\int_{\mathbb{R}} X_t \Diamond \dot{B}_t^H dt = \sum_{\alpha,k} \int_{\mathbb{R}} c_\alpha(t) M_+^H \widetilde{h}_k(t) dt \cdot \mathcal{H}_{\alpha+\varepsilon_k}(\omega). \tag{2.2.9}$$

First, replace the operator M_+^H in the last equality. Evidently,

$$\int_{\mathbb{R}} f(t) M_+^H g(t) dt = \int_{\mathbb{R}} M_-^H f(t) g(t) dt \tag{2.2.10}$$

for $f \in L_p(\mathbb{R}), g \in L_q(\mathbb{R})$ with $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \alpha = H + 1/2$. Moreover, $\widetilde{h}_k \in L_q(\mathbb{R})$ for any $q > 1$. Since $E\int_{\mathbb{R}} |X_t|^2 dt = \sum_\alpha \alpha! \int_{\mathbb{R}} c_\alpha^2(t) dt < \infty$, we can take $p = 2, q = \frac{1}{H}$ and obtain from (2.2.10) that

$$\int_{\mathbb{R}} c_\alpha(t) M_+^H \widetilde{h}_k(t) dt = \int_{\mathbb{R}} (M_-^H c_\alpha)(t) \widetilde{h}_k(t) dt. \tag{2.2.11}$$

Further,consider the formal expansion $Y_t = \sum_\alpha (M_-^H c_\alpha)(t) \mathcal{H}_\alpha(\omega)$. Again, from Corollary 2.2.1, the condition

$$E\int_{\mathbb{R}} Y_t^2 dt = \sum_\alpha \alpha! \int_{\mathbb{R}} |(M_-^H c_\alpha)(t)|^2 dt < \infty \tag{2.2.12}$$

ensures the equality

$$\int_{\mathbb{R}} Y_t \Diamond \dot{B}_t dt = \sum_{\alpha,k} \int_{\mathbb{R}} (M_-^H c_\alpha)(t) \widetilde{h}_k(t) dt \mathcal{H}_{\alpha+\varepsilon_k}(\omega). \tag{2.2.13}$$

So, we want to know when (2.2.12) holds and we need the equality $Y_t = (M_-^H X)(t)$. This follows from the equalities

$$((M_-^H X)(t), \mathcal{H}_\alpha(\omega))_{L_2(P)} = (M_-^H c_\alpha)(t) = M_-^H (X_t, \mathcal{H}(\omega))_{L_2(P)}, \tag{2.2.14}$$

if they hold for any $\alpha \in \mathcal{J}$. Equalities (2.2.14) can be reduced to

$$\int_\Omega \left(\int_t^\infty (x-t)^{\alpha-1} X_x(\omega) dx \right) \mathcal{H}_\alpha(\omega) dP$$

$$= \int_t^\infty (x-t)^{\alpha-1} \left(\int_\Omega X_x(\omega) \mathcal{H}_\alpha(\omega) dP \right) dx \qquad (2.2.15)$$

for a.a. $t \in \mathbb{R}$. In turn, the Fubini theorem can be applied to (2.2.15) in the case when

$$E \left(\int_t^\infty (x-t)^{\alpha-1} |X_x(\omega)| dx \right)^2 < \infty \ \ for \ \ a.a. \ \ t \in \mathbb{R}. \qquad (2.2.16)$$

because $E\mathcal{H}_\alpha^2(\omega) = \alpha! < \infty$. Evidently, the condition $E\int_\mathbb{R}((M_-^H|X|)(t))^2 dt < \infty$ ensures both (2.2.12) and (2.2.16). The proof now follows from (2.2.9), (2.2.11), (2.2.13) and (2.2.14). $\square$

2.3 Skorohod, Forward, Backward and Symmetric Integration w.r.t. fBm.

Taking into account the definition of the integral for nonrandom function w.r.t. fBm: $\int_\mathbb{R} f(t) dB_t^H = \int_\mathbb{R} (M_-^H f)(t) dB_t$, and Theorem 2.2.3, it is desirable to define the integral $\int_\mathbb{R} f(t) dB_t^H$ for stochastic integrands in a similar way, for more information we refer the reader to ([28]). Let the stochastic process $X_t = X_t(\omega)$ be such that

$$EX_t^2 < \infty \ \text{ for all } \ t \in \mathbb{R}.$$

Then X_t admits a Wiener-Itô chaos expansion

$$X_t = \sum_{n=0}^\infty \int_{\mathbb{R}^n} f_n(s_1, \dots, s_n, t) dB^{\otimes n}(s_1, \dots, s_n),$$

where the functions $f_n(\cdot) \in L_2(\mathbb{R}^n)$ and are symmetric in variables $(s_1, \dots, s_n)$, for $n = 0, 1, 2, \dots,\ t \in \mathbb{R}$. Let $\widehat{f_n}(s_1, \dots, s_n, s_{n+1})$ be the symmetrization of $f_n(s_1, \dots, s_n, s_{n+1})$ w.r.t $(n+1)$ variables $s_1, \dots, s_n, s_{n+1}$.

Definition 2.3.1. *Assume that*

$$\sum_{n=0}^\infty (n+1)! \|\widehat{f_n}\|_{L_2(\mathbb{R}^{n+1})} < \infty.$$

Then we say that the process X is Skorohod integrable, write $X \in Dom(\delta)$, denote the Skorohod integral as $\int_{\mathbb{R}} X_t \delta B_t$, and define it as $\int_{\mathbb{R}} X_t \delta B_t = \sum_{n=0}^{\infty} \int_{\mathbb{R}^{n+1}} \widehat{f}_n(s_1, \ldots, s_{n+1}) dB^{\otimes(n+1)}(s_1, \ldots, s_{n+1})$. The Skorohod integral belongs to $L_2(P)$,

$$E \int_{\mathbb{R}} X_t \delta B_t = 0, \;\; \text{and} \;\; E|\int_{\mathbb{R}} X_t \delta B_t|^2 = \sum_{n=0}^{\infty} (n+1)! \|\widehat{f}_n\|_{L_2(\mathbb{R}^{n+1})}.$$

Definition 2.3.2. *([3]) Let the stochastic process $X_t = X_t(\omega)$ be such that $(M_-^H X)(t)$ exists and belongs to $Dom(\delta)$. Then we define the Skorohod integral with respect to fBm B^H as*

$$\int_{\mathbb{R}} X_t \delta B_t^H = \int_{\mathbb{R}} (M_-^H X)(t) \delta B_t$$

for the underlying Wiener process B.

Theorem 2.3.1. *Let $M_-^H X \in Dom(\delta)$, $E \int_{\mathbb{R}} |X_t|^2 dt < \infty$ and $E \int_{\mathbb{R}} ((M_-^H |X|)(t))^2 dt < \infty$. Then*

$$\int_{\mathbb{R}} X_t \delta B_t^H = \int_{\mathbb{R}} X_t \diamondsuit \dot{B}_t^H dt.$$

Proof. According to ([12], Theorem 2.5.9), the condition $M_-^H X \in Dom(\delta)$ ensures the existence of $\int_{\mathbb{R}} (M_-^H X)(t) \diamondsuit \dot{B}_t dt$ and

$$\int_{\mathbb{R}} (M_-^H X)(t) \diamondsuit \dot{B}_t dt = \int_{\mathbb{R}} (M_-^H X)(t) \delta B_t = \int_{\mathbb{R}} X_t \delta B_t^H.$$

Further, according to Theorem 2.2.4,

$$\int_{\mathbb{R}} (M_-^H X)(t) \diamondsuit \dot{B}_t dt = \int_{\mathbb{R}} X_t \diamondsuit \dot{B}_t^H dt$$

Whence the proof follows. □

Definition 2.3.3. *Let $H \in (0,1)$. Let $(u_t)_{t \in [0,T]}$ be a process with integrable trajectories. The symmetric integral of u with respect to B_t^H is defined as*

$$\int_0^t u_s dB_s^{H,\circ} = P - \lim_{\varepsilon \to 0} (2\varepsilon)^{-1} \int_0^t u_s (B_{(s+\varepsilon)\wedge t}^H - B_{(s-\varepsilon)\wedge t}^H) ds,$$

Definition 2.3.4. *Let $H \in (0,1)$. Suppose that $(u_t)_{t\in[0,T]}$ is a process with integrable trajectories. The forward integral of u_t with respect to B_t^H is defined as*

$$\int_0^t u_s dB_s^{H,-} = P - \lim_{\varepsilon\to 0}(\varepsilon)^{-1}\int_0^t u_s(B^H_{(s+\varepsilon)\wedge t} - B^H_{(s)})ds. \tag{2.3.1}$$

The backward integral is defined as

$$\int_0^t u_s dB_s^{H,+} = P - \lim_{\varepsilon\to 0}(\varepsilon)^{-1}\int_0^T u_s(B^H_{(s-\varepsilon)\wedge t} - B^H_{(s)})ds.$$

Note that it is mentioned in ([21]) that, for $u \in C^\beta[0,T]$ with $\beta + H > 1$ all the integrals, symmetric, forward, backward, and pathwise coincide.

2.4 Stochastic Fubini Theorem for Stochastic Integrals w.r.t. Fractional Brownian Motion

In this section we prove the generalization of stochastic Fubini theorem for the Wiener integrals with respect to fBm .

Definition 2.4.1. *The nonrandom function $f : \mathbb{R} \to \mathbb{R}$ is called piecewise Hölder of order α on the interval $[T_1,T_2] \subset \mathbb{R}(f \in C^\alpha_{pw}[T_1,T_2])$, if there exists a finite set of disjoint subintervals $\{[a_i,b_i), 1 \le i \le N | \bigcup_{i=1}^N [a_i,b_i] \cup T_2 = [T_1,T_2]\}$ and the function $f \in C^\alpha[a_i,b_i)$ for $1 \le i \le N$.*

As before, we denote

$$\|f\|_{C^\alpha[a_i,b_i)} = \sup_{a_i\le t<b_i}|f(t)| + \sup_{a_i\le s<t<b_i}\frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

Definition 2.4.2. *For $f \in C^\alpha_{pw}[T_1,T_2]$, let*

$$\|f\|_{C^\alpha_{pw}[T_1,T_2]} = \max_{1\le i\le N}\|f\|_{C^\alpha[a_i,b_i)}.$$

Let $f \in C^\alpha[a,b], g \in C^\beta[a,b]$, with $\alpha + \beta > 1$. Then we know that the Riemann-Stieltjes integral exists, where

$$\int_a^b f(t)dg(t) := \lim_{|\pi_n|\to 0}\sum_{k=0}^{k_n-1} f(t_k^n)\Delta g(t_k^n), \tag{2.4.1}$$

where , $\pi_n = \{a = t_k^0 < t_k^1 < \ldots < t_k^{k_n} = b\}, \Delta g(t_k^n) = g(t_{k+1}^n) - g(t_k^n), \pi_n \subset \pi_{n+1}$.

Moreover, according to ([8], Theorem 2.1), there exist the sequences $\{f_n, g_n\} \subset C^{(1)}[a,b]$ such that $\|f_n - f\|_{C^\alpha[a,b]} \to 0, n \to \infty$.

We shall use some bounds for integrals involving Hölder functions. They are proved in [21].

Lemma 2.4.1. *Let* $f \in C^\alpha[a,b], g \in C^\beta[a,b], \alpha + \beta > 1, f_m, g_m \in C^1[a,b]$, $m \geq 1$ *and* $\|f_m - f\|_{C^\alpha[a,b]} \to 0, \|g_m - g\|_{C^\beta[a,b]} \to 0$, *as* $m \to \infty$. *Then*

1) $\int_a^b f(t)dg(t) = \lim_{m\to\infty} \int_a^b f_m(t)g'_m(t)dt$;

2) *the following estimate holds:*

$$\left|\int_a^b f(t)dg(t)\right| \leq C\|f\|_{C^\alpha[a,b]} \cdot \|g\|_{C^\beta[a,b]} \cdot ((b-a)^{1+\varepsilon} \vee (b-a)^\beta);$$

3) *if* $f(a) = 0$, *then*

$$\left|\int_a^b f(t)dg(t)\right| \leq C\|f\|_{C^\alpha[a,b]} \cdot \|g\|_{C^\beta[a,b]} \cdot (b-a)^{1+\varepsilon}, \tag{2.4.2}$$

where $0 < \varepsilon < \alpha + \beta - 1, C > 0$ *is a constant not depending on* α *and* β.

Lemma 2.4.2. *Let* f *be piecewise Hölder of order* $\beta > 1 - H$ *on the interval* $[a,b]$. *Then there exists the Riemann-Stieltjes integral*

$$\int_a^b f(u)dB_u^H = \sum_{i=1}^N \int_{a_i}^{b_i} f(u)dB_u^H$$

and for an arbitrary sequence π_n *of partitions of* $[a,b]$ *it can be represented as a limit*

$$\int_a^b f(u)dB_u^H = \lim_{|\pi_n|\to 0} \sum_{k=1}^{k_n} f(u_k^n)\Delta B_{u_k^n}^H$$

(*We suppose that* $\bigcup_{i=1}^N [a_i, b_i) = [a,b)$, $[a_i, b_i)$ *are disjoint and* $f \in C^\alpha[a_i, b_i)$).

Proof. Put $\pi_n^i := [a_i, b_i) \cap \pi_n$. Evidently, $|\pi_n^i| \leq |\pi_n|$. It follows from boundedness of f and continuity of B^H that

$$\sum_{j:u_j^n \in \pi_n^i} f(u_j^n)\Delta B_{u_j^n}^H \to \int_{a_i}^{b_i} f(u)dB_u^H$$

even in the case when π_n^i does not contain a_i or(and) b_i.

Therefore, $\sum_{k:u_k^n\in\pi_n} f(u_k^n)\Delta B^H_{u_k^n} = \sum_{i=1}^N \sum_{k:u_k^n\in\pi_n^i} f(u_k^n)\Delta B^H_{u_k^n}$
$\to \sum_{i=1}^N \int_{a_i}^{b_i} f(u)dB_u^H = \int_a^b f(u)dB_u^H$, as $|\pi_n|\to 0$. $\square$

Let $0<T_1<T_2, \Phi=\Phi(t,u,\omega) : \mathcal{P}_T := [T_1,T_2]^2\times\Omega\to\mathbb{R}$ be the random function measurable in all the variables.

Theorem 2.4.1. *Let there exist the set $\Omega'\subset\Omega$ such that $P(\Omega')=1$ and let for any $\omega\in\Omega'$ the function $\Phi(s,u,\omega)$ satisfy the conditions:*
1) $\forall s\in(T_1,T_2)\Phi(t,\cdot,\omega)$ *is piecewise Hölder of order $\beta>1-H$ in $u\in[T_1,T_2]$, and there exists $C=C(\omega)>0$ such that $\|\Phi(t,\cdot,\omega)\|_{C^\beta_{pw}[T_1,T_2]}\le C$*
2) *the function $\int_{T_1}^{T_2}\Phi(s,u,\omega)dB_u^H$ is Riemann integrable in the interval $[T_1,T_2]$. Then there exist the repeated integrals*

$$I_1=\int_{T_1}^{T_2}\left(\int_{T_1}^{T_2}\Phi(t,u,\omega)dB_u^H\right)dt \quad and \quad I_2=\int_{T_1}^{T_2}\left(\int_{T_1}^{T_2}\Phi(t,u,\omega)dt\right)dB_u^H$$

$I_1=I_2 \quad P-a.s.$

Proof. We fix $\omega\in\Omega'$ and omit ω throughout the proof. The integral $\int_{T_1}^{T_2}\Phi(t,u)dB_u^H$ exists according to Lemma 2.4.2 and condition 1); the repeated integral I_1 exists according to condition 2). Since $\Phi(t,\cdot)$ is piecewise Hölder, then from the evident bound $\int_{T_1}^{T_2}|\Phi(t,u_1)-\Phi(t,u_2)|ds\le C(T_2-T_1)|u_1-u_2|^\alpha$ we obtain that $\int_{T_1}^{T_2}\Phi(t,u)ds$ is piecewise Hölder of order α in $u\in[T_1,T_2]$. Further, since B^H is Hölder up to order $H>1/2$ and $\alpha+H>1$, the integral I_2 also exists. The integral I_1 can be presented as a limit of integral sums,

$$I_1=\lim_{|\pi_n|\to 0}\sum_{k=0}^{k_n-1}\int_{T_1}^{T_2}\Phi(t_k^n,u)dB_u^H\Delta t_k^n. \tag{2.4.3}$$

For any point $t_k^n\in\pi_n$, according to condition 1), there exists a finite number of points $\{u_{1,k}<u_{2,k}<\ldots<u_{l(k),k}\}$ such that $\Phi(\cdot,u)$ is Hölder between them. Denote

$$\{T_1=u_0<u_1<u_2<\ldots<u_{L(n)}=T_2\}$$
$$:=\bigcup_{k=1}^{k_n}\{u_{1,k}<u_{2,k}<\ldots<u_{l(k),k}\}\cup\{T_1,T_2\}.$$

For any interval $[u_i,u_{i+1}]$ we consider the sequence of partitions $\pi_{i,r}, r\ge 1$ of the form

$$\pi_{i,r}=\{u_i=u_{i,r}^{(0)}<u_{i,r}^{(1)}<\ldots<u_{i,r}^{(m_r)}=u_{i+1}\}, |\pi_{i,r}|\to 0, r\to\infty.$$

Then $\tilde{\pi}_r = \bigcup_{i=0}^{L(n)-1} \pi_{i,r} \cup \{T_1, T_2\} = \{T_1 = u_r^{(0)} < \ldots < u_r^{(N_r)} = T_2\}$ is a partition of interval $[T_1, T_2]$ w.r.t. argument u, its diameter

$|\tilde{\pi}_r| = \max_{1\le i\le L(n)-1} |\pi|_{i,r}$, and $|\tilde{\pi}_r| \to 0, r \to \infty$. Estimate the difference $|I_1 - I_2|$

$$|I_1 - I_2| \le \left| I_1 - \sum_{k=0}^{k_n-1} \sum_{j=0}^{N_r-1} \Phi(t_k^n, u_r^{(j)}) \Delta B^H_{u_r^{(j)}} \Delta t_k^n \right|$$

$$+ \left| I_2 - \sum_{j=0}^{N_r-1} \sum_{k=0}^{k_n-1} \Phi(t_k^n, u_r^{(j)}) \Delta t_k^n \Delta B^H_{u_r^{(j)}} \right| = \Delta_1^{n,r} + \Delta_2^{n,r}. \qquad (2.4.4)$$

Further,

$$\Delta_1^{n,r} \le \left| I_1 - \sum_{k=0}^{k_n-1} \int_{T_1}^{T_2} \Phi(t_k^n, u) dB_u^H \cdot \Delta t_k^n \right|$$

$$+ \sum_{k=0}^{k_n-1} \left| \int_{T_1}^{T_2} \Phi(t_k^n, u) dB_u^H - \sum_{j=0}^{N_r-1} \Phi(t_k^n, u_r^{(j)}) \Delta B^H_{u_r^{(j)}} \right| \Delta t_k^n.$$

Since Φ is piecewise Hölder, then, according to Lemma 2.4.2,

$$\left| \int_{T_1}^{T_2} \Phi(t_k^n, u) dB_u^H - \sum_{j=0}^{N_r-1} \Phi(t_k^n, u_r^{(j)}) \Delta B^H_{u_r^{(j)}} \right| \to 0, \;\; r \to \infty.$$

According to (2.4.3), $\left| I_1 - \sum_{k=0}^{k_n-1} \int_{T_1}^{T_2} \Phi(t_k^n, u) dB_u^H \cdot \Delta t_k^n \right| \to 0, \;\; r \to \infty.$ Therefore,

$$\lim_{n\to\infty} \lim_{r\to\infty} \Delta_1^{n,r} = 0. \qquad (2.4.5)$$

Further,

$$\Delta_2^{n,r} \le \left| I_2 - \sum_{j=0}^{N_n-1} \int_{T_1}^{T_2} \Phi(t, u_r^{(j)}) dt \cdot \Delta B^H_{u_r^{(j)}} \right|.$$

$$(2.4.6)$$

$$+ \left| \sum_{j=0}^{N_n-1} \sum_{k=0}^{k_n-1} \int_{t_k^n}^{t_{k+1}^n} (\Phi(t, u_r^{(j)}) - \Phi(t_k^n, u_r^{(j)})) dt \cdot \Delta B^H_{u_r^{(j)}} \right|.$$

The second term can be expanded as

$$\left|\sum_{k=0}^{k_n-1}\int_{t_k^n}^{t_{k+1}^n}\sum_{j=0}^{N_n-1}(\Phi(t,u_r^{(j)})-\Phi(t_k^n,u_r^{(j)}))\Delta B^H_{u_r^{(j)}}dt\right| \tag{2.4.7}$$

$$=\left|\sum_{k=0}^{k_n-1}\sum_{i=0}^{L(N)-1}\int_{t_k^n}^{t_{k+1}^n}\sum_{u_r^{(j)}\in\pi_{i,r}}(\Phi(t,u_r^{(j)})-\Phi(t_k^n,u_r^{(j)}))\Delta B^H_{u_r^{(j)}}dt\right|.$$

Since the function $\Phi(s,u)-\Phi(t_k^n,u)$ is Hölder on any interval $[u_i,u_{i+1})$ we have that

$$\lim_{|\pi_{i,r}|\to 0}\sum_{u_r^{(j)}\in\pi_{i,r}}\left(\Phi(t,u_r^{(j)})-\Phi(t_k^n,u_r^{(j)})\right)\Delta B^H_{u_r^{(j)}}$$

$$=\int_{u_i}^{u_{i+1}}\left(\Phi(t,u)-\Phi(t_k^n,u)\right)dB_u^H \tag{2.4.8}$$

Moreover, $\forall 0\le i\le L(n)-1$ the sequence $f_i^r(t,t_k^n)=\sum_{u_r^{(j)}\in\pi_{i,r}}(\Phi(t,u_r^{(j)})-\Phi(t_k^n,u_r^{(j)}))\Delta B^H_{u_r^{(j)}}$ has the integrable dominant. Indeed, we can use the bounds from ([8], Corollary 20), Lemma 2.4.1 and the boundedness of Hölder norms, and obtain that

$$|f_i^r(t,t_k^n)|\le\left|f_i^r(t,t_k^n)-\int_{u_r^{(j)}}^{u_{r+1}^{(j)}}(\Phi(t,u)-\Phi(t_k^n,u))dB_u^H\right|$$

$$+\left|\int_{u_r^{(j)}}^{u_{r+1}^{(j)}}(\Phi(t,u)-\Phi(t_k^n,u))dB_u^H\right|$$

$$\le C|\pi_{i,r}|^{\varepsilon}\cdot\|\Phi(t,\cdot)-\Phi(t_k^n,\cdot)\|_{C[u_r^{(j)},u_{r+1}^{(j)}]^{\beta'}}\cdot\|B^H\|_{C[u_r^{(j)},u_{r+1}]^{H'}}$$

$$+\left|\int_{u_r^{(j)}}^{u_{r+1}^{(j)}}\left(\Phi(t,u)-\Phi(t_k^n,u)\right)dB_u^H\right|. \tag{2.4.9}$$

$$\le C+\left|\int_{u_r^{(j)}}^{u_{r+1}^{(j)}}\left(\Phi(t,u)-\Phi(t_k^n,u)\right)dB_u^H\right|,$$

where $\beta' < \beta, H' < H$ and $\beta' + H' > 1$. Using the second statement of Lemma 2.4.1 and condition 1) of this theorem, we obtain the bound

$$\left|\int_{u_r^{(j)}}^{u_{r+1}^{(j)}} \left(\Phi(t,u) - \Phi(t_k^n,u)\right) dB_u^H\right|$$
$$\leq C\|\Phi(t,\cdot) - \Phi(t_k^n,\cdot)\|_{C_{pw}^{\alpha'}[T_1,T_2]} \cdot \|B^H\|_{C^{H'}[T_1,T_2]} \leq C. \tag{2.4.10}$$

Estimates (2.4.9) and (2.4.10) mean that we can use the Lebesgue dominant convergence theorem and obtain that

$$\lim_{r\to\infty} \int_{t_k^n}^{t_{k+1}^n} f_i^r(t,t_k^n)dt = \int_{t_k^n}^{t_{k+1}^n} \int_{u_i}^{u_{i+1}} \left(\Phi(t,u) - \Phi(t_k^n,u)\right) dB_u^H dt$$

where the integrand $\int_{u_i}^{u_{i+1}} \left(\Phi(t,u) - \Phi(t_k^n,u)\right) dB_u^H$ is measurable and bounded in t.
Therefore,

$$\lim_{r\to\infty} \sum_{k=0}^{k_n-1} \sum_{i=0}^{L(n)-1} \int_{t_k^n}^{t_{k+1}^n} \sum_{u_r^{(j)}\in\pi_{i,r}} \left(\Phi(t,u_r^{(j)}) - \Phi(t_k^n,u_r^{(j)})\right) \Delta B_{u_r^{(j)}}^H dt$$
$$= \sum_{k=0}^{k_n-1} \int_{t_k^n}^{t_{k+1}^n} \int_{T_1}^{T_2} \left(\Phi(t,u) - \Phi(t_k^n,u)\right) dB_u^H dt$$
$$= \int_{T_1}^{T_2} \left(\int_{T_1}^{T_2} \Phi(t,u)dB_u^H\right) dt - \sum_{k=0}^{k_n-1} \int_{T_1}^{T_2} \Phi(t_k^n,u)dB_u^H \Delta t_k^n. \tag{2.4.11}$$

According to condition 2) of this theorem, the integral $\int_{T_1}^{T_2} \Phi(t,u)dB_u^H$ is Riemann integrable in t, therefore

$$\lim_{n\to\infty} \sum_{k=0}^{k_n-1} \int_{T_1}^{T_2} \Phi(t_k^n,u)dB_u^H \Delta t_k^n = \int_{T_1}^{T_2} \left(\int_{T_1}^{T_2} \Phi(t,u)dB_u^H\right) dt. \tag{2.4.12}$$

From Lemma 2.4.1,

$$\left|I_2 - \sum_{r=0}^{L(n)-1} \int_{T_1}^{T_2} \Phi(t,u_r^{(j)})dt \cdot \Delta B_{u_r^{(j)}}^H\right| \to 0, as\ \ n \to \infty. \tag{2.4.13}$$

Now the proof follows from (2.4.4) – (2.4.13). $\square$

2.5 The Itô Formula for Fractional Brownian Motion

2.5.1 The Simplest Version

First, we present a very elegant proof of the Itô formula involving fBm from ([27]).

Lemma 2.5.1. *Let B^H be an fBm with $H \in (1/2, 1), F \in C^2(\mathbb{R})$. Then for any $t > 0$*

$$F(B_t^H) = F(0) + \int_0^t F'(B_u^H) dB_u^H.$$

Proof. The Taylor formula with the reminder term in the integral form gives us

$$F(x) = F(y) + F'(y)(x-y) + \int_y^x F''(u)(x-u)du.$$

Let the sequence of partitions $\pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = t\}$ $, |\pi_n| \to 0, \quad n \to \infty$. Then $F(B_t^H) - F(0) = \sum_{k=1}^{kn} [F(t_k^n) - F(t_{k-1}^n)] = \sum_{k=1}^{kn} F'(B_{t_{k-1}^n}^H)(B_{t_k^n}^H - B_{t_{k-1}^n}^H) + R_t^n$, where $R_t^n = \sum_{k=1}^{kn} \int_{B_{t_{k-1}^n}^H}^{B_{t_k^n}^H} F''(u)(B_{t_k^n}^H - u)du$. Further, $\sup_{0 \le u \le t} |F''(B_u^H)| < \infty$ a.s. and for $H \in (1/2, 1)$, and

$$P - \lim_{n \to \infty} \sum_{k=1}^{kn} |B_{t_k^n}^H - B_{t_{k-1}^n}^H|^2 = 0.$$

Therefore $|R_t^n| \le \frac{1}{2} \sup_{0 \le u \le t} |F''(B_u^H)| \sum_{k=1}^{kn} \left| B_{t_k^n}^H - B_{t_{k-1}^n}^H \right|^2 \xrightarrow{P} 0$. Even if we do not know that the limit of integral sums $\sum_{k=1}^{kn} F'(B_{t_{k-1}^n}^H)(B_{t_k^n}^H - B_{t_{k-1}^n}^H)$ exists (but we know it from Theorem 2.1.3) we can obtain this existence now and moreover

$$F(B_t^H) - F(0) = \int_0^t F'(B_u^H) dB_u^H.$$

□

2.5.2 The Itô Formula in Terms of Wick Integrals

The next result is a direct consequence of ([21], Theorems 2.3.8 and 2.7.3.)

Theorem 2.5.1. *Let the function* $F = F(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *be continuously differentiable in* t *and twice continuously differentiable in* x. *Let* $Y_t = \sum_{i=1}^m \sigma_i B_t^{H_i}$, $E|\frac{\partial F}{\partial x}(t,Y_t)|^{2+\varepsilon} < \infty$, $t > 0$ *for some* $\varepsilon > 0$, $E\sup_{0\le s\le t}\left[\left(\frac{\partial F}{\partial x}(s,Y_s)\right)^2 + \left(\frac{\partial^2 F}{\partial x^2}(s,Y_s)\right)^2\right] < \infty, t > 0$. *Then*

$$F(t,Y_t) - F(0,0) = \int_0^t \frac{\partial F}{\partial t}(s,Y_s)ds + \int_0^t \frac{\partial F}{\partial x}(s,Y_s)\Diamond dY_s$$

$$+\sum_{i,k=1}^m \sigma_i\sigma_k \tilde{C}_{H_i,H_k}(H_i+H_k)\int_0^t \frac{\partial^2 F}{\partial x^2}(s,Y_s)s^{H_i+H_k-1}ds. \tag{2.5.1}$$

2.5.3 The Itô Formula for $H \in (0,1/2)$

We use the integral representation of fbm via the underlying Wiener process B on the finite interval $[0,t]$:

$$B_t^H = \int_0^t m_H(t,s)dB_s$$

$$= C_H^{(6)}t^\alpha \int_0^t u^{-\alpha}(t-u)^\alpha dB_u - C_H^{(6)}\alpha\int_0^t s^{\alpha-1}\left(\int_0^s u^{-\alpha}(s-u)^{-\alpha}dB_u\right)ds.$$

Let the fonction $F \in C^3(\mathbb{R})$ and we want to expand $F(B_t^H)$. Note that $B_t^H = B_{t,t}^H$ where for $0 < z < t$, $B_{z,t}^H = C_H^{(6)}z^\alpha\int_0^z u^{-\alpha}(t-u)^\alpha dB_u$

$-C_H^{(6)}\alpha\int_0^z s^{\alpha-1}\left(\int_0^s u^{-\alpha}(s-u)^{-\alpha}dB_u\right)ds$. Therefore

$$F(B_t^H) = F(0) + \int_0^t F'(B_{z,t}^H)dzB_{z,t}^H + \frac{1}{2}(C_H^{(6)})^2\int_0^t F''(B_{z,t}^H)(t-z)^{2\alpha}dz$$

$$= F(0) + \alpha C_H^{(6)}\int_0^t F'(B_{z,t}^H)z^{\alpha-1}\int_0^z u^{-\alpha}(t-u)^\alpha dB_u dz$$

$$+C_H^{(6)}\int_0^t F'(B_{z,t}^H)(t-z)^\alpha dB_z$$

$$-\alpha C_H^{(6)}\int_0^t F'(B_{z,t}^H)z^{\alpha-1}\left(\int_0^z u^{-\alpha}(t-u)^{-\alpha}dB_u\right)dz.$$

$$+\frac{1}{2}(C_H^{(6)})^2\int_0^t F''(B_{z,t}^H)(t-z)^{2\alpha}dz \tag{2.5.2}$$

further

$$B^H_{z,t} = B^H_z + \alpha C^{(6)}_H z^\alpha \int_0^z u^{-\alpha} \int_z^t (v-u)^{\alpha-1} dv dB_u.$$
$$= B^H_z + \alpha C^{(6)}_H z^\alpha \int_z^t \int_0^z u^{-\alpha}(v-u)^{\alpha-1} dB_u dv \tag{2.5.3}$$

whence

$$F'(B^H_{z,t}) = F'(B^H_z) + \int_z^t F''\left(B^H_z + \alpha C^{(6)}_H z^\alpha \int_z^r \int_0^z u^{-\alpha}(v-u)^{\alpha-1} dB_u dv\right).$$
$$\times \alpha C^{(6)}_H z^\alpha \int_0^z u^{-\alpha}(r-u)^{\alpha-1} dB_u dr = F'(B^H_z) + \phi(F'', z, t), \tag{2.5.4}$$

and similar relation holds for $F''(B^H_z, t)$. But

$$\int_z^r \int_0^z u^{-\alpha}(v-u)^{\alpha-1} dB_u dv = \frac{1}{\alpha}\int_0^z u^{-\alpha}[(r-u)^\alpha - (z-u)^\alpha] dB_u. \tag{2.5.5}$$

Substituting (2.5.3) – (2.5.5) into (2.5.2), we obtain the following result.

Theorem 2.5.2. *Let* $H \in (0, 1/2), B^H$ *be an fBm with Hurst index* H, *represented as* $B^H_t = \int_0^t m_H(t,s) dB_s$. *Denote* $Y_{r,z} := C^{(6)}_H \int_0^z u^{-\alpha}(r-u)^\alpha dB_u, 0 \leq z \leq r, Y_z := Y_{z,z}$. *Then*

$$F(B^H_t) = F(0) + \int_0^t F'(B^H_z)\alpha z^{\alpha-1} Y_{t,z} dz + C^{(6)}_H \int_0^t F'(B^H_z)(t-z)^\alpha dB_z$$
$$-\alpha \int_0^t F'(B^H_z) z^{\alpha-1} Y_{t,z} dz + \frac{1}{2}(C^{(6)}_H)^2 \int_0^t F''(B^H_z)(t-z)^{2\alpha} dz + R_t,$$

where

$$R_t = \alpha \int_0^t \phi(F'', z, t)\alpha z^{\alpha-1} Y_{t,z} dz + C^{(6)}_H \int_0^t \phi(F''', z, t)(t-z)^\alpha dB_z$$
$$-\alpha \int_0^t \phi(F'', z, t) z^{\alpha-1} Y_{t,z} dz + \frac{1}{2}(C^{(6)}_H)^2 \int_0^t \phi(F''', z, t)(t-z)^{2\alpha} dz,$$

2.6 The Girsanov Theorem for fBm

Consider the kernel $l_H(t,s) = C^{(5)}_H s^{-\alpha}(t-s)^{-\alpha}, 0 < s < t$. Let $\mathcal{F}_t = \sigma\{B^H_s, 0 \leq s \leq t\} = \sigma\{B_s, 0 \leq s \leq t\}$, where B is underlying Wiener process in the representation

$$M^H_t = \int_0^t l_H(t,s) dB^H_s \quad , B_t = \hat{\alpha}\int_0^t s^\alpha dM^H_s.$$

Assume that the random process $\{\phi_t, t \geq 0\}$ is adapted to filtration $\mathcal{F}_t$ and satisfies

$$\int_0^t l_H(t,s)|\phi_s|ds < \infty, \;\; t > 0, \;\; P - a.s. \tag{2.6.1}$$

Assume also that we have the representation

$$\int_0^t l_H(t,s)\phi_s ds = \hat{\alpha} \int_0^t \delta_s ds, \;\; t > 0 \tag{2.6.2}$$

with some $\mathcal{F}_t$-adapted process δ satisfying

$$\int_0^t |\delta_s|ds < \infty, \;\; P - a.s., \;\; t > 0, \tag{2.6.3}$$

and

$$E \int_0^t s^{2\alpha}\delta_s^2 ds < \infty, \;\; t > 0. \tag{2.6.4}$$

Define a square-integrable martingale L by $L_t := \int_0^t s^\alpha \delta_s dB_s$.

Theorem 2.6.1. *Assume that we have* (2.6.1) – (2.6.4) *and the martingale L satisfies*

$$E\, exp\{L_t - 1/2\langle L\rangle_t\} = 1, \;\; t > 0.$$

Then the process $\widetilde{B}_t^H := B_t^H - \int_0^t \phi_s ds$ is an fBm with respect to measure Q, where the measure Q, is defined by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = exp\left\{L_t - \frac{1}{2}\langle L\rangle_t\right\}.$$

Proof. Note first that the integral

$$\widetilde{M}_t^H = \int_0^t l_H(t,s)d\tilde{B}_s^H = \int_0^t l_H(t,s)dB_s^H - \int_0^t l_H(t,s)\phi_s ds \tag{2.6.5}$$

exists, since both integrals exist as pathwise integrals (the first integral was studied in Section 1.8 and (2.6.2) ensures the existence of the second integral). Moreover, from (2.6.2) it follows that

$$\widetilde{M}_t^H = M_t^H - \tilde{\alpha}\int_0^t \delta_s ds = \tilde{\alpha}\left(\int_0^t s^{-\alpha}dB_s - \int_0^t \delta_s ds\right).$$

Evidently, $\left[\widetilde{M}^H\right]_t := P - \lim_{|\pi|\to 0}\sum_{t_i\in\pi}(\widetilde{M}^H_{t_i} - \widetilde{M}^H_{t_{i-1}})^2$ exists and equals $\left[\widetilde{M}^H\right]_t = t^{1-2\alpha}$. Therefore, for any $\theta \in \mathbb{R}$ we have for $\widehat{M}^H_t = \widehat{\alpha}\widetilde{M}^H_t$ that

$$\theta\widehat{M}^H_t - \frac{\theta^2}{2}\left[\widehat{M}^H\right]_t + L_t - \frac{1}{2}\langle L\rangle_t = \theta\int_0^t s^{-\alpha}dB_s - \theta\int_0^t \delta_s ds - \frac{\theta^2}{2}\frac{t^{1-2\alpha}}{1-2\alpha}$$

$$+\int_0^t s^{\alpha}\delta_s dB_s - \frac{1}{2}\int_0^t s^{2\alpha}\delta_s^2 ds = \int_0^t (\theta s^{-\alpha} + s^{\alpha}\delta_s)dB_s$$

$$-\frac{1}{2}\int_0^t (\theta^2 s^{-2\alpha} - 2\delta_s\theta + \delta_s^2 s^{2\alpha})ds =: R_t - \frac{1}{2}\langle R\rangle_t, \tag{2.6.6}$$

where R is a square-integrable martingale given by $R_t := \int_0^t(\theta s^{-\alpha} + s^{\alpha}\delta_s)dB_s$. But (2.6.6) means that the process

$$K_t := exp\left\{\theta\widehat{M}^H_t - \frac{\theta^2}{2}\left[\widehat{M}^H\right]_t + L_t - \frac{1}{2}\langle L\rangle_t\right\}$$

is a local $P-$ martingale. This implies, in turn, that the process $exp\left\{\theta\widehat{M}^H_t - \frac{\theta^2}{2}\left[\widehat{M}^H\right]_t\right\}$ is a local Q-martingale. From ([21], p.192), we can conclude that $\widehat{M}^H$ is a local Q-martingale with the angle bracket $\langle\widehat{M}^H\rangle_t = \int_0^t s^{-2\alpha}ds$ and so $\widetilde{M}_t = \widetilde{\alpha}\int_0^t s^{-\alpha}d\widetilde{B}_s$,where $\widetilde{B}$ is a standard Brownian motion with respect to Q (and is obtained from B by subtracting a drift). This means that

$$\int_0^t l_H(t,s)d\widetilde{B}^H_s = \widetilde{\alpha}\int_0^t s^{-\alpha}d\widetilde{B}_s. \tag{2.6.7}$$

Now, using two representations for $\widetilde{B}^H$, (2.6.5) and (2.6.7), we can obtain (1.8.8) for $\widetilde{B}^H$ and then conclude from ([21], Remark 1.8.2) that it is the fBm with respect to the measure Q. □

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