N° d'ordre :

Université de Saida– Dr. Moulay Tahar

Faculté des Sciences

Thèse

Présentée pour obtenir le diplôme de

Doctorat de 3ième Cycle

Spécialité : Probabilités et statistique

Filière : Mathématiques

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Thème :

Les propriétés du flot stochastique engendré par le modèle naturel dans les cas unidimensionnel et multidimensionnel.

Thèse soutenue le 31 / 03 / 2022 devant le jury composé de :

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Dedication

I dedicate my simple project with all respect and love to:

My dear father, may God bless him.

My mother who had been the wind beneath my wings until I completed this work.

My brothers for their help and encouragements.

My close friends for their contributions in this work and for their true friendship along the four years of study.

Everyone who gave me support.

Acknowledgments

First of all, Praise be to God who helped me to achieve this modest work.

My deepest thanks and great appreciation go to my supervisors, ***Pr.Abdeldjebbar Kandouci*** *and* ***Dr.Fatima Benziadi****, for their constant support, their patience, their motivation, their careful guidance and immense knowledge helped me to carry out this work.*

I want to express my great thanks to ***Pr. Rahmani Saadia*** for chairing the jury. I am also very grateful that **Pr. Chouaf Abdelhak** and **Pr.Guendouzi Toufik** have agreed to participate in my jury. I am very honored with their presence.

My sincere thanks also to all the professors of mathematics department and all the members of ***LMSSA*** *laboratory especially* ***Dr.Soumia Idrissi****.*

My special thanks to all colleagues who have helped me in this work. I also would like to thank everybody who was important to the successful realization of this thesis,

Contents

Résumé

Cette thèse s'intéresse à un modèle important de risque de crédit qui s'appelle le modèle à un défaut ou le modèle naturel qui est exprimé par une équation différentielle stochastique appelé l'équation naturelle, Cette équation joue un rôle important dans notre étude. Sous certaines hypothèses, la recherche rapportée dans cette étude est la régularité des trajectoires de flot stochastique engendré par l'équation naturelle dans le cas multidimensionnel basé sur le théorème de kolmogorov. Nous prouverons également la différentiabilité de flot stochastique engendré par l'équation naturelle par rapport à la valeur initiale dans le cas unidimensionnel en se basant sur les théorèmes de Burkholder-Davis-Gundy, Hôlder et Gronwall. Nous prouverons également la même propriété mais dans le cas multidimensionnel basé sur l'idée de Hiroshi Kunita.

Abstract

This thesis is interested with an important model of credit risk so-called the one-default or natural model which is expressed by a natural equation, this equation play an important role in our research. Under some assumptions, the research reported in this paper is the regularity of the trajectories of the stochastic flow generated by the natural equation in multidimensional case based on the Kolmogorov's theorem. Additionally, we will look at the differentiability of stochastic flow generated by the natural equation with respect to the initial value in one-dimensional case based on the Burkholder-Davis-Gundy , Hôlder and the Gronwall theorems . In addition to this we will prove the same property in multidimensional case based on the idea of Hiroshi Kunita.

The List Of Works

Publication:

Y.Khatir, A.Kandouci, F.Benziadi, On the trajectories of stochastic flow generated by the natural model in multidimensional case, Facta Universitatis'Series: Mathematics and Informatics. Vol 36, No 4(2021)879-893.

Submission:

1. Y.Khatir, A.Kandouci, F.Benziadi, The continuity of the solution of the natural equation in Multidimensional case.
2. Y.Khatir, A.Kandouci, F.Benziadi, On stochastic flows generated by one-default model in one-dimensional case.

Communications:

1. Oral presentation entitled: the properties of stochastic flow generated by one-default model in International conference on Financial Mathematics: Tools and Applications $28-29$ October 2019, Bejaia (Algeria).
2. Oral presentation entitled: the properties of stochastic flow generated by one-default model in multidimensional in 1^st International conference on pure and applied Mathematics, $26-27$ May 2021, Ouargla (Algeria).

Participations:

1. Talking about Mathematics organized at April 18^{th}, 2018 in saida.
2. The spring school organized from 22^{nd} to 26^{th} of April, 2018 on the backward stochastic differential equations and stochastic contrôl under the guests speakers: **Miss Agram Nacira and Msr Kebiri Omar**.

3. Presentation in LMSSA laboratory at September 02^{nd}, 2018 about the progress state of thesis entitled the properties of stochastic flow generated by one-default model.

4. The fall (autumn) school organized from 23^{rd} to 28^{th} of October, 2018 on statistical inference and information theory for Markovian and semi-Markovian processes under a Guest Speaker: **Msr Vlad Stefan Barbu**.

5. CIMPA school organized in Saida from 1^{st} to 9^{th} of march, 2019 about stochastic analysis and applications.

6. Presentation in LMSSA laboratory at September 28^{th}, 2019 about the progress state of thesis entitled the properties of stochastic flow generated by natural model.

7. Presentation in LMSSA laboratory at February 11^{th}, 2020 entitled: the properties of stochastic flow generated by one-default model (weekly seminar).

8. Scientific day organized at February 24^{th}, 2020 on statistics applied in honor of Professor **Mourid Tahar**.

9. Presentation in LMSSA laboratory at October 21^{st}, 2021 about the progress state of thesis entitled the properties of stochastic flow generated by one-default model.

10. Presentation in LMSSA laboratory at january 22^{nd}, 2022 entitled: the differentiability of stochastic flow generated by the natural model model in multidimensional case (weekly seminar).

Introduction

Theory of stochastic differential equations has been treated extensively, especially the geometric property of the solutions of $SDEs$ generated by Wiener process, Brownian Motion or a continuous semi-martingale which defines this solutions as stochastic flow of diffeomorphisms. Results may be found in papers of H-Kunita [31], Elworthy [13], P-Meyer [39], P-Protter [49], T-Fujiwara [21] etc. In 1973 J-Neveu [42] who's the first one which demonstrated a theorem of continuity of the solution according to the initial condition for classical SDE governed by terms in dBt and dt and Malliavin [37] studied the differentiability of the solutions of $SDEs$ according to the initial conditions (for classical type equations on manifolds) also M-Emmery [18] treated the general case of the weak injectivity but without the time reversal technique as well Yamada in collaboration with Ogura [45], they demonstrated the homeomorphism property in the one-dimensional case, and they gave a counter example which shows that the solution is strong but it's not injective if the coefficients are only $\alpha - H\ddot{o}$lderian. Bismut [12] studied the stochastic flow in the case where the initial condition itself is a stochastic process, and with the help of an interesting Ito's formula, he proved that if the initial condition is a semi-martingale, then the solution is a semi-martingale.

Since the beginning of 21^{st} century, X-Zhang [57] could treated the theory of stochastic flows without Lipschitz coefficients. In the other side, this theory has been developed by P-Andrey ([2],[3]) who demonstrated the sobolev differentiability of the solutions of SDE with reflection as well for SDE with lipschitz continuous coefficients. In another study [4] he proved in collaboration with O-V-Aryasova the differentiability of stochastic flow for SDE with discontinuous drift in multidimensional case. We also mention an other results as: E-Fedrizzi and F-Flandoli [19] obtained weakly differentiability of solutions of SDE with Nonregular Drift, Qian Lin [50] studied the differentiability of the solutions of SDE driven by G-Brownian Motion with respect to the initial data and parameters, Philip Protter [49] studied the properties of stochastic flows for SDE governed by semi-martingale with local lipschitz coefficients.

In this thesis, we will study the properties of stochastic flow generated by $\natural-$model in one-dimensional and multidimensional cases. This model is expressed by a stochastic differential equation so-called natural equation which based on a continuous local martingale and it is considered as one of the best way to describe the evolution of market after the default time τ.

For more details, see [25].

This thesis is divided into three chapters, The first one gives some the basic preliminary facts needed to establish our main results. Secondly, we present the theory of stochastic calculus on a local martingale, and we briefly recall some basic information about stochastic differential equations and their applications in finance field. In the second chapter, we present the definition and the properties of flow in deterministic and stochastic cases then we will look at a brief zoology of risks which is known as banks risks: Liquidity risk, Market risk, Operational risk and Credit risk, and we discus the main models of credit risk among them: Merton model, intensity model and density model. In the last chapter, the first section is concerning the main model of credit risk which is expressed by $\natural-$eqution, we will give a global description of this equation. The second section is devoted to the continuity property of the solution of the $\natural-$eqution with real values under the Lipschitz condition of the coefficients in multidimensional case based on the criterion of Kolmogorov. It is remain the third section which is the heart of this thesis, where we establish the differentiability property of the solution of the $\natural-$eqution with real values under the countinuous Lipschitz coefficients with respect to the initial data based on the Burkholder-Davis-Gundy and *Hölder* inequalities and the gronwall's lemma and we give our main result about the differentiability of the solution of the $\natural-$eqution but in multidimensional case under the same conditions i.e we prove the existence of the partial derivatives depending on the idea of R-M-Dudley, H-Kunita and F-Ledrappier [28]. This is our approach in this research.

Chapter 1

Generalities on the stochastic differential equations

The first chapter provides some background information and basic notions, stochastic calculus on the local martingale, stochastic differential equations and its applications in finance field .

1.1 The Kolmogorov's theorem and basic notions

There are several versions of Kolmogorov's theorem; we give here a quite general one.

Theorem 1.1.1 *Let (E, d) be a complete metric space, and let U^x be an E-valued random variable for all x dyadic rationales in $\mathbb{R}^n$. Suppose that for all x and y, we have $d(U^x, U^y)$ which is a random variable and that there exist strictly positive constants ε, C, β such that*

$$E\{d(U^x, U^y)^\varepsilon\} \leq C\|x - y\|^{n+\beta}.$$

Then for almost all ω the function $x \longmapsto U^x$ can be extended uniquely to a continuous function from R^n to E.

Proof:

We prove the theorem for the unit cube $[0, 1]^n$. Before the statement of the theorem we establish some notations. Let Δ denote the dyadic rational points of the unit cube $[0, 1]^n$ in R^n, and let Δ_m denote all $x \in \Delta$ whose coordinates are of the form $k2^{-m}, 0 \leq k \leq 2^m$. Two points x and

y in Δ_m are neighbors if $\sup_i |x^i y^i| = 2^{-m}$. We use Chebyshev's inequality on the inequality hypothesized to get

$$P\{d(U^x, U^y) \geq 2^{-\alpha m}\} \leq C 2^{\alpha \varepsilon m} 2^{-m(n+\beta)}.$$

Let

$$\Lambda_m = \{\omega : \exists \, \text{neighbors } x, y \in \Delta_m \text{ with } d(U^x(\omega), U^y(\omega)) \geq 2^{-\alpha m}\},$$

since each $x \in \Delta_m$ has at most 3^n neighbors, and the cardinality of Δ_m is 2^{mn}, we have

$$P(\Lambda_m) \leq c 2^{m(\alpha\varepsilon - \beta)},$$

where the constant $c = 3^n C$. Take α a sufficiently small so that $\alpha\varepsilon < \beta$. Then

$$P(\Lambda_m) \leq c 2^{-m\delta},$$

where $\delta = \beta - \alpha\varepsilon > 0$.
The Borel-Cantelli lemma then implies $P(\Lambda_m \text{ infinitely often}) = 0$. In other words, there exists an m_0 such that for $m \geq m_0$ and every pair (u, v) of points of Δ_m that are neighbors,

$$d(U^u, U^v) \leq 2^{-\alpha m}.$$

We now use the preceding to show that $x \longmapsto U^x$ is uniformly continuous on Δ and hence extendable uniquely to a continuous function on $[0, 1]^n$. To this end, let $x, y \in \Delta$ be such that $\|x - y\| \leq 2^{-k-1}$. We will show that $d(U^x, U^y) \leq c 2^{-\alpha k}$ for a constant c, and this will complete the proof. Without loss of generality assume $k \geq m_0$. Then $x = (x^1, ..., x^n)$ and $y = (y^1, ..., y^n)$ in Δ with $\|x - y\| \leq 2^{-k-1}$ have dyadic expansions of the form:

$$x^i = u^i + \sum_{j>k} a^i_j 2^{-j},$$

$$y^i = v^i + \sum_{j>k} b^i_j 2^{-j},$$

where a_j^i, b_j^i are each 0 or 1 and u,v are points of Δ_k which are either equal or neighbors. Next set $u_0 = u, u_1 = u_0 + a_{k+1}2^{-k-1}, u_2 = u_1 + a_{k+2}2^{-k-2}$,...
We also make analogous definitions for $v_0, v_1, v_2, \ldots$ then u_{i-1} and u_i are equal or neighbors in Δ_{k+i} each i, and analogously for v_{i-1} and v_i. Hence

$$d(U^x(\omega), U^u(\omega)) \leq \sum_{j=k}^{\infty} 2^{-\alpha j},$$

$$d(U^y(\omega), U^v(\omega)) \leq \sum_{j=k}^{\infty} 2^{-\alpha j},$$

and moreover

$$d(U^u(\omega), U^v(\omega)) \leq 2^{-\alpha k}.$$

The result now follows by the triangle inequality.

Lemma 1.1.1 ($Gronwall's\,lemma$). *Let $(a, b) \in R^2$ with $a < b$, φ and $\psi : [a, b] \longrightarrow R$ non-negative continuous functions, such that $\exists \rho \in R^+, \forall t \in [a, b], \varphi(t) \leq \rho + \int_a^t \varphi(s)\psi(s)ds$ then:*

$$\forall t \in [a, b], \varphi(t) \leq \rho \exp(\int_a^t \psi(s)ds).$$

Proof:
We assume $G : [a, b] \longrightarrow R$

$$u \longmapsto \left(\int_a^u \varphi(s)\psi(s)ds\right) \exp\left(-\int_a^u \psi(s)ds\right),$$

because φ and ψ are continuous functions, then G is the is continuously derivable on $[a, b]$ and $\forall u \in [a, b]$

$$\grave{G}(u) = \varphi(u)\psi(u) \exp\left(-\int_a^u \psi(s)ds\right) - \psi(u) \left(\int_a^u \varphi(s)\psi(s)ds\right) \exp\left(-\int_a^u \psi(s)ds\right),$$

$$\forall u \in [a,b],\, \grave{G}(u) = \psi(u) \exp\left(-\int_a^u \psi(s)ds\right)\left(\varphi(u) - \int_a^u \varphi(s)\psi(s)ds\right).$$

But, by hypothesis

$$\forall u \in [a,b], \varphi(u) \leq \rho + \int_a^u \varphi(s)\psi(s)ds.$$

So

$$\forall u \in [a,b], \grave{G}(u) \leq \rho\, \psi(u) \exp\left(-\int_a^u \psi(s)ds\right),$$

let $t \in [a,b]$, integrating this inequality for i from a and t:

$$G(t) - G(a) \leq \rho \int_a^t \psi(u) \exp\left(-\int_a^u \psi(s)ds\right) du.$$

By definition of G and as $G(a) = 0$:

$$\left(\int_a^t \varphi(s)\psi(s)ds\right) \exp\left(-\int_a^t \psi(s)ds\right) \leq \rho\left[-\exp\left(-\int_a^u \psi(s)ds\right)\right]_a^t$$

$$\leq -\rho \exp\left(-\int_a^t \psi(s)ds\right) + \rho \exp(0).$$

From where

$$\left(\int_a^t \varphi(s)\psi(s)ds\right) \leq -\rho + \rho \exp\left(\int_a^t \psi(s)ds\right),$$

and finally

$$\varphi(t) \leq \rho \exp\left(\int_a^t \psi(s)ds\right).$$

Theorem 1.1.2 (*Burkholder − Davis − Gundy*) (*BDG*). *Let $T > 0$ and ξ be a continuous local martingale such that $\xi_0 = 0$. For any $1 \leq p < \infty$, there exists positive constants c'_p, C'_p independent of T and $(\xi_t)_{0\leq t\leq T}$ such that,*

$$c'_p \mathbb{E}[< \xi >_T^{p/2}] \leq \mathbb{E}[(\xi_t^*)^p] \leq C'_p \mathbb{E}[< \xi >_T^{p/2}],$$

where $\xi_t^* = \sup_{0\leq t\leq T}|\xi_t|$.

Proof:

By stopping it is enough to prove the result for bounded ξ. Let $q \geq 2$. From Itô's formula we have

$$d|\xi_t|^q = q|\xi_t|^{q-1}sgn(\xi_t)d\xi_t + \frac{1}{2}q(q-1)|\xi_t|^{q-2}d<\xi>_t \quad (1.1)$$

$$= qsgn(\xi_t)|\xi_t|^{q-1}d\xi_t + \frac{1}{2}q(q-1)|\xi_t|^{q-2}d<\xi>_t . \quad (1.2)$$

As a consequence of the Doob's stopping theorem, we get that for every bounded stopping time τ,

$$\mathbb{E}\left[|\xi_\tau|^q/\mathcal{F}_0\right] \leq \frac{1}{2}q(q-1)\mathbb{E}\left[\int_0^\tau |\xi_t|^{q-2}d<\xi>_t /\mathcal{F}_0\right].$$

From the Lenglart's domination inequality, we deduce then that for every $\hat{k} \in (0,1)$,

$$\mathbb{E}[\sup_{0\leq t\leq T}|\xi_t|^q]^{\hat{k}} \leq \frac{2-\hat{k}}{1-\hat{k}}(\frac{1}{2}q(q-1))^{\hat{k}}\mathbb{E}\left[\left(\int_0^T |\xi_t|^{q-2}d<\xi>_t\right)^{\hat{k}}\right].$$

We now bound

$$\mathbb{E}\left[\left(\int_0^T |\xi_t|^{q-2}d<\xi>_t\right)^{\hat{k}}\right] \leq \mathbb{E}\left[\left(\sup_{0\leq t\leq T}|\xi_t|\right)^{\hat{k}(q-2)}\left(\int_0^T d<\xi>_t\right)^{\hat{k}}\right]$$

$$\leq \mathbb{E}\left[\left(\sup_{0\leq t\leq T}|\xi_t|\right)^{\hat{k}q}\right]^{1-\frac{2}{q}}\mathbb{E}\left[d<\xi>_T^{\frac{\hat{k}q}{2}}\right]^{\frac{2}{q}}.$$

As a consequence, we obtain:

$$\mathbb{E}[\sup_{0\leq t\leq T}|\xi_t|^q]^{\hat{k}} \leq \frac{2-\hat{k}}{1-\hat{k}}(\frac{1}{2}q(q-1))^{\hat{k}}\mathbb{E}\left[\left(\sup_{0\leq t\leq T}|\xi_t|\right)^{\hat{k}q}\right]^{1-\frac{2}{q}}\mathbb{E}\left[d<\xi>_T^{\frac{\hat{k}q}{2}}\right]^{\frac{2}{q}},$$

Letting $p = q\hat{k}$ yields the claimed result, that is

$$\mathbb{E}[\sup_{0\le t\le T} |\xi_t|^p] \le C'_p \mathbb{E}\left[d < \xi >_T^{\frac{p}{2}}\right].$$

We proceed now to the proof of the left hand side inequality. We have,

$$\xi_t^2 =< \xi >_t + 2\int_0^t \xi_s d\xi_s.$$

Therefore, we get

$$\mathbb{E}\left[d < \xi >_T^{\frac{p}{2}}\right] \le A_p \left(\mathbb{E}\left[\sup_{0\le t\le T} |\xi_t|^p\right] + \mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t \xi_s d\xi_s\right| \frac{p}{2}\right]\right),$$

by using the previous argument, we now have

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t \xi_s d\xi_s\right|^{\frac{p}{2}}\right] &\le B_p \mathbb{E}\left[\left(\int_0^T \xi_s^2 d < \xi >_s\right)^{\frac{p}{4}}\right] \\
&\le B_p \mathbb{E}\left[\left(\left(\sup_{0\le t\le T} |\xi_s|\right)^{\frac{p}{2}} < \xi >_T^{\frac{p}{4}}\right)\right] \\
&\le B_p \mathbb{E}\left[\left(\sup_{0\le t\le T} |\xi_s|\right)^{p}\right]^{\frac{1}{2}} \mathbb{E}\left[< \xi >_T^{\frac{p}{2}}\right]^{\frac{1}{2}}.
\end{aligned}$$

As a conclusion, we obtained

$$\mathbb{E}\left[d < \xi >_T^{\frac{p}{2}}\right] \le A_p \left(\mathbb{E}\left[\sup_{0\le t\le T} |\xi_t|^p\right] + B_p \mathbb{E}\left(\left(\sup_{0\le t\le T} |\xi_s|\right)^{p}\right]^{\frac{1}{2}} \mathbb{E}\left[< \xi >_T^{\frac{'}{p}2}\right]^{\frac{1}{2}}\right).$$

This is an inequality of the form $x^2 \le A_p(y^2 + B_p xy)$,which easily implies $c_p x^2 \le y^2$, thanks to the inequality $2xy \le \frac{1}{\hat{\delta}}x^2 + \hat{\delta}y^2$, with a conveniently chosen $\hat{\delta}$. $\square$

Theorem 1.1.3 (*Hölder Inequality*). *Let* $1 \le p, q \le \infty$ *so that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f, g : \mathbb{R}^d \longrightarrow \mathbb{R}$ *are Lebesgue measurable. Then*

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

Proof:

There are several proofs of Hölder's inequality; the main idea in the following is Young's inequality for products.

If $\|f\|_p = 0$, then f is zero $\mu-$almost everywhere, and the product fg is zero $\mu-$almost everywhere, hence the left-hand side of Hölder's inequality is zero. The same is true if $\|g\|_q = 0$. Therefore, we may assume $\|f\|_p > 0$ and $\|g\|_q > 0$ in the following.

$\|f\|_p = \infty$ or $\|g\|_q = \infty$ then the right-hand side of Hölder's inequality is infinite. Therefore, we may assume that $\|f\|_p$ and $\|g\|_q$ are in $(0, \infty)$.

If $p = \infty$ and $q = 1$, then $|fg| \leq \|f\|_\infty |g|$ almost everywhere and Hölder's inequality follows from the monotonicity of the Lebesgue integral. Similarly for $p = 1$ and $q = \infty$ Therefore, we may also assume $p, q \in (1, \infty)$.

Dividing f and g by $\|f\|_p$ and $\|g\|_q$, respectively, we can assume that

$$\|f\|_p = \|g\|_q = 1.$$

We now use Young's inequality for products, which states that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

for all nonnegative a and b, where equality is achieved if and only if $a^p = b^q$. Hence

$$|f(s)g(s)| \leq \frac{|f(s)|^p}{p} + \frac{|g(s)|^q}{q}, \qquad s \in S$$

Integrating both sides gives

$$\|fg\|_1 \leq \frac{\|f\|_p^p}{p} + \frac{\|g\|_q^q}{q} = \frac{1}{P} + \frac{1}{q} = 1,$$

which proves the claim.

Under the assumptions $p \in (1, \infty)$ and $\|f\|_p = \|g\|_q$, equality holds if and only if $|f|^p = |g|^q$ almost everywhere. More generally, if $\|f\|_p$ and $\|g\|_q$ are in $(0, \infty)$ then Hölder's inequality becomes an equality if and only if there exists real numbers $\sigma_1, \sigma_2 > 0$, namely

$$\sigma_1 = \|g\|_q^q, \qquad \sigma_2 = \|f\|_p^p,$$

Such that

$$\sigma_1\|f\|^p = \sigma_2 = |g|^q \qquad (*) \qquad\qquad \mu - almost\, everywhere,$$

the case $\|f\|_p = 0$ corresponds to $\sigma_2 = 0$ in $(*)$. The case $\|g\|_q = 0$ corresponds to $\sigma_1 = 0$ in $(*)$.

□

We also need the following propositions.

Proposition 1.1.1 *Let $p \geq 1$, there is a constant R, depending on T and P such that $\forall s \in [0,T], \forall x \in \mathbb{R}^n$*

$$\mathbb{E}[\sup_{0\leq t\leq T} |\xi_{s,t}^x|^p] \leq R(1 + |x|^p). \tag{1.3}$$

Proof:

We will demonstrate in the case $n = 1$. We start with the case $p \geq 2$. We fix s and x, we note ξ_t in place of $\xi_{s,t}^x$ for case of writing. In the following R is a constant depending on p and T but which does not depend on (s,x). We have firstly,

$$\sup_{0\leq t\leq T} |\xi_t|^p \leq \sup_{t\in[0,s]} |\xi_t|^p + \sup_{t\in[s,T]} |\xi_t|^p \leq |x|^p + \sup_{t\in[s,T]} |\xi_t|^p,$$

it suffices to establish the inequality $\mathbb{E}[\sup_{t\in[0,T]} |\xi_t|^p] \leq R(1 + |x|^p)$. As we do not know a priori if this quantity is finite or not, we introduce the stopping time $\varrho_n = \inf t \in [0,T], |x|^p > n$ and we take $n > |x|$ such that $\varrho_n > s$. the inequality $(a+b+c)^p \leq 3^{p-1}(a^p + b^p + c^p)$ supplies estimates, for any $l \in [s,T]$,

$$|\xi_{l\wedge\varrho_n}|^p \leq 3^{p-1}\left(|x|^p + \sup_{s\leq l\leq t}\left|\int_s^{l\wedge\varrho_n} b(r,\xi_r)dr\right|^p + \sup_{s\leq l\leq t}\left|\int_s^{l\wedge\varrho_n} \sigma(r,\xi_r)dW_r\right|^p\right)$$

$$\leq 3^{p-1}\left(|x|^p + \left(\int_s^{t\wedge\varrho_n} |b(r,\xi_r)|dr\right)^p + \sup_{s\leq l\leq t}|\int_s^{l\wedge\varrho_n} \sigma(r,\xi_r)dW_r|^p\right),$$

the BDG's inequality (1.1.2) leads to:

$$\mathbb{E}[\sup_{s\leq l\leq t\wedge\varrho_n} |\xi_l|^p] \leq R\left(|x|^p + \mathbb{E}\left[\left(\int_s^{t\wedge\varrho_n} |b(r,\xi_r)|dr\right)^p\right] + \mathbb{E}\left[\left(\int_s^{l\wedge\varrho_n} |\sigma(r,\xi_r)|^2 dr\right)^{\frac{p}{2}}\right]\right),$$

using the Hölder's inequality (1.1.3) ($\frac{p}{2} \geq 1$), noting p^* the conjugate of p and q that of $\frac{p}{2}$,

$$\mathbb{E}[\sup_{s\leq l\leq t\wedge\varrho_n} |\xi_l|^p] \leq R\left(|x|^p + T^{\frac{p}{p^*}}\mathbb{E}\left[\int_s^{t\wedge\varrho_n} |b(r,\xi_r)|^p dr\right] + T^{\frac{p}{2q}}\mathbb{E}\left[\int_s^{l\wedge\varrho_n} |\sigma(r,\xi_r)|^p dr\right]\right).$$

Furthermore, as b and σ are linear increase, we have:

$$\begin{array}{rcl}\mathbb{E}\left[\int_s^{t\wedge\varrho_n} |b(r,\xi_r)|^p dr\right] & \leq & \Im^p\mathbb{E}\left[\int_s^{t\wedge\varrho_n}(1+|\xi_r|)^p dr\right] \\ & \leq & R\left(1+\mathbb{E}\left[\int_s^{t\wedge\varrho_n}|\xi_r|^p dr\right]\right)\end{array}.$$

Therefore

$$\mathbb{E}\left[\int_s^{t\wedge\varrho_n} |b(r,\xi_r)|^p dr\right] \leq R\left(1+\mathbb{E}\left[\int_s^t \sup_{s\leq l\leq r\leq\varrho_n} |\xi_l|^p dr\right]\right),$$

and the same inequality is valid for the term σ. As q result, we obtain:

$$\mathbb{E}[\sup_{s\leq l\leq t\wedge\varrho_n} |\xi_l|^p] \leq R\left(1+|x|^p + \int_s^t \mathbb{E}\left[\sup_{s\leq l\leq r\leq\varrho_n} |\xi_l|^p\right] dr\right),$$

where R does not depend on n. Gronwall's lemma (1.1.1) then gives for all n,

$$\mathbb{E}[\sup_{s\leq l\leq t\wedge\varrho_n} |\xi_l|^p] \leq R(1+|x|^p),$$

we fact tender n to n to infinity and apply Fatou's lemma to get:

$$\mathbb{E}[\sup_{s\leq l\leq T} |\xi_l|^p] \leq R(1+|x|^p)$$

which completed the proof in the case $p \geq 2$. If now $1 \leq p \leq 2$ then $2p \geq 2$ and Hölder's inequality (1.1.3)given

$$\mathbb{E}[\sup_{s\leq l\leq T} |\xi_l|^p] \leq \left(\mathbb{E}[\sup_{s\leq l\leq T} |\xi_l|^{2p}]\right)^{\frac{1}{2}} \leq R^{\frac{1}{2}}(1+|x|^{2p})^{\frac{1}{2}}.$$

This leads to

$$\mathbb{E}[\sup_{s\leq l\leq T} |\xi_l|^p] \leq R^{\frac{1}{2}}(1+|x|^p).$$

This last inequality completes the proof of this proposition.

Proposition 1.1.2 *Let $2 \leq p < \infty$. There exists a constant R such that, for any $(s,x),(s',x')$ belonging to $[0,T]\times\mathbb{R}^n$,*

$$\mathbb{E}[\sup_{s\leq t\leq T}|\xi^x_{s,t}-\xi^{x'}_{s',t}|^p] \leq R(|x-x'|^p + |s-s'|^{\frac{p}{2}}(1+|x'|^p)). \tag{1.4}$$

Proof:

We fix (s,x) and (s',x'). trivially,

$$|\xi^x_{s,t}-\xi^{x'}_{s',t}|^p \leq 2^{p-1}(|\xi^x_{s,t}-\xi^{x'}_{s,t}|^p + |\xi^x_{s,t}-\xi^{x'}_{s',t}|^p),$$

so that we show the inequality to each of the previous two terms. Start with the first $|\xi^x_{s,t}-\xi^{x'}_{s,t}|^p$. There is no need to take a stopping time because the previous proposition tells us that the expectation of the sup in t is finite. We have

$$\sup_{t\in[0,T]}|\xi^x_{s,t}-\xi^{x'}_{s,t}|^p \leq \sup_{t\in[0,s]}|\xi^x_{s,t}-\xi^{x'}_{s,t}|^p + \sup_{t\in[s,T]}|\xi^x_{s,t}-\xi^{x'}_{s,t}|^p,$$

so that

$$\sup_{t\in[0,T]}|\xi^x_{s,t}-\xi^{x'}_{s,t}|^p \leq |x-x'|^p + \sup_{t\in[s,T]}|\xi^x_{s,t}-\xi^{x'}_{s,t}|^p.$$

Therefore, we are only interested in the second member of this inequality, for all $l\in[s,T]$, we have

$$|\xi^x_{s,l}-\xi^{x'}_{s,l}|^p \leq 3^{p-1}\left(|x-x'|^p + \left(\int_s^t |b(r,\xi^x_{s,r})-b(r,\xi^{x'}_{s,r})|dr\right)^p + \sup_{l\in[s,t]}\left|\int_s^t \sigma(r,\xi^x_{s,r})-\sigma(r,\xi^{x'}_{s,r})dW_r\right|^p\right),$$

BDG's and Hölder's inequalities lead to the inequality, noting p^* the conjugate of P,

$$\mathbb{E}\left[\sup_{l\in[s,t]}|\xi^x_{s,l}-\xi^{x'}_{s,l}|^p\right] \leq R(|x-x'|^p + T^{\frac{p}{p^*}}\mathbb{E}\left[\int_s^t |b(r,\xi^x_{s,r})-b(r,\xi^{x'}_{s,r})|^p dr\right]$$

$$+\mathbb{E}[(\int_s^t |\sigma(r,\xi^x_{s,r})-\sigma(r,\xi^{x'}_{s,r})|^2 dr)^{\frac{p}{2}}]),$$

using again the Hölder's inequality, noting q the conjugate of $\frac{p}{2}$, we obtain

$$\mathbb{E}\left[\left(\int_s^t |\sigma(r,\xi^x_{s,r})-\sigma(r,\xi^{x'}_{s,r})|^2 dr\right)^{\frac{p}{2}}\right] \leq T^{\frac{p}{2q}}\mathbb{E}\left[\int_s^t |\sigma(r,\xi^x_{s,r})-\sigma(r,\xi^{x'}_{s,r})|^p dr\right],$$

b and σ are Lipschitz, the previous inequality gives

$$\mathbb{E}\left[\sup_{l\in[s,t]}|\xi^x_{s,l}-\xi^{x'}_{s,l}|^p\right]\leq R\left(|x-x'|^p+\int_s^t\mathbb{E}\left[\sup_{l\in[s,t]}|\xi^x_{s,l}-\xi^{x'}_{s,l}|^p\right]dr\right),$$

Gronwall's lemma (1.1.1) then gives-changing R

$$\mathbb{E}\left[\sup_{l\in[s,t]}|\xi^x_{s,l}-\xi^{x'}_{s,l}|^p\right]\leq R|x-x'|^p.$$

It remains to study the term $\mathbb{E}\left[\sup_{t\in[0,T]}|\xi^{x'}_{s,t}-\xi^{x'}_{s',t}|^p\right]$. We assume without loss of generality that $s\leq s'$ and cutting into three parts,

$$\sup_{t\in[0,T]}|\xi^{x'}_{s,t}-\xi^{x'}_{s',t}|^p\leq\sup_{t\in[0,s]}|\xi^{x'}_{s,t}-\xi^{x'}_{s',t}|^p+\sup_{t\in[s,s']}|\xi^{x'}_{s,t}-\xi^{x'}_{s',t}|^p+\sup_{t\in[s',T]}|\xi^{x'}_{s,t}-\xi^{x'}_{s',t}|^p.$$

from which we deduce that

$$\sup_{t\in[0,T]}|\xi^{x'}_{s,t}-\xi^{x'}_{s',t}|^p\leq\sup_{t\in[s,s']}|\xi^{x'}_{s,t}-x'|^p+\sup_{t\in[s',T]}|\xi^{x'}_{s,t}-\xi^{x'}_{s',t}|^p,$$

For the first term of the right side of the previous inequality, we have

$$\mathbb{E}\left[\sup_{t\in[s,s']}|\xi^{x'}_{s,t}-x'|^p\right]\leq 2^{p-1}\left(\mathbb{E}\left[\left(\int_s^{s'}|b(r,\xi^{x'}_{s,r})|dr\right)^p\right]+\mathbb{E}\left[\sup_{t\in[s,s']}\left|\int_s^t\sigma(r,\xi^{x'}_{s,r})dW_r\right|^p\right]\right),$$

the Hölder's inequality (1.1.3) and the mark (1.3) give, using the linear increase of b,

$$\mathbb{E}\left[\left(\int_s^{s'}|b(r,\xi^{x'}_{s,r})|dr\right)^p\right]\leq(s-s')^p\mathbb{E}\left[\sup_{l\in[s,s']}|b(l,\xi^{x'}_{s,l})|^p\right]\leq RT^{\frac{p}{2}}|s-s'|^{\frac{p}{2}}(1+|x'|^p).$$

On the other hand, inequality of BDG (1.1.2) gives

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[s,s']}|\int_s^t\sigma(r,\xi^{x'}_{s,r})dW_r|^p\right] &\leq \mathbb{E}\left[\left(\int_s^t|\sigma(r,\xi^{x'}_{s,r})dr|^2\right)^{\frac{p}{2}}\right]\\ &\leq (s-s')^{\frac{p}{2}}\mathbb{E}\left[\sup_{l\in[s,s']}|\sigma(l,\xi^{x'}_{s,l})|^p\right]\end{aligned},$$

and because of the increase of σ and the estimate (1), we obtain

$$\mathbb{E}\left[\sup_{t\in[s,s']}|\int_s^t \sigma(r,\xi_{s,r}^{x'})dW_r|^p\right] \leq R|s-s'|^p(1+|x'|^p).$$

Finally

$$\mathbb{E}\left[\sup_{t\in[s,s']}|\xi_{s,t}^{x'}-\xi_{s',t}^{x'}|^p\right] \leq R|s-s'|^{\frac{p}{2}}(1+|x'|^p),$$

study to finish the term $\mathbb{E}\left[\sup_{t\in[s',T]}|\xi_{s,t}^{x'}-\xi_{s',t}^{x'}|^p\right]$. Note that, for $t\in[s',T]$,

$$\begin{array}{rcl}\xi_{s,t}^{x'} & = & \xi_{s',s}^{x'}+\int_{s'}^t b(r,\xi_{s,r}^{x'})dr+\int_{s'}^t \sigma(r,\xi_{s,r}^{x'})dW_r \\ & = & x'+\int_{s'}^t b(r,\xi_{s',r}^{x'})dr+\int_{s'}^t \sigma(r,\xi_{s',r}^{x'})dW_r.\end{array}$$

We have therefore, for any $l\in[s',t]$

$$|\xi_{s,t}^{x'}-\xi_{s',t}^{x'}|^p \leq 3^{p-1}\left(|\xi_{s,t}^{x'}-x'|^p+\left(\int_{s'}^t |b(r,\xi_{s,r}^{x'})-b(r,\xi_{s',r}^{x'})|dr\right)^p+\sup_{l\in[s',t]}\left|\int_{s'}^t \sigma(r,\xi_{s,r}^{x'})-\sigma(r,\xi_{s',r}^{x'})dW_r\right|^p\right),$$

using BDG's and Hölder's inequalities, and the bound (3), and the fact that b and σ are Lipschitz,

$$\begin{array}{rcl}\mathbb{E}\left[\sup_{l\in[s',t]}|\xi_{s,l}^{x'}-\xi_{s',l}^{x'}|^p\right] & \leq & R\left(|s-s'|^{\frac{p}{2}}(1+|x'|^p)+\mathbb{E}\left[\int_{s'}^t|\xi_{s,r}^{x'}-\xi_{s',r}^{x'}|^p dr\right]\right) \\ & \leq & R\left(|s-s'|^{\frac{p}{2}}(1+|x'|^p)+\mathbb{E}\left[\int_{s'}^t \sup_{l\in[s',r]}|\xi_{s,l}^{x'}-\xi_{s',l}^{x'}|^p dr\right]\right)\end{array},$$

Gronwall's lemma (1.1.1) applied to $r\to\sup_{l\in[s',r]}|\xi_{s,l}^{x'}-\xi_{s',l}^{x'}|^p$ then gives

$$\mathbb{E}\left[\sup_{l\in[s',r]}|\xi_{s,l}^{x'}-\xi_{s',l}^{x'}|^p\right] \leq R|s-s'|^{\frac{p}{2}}(1+|x'|^p).$$

which completed the proof.

1.2 Stochastic calculus on the local martingale

In this section we give some results on the theory of continuous time of local martingales which is a type of stochastic processes, satisfying the localized version of the martingale, it plays an important role in the theory of stochastic calculus.

We consider a complete probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ where $(\mathcal{F}_t, t \geq 0)$ a non-decreasing sub tribes family of the filtration $\mathbb{F}$. which satisfies the usual conditions and $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be a topological space with $\mathcal{B}(\mathbb{R}^d)$ is the topological $\sigma-$field. A mapping ξ from Ω into $\mathbb{R}^d$ which is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)-$measurable is called an $\mathbb{R}^d-$valued random variable or $d-$dimensional random variable.

Definition 1.2.1 *A $d-$dimensional Stochastic process $(\xi_t, t \geq 0)$ is a collection of $d-$dimensional random variables indexed by time t i.e $\xi_t : \Omega \to \mathbb{R}^d$. We can also fixe $w \in \omega$ and consider the map $\xi_t(w)$ on $[0, T]$. These maps are called the trajectories or sample paths of the process.*

Definition 1.2.2 *We say that $d-$dimensional Stochastic process ξ_t is adapted to $\mathcal{F}_t$ if ξ_t is $\mathcal{F}_t-$measurable for every t.*

In general, a process ξ_t is measurable if the mapping $(t, w) \in [0, T] \times \Omega \longrightarrow \xi_t \in \mathbb{R}^d$ is measurable from $\mathcal{B}([0, T]) \otimes \mathcal{F}_t$ into $(\mathbb{R}^d, \mathcal{R}^d)$.

Definition 1.2.3 *Let ξ_t and ξ'_t be tow stochastic processes, we say that ξ_t and ξ'_t are modifications of each other if and only if:*

$$\mathbb{P}(\{w \in \Omega : \xi_t(w) - \xi'_t(w)\}) = 1 \;\; \forall t \geq 0,$$

i.e $\forall t \geq 0$, there is a negligible part $\mathcal{N}_t$ such as $\forall w \in \mathcal{N}$, we have $\xi_t(w) - \xi'_t(w)$.

Definition 1.2.4 *Let ξ_t and ξ'_t be tow stochastic processes, we say that ξ_t and ξ'_t are indistinguishable if and only if:*

$$\mathbb{P}(\{w \in \Omega : \xi_t(w) - \xi'_t(w) \; \forall t \geq 0\}) = 1,$$

i.e there is a negligible part $\mathcal{N}_t$ such as $\forall w \in \mathcal{N}$ and $\forall t \geq 0$, we have $\xi_t(w) - \xi'_t(w)$.

Hence, there is an equivalence relation:

$$\xi \, \mathcal{R} \, \xi' \Leftrightarrow \xi \, and \, \xi' \; are \; indistinguishable.$$

Definition 1.2.5 ***(Martingales)***: *A real-valued process ξ is called a martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if:*

1. $\mathbb{E}[|\xi_t|] < \infty$ *for all* $t \in [0, T]$.

2. $\mathbb{E}(\xi_t/\mathcal{F}_s) = \xi_s$ *for all* $s \leq t$.

If property (2) *holds with* $\geq$ *(resp.* $\leq$*) instead of* $=$*, then* ξ *is called a submartingale (resp. supermartingale).*
An $d-$*dimensional Stochastic process* ξ *is an* $\mathcal{F}_t-$*Martingale if each of its components* ξ_t^i,
$t \geq 0.\, i = 1...d$ *is an* $\mathcal{F}_t-$*Martingale.*

Definition 1.2.6 ***(Local Martingale)*** *A process* ξ *is a local martingale if there is an increasing sequence of stopping times* T_n *with* $T_n \nearrow \infty$ *a.s.* $T_n < T$ *on* $\{T > 0\}$ *, furthermore* $\xi_{t\wedge T_n}$ *is a martingale uniformly integrable.*

Definition 1.2.7 ***(Semi-Martingale)*** *We say that the process* Z *is a continuous semi-martingale if it can be written as*

$$Z = \xi + A,$$

where ξ *is a continuous local martingale and* A *is a finite variation process, null at* 0*, continuous and adapted.*

Proposition 1.2.1 *[47]*
If ξ *is a local martingale null at* 0*, and if*

$$T_n = \{t \geq 0; |\xi_t| = 0\},$$

Then ξ^{T_N} *is a bounded martingale.*

Quadratic variation and covariation:

Theorem 1.2.1 *[39]*
Let ξ *be a continuous local martingale. We define its quadratic variation* $< \xi, \xi >$ *as an increasing*

continuous process null at 0 *such that* $\xi^2 - < \xi, \xi >$ *is a local martingale.*
Moreover

$$\sup_{s \leq t} |Q_s^{\triangle_n}(\xi) - < \xi, \xi >_s| \longrightarrow 0$$

in probability when $n \longrightarrow +\infty$.

Corollary 1.2.1 *[39]*
Let ξ, ζ *be tow continuous local martingales. We define quadratic covariation* $< \xi, \zeta >$ *as a unique continuous process with finite variation null at* 0 *such that* $\xi\zeta - < \xi, \zeta >$ *is a continuous local martingale. Moreover,* $\forall t$

$$\sup_{s \leq t} |\hat{Q}_s^{\triangle_n}(\xi, \zeta) - < \xi, \zeta >_s| \longrightarrow 0$$

in probability when $n \longrightarrow +\infty$,
where

$$\hat{Q}_s^{\triangle_n}(\xi, \zeta) = \sum_{t_i \in \triangle_n} (\xi_{t_{i+1}}^s - \xi_{t_i}^s)(\zeta_{t_{i+1}}^s - \zeta_{t_i}^s),$$

and $(\triangle_n)_n$ *is a sequence of subdivisions of* $[0, t]$ *such that* $| \triangle_n | \longrightarrow 0$.

Proposition 1.2.2 *[39]*
A continuous semi-martingale Z *with the decomposition* $Z = \xi + A$ *admits a quadratic variation noted* $< Z, Z >$ *such that*

$$< Z, Z > = < \xi, \xi > .$$

Stochastic integral with respect to a local martingale:

Let ξ be a continuous local martingale and $L^2_{loc}(\xi)$ a space of the measurable progressive processes H such that for all $t \geq 0$

$$\int_0^t H^2 d < \xi >_s < \infty.$$

Definition 1.2.8 *The process $H.\xi$ is called a stochastic integral of H by the local martingale ξ, where $H.\xi = \int_0^t H_s d\xi_s$.*

Proposition 1.2.3 *[39]*

For all $H \in L^2_{loc}(\xi)$, there is a unique continuous local martingale null at 0, noted $H.\xi$, such that for any continuous local martingale ζ

$$< H.\xi, \zeta >= H. < \xi, \zeta > .$$

Proposition 1.2.4 *[24] (**Kunita Watanabe inequality**)*

Let ξ, ζ be tow continuous local martingales and H, K tow measurable processes, then

$$\int_0^t |H_s||K_s||d < \xi, \zeta >_s | \leq \left(\int_0^t H_s^2 d < \xi, \xi >_s \right)^{1/2} \left(\int_0^t K_s^2 d < \zeta, \zeta >_s \right)^{1/2} .$$

Itô's Formula:

Theorem 1.2.2 *[24]*

Let ξ be a continuous local martingale and f a function of $\mathcal{C}^2(\mathbb{R})$. Then, we have for all $t \geq 0$

$$f(\xi_t) = f(\xi_0) + \int_0^t f'(\xi_s) d\xi_s + \frac{1}{2} \int_0^t f''(\xi_s) d < \xi >_s .$$

Differential notation:

$$df(\xi_t) = f'(\xi_s) d\xi_s + \frac{1}{2} f''(\xi_s) d < \xi >_s .$$

General Itô's Formula:

Theorem 1.2.3 *[24]*

Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ a function of $\mathcal{C}^2$ and ζ is a continuous semi-martingale with values in $\mathbb{R}^d$. Then , $F(\xi)$ is a semi-martingale and

$$F(\xi_t) = F(\xi_0) + \sum_{i=1}^{d} \int_0^t D_i F(\xi_s) d\xi_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t D_{ij} F(\xi_s) d < \xi^i, \xi^j >_s,$$

where D_iF and $D_{ij}F$ are the partial derivatives.

Differential notation:

$$F(\xi_t) = \sum_{i=1}^{d} D_iF(\xi_t)d\xi_t^i + \frac{1}{2}\sum_{i,j=1}^{d} D_{ij}F(\xi_t)d < \xi^i, \xi^j >_t .$$

1.3 Stochastic differential equations

Stochastic differential equations are considered as an exceptional effective structure to analyze and construct the stochastic models which play an important role to describe a various phenomena such as unstable stochastic prices or physical, biological and engineering systems. Moreover, stochastic differential equation links between probability theory and developed fields of ordinary and partial differential equations also that it's a differential equation in which one or more of its terms is a stochastic process. Therefore a resulting solution is also a stochastic process. On the other hand, it contains a white noise variable which is calculated as the derivative of Brownian motion or Wiener process.
The famous definition of stochastic differential equation was given by K.It$\hat{o}$: It has the form :

$$d\zeta_t = X_0(t,\zeta_t)dt + \sum_{k=1}^{m} X_k(t,\zeta_t)dB_t^k, \tag{1.5}$$

where $B_t^k,\ k = 1...m\ \ and\ \ t \in [0,T]$ be an multi-dimensional Brownian motion defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

1.3.1 Existence and uniqueness of the Solution of stochastic differential equation

The equation (1.5) has a unique solution ζ_t with valued in $\mathbb{R}^d$, if it is $\mathcal{F}_{s,t}-$adapted for each $s \leq t$ and satisfies

$$\zeta_t = x + \sum_{k=0}^{m} \int_s^t X_k(r,\zeta_r)dB_r^k, \tag{1.6}$$

where $\zeta_s = x$ is the initial condition and $\mathcal{F}_{s,t}$ is the least complete $\sigma-$field for all $B_v - B_u;\ s \leq u \leq v \leq t$ are measurable. Knowing that the existence and uniqueness of the solution is verified that the coefficients $X_0, ..., X_m$ are Lipschitz continuous i.e there is a positive constant M such that

$$|X_k(t,x) - X_k(t,y)| \leq M|x-y| \qquad k = 0, ..., m.$$

Proof:

The construction of the solution starting from x at time s was shown by the method of successive approximation. Define a sequence of $\mathcal{F}_{s,t}-$adapted continuous processes by induction:

$$\begin{aligned} \zeta_t^0 &= x \\ \zeta_t^n &= x + \sum_{k=0}^{m} \int_s^t X_k(r, \zeta_r^{n-1}) dB_r^k, \qquad n \geq 1. \end{aligned}$$

Then it holds

$$\zeta_t^{n+1} - \zeta_t^n = \sum_{k=0}^{m} \int_s^t \left(X_k(r, \zeta_r^n) - X_k(r, \zeta_r^{n-1}) \right) dB_r^k.$$

Thus for $p \geq 2$, we have

$$\mathbb{E}\left[\sup_{s \leq u \leq t} \left| \zeta_t^{n+1} - \zeta_t^n \right|^p \right] \leq (m+1)^p \sum_{k=0}^{m} \mathbb{E}\left[\sup_{s \leq u \leq t} \left| \int_s^t \left(X_k(r, \zeta_r^n) - X_k(r, \zeta_r^{n-1}) \right) dB_r^k \right|^p \right].$$

Using Doob's inequality and Burkholder's inequality, each term corresponding to $k \geq 1$ is dominated by

$$\begin{aligned} q^p \mathbb{E}\left[\left| \int_s^t \left(X_k(r, \zeta_r^n) - X_k(r, \zeta_r^{n-1}) \right) dB_r^k \right|^p \right] &\leq q^p C^p |t-s|^{\frac{p}{2}-1} \mathbb{E}\left[\int_s^t \left| X_k(r, \zeta_r^n) - X_k(r, \zeta_r^{n-1}) \right|^p dr \right] \\ &\leq q^p C^p |t-s|^{\frac{p}{2}-1} M^p \mathbb{E}\left[\int_s^t \left| \zeta_r^n - \zeta_r^{n-1} \right|^p dr \right]. \end{aligned}$$

The term corresponding to $k = 0$ is dominated by

$$q^p C^p |t-s|^{\frac{p}{q}} M^p \mathbb{E}\left[\int_s^t \left| \zeta_r^n - \zeta_r^{n-1} \right|^p dr \right].$$

Therefore, we obtain

$$\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\zeta_t^{n+1}-\zeta_t^n\right|^p\right]\leq c\mathbb{E}\left[\left|\int_s^t \zeta_r^n-\zeta_r^{n-1}dr\right|^p\right].$$

Denote

$$\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\zeta_t^{n+1}-\zeta_t^n\right|^p\right]=\phi_t$$

Then the above implies $\phi_t^n \leq c\int_s^t \phi_r^{n-1}dr$. By iteration, we get $\phi_t^n \leq \frac{c^n}{n!}T^n\phi_t^0$. Hence

$$\sum_{n=0}^{\infty}\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\zeta_t^{n+1}-\zeta_t^n\right|^p\right]^{\frac{1}{p}}\leq\sum_{n=0}^{\infty}\left[\frac{c^n}{n!}T^n\phi_t^0\right]^{\frac{1}{p}}<+\infty,$$

since $\phi_t^0<\infty$. Thus, (ζ_t^n) converges uniformly in $[s,t]$ a.s and in L^p-norm. Denote the limit as ζ_t, it's $\mathcal{F}_{s,t}-$adapted continuous process.

Moreover, $\int_s^t X_k(r,\zeta_r^n)dB_r^k$ converges to $\int_s^t X_k(r,\zeta_r)dB_r^k$ in L^p-norm. The convergence is valid for $k=0$ since the quadratic variation of $\int_s^t X_k(r,\zeta_r^n)-X_k(r,\zeta_r)dB_r^k$ converges to 0 in L^p-norm. Therefore ζ_t is a solution of the equation (1.5).

It remains to demonstrate the uniqueness of the solution. Suppose that ζ_t and $\widetilde{\zeta_t}$ are the solutions of (1.5) relative to the same brownian motion and initial condition x on the same probability space $(\Omega,\mathbb{F},\mathbb{P})$. Define $\tau_n=\{t>0,|\zeta_t|\geq n \ or\ |\widetilde{\zeta_t}|\geq n\}$. Then, it holds

$$\zeta_t^{\tau_n}-\widetilde{\zeta}_t^{\tau_n}=\sum_{k=0}^{n}\int_s^{t\wedge\tau_n}\left(X_k(r,\zeta_r^{\tau_n})-X_k(r,\widetilde{\zeta}_r^{\tau_n})\right)dB_r^k.$$

by similar calculation as above, we get

$$\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\zeta_t^{\tau_n}-\widetilde{\zeta}_t^{\tau_n}\right|^p\right]\leq c\mathbb{E}\left[\int_s^{t\wedge\tau_n}\left|\zeta_r^{\tau_n}-\widetilde{\zeta}_r^{\tau_n}\right|^p dr\right].$$

Denote $\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\zeta_t^{\tau_n}-\widetilde{\zeta}_t^{\tau_n}\right|^p\right]=\hat{\phi}_t$, where n is fixed. Therefore $\hat{\phi}_t\leq c\int_s^t\hat{\phi}_r dr$.

By Gronwall's lemma , we obtain $\hat{\phi}_t\equiv 0$. this shows $\zeta_r^{\tau_n}=\widetilde{\zeta}_r^{\tau_n}$. Since $\tau_n\nearrow\infty$, we have $\zeta_r=\widetilde{\zeta}_r$.

□

1.3.2 Applications Of Stochastic Differential Equations in Finance

Stochastic differential equation have many applications in different fields of sciences and technology, the first application is discovered in (1930) known by Ornstein-Uhlenbeck model of Brownian motion and its resulting solution is a Ornstein-Uhlenbeck process, until (1951) appeared another model by K.It$\widehat{o}$, after that this theory is widely used in different fields such as: physics, chemistry, biology, finance, economics, etc.
Financial field is the most domain which uses the theory of stochastic differential equation to solve many problems such as: credit risk, immunization risk of investment portfolios, the variation of the interest rates and exchange rates, the stock prices. Merton [39] is the most researcher market used the stochastic differential equation in finance for example: Black-Sholes-Merton model (1973) for calculate the value of a call or put option and Structural model (1974) for modeling a credit risk .

Stochastic modeling in asset prices

There are several stochastic differential equations which describe many problems in finance, one of the important problem is the specification of the stochastic process governing the behaviour of an asset, here the term asset describe the financial object whose value is known at present but can be changed in the future for example : shares in firm, commodities such as gold or oil. among this equations we have [56]:

1. **Geometric Brownian Motion**:
   In several studies, the market researchers assumed that the price of an asset followed a Gaussian process described by Itô's differential equation

$$dS_t = \lambda dt + \sigma dB_t \ \ \forall t \geq 0, \tag{1.7}$$

   where S_t is the price of the asset at time t, λ and σ are positives constants and B_t is one-dimensional brownian motion. Furthermore, the initial price is the constant S_0, such as

$$S_t = S_0 + \lambda t + \sigma B_t,$$

which is normally distributed with mean $S_0 + \lambda t$ and variance $\sigma^2 t$.

The price may be negative but this violates the condition of limited liability.

To overcome this weakness, many market researchers suggested the idea of modeling by geometric brownian motion.

2. **Mean reverting process**:

   The following equation is considered as a useful modeling asset prices

$$dS_t = \lambda(\mu - S_t)dt + \sigma S_t dB_t \ \ \forall t \geq 0. \tag{1.8}$$

   This is often used to model interest rate dynamics. If the drift $\lambda(\mu - S_t)$ is negative, this makes dS_t probably be negative and the price will decrease. In the other side, when the price S_t falls below μ, $\lambda(\mu - S_t)$ will be positive, this makes dS_t probably be positive and the price S_t will increase.

3. **Mean reverting Ornstein-Uhlenbeck process**:

$$dS_t = \lambda(\mu - S_t)dt + \sigma dB_t, \tag{1.9}$$

   This model is close to the previous model. Here, the diffusion term does not depend on the price S_t which may become negative.

4. **Square root process**:

   The following model is close to the geometric brownian motion model

$$dS_t = \lambda S_t dt + \sigma\sqrt{S_t}dB_t. \tag{1.10}$$

   In this model the price never becomes negative.

5. **Mean reverting Square root process**:

   The following model is the combining square root idea and mean reverting

$$dS_t = \lambda(\mu - S_t)dt + \sigma\sqrt{S_t}dB_t. \tag{1.11}$$

   Again this process will never be negative.

6. **Theta process**:

   Another useful model is theta process defined by the following stochastic differential equation

$$dS_t = \lambda S_t dt + \sigma S_t^{\theta} dB_t, \tag{1.12}$$

   where θ is constant no less than $0,5$. If $\theta = 1$ we observe that the equation (1.12) becomes the geometric brownian motion process and if $\theta = 0,5$ it becomes the square root process, in this case the price will never be negative. When $\theta > 0$, the price will remain positive.

7. **Mean reverting Theta process**:

   the following model is the combining idea of theta process and mean reverting process

$$dS_t = \lambda(\mu - S_t)dt + \sigma S_t^{\theta} dB_t, \tag{1.13}$$

   where $\theta > 0,5$. If $\theta = 1$ we get the mean reverting process. When $\theta = 0,5$ the equation (1.13) becomes the mean reverting square root process. In this model, the price will never become negative.

8. **Stochastic volatility**:

   In all previous models, the drift and the diffusion are considered as constants, the following model allows us to consider the volatility as a random given S_t:

$$dS_t = \lambda S_t dt + \sigma_t S_t dB_t, \tag{1.14}$$

   where λ is a positive constant as before, while the volatility σ_t is supposed to change over time. Precisely, σ_t is supposed to change according to Ornstein-Uhlenbeck process

$$d\sigma_t = -\beta \sigma_t dt + \delta d\widetilde{B}_t, \tag{1.15}$$

   with initial value σ_0, where β and δ are positives constants and $\widetilde{B}_t$ is another brownian motion independent of B_t.

Black-Sholes Model

After many efforts of market researchers to analyze the problems of options pricing . Fischer Black , Myron Sholes and Robert C. Merton developed in 1973 a new formula which was considered as one of the best ways of modeling fair prices of options and still used today by small town branches of brokerage firms such that it's used to estimate a value of price of European options depending on six variables; the current stock prices, expected dividends, the option's exercise price (Strike price), expected interest rates, expected volatility and the time to the option's expiry. This model is used only for pricing European options which simply means that the option can only be exercised at the maturity date such that a stock call option is a security which gives the owner the right to buy an underlying asset at a fixed strike price at the expiration time T and a put option is the right to sell an underlying asset at the strike price at the expiration time. The Black-Scholes formula (1973) estimates the fair value cost of a put or call options on stocks. It assumes the underlying stock price follows a geometric Brownian motion with constant volatility, the values of call price and put price are:

$$C = SN(d_1) - Ke^{-rt}N(d_2).$$

$$P = Ke^{-rt}N(-d_2) - SN(-d_1),$$

where
S : the price of the underlying stock.
K : the strike price.
r : the continuously compounded risk free interest rate.
t :the time in years until the expiration of the option.
N : the standard normal cumulative distribution function.
And

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}.$$

$$d_2 = d_1 - \sigma\sqrt{t},$$

where
σ: the implied volatility for the underlying stock.

One-default Model

Credit risk is inability to pay the debts of the borrowers in the due date of the contract which causes the financial losing to the lenders that's way it is important to cover this risk depending on some models among them; the structural model and intensity model [11]. The first model was initiated in (1974) by Merton , it is based on the modeling of the evolution of the company balance sheet. In this context, the default occurs if the debt issuer is unable to honor its obligations such as the default is considered as a predictable event and it is modeled as the first passage of a stochastic process by barrier. In contrast, in the intensity model the default is considered as an unpredictable event and it is modeled as the first jump of the time of homogeneous poisson process. The two approaches have been studied before the default time i.e it is conditioned to events where the default did not occurred but it is essential to analyze the impact of the default (after the default time). Janblanc Monique in collaboration with Shiqi Song [25] proposed a new model which describe the evolution of the market after a default time and it is based on the conditional law of a random time with respect to a reference filtration i.e there is a random time τ combined with a filtration $\mathbb{F}$ under $\mathbb{Q}$ a probability measure on an extension of $(\Omega, \mathbb{F}, \mathbb{P})$, such that the family of conditional expectations $dX_t^u = \mathbb{Q}[\tau \leq u/\mathcal{F}_t]$, $0 < u, t < \infty$, verify the following stochastic differential equation:

$$(\natural_u) = \begin{cases} dX_{u,t}^x = X_{u,t}^x \left(-\frac{e^{-\Lambda_t}}{1-Z_t} N_t + f(X_t - (1 - Z_t))dY_t\right), \, t \in [u, \infty[, \\ X_{u,u}^x = x, \end{cases}$$

This setting is called natural equation ($\natural-$equation) where x is the initial condition that is $\mathcal{F}_u-$measurable random variable, Λ is a continuous increasing process null at the origin, N is a continuous positive local martingale such that $0 < Z_t = N_t e^{-\Lambda_t} < 1, \ t > 0, \ Y$ is a continuous local martingale and f is a given function on $\mathbb{R}$ null at the origin satisfying lipschitz condition. We know that there exists a unique solution of the above equation [25] such that:

$$X_{u,t}^x = x + \int_u^t X_s^x \left(-\frac{e^{-\Lambda_s}}{1 - Z_s}\right) dN_s + \int_u^t X_s^x f(X_s - (1 - Z_s))dY_s, \, s \in [u, t].$$

Chapter 2

Stochastic flows and credit risk modeling

The second chapter presents the notion of stochastic flows and its properties. In addition to this, it contains some background on the credit risk modeling.

2.1 Stochastic flows of diffeomorphisms

2.1.1 Definition of the flow and elementary properties

In mathematics, The notion of flow is based on the study of ordinary differential equations which is founded in the study of continuous dynamic system. More formally, a flow is associated with the notion of vectors field i.e to a map f, at a point x of an open $\mathcal{D}$ of a Banach space E, the field defines an ordinary differential equation of the type:

$$\zeta_t(x)' = f(\zeta_t(x)). \tag{2.1}$$

If the function f is locally Lipschitzian, for each point $x \in \mathcal{D}$, there is a maximal unique solution $\zeta_t(x)$ of the differential equation according to the initial condition $\zeta_0(x) = x_0$. Seen as a function of two variables t *and* x which is called the flow of vectors field f.

Proposition 2.1.1 *[48] The map $\zeta_t(x)$ defined from Ω_0 into $\mathbb{R}^d$ is continuous, where Ω_0 is $\mathbb{R}\times\mathcal{D}$. For any $(t, x) \in \Omega_0$, there is a neighborhood $\mathcal{V}$ of x such as the map $\zeta_t(.)$ is defined and continuous on $\mathcal{V}$.*

Proposition 2.1.2 *[48] We suppose $\mathcal{D} = \mathbb{R}^d$, if f is C^1 and globally Lipschitzian, then the flow is defined on all $\mathbb{R} \times \mathbb{R}^d$ and the map*

$$\mathbb{R} \longrightarrow Diff(\mathbb{R}^d)$$

$$t \longrightarrow \zeta_t(.)$$

is homeomorphism from $\mathbb{R}^+$ into $Diff(\mathbb{R}^d)$, where $Diff(\mathbb{R}^d)$ is the set of diffeomorphisms maps defined from $\mathbb{R}^d$ into itself.

Proof:

For all $(s, t) \in \mathbb{R}^2$, we have $\zeta_s \circ \zeta_t = \zeta_t \circ \zeta_s = \zeta_{s+t}$, by the uniqueness of the solution of the above ordinary differential equation (Cauchy-Lipschitz theorem). Consequently, ζ_t is bijective and its inverse $(\zeta_t)^{-1}$ is continuous. By anticipating on the following paragraph, we also have ζ_t is continuously differentiable of inverse continuously differentiable. □

Differentiability with respect to the initial data:

We are interested with the dependence of the solution at the initial condition, more precisely on the differentiability of the flow $\zeta_t(x)$ with respect to x. Assuming provisionally the differentiability is assured and the derivatives with respect to t and x commute. By deriving the differential equation satisfied by ζ_t, we get

$$\frac{\partial}{\partial x_0}\frac{\partial}{\partial t}\zeta_t(x_0) = \frac{\partial}{\partial x_0}(f(\zeta_t(x_0))) = \frac{\partial f}{\partial x}(\zeta_t(x_0))\frac{\partial}{\partial x_0}\zeta_t(x_0),$$

it means also

$$\frac{d}{dt}(\frac{\partial \zeta_t}{\partial x_0}(x_0)) = \frac{\partial f}{\partial x}(\zeta_t(x_0))\frac{\partial \zeta_t}{\partial x_0}(x_0).$$

We denote that $\Psi_t = \frac{\partial \zeta_t}{\partial x_0}(x_0)$ which is a solution of the following matrix differential equation:

$$\begin{cases} \Psi_t' &= \frac{\partial f}{\partial x}(\zeta_t(x_0)).\Psi_t \\ \Psi_0 &= I_d \end{cases}$$

that is a variational equation associated with (2.1). it is a linear system of the type

$$\Phi_t' = A(t, x_0)\Phi_t,$$

where $A(t, x_0) = \frac{\partial f}{\partial x}(\zeta_t(x_0))$ does not only depend on the time t but also on the parameter x_0. Therefore, the solution exhibits itself depend on x_0, but not at t_0 (only $t - t_0$ counts), for this reason we denote it $S(t, x_0)$. Then, from the previous discussion, we have:

$$S(t, x_0) = \frac{\partial \zeta_t}{\partial x_0}((x_0)).$$

Now, We will show this equality in the proof of the following theorem.

Theorem 2.1.1 *[48] Let f be a continuous function defined from $\mathcal{D}$ into $\mathbb{R}^d$ and locally Lipschitzian, such that $\frac{\partial f}{\partial x}(x)$ exist and continuous on $\mathcal{D}$. Then, the flow of (2.1) is continuously differentiable map with respect to x and its derivative $\Psi_{x_0} = \frac{\partial \zeta_t}{\partial x_0}((x_0))$ verify the variational equation associated with (2.1):*

$$\begin{cases} \Psi_t'(x_0) & = \frac{\partial f}{\partial x}(\zeta_t(x_0)).\Psi_t(x_0) \\ \Psi_0'(x_0) & = I_d \end{cases}$$

Proof: We first show the existence of $\frac{\partial \zeta_t}{\partial x_0}((x_0))$ for all (t, x_0): to do this, it's enough to establish, for (t, x_0) fixed, that exists a function $\varpi(.)$ defined from $\mathbb{R}$ into itself and a radius $r > 0$ such that:

$$\forall \Delta x_0 \in B_r(0), \|\zeta_t(x_0 + \Delta x_0) - \zeta_t(x_0) - S(t, x_0)\Delta x_0\| = \|\Delta x_0\|\varpi(\|\Delta x_0\|), \qquad (2.2)$$

with

$$\lim_{\|\Delta x_0\| \to 0} \varpi(\|\Delta x_0\|) = 0.$$

For $s \in [0, t]$, we define $z(s) = \zeta_s(x_0) + S(s, x_0)\Delta x_0$: $z(.)$ is not a solution of the differential system (2.1), but can be seen as an approximate solution:

$$\begin{cases} z_t'(s) & = f(z(s)) - \delta(s) \\ z_0(0) & = x_0 + \Delta x_0 \end{cases}$$

where

$$\delta(s) = f(\zeta_s(x_0) + S(s, x_0)\Delta x_0) - f(\zeta_s(x_0)) - \frac{\partial f}{\partial x}(\zeta_t(x_0))S(s, x_0)\Delta x_0.$$

The solution $S(s, x_0)$ is continuous in the variable $s \in [0, t]$, it is bounded by a constant $\bar{M} > 0$, so that if $r = \rho/\bar{M}$, the function $z(s)$ does not come out of the compact cylinder :

$$K = \{x \in \mathbb{R}^d; \exists s \in [0, t], \|x - \zeta_s(x_0)\| \leq \rho\},$$

which is itself contained in Ω_0 for small enough ρ (the interval $[0, t]$ is indeed compact). The main idea of the proof is to apply the finite increments theorem to the function:

$$\Gamma(\Delta z) = f(z + \Delta z) - f(z) - f'(z)\Delta z,$$

it means

$$\|\Gamma(\Delta z) - \Gamma(0)\| \leq \sup_{0 \leq \hat{\mu} \leq 1} \|\Gamma'(\hat{\mu}\Delta z)\|\|\Delta z\|,$$

by using the uniform continuity of $f'(z)$ on the compact K, we obtain

$$\begin{aligned} \sup_{0 \leq \hat{\mu} \leq 1} \|\Gamma'(\hat{\mu}\Delta z)\| &= \sup_{0 \leq \hat{\mu} \leq 1} \|f'(z + \hat{\mu}\Delta z) - f'(z)\| \\ &= \Theta(\|\Delta z\|) \end{aligned}$$

where $\Theta(\|\Delta z\|)$ is a function which can be chosen monotonic in steps, independent of z and which tends towards 0 when $\|\Delta z\|$ tends towards 0. Finally, f is Lipschitzian (at least locally on K) of Lipschitz constant L. By Gronwall's lemma, we get

$$\|z(s) - \zeta_s(x_0 + \Delta x_0)\| \leq \Theta(\bar{M}\|\Delta x_0\|)\bar{M}\|\Delta x_0\|\frac{e^{Lt} - 1}{L},$$

which proves the validity of (2.2) with $\varpi(x) = \Theta(\bar{M}x)\bar{M}\frac{e^{Lt}-1}{L}$.

It remains to show that $S(t, x_0)$ is continuous with respect to (t, x_0). For $(t, x_0) \in \Omega_0$, let $\mathcal{D}$ is a compact neighborhood of x_0 such as $[0, t] \times \mathcal{D} \subset \Omega_0$, we define:

$$\ell = \sup_{s \in [0,t], x \in \mathcal{D}} \|f'(\zeta_s(x))\|.$$

Then, for all $\tilde{x_0} \in \mathcal{D}$, we have

$$S'(s,\tilde{x_0}) - S'(s,x_0) = f'(\zeta_s(\tilde{x}_0))(S(s,\tilde{x_0}) - S(s,x_0)) + \Delta(s),$$

where

$$\Delta(s) = f'(\zeta_s(x_0)) - f'(\zeta_s(\tilde{x}_0)).S(s,x_0),$$

if $\sup_{s\in[0,t]} \|\Delta(s)\| \leq \gamma$. Then (by Gronwall's lemma)

$$\sup_{s\in[0,t]} (S(s,\tilde{x_0}) - S(s,x_0)) \leq \frac{\gamma(e^{t\ell}-1)}{\ell}.$$

Consequently, $f'(\zeta_s(x_0))$ is continuous in x_0 and we can conclude the continuity of $S(s,x_0)$ with respect to x_0 and (t,x_0). □

Theorem 2.1.2 *[48] If f is of class C^k on $\mathcal{D}$, then $(t,x) \mapsto \zeta_t(x)$ is also of class C^k on Ω_0.*

Proof: By recurrence. □

2.1.2 Definition of stochastic flow and its properties

Stochastic flow of homeomorphisms(or simply a flow) is a continuous $\mathbb{R}^d-$valued random field $\zeta_{s,t}(x)(\omega) \equiv \zeta_{s,t}(.,\omega)$, $s,t \in [0,T], x \in \mathbb{R}^d$ defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ for almost all ω indexed by tow parameters s *and* t such that $s < t$, the first represents the initial time of the flow and the second represents the state of the flow and it is constructed by solving stochastic differential equation according at the initial condition such that it satisfies the following properties:

1. For any $x \in \mathbb{R}^d$, $\zeta_{s,t}(x)(\omega)$ is continuous.

2. The map $\zeta_{s,t}(x)(\omega) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a homeomorphism for any s,t.

3. $\zeta_{s,t}(x)(\omega)$ is $k-$times continuously differentiable with respect to x for all $s,t \in \mathbb{R}^d$.

4. $\zeta_{t,u}(\zeta_{s,t}(x))(\omega) = \zeta_{s,u}(x)(\omega)$ for any s,u,t and any x. and $\zeta_{s,s}(x)(\omega) = Id_{\mathbb{R}^d}$ for any s.

If additionally $\zeta_{s,t}(x)(\omega)$ satisfies also properties (2)and (3). It is called stochastic flow C^k-diffeomorphism.

2.1.3 Continuity of the solution with respect to the initial data.

The solution of equation (1.5) has a continuous connection with the initial condition, this is a key to construct the flow, the following theorem prove the continuity of $\zeta_{s,t}(x)$ in (s,t,x):

Theorem 2.1.3 *[19] (Kolmogorov's theorem)*
For any $P \geq 2$ there exists a positive constant C_P such that:

$$\mathbb{E}[|\zeta_{s,t}(x) - \zeta_{s',t'}(x')|^P] \leq C_P(|x-x'|^P + |t-r|^{P/2}),$$

holds for all $x, x' \in \mathbb{R}^d$, $s \in [0,T]$ and $t, t' \in [s,T]$ so that $t' < t$.
In particular, there exists a modification of the solution $\zeta_{s,t}(x)$ which is continuous in (s,t,x).
Furthermore, the solution $\zeta_{s,t}(x)$ is $(\alpha,\beta)-$Hölder continuous in (s,t,x), where β is an arbitrary number less than $1/2$ and α is an arbitrary number less than 1.

Proof:
If the following estimate is verified: there is a positive constant C^p such that

$$\mathbb{E}\left|\zeta_{s,t}(x) - \zeta_{s',t'}(x')\right| \leq C^p\left[|x-x'|^p + (1+|x|^p+|x'|^p)(|t-t'|^{\frac{p}{2}} + |s-s'|^{\frac{p}{2}})\right]. \tag{2.3}$$

Then by Kolmogorov's theorem, the solution $\zeta_{s,t}(x)$ has a continuous modification, it's immediate to satisfy $(\alpha,\beta) - H\ddot{o}lder$ continuous in (s,t,x), where $\alpha < 2p^{-1}(\frac{p}{2} - d)$ and $\beta < p^{-1}(\frac{p}{2} - d)$.
Now we prove the continuity of the quantity $\int_s^t X_k(r, \zeta_{s,r}(x) dB_r^k$.
Since the case $k = 0$ is obvious, we consider the case $k \geq 1$. Assume $s < s' < t < t'$. Then

$$\begin{aligned}\int_s^t X_k(r,\zeta_{s,r}(x))dB_r^k - \int_{s'}^{t'} X_k(r,\zeta_{s',r}(x'))dB_r^k &= \int_s^{s'} X_k(r,\zeta_{s,r}(x))dB_r^k + \int_{s'}^t [X_k(r,\zeta_{s,r}(x)) \\ &- X_k(r,\zeta_{s',r}(x'))]dB_r^k - \int_{t'}^t X_k(r,\zeta_{s',r}(x'))dB_r^k\end{aligned}$$

Therefore, L_p-estimates of the first and the third terms of the right side are given by

$$\mathbb{E}\left[\left|\int_t^{t'} X_k(r,\zeta_{s',r}(x'))\right|^p dB_r^k\right] \leq C_1^P|t-t'|^{\frac{p}{2}}(1+|x'|^p). \tag{2.4}$$

And L_p-estimate of the second term is given by:

$$\mathbb{E}\left[\left|\int_{s'}^{t}\left[X_k(r,\zeta_{s,r}(x))-X_k(r,\zeta_{s',r}(x'))\right]dB_r^k\right|^p\right]\leq C_2^P\left[|x-x'|^p+|t-t'|^{\frac{p}{2}}(1+|x'|^p)\right]. \quad (2.5)$$

Proof of (2.4) and (2.5) see [28].

Consequently, the expectation of the $p-$th power of the left side is dominated by the right side quantity of (2.3). Thus, the term $\int_s^t X_k(r,\zeta_{s,r}(x)dB_r^k$ have a same kind of continuity as that of $\zeta_{s,t}(x)$. □

2.1.4 Homeomorphism property.

- **One to one property:**

Lemma 2.1.1 *[19] For every fixed $s\in[0,T]$, we have: $\eta_t(x,y)=\frac{1}{|\zeta_{s,t}(x)-\zeta_{s,t}(y)|}$ for $s<t$. Then for any $a>2$ there exists a constant C_a such that for any $\delta>0$*

$$\mathbb{E}[|\eta_t(x,y)-\eta_{t'}(x',y')|^a]\leq C_a\delta^{-2a}(|x-x'|^a+|y-y'|^a+|t-t'|^{a/2})$$

holds for any $t,t'\in[s,T]$ and $|x-x'|>\delta,\quad |y-y'|>\delta$.

Theorem 2.1.4 *[19] The map $\zeta_{s,t}:\mathbb{R}^d\longrightarrow\mathbb{R}^d$ is one to one for any $t\in[s,T]$ almost surely.*

Proof:

By the kolmogorov's theorem and the lemma (2.1.1): for p large enough such that $p/2>2(d+1)$ we obtain that $\eta_t(x,y)$ is continuous in the domain $\{(s,t,x,y)/s<t,|x-y|\geq\delta\}$, it's also continuous in the domain $\{(s,t,x,y)/s<t,x\neq y\}$, since δ is arbitrary. This is lead to the map $\zeta_{s,t}:\mathbb{R}^d\longrightarrow\mathbb{R}^d$ is one to one for any $0<s<t\leq T$ a.s.

- **Onto property:**

Lemma 2.1.2 *[19] Let $\widetilde{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^d$; set $\widetilde{x} = x/|x|^2$ for $x \in \mathbb{R}^d \backslash 0$ and Define for every $s < t$:*

$$\theta_t(\widetilde{x}) = \begin{cases} \frac{1}{1+|\zeta_{s,t}(x)|} \ if \ \widetilde{x} \in \mathbb{R}^d \\ \\ 0 \ if \ \widetilde{x} = 0 \end{cases}$$

Then, for any $a \geq 0$ there exists a constant C_a such that:

$$\mathbb{E}[|\theta_t(\widetilde{x}) - \theta_{t'}(\widetilde{y})|^a] \leq C_a(|\widetilde{x} - \widetilde{y}|^a + |t - t'|^{a/2}).$$

Theorem 2.1.5 *[19] The map $\zeta_{s,t} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is onto for any $t \in [s, T]$ almost surely.*

Proof:

From the lemma (2.1.2) : for p large enough such that $p > 2(d + 3)$, and by applying Kolmogorov's theorem we get that $\theta_t(\widetilde{x})$ is continuous at $\widetilde{x} = 0$. Therefore, $\zeta_{s,t}(., \omega)$ can be extended to a continuous map from $\widetilde{\mathbb{R}}^d$ into itself for any $t \in [s, T]$ almost surely. The extension $\widetilde{\zeta}_{s,t}(., \omega)$ is continuous in (s, t, x) almost surely. For all w such that $\widetilde{\zeta}_{s,t}(., \omega)$ is continuous, it's homotopically equivalent to the identity map $\zeta_{s,s}(., \omega)$. Then, it's onto map by the famous theorem of homopotic theory. Since $\widetilde{\zeta}^{\infty}_{s,t}(., \omega) = \infty$, the restriction of $\widetilde{\zeta}_{s,t}(., \omega)$ in $\mathbb{R}^d$ is also onto.

We will summarize the previous results: The continuous map $\zeta_{s,t}(., \omega) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is one to one and onto. Therefore the inverse map $\zeta^{-1}_{s,t}(., \omega) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is continuous, one to one and onto.

Proof:

Since the map $\zeta_{s,t}(., \omega)$ is continuous, one to one and onto from the previous results. Then the inverse map $\zeta^{-1}_{s,t}(., \omega)$ is well defined, one to one and onto. We suppose that it is also continuous. In fact, the map $\zeta_{s,t}(., \omega)$ is continuous and one to one from the compact space $\widetilde{\mathbb{R}}^d$ into itself, it is a closed map. Therefore, the inverse map $\zeta_{s,t}^{-1}(.,\omega)$ is continuous, and so is its restriction to $\mathbb{R}^d$.

2.1.5 Differentiability of the solution with respect to the initial data.

Let $C_g^{k,\alpha}$ be the space of globally Lipschitz functions f on $\mathbb{R}^d$ which are Hölder continuous of order α and $k-th$ continuously differentiable with $0<\alpha<1$ and $k \in \mathbb{N}$.

Theorem 2.1.6 *[28] suppose that coefficients $X_0, ..., X_m$ are $(C_g^{1,\alpha})$ functions for some $\alpha > 0$ and their first derivatives are bounded. Then the solution $\zeta_{s,t}(x)$ is a $C^{1,\beta}$ of x for any β less than α for each $s<t$ a.s. Moreover, the derivative $\frac{\partial \zeta_{s,t}^x}{\partial x_l}$ satisfies the following stochastic differential equation for all (s,t,x):*

$$\frac{\partial \zeta_{s,t}^x}{\partial x_l} = e_l + \sum_{k=0}^{m} \int_s^t X_k'(r, \zeta_{s,r}(x)) \frac{\partial \zeta_{s,t}^x}{\partial x_l} dB_r^k. \tag{2.6}$$

Where $X_k'(r,\zeta_{s,r}(x)) = \frac{\partial X_k^i(r,x)}{\partial x_j}$ and e_l is the unit vector.

Proof: (see [28]) For $y \in \mathbb{R} \setminus 0$:

$$\psi_{s,t}(x,y) = \frac{1}{y}[\zeta_{s,t}^{x+ye_l} - \zeta_{s,t}^x].$$

Then the existence of the partial derivative $\frac{\partial \zeta_{s,t}^x}{\partial x_l}$ for any s,t,x can be proved if $\psi_{s,t}(x,y)$ has a continuous extension at $y=0$ for any s,t,x based on the following estimate and Kolmogorov's theorem : for any $p>2$, there exits a positive constant C_3^p such that:

$$\begin{aligned} \mathbb{E}|\psi_{u,t}(x,y) - \psi_{s',t'}(x',y')|^p &\leq C_3^p[|x-x'|^{\alpha p} + |y-y'|^{\alpha p} \\ &+ (1+|x|+|x'|)^{\alpha p}(|s-s'|^{\frac{\alpha p}{2}} + |t-t'|^{\frac{\alpha p}{2}})]. \end{aligned} \tag{2.7}$$

We first show the boundedness of $\mathbb{E}|\psi_{s,t}(x,y)|^P$, by the mean value of theorem, it holds

$$\psi_{s,t}(x,y) = e_l + \sum_{k=0}^{m} \int_s^t \{\int_0^1 X_k'(r, \zeta_{s,r}(x) + v(\zeta_{s,r}(x+ye_l) - \zeta_{s,r}(x)))dv\} \times \psi_{s,r}(x,y) dB_r^k. \tag{2.8}$$

therefore we have

$$\begin{aligned}\mathbb{E}|\psi_{s,t}(x,y)|^P &\leq (m+2)^P(1+\sum_{k=0}^{m}\mathbb{E}[|\int_s^t\{\int_0^1 X_k'(r,\zeta_{s,r}(x)+v(\zeta_{s,r}(x+ye_l)-\zeta_{s,r}(x)))dv\} \\ &\times \psi_{s,r}(x,y)dB_r^k|^P]). \qquad (2.9)\end{aligned}$$

using BDG's inequality, we have for $k \geq 1$

$$\mathbb{E}[|\int_s^t\{\int_0^1 X_k'(r,\zeta_{s,r}(x)+v(\zeta_{s,r}(x+ye_l)-\zeta_{s,r}(x)))dv\}\times\psi_{s,r}(x,y)dB_r^k|^P]$$

$$\leq C_4^P|t-s|^{\frac{P}{2}-1}\mathbb{E}[|\int_s^t\{\int_0^1 X_k'(r,\zeta_{s,r}(x)+v(\zeta_{s,r}(x+ye_l)-\zeta_{s,r}(x)))dv\}\times\psi_{s,r}(x,y)|^P dr]$$

$$\leq C_4^P|t-s|^{\frac{P}{2}-1}||X_k'||\int_s^t\mathbb{E}|\psi_{s,r}(x,y)|^P dr,$$

where $||X_k'|| = \sup_{(r,x)|X_k'(r,x)|}$ and $|A|$ denotes the norm of the matrix $A = (a_{ij})$ defined by $|A| = \sqrt{\sum_{i,j} a_{ij}^2}$. Similar estimate is valid for $k = 0$.

Then from (2.9), we obtain

$$\mathbb{E}|\psi_{s,t}(x,y)|^P \leq C_5^P + C_6^P\int_s^t\mathbb{E}|\psi_{s,r}(x,y)|^P dr,$$

where constants $C_5^P \, and \, C_6^P$ do not depend on s,t,x,y. Therefore by Gronwall's lemma, we see that $\mathbb{E}|\psi_{s,t}(x,y)|^P$ is bounded.

We next show (2.7) in case $t = t'$. We assume $s < s' \leq t$. The other cases will be treated similarly. Note that $\psi_{s,t}(x,y) - \psi_{s',t}(x',y')$ is a difference between the following terms

$$\int_s^{s'}\{\int_0^1 X_k'(r,\zeta_{s,r}(x)+v(\zeta_{s,r}(x+ye_l)-\zeta_{s,r}(x)))dv\}\times\psi_{s,r}(x,y)dB_r^k \qquad (2.10)$$

$$\begin{aligned}&\int_{s'}^t[\{\int_0^1 X_k'(r,\zeta_{s,r}(x)+v(\zeta_{s,r}(x+ye_l)-\zeta_{s,r}(x)))dv\}\times\psi_{s,r}(x,y) \\ -\ &\{\int_0^1 X_k'(r,\zeta_{s',r}(x')+v(\zeta_{s',r}(x'+y'e_l)-\zeta_{s,r}(x')))dv\}\times\psi_{s,r}(x',y')]dB_r^k \qquad (2.11)\end{aligned}$$

Using BDG's inequality (1.1.2), the expectation of p-th power of (2.10) is estimated in case $k \geq 1$ as:

$$\mathbb{E}[|\int_s^{s'}\{\int_0^1 X_k'(r,\zeta_{s,r}(x)+v(\zeta_{s,r}(x+ye_l)-\zeta_{s,r}(x)))dv\} \times \psi_{s,r}(x,y)dB_r^k|^P]$$

$$\leq C_7^P|s'-s|^{\frac{P}{2}-1}||X_k'||^P\int_s^t \mathbb{E}|\psi_{s,r}(x,y)|^P dr$$

which is dominated by $C_8^P|s-s'|^{\frac{P}{2}}$ by the argument of the previous paragraph.

We will calculate the expectation of p-th power of (2.11). Note that (2.11) is $integrant[B]$ which is estimated as:

$$\begin{aligned}
|integrant[B]| &\leq \int_0^1 |X_k'(r,\zeta_{s,r}(x)+vy\psi_{s,r}(x,y))|dv \times |\psi_{s,r}(x,y)-\psi_{s',r}(x',y')| \\
&+ \int_0^1 |X_k'(r,\zeta_{s,r}(x)+vy\psi_{s,r}(x,y))-X_k'(r,\zeta_{s',r}(x')+vy'\psi_{s',r}(x',y'))|dv \times |\psi_{s',r}(x',y')| \\
&\leq ||X_k'|||\psi_{s,r}(x,y)-\psi_{s',r}(x',y')| + L\int_0^1[(1-v)^\alpha|\zeta_{s,r}(x)-\zeta_{s',r}(x')|^\alpha \\
&+ v^\alpha|\zeta_{s,r}(x,ye_l)-\zeta_{s',r}(x',y'e_l)|^\alpha]dv \times |\psi_{s',r}(x',y')| \\
&\leq ||X_k'|||\psi_{s,r}(x,y)-\psi_{s',r}(x',y')| + L|\zeta_{s,r}(x)-\zeta_{s',r}(x')|^\alpha \times |\psi_{s',r}(x',y')| \\
&+ L|\zeta_{s,r}(x,ye_l)-\zeta_{s',r}(x',y'e_l)|^\alpha \times |\psi_{s',r}(x',y')|,
\end{aligned}$$

where L is Hölder constant; $|X_k'(r,x)-X_k'(r,x')| \leq L|x-x'|^\alpha$. Therefore by BDG's inequality, we get

$$\begin{aligned}
C^{(P)^{-1}}\mathbb{E}[\int_{s'}^t |integrant[B]dB_r^k|^P] &\leq |t-s'|^{\frac{P}{2}-1}\int_{s'}^t \mathbb{E}[|integrant[B]|^P]dr \\
&\leq |t-s'|^{\frac{P}{2}-1}3^P(||X_k'||^P\int_{s'}^t \mathbb{E}[|\psi_{s,r}(x,y)-\psi_{s',r}(x',y')|^P]dr \\
&+ L^P(\int_{s'}^t \mathbb{E}[|\zeta_{s,r}(x)-\zeta_{s',r}(x')|^{2\alpha P}]^{\frac{1}{2}} \times \mathbb{E}[|\psi_{s',r}(x',y')|^{2P}]^{\frac{1}{2}}dr \\
&+ \int_{s'}^t \mathbb{E}[|\zeta_{s,r}(x,ye_l)-\zeta_{s',r}(x',y'e_l)|^{2\alpha P}]^{\frac{1}{2}} \times \mathbb{E}[|\psi_{s',r}(x',y')|^{2P}]^{\frac{1}{2}}dr))
\end{aligned}$$

Apply theorem 2.1 (See [28] page 211) to $\mathbb{E}|\zeta_{s,r}(x)-\zeta_{s',r}(x')|^{\alpha P}$. Then the above is dominated by

$$C_8^P[(1+|x|+|x'|)^{\alpha P}|s-s'|^{\frac{\alpha P}{2}}+|x-x'|^{\alpha P}+|y-y'|^{\alpha P}]+C_9^P\int_{s'}^{t}\mathbb{E}[|\psi_{s,r}(x,y)-\psi_{s',r}(x',y')|^P]dr$$

summing up these calculations for (2.10) and (2.11), We obtain

$$\begin{aligned}\mathbb{E}[|\psi_{s,r}(x,y)-\psi_{s',r}(x',y')|^P] &\leq C_{10}^P[(1+|x|+|x'|)^{\alpha P}|s-s'|^{\frac{\alpha P}{2}}+|x-x'|^{\alpha P}+|y-y'|^{\alpha P}]\\ &+ C_{11}^P\int_{s'}^{t}\mathbb{E}[|\psi_{s,r}(x,y)-\psi_{s',r}(x',y')|^P]dr\end{aligned}$$

By Gronwall's lemma , We have

$$\mathbb{E}[|\psi_{s,r}(x,y)-\psi_{s',r}(x',y')|^P] \leq C_{10}^P[(1+|x|+|x'|)^{\alpha P}|s-s'|^{\frac{\alpha P}{2}}+|x-x'|^{\alpha P}+|y-y'|^{\alpha P}]\times C_{11}^P\exp(t-t')$$

This proves (2.7) in case $t=t'$. It remains to prove (2.7) for $t\neq t'$. Assuming $t<t'$, we have

$$\begin{aligned}\psi_{s,r}(x,y)-\psi_{s',r}(x',y') &= \psi_{s,r}(x,y)-\psi_{s',r}(x',y')\\ &- \sum_{k=0}^{m}\int_t^{t'}\{\int_0^1 X_k'(r,\zeta_{s,r}(x)+v(\zeta_{s,r}(x+ye_l)-\zeta_{s,r}(x)))dv\}\times\psi_{s,r}(x,y)dB_r^k\end{aligned}$$

it holds

$$\begin{aligned}&C^{(P)^{-1}}\mathbb{E}[|\int_t^{t'}\{\int_0^1 X_k'(r,\zeta_{s,r}(x)+v(\zeta_{s,r}(x+ye_l)-\zeta_{s,r}(x)))dv\}\times\psi_{s,r}(x,y)dB_r^k|^P]\\ \leq\ & |t'-t|^{\frac{P}{2}-1}\mathbb{E}[\int_t^{t'}|\{\int_0^1 X_k'(r,\zeta_{s,r}(x)+v(\zeta_{s,r}(x+ye_l)-\zeta_{s,r}(x)))dv\}\times\psi_{s,r}(x,y)|^P]dr\\ \leq\ & |t'-t|^{\frac{P}{2}-1}||X_k'||^P\int_t^{t'}\mathbb{E}|\psi_{s,r}(x,y)|^P dr\\ \leq\ & C_{12}^P|t'-t|^{\frac{P}{2}}\end{aligned}$$

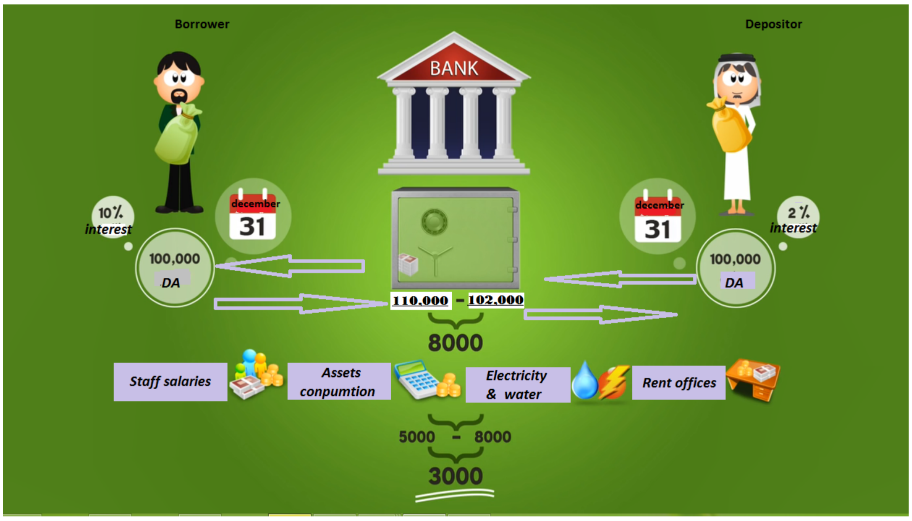
Therefore we get the desired estimation (2.7). The proof is complete.

Proof of theorem 2.1.6: by Kolomorov's theorem, $\psi_{s,r}(x, y)$ has continuous extension at $y = 0$. This means that $\zeta_{s,r}(x)$ is continuously differentiable in the domain $\{(s, t, x)|s < t, x \in \mathbb{R}^d\}$ and the derivative $\partial_l \zeta_{s,r}(x)$ is $\beta-$Hölder continuous for any $\beta < \alpha$. Let y tends to 0 in (2.8). Then we obtain (2.6). The proof is complete.

2.2 Credit risk modeling

2.2.1 A brief zoology of risks.

The main activity of commercial banks is to receive money from the depositors in deposits form that are loaned out to borrowing customers in loans form such that the banks pay interest to depositors as a cost of obtaining funds. In other side, the banks earn more interest than the loan proceeds which represents the main income, the bank profits arises from the difference between the tow interests.



But it may happens what affects this profit or exposes the bank to loss, what is known as bank risks, there are traditionally four main types of financial risks: liquidity risk, market risk (currency fluctuation risk, devaluation risk, high interest rate risk, low interest rate risk), operational risk and credit risk. Moreover, there are a non-commercial risk and Reputation risk.

1. **Liquidity risk:** For a company, this is the risk of not being able to mobilize enough liquidity at a given time to be able to meet its commitments.

2. **Market risk:** Can be defined as the risk of loss linked to variations in market conditions (currency fluctuation , devaluation , high interest rate, low interest rate).

3. **Operational risk:** In this category are grouped, for example, the risks of fraud, operator errors, system failures, etc...

4. **Credit risk:** Our ambition here is to recall some "obvious facts" on credit risk or the default risk, in order to understand the credit risk, perhaps it is necessary to go back to the source of this risk, namely the company that borrowed capital, either through a traditional bank loan, or issuing bonds on the market. the default occurs when the company is unable to meet its commitments to its debts at the time of maturity, causing a financial loss to the lender. From the point of view of Financial Mathematics, this expression may carry a broader meaning. It can refer to the company's default risk, the risk of contagion in a financial crisis, or the risk associated with a sudden change in the value of a portfolio of assets, the consequence of an event that is not necessarily resulting from the financial transactions activity (such as category transition in a credit note, natural disaster, fraud, etc.). A common thread of these risks is that the price flows of financial products before the counterparty event do not give all the information about credit risk. We can distinguish two types of credit risk: counterparty risk and reference risk. For a given issuer, this risk may materialize in the form:

(a) A change in its rating (upgrade or downgrade) such as that issued by the big agencies of ratings: Moody's, Standard and Poor's.

(b) A change in its credit spread.

(c) A credit event such as default of payment or restructuring of its debt.

These three risks are correlated. A sharp increase in the level of the issuer spread increases the probability of a credit event. Likewise, a change in rating strongly influences the probability of an issuer defaulting.
When a party A enters into a relationship with a counterparty B via a financial instrument, it may be subject to the risk that B will be unable to honor its commitments. For example, if A is in possession of a bond issued by B, he runs the risk that at maturity B cannot repay him the invested capital. We say in this case of unilateral counterparty risk since B is not subject to the credit risk of A. If A and B are the two counterparties of a swap, they are both subject to the counterparty risk: we then say bilateral counterparty risk. Now assume that the quality of the signature of counterparties A and B is of infinite quality (so that the bilateral counterparty risk is zero). Parties A and B can enter into a contract that involves the credit risk of a third counterparty C (a credit swap whose payoff depends on the occurrence of a credit event is an example of such a contract). The credit risk associated with C is called the reference risk. We will see that the purpose of credit derivatives is to transfer this reference risk. It is not always easy to distinguish between these risks: the credit risk associated with changes in default swap credit spreads can be considered as market risk. The portfolios of derivative products (OTC) are submissive to the market risk, but they are also exposed to counterparty risk.

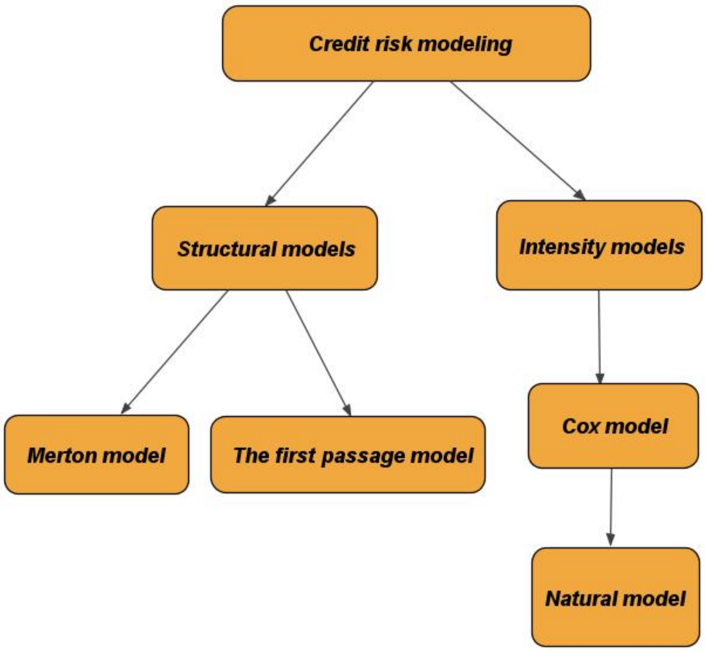
In addition, the banks may be exposed to earthquake, theft or fire risks. This is known as non-commercial risks, or the bank may be involved in operations that may damage its reputation and may lose its customers, and this is called reputation risks.

2.2.2 The main models of credit risk

Credit risk can be defined, as a first approximation, as the risk of loss linked to a change in the quality of a counterparty's signature. All financial firms (also a market participants) accumulate a big quantity of credit risk: either directly through their debt portfolios or indirectly in the form of counterparty risk in their asset portfolios and of derivative products (OTC).
The challenge of modeling this risk is therefore very important: it is about being able to:

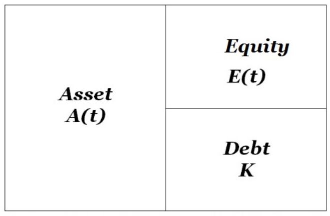
- Measure the credit risk contained in the portfolios.
- Evaluation of financial instruments sensitive to credit risk. More generally, any instrument exposed to this risk (counterparty risk), It is important to be able to control exposure to counterparty credit risk by counterparty also the evolution of this exposure by geographic and industrial sectors. Such practices make it possible to reduce the risk of concentration.

There are two important approaches in the classical literature to analyze and cover the credit risk: structural models and intensity models.



1. **Structural models:**
   Under structural models, a default event occurs if the borrower (firm) is unable to honor its

commitments and risky zero coupons appear as derivatives on the value of the firm issuing this debt. These models require strong assumptions on the dynamics of the firm's asset, its debt and how its capital is structured.
The so-called Merton Model was initiated by Merton in (1974) [35] gives a simplified presentation of the company's passive, composed only of bond debt and the equity $E(t)$.



Financial sructure of the firm in Merton (1974)

The main idea of this model is based on modeling the evolution of the total value of a firm's assets A_t which follows a geometric brownian motion. this model suppose that debt consists of a single zero-coupon with a value K and maturity T. At maturity, the default appears if the debt is greater than the total value of the assets such that the default time is considered as a predictable event and it is modeled as the first passage of a stochastic process through a barrier. The process can be the value of an asset which is considered as the main point of this model. Merton propose a dynamic of Black-Scholes type for the value of the company's assets:

$$\frac{dA_t}{A_t} = \alpha dt + \sigma dW_t. \tag{2.12}$$

On the other hand, he assumes that the risk-free rate r is constant. Assuming the absence of an arbitrage opportunity, the theoretical price of the risk debt is calculated by the option pricing of Black-Sholes formula.

$$K(A_t, t) = K.e^{-r(T-t)} - (K.e^{-r(T-t)}N(-d_2) - A_tN(-d_1)),$$

where N is the distribution function of the reduced central normal law.
$d_1 = \frac{\frac{1}{2}\sigma^2 T - \log(K.e^{-r(T-t)}/A_t)}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.
This equation can be rewritten by:

$$K(A_t, t) = e^{-r(T-t)}.K.N(d_2) + A_t(1 - N(d_1)).$$

According to these two formulas, we have two remarks. First, if the option is far out of the money, the debt behaves like a risk-free bond. Inversely, if the option is in the money, the value of the debt is very sensitive to the volatility of the asset. Secondly, we see when the maturity of the bond tends towards zero, the rate spread also tends towards zero. Finally, this model shows the relationship between market risk and credit risk through the accounting relationship that exists between the value of the asset, the debt and the share, the decrease in asset value implies an increase in the probability of default, and credit risk. The Merton model is only a starting point for studying credit risk, and is obviously far from realistic:

- The non-stationary structure of the debt that leads to the termination of operations on a fixed date, and default can only happen on that date. extended the Merton model to the case of bonds of different maturities.
- It is incorrect to assume that the firm value is tradeable. In fact, the firm value and its parameters is not even directly observed.
- Interest rates should certainly be taken to be stochastic: this is not a serious drawback, and its generalization was included in Merton's original paper.
- The short end of the yield spread curve in calibrated versions of the Merton model typically remains essentially zero for months, in strong contradiction with observations.

The so-called first passage models extend the Merton framework by allowing default to happen at intermediate times i.e instead of admitting only the possibility of default at maturity time T, this model suppose that the default occurs at the time that the firm's asset value go down below a certain time dependent barrier K_t, this allows to bondholders to exercice a safety covenant i.e to liquidate the firm if at any time its value drops below the specified threshold K_t. Thus, the default time is given as the stopping time.

Structural models are widely used by practitioners. To be convinced of this, it's enough to mention the company Moody's KMV (www.moodyskmv.com) which has developed a complete range of financial services based on this model. Thus, they offer their customers the analysis tools and provide them with data (estimations of default probability) obtained from their model.

2. **Intensity models:**

The initial motivation for intensity models is to describe the default times in a more "surprise" way i.e the default time is considered as a unpredictable event as well it is modeled as the first time of jump of poisson process of intensity λ i.e the default time τ follows an exponential law of parameter λ therefore we have:

$$\mathbb{P} = (\tau > t) = e^{-\lambda t}, \quad \mathbb{E}(\tau) = \frac{1}{\lambda}, \quad \mathbb{P}(\tau \in (t, t + \Delta t)/\tau > t) = \lambda \Delta t + \circ(\Delta t)$$

In practice, this default intensity is linked to an economic variables number (such as interest rates) and / or variables linked to the company (such as its rating). This models are widely used for the valuation of credit derivatives.

We can further develop the model to let it capture credit spread volatility, We generalize the Poisson process in the case where the intensity is allowed to be random i.e it is a deterministic function or a stochastic process therefore an in-homogeneous Poisson process with stochastic intensity λ_s is called Cox process. This generalization is called Cox model such that the default time is modeled as a first jump time of Cox process.

All the models mentioned above have been studied before the default time i.e conditioned on events where the default did not occur but it is important to study what happens after the default time and it is essential to analyze the impact of default on the counterparty and the rest of the market, these models did not adequate these problems. In a series of works in collaboration with J.Monique and N.Elkaroui, they are represented a new approach which describe the market after the default and it is based on the conditional distribution of default time with respect to the reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t>0}$ which represents market information that is not directly related to the default event, it is known by The density approach which adequate naturally in the framework of the theory of progressive enlargement of filtration with an emphasis on the set "after-default".

3. **Modeling in the case of the one-default:**
We consider in this paragraph the problem of modeling a financial market that contains the default risk of an asset. In this context, we can already observe the important ideas in our modeling approach and its links with classical models. Intuitively, the information of the default of the asset can lead to changes in the values of financial products in the market. Therefore, the study of market filtration is often more complicated than that in classical problems.
We consider a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$ which is modeled the financial market and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration that describes general market information which is not directly related to the default event. We suppose that a filtration $\mathbb{F}$ satisfies the usual conditions. In practice, the filtration $\mathbb{F}$ is often assumed to be generated by a Brownian motion or a Lévy process. We denote τ the default time of the asset, which is a strictly positive random variable. The observation at time t of default or not of the asset is modeled by the tribe $\sigma(\tau \wedge t)$. We denote a filtration $(\sigma(\tau \wedge t))_{t\geq 0}$ by $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ return usual. Then, the market information is modeled by the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, the progressive enlargement of filtration $\mathbb{F}$ by $\mathbb{D}$. In other words, we have $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ for all $t > 0$. It appears that, once know the conditional distribution of default time with respect to filtration $\mathbb{F}$ (which is a process in random measures), the evaluation of financial products is based only on the stochastic calculus with respect to the filtration $\mathbb{F}$. This method gives us flexibility to apply various tools of financial mathematics (optimization, representation of martingales, etc.). Classical studies are mainly carried out on the set "before-default" $\{t < \tau\}$, we are also interested by the "after-default" set $\{t \geq \tau\}$.
The enlargement theory of filtration gives explicit links between the processes $\mathbb{F}$ and $\mathbb{G}-$adapted (resp. predictable). Indeed, any process ξ_t $\mathbb{G}-$adapted can be written by the following form:

$$\xi_t = \xi_t^0 1\!\!1_{\{\tau > t\}} + \xi_t^1(\tau) 1\!\!1_{\{\tau \leq t\}} \quad t \geq 0,$$

where ξ_t^0 is $\mathbb{F}-$adapted process and $\xi_t^1(.)$ is $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}^+)-$adapted process, $\mathcal{B}(\mathbb{R}^+)$ is the Borelian tribe of $\mathbb{R}^+$. Likewise, any process χ_t $\mathbb{G}-$predictable can be written by the following form:

$$\chi_t = \chi_t^0 1\!\!1_{\{\tau\geq t\}} + \chi_t^1(\tau)1\!\!1_{\{\tau<t\}}\ t\geq 0,$$

where χ_t^0 is $\mathbb{F}-$predictable process and $\chi_t^1(.)$ is $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)-$measurable function, $\mathcal{P}(\mathbb{F})$ is the predictable tribe of filtration $\mathbb{F}$.

We first study the conditional laws of default with respect to the different filtrations. For all $t\geq 0$, we denote the conditional law τ with respect to the tribe $\mathcal{G}_t$ by $\mu_t^{\mathbb{G}}$, it is a random $\mathcal{G}_t-$measure. By definition, the integral of a bounded or positive Borelian function f with respect to $\mu_t^{\mathbb{G}}$ is the conditional expectation of $f(\tau)$ with respect to a filtration $\mathcal{G}_t$. Furthermore, the family of random variables $(\int f d\mu_t^{\mathbb{G}})_{t\geq 0}$ defines a $\mathbb{G}-$martingale. We can also consider $(\mu_t^{\mathbb{G}})_{t\geq 0}$ as a $\mathbb{G}-$martingale in random measure. Likewise, we denote the conditional law τ with respect to the tribe $\mathcal{F}_t$ by $\mu_t^{\mathbb{F}}$. We can establish an explicit link between the random measures $\mu_t^{\mathbb{G}}$ and $\mu_t^{\mathbb{F}}$ (on $\mathbb{R}^+$):

$$\mu_t^{\mathbb{G}}(du) = \frac{\mu_t^{\mathbb{F}}(1\!\!1_{\{\tau>t\}}.du)}{\mu_t^{\mathbb{F}}(1\!\!1_{\{\tau>t\}})}1\!\!1_{\{\tau>t\}} + 1\!\!1_{\{\tau\leq t\}}\delta_\tau(du),$$

where δ_τ is Dirac measure at τ.

To calculate the conditional expectations with respect to $\mathcal{G}_t$, it is useful to extend $\mu_t^{\mathbb{F}}$ in a random $\mathcal{F}_t-$measure on the measurable space $(\Omega\times\mathbb{R}^+, \mathcal{F}\otimes\mathcal{B}(\mathbb{R}^+))$, which sends any positive $(\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^+))-$ measurable function $Z_t(.)$ to:

$$\int_{\mathbb{R}^+} Z_t(x)\mu_t^{\mathbb{F}}(dx) = \mathbb{E}[Z_t(\tau)/\mathcal{F}_t].$$

we still use the expression $\mu_t^{\mathbb{F}}$ to establish the extended random measure and if $\chi_t(.)$ is $(\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^+))-$ measurable function on $\Omega\times\mathbb{R}^+$ which is positive or bounded, we denote the random measure by $\mu_t^{\mathbb{F}}(\chi_t(x).dx)$ such as:

$$\int_{\mathbb{R}^+} Z_t(x)\mu_t^{\mathbb{F}}(\chi_t(x).dx) = \int_{\mathbb{R}^+} Z_t(x)\chi_t\mu_t^{\mathbb{F}}(dx).$$

With these notations, we have established the following result:

Theorem 2.2.1 *[35] Let $T > 0$ be a real number that denotes a maturity, For all $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+))-$ measurable function $\chi_T(.)$ which is positive or bounded, we have*

$$\mathbb{E}_{\mathbb{P}}[\chi_T(\tau)/\mathcal{G}_t] = \int_{\mathbb{R}^+} \frac{\mathbb{E}[\mu_T^{\mathbb{F}}(\chi_T(x).dx)/\mathcal{F}_t]}{\mu_t^{\mathbb{F}}(dx)} d\mu_t^{\mathbb{G}},$$

where the quotient denotes the derivative in the Radon-Nikodym sense of two random $\mathcal{F}_t-$measures on the measurable space $(\Omega \times \mathbb{R}^+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+))$.

This result shows that, to evaluate the prices of derivatives sensitives products to counterparty risk in τ, it's enough to model the conditional distribution of random time with respect to the filtration $\mathbb{F}$, which is in general simpler than that with respect to the filtration $\mathbb{G}$. In the following paragraphs, we present the roles played by this theorem in different contexts, in addition to this the comparison with classical approaches.

Density hypothesis:

In the classical literature, it is standard to bring back the study of the $\mathbb{G}-$compensator from the default process $(\mathbb{1}_{\{\tau \leq t\}})_{t \geq 0}$ at the Doob-Meyer decomposition of the $\mathbb{F}-$supermartingale of survival $(\mathbb{P}(\tau > t/\mathcal{F}_t))_{t \geq 0}$. Based on this idea, the modeling of the $\mathbb{F}-$intensity of default is frequent in the study of credit risk, and it is particularly convenient when the derivative stops at the time of default τ. Recall that the $\mathbb{G}-$intensity of default τ is the $\mathbb{G}-$adapted process $\lambda^{\mathbb{G}}$, if $\mathbb{G}-$intensity exists, then there is $\mathbb{F}-$adapted process $\lambda^{\mathbb{F}}$ such as $\lambda_t^{\mathbb{G}} = \mathbb{1}_{\{t < \tau\}} \lambda_t^{\mathbb{F}}$, the process $\lambda^{\mathbb{F}}$ is called $\mathbb{F}-$intensity of τ which gives a little information on the impacts of the default event on the filtration $\mathbb{F}$ after the default time, unless if certain assumptions of conditional independence are required. Thus, this kind of models are not sufficient to study the derivatives products which continue to exist after default time.

We assume that the default time τ verifies the density hypothesis i.e. the conditional law of τ with respect to $\mathcal{F}_t$ admits a density with respect to the Lebesgue measure. In other words, there is a family of functions $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+)-$measurables $(w, \theta) \mapsto \alpha_t(\theta)$ such as:

$$\mathbb{P}(\tau \in d\theta/\mathcal{F}_t) = \alpha_t(\theta) d\theta \;\; t \geq 0,$$

We can distinguish the conditional densities for "before-default", we have $t \leq \theta$ and for "after-default" we have $t > \theta$. Indeed, there is a close link between the density before-default and the $\mathbb{F}-$intensity, while the density after-default plays a particularly important role on the studies of market after the default event.
The existence of $\mathbb{F}-$intensity is verified under the density hypothesis, and in this case the $\mathbb{F}-$intensity can be computed explicitly as:

$$\lambda_t^{\mathbb{F}} = \frac{\alpha_t(t)}{S_t} = \frac{\alpha_t(t)}{\int_t^\infty \alpha_t(\theta)d\theta},$$

where S is the Azema super-martingale $S_t = (\mathbb{P}(\tau > t/\mathcal{F}_t))_{t\geq O}$. In particular, the intensity only depends on the before-default part of the conditional densities. Inversely, for all $\theta \geq t$, the conditional density before-default is given by the following form: $\alpha_t(\theta) = \mathbb{E}[\lambda_\theta^{\mathbb{G}}/\mathcal{F}_t]$. However, the density after-default cannot be deduced from the intensity without supplementary assumptions.
Under the density hypothesis, the random measure $\mu_t^{\mathbb{F}}$ is written as $\mu_t^{\mathbb{F}}(dx) = \alpha_t(x)dx$. In particular, the conditional expectation mentioned in the previous theorem admits an explicit form as follows.

Proposition 2.2.1 *[35] Under the density hypothesis, for any $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+)-$measurables function $\chi_T(.)$ that is positive or bounded, we have :*

$$\mathbb{E}[\chi_T(\tau)/\mathcal{G}_t] = \frac{[\int_t^\infty \mathbb{E}\chi_T(\theta)\alpha_T(\theta)/\mathcal{F}_t]d\theta}{S_t}1\!\!1_{\{\tau>t\}} + \frac{\mathbb{E}[\chi_T(\theta)\alpha_T(\theta)/\mathcal{F}_t]}{\alpha_t(\theta)} |_{\theta=\tau} 1\!\!1_{\{\tau\leq t\}},$$

where S_t is the conditional probability of survival $\mathbb{P}(\tau \geq t/\mathcal{F}_t)$.

The super-martingale S admits a multiplicative Doob-Meyer decomposition of the form $S_t = \exp(-\int_0^t \lambda_s^{\mathbb{F}}ds)J_t$, where J_t is a local martingale with respect to $\mathbb{F}$. Under the immersion hypothesis ((H) hypothesis) which assumes that all $\mathbb{F}-$martingale is a $\mathbb{G}-$martingale, the process S is decreasing, then $J_t = 1$. The immersion hypothesis also implies that $\alpha_t(\theta) = \alpha_\theta(\theta)$ for all $t \geq \theta$, we have

$$\mathbb{E}[\chi_T(\tau)/\mathcal{G}_t]1\!\!1_{\{\tau\leq t\}} = \mathbb{E}[\chi_T(\theta)/\mathcal{F}_t] |_{\theta=\tau} 1\!\!1_{\{\tau\leq t\}},$$

this formula on the set $\{\tau \leq t\}$ completely ignores the conditional law of τ. Thus, the immersion hypothesis, which is often assumed in the studies before-default , becomes inadequate to analyze the impact of a default on a derivative product that exists after the default.

Theorem 2.2.2 *[35] Let χ^0 and $\chi_t(.)$ be tow processes which are $\mathbb{F}-$adapted (resp.$\mathbb{F} \otimes \mathcal{B}(\mathbb{R}^+)-$adapted):*

(a) The $\mathbb{G}-$adapted process χ^{bd} is $\mathbb{G}-$martingale (local) if and only if the $\mathbb{F}-$adapted process $(\chi_t^0 S_t \int_0^t \chi_s^1(s)\alpha_s(s)ds)_{t\geq 0}$ is $\mathbb{F}-$martingale (local) or the process $(J_t(\chi_t^0 + \int_0^t(\chi_s^1(s) - \chi_s^0)\lambda_s^{\mathbb{F}} ds))_{t\geq 0}$ is $\mathbb{F}-$martingale (local).

(b) The $\mathbb{G}-$adapted process χ^{ad} is $\mathbb{G}-$martingale (local) if and only if, for all $\theta \geq 0$, the $\mathbb{F}-$adapted process $(\chi_t^1(\theta)\alpha_t(\theta))_{t\geq\theta}$ is $\mathbb{F}-$martingale (local).

Where $\chi^{bd} = \chi_t^0 1\!\!1_{\{\tau > t\}} + \chi_t^1(\tau)1\!\!1_{\{\tau\leq t\}}$ and $\chi^{ad} = (\chi_t^1(\tau) - \chi_t^1(\tau))1\!\!1_{\{\tau\leq t\}}$.

Change of probability:

Now we discuss the problem of change of probabilities under the density hypothesis. we assume that τ admits a family of conditional densities $(\alpha_t(.))_{t\geq 0}$ with respect to a filtration $\mathbb{F}$ under the probability $\mathbb{P}$. Let $\mathbb{Q}$ a probability measure equivalent to $\mathbb{P}$ by the Radon-Nikodym derivative with respect to $\mathbb{P}$ on $\mathcal{G}_t$ is given by a $\mathcal{G}_t-$measurable random variable $Q_t^{\mathbb{G}} = q_t 1\!\!1_{\{\tau>t\}} + q_t(\tau)1\!\!1_{\{\tau\leq t\}}$ which is $\mathbb{G}-$martingale with $Q_0^{\mathbb{G}} = 1$.

By the previous theorem, we get that the process $(q_t S_t \int_0^t q_s(s)\alpha_s(s)ds)_{t\geq 0}$ is $\mathbb{F}-$martingale and $(\chi_t(\theta)\alpha_t(\theta))_{t\geq 0}$ is $\mathbb{F}-$martingale for all $\theta \geq 0$. The density hypothesis of random time τ is still verified under the new probability $\mathbb{Q}$ and The conditional density $\alpha^{\mathbb{Q}}$ of τ on the filtration $\mathbb{F}$ is given by the following formulas:

$$\alpha_t^{\mathbb{Q}}(\theta) = \alpha_t(\theta) = \frac{q_t(\theta)}{Q_t^{\mathbb{F}}}, \quad t \geq \theta$$

$$\alpha_t^{\mathbb{Q}}(\theta) = \mathbb{E}_{\mathbb{Q}}[\alpha_\theta^{\mathbb{Q}}(\theta)/\mathcal{F}_t] = \frac{1}{Q_t^{\mathbb{F}}}\mathbb{E}_{\mathbb{P}}[\alpha_\theta(\theta)q_\theta(\theta)], \quad t < \theta$$

where $Q_t^{\mathbb{F}}$ is a conditional expectation of $Q_t^{\mathbb{G}}$ with respect to $\mathcal{F}_t$:

$$Q_t^{\mathbb{F}} = \mathbb{E}_{\mathbb{P}}[Q_t^{\mathbb{G}}/\mathcal{F}_t] = q_t S_t + \int_0^t q_t(u)\alpha_t(u)du.$$

The $\mathbb{F}-$intensity of default under the probability $\mathbb{Q}$ is given by:

$$\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}} \frac{q_t(t)}{q_t} \quad t \geq 0,$$

and the Azéma super-martingale of survival becomes $S^{\mathbb{Q}} = qS/Q^{\mathbb{F}}$.

The change of probabilities is an important tool in conditional density modeling. With this method, we can systematically propose conditional $\mathbb{F}-$density models from a model that is relatively simple. We can start with a standard intensity model (the Cox model) where the immersion hypothesis is satisfied under the initial probability $\mathbb{P}$. We want to construct a new probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and which coincides with $\mathbb{P}$ on the tribe $\mathcal{G}_t$. This change of probabilities does not modify the before-default part of the model:

The $\mathbb{F}-$intensity $\lambda^{\mathbb{F}}$ and the density after-default $\alpha_t(\theta)$, $\ t \leq \theta$ remain unchanged. This is possible by taking the Radon-Nikodym derivative of the form:

$$Q_t^{\mathbb{G}} = 1\!\!1_{\{\tau > t\}} + q_t(\tau) 1\!\!1_{\{\tau \leq t\}},$$

where $(q_t(\theta), t \geq \theta)$ is a family of positive $(\mathbb{P},\mathbb{F})-$martingales such as $q_\theta(\theta) = 1$ for all $\theta \geq 0$. Furthermore, the processes $(S_t/S_t^{\mathbb{Q}}, t \geq 0)$ and $(\alpha_t^{\mathbb{Q}}(\theta)S_t/S_t^{\mathbb{Q}}, t \geq 0)$ are $(\mathbb{P},\mathbb{F})-$martingales. On the other hand, the immersion property is not necessarily kept under the new probability $\mathbb{Q}$.

In the next chapter, we define the natural model which is expressed by $\natural-$equation (1.3.2).

Chapter 3

The properties of the solution of the natural model

The third chapter is the heart of our research, it contains the description of the natural equation in one-dimensional and multidimensional cases, the regularity of the solution of the natural equation in multidimensional case and the differentiability property of solution with respect to the initial data in one-dimensional and multidimensional cases.

3.1 Description of natural model

In this section, we are interested to the main model of credit risk modeling such that it is the only one in which the conditional laws of τ with respect to $\mathbb{F}$ are defined by a system of dynamic equations as well it is considered as one of the best ways to represent the evolution of financial market after the default time. The knowledge of market evolution is a valuable property. This evolution form of the natural model ($\natural$-model) had allowed to establish the so-called enlargement of filtration formula [25] i.e the $\natural$-model is based on the construction of a survival probability $\mathbb{Q}$ and the random time τ on an extension of $(\Omega, \mathbb{F}, \mathbb{P})$, such that the survival probability satisfies $\mathbb{Q}[\tau > t|\mathcal{F}_t] = Ne^{-\Lambda_t}$ where Λ is an $\mathbb{F}$-adapted increasing continuous process and N is $(\mathbb{P}, \mathbb{F})$-local martingale. Therefore, this model is equipped with enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ where $\mathcal{G}_t = \mathcal{F}_t \vee (\tau \wedge t)$ such that all $(\mathbb{P}, \mathbb{F})$-martingales remains $\mathbb{G}$-semi-martingale. We recall that the formula of enlargement of filtration is essential, when the no-arbitrage price valuation is considered in an one-default model ([1],[20], [53]).As much as the enlargement of filtration formula is universally valid before the default time τ , for a long time, the part of the enlargement of filtration formula after τ was merely proved for the honest time model or the initial time model. The $\natural-$models constitute the third family of models where the enlargement of filtration formula is valid on the whole $\mathbb{R}^+$. In addition, the enlargement of filtration formula in the $\natural-$model has a richer structure than that of honest time model, and has a more accurate expression than that of the initial time model.

The usefulness of an one-default model is conditional upon the way that the conditional laws of τ can be computed with respect to the filtration $\mathbb{F}$. The most used examples of random times, therefore, are the independent time, the Cox time, the honest time, the pseudo stopping time, the initial time,etc (for example [9], [10], [17], [26], [27], [43], [44]). In the paper [25] a new class of random times has been introduced. Precisely, it is proved that, for any continuous increasing process Λ null at the origin, for any continuous non-negative local martingale N such that $0 < Ne^{-\Lambda_t} < 1$, $t > 0$, for any continuous local martingale Y and for any Lipschitz function f on $\mathbb{R}$ null at origin, there is a random variable τ such that the family of conditional expectations $dX_t^u = \mathbb{Q}[\tau \leq u/\mathcal{F}_t]$, $0 < u, t < \infty$,verify the $\natural$-equation:

$$(\natural_u) = \begin{cases} dX^x_{u,t} = X^x_{u,t}\left(-\frac{e^{-\Lambda_t}}{1-Z_t}dN_t + f(X_t - (1-Z_t))dY_t\right), t \in [u,\infty), \\ X^x_{u,u} = x, \end{cases}$$

where x is the initial condition that is $\mathcal{F}_u-$measurable random variable.

The $\natural-$equation which displays the evolution of the defaultable market, it is a prosperous system of parameters (Z, Y, f), where the parameter Z determines the default intensity. The parameters Y and f describe the evolution of the market after the default time τ. Such a system of parameters sets up a propitious framework for infering the market behavior and for calibrating the financial data. We believe that the $\natural$-model can be a useful instrument to modeling financial market.

Theorem 3.1.1 *[25] Let $0 < u < \infty$. The $\natural-$equation has a unique solution X for each given initial value x. Moreover, if $X_u \leq 1 - Z_u$, then X is bounded by $(1 - Z)$ on $[u, \infty)$.*

Proof:

For the existence and the uniqueness of the solution of the $\natural-$equation we refer to Protter[49].

To see that the solution is bounded by on $[u, \infty)$, we introduce the process $\Delta = X - (1 - Z)$. In order to prove that the local time $L^0(\Delta)$ is identically null, it's enough to apply the following lemma.

Lemma 3.1.1 *[25] Let $0 < u < \infty$. Let X be a $(\mathbb{P}, \mathbb{F})-$local martingale on $[u, \infty)$ such that $X_u \leq (1 - Z_u)$. Then, $X_t \leq (1 - Z_t)$ on $[u, \infty)$ if and only if the local time at zero $L^0(X - (1 - Z))$ of $X - (1 - Z)$ on $[u, \infty)$ is identically null. Here, the local time is taken a right continuous in $a \longrightarrow L_t^a(X - (1 - Z))$.*

To do this, we calculate $< \Delta >$ using the fact that, from Itô's calculus

$$d\Delta_t = -\Delta_t \frac{e^{-\Lambda_t}}{1 - Z_t} dN_t + X_t f(\Delta_t) dY_t - Z_t d\Lambda_t$$

Therefore,

$$\begin{aligned} d < \Delta >_t &= \Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t} \right)^2 d < N >_t + X_t^2 f(\Delta_t)^2 d < Y >_t - 2\Delta_t \frac{e^{-\Lambda_t}}{1 - Z_t} X_t f(\Delta_t) d < N, Y >_t \quad (1) \\ &\leq 2\Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t} \right)^2 d < N >_t + 2X_t^2 f(\Delta_t)^2 d < Y >_t \\ &\leq 2\Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t} \right)^2 d < N >_t + 2X_t^2 K^2 \Delta_t^2 d < Y >_t \end{aligned}$$

From this, we can write

$$\int_0^t 1\!\!1_{\{0 < \Delta_s < \varepsilon\}} \frac{1}{\Delta_s^2} d < \Delta >_s < \infty, \, \varepsilon > 0, \, 0 < t < \infty.$$

According to Revuz-Yor[51], $L^0(\Delta) \equiv 0$. The proof is completed. □

Theorem 3.1.2 *[25] Let $0 < u < \infty$. Let X, L are tow solutions of the $\natural-$equation with initial conditions $X_u = x < y = L_u$. Then, $X_t < L_t$ for all $u \leq t \leq \infty$.*

Proof:

Let $L^0(X - L)$ denote the local time at zero of $X - L$. Denoting $\Delta = X - (1 - Z)$ and

$\Delta^L = L - (1 - Z)$, we have

$$d(X_t - L_t) = (X_t - L_t)\left(-\frac{e^{-\Lambda_t}}{1 - Z_t}\right) dN_t + (X_t f(\Delta_t) - L_t f(\Delta_t^L))dY_t$$

So, using the same computation as in (1), we obtain

$$d < X - L >_t \leq 2(X_t - L_t)^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t}\right)^2 d < N >_t + 2(X_t f(\Delta_t) - L_t f(\Delta_t^L))^2 d < Y >_t .$$

Then, using the fact that

$$X_t f(\Delta_t) - L_t f(\Delta_t^L) = (X_t - L_t) f(\Delta_t) + L_t (f(\Delta_t) - f(\Delta_t^L)),$$

we obtain

$$d < X - L >_t \leq 2(X_t - L_t)^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t}\right)^2 d < N >_t + 4(X_t - L_t)^2 f(\Delta_t)^2 d < Y >_t + 4L_t^2 (f(\Delta_t) - f(\Delta_t^L))^2 d < Y >_t$$

$$\leq 2(X_t - L_t)^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t}\right)^2 d < N >_t + 4(X_t - L_t)^2 f(\Delta_t)^2 d < Y >_t + 4L_t^2 K^2 (X_t - L_t)^2 d < Y >_t .$$

It yields that

$$\int_0^t 1\!\!1_{\{0 < X_s - L_s < \epsilon\}} \frac{1}{(X_s - L_s)_s^2} d < X - L >_s < \infty, \; 0 < \epsilon < \infty, \; 0 < t < \infty.$$

Thus $L^0(X - L)$ is identically null. The asserted property is proved. $\square$

3.1.1 Multidimensional version of $\natural-$equation

According to the paper [6], we define the multidimensional version of $\natural-$equation . On a probability space $(\Omega, (\mathbb{F})_{t\geq 0}, \mathbb{P})$. We have:

$$(\natural_u) = \begin{cases} dX^1_{u,t}(x) = X^1_{u,t}(x)\left(-\frac{e^{-\Lambda^1_t}}{1-Z^1_t}dN^1_t + F_{11}dY^1_t + ... + F_{1d}dY^n_t\right) \\ . \qquad\qquad . \\ . \qquad\qquad . \\ . \qquad\qquad . \\ dX^d_{u,t}(x) = X^d_{u,t}(x)\left(-\frac{e^{-\Lambda^d_t}}{1-Z^d_t}dN^d_t + F_{n1}dY^1_t + ... + F_{nd}dY^n_t\right) \end{cases}$$

Then

$$(\natural_u) = \begin{cases} dX_{u,t}(x) = X_t(x)\left(-\frac{e^{-\Lambda_t}}{1-Z_t}dN_t + F(X_t(x) - (1-Z_t))dY_t\right), \, t \in [u, \infty[, \\ X_{u,u}(x) = x, \end{cases}$$

Where $X_{u,t}(x) = (X^1_{u,t}(x), ..., X^d_{u,t}(x))^T$, $-\frac{e^{-\Lambda_t}}{1-Z_t} = \left(-\frac{e^{-\Lambda^1_t}}{1-Z^1_t}, ..., -\frac{e^{-\Lambda^d_t}}{1-Z^d_t}\right)^T$, $x = (x^1, ..., x^d)^T$ the initial condition and:

$$F = \begin{pmatrix} F_{11} & . & . & . & F_{1d} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ F_{n1} & . & . & . & F_{nd} \end{pmatrix}$$

is such that:

$$|F^i_j(x) - F^i_j(y)| \leq \tilde{L}|x-y|, \, \forall\, x, y \in R^d, \, 1 \leq i \leq d, \, 1 \leq j \leq r$$

holds for all indices i, j, where $F^i_j(x)$ is the i-th component of the vector function $F_j(x)$. Then for a given point x of R^d, the $(\natural_u)$-equation has a unique solution such that $X_u = x$. We denote it as $X_t(x)$ or $X_t(x, \omega)$.

$$X^u_t = x + \int_u^t X_s\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)dN_s + \int_u^t X_s \sum_{i=1}^d \sum_{j=1}^n F^{ij}(X_s - (1-Z_s))dY^j_s, \, s \in [u, t].$$

There are several works on the solution of $\natural-$equetion such as: F.Benziadi and A.Kandouci demonstrated the continuity property of the solution in one-dimensional case [7] as well they proved the homeomorphism property of the solution in one-dimensional and multi-dimensional cases ([6], [8]).

3.2 The continuity of the solution of the natural model in the multidimensional case

This section is mainly concerned the continuity property of the stochastic flow generated by the natural model. More precisely, we demonstrate the continuity of the trajectories of the solution in a multi-dimensional case, based on the criterion of Kolmogorov. This is the main motivation of our research. The property appears important in the theory of stochastic differential equations and especially in stochastic flows. The continuity of the stochastic flow has been studied by Philip E.Protter (see [49]) for a general system of equations in the form:

$$X_t^x = H_t^x + \int_0^t F(X^x)_{s_-} dZ_s,$$

where X_t^x and H_t^x are column vectors in R^n, Z is a column vector of m semi-martingales, and F is an $n \times m$ matrix. His study is a direct application of Kolmogorov's theorem .

The same study was also done by H.Kunita (see [32]) but with a detailed proof, for a general system of equations of the form:

$$\xi_{st}^m(x) = x + \sum_{k=0}^{m} \int_0^t V_k(r, \xi_{sr}(x)) dB_r^k,$$

where V_k is a family of vector fields on R^d and B^k is a family of standard Brownian motions, such that this result has been based on Kolmogorov's theorem , Itô's formula and the inequality of Burkholder-Davis-Gundy.

A technical proof based on the use of the Kolmogorov's theorem has been also done by G.Barles and Bernt Oksendal (see [5, 46]), for a general system:

$$X_t = Z + \int_0^t \alpha(r, X_r) dr + \int_0^t \sigma(r, X_r) dW_r,$$

where α and σ are measurable functions, Z is a square integrable random variable and W is a d-dimensional Brownian motion.

3.2.1 Main result

We recall the $\natural$-equation announced above in higher dimensions. Let $(\Lambda_1, ..., \Lambda_d)$ be a d-dimensional continuous increasing process null at the origin, and a d-dimensional continuous non-negative local martingale N such that $Z = N\,e^{-\Lambda}$ with $0 < Z < 1$, $t > 0$ and $Z(t,\omega) = (Z_1(t,\omega), ..., Z_d(t,\omega))$ denotes the default intensity. Let F be continuous, Lipschitz mapping from R^d into itself and $Y(t,\omega) = (Y_1(t,\omega), ..., Y_n(t,\omega))$ denote a n-dimensional continuous local martingale defined on a probability space $(\Omega, F = (F_t)_{t\geq 0}, P)$. We consider the $\natural$-equation in multi-dimensional case :

$$(\natural_u) = \begin{cases} dX_{u,t}(x) = X_t(x)\left(-\frac{e^{-\Lambda_t}}{1-Z_t} dN_t + F(X_t(x) - (1-Z_t))dY_t\right), \, t \in [u, \infty), \\ X_{u,u}(x) = x, \end{cases}$$

and its solution

$$X_t^u = x + \int_u^t X_s\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right) dN_s + \int_u^t X_s \sum_{i=1}^{d}\sum_{j=1}^{n} F^{ij}(X_s - (1-Z_s))dY_s^j, \, s \in [u,t].$$

We know that the quantity $F_j^i(X_s - (1-Z_s))$ is bounded because F is a Lipschitz function, but we do not know a priori if the quantity $\left(-\dfrac{e^{-\Lambda_s}}{1-Z_s}\right)$ is finite or not; we introduce the stopping time $\tau_n = \inf\{t, \, 1 - Z_t < \frac{1}{n}\}$, therefore, we assume the process $\tilde{X}$ instead of X:

$$d\tilde{X}_t = \tilde{X}_t\left(-\frac{e^{-\Lambda_t}}{1-Z_{t\wedge\tau_n}} dN_t + \sum_{i=1}^{d}\sum_{j=1}^{n} F_j^i\,(\tilde{X}_t - (1-Z_t))dY_t^j\right),$$

such as $\tilde{X}_t = X_t$, $\forall t \leq \tau_n$, $n \in N$.

Theorem 3.2.1 *For $t \in [u, \infty)$, the solution of the $\natural$-equation starting at time u is continuous in (u, t, x) under the following hypotheses:*

hypothesis 1*: We keep the same naturel model, but we assume that all the processes indicated in the $\natural$-equation take real values. Thus, we impose that the coefficients of this equation are Lipschitz continuous.*

hypothesis 2*: We always assume the hypothesis mentioned in [7], which denoted that the stochastic integral $\int_u^t \frac{e^{-\Lambda_s}}{1-Z_s} dN_s$, $u \leq t < \infty$, exists and defines a local martingale.*

Now we show the continuity of the solution of the $\natural$-equation by applying the theorem of Kolmogorov (1.1.1) and the lemma of Gronwall (1.1.1).

So, if $x = y$ the inequality is clearly satisfied for any constant $\tilde{K}^{(2)}_{p,T}$. We shall assume $x \neq y$. Let $\tilde{\varepsilon}$ be an arbitrary positive number and:

$$\sigma_{\tilde{\varepsilon}} = \inf\{t > 0,\, |\tilde{X}^u_t(x) - \tilde{X}^u_t(y)| < \tilde{\varepsilon}\}.$$

We denote $A_t = \tilde{X}^u_t(x) - \tilde{X}^u_t(y)$, and we shall apply Itô's formula to the function $f(z) = |z|^p$. Then we have for $t < \tilde{\varepsilon}$;

$$\tilde{X}^u_t(x) = x + \int_u^t \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \int_u^t \sum_{i=1}^d \sum_{j=1}^n \tilde{X}_s F^i_j \left(\tilde{X}_s - (1 - Z_s) \right) dY^j_s.$$

$$d\tilde{X}_t = \tilde{X}_t \left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}} \right) dN_t + \sum_{i=1}^d \sum_{j=1}^n \tilde{X}_t F^i_j \left(\tilde{X}_t - (1 - Z_t) \right) dY^j_t.$$

$$\left| \tilde{X}^u_t(x) - \tilde{X}^u_t(y) \right|^p - |x - y|^p = \sum_{i,j} \int_u^t \frac{\partial f}{\partial z_i} \left(\tilde{X}^u_t(x) - \tilde{X}^u_t(y) \right) \times$$

$$\left(\tilde{X}_s(x) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \tilde{X}_s(x) F^i_j \left(\tilde{X}_s(x) - (1 - Z_s) \right) dY^j_s \; - \right.$$

$$\left. \tilde{X}_s(y) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \tilde{X}_s(y) F^i_j \left(\tilde{X}_s(y) - (1 - Z_s) \right) dY^j_s \right) +$$

$$\frac{1}{2}\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\times$$

$$\left(\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(x)F_k^i\left(\tilde{X}_s(x)-(1-Z_s)\right)dY_s^k-\right.$$

$$\left.\tilde{X}_s(y)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(y)F_k^i\left(\tilde{X}_s(y)-(1-Z_s)\right)dY_s^k\right)\times$$

$$\left(\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(x)F_l^j\left(\tilde{X}_s(x)-(1-Z_s)\right)dY_s^l-\right.$$

$$\left.\tilde{X}_s(y)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\tilde{X}_s(y)F_l^j\left(\tilde{X}_s(y)-(1-Z_s)\right)dY_s^l\right).$$

$$\left|\tilde{X}_t^u(x)-\tilde{X}_t^u(y)\right|^p-|x-y|^p=\sum_{i,j}\int_u^t \frac{\partial f}{\partial z_i}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\times$$

$$\left[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\left(\tilde{X}_s(x)F_j^i\left(\tilde{X}_s(x)-(1-Z_s)\right)-\right.\right.$$

$$\left.\left.\tilde{X}_s(y)F_j^i\left(\tilde{X}_s(y)-(1-Z_s)\right)\right)dY_s^j\right]+\frac{1}{2}\sum_{i,j,k,l}\int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\times$$

$$\left[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\left(\tilde{X}_s(x)F_k^i\left(\tilde{X}_s(x)-(1-Z_s)\right)-\right.\right.$$

$$\left.\left.\tilde{X}_s(y)F_k^i\left(\tilde{X}_s(y)-(1-Z_s)\right)\right)dY_s^k\right]\times$$

$$\left[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\left(\tilde{X}_s(x)F_l^j\left(\tilde{X}_s(x)-(1-Z_s)\right)-\right.\right.$$

$$\tilde{X}_s(y)F_l^j\left(\tilde{X}_s(y)-(1-Z_s)\right)\right)dY_s^l\Big].$$

$$=\tilde{I}_t+\tilde{J}_t$$

$$\tilde{I}_t=\sum_{i,j}\int_u^t\frac{\partial f}{\partial z_i}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\times$$

$$\Big[\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\left(\tilde{X}_s(x)F_j^i\left(\tilde{X}_s(x)-(1-Z_s)\right)-\right.$$

$$\tilde{X}_s(y)F_j^i\left(\tilde{X}_s(y)-(1-Z_s)\right)\right)dY_s^j\Big].$$

Noting

$$V_j^i(\tilde{X}_s^x)=\tilde{X}_s(x)F_j^i\left(\tilde{X}_s(x)-(1-Z_s)\right),$$

$$V_j^i(\tilde{X}_s^y)=\tilde{X}_s(y)F_j^i\left(\tilde{X}_s(y)-(1-Z_s)\right),$$

such that

$$\left|V_j^i(\tilde{X}_s^x)-V_j^i(\tilde{X}_s^y)\right|\leq\tilde{L}\left|\tilde{X}_s^x-\tilde{X}_s^y\right|$$

And

$$\frac{\partial f}{\partial z_i}=p|z|^{p-2}z_i.$$

We put

$$\tilde{I}_t=\tilde{I}_t^1+\tilde{I}_t^2,$$

such that

$$\tilde{I}_t^1=\sum_{i,j}\int_u^t\frac{\partial f}{\partial z_i}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s$$

$$\tilde{I}_t^2 = \sum_{i,j} \int_u^t \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \left(V_j^i(\tilde{X}_s^x) - V_j^i(\tilde{X}_s^y) \right) dY_s^j$$

For $\tilde{I}_t^1$, we have:

$$\sum_i \left| \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right) \right| \leq |p||z|^{p-2}|z_i|\sqrt{d} \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|$$

$$\leq |p|\sqrt{d} \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p$$

Therefore,

$$\tilde{I}_t^1 \leq |p|\sqrt{d} \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds \times \int_u^t -\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} dN_s$$

Noting

$Q_t = \int_u^t -\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} dN_s$, it is a local martingale (so called the hypothesis $H_Y(C)$ [7]).

So

$$\tilde{I}_t^1 \leq |p|n\sqrt{d}\; Q_t \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds$$

For $\tilde{I}_t^2$, we have:

$$\sum_i \left| \frac{\partial f}{\partial z_i} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \left(V_j^i(\tilde{X}_s^x) - V_j^i(\tilde{X}_s^y) \right) \right|$$

$$\leq |p||z|^{p-2}|z_i|\sqrt{d}\, \tilde{L} \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|$$

$$\leq |p|\sqrt{d}\, \tilde{L} \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p$$

Therefore,

$$\tilde{I}_t^2 \leq |p|\sqrt{d}\; n\, \tilde{L} \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds$$

So, we have

$$\begin{aligned}
\tilde{I}_t = \tilde{I}_t^1 + \tilde{I}_t^2 &\leq |p| n\sqrt{d}\; Q_t \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds \\
&\quad + |p|\sqrt{d}\; n\, \tilde{L} \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds \\
&\leq |p| n\sqrt{d} \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds \,(Q_t + \tilde{L})
\end{aligned}$$

Therefore, we have

$$\left|E\, \tilde{I}_{t\wedge\sigma_{\tilde{\varepsilon}}}\right| \leq |p| n\sqrt{d}\,(Q_{t\wedge\sigma_{\tilde{\varepsilon}}} + \tilde{L}) \int_u^t E\left|\tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(x) - \tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p ds \tag{3.1}$$

Next,

$$\begin{aligned}
\tilde{J}_t &= \frac{1}{2} \sum_{i,j,k,l} \int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \\
&\times \left[\left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \left(V_k^i(\tilde{X}_s^x) - V_k^i(\tilde{X}_s^y)\right) dY_s^k\right] \\
&\times \left[\left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \left(V_l^j(\tilde{X}_s^x) - V_l^j(\tilde{X}_s^y)\right) dY_s^l\right]
\end{aligned}$$

$$\begin{aligned}
\tilde{J}_t &= \frac{1}{2} \sum_{i,j,k,l} \int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \\
&\times \left[\left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)^2 \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s + \left(V_k^i(\tilde{X}_s^x) - V_k^i(\tilde{X}_s^y)\right)\right. \\
&\times \left(V_l^j(\tilde{X}_s^x) - V_l^j(\tilde{X}_s^y)\right) dY_s^k dY_s^l \; + \left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) \\
&\times \left(V_l^j(\tilde{X}_s^x) - V_l^j(\tilde{X}_s^y)\right) \times dN_s\, dY_s^l + \left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) \\
&\times \left.\left(V_k^i(\tilde{X}_s^x) - V_k^i(\tilde{X}_s^y)\right) dN_s\, dY_s^k\right]
\end{aligned}$$

Noting $\tilde{J}_t = \frac{1}{2}\left[\tilde{J}_t^1 + \tilde{J}_t^2 + \tilde{J}_t^3 + \tilde{J}_t^4\right]$ such that:

$$\tilde{J}_t^1 = \sum_{i,j,k,l} \int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \times \left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)^2$$

$$\times \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s$$

$$\tilde{J}_t^2 = \sum_{i,j,k,l} \int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \times \left(V_k^i(\tilde{X}_s^x) - V_k^i(\tilde{X}_s^y)\right)$$

$$\times \left(V_l^j(\tilde{X}_s^x) - V_l^j(\tilde{X}_s^y) \right) dY_s^k dY_s^l$$

$$\tilde{J}_t^3 = \sum_{i,j,k,l} \int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right)$$

$$\times \left(V_l^j(\tilde{X}_s^x) - V_l^j(\tilde{X}_s^y) \right) dN_s \, dY_s^l$$

$$\tilde{J}_t^4 = \sum_{i,j,k,l} \int_u^t \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right)$$

$$\times \left(V_k^i(\tilde{X}_s^x) - V_k^i(\tilde{X}_s^y) \right) dN_s \, dY_s^k$$

and note that

$$\frac{\partial^2 f}{\partial z_i \partial z_j} = p|z|^{p-2}\delta_{ij} + p(p-2)|z|^{p-4} z_i z_j$$

where δ_{ij} is the Kronecker's delta, then. For $\tilde{J}_t^1$, we have

$$\left| \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right)^2 \right|$$

$$\leq \left| \left(p|z|^{p-2}\delta_{ij} d + p(p-2)|z|^{p-4} z_i z_j \right) \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right)^2 \right|$$

$$\leq |p| \left(|p-2| + d \right) \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p$$

Therefore,

$$\tilde{J}_t^1 \leq |p| \left(|p-2| + d \right) \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds \int_u^t \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right)^2 dN_s dN_s$$

The hypothesis $H_Y(C)$ is always assumed, so

$$\tilde{J}_t^1 \leq |p| \left(|p-2| + d \right) Q_t^2 \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds$$

For $\tilde{J}_t^2$, we have

$$\left|\sum_{i,j}\frac{\partial^2 f}{\partial z_i\partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\times\left(V_k^i(\tilde{X}_s^x)-V_k^i(\tilde{X}_s^y)\right)\left(V_l^j(\tilde{X}_s^x)-V_l^j(\tilde{X}_s^y)\right)\right|$$

$$\leq\left|\left(p|z|^{p-2}\delta_{ij}d+p(p-2)|z|^{p-4}z_iz_j\right)\tilde{L}^2\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)^2\right|$$

$$\tilde{J}_t^2\leq|p|\left(|p-2|+d\right)\tilde{L}^2\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p$$

So

$$\tilde{J}_t^2\leq|p|\left(|p-2|+d\right)\tilde{L}^2\,n^2\int_u^t\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds$$

For $\tilde{J}_t^3$, we have

$$\left|\sum_{i,j}\frac{\partial^2 f}{\partial z_i\partial z_j}\left(\tilde{X}_s^u(x)-\tilde{X}_s^u(y)\right)\times\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\left(V_l^j(\tilde{X}_s^x)-V_l^j(\tilde{X}_s^y)\right)\right|$$

$$\leq\left|\left(p|z|^{p-2}\delta_{ij}d+p(p-2)|z|^{p-4}z_iz_j\right)\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\tilde{L}\,r\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\right|$$

$$\leq|p|\left(|p-2|+d\right)\tilde{L}\,n\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p$$

The hypothesis $H_Y(C)$ is always assumed, so

$$\tilde{J}_t^3\leq|p|\left(|p-2|+d\right)\tilde{L}\,n\,Q_t\int_u^t\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds$$

For $\tilde{J}_t^4$, we have also

$$\tilde{J}_t^4\leq|p|\left(|p-2|+d\right)\tilde{L}\,n\,Q_t\int_u^t\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds$$

$$\tilde{J}_t=\frac{1}{2}\left[\tilde{J}_t^1+\tilde{J}_t^2+\tilde{J}_t^3+\tilde{J}_t^4\right]$$

$$\leq\frac{1}{2}\left[|p|\left(|p-2|+d\right)\,Q_t^2\int_u^t\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds\right.$$

$$+|p|\,(|p-2|+d)\,\tilde{L}^2\,n^2\int_u^t\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds$$

$$+2|p|\,(|p-2|+d)\,\tilde{L}\,n\,Q_t\int_u^t\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds\Big]$$

$$\leq\frac{1}{2}\left[|p|\,(|p-2|+d)\int_u^t\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds\left(Q_t^2+\tilde{L}^2\,n^2+2\tilde{L}\,n\,Q_t\right)\right]$$

Therefore,

$$\left|E\,\tilde{J}_{t\wedge\sigma_{\tilde{\varepsilon}}}\right|\leq\frac{1}{2}|p|\,(|p-2|+d)\left(Q_t+n\,\tilde{L}\right)^2\int_u^t E\left|\tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(x)-\tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p ds. \tag{3.2}$$

Summing up these two inequalities 3.1 and 3.2, we obtain

$$E\left|\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(x)-\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p\leq|x-y|^p+\tilde{C}_p\int_u^t E\left|\tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(x)-\tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p ds,$$

where $\tilde{C}_p$ is a positive constant.

By Grönwall's inequality we have

$$E\left|\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(x)-\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p\leq K^{(2)}_{p,u}\,|x-y|^p,\quad u\leq t\leq\infty$$

such that

$$K^{(2)}_{p,u}\,|x-y|^p=\exp(\tilde{C}_p u).$$

Letting $\tilde{\varepsilon}$ tend to 0, we have

$$E\left|\tilde{X}^u_{t\wedge\sigma}(x)-\tilde{X}^u_{t\wedge\sigma}(y)\right|^p\leq K^{(2)}_{p,u}\,|x-y|^p,$$

where σ is the first time such that $\tilde{X}^u_t(x)=\tilde{X}^u_t(y)$. However, we have $\sigma=\infty$ a.s , since otherwise the left hand side would be infinity if $p<0$. The proof is complete. □

3.3 the differentiability of the solution of the natural model in One-dimensional and multidimensional cases

The differentiability property has been treated by several mathematicians for different stochastic differential equations under different conditions such as H-Kunita [34] proved that the solution of stochastic differential equation based on lévy processes is differentiable with respect to the initial state if coefficients of the equation are smooth. However, the homeomorphic property or the diffeomorphic property is not always satisfied owing to the behavior of jumps. Therefore, he showed that the solution defines a stochastic flow of homeomorphisms, if it makes a "homeomorphic" jump such that the most materials of this results have been chosen from the joint works with T-Fujiwara [22]. The same Property has been studied by A.Stefano [54] such that he demonstrated the existence of stochastic flow of class $C^{1,\alpha}$ for one-dimensional stochastic differential equation with discontinuous drift.In this chapter, we will be mainly concerned the differentiability of the solution of the natural model with respect to the initial value in one-dimensional and multidimensional cases under the Lipschitz and continuous coefficients.

3.3.1 The differentiability of the solution of the one-default model in one-dimensional case

We recall the $\natural-$equation in one-dimensional case:

$$(\natural_u) = \begin{cases} dX^x_{u,t} = X^x_{u,t}\left(-\frac{e^{-\Lambda_t}}{1-Z_t}dN_t + f(X_t - (1-Z_t))dY_t\right), \, t \in [u,\infty), \\ X^x_{u,u} = x, \end{cases}$$

where x is the initial condition that is $\mathcal{F}_u-$measurable random variable.
This equation has a unique continuous solution such that:

$$X^x_{u,t} = x + \int_u^t X^x_s \left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right) dN_s + \int_u^t X^x_s f(X_s - (1-Z_s))dY_s, \, s \in [u,t],$$

We will look at the differentiability of $X^x_{u,t}$ with respect to the initial data, Precisely, We show the existence of the derivative term in the initial value basing on the gronwall's lemma , Itô's isometry and Burkholder-Davis-Gundy and *Hölder* inequalities. This is our path in our research.

Main result

we have the quantity $f(X_s - (1-Z_s))$ is bounded because of f is a Lipschitz function. But we don't know whether the quantity $\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)$ is finite or not that's why we introduce the stopping time $\tau_n = inf\{t, 1 - Z_t < \frac{1}{n}\}$ Therefore, we suppose the process $\hat{X}^x_{u,t}$ instead of $X^x_{u,t}$

$$\hat{X}^x_{u,t} = x + \int_u^t \hat{X}^x_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \int_u^t \hat{X}^x_s f(\hat{X}^x_s - (1 - Z_s)) dY_s,$$

such as $\hat{X}^x_{u,t} = X^x_{u,t}, \forall t \leq \tau_n, n \in \mathbb{N}$.

Theorem 3.3.1 *For $t \in [u, \infty)$, the solution $X^x_{u,t}$ is differentiable, it means: for evrey $x \in \mathbb{R}$, the limit:*

$$\lim_{h\longrightarrow 0} \frac{\hat{X}^{x+h}_t - \hat{X}^x_t}{h},$$

exists in $L^p(\Omega \times [u, t]; \mathbb{R})$.

Proof: In order to show the existence of the derivative term, it's enough to prove the existence of the above limit in $L^p(\Omega \times [u, t]; \mathbb{R})$.

We denote $Z^h_t = \frac{\bar{X}_t}{h}$ where $\bar{X}_t = \hat{X}^{x+h}_t - \hat{X}^x_t$ and $M_t = \left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}} \right)$.

Firstly for $p = 1$: so we have

$$\begin{aligned} Z^h_t &= \frac{1}{h}\left[\left((x + h) + \int_u^t \hat{X}^{x+h}_s M_s dN_s + \int_u^t \hat{X}^{x+h}_s f(\hat{X}^{x+h}_s - (1 - Z_s)) dY_s \right) \right] \\ &- \frac{1}{h}\left[\left(x + \int_u^t \hat{X}^x_s M_s dN_s + \int_u^t \hat{X}^x_s f(\hat{X}^x_s - (1 - Z_s)) dY_s \right) \right], \end{aligned} \tag{3.3}$$

so

$$Z^h_t = 1 + \frac{1}{h}\left[\int_u^t \bar{X}_t M_s dN_s + \int_u^t \hat{X}^{x+h}_s f(\hat{X}^{x+h}_s - (1 - Z_s)) - \hat{X}^x_s f(\hat{X}^x_s - (1 - Z_s)) dY_s \right]. \tag{3.4}$$

Noting

$$I_1 = \int_u^t \bar{X}_t M_s dN_s$$

$$I_2 = \int_u^t \hat{X}^{x+h}_s f(\hat{X}^{x+h}_s - (1 - Z_s)) - \hat{X}^x_s f(\hat{X}^x_s - (1 - Z_s)) dY_s$$

Then BDG's inequality, yields

$$\mathbb{E}\left[\sup_{u\leq t<\infty}|I_1|\right] \leq C_{13}^{(P)}\mathbb{E}\left[\left(\int_u^t |\bar{X}_s|^2|M_s|^2 ds\right)^{1/2}\right], \tag{3.5}$$

and by Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}\left[\sup_{u\leq t<\infty}|I_1|\right] \leq C_{13}^{(P)}\mathbb{E}\left[\sup_{u\leq t<\infty}|\bar{X}_t|\left(\int_u^t |M_s|^2 ds\right)^{1/2}\right], \tag{3.6}$$

and by following, we have: $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$

$$\mathbb{E}\left[\sup_{u\leq t<\infty}|I_1|\right] \leq C_{14}^{(P)}\left(\mathbb{E}\left[\sup_{u\leq t<\infty}|\bar{X}_t|^2\right] + \mathbb{E}\left[\int_u^t |M_s|^2 ds\right]\right). \tag{3.7}$$

Then, by virtue of proposition (1.1.2). We obtain for some constants $R_1, \mathcal{C}$

$$\begin{aligned}\mathbb{E}\left[\sup_{u\leq t<\infty}|I_1|\right] &\leq C_{14}^{(P)}(R_1h^2+\mathcal{C}) \\ &\leq C_{15}^{(P)},\end{aligned} \tag{3.8}$$

where $C_{15}^{(P)} = C_{14}^{(P)}(R_1h^2+\mathcal{C})$ and $\mathbb{E}\left[\int_u^t |M_s|^2 ds\right] < \infty$.

For all $\epsilon > 0$ and $a, b \geq 0$, from $ab \leq \frac{a^2}{\epsilon 2} + \epsilon\frac{b^2}{2}$, it follows that

$$I_2 \leq \frac{1}{2}\int_u^t \frac{(\hat{X}_s^{x+h})^2 - (\hat{X}_s^x)^2}{\epsilon} + \epsilon\left(f(\hat{X}_s^{x+h} - (1-Z_s))^2 - f(\hat{X}_s^x - (1-Z_s))^2\right) dY_s.$$

Noting

$$f(\hat{X}_s^{x+h} - (1-Z_s)) = \theta(\hat{X}_s^{x+h})$$

$$f(\hat{X}_s^x - (1-Z_s)) = \theta(\hat{X}_s^x)$$

Therefore

$$\theta(\hat{X}_s^{x+h})^2 - \theta(\hat{X}_s^x)^2 = \left(\theta(\hat{X}_s^{x+h}) - \theta(\hat{X}_s^x)\right)\left(\theta(\hat{X}_s^{x+h}) + \theta(\hat{X}_s^x)\right),$$

and since the function f is Lipschitzian, there's a real positive constant k, such that

$$|\theta(\hat{X}_s^{x+h}) - \theta(\hat{X}_s^x)| \le k|\hat{X}_s^{x+h} - \hat{X}_s^x|,$$

so

$$|I_2| \le \frac{1}{2\epsilon}\int_u^t |\hat{X}_s^{x+h}|^2 + |-\hat{X}_s^x|^2 dY_s + k\frac{\epsilon}{2}\int_u^t |\hat{X}_s^{x+h} - \hat{X}_s^x||\theta(\hat{X}_s^{x+h}) + \theta(\hat{X}_s^x)|dY_s.$$

Thus, from BDG's inequality, it follows that, for some constant $C_\epsilon, C_{\epsilon,k} \in \mathbb{R}^+$:

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le t<\infty}|I_2|\right] &\le C_\epsilon\left(\mathbb{E}\left[\left(\int_u^t |\hat{X}_s^{x+h}|^4 ds\right)^{1/2}\right] + \mathbb{E}\left[\left(\int_u^t |\hat{X}_s^x|^4 ds\right)^{1/2}\right]\right)\\ &+ C_{\epsilon,k}\mathbb{E}\left[\left(\int_u^t |\hat{X}_s^{x+h} - \hat{X}_s^x|^2|\theta(\hat{X}_s^{x+h}) + \theta(\hat{X}_s^x)|^2 ds\right)^{1/2}\right]. \qquad (3.9)\end{aligned}$$

Using Cauchy-Schwarz inequality, and for some constants $C_\epsilon, C_{\epsilon,k}, \alpha \in \mathbb{R}^+$

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le t<\infty}|I_2|\right] &\le C_\epsilon T_1^{1/2}\left(\mathbb{E}\left[\int_u^t |\hat{X}_s^{x+h}|^2 ds\right] + \mathbb{E}\left[\int_u^t |\hat{X}_s^x|^2 ds\right]\right)\\ &+ C_{\epsilon,k}\alpha\mathbb{E}\left[\int_u^t |\hat{X}_s^{x+h} - \hat{X}_s^x| ds\right], \qquad (3.10)\end{aligned}$$

where $|\theta(\hat{X}_s^{x+h}) + \theta(\hat{X}_s^x)| = \alpha$,

Therefore

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le t<\infty}|I_2|\right] &\le C_\epsilon T_1^{1/2}\left(\int_u^t \mathbb{E}\left[\sup_{u\le s<\infty}|\hat{X}_s^{x+h}|^2 ds\right] + \int_u^t \mathbb{E}\left[\sup_{u\le s<\infty}|\hat{X}_s^x|^2 ds\right]\right)\\ &+ C_{\epsilon,k}\alpha\int_u^t \mathbb{E}\left[\sup_{u\le s<\infty}|\hat{X}_s^{x+h} - \hat{X}_s^x|\right] ds. \qquad (3.11)\end{aligned}$$

By the proposition (4),we get for some constants $k_1, k_2 \in \mathbb{R}^+$ and $\forall x \in \mathbb{R}$

$$\mathbb{E}\left[\sup_{u\le t<\infty}|I_2|\right] \le C_\epsilon T_1^{1/2}(k_1(1+|x+h|^2) - k_2(1+|x|^2)) + C_{\epsilon,k}\alpha\int_u^t \mathbb{E}\left[\sup_{u\le s<\infty}|\hat{X}_s^{x+h} - \hat{X}_s^x| ds\right]; \qquad (3.12)$$

We denote

$$\mu = C_\epsilon T_1^{1/2}(k_1(1+|x+h|^2)+k_2(1+|x|^2))$$

$$\nu = C_{\epsilon,k}\alpha$$

From (3.4), (3.8)and (3.12),we have

$$\mathbb{E}\left[\sup_{u\le t<\infty}|Z_t^h|\right] \le 1+\frac{1}{h}(C_{15}^{(P)}+\mu)+\nu\int_u^t \mathbb{E}\left[\sup_{u\le s<\infty}|Z_s^h|\right]ds. \tag{3.13}$$

Then Gronwall's lemma , yields

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le t<\infty}|Z_t^h|\right] &\le \delta\exp\left(\nu\int_u^t ds\right)\\ &\le \delta',\end{aligned} \tag{3.14}$$

where

$$\delta = 1+\frac{1}{h}(C_{15}^{(P)}+\mu)$$

$$\delta' = \delta\exp\left(\nu\int_u^t ds\right)$$

Letting $h\to 0$, we get

$$\lim_{h\to 0}\mathbb{E}\left[\sup_{u\le t<\infty}|Z_t^h|\right]\le\delta'.$$

□

Secondly in the case $p=2$. We start by the first term I_1, by Itô's isometry, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le t<\infty}|I_1|^2\right] &\le \mathbb{E}\left[\left(\sup_{u\le t<\infty}\int_u^t|\bar{X}_t||M_s|dN_s\right)^2\right]\\ &\le \mathbb{E}\left[\sup_{u\le t<\infty}\int_u^t|\bar{X}_t|^2|M_s|^2ds\right]\\ &\le \mathbb{E}\left[\sup_{u\le t<\infty}|\bar{X}_t|^2\int_u^t|M_s|^2ds\right],\end{aligned} \tag{3.15}$$

and by following, we have: $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$

$$\mathbb{E}\left[\sup_{u\leq t<\infty} |I_1|^2\right] \leq \frac{1}{2}\left(\left(\mathbb{E}\left[\sup_{u\leq t<\infty} |\bar{X}_t|^2\right]\right)^2 + \left(\mathbb{E}\left[\int_u^t |M_s|^2 ds\right]\right)^2\right). \tag{3.16}$$

Then by the proposition (1.1.2). We obtain for some constants $R_2, \mathcal{C}$

$$\mathbb{E}\left[\sup_{u\leq t<\infty} |I_1|^2\right] \leq \frac{1}{2}(R_2 h^4 + \mathcal{C}^2). \tag{3.17}$$

Let's move to second term I_2. we have

$$I_2 \leq \frac{1}{2}\int_u^t \frac{(\hat{X}_s^{x+h})^2 - (\hat{X}_s^x)^2}{\epsilon} + \epsilon\left(\theta(\hat{X}_s^{x+h})^2 - \theta(\hat{X}_s^x)^2\right) dY_s,$$

and since the function f is Lipschitzian, there exist a real positive constant k, such that

$$|\theta(\hat{X}_s^{x+h}) - \theta(\hat{X}_s^x)| \leq k|\hat{X}_s^{x+h} - \hat{X}_s^x|$$

so

$$|I_2| \leq \frac{1}{2\epsilon}\int_u^t |\hat{X}_s^{x+h}|^2 + |\hat{X}_s^x|^2 dY_s + \alpha k\frac{\epsilon}{2}\int_u^t |\hat{X}_s^{x+h} - \hat{X}_s^x| dY_s.$$

by It$\hat{o}$'s isometry, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{u\leq t<\infty} |I_2|^2\right] &\leq C_\epsilon\left(\mathbb{E}\left[\left(\int_u^t \sup_{u\leq s<\infty} |\hat{X}_s^{x+h}|^2 dY_s\right)^2\right] + \mathbb{E}\left[\left(\int_u^t \sup_{u\leq s<\infty} |\hat{X}_s^x|^2 dY_s\right)^2\right]\right)\\ &+ C_{\epsilon,k}\alpha\mathbb{E}\left[\left(\int_u^t \sup_{u\leq s<\infty} |\hat{X}_s^{x+h} - \hat{X}_s^x| dY_s\right)^2\right]. \end{aligned}\tag{3.18}$$

Therefore

$$\begin{aligned}\mathbb{E}\left[\sup_{u\leq t<\infty} |I_2|^2\right] &\leq C_\epsilon\left(\int_u^t \mathbb{E}\left[\sup_{u\leq s<\infty} |\hat{X}_s^{x+h}|^4\right] ds + \int_u^t \mathbb{E}\left[\sup_{u\leq s<\infty} |\hat{X}_s^x|^4\right] ds\right)\\ &+ C_{\epsilon,k}\alpha\int_u^t \mathbb{E}\left[\sup_{u\leq s<\infty} |\hat{X}_s^{x+h} - \hat{X}_s^x|^2\right] ds. \end{aligned}\tag{3.19}$$

Then by the proposition (4), we obtain for some constants $L_1, L_2 \in \mathbb{R}^+$ and $\forall x \in \mathbb{R}$

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le t<\infty}|I_2|^2\right] &\le C_\epsilon(L_1(1+|x+h|^4)+L_2(1+|x|^4)) \\ &+ C_{\epsilon,k}\alpha\int_u^t \mathbb{E}\left[\sup_{u\le s<\infty}|\hat{X}_s^{x+h}-\hat{X}_s^{x}|^2 ds\right]. \qquad (3.20)\end{aligned}$$

We denote

$$a_1 = C_\epsilon L_1(1+|x+h|^4)$$
$$a_2 = L_2(1+|x|^4)$$
$$a_3 = C_{\epsilon,k}\alpha$$

From (3.4), (3.17)and (3.19),we have

$$\mathbb{E}\left[\sup_{u\le t<\infty}|Z_t^h|^2\right] \le 1+\frac{1}{h^2}(\frac{1}{2}(R_2h^4+\mathcal{C}^2)+a_1+a_2)+a_3\int_u^t \mathbb{E}\left[\sup_{u\le s<\infty}|Z_s^h|^2\right] ds. \qquad (3.21)$$

Then Gronwall's lemma , yields

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le t<\infty}|Z_t^h|^2\right] &\le \beta_1\exp\left(a_3\int_u^t ds\right) \\ &\le \beta_2, \qquad (3.22)\end{aligned}$$

where $\beta_1 = 1+\frac{1}{h}(\frac{1}{2}(R_2h^4+\mathcal{C}^2)+a_1+a_2)$ and $\beta_2 = \beta_1\exp\left(a_3\int_u^t ds\right)$.

Letting $h\to 0$, we get

$$\lim_{h\to 0}\mathbb{E}\left[\sup_{u\le t<\infty}|Z_t^h|^2\right] \le \beta_2.$$

□

It remains to study the case $p>2$. For the first term, using BDG's and Hölder inequalities, noting q^* the conjugate of $\frac{p}{2}$

$$\mathbb{E}|\sup_{u\le t<\infty} I_1|^p \le C_{16}^{(p)}\mathbb{E}\left[\left(\int_u^t |\bar{X}_s|^2|M_s|^2 ds\right)^{\frac{p}{2}}\right]$$

$$\leq \quad C_{16}^{(p)}(t-u)^{\frac{p}{2q^*}}\mathbb{E}\left[\sup_{u\leq t<\infty}|\bar{X}_s|^p\int_u^t|M_s|^p ds\right]. \qquad (3.23)$$

So we have: $ab\leq\frac{a^2}{2}+\frac{b^2}{2}$

$$\mathbb{E}|\sup_{u\leq t<\infty}I_1|^p\leq\frac{1}{2}C_{16}^{(p)}(t-u)^{\frac{p}{2q^*}}\left(\mathbb{E}\left[\sup_{u\leq t<\infty}|\bar{X}_s|^p\right]^2+\mathbb{E}\left[\int_u^t|M_s|^p ds\right]^2\right), \qquad (3.24)$$

depending on the proposition (1.1.2). We obtain

$$\begin{aligned}\mathbb{E}|\sup_{u\leq t<\infty}I_1|^p &\leq \frac{1}{2}C_{16}^{(p)}(t-u)^{\frac{p}{2q^*}}(R_3h^{2p}+\tilde{C}^2)\\ &\leq C_{17}^{(p)}, \qquad (3.25)\end{aligned}$$

where $C_{17}^{(p)}=\frac{1}{2}C_{16}^{(p)}(t-u)^{\frac{p}{2q^*}}(R_3h^{2p}+\tilde{C}^2)$.

it remains the second term I_2. Using again BDG's and Hölder inequalities, noting q^* the conjugate of $\frac{p}{2}$

$$\begin{aligned}\mathbb{E}|\sup_{u\leq t<\infty}I_2|^p &\leq C_{p,\epsilon}\mathbb{E}\left[\int_u^t\sup_{u\leq s<\infty}|\hat{X}_s^{x+h}|^4+\sup_{u\leq s<\infty}|\hat{X}_s^x|^4 ds\right]^{\frac{p}{2}}+C_{\epsilon,k,p}\alpha\int_u^t\mathbb{E}\left[\sup_{u\leq s<\infty}|\hat{X}_s^{x+h}-\hat{X}_s^x|^2 ds\right]^{\frac{p}{2}}\\ &\leq C_{p,\epsilon}(t-u)^{\frac{p}{2q^*}+1}\mathbb{E}\left[\sup_{u\leq s<\infty}|\hat{X}_s^{x+h}|^{2p}\right]+\mathbb{E}\left[\sup_{u\leq s<\infty}|\hat{X}_s^x|^{2p}ds\right]\\ &+ C_{\epsilon,k,p}\alpha(t-u)^{\frac{p}{2q^*}}\int_u^t\mathbb{E}\left[\sup_{u\leq s<\infty}|\hat{X}_s^{x+h}-\hat{X}_s^x|^p ds\right], \qquad (3.26)\end{aligned}$$

depending on the proposition (4), we get

$$\begin{aligned}\mathbb{E}|\sup_{u\leq t<\infty}I_2|^p &\leq C_{p,\epsilon}(t-u)^{\frac{p}{2q^*}+1}(K_1(1+|x+h|^{2p})+K_2(1+|x|^{2p}))\\ &+ C_{\epsilon,k,p}\alpha(t-u)^{\frac{p}{2q^*}}\int_u^t\mathbb{E}\left[\sup_{u\leq s<\infty}|\hat{X}_s^{x+h}-\hat{X}_s^x|^p ds\right]. \qquad (3.27)\end{aligned}$$

Thus, from (3.4), (3.25)and (3.27),we have

$$\mathbb{E}|\sup_{u\leq t<\infty} Z_t^h|^p \leq 1+\frac{1}{h^p}(C_{17}^{(p)}+B_1)+B_2\int_u^t \mathbb{E}\left[\sup_{u\leq s<\infty}|Z_t^h|^p ds\right], \tag{3.28}$$

where

$$B_1 = C_{p,\epsilon}(t-u)^{\frac{p}{2q^*}+1}(K_1(1+|x+h|^{2p})+K_2(1+|x|^{2p}))$$

$$B_2 = C_{\epsilon,k,p}\alpha(t-u)^{\frac{p}{2q^*}},$$

and Gronwall's lemma , yields

$$\begin{aligned}\mathbb{E}|\sup_{u\leq t<\infty} Z_t^h|^p &\leq (1+\frac{1}{h^p}(C_{17}^{(p)}+B_1))\exp(B_2(t-u)) \\ &\leq B_3, \end{aligned}\tag{3.29}$$

where $B_3 = (1+\frac{1}{h^p}(C_{17}^{(p)}+B_1))\exp(B_2(t-u))$.

Letting $h \longrightarrow 0$, we get

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{u\leq t<\infty}|Z_t^h|^p\right] \leq B_3.$$

The proof is complete.

3.3.2 The differentiability of the solution of the natural model in multidimensional case

the differentiability property is important in the theory of stochastic flow which has been proved by Olga.V. Aryasova and Andrey.Yu. Pilipenko [4] for general system in the form:

$$\begin{cases} d\zeta_t(x) = a(\zeta_t(x))dt + dw_t, \\ \zeta_0(x) = x, \end{cases}$$

Where $x \in \mathbb{R}^d$, $d \geq 1$, $(w_t)_{t\geq 0}$ is a $d-$dimensional Wiener process, $a = (a^1, ..., a^d)$ is a bounded measurable mapping from $\mathbb{R}^d \, to \, \mathbb{R}^d$.

This equation has a unique strong solution (see [4]). The differentiability of this solution with respect to initial data is given in the following theorem.

Theorem 3.3.2 *Let $a : \mathbb{R}^d \to \mathbb{R}^d$ be such that for all $1 \leq i \leq d$, a^i is a function of bounded variation on $\mathbb{R}^d$. Put $\mu^{ij} = \frac{\partial a^i}{\partial x_j}$. Assume that the measures $|\mu^{ij}|, 1 \leq i, j \leq d$, belong to Kato's class. Let $\phi_t(x), t \geq 0$, be a solution to the integral equation*

$$\phi_t(x) = E + \int_0^t dA_s(\zeta(x))\phi_s(x), \tag{3.30}$$

Where E is $d \times d-$identity matrix, the integral on the right-hand side of (3.30)is the Lebesgue-Stieltjes integral with respect to the continuous function of bounded variation $t \to A_t(\zeta(x))$. Then $\phi_t(x)$ is the derivative of $\zeta_t(x)$ in L^p-sense : for all $p > 0$, $x \in \mathbb{R}^d$, $h \in \mathbb{R}^d$, $t > 0$,

$$\mathbb{E}\left\|\frac{\zeta_t(x+h) - \zeta_t(x)}{\epsilon} - \phi_t(x)h\right\|^p \to 0,\, \epsilon \to 0, \tag{3.31}$$

where $\|.\|$ is a norm in the space $\mathbb{R}^d$. Moreover,

$$\mathbb{P}\{\forall t \geq 0 : \zeta_t(.) \in W^1_{p,loc}(\mathbb{R}^d, \mathbb{R}^d),\, \nabla\zeta_t(x) = \phi_t(x)\, for\, \lambda - a.a.x\} = 1,$$

where λ is the Lebesgue measure on $\mathbb{R}^d$.

Proof:
Define approximating equations by (3.3)(see [4]). Where $a_n, n \geq 1$ are determined by (3.1) see [4] . From Lemma (3.1) (see [4]) and the dominated convergence theorem we get the relation

$$\mathbb{E} \sup_{t\in[0,T]} \int_U |\zeta^i_{n,t}(x) - \zeta^i_t(x)|^p dx \to 0,\, n \to \infty,$$

valid for any bounded domain $U \in \mathbb{R}^d$, $T > 0$, $p \geq 1\, and\, 1 \leq i \leq d$. So for each $1 \leq i \leq d$ there exists a subsequence $\{n^i_k, k \geq\}$ such that

$$\sup_{t\in[0,T]} \int_U |\zeta^i_{n^i_k,t}(x) - \zeta^i_t(x)|^p dx \to 0,\, a.s.\, as\, k \to \infty.$$

Without loss of generality we can suppose that

$$\sup_{t\in[0,T]} \int_U |\zeta^i_{n,t}(x) - \zeta^i_t(x)|^p dx \to 0,\, a.s.\, as\, n \to \infty, \tag{3.32}$$

Arguing similarly and taking into account Lemma (4.1) (see [4]) we arrive at the relation

$$\sup_{t\in[0,T]}\int_U |\phi^{ij}_{n,t}(x)-\phi^{ij}_t(x)|^p dx \to 0,\ n\to\infty,\ almost\, surely, \tag{3.33}$$

which is fulfilled for all $1\le i,j\le d$, $p\ge 0$.

Since the Sobolev space is a Banach space, the relations (3.32) and (3.33) mean that $\phi_t(x)$ is the matrix of derivatives of the solution.

Let us verify (1.6). We have for all $x,h\in\mathbb{R}^d$, $\alpha\in\mathbb{R}$,

$$\zeta_{n,t}(x+\alpha h)=\zeta_{n,t}(x)+\int_0^\alpha \phi_{n,t}(x+uh)du.$$

It follows from Lemmas (3.1) and (4.1) (see [4]) that

$$\zeta_t(x+\alpha h)=\zeta_t(x)+\int_0^\alpha \phi_t(x+uh)du. \tag{3.34}$$

The lemma (5.1) (see [4]) implies the relation

$$\forall x'_0\in\mathbb{R}^d:\phi_t(x')\to\phi_t(x'_0), x'\to x'_0, \tag{3.35}$$

in probability and hence in all L^p. This completes the proof of the Theorem, as (3.34) and (3.35) implies (3.31).

The same study was also done by Philip E. Protter [49] for general system of equation in form:

$$\begin{aligned}&\varphi^i_t = x_i+\textstyle\sum_{\alpha=1}^m\int_0^t f^i_\alpha(\varphi_{s^-})dZ^\alpha_s\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (D)\\ &D^i_{kt}=\delta^i_k+\textstyle\sum_{\alpha=1}^m\sum_{j=1}^n\int_0^t \frac{\partial f^i_\alpha}{\partial x_j}(\varphi_{s^-})D^j_{ks}dZ^\alpha_s,\end{aligned}$$

$(1\le i\le n)$ where D denotes an $n\times n$ matrix-valued process and $\delta^i_k=1$ if $i=k$ and 0 otherwise (Kronecker's delta). A convenient convention, sometimes called the Einstein convention, is to leave the summations implicit. Thus the system of equations (D) can be alternatively written as

$$\varphi_t^i = x_i + \int_0^t f_\alpha^i(\varphi_{s^-})dZ_s^\alpha$$

$$(D)$$

$$D_{kt}^i = \delta_k^i + \int_0^t \frac{\partial f_\alpha^i}{\partial x_j}(\varphi_{s^-})D_{ks}^j dZ_s^\alpha,$$

Theorem 3.3.3 *Let Z be as in (H_1) and let the functions (f_α^i) in (H_2) have locally Lipschitz first partial derivatives. Then for almost all w there exists a function $\varphi(t,w,x)$ which is continuously differentiable in the open set $\{x : \rho(x,w) > t\}$, where ρ is the explosion time (see [49]. theorem 38). If (f_α^i) are globally Lipschitz then $\rho = \infty$. Let $D_k(t,w,x) \equiv \frac{\partial}{\partial x_k}\varphi(t,w,x)$. Then for each x the process $(\varphi(.,w,x), D(.,w,x))$ is identically càdlàg, and it is the solution of equations (D) on $[0, \rho(x,.)]$.*

Proof:(see [49]) We will give the proof in several steps. in Step 1 we will reduce the problem to one where the coefficients are globally Lipschitz. We then resolve the first system $(for\ \varphi)$ of (D). In second step we will show that, given φ, there exists a "nice" solution D of the second system of equations, which depends continuously on x, and in the third step we will show that D_k^i is partial derivative in x_k of φ^i in the distributional sense. Then since it is continuous $(in\ x)$, we can conclude that is the true partial derivatives.

Our approach to the $\natural-$model

This section contains the main result which is concerning the differentiability of the solution of the natural equation with $\mathbb{R}^d-$ values with respect to the initial data. We recall the solution of this equation:

$$X_t^u = x + \int_u^t X_s\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)dN_s + \int_u^t X_s\sum_{i=1}^d\sum_{j=1}^n F^{ij}(X_s - (1-Z_s))dY_s^j,\ s\in[u,t]$$

We introduce the stopping time $\tau_n = inf\{t, 1 - Z_t < \frac{1}{n}\}$ on the quantity $\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)$ (because we don't know if it's finite or not). Therefore, we assume the process $\widetilde{X}^x_{u,t}$ instead of $X^x_{u,t}$

$$\widetilde{X^u_t} = x + \int_u^t \widetilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \widetilde{X}_s \sum_{i=1}^{d}\sum_{j=1}^{n} F^{ij}(\widetilde{X}_s - (1 - Z_s))dY^j_s,\ s \in [u, t],$$

Theorem 3.3.4 *The solution X^u_t is continuously differentiable for any (u, t, x). Precisely, For $y \in \mathbb{R}^*$, the following partial derivatives:*

$$\theta_{u,t}(x, y) = \frac{\partial \widetilde{X}^x_{u,t}}{\partial x_k} = \frac{1}{y}[\widetilde{X}^{x+ye_k}_{u,t} - \widetilde{X}^x_{u,t}],$$

has a continuous extension at $y = 0$ for any (u, t, x), where e_k is the unit vector $(0, ..., 0, 1, 0, ..., 0)$ for $k = 1...d$.

In order to prove the differentibility property, it's enough to apply the following estimate and Kolmogorov's theorem: for any $p > 2$, there exits a positive constant C^p such that:

$$\begin{aligned}\mathbb{E}|\theta_{u,t}(x, y) - \theta_{u',t'}(x', y')|^p \quad &\leq \quad C^p[|x - x'|^{\alpha p} + |y - y'|^{\alpha p} + (1 + |x| + |x'|)^{\alpha p} \\ &\qquad (|u - u'|^{\frac{\alpha p}{2}} + |t - t'|^{\frac{\alpha p}{2}})]. \end{aligned} \tag{3.36}$$

We also need the following lemma.

Lemma 3.3.1 *Let $T > 0$ and p be any real number. Then there is a positive constant $C_{p,T}$ such that $\forall x, y \in \mathbb{R}^d$ and $\forall t \in [0; T]$,*

$$\mathbb{E}|J_t(x) - J_s(y)|^p \leq C_{p,T}|x - y|^p$$

Proof:If $x = y$, the inequality is clearly satisfied for any positive constant $C_{p,T}$. We shall assume $x \neq y$. Let ϵ be an arbitrary positive number and $\sigma_\epsilon = \inf\{t > 0, |J_t(x) - J_s(y)| < \epsilon\}$ We shall apply Itô's formula to $f(z) = |z|^p$. Then we have for $t < \sigma_\epsilon$,

$$
\begin{aligned}
|J_t(x) - J_s(y)|^p - |x-y|^p &= \sum_{i,j} \int_0^t \frac{\partial f}{\partial z_i}(J_s(x) - J_s(y))(G^i_j(J_s(x)) - G^i_j(J_s(y)))dH^j_s \\
&+ \frac{1}{2} \sum_{i,j,k,l} \int_0^t \frac{\partial^2 f}{\partial z_i \partial z_j}(J_s(x) - J_s(y))(G^i_k(J_s(x)) - G^i_k(J_s(y))) \\
&\times (G^i_l(J_s(x)) - G^i_l(J_s(y)))d < H^k, H^l >_s \\
&= I_t + S_t \qquad (3.37)
\end{aligned}
$$

Note $\frac{\partial f}{\partial z_i} = p|z|^{p-2} z_i$ and apply Lipschitz inequality. Then

$$
\left| \sum_i \int_0^t \frac{\partial f}{\partial z_i}(J_s(x) - J_s(y))(G^i_j(J_s(x)) - G^i_j(J_s(y))) \right| \leq |p|\sqrt{d}L|J_t(x) - J_s(y)|^p
$$

Therefore we have

$$
|\mathbb{E} I_{t\wedge\sigma_\epsilon}| \leq |p| r \sqrt{d} L \int_0^t |J_{s\wedge\sigma_\epsilon}(x) - J_{s\wedge\sigma_\epsilon}(y)|^p ds
$$

Next, note that

$$
\frac{\partial^2 f}{\partial z_i \partial z_j} = p|z|^{p-2}\delta_{ij} + p(p-2)|z|^{p-4} z_i z_j,
$$

where δ_{ij} is the Kronecker's delta. Then

$$
\left| \sum_{i,j} \int_0^t \frac{\partial^2 f}{\partial z_i \partial z_j}(J_s(x) - J_s(y))(G^i_k(J_s(x)) - G^i_k(J_s(y)))(G^i_l(J_s(x)) - G^i_l(J_s(y))) \right|
$$

$$
\leq |p|(|p-2| + d)L^2 |J_t(x) - J_s(y)|^p
$$

Therefore

$$
|\mathbb{E} S_{t\wedge\sigma_\epsilon}| \leq \frac{1}{2} r^2 |p|(|p-2| + d)L^2 \int_0^t |J_{s\wedge\sigma_\epsilon}(x) - J_{s\wedge\sigma_\epsilon}(y)|^p ds
$$

Summing up these two inequalities, we obtain

$$
\mathbb{E}|J_{t\wedge\sigma_\epsilon}(x) - J_{t\wedge\sigma_\epsilon}(y)|^p \leq |x-y|^p + C'_P \int_0^t |J_{s\wedge\sigma_\epsilon}(x) - J_{s\wedge\sigma_\epsilon}(y)|^p ds
$$

where C_p' is a positive constant. By Grnonwall's inequality,

$$\mathbb{E}|J_{t\wedge\sigma_\epsilon}(x) - J_{t\wedge\sigma_\epsilon}(y)|^p \leq C_{P,T}|x-y|^p \, \forall t \in [0,T],$$

where $C_{P,T} = \exp(C_P'T)$. Letting ϵ tend to 0,we have

$$\mathbb{E}|J_{t\wedge\sigma}(x) - J_{t\wedge\sigma}(y)|^p \leq C_{P,T}|x-y|^p \, \forall t \in [0,T],$$

where σ is the first time t such that $J_t(x) = J_t(y)$. However, we have $\sigma = \infty$ a.s. since otherwise the left hand side would be infinity if $p < 0$. The proof is complete.

Proof of the theorem (3.3.4):

Firstly we show the boundedness of $\mathbb{E}|\theta_{u,t}(x,y)|^p$, we have:

$$\theta_{u,t}(x,y) = \frac{1}{y}[\widetilde{X}_{u,t}^{x+ye_k} - \widetilde{X}_{u,t}^{x}]$$

We denote

$$M_t = -\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}$$

$$\widetilde{F}^{ij}(\widetilde{X}_t^{x+ye_k}) = \widetilde{X}_t^{x+ye_k} F^{ij}(\widetilde{X}_t^{x+ye_k} - (1 - Z_t))$$

$$\widetilde{F}^{ij}(\widetilde{X}_t^{x}) = \widetilde{X}_t^{x} F^{ij}(\widetilde{X}_t^{x} - (1 - Z_t))$$

So

$$\theta_{u,t}(x,y) = e_k + \frac{1}{y}\left[\int_u^t \widetilde{X}_s^{x+ye_k} - \widetilde{X}_s^{x} M_s dN_s + \sum_{i=1}^d \sum_{j=1}^n \int_u^t \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x}) dY_s^j\right] \quad (3.38)$$

Then

$$\begin{aligned}
\mathbb{E}|\theta_{u,t}(x,y)|^p &\leq 1 + \frac{1}{y}\mathbb{E}\left|\int_u^t \widetilde{X}_s^{x+ye_k} - \widetilde{X}_s^{x} M_s dN_s\right|^p \\
&+ \frac{1}{y}\sum_{i=1}^d \sum_{j=1}^n \mathbb{E}\left|\int_u^t \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x}) dY_s^j\right|^p \quad (3.39)
\end{aligned}$$

Using BDG'S inequality, we have

$$\begin{aligned}
\mathbb{E}|\theta_{u,t}(x,y)|^p &\leq 1 + C_1^p \mathbb{E}\left[\int_u^t |\theta_{r,s}(x,y)|^2 |M_s|^2 ds\right]^{\frac{p}{2}} \\
&+ C_1^p \frac{1}{y} \sum_{i=1}^{d} \sum_{j=1}^{n} \mathbb{E}\left[\int_u^t |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^x)|^2 ds\right]^{\frac{p}{2}} \qquad (3.40)
\end{aligned}$$

Now we apply the hölder inequality, noting q the conjugate of $\frac{p}{2}$,

$$\begin{aligned}
\mathbb{E}|\theta_{u,t}(x,y)|^p &\leq 1 + (t-u)^{\frac{p}{2q}} C_1^p \mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \\
&+ (t-u)^{\frac{p}{2q}} C_1^p \frac{1}{y} \sum_{i=1}^{d} \sum_{n}^{j=1} \mathbb{E}\left[\int_u^t |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^x)|^p ds\right] \qquad (3.41)
\end{aligned}$$

And as $\widetilde{F}^{ij}$ is Lipschitz, we have

$$|\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s)| \leq k_1 |\widetilde{X}_s^{x+ye_k} - \widetilde{X}_s^x|$$

Therefore

$$\begin{aligned}
\mathbb{E}|\theta_{u,t}(x,y)|^p &\leq 1 + (t-u)^{\frac{p}{2q}} C_1^p \mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \\
&+ (t-u)^{\frac{p}{2q}} k_1 C_1^p \mathbb{E}\left[\int_u^t |\theta_{r,s}(x,y)|^p ds\right] \qquad (3.42)
\end{aligned}$$

and by following, we have: $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$

$$\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \leq \frac{1}{2}\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^{2p}\right] + \frac{1}{2}\left[\int_u^t \mathbb{E}|M_s|^p ds\right]^2 \qquad (3.43)$$

Then the proposition(1.1.2), yields for any $x \in \mathbb{R}^d$ and a constant c'

$$\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \leq \frac{1}{2}c' + \frac{1}{2}\left[\int_u^t \mathbb{E}|M_s|^p ds\right]^2 \qquad (3.44)$$

Furthermore, we have the quantity $\mathbb{E}\left[\int_u^t |M_s|^p ds\right] < \infty$

Next, note that $\mathbb{E}\left[\int_u^t |M_s|^p ds\right] = \overline{R}$. Then

$$\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \leq C_2^p + C_3^p \overline{R}^2 \tag{3.45}$$

Where $\frac{1}{2}c'(t-u)^{\frac{p}{2q}} C_1^p = C_2^p \, and \, \frac{1}{2}(t-u)^{\frac{p}{2q}} C_1^p = C_3^p$ As a result

$$\mathbb{E}|\theta_{u,t}(x,y)|^p \leq C_4^p + C_5^p \int_u^t \mathbb{E}|\theta_{r,s}(x,y)|^p ds \tag{3.46}$$

Where $C_4^p = C_2^p + C_3^p \overline{R}^2$ and $C_5^p = (t-u)^{\frac{p}{2q}} k_1 C_1^p$.

Therefore by Gronwall's lemma , we get

$$\mathbb{E}|\theta_{u,t}(x,y)|^p \leq C_4^p \exp(C_5^p (t-u)) \tag{3.47}$$

consequently $\mathbb{E}|\theta_{u,t}(x,y)|^p$ is bounded.

Secondly we prove the estimate (3.36). In case $t = t'$, we suppose that $u < u' < t$. other cases will be proven in the same way. Then we have

$$\begin{aligned} \theta_{u,t}(x,y) - \theta_{u',t}(x',y') &= \int_u^{u'} \theta_{r,s}(x,y) - \theta_{r',s}(x',y') M_s dN_s + \frac{1}{y} \sum_{i=1}^d \sum_{j=1}^n \int_u^{u'} \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) \\ &- \widetilde{F}^{ij}(\widetilde{X}_s^x) - \widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'}) dY_s^j \end{aligned} \tag{3.48}$$

Noting

$$\widetilde{I}_1 = \int_u^{u'} \theta_{r,s}(x,y) - \theta_{r',s}(x',y') M_s dN_s$$

$$\widetilde{I}_2 = \frac{1}{y} \sum_{i=1}^d \sum_{j=1}^n \int_u^{u'} \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^x) - \widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'}) dY_s^j$$

So

$$\mathbb{E}|\widetilde{I}_1|^p = \mathbb{E}\left|\int_u^{u'} \theta_{r,s}(x,y) - \theta_{r',s}(x',y')M_s dN_s\right|^p \tag{3.49}$$

The BDG's inequality leads to:

$$\mathbb{E}|\widetilde{I}_1|^p \leq C_6^p \mathbb{E}\left[\int_u^{u'} |\theta_{r,s}(x,y) - \theta_{r',s}(x',y')|^2 |M_s|^2 ds\right]^{\frac{p}{2}} \tag{3.50}$$

using Hölder's inequality, noting q^* the conjugate of $\frac{p}{2}$,

$$\mathbb{E}|\widetilde{I}_1|^p \leq (u'-u)^{\frac{p}{2q^*}} C_6^p \mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y) - \theta_{u',t}(x',y')|^p \int_u^{u'} |M_s|^p ds\right] \tag{3.51}$$

and by following, we have $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$

$$\mathbb{E}|\widetilde{I}_1|^p \leq (u'-u)^{\frac{p}{2q^*}} C_7^p \left[\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y) - \theta_{u',t}(x',y')|^{2p}\right] + \left[\int_u^{u'} \mathbb{E}|M_s|^p ds\right]^2\right] \tag{3.52}$$

Then the proposition (1.1.2), gives

$$\mathbb{E}|\widetilde{I}_1|^p \leq (u'-u)^{\frac{p}{2q^*}} C_7^p \left[R_1|y-y'|^{2p} + \overline{R}_1^2\right] \tag{3.53}$$

Where $C_7^p = \frac{1}{2}C_6^p$.

it remains to study the term $\widetilde{I}_2$.

$$|\widetilde{I}_2| \leq \frac{1}{y}\sum_{i=1}^{d}\sum_{j=1}^{n}\int_u^{u'} |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x})| + |-\widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'})| dY_s^j \tag{3.54}$$

Using again the $BDG's$ inequality, we obtain

$$\begin{aligned}\mathbb{E}|\widetilde{I}_2|^p &\leq \frac{1}{y}C_8^p \sum_{i=1}^{d}\sum_{j=1}^{n}\mathbb{E}[\int_u^{u'} |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x})|^2 \\ &+ |-\widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'})|^2 ds]^{\frac{p}{2}}\end{aligned} \tag{3.55}$$

applying Hölder's inequality, noting q^* the conjugate of $\frac{p}{2}$, we have

$$\begin{aligned}\mathbb{E}|\widetilde{I}_2|^p &\leq \frac{1}{y}C_8^p(u'-u)^{\frac{p}{2q^*}}\sum_{i=1}^{d}\sum_{j=1}^{n}\int_u^{u'}\mathbb{E}|\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k})-\widetilde{F}^{ij}(\widetilde{X}_s^{x})|^p \\ &+ \mathbb{E}|\widetilde{F}^{ij}(\widetilde{X}_s^{x'})-\widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k})|^p ds \qquad (3.56)\end{aligned}$$

We have always $\widetilde{F}$ is Lipschitz :

$$\mathbb{E}|\widetilde{I}_2|^p \leq \frac{1}{y}C_8^p(u'-u)^{\frac{p}{2q^*}}k_1\int_u^{u'}\mathbb{E}|\widetilde{X}_s^{x+ye_k}-\widetilde{X}_s^{x}|^p+\mathbb{E}|\widetilde{X}_s^{x'}-\widetilde{X}_s^{x'+y'e_k}|^p ds \qquad (3.57)$$

Thus, by the lemma 3.3.1, We get

$$\mathbb{E}|\widetilde{I}_2|^p \leq \frac{1}{y}K_{p,T}^1C_8^p(u'-u)^{\frac{p}{2q^*}+1}k_1(|y|^p+|y'|^p) \qquad (3.58)$$

From (3.53)and (3.58), we obtain

$$\mathbb{E}|\theta_{u,t}(x,y)-\theta_{u',t}(x',y')|^p \leq C_9^p(u'-u)^{\frac{p}{2q^*}} \qquad (3.59)$$

Where $C_9^p = C_7^p(R_1|y-y'|^{2p}+\overline{R}_1^2)+\frac{1}{y}K_{p,T}^1C_8^p(u'-u)k_1(|y|^p+|y'|^p)$.

It remains Kolmogorov's lemma, we denote $G=\theta_{u,t}(x,y)-\theta_{u',t'}(x',y')$ and simply applying It$\widehat{o}$'s formula to the function $f(G)=|G|^p$ for $t=t'$, we obtain

$$|G|^p = \sum_{i,j}\int_u^{u'}\frac{\partial f}{\partial G_i}(G)dG_s+\frac{1}{2}\sum_{i,j}\int_u^{u'}\frac{\partial^2 f}{\partial G_iG_j}(G)d<G^i,G^j>_s$$

noting

$$\widehat{I} = \sum_{i,j}\int_u^{u'}\frac{\partial f}{\partial G_i}(G)dG_s$$

$$\overline{I} = \frac{1}{2}\sum_{i,j}\int_u^{u'}\frac{\partial^2 f}{\partial G_iG_j}(G)d<G^i,G^j>_s$$

such that

$$\widehat{I} = \sum_{i,j} \int_u^{u'} \frac{\partial f}{\partial G_i}(G) \left[G_s M_s dN_s + \frac{1}{y} \widetilde{F}^{ij}(\widetilde{X}^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}^{x}) - \widetilde{F}^{ij}(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}^{x'}) dY_s^j \right]$$

Then

$$\begin{aligned} \overline{I} &= \sum_{i,j,h,l} \int_u^{u'} \frac{\partial^2 f}{\partial G_i G_j}(G) \left[G_s M_s dN_s + \frac{1}{y} \widetilde{F}_l^{i}(\widetilde{X}^{x+ye_k}) - \widetilde{F}_l^{i}(\widetilde{X}^{x}) - \widetilde{F}_l^{i}(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_l^{i}(\widetilde{X}^{x'}) dY_s^l \right] \\ &\times \left[G_s M_s dN_s + \frac{1}{y} \widetilde{F}_h^{j}(\widetilde{X}^{x+ye_k}) - \widetilde{F}_h^{j}(\widetilde{X}^{x}) - \widetilde{F}_h^{j}(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_h^{j}(\widetilde{X}^{x'}) dY_s^h \right] \end{aligned}$$

For $\widehat{I}$, we denote

$$\frac{\partial f}{\partial G_i}(G) = |p||G|^{P-1}$$

$$\widehat{I}_1 = \sum_i \int_u^{u'} \frac{\partial f}{\partial G_i}(G) G_s M_s dN_s$$

$$\widehat{I}_2 = \sum_i \int_u^{u'} \frac{\partial f}{\partial G_i}(G) \frac{1}{y} \widetilde{F}^{ij}(\widetilde{X}^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}^{x}) - \widetilde{F}^{ij}(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}^{x'})$$

So, we have

$$\sum_i \left| \frac{\partial f}{\partial G_i}(G) G_s \right| \leq d|p||G|^{P-1}|G_s| \tag{3.60}$$

Then

$$|\widehat{I}_1| \leq d|p| \int_u^{u'} |G_s|^P ds \times \int_u^{u'} M_s dN_s \tag{3.61}$$

noting $\varphi_t = \int_u^{u'} M_s dN_s$, it's a local martingale (see [25])

$$|\widehat{I}_1| \leq d|p| \varphi_t \int_u^{u'} |G_s|^P ds \tag{3.62}$$

And we have $\widetilde{F}^{ij}(\widetilde{X}^{x})$ is Lipschitz function, therfore

$$\sum_{i}\left|\frac{\partial f}{\partial G_i}(G)\frac{1}{y}\widetilde{F}^{ij}(\widetilde{X}^{x+ye_k})-\widetilde{F}^{ij}(\widetilde{X}^{x})-\widetilde{F}^{ij}(\widetilde{X}^{x'+y'e_k})+\widetilde{F}^{ij}(\widetilde{X}^{x'})\right|\leq d\,k_1\,|p|\,|G|^P \tag{3.63}$$

Then

$$|\widehat{I}_2|\leq d\,n\,k_1\,|p|\int_u^{u'}|G_s|^P ds \tag{3.64}$$

From (3.62) and (3.64), we get

$$|\widehat{I}|\leq d\,|p|(\varphi_t+n\,k_1)\int_u^{u'}|G_s|^P ds \tag{3.65}$$

For $\overline{I}$, we denote

$$\overline{I}_1=\sum_{i,j,h,l}\int_u^{u'}\frac{\partial^2 f}{\partial G_i G_j}(G)(G_s)^2(M_s)^2 dN_s dN_s \tag{3.66}$$

$$\overline{I}_2=\frac{1}{y}\sum_{i,j,h,l}\int_u^{u'}\frac{\partial^2 f}{\partial G_i G_j}(G)G_s M_s\widetilde{F}^i_l(\widetilde{X}^{x+ye_k})-\widetilde{F}^i_l(\widetilde{X}^{x})-\widetilde{F}^i_l(\widetilde{X}^{x'+y'e_k})+\widetilde{F}^i_l(\widetilde{X}^{x'})dN_s dY^l_s$$

$$\overline{I}_3=\frac{1}{y}\sum_{i,j,h,l}\int_u^{u'}\frac{\partial^2 f}{\partial G_i G_j}(G)G_s M_s\widetilde{F}^j_h(\widetilde{X}^{x+ye_k})-\widetilde{F}^j_h(\widetilde{X}^{x})-\widetilde{F}^j_h(\widetilde{X}^{x'+y'e_k})+\widetilde{F}^j_h(\widetilde{X}^{x'})dN_s dY^h_s$$

$$\begin{aligned}\overline{I}_4 &= \frac{1}{y^2}\sum_{i,j,h,l}\int_u^{u'}\frac{\partial^2 f}{\partial G_i G_j}(G)\left[\widetilde{F}^i_l(\widetilde{X}^{x+ye_k})-\widetilde{F}^i_l(\widetilde{X}^{x})-\widetilde{F}^i_l(\widetilde{X}^{x'+y'e_k})+\widetilde{F}^i_l(\widetilde{X}^{x'})\right]\\ &\times \left[\widetilde{F}^j_h(\widetilde{X}^{x+ye_k})-\widetilde{F}^j_h(\widetilde{X}^{x})-\widetilde{F}^j_h(\widetilde{X}^{x'+y'e_k})+\widetilde{F}^j_h(\widetilde{X}^{x'})\right]dY^l_s dY^h_s\end{aligned}$$

And note that

$$\frac{\partial^2 f}{\partial G_i G_j}(G)=p(p-1)|G|^{p-2}$$

Then for $\overline{I}_1$, we have

$$\sum_{i,j,h,l}\left|\frac{\partial^2 f}{\partial G_i G_j}(G)(G_s)^2\right|\leq d\,|p|\,|p-1|\,|G|^{p-2}|G|^2 \tag{3.67}$$

So

$$|\overline{I}_1| \leq d\,|p|\,|p-1| \int_u^{u'} |G_s|^p M_s^2 dN_s dN_s \tag{3.68}$$

$\int_u^{u'} M_s dN_s$ is always a local martingale, so

$$|\overline{I}_1| \leq d\,|p|\,|p-1|\,\varphi_t^2 \int_u^{u'} |G_s|^p ds \tag{3.69}$$

For $\overline{I}_2$, we have

$$\sum_{i,j,h,l} \frac{1}{y} \left| \frac{\partial^2 f}{\partial G_i G_j}(G) G_s \widetilde{F}_l^i(\widetilde{X}^{x+ye_k}) - \widetilde{F}_l^i(\widetilde{X}^x) - \widetilde{F}_l^i(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_l^i(\widetilde{X}^{x'}) \right| \leq d\,n\,k_1 |p|\,|p-1|\,|G|^{p-2}|G_s|^2$$

Therefore we get

$$|\overline{I}_2| \leq d\,n\,k_1 |p|\,|p-1|\,\varphi_t^2 \int_u^{u'} |G_s|^p ds \tag{3.70}$$

For $\overline{I}_3$, we have

$$|\overline{I}_3| \leq d\,n\,k_1 |p|\,|p-1|\,\varphi_t^2 \int_u^{u'} |G_s|^p ds \tag{3.71}$$

For $\overline{I}_4$, we have

$$\overline{I}_4 \leq d\,n\,k_1^2 |p|\,|p-1| \int_u^{u'} |G_s|^p ds \tag{3.72}$$

Then we have

$$\overline{I} = \frac{1}{2}\left[\overline{I}_1 + \overline{I}_2 + \overline{I}_3 + \overline{I}_4\right] \tag{3.73}$$

Such that

$$\overline{I} \leq \frac{1}{2}\,(2\,n\,k_1\,\varphi_t + \varphi_t^2 + n\,k_1^2)\,d\,|p|\,|p-1| \int_u^{u'} |G_s|^p ds \tag{3.74}$$

From these two inequalities (3.65) and (3.74), we get

$$|G|^p \leq d\,|p|\,(\frac{1}{2}\,|p-1|\,(2\,n\,k_1\,\varphi_t + \varphi_t^2 + n\,k_1^2) + \varphi_t + n\,k_1) \int_u^{u'} |G_s|^p ds \tag{3.75}$$

Therefore

$$\mathbb{E}|G|^p \leq C_{10}^p \int_u^{u'} \mathbb{E}|G_s|^p ds \tag{3.76}$$

By Grönwall's inequality we have

$$\mathbb{E}|G|^p \leq C_{11}^p \tag{3.77}$$

Where C_{11}^p is $\exp(C_{10}^p(u'-u))$.

The proof is completed.

Conclusion

The main purpose of this monograph is to study the behavior of the trajectories of the solution of the natural model which is considered as the best model of credit risk. These results give a best contribution in a finance field. we have shown the continuity of stochastic flow generated be the one-default model in multidimensional case as well under some hypothesis, we have proved the differentiability property with respect to the initial data in one-dimensional and multidimensional cases.

In our future works, we will try to make the application of our theoretical results and study an other properties of stochastic flow under different conditions and hypothesis.

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خصائص التدفق العشوائي الناتج عن النموذج الطبيعي في الحالة أحادية البعد ومتعددة الأبعاد.

الملخص:

هذه الأطروحة مهتمة بنموذج هام لمخاطر الائتمان يسمى نموذج الفردي أو الطبيعي الذي يعبر عنه بمعادلة تفاضلية عشوائية مسماة بالمعادلة الطبيعية , تلعب هذه المعادلة دورا مهما في بحثنا.
في ظل بعض الفرضيات, البحث الوارد في هذه الدراسة هو انتظام مسارات التدفقات العشوائية في الحالة متعددة الأبعاد بناءا على نظرية كولموغوروف.
سننظر أيضا في تمايز التدفقات العشوائية الناتجة عن المعادلة الطبيعية فيما يتعلق بالقيمة الأولية في الحالة أحادية البعد استنادا على النظريات التالية : بورخولدر- دفي- غاندي, اولدر و غرونوال.
كذلك سنثبت نفس الخاصية و لكن في حالة متعددة الأبعاد اعتمادا على فكرة: كونيتا.

الكلمات المفتا حية :

المعادلة التفاضلية العشوائية, التدفقات العشوائية, مخاطر الائتمان, تباين الأشكال, المعادلة الطبيعية

« *Les propriétés du flot stochastique engendré par le modèle naturel dans le cas unidimensionnel et multidimensionnel.* »

Résumé :

Cette thèse s'intéresse à un modèle important de risque de crédit qui s'appelle le modèle à un défaut ou le modèle naturel qui est exprimé par une équation différentielle stochastique appelé l'équation naturelle, Cette équation joue un rôle important dans notre étude.
Sous certaines hypothèses, la recherche rapportée dans cette étude est la régularité des trajectoires de flot stochastique engendré par l'équation naturelle dans le cas multidimensionnel basé sur le théorème de kolmogorov.
Nous prouverons également la différentiabilité de flot stochastique engendré par l'équation naturelle par rapport à la valeur initiale dans le cas unidimensionnel en se basant sur les théorèmes de Burkholder-Davis-Gundy, Hôlder et Gronwall.
Nous prouverons également la même propriété mais dans le cas multidimensionnel basé sur l'idée de *Hiroshi* Kunita.

Mots clés :

Equation différentielle stochastique, flot stochastique, risque de crédit, difféomorphism, équation naturelle.

« *The properties of stochastic flow generated by the natural model in one-dimensional and multi-dimensional cases* »

Abstract :

This thesis is interested with an important model of credit risk so-called the one-default or natural model which is expressed by a natural equation, this equation play an important role in our research.
Under some assumptions, the research reported in this monograph is the regularity of the trajectories of the stochastic flow generated by the natural equation in multidimensional case based on the Kolmogorov's theorem.
Additionally, we will look at the differentiability of stochastic flow generated by the natural equation with respect to the initial value in one-dimensional case based on the Burkholder-Davis-Gundy , Hôlder and the Gronwall theorems . In addition to this we will prove the same property in multidimensional case based on the idea of *Hiroshi* Kunita.

Keywords:

Stochastic differential equation, stochastic flow, credit risk, Diffeomorphism, natural equation .