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Dedication

This thesis is dedicated to my parents.

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First of all, I wish to praise Allah the Almighty and the most Merciful for giving me the patience to accomplish this work.

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ملخص

نقوم في هذه الأطروحة بدراسة سلوك نفاذ صبر الزبائن في مختلف أنظمة طوابير الانتظار لعدة أسباب من بينها حالة الخادم، نوعية الخدمة، طول وقت الانتظار، إلخ. أولا تحصلنا على احتمالات في حالة الاستقرار لنظام طابور الانتظار $M/M/2/N$ مع خادمين غير متجانسين، تغذية راجعة، إجازة الخوادم، تقديم الخدمة خلال فترة إجازة، العزوف، تنصل الزبائن المتعلق بحالة الخادم و الاحتفاظ بالزبائن المتنصلين، باستخدام طريقة المتغير الإضافي و الطريقة التراجعية. ثانيا استخدمنا طريقة تراجعية من أجل الحصول على الحل لنظام طابور الانتظار ذو قدرة استيعاب غير منتهية مع اختلاف الإجازات، إعاقة الإجازات، العزوف، التنصل والاحتفاظ بالزبائن المتنصلين خلال فترة الاشتغال. ثالثا، نحلل سلوك نفاذ صبر الزبائن (التنصل)، مدى تأثير الاحتفاظ و التغذية الراجعة في أنظمة طوابير الانتظار $M/M/1$ مع إجازة وحيدة، $M^X/M/1$ مع K إجازة متتالية، $M^X/M/c$ مع K إجازة متتالية و تقديم الخدمة خلال إجازات، $M^X/M/c$ مع إجازة وحيدة و مع إجازة متعددة، حيث أن الخوادم في النظام الأول و الثاني و الرابع غير مسموح أن يأخذوا إجازات عند شغور النظام إلا بعد فترة انتظار عشوائية. نقوم بدراسة و تحليل حالة المراوحة من أجل هذه الأنظمة باستخدام وظيفة توليد الاحتمال. إضافة إلى ذلك، نقوم باشتقاق مختلف مقاييس الأداء و تقديم التحليل الاقتصادي لمختلف النماذج المدروسة في هذه الأطروحة. إضافة إلى ذلك، نقوم بدراسة تحسينية لنظام طابور الانتظار الرابع والسادس باستخدام طريقتي PSO و QFSM.

كلمات مفتاحية :

أنظمة طوابير الانتظار، نفاذ صبر الزبائن ، العزوف ، التنصل ، حالة الاستقرار.

Abstract

In this thesis, we analyze the impatient behaviour in different queueing systems due to different factors including server state, quality of service, waiting time, etc. Firstly, we obtain the steady-state probabilities for an $M/M/2/N$ queueing system with two heterogeneous servers, feedback, vacation, working vacation, balking, reneging which depends on server state and retention of reneged customers, using supplementary variable and recursive techniques. Secondly, we use the recursive method to establish the solution of an infinite capacity queueing system with differentiated vacations, vacation interruption, balking, reneging during the busy period and retention of reneged customers. Thirdly, we analyze the impatient behavior (reneging), the impact of retention of reneged customers and feedback in an $M/M/1$ queueing system with single vacation and waiting server, $M^X/M/1$ with waiting server and K-variant vacations, $M^X/M/c$ with K-variant working vacations as well as $M^X/M/c$ with waiting servers and both single multiple vacation policies. We establish the stationary analysis for these queueing systems using the probability generating function. In addition, we derive useful performance measures and present the economic analysis of the different models presented in this thesis. In addition, we study the optimization of the fourth and sixth queueing systems using the PSO and QFSM methods.

Keywords:

Queueing systems, impatient customers, balking, reneging, stability.

Résumé

Dans cette thèse nous analysons le comportement d'impatience dans différents systèmes de files d'attente, due aux différents facteurs notamment l'état de serveur, qualité de service, le temps d'attente, etc. Dans un premier lieu, nous obtenons les probabilités d'état stable pour un système de file d'attente $M/M/2/N$ avec deux serveurs hétérogènes, feedback, vacances, service pendant les vacances, dérobade, abandon qui dépendent de l'état du serveur et rétention des clients abandonnés, en utilisant la méthode de variable supplémentaire et la récursivité. En second lieu, nous utilisons la méthode récursive afin d'établir la solution d'un système de file d'attente de capacité infinie avec des vacances différentiées, interruption de vacances, dérobade, abandon pendant la période d'occupation et rétention des clients abandonnés. En troisième lieu, nous analysons le comportement d'impatience (abandon), l'impact des rétentions et du feedback dans les systèmes de files d'attente $M/M/1$ avec vacance unique, $M^X/M/1$ avec K vacances consécutives, $M^X/M/c$ avec K vacances consécutives et services pendant les vacances, $M^X/M/c$ avec vacances uniques et multiples, où les serveurs dans le premier, deuxième et quatrième systèmes sont autorisés à prendre des vacances chaque fois que le système est vide après une période d'attente aléatoire . Nous établissons l'analyse stationnaire pour ces systèmes de files d'attente en utilisant la fonction génératrice des probabilités. En outre, Nous dérivons importantes mesures de performance et présentons l'analyse économique des différents modèles présentés dans cette thèse. En outre, nous étudions dans cette thèse l'optimisation du quatrième, et sixième système de files d'attente en utilisant les méthodes PSO et QFSM.

Mots clés:

Systèmes de files d'attente, clients impatients, dérobade, abandon, stabilité.

List of works

During the preparation of this PhD thesis, a number of research works and oral/poster presentations have been carried out.

List of research works

1 Bouchentouf, A. A. and Guendouzi, A. (2019a). Cost optimization analysis for an $M^X/M/c$ vacation queueing system with waiting servers and impatient customers, *SeMA Journal*, **76**(2), 309–341.

2 Bouchentouf, A. A. and Guendouzi, A. (2020a). The $M^X/M/c$ Bernoulli feedback queue with variant multiple working vacations and impatient customers: Performance and economic analysis, *Arabian journal of mathematics*, **9**(2), 309–327.

3 Bouchentouf, A. A. and Guendouzi, A. (2020b). Single server batch arrival Bernoulli feedback queueing system with waiting server, K-variant vacations and impatient customers, *submitted*.

4 Bouchentouf, A. A., Guendouzi, A. and Kandouci, A. (2019b). Performance and economic study of heterogeneous $M/M/2/N$ feedback queue with working vacation and impatient customers, *ProbStat Forum*, **12**(1), 15–35.

5 Bouchentouf, A. A., Guendouzi, A. and Kandouci, A. (2018). Performance and economic analysis of Markovian Bernoulli feedback queueing system with vacations, waiting server and impatient customers, *Acta Universitatis Sapientiae, Mathematica*, **10**(2), 218–241.

6 Bouchentouf, A. A. and Guendouzi, A. (2018). Sensitivity analysis of multiple vacation feedback queueing system with differentiated vacations, vacation interruptions and impatient customers, *International Journal of Applied Mathematics and Statistics*, **57**(6), 104–121.

List of presentations

1 Guendouzi, A., Bouchentouf, A. A. and Kandouci, A. (April 09 - 13, 2018). Performance analysis of two-heterogeneous queueing system with multiple working vacation and impatient customers, *Poster presentation at the International Workshop on Perspectives, On High-dimensional Data Analysis (HDDA)-VIII*, organized by Caddy Ayyad University of Marrakesh, Morocco.

2 Guendouzi, A., Bouchentouf, A. A. and Kandouci, A. (February 25 - 27, 2019). Analysis of infinite-buffer Markovien feedback queue with vacations, waiting server and impatient customers, *Oral communication at the* 9^{th} *edition of the colloquium "Trends in Mathematical Applications in Tunisia Algeria Morocco"(TAMTAM)*, organized by Abou Bekr Belkaid University of Tlemcen, Algeria.

3 Guendouzi, A., Bouchentouf, A. A. (November 21 – 24, 2019). On a Markovian multiserver vacation queueing system with waiting servers and impatient customers, *Poster presentation at : Onzième Rencontre d'Analyse Mathématique et Applications (RAMA-11)*, organized by Djillali Liabes University of Sidi Bel Abbes, Algeria.

Others

- Participation in the Mathematics meetings of Rouen RMR-2019 on Numeric simulation and Applications, organized by University of Rouen, France, from 19 to 21 June 2019.
- Participation in the CIMPA Research School CIMPA'ASA-2019 on Stochastic Analysis and Applications, organized by Dr Moulay Tahar University of Saida, Algeria, from 01 to 09 March 2019.
- Member of the PRFU Research Project number: C00L03UN200120180002, Functional statistics and queueing models, approved on 01/01/2018.

Contents

Chapter 1

Introduction

Queueing theory is an prominent branch of mathematics with applied probability, calculus, complex analysis, statistical distribution, matrix theory, etc. This theory is a tool for analyzing various components of a queueing system and evaluating mathematical results for various performance measures. The results of the queueing theory are necessary to obtain the characteristics of the system and to evaluate the effect of the modifications of the model. It helps to provide the vital information required by a decision maker by anticipating various queueing features, such as the mean waiting time, the mean number of customers in the queue, etc.

Queueing systems with impatient customers (balking and reneging) have been the subject of many studies over the last three decades. The analysis of these models has emerged as an significant area of queueing theory. These systems have wide applicability in many real life situations as telecommunication systems, computer networks, production and manufacturing systems, aircraft waiting for landing at a busy airport and other stochastic systems.

The objective of this thesis is to study different queueing systems with impatient customers wherein these later tend to leave the queue for different reasons including the temporary unavailability of the server (being in vacation period) for a certain period of time and other reasons associated with dissatisfaction of the customers with the service time. This type of abandonment is visible not only in human queue situations but also in many practical real-world systems including call centers, computer and telecommunication systems, production and manufacturing industries, and so on. The thesis contains a detailed analysis of the stochastic processes underlying these models.

The remainder of this chapter is organized as follows: In Section 1.1, we provide a literature review on impatient customers in queueing models. In Sections 1.2 and 1.3, we give a fairly broad set of results on vacation queues and customers's impatience in vacation queueing systems, respectively. Then, Section 1.4 we present different solution methods for queueing models with impatient customers and/or vacation. Section 1.5 is dedicated to presented the concept of feedback queues. Finally, we present the contribution and outline of the thesis in Sections 1.6 and 1.7, respectively.

1.1 Impatient customers in queueing models

1.1.1 Impatient behaviour

The main objective of the study of queueing systems with impatient customers (irritation/dislike with everything causing delay) is to understand the real-world situations as well as possible. Thus, the impatient customer acts are involved in the analysis of queueing systems to model reality accurately. To characterize impatient customer behavior, there are two terminologies employed in queueing system, that is, balking, defined as deciding not to join the queue at all, and reneging, defined as joining the queue but leaving without being served. Furthermore, there exists another term associated with the act of impatience, that is retrial, defined as join orbit (the virtual pool of customers) after balking or reneging and repeats its request after random period of time. The analysis of impatient behaviour in queueing theory is potentially very valuable research area as diverse real situations in important industries and service systems can be formulated as queueing systems with impatient customers. In this chapter we are limited to some which have been the basis of diverse research results in the literature while focusing on those presented in this thesis, namely, balking and reneging.

– Balking: The customer faces the decision to join the queue or not when the server is not inactive. The principal factor that leads the customer to decide whether or not to join the system is the waiting time before he experiences the service. Nevertheless, the customer always makes his decision according to the length of the queue since information on the waiting time is imperceptible.

– Reneging: A customers may give up after a waiting time but do not leave on arrival if the queues size exceeds a certain number. On the other hand, a customers who has joined the queue may decide to leave the system if it seems that the time spent can exceed the maximum waiting time T (threshold value) available to him.

– Retrial: The customer who may balk at interning the system or renege from it may join the virtual customer pool, called orbit, and repeat its request after a random period of time. The probability that either balked and reneged customers will join the orbit may depend on the number of customers in service. Indeed, queueing systems with repeated attempts occur in different practical situations including telephone systems, retrial shopping queue, random access protocols in digital communication networks, priority queues computer systems and communication systems, etc.

1.1.2 Literature Review

1.1.2.1 On balking behaviour

During the past decades, notable research works on balked customers in queueing systems have been done due to their wide application in real-life situations such as real-time telecommunication, computer networks with packet loss, hospital emergency rooms' handling of the critical patients and perishable goods storage inventory systems, etc. A body of pioneering researches within the queueing systems literature which consider the balking feature include Haight (1957), Finch (1959), Haight (1960), Ancker and Gafarian (1963b), Singh (1970), and Yechiali (1971). Later, Kumar et al. (1993) obtained the transient solution for the system size of an $M/M/1$ queueing model where balking occurs when the system size equals or exceeds a certain threshold value. Later, Liu and Kulkarni (2006) presented the explicit solution of the steady state equation for a $M/PH/1$ queue with workload dependent balking by reducing the integral equation of the steady state workload distribution to a differential equation with constant coefficients for Phase-type service time distribution. Liu and Kulkarni (2008b) extended the work in Liu and Kulkarni (2006) by considering the busy period analysis for $M/PH/1$ queues with workload dependent balking. They developed an alternative method to study the first passage time in this queue via fluid models. Lozano and Moreno (2008) carried out the economic analysis of an $Geo/Geo/1$ with finite and infinite buffer queueing system for two variants balking scenarios which are the constant and discouraged rate policies, authors proved the ergodicity condition and obtained the closed-form expression for the stationary distribution of the system size. Then, Manoharan and Jose (2011) dealt with a single-server Markovian queueing system with customers' impatience in the form of balking. They determined the mean and variance associated to the stationary distribution of the system size and discussed the maximum likelihood estimation of the balking parameter. Goswami (2014) gave an analysis of the multi-server Markovain queueing system with balking. The positive recurrence conditions of the $Geo/Geo/m$ and $M/M/m$ queues has been given. Recently, Wang et al. (2018) discussed the equilibrium strategies in infinite space single server Markovian queue with a pay-for-priority option and balking.

1.1.2.2 On reneging behaviour

There is a board literature in queueing theory which studies the problems of reneging. The pioneering research papers on the subject include Haight (1959), Finch (1960), Daley (1965), Stanford (1979), Boxma and de Waal (1994), Garnett et. (2002), Choi et al. (2004), and Choudhury (2008). Since then, abundant literature has been given in different queues with reneging. Xiong et al. (2008) dealt with an $M/G/1$ queueing system with deterministic reneging. Then, Xiong and Altiok (2009) extended the paper to a $M/G/n$ queueing model. The infinite space Markovian single server queueing system with pre-emptive priority and Poisson reneging has been addressed in Nasrallah (2009). Perel and Yechiali (2010) considered different Markovian queueing models, namely, $M/M/1$, $M/M/c$ and $M/M/\infty$ queues in fast and slow phases Markovian random environment, with impatient customers. After that, Kapodistria (2011) treated single and multi-server queueing models with synchronized reneging. Later, Singh et al. (2016) investigated the study of finite capacity and finite population $M/M/1/K$ and $M/M/1/K/L$ queues at which the server works in fast and slow random environments, depending on the status of service system. In this paper, authors considered reneging if the server does not change its state before the impatience timer expires. For more details on reneging behaviour in queueing system, excellent survey on the subject can be found in Ward (2012).

1.1.2.3 On balking and reneging behaviours

Balking and reneging together in the queueing systems have been discussed by many researchers, the pioneering works on the topic are given by Ancker and Gafarian (1963a,1963b), Rao (1965,1967), Montazer-Haghighi et al. (1986), Abou-El-Ata (1991), and Jain and Singh (2002). Later, the $M/M/c/N$ queueing system with balking and reneging concepts was analyzed by Abou-El-Ata and Hariri (1992) by giving an explicit form of the steady state distribution of the number of units in the system and the expected number of units in both the system and queue. In Wang and Chang (2002), authors focused on the cost analysis of an finite $M/M/R$ queue with balking, reneging and server breakdowns. After that, Liu and Kulkarni (2008a) presented the exact analysis and approximations for a $M/G/s$ queueing system. Shin and Choo (2009) considered the infinite space multi-server Markovian queue with balking, reneging and retrials. Al-Seedy et al. (2009) gave the analysis of a multi-server Markovian queueing model with balking and reneging, where a customer has a constant probability of balking independent of the queue size while he may renege with regard to negative exponential distribution. Then, Choudhury and Medhi (2011c) dealt with balking and reneging behaviours in Markovian multi-server queueing system. Ammar et al. (2012) obtained the transient solution of the $M/M/1/N$ queueing system with discouraged arrivals and reneging. Later, Ammar et al. (2013) considered the busy period study of a $M/M/1$ queueing system with balked and reneged customers. The literature on balking and reneging is extensive, we limit the reference body to the most important research works on the subject by refereing to Gupta et al. (2008,2009), Wu and Ke (2010), Choudhury and Medhi (2011a,2011b), and references therein.

1.1.2.4 On retention of reneged customers

Customer's impatience leads to loss of potential customers and has a very bad effect on the revenue generation of a firm. Thus, many firms employ various strategies to retain a reneging customer and they manage to do it with some probability. For that reason, the concept of customer retention assumes a great importance for the business management. Customer retention is the key issue in the organizations facing the problem of customer's impatience. Firms are employing a number of customer retention strategies to maintain their businesses. An impatient customer (due to reneging) may be convinced to stay in service system for his service by utilizing certain convicting mechanisms, like increasing the service rates or adding a supplementary service channels in the system Bouchentouf et al. (2019).

Although many research papers have been dealt with queueing models with impatience, while literature on customers retention is limited. Basic idea of retention of reneged customers has been developed by Kumar and Sharma (2012a,2012b). Then, Kumar and Sharma (2013) considered a finite capacity multi-server queueing model with reneging and retention of reneged customers. This work has been extended to a general case, where Kumar (2013) incorporated the balking behaviour. Then, Bouchentouf et al. (2014) dealt with heterogeneous two server queueing model with balking, reneging and retention of reneged customers. Later, Madheswari et al. (2016) discussed the retention mechanism of reneged customers for a retrial queueing system with impatience. Yang and Wu (2017) considered the retention of reneged customers in a finite-capacity Markovian queueing system with working breakdowns. Recently, Kumar and Sharma (2018a) presented the transient analysis of an infinite buffer multi-server queuing system with balking, reneging and retention of reneged customers. More papers on retention of reneging can be found in Kumar (2016), Som and Seta (2017), Laxmi and Kassahun (2017), and Kumar and Sharma (2018b).

1.2 Vacation queues

In the contemporary world, competition between service providers is tough enough. Thus, to survive, service systems must be managed efficiently and economically. The demand for service varies predominantly wherein there may be periods of low flow of customers. During such period, it may not be economical from the system point of view to keep inactive servers in the system. On the other hand, no system can afford to lose its customers. It is therefore necessary to find a balance between the two uttermost situations. It is from this point that the study of queueing systems with vacation and working vacation has been taken into consideration. In addition, other situations can lead to server vacation including system maintenance, system failure, resource sharing, and so on.

1.2.1 Different types of vacation policies

Various kind of vacation policies has been seen in the literature:

– Single vacation queueing models: The server goes on vacation at the end of each busy period and returns immediately after the vacation period is over, even if the system is empty at that time.

– Multiple vacation queueing models: The server takes a sequence of vacations until he finds the system nonempty at a vacation completion instant.

– Working vacation queues: The server works continuously in the this period with lower rate.

– Gated vacation queues: The server places a gate behind the last waiting customer and serves only the ones who are within the gate according to certain rules.

– Limited service discipline: The server goes on vacation after serving K consecutive customers or after a time length t or if he is idle.

– Exhaustive service discipline: The server is serving customers until the system is empty, after which he takes a vacation of a random length of time.

– Differentiated vacation queues: The server may takes two types of vacation, the first one is taken after the server has finished serving at least one customer and the second type is taken when the server has just returned from the previous vacation to find an empty space.

– K-variant vacation queues: At the vacation completion instant, if there exist some customers in the queue, the server switches to busy period, otherwise, he is permitted to take K vacations sequentially. When the K consecutive vacations are complete, the server returns to the busy period and stay there till the arriving of new customers.

1.2.2 Literature review

Vacation queueing systems are that models at which the server may become unavailable for a random period of time when there are no waiting customers in the queue at a service completion instant. The idea of server vacation was proposed at the first time by Miller (1964) in his Phd thesis. In the past four decades, these queueing models have been well described and successfully applied in many areas such as computer and communication network systems, flexible manufacturing systems, service systems, and so forth. Several excellent surveys, monographs and books on the subject have been done by Doshi (1986), Takagi (1991,1993), Tian and Zhang (2006), Ke et al. (2010), Upadhyaya (2016), and references therein.

The literature on queueing models with vacation is abundant. In what follows, we are limited to some which have been the basis of various research results including those presented in this thesis.

On single and multiple vacation queues

Single and multiple vacation queueing models have attracted a considerable attention of many researchers. Levy and Yechiali (1975) considered the vacation time for some additional work and supposed that the idle time of the server is not completely lost. In their paper, authors gave the mean queue length and the Laplace transform of the waiting time of an arbitrary arrival in $M/G/1$ queueing model. Then, Levy and Yechiali (1976) analyzed an infinite-buffer multi-server Markovian queueing system with multiple and single asynchronous vacation policies. An $M/G/1$ queueing model with vacation times has been studied by Van der Duyn Schouten (1978). Fuhrmann (1984) presented a note on the $M/G/1$ queue with server vacations. Later, Baba (1986) treated the $M/G/1$ queue with group arrival and vacation time, he obtained the steady state distribution of the queue size at an arbitrary time. Servi (1986) dealt with vacation times in an $D/G/1$ queueing model. Then, Kella and Yechiali (1988) gave the analysis for the $M/G/1$ queueing system with the preemptive and non-preemptive priority regimes under single and multiple server vacations. An $GI/M/1$ queue with exhaustive service and multiple exponential vacation were considered in Tian (1989). Chatterjee and Mukherjee (1990) studied a $GI/M/1$ queue with multiple vacation policy. Kao and Narayanan (1991) dealt with an $M/M/N$ queueing model with asynchronous multiple vacation policies. Later, Afthab Begum and Nadarajan (1997) studied an $M/M/s$ queueing system with multiple vacation at which at least r servers should be always available in the system whatever busy or idle for service and only the remaining $s-r$ servers are permitted to go for vacation, when the there are no customers waiting in the queue. After that, Alfa (2003) presented the analysis of a single server vacation queueing models with single arrivals and non-batch service. Zhang and Tian (2003) discussed a multi-server $M/M/c$ queueing model with a single and multiple vacation policies for some idle servers. They obtained the stationary distribution of the queueing systems. In addition, they established conditional stochastic decomposition properties for the waiting time and the queue length given that all servers are busy. Recently, Saffer and Yue (2015) considered a $M/G/1$ multiple vacation queueing model with balking for a class of disciplines. Then, Ke et al. (2019) analyzed an $M/M/c$ balking retrial queueing system with both single and multiple vacation policies.

On working vacation queues

In the classical vacation queues, the basis of the research is the supposition that during vacation period, the server completely stops a service. Nevertheless, there are many situations in which the server does not remain completely idle during the vacation but provides a service to the waiting customers at a lower rate. The idea of this concept was first used by Servi and Finn (2002), where they studied an $M/M/1$ queueing model with multiple working vacation policy. Since then, the studies on this type of vacation have received an increasing interest. Research papers on single working vacation policies are extensive, the most important papers given on the subject include Wu and

Takagi (2006), Li and Tian (2008), Lin and Ke (2009), and Zhang and Hou (2012). Analyses of multiple working vacation queueing models showed a significant interest, some preeminent articles include Baba (2005), Li et al. (2007), Baba (2012), Laxmi et al. (2013b), Selvaraju and Goswami (2013), and Yu et al. (2017).

On K-variant working vacation queues

Variant of multiple vacation policy is relatively recent compared to the vacation types presented above at which a server is allowed to take a certain fixed number of consecutive vacations, if the system remains empty at the end of a vacation. Literature related to this kind of vacation can be found in Zhang and Tian (2001), Ke (2007), Ke et al. (2010), Wang et al. (2011), Yue et al. (2014), and Laxmi and Rajesh (2016).

On differentiated vacation queues

Differentiated vacation models can be applied in hospital emergency room operation where a type 1 vacation is used to set up the room for the next wave of patients, getting the equipment ready and performing any cleanups and sterilization. Similarly, a type 2 vacation can be used to give the emergency room personnel some actual rest, given that the necessary cleanup and setup preparation for the room have been done.

The concept of queueing system with differentiated vacations was initiated by Ibe and Isijola (2014). Then, Isijola and Ibe (2014) extended their work by considering vacation interruption. Further, Ibe and Isijola (2015) considered single server Markovian queueing model with differentiated multiple vacations and vacation-dependent service rates. Later, the model given in Ibe and Isijola (2014) were extended by Gupta et al. (2016), authors considered a deterministic service time. Recently, Vijayashree and Janani (2018) determined the time-dependent result for an $M/M/1$ queueing system with differentiated vacations.

On vacation queues with waiting server

Vacation queueing models with waiting server are seen in many real-life situations including postoffice, banks, etc, where the server waits a certain period of time before taking a vacation even though the system is empty.

The concept of waiting server was introduced by Boxma et al. (2002), where they considered a $M/G/1$ queueing model with waiting server under single and multiple vacation policies. Yechiali (2004) generalized this study to the case when customers arrive in batches of random size ($M^X/G/1$ queue). Other references on this subject are found in Padmavathy et al. (2011), Sudhesh and Azhagappan (2016), and Suranga Sampath and Jicheng (2018).

1.3 On customers' impatience in vacation queues

Eminent research papers have been done on different vacation queueing systems with impatient customers. In what follows, we cite the most resent works. Yue et al. (2006) presented the analysis of a finite-buffer multi-server Markovian queueing model with balking, reneging and synchronous vacations of partial servers. The impatient behaviour of the $M/M/1$ and $M/G/1$ queueing models with waiting server timer and server vacation were analyzed by Padmavathy et al. (2011). Then, Yue et al. (2012) dealt with customers' impatience in an $M/M/1$ queue with working vacations. A discrete-time single server queue with balking and multiple working vacations was discussed in Laxmi et al. (2013). After that, Goswami (2014) presented a study of balking and reneging in finite-buffer discrete-time single server queue with single and multiple working vacations. Goswami and Selvaraju (2016) considered a $PH/M/c$ queueing system with multiple working vacations and impatient customers. Later, the analysis of batch arrival queue with variant working vacations and reneging were done in Laxmi and Rajesh (2017). Recently, Suranga Sampath and Jicheng (2018) studied the behavior of the impatient customers in an $M/M/1$ queue with waiting server subject to differentiated vacations policy. In addition, Azhagappan (2018) analyzed the transient behaviour of an $M/M/1$ queue with working vacation, variant reneging behavior and waiting server. For recent research papers on customers' impatience in vacation queues with waiting servers, we cite Sudhesh and Azhagappan (2016), Ammar (2017), and Suranga Sampath and Jicheng (2018).

1.4 Solution methods for queueing models with impatient customers and/or vacations

In recent decades, many efforts have focused on the investigation of queueing systems with impatient customers and/or vacations. In this sense, we mention in particular the use of

1.4.1 Recursive method

The recursive method is the most used method in queueing literature. Several authors employed this technique in their analysis for vacation queueing models, let's cite for instance, Chao and Zhao (1998), Chao and Rahman (2006), Ibe and Isijola (2014), and references therein. The recursive method was also extensively utilized while studying queues with impatience. El-Paoumy and Nabwey (2011) analyzed an $M/M/2/N$ queueing system with balking function and exponential reneging time of the impatient customer. The steady-state probabilities and some performance measures of the system in closed-form were presented. Later, Kumar and Sharma (2014) discussed the $M/M/c/N$ queue with balking and retention of reneged customers, the exact expressions of the steady-state probabilities were obtained. Very recently, Bouchentouf and Messabihi (2018a,2018b) studied heterogeneous servers Markovian queueing systems with impatient customers, the system size distribution was obtained, the performance measures and economic model where carried out.

1.4.2 Matrix analytic method

Matrix analytic method has been successfully applied in the vacation queueing system with impatience. One of practical advantages of this method is that the elementary matrix operations may easily be programmed for a high-speed computer. Yue et al. (2006) presented the optimization study of $M/M/1/N$ queueing system with multiple vacation of server, balking and reneging. In their work, authors developed the equations and derived the matrix form solution of the steady state probabilities. Further, the performance analysis of $GI/M/1$ queue with working vacations and vacation interruption has been discussed by Li et al. (2008). Later, Ammar et al. (2012) used the computable matrix technique to derive the transient distribution of the system length in single-server Markovian queue with discouraged arrivals and reneging. In addition, Liou (2015) dealt with infinite capacity Markovian queue with a single unreliable service station subject to working breakdowns and impatient customers. The matrix-analytic method has been used to compute the system size probabilities in the steady-state. Recently, in Afroun et al. (2018), the stationary analysis of a multiple vacation finite capacity Markovian queueing system with Bernoulli feedback, balking, reneging and retention of the impatient customers under server breakdown and repair has been established.

1.4.3 Transform method

This method involves the use of the Laplace-Stieltjes transform and the probability generating function techniques in the queueing problem analysis. Altman and Yechiali (2006) presented an analysis of the single and multi-server queueing models ($M/M/1$, $M/G/1$ and $M/M/c$ queues) with impatient customers for both multiple and single vacation policies, using the probability generating function (PGF). In Banik et al. (2007), the Laplace-Stieltjes transform is applied in the analytic analysis and computation of the $GI/M/1/N$ queueing model with multiple working vacations. Then, an $M/M/\infty$ queueing model with impatient customers has been studied by Altman and Yechiali (2008), authors obtained various performance measures such as the PGF of the stationary probabilities and the mean cycle time. Further, Boxma and Prabhu (2011) analyzed an $M/G/1$ queue with customer impatience and arrival adapted process, they derived the Laplace-Stieltjes transform of the joint stationary workload and arrival intensity process. In Yue et al. (2012), the authors considered a single server Markovian queueing system with multiple working vacations and impatient customers. They gave the probability generating functions of customers number in the system when the server is in normal busy and working vacation periods. Laxmi and Jyothsna (2013) presented the analysis of finite buffer renewal input queue with balking and multiple working vacations. Then, Laxmi and Rajesh (2016) examined a batch arrival infinite-buffer single server Markovain queueing system under the variant working vacations policy. The authors derived the probability generating function of the steady-state probabilities and obtained the closed form expressions of the system size when the server is in different states. After that, Laxmi and Rajesh (2017) extended their model given in Laxmi and Rajesh (2016) by including the impatience behavior. They used the same methodology for the analysis of the system. Recently, Bouchentouf and Yahiaoui (2017) carried out the stationary analysis of an $M/M/1$ queue with multiple working vacations, Bernoulli schedule vacation interruption, feedback, reneging and retention of reneged customers using the probability generating function.

1.4.4 Supplementary variable technique

The supplementary variable technique introduced by Cox (1955) plays an important role in the analysis of non-Markovian queueing systems. Many studies are based on this technique. Baba (1986) considered an $M^X/G/1$ queue with vacation. For the analysis, he used the remaining service time for the customers in service and the remaining vacation time for the server on vacation as supplementary variables to derive the queue size distribution at an arbitrary time. He also determined the waiting time and busy period distributions. Then, this model were extended by Choudhury (2002) by considering the elapsed service and vacation times as supplementary variables. The steady-state behaviour of the queue distribution has been established under single and multiple vacation policies. Later, Zhang and Hou (2012) carried out the stationary analysis of an $M/G/1$ queueing system under single working vacation policy of the server. The supplementary variable and the matrix-analytic techniques have been used to obtain the stationary differential equations and their solutions. Recently, Goswami (2015) computed the steady state system size distribution at pre-arrival and arbitrary epochs of an $GI/M/1/N$ queue with balking, reneging and working vacation using the supplementary variable and recursive methods. Various performance measures of the model were evaluated.

1.4.5 Embedded Markov chain technique

Embedded Markov chain method is applied in many queueing problems when the distributions of the inter-arrival time, the service time or the vacation time do not get the memoryless property. This technique was given for the first time by Kendall (1951) and it was employed in many research papers, for instance, Chatterjee and Mukherjee (1990) explored the equilibrium probability distributions of system size at pre-arrival and at random epochs separately for a $GI/M/1$ queue with server vacation by applying the embedded Markov chain technique. In addition, Baba (2005) used this technique to present serval results such as the steady-state distributions for the number of customers in the system at both arrival and arbitrary epochs as well as the sojourn time for an arbitrary customer of an $GI/M/1$ queue with multiple working vacations. Afterward, Chae et al. (2009) gave the steady-state distributions for customers number in the system of both $GI/M/1$ and $GI/Geo/1$ queues with single working vacations. In recent time, Laxmi and Jyothsna (2013) investigated the finite-buffer renewal input queue with balking and multiple working vacation. The system size distributions at pre-arrival and arbitrary epochs have been obtained by using the embedded Markov chain and supplementary variable techniques.

1.4.6 Stochastic decomposition

This technique is very powerful in the study of the queueing models with vacation and working vacation. Fuhrmann and Cooper (1985) used the stochastic decomposition to show that the stationary distribution of the number of customers in an $M/G/1$ queue with generalized server vacation is a convolution of the distribution functions of two independent positive random variables, at which one of these is the stationary distribution of the number of customers in an $M/G/1$ queue without server vacations. In Shanthikumar (1988), an analytic proof of that result were done for more general systems which allow bulk customer arrivals and certain variations with reneging, balking, and in arrival rate depending on the state of the system. Then, Artalejo (1997) developed the analysis of a $M/G/1$ queueing model with constant attempts and server vacations including ergodicity, limiting behaviour, stochastic decomposition and optimal control. Zhang and Tian (2003) investigated the $M/M/c$ queueing model with vacations of the servers, the conditional stochastic decomposition property of the queue length and waiting time in the system were demonstrated. Further, Liu et al. (2007) gave the stochastic decomposition of an $M/M/1$ queue with working vacation. Then, in Li et al. (2009), authors were based on the stochastic decomposition structure of the queue length and waiting time in the $M/G/1$ queue with working vacation and used the embedded Markov chain as well as the matrix analytic techniques to determine the distribution of the stationary queue length at departure epochs and the distribution of the waiting time. Later, Selvaraju and Goswami (2013) considered a working vacation Markovain queueing model with impatient customers. They presented the stochastic decomposition properties for both single and multiple working vacation cases.

1.5 Queueing models with feedback

The feedback in the queueing systems is the situation when the customers are not satisfied with the initial service due to inappropriate quality of the previous one or when they are requiring another regular service. In this situation, customers may return to the system to satisfy their needs. These cases can be observed in super markets, hospital management, post offices, and banks etc. Interesting examples may be found in computer communication, where the transmission of a protocol data unit can be from time to time repeated because of occurrence of an error, this mostly arises because of dissatisfying quality of service. Another example of feedback queue is a rework in industrial operations at which units that do not comply with certain specifications are placed in the production line and are again subject to rework. So, if the service is incomplete or unsatisfactory, the customer can either leave the system definitively with some probability or rejoin the end of the queue with a complementary probability. This is known as Bernoulli feedback.

Takacs (1963) was the first who introduced a queue with feedback, he obtained the Laplace-Stieltjes transform and the closed-form expressions of the first two moments of the distribution function of the total time spent in the system by a customer. Since then, several authors investigated queueing systems subject to feedback. D'Avignon and Disney (1976) gave the analysis of single server queues with state dependent feedback. Later, Choi et al. (1998) investigated a $M/M/c$ retrial queue with geometric loss and feedback. Later, Santhakumaran and Thangaraj (2000) studied the $M/M/1$ queue with impatient and Bernoulli feedback. The study of an $M/M/1$ feedback queue with catastrophes have been examined by Thangaraj and Vanitha (2009), the authors derived transient and stationary distributions of the system size by using continued fractions method. Further, Bouchentouf and Belarbi (2013) presented the performance evaluation of two retrial queueing system with balking and feedback. Then, a cost analysis and optimization study of an $M/M/1/N$ feedback queue with retention of reneged customers have been discussed by Kumer et al. (2014). Furthermore, Ayyappan and Shyamala (2016) analyzed a single server queueing system with Bernoulli feedback and Bernoulli server vacation, random breakdowns, where the customers arrive in batches according to a Poisson process, the service and vacation times have a general distributions, authors obtained the probability generating function of transient solutions of the system and presented the analysis of the steady state. Recently, Vijayalakshmi and Kalidass (2018) considered a Markovian queueing model with geometric abandonments and Bernoulli feedbacks, they derived the steady state probabilities, and gave some important performance measures of the system.

1.6 Summary of results established in this thesis

▸ **First Result: Performance and economic study of heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations and impatient customers.**

In this work we present a study of heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, balking, reneging and retention of reneged customers, at which the impatience timers of customers in the system depend on the state of the servers. The current study has a large application in many real world systems as telecommunication networks, call centers and production-inventory systems, where the assumption of impatience timers depending on server's states takes place, when arriving customers can not have an information about the servers. It is supposed that the inter-arrival times are i.i.d r.v with cumulative distribution function $A(u)$, probability density function $a(u)$, $u \geq 0$, Laplace-Stieltjes transform (L.S.T.) $A^*(\varpi)$ and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$, where $h^{(1)}(0)$ is the first derivative of $h(\varpi)$ evaluated at $\varpi = 0$. The capacity of the system is taken finite N and the customers are served on a FCFS discipline. The considered queueing system consists of two heterogeneous servers, namely, server 1 and server 2. The service times follow an exponential distribution with parameters μ_1 and μ_2, respectively, where $\mu_2 \leq \mu_1$. We suppose that once server 2 becomes idle and finds the queue imply, he leaves for an exponential working vacation with parameter ϕ. During this period, he serves the waiting customers at a lower rate than the normal service rate which is assumed to be exponentially distributed with parameter ν. Then, whenever the working vacation is finished, if there are customers waiting in the queue, he switches to normal working level, otherwise he continues the vacation. The server 1 is always available in the system. Moreover, an external arriving customer who finds i customers in the system decides either to join the queue with probability $b_i = 1 - \frac{i}{N^2}$ or balk with probability $\overline{b}_i = 1 - b_i = \frac{i}{N^2}$, with $b_0 = b_1 = 1$, $0 \leq b_{i+1} \leq b_i \leq 1$, $2 \leq i \leq N-1$, and $b_N = 0$. Fuhrer, If i customers are present in the system, one of the $(i-2)$ waiting customers in the queue may renege, such that whenever a customer arrives at the system and finds the server 2 on working vacation (resp. on normal busy period), he activates an impatience timer T_1 (resp. T_2,) which is exponentially distributed with parameter ξ_1 (resp. ξ_2). If the customer's service has not started before the customer's timer expires, the customer leaves the queue. So, the customer's average reneging rate is done by $(i-2)\xi_1$ (resp. $(i-2)\xi_2$) when server 2 is on working vacation (resp. on normal busy period), $2 \leq i \leq N$. We suppose that impatience timers are i.i.d r.v and independent of the number of waiting customers. Further, each reneged customer may be retained in the system with some probability α', and may leave the queue definitively with probability α. After getting incomplete or unsatisfactory service either from working vacation service or normal busy service, with probability β', a customer may rejoin the end of the queue as a Bernoulli feedback customer to receive another regular service. Otherwise, he leaves the system definitively with probability β, where $\beta' + \beta = 1$. Finally, it is assumed that the inter-arrival times, service times and vacation times are assumed to be independent.

Let $N_s(t)$ denote the number of customers in the system at time t. Let $I(t)$ denote the remaining inter-arrival time at time t for the next arrival.

Let

$$S(t) = \begin{cases} 0, & \text{when server 2 is idle during working vacation (WV) period;} \\ 1, & \text{when server 2 is busy during working vacation (WV) period;} \\ 2, & \text{when server 2 is busy during normal busy period,} \end{cases}$$

be the state of the system.

We use the supplementary variable and recursive techniques to obtain the steady-state probabilities of the system following the same method given in Laxmi and Jyothsna (2015) . To get the system length distributions at arbitrary epoch, the differential difference equations using the remaining inter-arrival time as the supplementary variable are developed.

The joint probabilities are as

$$\pi_{i,0}(u,t)du = \mathbb{P}(N_s(t) = i, u \leq I(t) < u + du, S(t) = 0), u \geq 0, i = 0, 1,$$

$$\pi_{i,j}(u,t)du = \mathbb{P}(N_s(t) = i, u \leq I(t) < u + du, S(t) = j), u \geq 0, j = 1, 2,$$

$$1 \leq i \leq N.$$

Thus

$$\pi_{i,0}(u) = \lim_{t \to \infty} \pi_{i,0}(u,t), \; i = 0, 1, \; \pi_{i,j}(u) = \lim_{t \to \infty} \pi_{i,j}(u,t), \; j = 1, 2, \; 1 \leq i \leq N.$$

The L.S.T. of the steady-state probabilities are given as

$$\pi^*_{i,0}(\varpi) = \int_0^\infty e^{-\varpi u} \pi_{i,0}(u) du, \; i = 0, 1, \; \pi^*_{i,j}(\varpi) = \int_0^\infty e^{-\varpi u} \pi_{i,j}(u) du,$$

$$j = 1, 2, \; 1 \leq i \leq N.$$

Let $\pi_{i,j} = \pi^*_{i,j}(0)$ be the probability of i customers in the system when the server is in state j at an arbitrary epoch.

The pre-arrival epoch probabilities are as

$$\pi^{-}_{i,j} = \lim_{t\to\infty} \mathbb{P}(N_s(t) = i, S(t) = j/I(t) = 0)$$

with $\pi^{-}_{i,0}$, $i = 0,1$ and $\pi^{-}_{i,j}$, $j = 1,2$; $1 \le i \le N$.

– The steady-state probabilities of the system are as

$$\begin{aligned}
\pi_{N,1} &= \frac{\lambda}{\zeta_N}\left(1-\frac{N-1}{N^2}\right)\pi^{-}_{N-1,1},\\
\pi_{i,1} &= \left(\frac{\zeta_{i+1}-\phi}{\zeta_i}\right)\pi_{i+1,1}+\frac{\lambda}{\zeta_i}\left(\left(1-\frac{i-1}{N^2}\right)\pi^{-}_{i-1,1}-\left(1-\frac{i}{N^2}\right)\pi^{-}_{i,1}\right), i = N-1,\ldots,3,\\
\pi_{2,1} &= \left(\frac{\zeta_3-\phi}{\zeta_2}\right)\pi_{3,1}+\frac{\lambda}{\zeta_2}\left(\pi^{-}_{1,0}+\pi^{-}_{1,1}-\left(1-\frac{2}{N^2}\right)\pi^{-}_{2,1}\right),\\
\pi_{1,1} &= \left(\frac{\beta\mu_1}{\phi+\beta\nu}\right)\pi_{2,1}-\left(\frac{\lambda}{\phi+\beta\nu}\right)\pi^{-}_{1,1},\\
\pi_{N,2} &= \frac{\phi}{\theta_N}\pi_{N,1}+\frac{\lambda}{\theta_N}\left(1-\frac{N-1}{N^2}\right)\pi^{-}_{N-1,2},\\
\pi_{i,2} &= \left(\frac{\theta_{i+1}}{\theta_i}\right)\pi_{i+1,2}+\frac{\phi}{\theta_i}\pi_{i,1}+\frac{\lambda}{\theta_i}\left(\left(1-\frac{i-1}{N^2}\right)\pi^{-}_{i-1,2}-\left(1-\frac{i}{N^2}\right)\pi^{-}_{i,2}\right), i = N-1,\ldots,2,\\
\pi_{1,2} &= \frac{\mu_1}{\mu_2}\pi_{2,2}+\frac{\phi}{\beta\mu_2}\pi_{1,1}-\frac{\lambda}{\beta\mu_2}\pi^{-}_{1,2},\\
\pi_{1,0} &= \frac{\nu}{\mu_1}\pi_{2,1}+\frac{\mu_2}{\mu_1}\pi_{2,2}+\frac{\lambda}{\beta\mu_1}\left(\pi^{-}_{0,0}-\pi^{-}_{1,0}\right),
\end{aligned}$$

where

$$\pi_{0,0} = 1-\pi_{1,0}-\sum_{i=1}^{N}(\pi_{i,1}+\pi_{i,2}).$$

– Useful system characteristics include:

. The mean number of customers in the system.

$$L_s = \pi_{1,0}+\sum_{i=1}^{N} i(\pi_{i,1}+\pi_{i,2}).$$

. The mean number of customers waiting for service.

$$L_q = \sum_{i=2}^{N} (i-2)(\pi_{1,1} + \pi_{i,2}).$$

. The mean waiting time of customers in the system.

$$W_s = \frac{L_s}{\lambda'}, \text{ where } \lambda' = \lambda(1 - (\pi_{N,1} + \pi_{N,2})) \text{ is the effective arrival rate.}$$

. The mean rate of joining the system.

$$J_s = \lambda(\pi_{0,0} + \pi_{1,0} + \pi_{1,1} + \pi_{1,2}) + \sum_{i=2}^{N} \lambda\left(1 - \frac{i}{N^2}\right)(\pi_{i,1} + \pi_{i,2}).$$

. The probability that server 2 is idle, in working vacation period and in normal busy period, respectively.

$$P_{idle} = \sum_{i=0}^{1} \pi_{i,0}, \; P_w = \sum_{i=1}^{N} \pi_{i,1}, \text{ and } P_b = \sum_{i=1}^{N} \pi_{i,2}.$$

. The average balking rate.

$$B_r = \frac{\lambda}{N^2} \sum_{i=1}^{N} i(\pi_{i,1} + \pi_{i,2})$$

. The average reneging rates during busy period and working vacation period, respectively.

$$R_{ren1} = \alpha\xi_1 \sum_{i=2}^{N} (i-2)\pi_{i,1} \text{ and } R_{ren2} = \alpha\xi_2 \sum_{i=2}^{N} (i-2)\pi_{i,2}.$$

. The average retention rates during busy period and working vacation period, respectively.

$$R_{ret1} = \alpha'\xi_1 \sum_{i=2}^{N} (i-2)\pi_{i,1} \text{ and } R_{ret2} = \alpha'\xi_2 \sum_{i=2}^{N} (i-2)\pi_{i,2}.$$

Based on the steady-state distribution of the system size, explicit performance measures are derived and a cost model is developed for the costs incurred in the considered queueing system. Moreover, numerical examples are presented.

► **Second result: Sensitivity analysis of feedback multiple vacation queueing system with differentiated vacations, vacation interruptions and impatient customers.**

In this work, we consider an $M/M/1$ queueing system with Bernoulli feedback, multiple vacations, differentiated vacations, vacation interruptions, balking, reneging and retention of reneged customers. Customers arrive into the system according to a Poisson process with arrival rate λ, the service time is supposed to be exponentially distributed with mean $1/\mu$. The service discipline is FCFS and there is an infinite space for customers to wait. Two types of vacation are considered: type 1 vacation that may be taken after a busy period where at least one customer is served, and type 2 vacation which is taken when the server comes back from any vacation (either a type 1 vacation or a type 2 vacation) and finds the system empty. The period of a type 1 (resp. type 2) vacation is assumed to be exponentially distributed with rate ϕ_1 (resp. ϕ_2).
The server's vacation can be interrupted when the number of customers in the system reaches n_1 (resp. n_2) when the server is on type 1 (resp. type 2) vacation. Moreover, we suppose that $n_1 > n_2$, as we desire that the server will be interrupted earlier when he takes a vacation after zero busy period than when he takes a vacation after having a non-zero busy period.
Whenever a customer arrives at the system and finds the server busy, he activates an impatience timer T, exponentially distributed with parameter ξ, if the customer's service has not been completed before the customer's timer expires, the customer may leave the system. We suppose that the customers timers are independent and identically distributed random variables, independent of the size of the queue at that time. Further, using a certain mechanism, a reneged customer may abandon the system without getting service with probability α and can be retained in the system with probability α', where $\alpha + \alpha' = 1$.
A customer who on arrival finds at least one customer in the system, either decides to join the queue with probability θ or balk with probability θ', where $\theta + \theta' = 1$. If the service is incomplete or unsatisfactory, the customer can either leave the system definitively with probability β or rejoin the end of the queue with probability β', where $\beta + \beta' = 1$.

The steady-state analysis of the system is carried out using a recursive method. Let $L(t)$ be the number of customers in the system at time t, and $J(t)$ denotes the state of the server at time t such that

$$J(t) = \begin{cases} 0, & \text{if the server is on busy period;} \\ 1, & \text{if the server is on type 1 vacation;} \\ 2, & \text{if the server is on type 2 vacation.} \end{cases}$$

The process $\{(L(t), J(t)), t \geq 0\}$ is a continuous-time Markov process with state space $\Omega = \{(n,0) : n \geq 1\} \cup \{(n,j) : n \geq 0, j = 1,2\}$.

Let $P_{n,j} = \lim\limits_{t\to\infty} P\{L(t) = n, J(t) = j, \ n \geq 0, j = \overline{0,2}\}$ denote the steady-state probabilities of the system.

– The analytic expressions of the steady-state probabilities are given by

$$P_{0,1} = \omega_1 P_{1,0}, \qquad \text{where} \qquad \omega_1 = \left(\frac{\beta\mu + \alpha\xi}{\lambda + \phi_1}\right),$$

$$P_{k,1} = \delta_1 \beta_1^k P_{1,0}, \ k = \overline{1, n_1 - 1}, \qquad \text{where} \qquad \delta_1 = \frac{\omega_1}{\theta}, \text{ and } \beta_1 = \frac{\theta\lambda}{\theta\lambda + \phi_1}, \tag{1.1}$$

$$P_{0,2} = \omega_2 P_{1,0}, \qquad \text{where} \qquad \omega_2 = \frac{\phi_1}{\lambda}\omega_1.$$

$$P_{k,2} = \delta_2 \beta_2^k P_{1,0}, \ k = \overline{1, n_2 - 1}, \qquad \text{where} \qquad \delta_2 = \frac{\omega_2}{\theta}, \ \beta_2 = \frac{\theta\lambda}{\theta\lambda + \phi_2}. \tag{1.2}$$

$$P_{k,0} = \frac{\theta\lambda}{\beta\mu + k\alpha\xi}\left\{(\theta\lambda)^{k-2}\prod_{i=2}^{k-1}\left(\frac{1}{\beta\mu + i\alpha\xi}\right) + \delta_1\beta_1^{k-1} + \delta_2\beta_2^{k-1} + \delta_1\sum_{i=1}^{k-2}\frac{(\theta\lambda)^{k-(i+1)}}{\prod\limits_{j=i+1}^{k-1}\left(\beta\mu + j\alpha\xi\right)}\beta_1^i \right.$$
$$\left. + \delta_2\sum_{i=1}^{k-2}\frac{(\theta\lambda)^{k-(i+1)}}{\prod\limits_{j=i+1}^{k-1}\left(\beta\mu + j\alpha\xi\right)}\beta_2^i\right\} P_{1,0}, \qquad k = \overline{2, n_2 - 1}. \tag{1.3}$$

$$P_{n_2,0} = \frac{\theta\lambda}{\beta\mu + n_2\alpha\xi}\left(P_{n_2-1,0} + P_{n_2-1,1} + P_{n_2-1,2}\right) = A(n_2, 0)P_{1,0},$$

where

$$A(n_2,0)=\left(\frac{\theta\lambda}{\beta\mu+n_2\alpha\xi}\right)\left\{\frac{\theta\lambda}{\beta\mu+(n_2-1)\alpha\xi}\left\{(\theta\lambda)^{n_2-3}\prod_{i=2}^{n_2-2}\left(\frac{1}{\beta\mu+i\alpha\xi}\right)+\delta_1\beta_1^{n_2-2}+\delta_2\beta_2^{n_2-2}\right.\right.$$

$$\left.\left.+\delta_1\sum_{i=1}^{n_2-3}\frac{(\theta\lambda)^{n_2-(i+1)}}{\prod_{j=i+1}^{n_2-2}\left(\beta\mu+j\alpha\xi\right)}\beta_1^i+\delta_2\sum_{i=1}^{n_2-3}\frac{(\theta\lambda)^{n_2-(i+1)}}{\prod_{j=i+1}^{n_2-2}\left(\beta\mu+j\alpha\xi\right)}\beta_2^i\right\}+\delta_1\beta_1^{n_2-1}+\delta_2\beta_2^{n_2-1}\right\}.$$

$$P_{k,0}=\left\{\frac{(\theta\lambda)^{k-n_2}}{\prod_{i=1}^{k-n_2}\left(\beta\mu+(n_2+i)\alpha\xi\right)}A(n_2,0)+\delta_1\sum_{j=0}^{k-n_2}\frac{(\theta\lambda)^{k-n_2-j}}{\prod_{i=j+1}^{k-n_2}\left(\beta\mu+(n_2+i)\alpha\xi\right)}\beta_1^{n_2+j}\right\}P_{1,0},\quad k=\overline{n_2,n_1-1}. \tag{1.4}$$

$$P_{k,0}=\Phi A(n_1,0)P_{1,0},\ \text{with}\ \Phi=\frac{(\theta\lambda)^{k-n_1}}{\prod_{j=1}^{k-n_1}\left(\beta\mu+(n_1+j)\alpha\xi\right)},\qquad k=n_1,n_1+1,\ldots. \tag{1.5}$$

where

$$A(n_1,0)=\left(\frac{\theta\lambda}{\beta\mu+n_1\alpha\xi}\right)\left\{\frac{(\theta\lambda)^{n_1-n_2-1}}{\prod_{i=1}^{n_1-n_2-1}\left(\beta\mu+(n_2+i)\alpha\xi\right)}A(n_2,0)\right.$$

$$\left.+\delta_2\sum_{j=0}^{n_1-n_2-1}\frac{(\theta\lambda)^{n_1-n_2-j-1}}{\prod_{i=j+1}^{n_1-n_2-1}\left(\beta\mu+(n_2+i)\alpha\xi\right)}\beta_1^{n_2+j}+\delta_1\beta_1^{n_1-1}\right\},$$

and

$$P_{1,0}=\frac{1}{1+B_1+B_2+B_3+B_4+B_5},$$

where

$$B_1 = \sum_{k=2}^{n_2-1} \frac{\theta\lambda}{\beta\mu + k\alpha\xi} \Bigg\{ (\theta\lambda)^{k-2} \prod_{i=2}^{k-1} \left(\frac{1}{\beta\mu + i\alpha\xi} \right) + \delta_1 \beta_1^{k-1} + \delta_2 \beta_2^{k-1}$$

$$+ \delta_1 \sum_{i=1}^{k-2} \frac{(\theta\lambda)^{k-(i+1)}}{\prod_{j=i+1}^{k-1} \left(\beta\mu + j\alpha\xi \right)} \beta_1^i + \delta_2 \sum_{i=1}^{k-2} \frac{(\theta\lambda)^{k-(i+1)}}{\prod_{j=i+1}^{k-1} \left(\beta\mu + j\alpha\xi \right)} \beta_2^i \Bigg\},$$

$$B_2 = \sum_{k=n_2}^{n_1-1} \left\{ \frac{(\theta\lambda)^{k-n_2}}{\prod_{i=1}^{k-n_2} \left(\beta\mu + (n_2+i)\alpha\xi \right)} A(n_2, 0) + \delta_1 \sum_{j=0}^{k-n_2} \frac{(\theta\lambda)^{k-n_2-j}}{\prod_{i=j+1}^{k-n_2} \left(\beta\mu + (n_2+i)\alpha\xi \right)} \beta_1^{n_2+j} \right\},$$

$$B_3 = \sum_{k=n_1}^{+\infty} \frac{(\theta\lambda)^{k-n_1}}{\prod_{j=1}^{k-n_1} \left(\beta\mu + (n_1+j)\alpha\xi \right)} A(n_1, 0), \; B_4 = \omega_1 + \delta_1 \sum_{k=1}^{n_1-1} \beta_1^k, \text{ and } B_5 = \omega_2 + \delta_2 \sum_{k=1}^{n_2-1} \beta_2^k.$$

– Useful performance measures of the system that are of general interest.

. The probability that the server is in busy period.

$$P_B = P_{1,0} + \sum_{k=2}^{n_2-1} P_{k,0} + \sum_{k=n_2}^{n_1-1} P_{k,0} + \sum_{k=n_1}^{+\infty} P_{k,0}.$$

. The probability that the server is on vacation.

$$P_V = P_{V_1} + P_{V_2} = \sum_{k=0}^{n_1-1} P_{k,1} + \sum_{k=0}^{n_2-1} P_{k,2} = 1 - P_B,$$

where P_{V_1} and P_{V_2} are the probabilities that the server is on type 1 vacation and type 2 vacation, respectively.

. The average number of customers in the system.

$$L_s = P_{1,0} + \sum_{k=2}^{n_2-1} kP_{k,0} + \sum_{k=n_2}^{n_1-1} kP_{k,0} + \sum_{k=n_1}^{+\infty} kP_{k,0} + \sum_{k=0}^{n_1-1} kP_{k,1} + \sum_{k=0}^{n_2-1} kP_{k,2}.$$

- The average number of customers in the queue.

$$L_q = \sum_{k=2}^{n_2-1}(k-1)P_{k,0} + \sum_{k=n_2}^{n_1-1}(k-1)P_{k,0} + \sum_{k=n_1}^{+\infty}(k-1)P_{k,0} + \sum_{k=1}^{n_1-1}kP_{k,1} + \sum_{k=1}^{n_2-1}kP_{k,2}.$$

- The average reneging rate.

$$R_{ren} = \alpha\xi\left(P_{1,0} + \sum_{k=2}^{n_2-1}kP_{k,0} + \sum_{k=n_2}^{n_1-1}kP_{k,0} + \sum_{k=n_1}^{+\infty}kP_{k,0}\right).$$

- The average retention rate.

$$R_{ret} = \alpha'\xi\left(P_{1,0} + \sum_{k=2}^{n_2-1}kP_{k,0} + \sum_{k=n_2}^{n_1-1}kP_{k,0} + \sum_{k=n_1}^{+\infty}kP_{k,0}\right).$$

- The average balking rate.

$$B_r = \theta'\lambda\left(P_{1,0} + \sum_{k=2}^{n_2-1}P_{k,0} + \sum_{k=n_2}^{n_1-1}P_{k,0} + \sum_{k=n_1}^{+\infty}P_{k,0} + \sum_{k=1}^{n_1-1}P_{k,1} + \sum_{k=1}^{n_2-1}P_{k,2}\right).$$

- The expected number of customers served per unit of time.

$$E_{cs} = \beta\mu\left(P_{1,0} + \sum_{k=2}^{n_2-1}kP_{k,0} + \sum_{k=n_2}^{n_1-1}kP_{k,0} + \sum_{k=n_1}^{+\infty}kP_{k,0}\right).$$

Using the steady-state probabilities, we obtained useful performance measures of the system and developed a cost model. An extensive numerical analysis is done.

▶ **Third result: Performance and economic analysis of Markovian Bernoulli feedback queueing system with vacations, waiting server and impatient customers.**

In this work, we treat a $M/M/1$ vacation queueing system with Bernoulli feedback, waiting server, reneging, and retention of reneged customers, at which once the busy period ended the server waits a random duration of time before beginning on a vacation. The impatience timers of customers depend on the server's states. The analysis presented has a large application especially when we deal with a human behavior, examples can be found in post offices, banks, hospitals, etc. Another great scope of this investigation concerns the supposition that customers may be impatient because of state of the server, where the customer may become impatient due to the long wait in the queue even if the server is present in the system, another example when the customer may leave the system during busy period is when he cannot see the server state, these situations can be found in telecommunication systems, call centers and production inventory systems. Consider a $M/M/1$ vacation queueing model, customers arrive into the system according to a Poisson process with arrival rate λ, and service time is assumed to be exponentially distributed with parameter μ. The service discipline is FCFS and there is infinite space for customers to wait. When the busy period is finished the server waits a random duration of time before beginning on a vacation. This waiting duration is exponentially distributed with parameter η. If the server comes back from a vacation to an empty system he waits passively the first arrival, then he begins service. Otherwise, if there are customers waiting in the queue at the end of a vacation, the server starts immediately a busy period. That is single vacation policy. The period of vacation has an exponential distribution with parameter γ. Whenever a customer arrives at the system and finds the server on vacation (respectively. busy), he activates an impatience timer T_0 (respectively. T_1), which is exponentially distributed with parameter ξ_0 (respectively. ξ_1). If the customer's service has not been completed before the impatience timer expires, the customer may abandon the queue. We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers. Each reneged customer may leave the system without getting service with probability α and may be retained in the system with probability $\alpha' = (1-\alpha)$. After completion of each service, the customer can either leave the system definitively with probability β or return to the system and join the end of the queue with probability β', where $\beta + \beta' = 1$.

Let $L(t)$ be the number of customers in the system at time t, and $J(t)$ denotes the state of the server at time t such that

$$J(t) = \begin{cases} 1, & \text{when the server is in a busy period;} \\ 0, & \text{otherwise.} \end{cases}$$

The process $\{(L(t);J(t)); t \geq 0\}$ is a continuous-time Markov process with state space

$$\Omega = \{(j,n) : j = 0,1,\ n = 0,1,...\}.$$

Let $P_{j,n} = \lim_{t\to\infty} P\{J(t) = j, L(t) = n\},\ j = 0,1, n = 0,1,...,\ (j,n) \in \Omega$, denote the system state probabilities.

– Using the probability generating function (PGF), we get

1. The steady-state probability $P_{0,.}$:

$$P_{0,.} = \left(\frac{\gamma \alpha \xi_0 + \delta_1 K_0(1)(1-\gamma)}{\gamma K_0(1)} \right) P_{0,0}. \tag{1.6}$$

2. The steady-state probability $P_{1,.}$:

$$\begin{aligned} P_{1,.} &= e^{\frac{\lambda}{\alpha\xi_1}} \left(\frac{\gamma}{\lambda+\eta} \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) - \frac{\gamma}{\alpha\xi_1} K_3(1) \right. \\ &\left. + \frac{\beta\mu + \alpha\xi_1}{\lambda+\eta} \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) \left(\frac{\alpha\xi_0 - \delta_1 K_0(1)}{\delta_2 K_0(1)} \right) \right) P_{0,0}, \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} P_{0,0} = &\left\{ \frac{\delta_1\delta_2 K_0(1) + \delta_2(\alpha\xi_0 - \delta_1 K_0(1))}{\gamma\delta_2 K_0(1)} + e^{\frac{\lambda}{\alpha\xi_1}} \left[\left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) \right. \right. \\ &\left. \left. \left(\frac{\gamma}{\lambda+\eta} + \left(\frac{\beta\mu+\alpha\xi_1}{\lambda+\eta} \left(\frac{\alpha\xi_0 - \delta_1 K_0(1)}{\delta_2 K_0(1)} \right) \right) \right) - \frac{\gamma}{\alpha\xi_1} K_3(1) \right] \right\}^{-1}, \end{aligned} \tag{1.8}$$

$$K_0(1) = \int_0^1 (1-s)^{\frac{\gamma}{\alpha\xi_0}-1} e^{-\frac{\lambda}{\alpha\xi_0}s} ds,$$

$$K_1(1) = \int_0^1 s^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds, \;\; K_2(1) = \int_0^1 (1-s)^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds,$$

and

$$K_3(1) = \int_0^1 \left(1 - \frac{K_0(s)}{K_0(1)} \right) s^{\frac{\beta\mu}{\alpha\xi_1}} (1-s)^{-\left(\frac{\gamma}{\alpha\xi_0}+1\right)} e^{\left(\frac{\lambda}{\alpha\xi_0} - \frac{\lambda}{\alpha\xi_1}\right)s} ds.$$

– The performance measures that are of general interest include:

. The probability that the server is in busy period, on vacation, and idle during busy period, respectively.

$$P_B = P_{1,.},\; P_V = 1 - P_B, \text{ and } P_I = P_{1,0}.$$

. The average number of customers in the system when the server is taking vacation.

$$\mathbb{E}(L_0) = \left(\frac{\lambda}{\gamma + \alpha\xi_0}\right) P_{0,.}.$$

. The average number of customers in the system when the server is in busy period.

$$\mathbb{E}(L_1) = \left(\frac{\lambda - \beta\mu}{\alpha\xi_1}\right) P_{1,.} + \frac{\gamma}{\alpha\xi_1}\mathbb{E}(L_0) + \frac{\beta\mu}{\alpha\xi_1(\lambda+\eta)}\left(\gamma + \frac{(\beta\mu + \alpha\xi_1)(\alpha\xi_0 - \delta_1 K_0(1))}{\delta_2 K_0(1)}\right) P_{0,0}.$$

. The average number of customers in the system.

$$\mathbb{E}(L) = \mathbb{E}(L_0) + \mathbb{E}(L_1).$$

. The average number of customers in the queue.

$$\mathbb{E}(L_q) = \mathbb{E}(L) - (P_{1,.} - P_{1,0}).$$

. The mean waiting time of a customer in the system.

$$W_s = \frac{\mathbb{E}(L_0) + \mathbb{E}(L_1)}{\lambda} = \frac{\mathbb{E}(L)}{\lambda}.$$

. The expected number of customers served per unit of time.

$$E_{cs} = \beta\mu(P_{1,.} - P_{1,0}).$$

. The average rate of reneging (resp. retention) during vacation period.

$$R_{ren_0} = \alpha\xi_0\mathbb{E}(L_0), \;\; R_{ret_0} = (1-\alpha)\xi_0\mathbb{E}(L_0).$$

. The average rate of reneging (resp. retention) during busy period.

$$R_{ren_1} = \alpha\xi_1\mathbb{E}(L_1), \;\; R_{ret_1} = (1-\alpha)\xi_1\mathbb{E}(L_1).$$

. The average rate of abandonment of a customer due to impatience.

$$R_{ren} = R_{ren_0} + R_{ren_1}.$$

. The average rate of retention of impatient customers.

$$R_{ret} = R_{ret_0} + R_{ret_1}.$$

Based on the steady-state probabilities of the queueing model, explicit expressions of useful measures of effectiveness as well as a cost model are derived. The impact of diverse parameters on the performance measures of the system has been shown. Further, an economic analysis of the model is carried out.

▶ **Fourth result: Single server batch arrival Bernoulli feedback queueing system with waiting server, K-variant vacations and impatient customers.**

This work deals with a $M/M/1$ queueing system where customers arrive in batches according to a Poisson process with rate λ. Let X be the batch size random variable of the arrival with probability mass function $P(X = l) = b_l$, $l = 1, 2, \cdots$. The service time is supposed to be exponentially distributed with mean $1/\mu$. The customers are served on FCFS discipline. Once the busy period is ended the server waits a random period before taking a vacation, this waiting time follows an exponential distribution with parameter η. At the end of the vacation period, if the system is still empty, the server returns to the vacation. The server is allowed to take K of successive vacations. When the K consecutive vacations are complete, the server returns to busy period and depending on the arriving batch of customers, he stays idle or busy. The period of a vacation follows an exponential distribution with mean $1/\phi$. During vacation period, each incoming customer starts up an impatience timer independently of the other customers in the system, which is supposed to be exponentially distributed with parameter ξ. The impatient customers may leave the system with probability α and they can be retained in the system with probability $\alpha' = 1 - \alpha$. After completion of each service, the customer may decide either to leave the system with probability β or return and join the tail of the queue with probability β', where $\beta + \beta' = 1$.

The inter-arrival times, service times and vacation times are mutually independent.

Let $L(t)$ denote the number of customers in the system and $S(t)$ be the status of the server at time t, such that

$$S(t) = \begin{cases} j, & \text{when the server is taking the } (j+1)^{th} \text{ vacation at time } t, \\ & j = \overline{0, K-1}; \\ K, & \text{the server is in busy period at time } t. \end{cases}$$

The bi-variate $\{(L(t); S(t)); t \geq 0\}$ represents two dimensional infinite state continuous-time Markov chain with state space $\Omega = \{(n, j) : n \geq 0, j = \overline{0, K}\}$.

Let $P_{n,j} = \lim_{t \to \infty} P\{L(t) = n, S(t) = j\}$, $n \geq 0, j = \overline{0, K}$ denote the system state probabilities of the process $\{(L(t), S(t)), t \geq 0\}$.

– Under the stability condition $\lambda E(X) < \beta\mu$,

1. The steady-state-probabilities $P_{n,j}$ are given as

$$P_{.,j} = \sum_{n=0}^{\infty} P_{n,j} = A^{j-1} P_{0,0}, \qquad j = \overline{0, K-1}, \tag{1.9}$$

2. The steady-state-probabilities $P_{n,K}$ are given as

$$P_{.,K} = \sum_{n=0}^{\infty} P_{n,K} = \frac{1}{\beta\mu - \lambda B'(1)} \left\{ \frac{\phi\lambda B'(1)}{\alpha\xi + \phi} \frac{1 - A^K}{A(1-A)} + \frac{\beta\mu\alpha\xi}{\eta C} \right\} P_{0,0}, \tag{1.10}$$

where

$$P_{0,0} = \left\{ \frac{\beta\mu\alpha\xi}{\eta C(\beta\mu - \lambda B'(1))} + \frac{1 - A^K}{A(1-A)} \left(\frac{\phi\lambda B'(1)}{(\beta\mu - \lambda B'(1))(\alpha\xi + \phi)} + 1 \right) \right\}^{-1},$$

such that

$$A = \frac{\phi C}{\alpha\xi},$$

with

$$C = \int_0^1 e^{\frac{\lambda}{\alpha\xi} H(x)} (1-x)^{\frac{\phi}{\alpha\xi} - 1} dx \quad \text{and} \quad H(z) = \int_0^z \frac{B(x) - 1}{1 - x} dx,$$

where $B(x)$ is the probability generating function of the batch arrival size X and $B'(1) = E(X)$ is the first moment of random variable X.

– The indices that are of general interest for the evaluation of the performances of our system include:

. The probability that server is idle during busy period, in vacation period, and serving customers during busy period, respectively.

$$P_{0,K} = \frac{\alpha\xi}{\eta C} P_{0,0}, \; P_v = \frac{1 - A^K}{A(1-A)} P_{0,0}, \text{ and } P_b = 1 - P_v - P_{0,K}.$$

. The mean system size when the server is on vacation.

$$E[L_V] = \frac{\lambda B'(1)}{\alpha\xi + \phi} \frac{1 - A^K}{A(1-A)} P_{0,0}.$$

. The mean system size when the server is in busy period.

$$E[L_K] = \left[\frac{\phi\lambda B'(1)}{(2\alpha\xi+\phi)(\beta\mu-\lambda B'(1))} + \frac{\phi(2\beta\mu+\lambda B''(1))}{2(\beta\mu-\lambda B'(1))^2}\right]E[L_V]$$
$$+\frac{\beta\mu\lambda(2B'(1)+B''(1))}{2(\beta\mu-\lambda B'(1))^2}P_{0,K},$$

where $B''(1)$ is the second moment of random variable X.

. The mean system size.

$$E[L] = E[L_V] + E[L_K].$$

. The mean queue length.

$$E[L_q] = E[L] - \left[1 - \sum_{j=0}^{K} P_{0,j}\right].$$

. The mean number of customers served per unit time.

$$N_s = \beta\mu P_b.$$

. The average rates of reneging and retention, respectively.

$$R_a = \alpha\xi E[L_V] \text{ and } R_e = (1-\alpha)\xi E[L_V].$$

Based on the results given above, useful system characteristics are derived. Further, the cost model is developed. Then, we considered the cost optimization problem under a given cost structure via quadratic fit search method (QFSM) and particle swarm optimization (PSO). We showed via numerical experiments that both methods give identical results, but the convergence is faster in PSO algorithm.

► **Fifth result: The $M^X/M/c$ Bernoulli feedback queue with variant multiple working vacations and impatient customers: Performance and economic analysis.**

This work considers a $M^X/M/c$ queueing system with batch arrival, variant of working vacations, Bernoulli feedback, impatient customers which depend on the states of the servers and retention of reneged customers. Customers arrive in batches according to a Poisson process with rate λ. The arrival batch size X is a random variable with probability mass function $P(X = l) = b_l;\ l = 1, 2, \ldots$. The service times during normal busy period follow an exponential distribution with mean $1/\mu$. During the vacation time, the service is provided according to an exponential distribution with mean $1/\eta$,

such that $\mu \geq \eta$. The queueing system consists of c servers such that all the servers go for working vacation and vacation time synchronously once the system becomes empty, and they also return to the system as one at the same time. If the servers comeback from working vacation and vacation period to find an empty queue, they immediately leave all together for another vacation and working; otherwise, they return to serve the queue. Vacation and working vacation periods are assumed to be exponentially distributed with mean $1/\phi$. The servers are allowed to take all together K vacations sequentially. When the K consecutive working vacations are complete, the servers returns to busy period and depending on the arriving batch of customers, they stay idle or busy with the next arrivals. Whenever a customer arrives at the system and finds the servers on vacation or working vacation (resp. busy) period, he activates an impatience timer T_1 (resp. T_2), which is exponentially distributed with parameter ξ_1 (respectively. ξ_2). If the customer's service has not been completed before the customer's timer expires, this later may leave the system. The customers timers are independent and identically distributed random variables and independent of the number of waiting customers. Each impatient customer may abandon the system with probability α and can be retained in the queue with complementary probability $\alpha' = (1-\alpha)$. If the service is uncomplete, or unsatisfactory, the customer can either leave the system definitively with probability β or rejoin the end of the queue of the system for another service with probability β', where $\beta + \beta' = 1$. Note that, both customers, the newly arrived and those that are fed back are served in order in which they join the tail of the primary queue.
The inter-arrival times, working vacation and vacation periods, normal busy period are mutually independent.

Let $N(t)$ denote the number of customers in the system at time t, and let $\kappa(t)$ be the status of the servers at time t.

$$\kappa(t) = \begin{cases} j, & \text{the servers are taking the } (j+1)^{th} \text{ vacation at time } t,\ j = \overline{0,1,K-1}, \\ K, & \text{the servers are idle or busy at time } t. \end{cases}$$

The bi-variate process $\{(N(t),\kappa(t)),\ t \geq 0\}$ represents two dimensional infinite state Markov chain in continuous time with state space

$$\Omega = \{(n,j):\ n \geq 0;\ j = \overline{0,K}\}.$$

Let $P_{n,j} = \lim_{t\to\infty} P(N(t) = n; \kappa(t) = j),\ n \geq 0;\ j = \overline{0,K}$, be the steady-state probabilities of the process $\{(N(t);\kappa(t)); t \geq 0\}$.

– The steady-state probabilities of the queueing model are as

$$\sum_{j=0}^{K-1} P_{\cdot,j} = \left\{ \frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1-C^{K-1}}{1-C} \right\} P_{0,0},$$

and

$$P_{\cdot,K} = e^{-\frac{\lambda}{\alpha\xi_2}H(1)} \left\{ -\frac{\phi}{\alpha\xi_2}\left(K_4(1) - \left(\frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1-C^{K-1}}{1-C} \right) K_5(1) \right) + \frac{\beta\mu}{\alpha\xi_2} K_6(1) \right\} P_{0,0},$$

where

$$P_{0,0} = \left\{ e^{-\frac{\lambda}{\alpha\xi_2}H(1)} \left\{ -\frac{\phi}{\alpha\xi_2}\left(K_4(1) - \left(\frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1-C^{K-1}}{1-C} \right) K_5(1) \right) + \frac{\beta\mu}{\alpha\xi_2} K_6(1) \right\} + \left(\frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1-C^{K-1}}{1-C} \right) \right\}^{-1},$$

with

$$C = \frac{\phi K_2(1) - \beta\nu K_3(1)}{\beta\nu K_0(1)}, \quad \theta_1 = \frac{\beta\nu K_0(1)}{(\beta\mu + \alpha\xi_2)K_2(1) - \beta\nu K_1(1)},$$

$$K_0(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi_1}H(x)} (1-x)^{\frac{\phi}{\alpha\xi_1}} x^{\frac{c\beta\nu}{\alpha\xi_1}-1} Q_0(x)dx, \quad K_1(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi_1}H(x)} (1-x)^{\frac{\phi}{\alpha\xi_1}} x^{\frac{c\beta\nu}{\alpha\xi_1}-1} Q_1(x)dx,$$

$$K_2(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi_1}H(x)} (1-x)^{\frac{\phi}{\alpha\xi_1}-1} x^{\frac{c\beta\nu}{\alpha\xi_1}} dx, \quad K_3(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi_1}H(x)} (1-x)^{\frac{\phi}{\alpha\xi_1}} x^{\frac{c\beta\nu}{\alpha\xi_1}-1} Q_2(x)dx,$$

$$K_4(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi_2}H(x)} x^{\frac{c\beta\mu}{\alpha\xi_2}} (1-x)^{-1} \Psi(x)dx,$$

$$K_5(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi_2}H(x)} x^{\frac{c\beta\mu}{\alpha\xi_2}} (1-x)^{-1} dx, \quad K_6(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi_2}H(x)} x^{\frac{c\beta\mu}{\alpha\xi_2}-1} Q_3(x)dx,$$

$$\Psi(x) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(x)}}{(1-x)^{\frac{\phi}{\alpha\xi_1}}x^{\frac{c\beta\nu}{\alpha\xi_1}}}\left\{\frac{\beta\nu K_0(x)+(\beta\nu K_1(x)-(\beta\mu+\alpha\xi_2)K_2(x))\theta_1}{\alpha\xi_1}\right.$$

$$\left.+\frac{C}{\alpha\xi_1}\left(\beta\nu K_0(x)+\frac{\beta\nu K_3(x)-\phi K_2(x)}{C}\right)\left(\frac{1-C^{K-1}}{1-C}\right)\right\},$$

$$Q_0(x) = \sum_{n=0}^{c-1}(c-n)\gamma_n x^n,\ Q_1(x) = \sum_{n=0}^{c-1}(c-n)\varphi_n x^n,$$

$$Q_2(x) = \sum_{n=0}^{c-1}(c-n)\omega_n x^n,\ \text{and } Q_3(x) = \sum_{n=0}^{c-1}(c-n)\theta_n x^n,$$

such that

$$\gamma_n = \begin{cases} 1, & \text{if } n=0; \\ \dfrac{\lambda+\phi}{\beta\nu+\alpha\xi_1}, & \text{if } n=1. \\ \psi_{n-1}\gamma_{n-1} - \dfrac{A}{n}\displaystyle\sum_{i=1}^{n-1} b_i\gamma_{n-1-i} & \text{if } 2\le n\le c-1. \end{cases}$$

$$\varphi_n = \begin{cases} 0, & \text{if } n=0; \\ -\dfrac{\beta\mu+\alpha\xi_2}{\beta\nu+\alpha\xi_1}, & \text{if } n=1. \\ \psi_{n-1}\varphi_{n-1} - \dfrac{A}{n}\displaystyle\sum_{i=1}^{n-1} b_i\varphi_{n-1-i} & \text{if } 2\le n\le c-1. \end{cases}$$

$$\omega_n = \begin{cases} 0, & \text{if } n=0; \\ -\dfrac{\phi}{\beta\nu+\alpha\xi_1}, & \text{if } n=1. \\ \psi_{n-1}\omega_{n-1} - \dfrac{A}{n}\displaystyle\sum_{i=1}^{n-1} b_i\omega_{n-1-i} & \text{if } 2\le n\le c-1. \end{cases}$$

$$\theta_n = \begin{cases} \theta_0, & \text{if } n=0; \\ \theta_1, & \text{if } n=1. \\ \sigma_{n-1}\theta_{n-1} - \dfrac{B}{n}\displaystyle\sum_{i=1}^{n-1} b_i\theta_{n-1-i} - \dfrac{E}{n}(\gamma_{n-1}H(K)+\omega_{n-1}h(K)) & \text{if } 2\le n\le c-1. \end{cases}$$

where

$$\sigma_n = \frac{\lambda + n(\beta\mu + \alpha\xi_2)}{(n+1)(\beta\mu + \alpha\xi_2)},\ A = \frac{\lambda}{\beta\nu + \alpha\xi_1},\ B = \frac{\lambda}{\beta\mu + \alpha\xi_2},\ E = \frac{\phi}{\beta\mu + \alpha\xi_2},$$

$$H(K) = \frac{1 - C^K}{1 - C}, h(K) = \frac{1 - C^K}{C(1 - C)},\ H(1) = \int_0^1 \frac{B(x) - 1}{1 - x} dx, \text{and}\ \psi_n = \frac{\lambda + \phi + n(\beta\nu + \alpha\xi_1)}{(n+1)(\beta\nu + \alpha\xi_1)}.$$

– Measures of effectiveness of the considered queueing system include:

. The average number of customers in the system.

$$E(L) = E(L_{WV}) + E(L_K).$$

. The average number of customers in the system during working vacation period.

$$E(L_{WV}) = \left\{ \frac{\lambda B'(1) - c\beta\nu}{\alpha\xi_1 + \phi} \left(\frac{\beta\mu + \alpha\xi_2}{\phi} \right) \theta_1 + \frac{\beta\nu(Q_0(1) + \theta_1 Q_1(1))}{\alpha\xi_1 + \phi} \right.$$
$$\left. + \left(\left(\frac{1 - C^{K-1}}{C(1 - C)} \right) \frac{\lambda B'(1) + \beta\nu(Q_0(1)C - Q_2(1) - c)}{\alpha\xi_1 + \phi} \right) \right\} P_{0,0},$$

where

$$Q_0(1) = \sum_{n=0}^{c-1} (c - n)\gamma_n,\ Q_1(1) = \sum_{n=0}^{c-1} (c - n)\varphi_n, \text{ and } Q_2(1) = \sum_{n=0}^{c-1} (c - n)\omega_n,$$

with $B'(1)$ is the first moment of random variable X.

. The average number of customers in the system during normal busy period.

$$E(L_K) = \frac{\lambda B'(1) - \beta\mu}{\alpha\xi_2} G_K(1) + \frac{\phi}{\alpha\xi} E(L_{WV}) + \frac{\beta\mu}{\alpha\xi_2} Q_3(1) P_{0,0}, \tag{1.11}$$

where

$$Q_3(1) = \sum_{n=0}^{c-1} (c - n)\theta_n.$$

. The mean queue length.

$$L_q = E(L) - c + \left\{ \left(Q_0(1) + \frac{Q_2(1)}{C} \right) \left(\frac{1 - C^K}{1 - C} \right) + Q_3(1) \right\} P_{0,0}.$$

- The mean expected number of customers served per unit time.

$$N_s = c\beta\left(\mu(P_b + P_{0,K}) + \nu P_{wv}\right) + \beta\left(\mu Q_3(1) + \nu(Q_0(1)H(K) + Q_2(1)h(K))\right) P_{0,0}.$$

- The probability that the servers are in working vacation period.

$$P_{WV} = \left\{ \frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1 - C^{K-1}}{1 - C} \right\} P_{0,0}.$$

- The probability that the servers are idle during working vacation.

$$P_{idle} = \sum_{j=0}^{K-1} P_{0,j} = \frac{1 - C^K}{1 - C} P_{0,0}.$$

- The probability that the servers are busy during normal busy period.

$$P_{busy} = 1 - P_{0,K} - P_{WV}.$$

- The average rate of reneging (resp. retention)

$$R_a = \alpha\xi_1 E(L_{WV}) + \alpha\xi_2 E(L_K), \;\; R_e = (1-\alpha)\xi_1 E(L_{WV}) + (1-\alpha)\xi_2 E(L_K).$$

► **Sixth result: Cost optimization analysis for an $M^X/M/c$ vacation queueing system with waiting servers and impatient customers.**

In this investigation, we consider an $M^X/M/c$ Bernoulli feedback queueing system under single and multiple vacation policies. Customers arrive in batches according to a Poisson process with rate λ. The sizes of successive arriving batches are i.i.d. r.v X_1, X_2,... distributed with probability mass function $P(X = l) = b_l;\; l = 1, 2, 3, \ldots$. The customers are served on a First-Come First-Served (FCFS) queue discipline. The service times follow exponential distribution with mean $1/\mu$. When the busy period is finished the servers wait a random duration of time before beginning on a vacation. This waiting duration is exponentially distributed with mean $1/\eta$. The queueing model consists of c servers. Synchronous vacation policy is considered; once the system is empty, all the servers leave for a vacation simultaneously, and they return to the system as one at the same time. Further, both single and multiple vacation policies are considered. Vacation periods follow an exponential distribution with mean $1/\phi$. In addition, if the servers are unavailable due to vacation, a batch of customers activates an independent impatience timer T, with exponentially distributed duration, with mean $1/\xi$. If

T expires while the servers are still on vacation, the customers may leave the system. Further, impatient customers may abandon the system, with probability α, and can be retained in the queue, with complementary probability $(1-\alpha)$. Moreover, If the service is uncomplete or unsatisfactory, the customers can either leave the system definitively, with probability β, or comeback to the system for another service, with complementary probability $(1-\beta)$. The system is stable under the condition $\rho = \frac{\lambda E(X)}{c\beta\mu} < 1$, where $E(X)$ is the mean of a batch of arrivals. We suppose that the inter-arrival times, batch sizes, server waiting times, vacation times, service times and impatience times are independent of each other.

Let $\{L(t); t \geq 0\}$ be the number of customers in the system at time t, and $S(t)$ be the state of servers at time t, where $S(t)$ is defined as follows:

$$S(t) = \begin{cases} 1, & \text{when the servers are in busy period at time } t; \\ 0, & \text{when the servers are in vacation period at time } t. \end{cases}$$

Then, let $\{(S(t), L(t)); t \geq 0\}$ be a two-dimensional continuous Markov process with state space

$$\Omega = \{(s,n) : s = 0,1, \ \ n = 0,1,...\}.$$

Let

$$P_{s,n} = \lim_{t\to\infty} P\{S(t) = s, L(t) = n\}, s = 0,1, n = 0,1,...,$$

denote the system steady-state probabilities.

1. The steady-state probabilities of the queueing system under multiple vacation policy (MVP) are given as

$$P_{0,.} = \frac{\alpha\xi}{\phi K(1)} P_{0,0},$$

and

$$P_{1,.} = \frac{\phi G_0'(1) + \beta\mu R(1) P_{0,0}}{c\beta\mu - \lambda B'(1)},$$

where

$$P_{0,0} = \left\{ \frac{\beta\mu}{c\beta\mu - \lambda B'(1)} R(1) + \frac{\alpha\xi}{\phi K(1)} \left(\frac{\phi\lambda B'(1)}{(c\beta\mu - \lambda B'(1))(\alpha\xi + \phi)} + 1 \right) \right\}^{-1},$$

and

$$G_0'(1) = \frac{\alpha\xi\lambda B'(1)}{(\alpha\xi+\phi)\phi K(1)} P_{0,0}.$$

1.1. Performance measures of the considered queueing system under MVP are as

. The mean system size.

$$E[L] = E[L_0] + E[L_1].$$

. The mean system size when the servers are in vacation period.

$$E[L_0] = \frac{\alpha\xi\lambda B'(1)}{(\alpha\xi+\phi)\phi K(1)} P_{0,0}.$$

. The mean system size when the servers are in busy period.

$$E[L_1] = \left(\frac{\phi(2c\beta\mu + \lambda B''(1))}{2(c\beta\mu - \lambda B'(1))^2} + \frac{\lambda\phi B'(1)}{(c\beta\mu - \lambda B'(1))(2\alpha\xi+\phi)}\right) E[L_0]$$

$$+\left(\frac{\beta\mu\lambda(2B'(1)+B''(1))}{2(c\beta\mu - \lambda B'(1))^2} R(1) + \frac{\beta\mu}{c\beta\mu - \lambda B'(1)} R'(1)\right) P_{0,0}$$

$$+\frac{\lambda\phi B''(1)}{2(c\beta\mu - \lambda B'(1))(2\alpha\xi+\phi)} G_0(1),$$

with

$$R'(1) = \sum_{n=1}^{c-1} n(c-n)\theta_n,$$

where $B'(1)$ and $B''(1)$ are first and second moments of random variable X, respectively.

. The mean number of customers in the queue.

$$E[L_q] = E[L] - c(1-P_v) + R(1)P_{0,0}.$$

. The probability that the servers are in vacation period, idle during busy period, serving customers during busy period, respectively.

$$P_v = \frac{\alpha\xi}{\phi K(1)} P_{0,0},\ P_e = \frac{\alpha\xi - \phi K(1)}{\eta K(1)} P_{0,0},\ \text{and } P_b = 1 - P_v - P_e.$$

. The mean number of customers served per unit time.

$$N_s = \beta\mu\big(c(P_b + P_e) + R(1)P_{0,0}\big).$$

. The average rate of abandonment of customers due to impatience.

$$R_a = \alpha\xi E[L_0].$$

. The average retention rate of impatient customers.

$$R_e = (1-\alpha)\xi E[L_0].$$

2. The steady-state probabilities of the queueing system under single vacation policy (SVP) are given as

$$P_{0,.} = \frac{\alpha\xi}{\phi K(1)} P_{0,0},$$

and

$$P_{1,.} = \left\{ \frac{\alpha\xi\lambda B'(1)}{K(1)(\alpha\xi+\phi)(c\beta\mu-\lambda B'(1))} + \frac{\beta\mu}{c\beta\mu-\lambda B'(1)} Q(1) \right\} P_{0,0},$$

with

$$P_{0,0} = \left\{ \frac{\alpha\xi}{\phi K(1)} \left(1 + \frac{\phi\lambda B'(1)}{(c\beta\mu-\lambda B'(1))(\alpha\xi+\phi)} \right) + \frac{\beta\mu}{c\beta\mu-\lambda B'(1)} Q(1) \right\}^{-1},$$

where

$$R(1) = \sum_{n=0}^{c-1} (c-n)\theta_n,\ K(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi}H(x)} (1-x)^{\frac{\phi}{\alpha\xi}-1} dx,\ Q(1) = \sum_{n=0}^{c-1} (c-n) M_n,$$

and

$$\theta_n = \begin{cases} \theta_0, & \text{if } n = 0, \\ \frac{\lambda+\eta}{\beta\mu}\theta_0, & \text{if } n = 1, \\ \rho_{n-1}\theta_{n-1} - \dfrac{\phi}{n\beta\mu}\omega_{n-1} - \dfrac{\lambda}{n\beta\mu}\displaystyle\sum_{i=1}^{n-1} b_i \theta_{n-1-i}, & \text{if } 2 \le n \le c-1, \end{cases}$$

$$M_n = \begin{cases} M_0, & \text{if } n = 0, \\ M_1, & \text{if } n = 1, \\ \rho_{n-1}M_{n-1} - \dfrac{\phi}{n\beta\mu}\gamma_{n-1} - \dfrac{\lambda}{n\beta\mu}\displaystyle\sum_{i=1}^{n-1} b_i M_{n-1-i}, & \text{if } 2 \le n \le c-1. \end{cases}$$

with

$$\rho_{n-1} = \frac{\lambda + (n-1)\beta\mu}{n\beta\mu}, \; M_0 = \frac{\alpha\xi}{\eta K(1)}, \text{ and } M_1 = \frac{\alpha\xi(\lambda+\eta)}{\eta\beta\mu K(1)} - \frac{\phi}{\beta\mu}.$$

2.1. Performance measures of the considered queueing system under SVP include

. The mean system size.

$$E[L] = E[L_0] + E[L_1].$$

. The mean system size when the servers are on vacation.

$$E[L_0] = \frac{\alpha\xi\lambda B'(1)}{\phi K(1)(\alpha\xi+\phi)} P_{0,0}.$$

. The mean system size when the servers are on busy period.

$$\begin{aligned} E[L_1] = &\left(\frac{\phi(2c\beta\mu + \lambda B''(1))}{2(c\beta\mu - \lambda B'(1))^2} + \frac{\lambda\phi B'(1)}{(c\beta\mu - \lambda B'(1))(2\alpha\xi+\phi)}\right) E[L_0] \\ &+ \left(\frac{\beta\mu\lambda(2B'(1) + B''(1))}{2(c\beta\mu - \lambda B'(1))^2} Q(1) + \frac{\beta\mu}{c\beta\mu - \lambda B'(1)} Q'(1)\right) P_{0,0} \\ &+ \frac{\lambda\phi B''(1)}{2(c\beta\mu - \lambda B'(1))(2\alpha\xi+\phi)} G_0(1), \end{aligned}$$

with

$$Q'(1) = \sum_{n=1}^{c-1} n(c-n) M_n.$$

. The mean number of customers in the queue.

$$E[L_q] = E[L] - c(1-P_v) + Q(1)P_{0,0}.$$

. The probability that the servers are in vacation period, idle during busy period, and serving customers during busy period, respectively.

$$P_v = \frac{\alpha\xi}{\phi K(1)} P_{0,0}, \; P_e = \frac{\alpha\xi}{\eta K(1)} P_{0,0}, \text{ and } P_b = 1 - P_v - P_e.$$

. The mean number of customers served per unit time.

$$N_s = \beta\mu \sum_{n=0}^{c-1} nP_{1,n} + c\beta\mu \sum_{n=c}^{\infty} P_{1,n} = \beta\mu\big(c(P_b + P_e) + Q(1)P_{0,0}\big).$$

. The average rate of reneging (resp. retention).

$$R_a = \alpha\xi E[L_0], \;\; R_e = (1-\alpha)\xi E[L_0].$$

1.7 Outline of the thesis

This thesis consists of seven chapters including the introductory chapter.

Chapter 2 deals with a heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, balking, reneging and retention of reneged customers at which the impatience timer of a customer in the system depends on the server's states. The steady-state probabilities of the considered queueing model are established using supplementary variable and recursive techniques. In addition, useful performance measures of the system are derived. Further, a model for the costs incurred is developed in order to study the economic analysis of the queueing system. Then, a sensitive numerical analysis for this model is carried out with respect to all system parameters.

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Chapter 3 analyzes an infinite-buffer single-server queueing system with Bernoulli feedback, multiple vacations, differentiated vacations, vacation interruptions and impatient customers (balking and reneging) at which two types of vacation are considered, namely, type 1 vacation taken after the busy period and type 2 vacation considered when the server comes back from a vacation and finds the system empty. Both vacations may be interrupted when the number of customers in the system reaches a predefined threshold (each type of vacation has a different threshold). Via certain mechanism, reneged customers may be retained in the system. Explicit expressions of the steady-state probabilities are obtained via recursive method. Then, a cost model is developed. In addition, a sensitivity analysis through numerical experiments are carried out.

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In chapter 4, we consider a single-server Markovian queueing system with Bernoulli feedback, single and multiple vacation policies, waiting server and impatient customers at which once the system is empty the sever waits for a random amount of time before he leaves for a vacation. In addition, the impatience time of a customer depends on the states of the server. Via certain mechanism, impatient customer may be retained in the system. The explicit expressions for the steady-state probabilities of the queueing model has been established using the probability generating function (PGF). After that, useful performance measures of the system are derived and a cost model is developed. Finally, an extensive numerical study is illustrated.

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Chapter 5 studies an infinite capacity batch arrival single server Markovian Bernoulli feedback queueing system with waiting server, K-variant vacations and impatient customers. We obtain the probability generating function of the steady state of the queueing system and derive important performance measures of the queueing model. Then, we develop a cost model for the queueing system in order to determine the optimal values of service rate and to minimize the total expected cost per unit time. For this, we adopt QFSM and PSO algorithm to implement the optimization tasks.

This chapter has been submitted.

In chapter 6, we present the analysis of an $M^X/M/c$ Bernoulli feedback queueing model with variant of multiple working vacations, reneging which depend on the states of the servers and retention of reneged customers. We establish the steady-state probabilities of the queueing model using probability generating functions (PGFs) and obtain various system characteristics. Then, the cost profit model is performed. In addition, an economic analysis of the system is carried out.

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In chapter 7, we deal with an infinite buffer multi-server vacation queueing system with batch arrival, Bernoulli feedback and waiting servers wherein customers may renege during vacation period. The reneged customers can be retained in the system, via certain strategy. Both multiple and single vacation policies are considered. The steady-state probabilities of the queueing system are found through probability generating functions (PGFs). Then, useful performance measures of the queueing system are derived. The cost profit analysis of the model is developed. Further, we perform the optimization of the model using quadratic fit search method (QFSM) in order to minimize the total expected cost of the system with respect to the service rate.

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Chapter 2

Performance and economic study of heterogeneous $M/M/2/N$ feedback queue with working vacation and impatient customers

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Performance and economic study of heterogeneous $M/M/2/N$ feedback queue with working vacation and impatient customers

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Abstract. This paper presents the analysis of a heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, balking, reneging and retention of reneged customers. We suppose that the impatience timer of a customer in the system depends on the server's states. The steady-state probabilities of the model are obtained. Various performance measures of the model have been discussed. Then, we develop a model for the costs incurred and carry out a sensitive analysis for this queueing system with respect to all system parameters. Further, numerical illustrations have been presented.

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Keywords: Queueing models. Heterogeneity. Working vacation. Impatient customers. Bernoulli feedback.

2.1 Introduction

Recent decades have seen an increasing interest in queueing systems with customer's impatience because of their great advantage in many real life applications such as situations involving impatient telephone switchboard customers, inventory systems with storage of perishable goods, business and industry etc. The readers can be referred to Gupta et al. (2008,2009), Boxma et al. (2010), Choudhury and Medhi (2011), Jose and Manoharan (2011,2014), Kumar and Sharma (2014a,2014b), Bouchentouf et al. (2014) and references therein.

Queueing models with vacation and working vacation have gained the interest of many researchers in the last three decades, due to their wide range of applications, especially in the communication and the manufacturing systems. Altman and Yechiali (2008) analyzed the infinite-server queues with system's additional tasks and impatient customers, both multiple and single U-task scenarios are studied considering both exponentially and generally distributed task and impatience times. Madhu and Anamika (2010) considered a working vacation queueing model with multiple types of server breakdowns, via a matrix geometric approach the stationary queue length distribution is computed and various performance indices are established. Laxmi et al. (2013) presented the analysis of a finite buffer $M/M/1$ queue with multiple and single working vacations. Then, Goswami (2014) analyzed a queueing system with Bernoulli schedule working vacations, vacation interruption and impatient customers. Abidini et al. (2016) presented an analysis and an optimization of vacation and polling models with retrials. Panda and Goswami (2016) established an equilibrium balking strategies in renewal input queue with bernoulli-schedule controlled vacation and vacation interruption. Later, Bouchentouf and Yahiaoui (2017) presented an analysis of a Markovian feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption, where customers' impatience is due to the servers' vacation.

Recently, there has been growing interest in the study of multiserver queues with vacation. For instance, Yue and Yue (2010) considered heterogeneous two-server network system with balking and a Bernoulli vacation schedule. An $M/M/2$ queueing system with heterogeneous servers including one with working vacation has been analyzed by Krishnamoorthy and Sreenivasan (2012). Ammar (2014) investigated the transient analysis of a two-heterogeneous servers queue with impatient behavior, the explicit solution for the considered model has been obtained. Later, Laxmi and Jyothsna (2015) presented the analysis of a renewal input multiple working vacations queue with balking, reneging and heterogeneous servers. Using supplementary variable and recursive techniques, the steady-state probabilities of the model are obtained. Recently, the cost optimization analysis for an $M^X/M/c$ vacation queueing system with waiting servers and impatient customers has been given by Bouchentouf and Guendouzi (2019).

In this paper we present a heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, and impatient customers. In this work we extend the analytical results of the model given in Laxmi and Jyothsna (2015) to the case where the impatience timer of customers in the system depend on the server's states, moreover the concept of feedback and retention of reneged customers is incorporated.

The rest of the paper is organized as follows, in Section 2, we give a detailed description of the model. In Section 3, the steady-state probabilities of the model are obtained using supplementary variable and recursive techniques. In Section 4, various performance measures of the model are presented. In Section 5, we develop the cost model, then, numerical results are illustrated in Section 6. Finally, conclusion and some future aspects of research done are stated in Section 7.

2.2 The model

Consider a heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, balking, server's states-dependent reneging and retention of reneged customers.

• The inter-arrival times are assumed to be independent and identically distributed random variables with cumulative distribution function $A(u)$, probability density function $a(u)$, $u \geq 0$, Laplace-Stieltjes transform (L.S.T.) $A^*(\varpi)$ and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$, where $h^{(1)}(0)$ denotes the first derivative of $h(\varpi)$ evaluated at $\varpi = 0$.

• There exist two heterogeneous servers, server 1 and server 2. The service times are supposed to be exponentially distributed with parameters μ_1 and μ_2, respectively, with $\mu_2 \leq \mu_1$. Whenever server 2 becomes idle and there are no waiting customers in the queue, he leaves for an exponential working vacation 'WV' with parameter ϕ. During a WV, server 2 serves the waiting customers at a rate lower than the normal service rate which is assumed to be exponentially distributed with parameter ν. At the end of vacation period, if there are customers waiting in the queue, server 2 switches to normal working level, otherwise he continues the vacation. Moreover, it is supposed that server 1 is always available in the system.

• The capacity of the system is taken finite N, and the customers are served on a FCFS discipline.

• An arriving customer who finds i customers in the system decides either to join the queue with probability $b_i = 1 - \frac{i}{N^2}$ or balk with probability $\overline{b}_i = 1 - b_i = \frac{i}{N^2}$. Suppose that $b_0 = b_1 = 1$, $0 \leq b_{i+1} \leq b_i \leq 1$, $2 \leq i \leq N-1$, and $b_N = 0$.

• If there are i customers in the system, one of the $(i-2)$ waiting customers in the queue may renege. Whenever a customer arrives at the system and finds the server 2 on working vacation (resp. on normal busy period), he activates an impatience timer T_1 (respectively. T_2,) which is exponentially distributed with parameter ξ_1 (resp. ξ_2). If the customer's service has not begun before the customer's timer expires, the customer abandons the queue. Thus, customer's average reneging rate is given by $(i-2)\xi_1$ (resp. $(i-2)\xi_2$) when server 2 is on working vacation (resp. on normal busy period), $2 \le i \le N$. We assume that impatience timers are independent and identically distributed random variables and independent of the number of waiting customers.

• Using certain mechanism, each reneged customer may leave the queue definitively with probability α or may be retained in the system with complimentary probability α'.

• After getting incomplete or unsatisfactory service either from working vacation service or normal busy service, with probability β', a customer may rejoin the system as a Bernoulli feedback customer to receive another regular service. Otherwise, he leaves the system definitively, i.e. with probability β, where $\beta' + \beta = 1$.

• The inter-arrival times, service times and vacation times are assumed to be independent.

2.3 Steady-State Solution

In this section, the distributions of the steady-state of the system will be obtained following the same method given in Laxmi and Jyothsna (2015) . Thus, using the supplementary variable and recursive techniques the steady-state probabilities will be derived. To get the system length distributions at arbitrary epoch, the differential difference equations using the remaining inter-arrival time as the supplementary variable will be developed.

Let $N_s(t)$ be the number of customers in the system at time t. And let $I(t)$ be the remaining inter-arrival time at time t for the next arrival.

Let

$$S(t) = \begin{cases} 0, & \text{when server 2 is idle during working vacation (WV) period;} \\ 1, & \text{when server 2 is busy during working vacation (WV) period;} \\ 2, & \text{when server 2 is busy during normal busy period.} \end{cases}$$

Then, the joint probabilities are presented as

$$\pi_{i,0}(u,t)du = \mathbb{P}(N_s(t) = i, u \le I(t) < u + du, S(t) = 0), u \ge 0, i = 0,1,$$

$$\pi_{i,j}(u,t)du = \mathbb{P}(N_s(t) = i, u \le I(t) < u + du, S(t) = j), u \ge 0, j = 1,2,$$

$$1 \le i \le N.$$

Thus

$$\pi_{i,0}(u) = \lim_{t\to\infty} \pi_{i,0}(u,t),\ \ i = 0,1,\ \ \pi_{i,j}(u) = \lim_{t\to\infty} \pi_{i,j}(u,t),\ \ j = 1,2,\ \ 1 \le i \le N.$$

The L.S.T. of the steady-state probabilities are given as

$$\pi^*_{i,0}(\varpi) = \int_0^\infty e^{-\varpi u}\pi_{i,0}(u)du,\ i = 0,1,\ \ \pi^*_{i,j}(\varpi) = \int_0^\infty e^{-\varpi u}\pi_{i,j}(u)du,$$

$j = 1,2,\ 1 \le i \le N.$

Let $\pi_{i,j} = \pi^*_{i,j}(0)$ be the probability of i customers in the system when the server is in state j at an arbitrary epoch.

The system of differential difference equations at steady-state is given as follows:

$$-\pi^{(1)}_{0,0}(u) = \beta\mu_1\pi_{1,0}(u) + \beta\nu\pi_{1,1}(u) + \beta\mu_2\pi_{1,2}(u), \tag{2.1}$$

$$-\pi^{(1)}_{1,0}(u) = -\beta\mu_1\pi_{1,0}(u) + \beta\nu\pi_{2,1}(u) + \beta\mu_2\pi_{2,2}(u) + a(u)\pi_{0,0}(0), \tag{2.2}$$

$$-\pi^{(1)}_{1,1}(u) = -(\phi + \beta\nu)\pi_{1,1}(u) + \beta\mu_1\pi_{2,1}(u), \tag{2.3}$$

$$\begin{aligned} -\pi^{(1)}_{2,1}(u) = &-\Big(\beta(\mu_1 + \nu) + \phi\Big)\pi_{2,1}(u) + \Big(\beta(\mu_1 + \nu) + \alpha\xi_1\Big)\pi_{3,1}(u) \\ &+a(u)\Big(\pi_{1,0}(0) + \pi_{1,1}(0) + \tfrac{2}{N^2}\pi_{2,1}(0)\Big), \end{aligned} \tag{2.4}$$

$$-\pi^{(1)}_{i,1}(u) = -\Big(\beta(\mu_1+\nu)+\phi+(i-2)\alpha\xi_1\Big)\pi_{i,1}(u)$$
$$+\Big(\beta(\mu_1+\nu)+(i-1)\alpha\xi_1\Big)\pi_{i+1,1}(u) \tag{2.5}$$
$$+a(u)\Big(\Big(1-\tfrac{i-1}{N^2}\Big)\pi_{i-1,1}(0)+\tfrac{i}{N^2}\pi_{i,1}(0)\Big),\ 3\le i\le N-1,$$

$$-\pi^{(1)}_{N,1}(u) = -\Big(\beta(\mu_1+\nu)+\phi+(N-2)\alpha\xi_1\Big)\pi_{N,1}(u)$$
$$+a(u)\Big(\Big(1-\tfrac{N-1}{N^2}\Big)\pi_{N-1,1}(0)+\pi_{N,1}(0)\Big), \tag{2.6}$$

$$-\pi^{(1)}_{1,2}(u) = -\beta\mu_2\pi_{1,2}(u)+\phi\pi_{1,1}(u)+\beta\mu_1\pi_{2,2}(u), \tag{2.7}$$

$$-\pi^{(1)}_{i,2}(u) = -\Big(\beta(\mu_1+\mu_2)+(i-2)\alpha\xi_2\Big)\pi_{i,2}(u)+\phi\pi_{i,1}(u)$$
$$+\Big(\beta(\mu_1+\mu_2)+(i-1)\alpha\xi_2\Big)\pi_{i+1,2}(u) \tag{2.8}$$
$$+a(u)\Big(\Big(1-\tfrac{i-1}{N^2}\Big)\pi_{i-1,2}(0)+\tfrac{i}{N^2}\pi_{i,2}(0)\Big),\ 2\le i\le N-1,$$

$$-\pi^{(1)}_{N,2}(u) = -\Big(\beta(\mu_1+\mu_2)+(N-2)\alpha\xi_2\Big)\pi_{N,2}(u)+\phi\pi_{N,1}(u)$$
$$+a(u)\Big(\Big(1-\tfrac{N-1}{N^2}\Big)\pi_{N-1,2}(0)+\pi_{N,2}(0)\Big), \tag{2.9}$$

Now, define $\zeta_i = \beta(\mu_1+\nu)+\phi+(i-2)\alpha\xi_1$, and $\theta_i = \beta(\mu_1+\mu_2)+(i-2)\alpha\xi_2$ for $2\le i\le N$.

Multiplying Equations (2.1)-(2.9) by $e^{-\varpi u}$ and integrating over u from 0 to ∞, we get

$$-\varpi\pi^*_{0,0}(\varpi) = -\pi_{0,0}(0)+\beta\mu_1\pi^*_{1,0}(\varpi)+\beta\nu\pi^*_{1,1}(\varpi)+\beta\mu_2\pi^*_{1,2}(\varpi), \tag{2.10}$$

$$(\beta\mu_1-\varpi)\pi^*_{1,0}(\varpi) = -\pi_{1,0}(0)+\beta\nu\pi^*_{2,1}(\varpi)+\beta\mu_2\pi^*_{2,2}(\varpi)+A^*(\varpi)\pi_{0,0}(0), \tag{2.11}$$

$$(\phi + \beta\nu - \varpi)\pi^*_{1,1}(\varpi) = -\pi_{1,1}(0) + \beta\mu_1\pi^*_{2,1}(\varpi), \tag{2.12}$$

$$\begin{aligned}(\zeta_2 - \varpi)\pi^*_{2,1}(\varpi) &= -\pi_{2,1}(0) + (\zeta_3 - \phi)\pi^*_{3,1}(\varpi) \\ &\quad + A^*(\varpi)\bigg(\pi_{1,0}(0) + \pi_{1,1}(0) + \tfrac{2}{N^2}\pi_{2,1}(0)\bigg),\end{aligned} \tag{2.13}$$

$$\begin{aligned}(\zeta_i - \varpi)\pi^*_{i,1}(\varpi) &= -\pi_{i,1}(0) + (\zeta_{i+1} - \phi)\pi^*_{i+1,1}(\varpi) \\ &\quad + A^*(\varpi)\bigg((1 - \tfrac{i-1}{N^2})\pi_{i-1,1}(0) + \tfrac{i}{N^2}\pi_{i,1}(0)\bigg),\end{aligned} \tag{2.14}$$

$$(\zeta_N - \varpi)\pi^*_{N,1}(\varpi) = -\pi_{N,1}(0) + A^*(\varpi)\bigg((1 - \frac{N-1}{N^2})\pi_{N-1,1}(0) + \pi_{N,1}(0)\bigg), \tag{2.15}$$

$$(\beta\mu_2 - \varpi)\pi^*_{1,2}(\varpi) = -\pi_{1,2}(0) + \phi\pi^*_{1,1}(\varpi) + \beta\mu_1\pi^*_{2,2}(\varpi), \tag{2.16}$$

$$\begin{aligned}(\theta_i - \varpi)\pi^*_{i,2}(\varpi) &= -\pi_{i,2}(0) + \phi\pi^*_{i,1}(\varpi) + \theta_{i+1}\pi^*_{i+1,2}(\varpi) \\ &\quad + A^*(\varpi)\bigg(\Big(1 - \tfrac{i-1}{N^2}\Big)\pi_{i-1,2}(0) + \tfrac{i}{N^2}\pi_{i,2}(0)\bigg),\end{aligned} \tag{2.17}$$

$$\begin{aligned}(\theta_N - \varpi)\pi^*_{N,2}(\varpi) &= -\pi_{N,2}(0) + \phi\pi^*_{N,1}(\varpi) \\ &\quad + A^*(\varpi)\bigg(\Big(1 - \tfrac{N-1}{N^2}\Big)\pi_{N-1,2}(0) + \pi_{N,2}(0)\bigg).\end{aligned} \tag{2.18}$$

Next, adding Equations (2.10)-(2.18), we get

$$\begin{aligned}-A^*(\varpi)\bigg(\sum_{i=0}^{1}\pi_{i,0}(0) + \sum_{i=1}^{N}(\pi_{i,1}(0) + \pi_{i,2}(0))\bigg) &= \\ \varpi\bigg(\sum_{i=0}^{1}\pi^*_{i,0}(\varpi) &+ \sum_{i=1}^{N}(\pi^*_{i,1}(\varpi) + \pi^*_{i,2}(\varpi))\bigg),\end{aligned}$$

Then, taking limit as $\varpi \longrightarrow 0$ and using the normalization condition we obtain

$$\sum_{i=0}^{1}\pi_{i,0}(0) + \sum_{i=1}^{N}(\pi_{i,1}(0) + \pi_{i,2}(0)) = \lambda. \tag{2.19}$$

Next, we have to derive the steady-state probabilities at pre-arrival epoch, to this end we shall establish the relations between system length distributions at arbitrary and pre-arrival epochs. Firstly, we have to connect the pre-arrival epoch probabilities $\pi^-_{i,j} = \lim_{t\to\infty} \mathbb{P}(N_s(t) = i, S(t) = j/I(t) = 0)$ ($\pi^-_{i,0}$, $i = 0,1$ and $\pi^-_{i,j}$, $j = 1,2$; $1 \le i \le N$,) with the rate probabilities $\pi_{i,0}(0)$ and $\pi_{i,j}(0)$, respectively.

Via Baye's theorem on conditional probabilities, we obtain

$$\pi^-_{i,j} = \frac{1}{\lambda}\pi_{i,j}(0), j = 0,\ i = 0,1;\ j = 1,2;\ 1 \le i \le N. \tag{2.20}$$

Putting $\varpi = \zeta_N$ in Equation (2.15), we obtain

$$\pi_{N-1,1}(0) = \psi_{N-1}\pi_{N,1}(0), \tag{2.21}$$

such that $\psi_{N-1} = \dfrac{(1 - A^*(\zeta_N))N^2}{A^*(\zeta_N)(N^2 - N + 1)}$.

Substituting Equation (2.21) in Equation (2.15), we get

$$(\zeta_N - \varpi)\pi^*_{N,1}(\varpi) = \left(A^*(\varpi)\left(\left(1 - \frac{N-1}{N^2}\right)\psi_{N-1} + \psi_N\right) - \psi_N\right)\pi_{N,1}(0), \tag{2.22}$$

with $\psi_N = 1$.

For $\varpi \ne \zeta_N$, we have

$$\pi^*_{N,1}(\varpi) = \frac{(A^*(\varpi)((1 - \frac{N-1}{N^2})\psi_{N-1} + \psi_N) - \psi_N)}{(\zeta_N - \varpi)}\pi_{N,1}(0) \tag{2.23}$$

Differentiating Equation (2.22) with respect to ϖ and taking $\varpi = \zeta_N$, we get

$$\pi^*_{N,1}(\zeta_N) = -A^{*(1)}(\zeta_N)\left(\left(1 - \frac{N-1}{N^2}\right)\psi_{N-1} + \psi_N\right)\pi_{N,1}(0) \tag{2.24}$$

Differentiating (2.22) with respect to ϖ successively l times, we obtain

$$(\zeta_N - \varpi)\pi^{*(l)}_{N,1}(\varpi) - l\pi^{*(l-1)}_{N,1}(\varpi) = A^{*(l)}(\varpi)\left(\left(1 - \frac{N-1}{N^2}\right)\psi_{N-1} + \psi_N\right)\pi_{N,1}(0). \tag{2.25}$$

From Equations (2.23)-(2.25), we get

$$\pi^*_{N,1}(\varpi) = \varsigma_{N,\varpi}\pi_{N,1}(0),$$

where

$$\varsigma_{N,\varpi} = \begin{cases} \dfrac{A^*(\varpi)((1-\frac{N-1}{N^2})\psi_{N-1}+\psi_N)-\psi_N}{(\zeta_N-\varpi)}, & \text{if } \varpi \neq \zeta_N; \\ -A^{*(1)}(\varpi)((1-\dfrac{N-1}{N^2})\psi_{N-1}+\psi_N), & \text{if } \varpi = \zeta_N, \end{cases}$$

with

$$\varsigma^{(l)}_{N,\varpi} = \begin{cases} \dfrac{A^{*(l)}(\varpi)((1-\frac{N-1}{N^2})\psi_{N-1}+\psi_N)+l\varsigma^{(l-1)}_{N,\varpi}}{(\zeta_N-\varpi)}, & \text{if } \varpi \neq \zeta_N; \\ \dfrac{-A^{*(l+1)}(\varpi)((1-\frac{N-1}{N^2})\psi_{N-1}+\psi_N)}{l+1}, & \text{if } \varpi = \zeta_N, \end{cases}$$

such that $\varsigma^{(l)}_{N,\varpi}$ denotes the l^{th} derivative of $\varsigma_{N,\varpi}$ with respect to ϖ.

For $i = N-1$, taking $\varpi = \zeta_{N-1}$ in Equation (2.14) and using Equation (2.21), we obtain

$$\pi_{N-2,1}(0) = \psi_{N-2}\pi_{N,1}(0), \tag{2.26}$$

with $\psi_{N-2} = \dfrac{(\psi_{N-1}-(\zeta_N-\phi)\varsigma_{N,\zeta_{N-1}}-A^*(\zeta_{N-1})\frac{N-2}{N^2}\psi_{N-1})N^2}{A^*(\zeta_{N-1})(N^2-N+2)}$.

Next, substituting Equation (2.26) in Equation (2.14) for $i = N-1$, we obtain

$$\pi^*_{N-1,1}(\varpi) = \varsigma_{N-1,\varpi}\pi_{N,1}(0),$$

where

$$\varsigma_{N-1,\varpi} = \begin{cases} \dfrac{A^*(\varpi)((1-\frac{N-2}{N^2})\psi_{N-2}+\frac{N-1}{N^2}\psi_{N-1})+(\zeta_N-\phi)\varsigma_{N,\varpi}-\psi_{N-1}}{(\zeta_{N-1}-\varpi)}, & \text{if } \varpi \neq \zeta_{N-1}; \\ -(A^{*(1)}(\varpi)((1-\dfrac{N-2}{N^2})\psi_{N-2}+\dfrac{N-1}{N^2}\psi_{N-1})+(\zeta_N-\phi)\varsigma^{(1)}_{N,\varpi}), & \text{if } \varpi = \zeta_{N-1}, \end{cases}$$

with

$$\varsigma^{(l)}_{N-1,\varpi} = \begin{cases} \dfrac{A^{*(l)}(\varpi)((1-\frac{N-2}{N^2})\psi_{N-2}+\frac{N-1}{N^2}\psi_{N-1})+(\zeta_N-\phi)\varsigma^{(l)}_{N,\varpi}+l\varsigma^{(l-1)}_{N-1,\varpi}}{(\zeta_{N-1}-\varpi)}, & \text{if } \varpi \neq \zeta_{N-1}; \\ -\dfrac{A^{*(l+1)}(\varpi)((1-\frac{N-2}{N^2})\psi_{N-2}+\frac{N-1}{N^2}\psi_{N-1})+(\zeta_N-\phi)\varsigma^{(l+1)}_{N,\varpi}}{l+1}, & \text{if } \varpi = \zeta_{N-1}. \end{cases}$$

In the same way, for $i = N-2, N-3, ..., 3$ in Equation (2.14), it yields

$$\pi_{i-1,1}(0) = \psi_{i-1}\pi_{N,1}(0),\ i = N-2, N-3, ..., 3. \tag{2.27}$$

where

$$\psi_{i-1} = \frac{(\psi_i - (\zeta_{i+1} - \phi)\varsigma_{i+1,\zeta_i} - A^*(\zeta_i)\frac{i}{N^2}\psi_i)N^2}{A^*(\zeta_i)(N^2 - i - 1)}, \qquad i = N-2, N-3, ..., 3,$$

and

$$\pi^*_{i,1}(\varpi) = \varsigma_{i,\varpi}\pi_{N,1}(0),\ i = N-2, N-3, ..., 3,$$

where

$$\varsigma_{i,\varpi} = \begin{cases} \dfrac{A^*(\varpi)((1-\frac{i-1}{N^2})\psi_{i-1} + \frac{i-1}{N^2}\psi_i) + (\zeta_{i+1} - \phi)\varsigma_{i+1,\varpi} - \psi_i}{(\zeta_i - \varpi)}, & \text{if } \varpi \neq \zeta_i; \\ -(A^{*(1)}(\varpi)((1 - \dfrac{i-1}{N^2})\psi_{i-1} + \dfrac{i-1}{N^2}\psi_i) + (\zeta_{i+1} - \phi)\varsigma^{(1)}_{i+1,\varpi}), & \text{if } \varpi = \zeta_i, \end{cases}$$

with

$$\varsigma^{(l)}_{i,\varpi} = \begin{cases} \dfrac{A^{*(l)}(\varpi)((1-\frac{i-1}{N^2})\psi_{i-1} + \frac{i-1}{N^2}\psi_i) + (\zeta_{i+1} - \phi)\varsigma^{(l)}_{i+1,\varpi} - l\varsigma^{(l-1)}_{i,\varpi}}{(\zeta_i - \varpi)}, & \text{if } \varpi \neq \zeta_i; \\ -\dfrac{(\zeta_{i+1} - \phi)\varsigma^{(l+1)}_{i+1,\zeta_i} + A^{*(l+1)}(\varpi)((1-\frac{i-1}{N^2})\psi_{i-1} + \frac{i-1}{N^2}\psi_i)}{l+1}, & \text{if } \varpi = \zeta_i. \end{cases}$$

Taking $\varpi = \zeta_2$ in Equation (2.13), we find

$$\pi_{1,1}(0) = \psi_1\pi_{N,1}(0) + \omega\pi_{1,0}(0), \tag{2.28}$$

where

$$\psi_1 = \frac{\psi_2 - (\zeta_3 - \phi)\varsigma_{3,\zeta_2} - A^*(\zeta_2)\frac{2}{N^2}\psi_2}{A^*(\zeta_2)} \text{ and } \omega = -\frac{A^*(\zeta_2)}{A^*(\zeta_2)} = -1.$$

Now, substituting Equation (2.28) in Equation (2.13), we obtain

$$\pi^*_{2,1}(\varpi) = \varsigma_{2,\varpi}\pi_{N,1}(0),$$

where

$$\varsigma_{2,\varpi} = \begin{cases} \dfrac{-\psi_2 + (\zeta_3 - \phi)\varsigma_{3,\varpi} + A^*(\varpi)(\psi_1 + \frac{2}{N^2}\psi_2)}{(\zeta_2 - \varpi)}, & \text{if } \varpi \neq \zeta_2; \\ -((\zeta_3 - \phi)\varsigma_{3,\varpi}^{(1)} + A^{*(1)}(\varpi)(\psi_1 + \dfrac{2}{N^2}\psi_2), & \text{if } \varpi = \zeta_2, \end{cases}$$

with

$$\varsigma_{2,\varpi}^{(l)} = \begin{cases} \dfrac{(\zeta_3 - \phi)\varsigma_{3,\varpi}^{(l)} + A^{*(l)}(\varpi)(\psi_1 + \frac{2}{N^2}\psi_2) - l\varsigma_{2,\varpi}^{(l-1)}}{(\zeta_2 - \varpi)}, & \text{if } \varpi \neq \zeta_2; \\ -\dfrac{(\zeta_3 - \phi)\varsigma_{3,\varpi}^{(l+1)} + A^{*(l+1)}(\varpi)(\psi_1 + \frac{2}{N^2}\psi_2)}{l+1}, & \text{if } \varpi = \zeta_2. \end{cases}$$

From Equation (2.12), we have

$$\pi_{1,1}^*(\varpi) = \varsigma_{1,\varpi}\pi_{N,1}(0) + \tau_{1,\varpi}\pi_{1,0}(0)),$$

where

$$\varsigma_{1,\varpi} = \begin{cases} \dfrac{\beta\mu_1\varsigma_{2,\varpi} - \psi_1}{(\phi + \beta\nu - \varpi)}, & \text{if } \varpi \neq \phi + \beta\nu; \\ -\beta\mu_1\varsigma_{2,\varpi}^{(1)}, & \text{if } \varpi = \phi + \beta\nu. \end{cases} ; \ \varsigma_{1,\varpi}^{(l)} = \begin{cases} \dfrac{\beta\mu_1\varsigma_{2,\varpi}^{(l)} - l\varsigma_{1,\varpi}^{(l-1)}}{(\phi + \beta\nu - \varpi)}, & \text{if } \varpi \neq \phi + \beta\nu; \\ -\dfrac{\beta\mu_1\varsigma_{2,\varpi}^{(l+1)}}{l+1}, & \text{if } \varpi = \phi + \beta\nu. \end{cases}$$

$$\tau_{1,\varpi} = \begin{cases} -\dfrac{\omega}{(\phi + \beta\nu - \varpi)}, & \text{if } \varpi \neq \phi + \beta\nu; \\ 0, & \text{if } \varpi = \phi + \beta\nu. \end{cases} ; \ \tau_{1,\varpi}^{(l)} = \begin{cases} \dfrac{l\tau_{1,\varpi}^{(l-1)}}{(\phi + \beta\nu - \varpi)}, & \text{if } \varpi \neq \phi + \beta\nu; \\ 0, & \text{if } \varpi = \phi + \beta\nu. \end{cases}$$

Putting $\theta_N = \varpi$ in Equation (2.18) and using $\pi_{N,1}^*(\varpi)$, we obtain

$$\pi_{N-1,2}(0) = \eta_{N-1}\pi_{N,2}(0) + \gamma_{N-1}\pi_{N,1}(0), \tag{2.29}$$

where

$$\eta_{N-1} = \frac{1 - A^*(\theta_N)N^2}{A^*(\theta_N)(N^2 - N - 1)}, \text{ and } \gamma_{N-1} = -\frac{\phi\varsigma_{N,\theta_N}N^2}{A^*(\theta_N)(N^2 - N - 1)}.$$

Substituting Equation (2.29) in Equation (2.18), we get

$$\pi_{N,2}^*(\varpi) = \rho_{N,\varpi}\pi_{N,2}(0) + \chi_{N,\varpi}\pi_{N,1}(0),$$

where

$$\rho_{N,\varpi} = \begin{cases} \dfrac{A^*(\varpi)((1-\frac{N-1}{N^2})\eta_{N-1}+\eta_N)-\eta_N}{\theta_N-\varpi}, & \text{if } \theta_N \neq \varpi; \\ -A^{*(1)}(\varpi)((1-\dfrac{N-1}{N^2})\eta_{N-1}+\eta_N), & \text{if } \theta_N = \varpi. \end{cases}$$

$$\chi_{N,\varpi} = \begin{cases} \dfrac{\phi\varsigma_{N,\varpi}+A^*(\varpi)(1-\frac{N-1}{N^2})\gamma_{N-1}}{\theta_N-\varpi}, & \text{if } \theta_N \neq \varpi; \\ -(\phi\varsigma^{(1)}_{N,\varpi}+A^{*(1)}(\varpi)(1-\dfrac{N-1}{N^2})\gamma_{N-1}), & \text{if } \theta_N = \varpi, \end{cases}$$

with

$$\rho^{(l)}_{N,\varpi} = \begin{cases} \dfrac{A^{*(l)}(\varpi)((1-\frac{N-1}{N^2})\eta_{N-1}+\eta_N)+l\rho^{(l-1)}_{N,\varpi}}{(\theta_N-\varpi)}, & \text{if } \theta_N \neq \varpi; \\ -\dfrac{A^{*(l+1)}(\theta_N)((1-\frac{N-1}{N^2})\eta_{N-1}+\eta_N)}{l+1}, & \text{if } \theta_N = \varpi, \end{cases}$$

$$\chi^{(l)}_{N,\varpi} = \begin{cases} \dfrac{\phi\varsigma^{(l)}_{N,\varpi}+A^{*(l)}(\varpi)(1-\frac{N-1}{N^2})\gamma_{N-1}+l\chi^{(l-1)}_{N,\varpi}}{\theta_N-\varpi}, & \text{if } \theta_N \neq \varpi; \\ -\dfrac{\phi\varsigma^{(l+1)}_{N,\theta_N}+A^{*(l+1)}(\theta_N)(1-\frac{N-1}{N^2})\gamma_{N-1}}{l+1}, & \text{if } \theta_N = \varpi, \end{cases}$$

$\eta_N = 1$ and $\gamma_N = 0$.

In the same manner, we obtain $\pi_{i,2}(0)$ and $\pi^*_{i,2}(\varpi)$ using Equation (2.17), thus

$$\pi_{i-1,2}(0) = \eta_{i-1}\pi_{N,2}(0)+\gamma_{i-1}\pi_{N,1}(0), \ 2 \leq i \leq N-1, \tag{2.30}$$

with

$$\eta_{i-1} = N^2\frac{\eta_i-\theta_{i+1}\rho_{i+1,\theta_i}-A^*(\theta_i)\frac{i}{N^2}\eta_i}{A^*(\theta_i)(N^2-i+1)}.$$

$$\gamma_{i-1} = N^2\frac{\gamma_i-\theta_{i+1}\chi_{i+1,\theta_i}-A^*(\theta_i)\frac{i}{N^2}\gamma_i-\phi\varsigma_{i,\theta_i}}{A^*(\theta_i)(N^2-i+1)}.$$

Substituting Equation (2.30) in Equation(2.17)

$$\pi^*_{i,2}(\varpi) = \rho_{i,\varpi}\pi_{N,2}(0)+\chi_{i,\varpi}\pi_{N,1}(0),$$

with

$$\rho_{i,\varpi} = \begin{cases} \dfrac{-\eta_i + \theta_{i+1}\rho_{i+1,\varpi} + A^*(\varpi)((1-\frac{i-1}{N^2})\eta_{i-1} + \frac{i}{N^2}\eta_i)}{(\theta_i - \varpi)}, & \text{if } \theta_i \neq \varpi; \\ -(\theta_{i+1}\rho^{(1)}_{i+1,\varpi} + A^{*(1)}(\varpi)((1-\dfrac{i-1}{N^2})\eta_{i-1} + \dfrac{i}{N^2}\eta_i)), & \text{if } \theta_i = \varpi. \end{cases}$$

$$\chi_{i,\varpi} = \begin{cases} \dfrac{-\gamma_i + \phi\varsigma_{i,\varpi} + \theta_{i+1}\chi_{i+1,\varpi} + A^*(\varpi)((1-\frac{i-1}{N^2})\gamma_{i-1} + \frac{i}{N^2}\gamma_i)}{\theta_i - \varpi}, & \text{if } \theta_i \neq \varpi; \\ -(\phi\varsigma^{(1)}_{i,\varpi} + \theta_{i+1}\chi^{(1)}_{i+1,\varpi} + A^{*(1)}(\varpi)((1-\dfrac{i-1}{N^2})\gamma_{i-1} + \dfrac{i}{N^2}\gamma_i)), & \text{if } \theta_i = \varpi, \end{cases}$$

where

$$\rho^{(l)}_{i,\varpi} = \begin{cases} \dfrac{\theta_{i+1}\rho^{(l)}_{i+1,\varpi} + A^{*(l)}(\varpi)((1-\frac{i-1}{N^2})\eta_{i-1} + \frac{i}{N^2}\eta_i) + l\rho^{(l-1)}_{i,\varpi}}{(\theta_i - \varpi)}, & \text{if } \theta_i \neq \varpi; \\ -\dfrac{\theta_{i+1}\rho^{(l+1)}_{i+1,\varpi} + A^{*(l+1)}(\varpi)((1-\frac{i-1}{N^2})\eta_{i-1} + \frac{i}{N^2}\eta_i)}{l+1}, & \text{if } \theta_i = \varpi. \end{cases}$$

$$\chi^{(l)}_{i,\varpi} = \begin{cases} \dfrac{\phi\varsigma^{(l)}_{i,\varpi} + \theta_{i+1}\chi^{(l)}_{i+1,\varpi} + A^{*(1)}(\varpi)((1-\frac{i-1}{N^2})\gamma_{i-1} + \frac{i}{N^2}\gamma_i) + l\chi^{(l-1)}_{i,\varpi}}{\theta_i - \varpi}, & \text{if } \theta_i \neq \varpi; \\ -\dfrac{\phi\varsigma^{(l+1)}_{i,\varpi} + \theta_{i+1}\chi^{(l+1)}_{i+1,\varpi} + A^{*(l+1)}(\varpi)((1-\frac{i-1}{N^2})\gamma_{i-1} + \frac{i}{N^2}\gamma_i)}{l+1}, & \text{if } \theta_i = \varpi. \end{cases}$$

Putting $\varpi = \beta\mu_1$ in Equation(2.11), we get

$$\pi_{0,0}(0) = \varepsilon_0\pi_{1,0}(0) + \sigma_0\pi_{N,1}(0) + \triangle_0\pi_{N,2}(0), \tag{2.31}$$

where $\varepsilon_0 = \dfrac{1}{A^*(\beta\mu_1)}$, $\sigma_0 = \dfrac{-\beta\nu\varsigma_{2,\beta\mu_1} - \beta\mu_2\chi_{2,\beta\mu_1}}{A^*(\beta\mu_1)}$, and $\triangle_0 = \dfrac{-\beta\mu_2\rho_{2,\beta\mu_1}}{A^*(\beta\mu_1)}$

Now, let $\varpi = \phi + \beta\nu$ and use (2.30) we get

$$\pi_{1,0}(0) = \kappa_1\pi_{N,1}(0), \tag{2.32}$$

where $\kappa_1 = \psi_1 - \beta\mu_1\varsigma_{2,\phi+\beta\nu}$.

Putting $\beta\mu_2 = \varpi$ in Equation (2.16)

$$\pi_{N,2}(0) = \kappa_2\pi_{N,1}(0), \tag{2.33}$$

where $\kappa_2 = \dfrac{\phi\varsigma_{1,\beta\mu_2} + \beta\mu_1\chi_{2,\beta\mu_2} + \phi\kappa_1\tau_{1,\beta\mu_2} - \gamma_1}{\eta_1 - \beta\mu_1\rho_{2,\beta\mu_2}}$

From Equations (2.19),(2.21), and (2.26)-(2.33), it yields

$$\pi_{N,1}(0) = \lambda\left(\kappa_1\varepsilon_0 + \sigma_0 + \kappa_2\triangle_0 + \psi_1 + \sum_{i=2}^{N}\psi_i + \sum_{i=2}^{N}(\gamma_i + \kappa_2\eta_i)\right)^{-1}.$$

Now, from the rate probabilities ($\pi_{i,j}(0)$) using Equation (2.20) the pre-arrival epoch probabilities ($\pi^-_{i,j}$) can be derived easily.

Next, setting $\varpi = 0$ in the Equations (2.11)-(2.18) and using (2.20). We obtain after slight simplification.

$$\begin{aligned}
\pi_{N,1} &= \frac{\lambda}{\zeta_N}\left(1 - \frac{N-1}{N^2}\right)\pi^-_{N-1,1} \\
\pi_{i,1} &= \left(\frac{\zeta_{i+1} - \phi}{\zeta_i}\right)\pi_{i+1,1} + \frac{\lambda}{\zeta_i}\left(\left(1 - \frac{i-1}{N^2}\right)\pi^-_{i-1,1} - \left(1 - \frac{i}{N^2}\right)\pi^-_{i,1}\right), i = N-1, ..., 3 \\
\pi_{2,1} &= \left(\frac{\zeta_3 - \phi}{\zeta_2}\right)\pi_{3,1} + \frac{\lambda}{\zeta_2}\left(\pi^-_{1,0} + \pi^-_{1,1} - \left(1 - \frac{2}{N^2}\right)\pi^-_{2,1}\right) \\
\pi_{1,1} &= \left(\frac{\beta\mu_1}{\phi + \beta\nu}\right)\pi_{2,1} - \left(\frac{\lambda}{\phi + \beta\nu}\right)\pi^-_{1,1} \\
\pi_{N,2} &= \frac{\phi}{\theta_N}\pi_{N,1} + \frac{\lambda}{\theta_N}\left(1 - \frac{N-1}{N^2}\right)\pi^-_{N-1,2} \\
\pi_{i,2} &= \left(\frac{\theta_{i+1}}{\theta_i}\right)\pi_{i+1,2} + \frac{\phi}{\theta_i}\pi_{i,1} + \frac{\lambda}{\theta_i}\left(\left(1 - \frac{i-1}{N^2}\right)\pi^-_{i-1,2} - \left(1 - \frac{i}{N^2}\right)\pi^-_{i,2}\right), i = N-1, ..., 2 \\
\pi_{1,2} &= \frac{\mu_1}{\mu_2}\pi_{2,2} + \frac{\phi}{\beta\mu_2}\pi_{1,1} - \frac{\lambda}{\beta\mu_2}\pi^-_{1,2} \\
\pi_{1,0} &= \frac{\nu}{\mu_1}\pi_{2,1} + \frac{\mu_2}{\mu_1}\pi_{2,2} + \frac{\lambda}{\beta\mu_1}\left(\pi^-_{0,0} - \pi^-_{1,0}\right)
\end{aligned}$$

Finally, the explicit expressions of $\pi_{0,0}$ can be computed by using the normalization condition, that is,

$$\pi_{0,0} = 1 - \pi_{1,0} - \sum_{i=1}^{N}(\pi_{i,1} + \pi_{i,2}).$$

2.4 Measures of Performance

– The mean number of customers in the system.

$$L_s = \pi_{1,0} + \sum_{i=1}^{N} i(\pi_{i,1} + \pi_{i,2}).$$

– The mean number of customers waiting for service.

$$L_q = \sum_{i=2}^{N} (i-2)(\pi_{1,1} + \pi_{i,2}).$$

– The mean waiting time of customers in the system.

$$W_s = \frac{L_s}{\lambda'}, \text{ where } \ \lambda' = \lambda(1-(\pi_{N,1} + \pi_{N,2})) \text{ is the effective arrival rate.}$$

– The mean rate of joining the system.

$$J_s = \lambda(\pi_{0,0} + \pi_{1,0} + \pi_{1,1} + \pi_{1,2}) + \sum_{i=2}^{N} \lambda\left(1 - \frac{i}{N^2}\right)(\pi_{i,1} + \pi_{i,2}).$$

– The probability that server 2 is idle, in working vacation period and in normal busy period, respectively.

$$P_{idle} = \sum_{i=0}^{1} \pi_{i,0}; \ P_w = \sum_{i=1}^{N} \pi_{i,1}; \ P_b = \sum_{i=1}^{N} \pi_{i,2}.$$

– The average balking rate.

$$B_r = \frac{\lambda}{N^2} \sum_{i=1}^{N} i(\pi_{i,1} + \pi_{i,2})$$

– The average reneging rates during busy period and working vacation period, respectively.

$$R_{ren1} = \alpha\xi_1 \sum_{i=2}^{N} (i-2)\pi_{i,1}, \ R_{ren2} = \alpha\xi_2 \sum_{i=2}^{N} (i-2)\pi_{i,2}.$$

– The average retention rates during busy period and working vacation period, respectively.

$$R_{ret1} = \alpha'\xi_1 \sum_{i=2}^{N}(i-2)\pi_{i,1}, \; R_{ret2} = \alpha'\xi_2 \sum_{i=2}^{N}(i-2)\pi_{i,2}.$$

2.5 Economic analysis

In this section, we develop a model for the costs incurred in the queueing system under consideration using the following symbols:

- C_1 : Cost per unit time when server 2 is on normal busy period.
- C_2 : Cost per unit time when server 2 is on working vacation period.
- C_3 : Cost per unit time when server 2 is idle during working vacation.
- C_4 : Cost per unit time when a customer joins the queue and waits for service.
- C_5 : Cost per unit time when a customer balks.
- C_6 : Cost per service per unit time during busy period.
- C_7 : Cost per service per unit time during working vacation period.
- C_8 : Cost per unit time when a customer reneges during the working vacation period of server 2.
- C_9 : Cost per unit time when a customer reneges during normal busy period of server 2.
- C_{10} : Cost per unit time when a customer is retained during the working vacation period of server 2.
- C_{11} : Cost per unit time when a customer is retained during normal busy period of server 2.
- C_{12} : Cost per unit time when a customer returns to the system as a feedback customer.
- C_{13} : Fixed server purchase cost per unit.

Let
R be the revenue earned by providing service to a customer.
Γ be the total expected cost per unit time of the system.
Δ be the total expected revenue per unit time of the system.
Θ be the total expected profit per unit time of the system.
Thus

$$\Gamma = C_1 P_b + C_2 P_w + C_3 P_{idle} + C_4 L_q + C_5 B_r + C_8 R_{ren1} + C_9 R_{ren2}$$

$$+ C_{10} R_{ret1} + C_{11} R_{ret2} + (\mu_1 + \mu_2) C_6 + \nu C_7 + \beta'(\mu_1 + \mu_2 + \nu) C_{12} + 2C_{13}.$$

The total expected revenue per unit time of the system is given by:

$$\Delta = R\Big(\mu_1 \pi_{1,0} + (\mu_1 + \nu) P_w + (\mu_1 + \mu_2) P_b\Big)$$

Now, the total expected profit is presented as

$$\Theta = \Delta - \Gamma.$$

2.6 Numerical analysis

2.6.1 Effect of different parameters on the performance measures of the system

Case 1: Effect of arrival rate (λ).

We check the behavior of the system characteristics for various values of (λ) by keeping all other variables fixed. Put $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\alpha = 0.4$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\alpha = 0.4$, $\beta = 0.6$, and $N = 5$.

According to Table 2.1, we observe that along the increasing of the arrival rate λ, B_r, L_s, J_s, P_b, R_{ren1}, R_{ren2}, R_{ret1}, R_{ret2}, P_w all increase. While P_{idle} decreases monotonically. This is due to the fact that along the increases of the arrival rate, the queue of the system becomes large, thus, the normal busy period becomes significant, while the probability that the server 2 becomes idle P_{idle} decreases. Furthermore, the average balking rate increases with λ because of the size of the system.

Table 2.1: Variation in system performance measures vs. λ .

λ	1,4	2,2	3	3,8	4,2	4,8
L_s	1.14991	1.91543	2.59842	3.12838	3.33741	3.59305
J_s	1.34147	1.95358	2.38493	2.66080	2.75654	2.86360
B_r	0.05852	0.24641	0.61506	1.13919	1.44346	1.93639
R_{ren1}	0.00231	0.01641	0.02729	0.03161	0.03187	0.03077
R_{ren2}	0.01089	0.05901	0.12314	0.18468	0.21173	0.24702
R_{ret1}	0.00347	0.02462	0.04094	0.04742	0.04780	0.04616
R_{ret2}	0.01634	0.08852	0.18472	0.27702	0.31759	0.37054
W_s	0.07773	0.43723	0.88339	1.28599	1.45611	1.67216
P_{idle}	0.58306	0.34822	0.19872	0.11374	0.08692	0.05907
P_w	0.16708	0.20504	0.19084	0.15799	0.14092	0.11745
P_b	0.24986	0.44674	0.61044	0.72827	0.77216	0.82348

Case 2: Effect of service rates $(\mu_1),(\mu_2)$ and (ν).

We examine the behavior of the characteristics of the system for various values of $(\mu_1),(\mu_2)$ and (ν), respectively, by keeping all other variables fixed. To this end, we consider the following cases

- $\lambda = 2.5$, $\mu_2 = 1.9$, $\nu = 1.4$, $\beta = 0.6$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\alpha = 0.4$, $\phi = 1.2$, and $N = 5$.
- $\lambda = 2.5$, $\mu_1 = 3$, $\nu = 1.4$, $\beta = 0.6$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\alpha = 0.4$, $\phi = 1.2$, and $N = 5$.
- $\lambda = 2.5$, $\mu_1 = 3$, $\mu_2 = 2.5$, $\beta = 0.6$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\alpha = 0.4$, $\phi = 0.5$, and $N = 5$.

Table 2.2: Variation in system performance measures vs. μ_1 .

μ_1	2.1	2.5	2.9	3.3	3.5	3.7
L_s	2.59636	2.37738	2.18182	2.00876	1.93012	1.85638
J_s	1.97894	2.06061	2.12774	2.18275	2.20640	2.22782
B_r	0.52105	0.43938	0.37225	0.31724	0.29359	0.27217
R_{ren1}	0.00409	0.00379	0.00339	0.00293	0.00269	0.00245
R_{ren2}	0.06321	0.05109	0.04092	0.03246	0.02879	0.02546
R_{ret1}	0.00613	0.00568	0.00509	0.00440	0.00404	0.00367
R_{ret2}	0.09482	0.07664	0.06139	0.04869	0.04319	0.03819
W_s	0.89249	0.73355	0.59642	0.47920	0.42735	0.37959
P_{idle}	0.20634	0.24159	0.27501	0.30622	0.32095	0.33510
P_w	0.17391	0.18698	0.19688	0.20405	0.20676	0.20896
P_b	0.61974	0.57143	0.52811	0.48972	0.47228	0.45594

From Tables 2.2–2.3–2.4, we observe that

– with the increases of μ_1, μ_2 and ν, B_r decreases, while J_s increases, as it should be. Therefore, customers are served faster with μ_1, μ_2 and ν. This implies a decrease in the mean number of customers in the system L_s, in the probability that the server 2 is on normal busy period P_b and in the mean waiting time W_s. Consequently, the probability that the server 2 becomes idle P_{idle} increases with the service rates.

– the probability of working vacation of server 2, P_w increases with both μ_1 and μ_2 because customers are served faster. Then, the mean system size decreases, hence,

Table 2.3: Variation in system performance measures vs. μ_2 .

μ_2	1.7	1.9	2.1	2.3	2.5	2.7
L_s	2.20418	2.13650	2.07650	2.02313	1.97550	1.93286
J_s	2.11974	2.14254	2.16232	2.17956	2.19464	2.20789
B_r	0.38025	0.35745	0.33767	0.32043	0.30535	0.29210
R_{ren1}	0.00306	0.00328	0.00348	0.00367	0.00384	0.00399
R_{ren2}	0.04224	0.03865	0.03550	0.03273	0.03027	0.02809
R_{ret1}	0.00459	0.00492	0.00523	0.00550	0.00576	0.00599
R_{ret2}	0.06336	0.05798	0.05326	0.04909	0.04541	0.04214
W_s	0.60457	0.56533	0.53104	0.50096	0.47448	0.45108
P_{idle}	0.26413	0.28303	0.30013	0.31564	0.32975	0.34260
P_w	0.18537	0.19891	0.21123	0.22244	0.23268	0.24204
P_b	0.55050	0.51806	0.48864	0.46192	0.43757	0.41536

Table 2.4: Variation in system performance measures vs. ν .

ν	1.3	1.5	1.7	1.9	2.1	2.3
L_s	2.10375	2.05437	2.00692	1.96140	1.91775	1.87594
J_s	2.15324	2.16818	2.18221	2.19538	2.20775	2.21936
B_r	0.34675	0.33181	0.31778	0.30461	0.29224	0.28063
R_{ren1}	0.00968	0.00911	0.00857	0.00807	0.00760	0.00716
R_{ren2}	0.02424	0.02337	0.02255	0.02176	0.02101	0.02029
R_{ret1}	0.01453	0.01367	0.01286	0.01211	0.01141	0.01075
R_{ret2}	0.03636	0.03506	0.03382	0.03264	0.03151	0.03044
W_s	0.54523	0.52009	0.49635	0.47394	0.45281	0.43290
P_{idle}	0.29148	0.30771	0.32362	0.33918	0.35437	0.36917
P_w	0.38610	0.37849	0.37095	0.36351	0.35618	0.34897
P_b	0.32242	0.31380	0.30543	0.29731	0.28945	0.28186

the server 2 switches to vacation period. On the other hand, P_w decreases with ν, as intuitively expected.

– when μ_1 and ν increase, the average reneging rates during working vacation and during normal busy period R_{ren1} and R_{ren2}, average retention rates in working vacation and during normal busy period R_{ret1} and R_{ret1} decrease. This agree absolutely with our intuition. While when μ_2 increases, R_{ren2} and R_{ret2} decrease because customers are served faster, thus, the size of the system will be reduced, hence, server 2 goes on vacation. Consequently, the probability of working vacation increases which leads to an increase in the average reneging and retention rates R_{ren1} and R_{ret1} respectively.

Case 3: Effect of reneging rates (ξ_1) and (ξ_2).

We check the behavior of the performance measures of the system for various values of (ξ_1) and (ξ_2), respectively by keeping all other variables fixed. Let

- $\lambda = 3.5$, $\mu_1 = 2.1$, $\mu_2 = 1.7$, $\nu = 1.3$, $\beta = 0.6$, $\xi_2 = 2$, $\alpha = 0.6$, $\phi = 0.1$, and $N = 5$.

- $\lambda = 3.5$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\beta = 0.6$, $\xi_1 = 1$, $\alpha = 0.6$, $\phi = 1.2$, and $N = 5$.

According to Tables 2.5–2.6, we observe that

Table 2.5: Variation in system performance measures vs. ξ_1 .

ξ_1	3.5	3.7	3.9	4.1	4.3	4.5
L_s	2.38620	2.36189	2.33912	2.31773	2.29763	2.27868
J_s	3.04204	3.05417	3.06521	3.07529	3.08453	3.09301
B_r	0.45795	0.44582	0.43478	0.42470	0.41546	0.40698
R_{ren1}	0.86827	0.87516	0.88040	0.88415	0.88655	0.88775
R_{ren2}	0.21529	0.21313	0.21109	0.20917	0.20736	0.20564
R_{ret1}	0.57884	0.58344	0.58693	0.58943	0.59103	0.59183
R_{ret2}	0.14353	0.14208	0.14072	0.13944	0.13824	0.13709
W_s	0.59288	0.57183	0.55215	0.53372	0.51642	0.50016
P_{idle}	0.15090	0.15327	0.15554	0.15769	0.15974	0.16170
P_w	0.66838	0.66710	0.66586	0.66468	0.66355	0.66247
P_b	0.18072	0.17963	0.17860	0.17762	0.17671	0.17583

Table 2.6: Variation in system performance measures vs. ξ_2 .

ξ_2	3.5	3.7	3.9	4.1	4.3	4.5
L_s	2.19085	2.17338	2.15715	2.14203	2.12791	2.11470
J_s	3.09108	3.09919	3.10654	3.11320	3.11926	3.12479
B_r	0.40891	0.40080	0.39345	0.38679	0.38073	0.37520
R_{ren1}	0.10338	0.10446	0.10547	0.10642	0.10733	0.10818
R_{ren2}	0.69007	0.69389	0.69657	0.69822	0.69896	0.69889
R_{ret1}	0.06892	0.06964	0.07031	0.07095	0.07155	0.07212
R_{ret2}	0.46004	0.46259	0.46438	0.46548	0.46597	0.46592
W_s	0.50091	0.48666	0.47347	0.46121	0.44980	0.43915
P_{idle}	0.22108	0.22338	0.22554	0.22758	0.22951	0.23134
P_w	0.26548	0.26824	0.27084	0.27330	0.27561	0.27780
P_b	0.51344	0.50838	0.50362	0.49912	0.49488	0.49086

– with the increases of reneging rates ξ_1 and ξ_2, the characteristics L_s, W_s and B_r decrease, while J_s increases, as intuitively expected.

– along the increasing of ξ_1, the average reneging rate during working vacation R_{ren1} increases, while the average rate of reneging in the normal busy period of server 2, R_{ren2} decreases.

– along the increasing of ξ_1, the probability of working vacation P_w, the probability of normal busy period P_b decrease because of the size of the system which become small due to reneging. Consequently, the probability that the server 2 becomes idle P_{idle} increases with ξ_1.

– when the reneging rate ξ_2 increases, the average reneging rate during normal busy period R_{ren2} and the average rate of reneging in the busy period of server 2 during his vacation R_{ren1} increase.

– the increases of ξ_2 implies a decreasing of P_b and an increasing of P_w, which can be explained by the fact that when reneging rate increases in the normal busy period of server 2, more customers are lost. Thus, server 2 goes on vacation, consequently P_w and P_{idle} increase.

Case 4: Effect of vacation rate (ϕ).

We study the behavior of the performance measures of the system for various values of (ϕ) by keeping all other variables fixed. Put $\lambda = 3$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\beta = 0.6$, $\xi_1 = 0.1$, $\xi_2 = 0.5$, $\alpha = 0.6$, and $N = 5$.

Table 2.7: Variation in system performance measures vs. ϕ .

ϕ	0.5	0.7	0.9	1.1	1.3	1.5
L_s	2.60098	2.56891	2.54766	2.53259	2.52136	2.51270
J_s	2.38962	2.40830	2.42040	2.42879	2.43490	2.43952
B_r	0.61037	0.59169	0.57959	0.57120	0.56509	0.56047
R_{ren1}	0.01925	0.01413	0.01087	0.00864	0.00705	0.00586
R_{ren2}	0.16768	0.18586	0.19731	0.20502	0.21048	0.21448
R_{ret1}	0.01283	0.00942	0.00724	0.00576	0.00470	0.00391
R_{ret2}	0.11179	0.12391	0.13154	0.13668	0.14032	0.14299
W_s	0.87984	0.85509	0.83889	0.82753	0.81917	0.81278
P_{idle}	0.19442	0.19954	0.20307	0.20567	0.20767	0.20927
P_w	0.34910	0.28653	0.24394	0.21293	0.18926	0.17055
P_b	0.45647	0.51392	0.55298	0.58139	0.60305	0.62017

From Table 2.7, we remark that along the increasing of the vacation rate ϕ, L_s and W_s decrease. Therefore, the average balking rate B_r decreases, while the average rate of joining the system J_s increases with ϕ. Further, the increase in vacation rate implies that P_b increases, wherein, the probability that the system goes on working vacation P_w decreases. This implies an increase in the mean number of customers served. Therefore, the probability that the server 2 becomes idle P_{idle} increases. Further, with the increases of ϕ, R_{ren1} and R_{ret1} (resp. R_{ren2} and R_{ret2}) decreases (resp. increase), as intuitively expected.

Case 5: Effect of non-feedback probability (β).

We examine the behavior of the performance measures of the system for various values of (β) by keeping all other variables fixed. Put $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\alpha = 0.4$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\alpha = 0.4$, $\lambda = 3$, and $N = 5$.

Thought Table 2.8, we see that when the non-feedback probability β increases, L_s and W_s decrease, this results in the decreasing of the average balking rate B_r and in the increasing of the average rate of joining the system J_s. Moreover, along the increases of the non-feedback probability, R_{ren1} and R_{ret1} increase, while R_{ren2} and R_{ret2} decreases. Further, obviously, the probability of normal busy period P_b decreases, the probability that the system is on working vacation P_w and the probability that the server is idle P_{idle} increase, as it should be.

Table 2.8: Variation in system performance measures vs. β .

β	0.1	0.3	0.5	0.7	0.9	1
L_s	4.54732	3.81878	2.97531	2.27287	1.77073	1.58122
J_s	0.86754	1.63389	2.19464	2.52636	2.70701	2.76395
B_r	2.13245	1.36610	0.80535	0.47363	0.29298	0.23604
R_{ren1}	0.00085	0.01173	0.02482	0.02695	0.02132	0.01758
R_{ren2}	0.40725	0.29047	0.16948	0.08786	0.04352	0.03040
R_{ret1}	0.00128	0.01760	0.03724	0.04042	0.03198	0.02637
R_{ret2}	0.61088	0.43570	0.25423	0.13179	0.06529	0.04560
W_s	2.54892	1.86435	1.16273	0.66146	0.36090	0.26329
P_{idle}	0.00141	0.03512	0.13419	0.26505	0.38866	0.44283
P_w	0.00291	0.05196	0.14789	0.22317	0.25612	0.26039
P_b	0.99568	0.91292	0.71792	0.51177	0.35522	0.29678

Case 6: Effect of non-retention probability (α).

We examine the behavior of the characteristics of the system for various values of (α) by keeping all other variables fixed. We take $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\alpha = 0.4$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\lambda = 3$, $\beta = 0.6$, and $N = 5$.

Table 2.9: Variation in system performance measures vs. α .

α	0,1	0.3	0.5	0.7	0.9	1.0
L_s	2.72298	2.63780	2.56099	2.49153	2.42851	2.39916
J_s	2.30217	2.35932	2.40879	2.45182	2.48939	2.50639
B_r	0.69782	0.64067	0.59120	0.54817	0.51060	0.49360
R_{ren1}	0.00697	0.02064	0.03379	0.04632	0.05815	0.06379
R_{ren2}	0.03488	0.09622	0.14786	0.19138	0.22809	0.24423
R_{ret1}	0.06277	0.04816	0.03379	0.01985	0.00646	0.00000
R_{ret2}	0.31400	0.22452	0.14786	0.08202	0.02534	0.00000
W_s	0.98850	0.91654	0.85196	0.79381	0.74127	0.71689
P_{idle}	0.18516	0.19437	0.20289	0.21078	0.21809	0.22155
P_w	0.17945	0.18723	0.19428	0.20069	0.20653	0.20924
P_b	0.63539	0.61840	0.60283	0.58853	0.57538	0.56921

Through Table 2.9, we remark that when the non-retention probability α increases, the size of the system L_s, the mean waiting time W_s and the average balking rate B_r decrease, while the probability that customers join the system increases. Moreover, the average reneging rates R_{ren1} and R_{ren2} increase with α, while average retention rates R_{ret1} and R_{ret2} decrease with the increasing of α, which absolutely agree with our intuition. This implies that the probability of normal busy period P_b decreases, consequently, the probability of working vacation P_w, and of idle period of server 2 P_{idle} increase.

Case 7: Effect of system capacity (N).

We analyze the behavior of the performance measures of the system for various values of (N) by keeping all other variables fixed. Let $\lambda = 3$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\beta = 0.6$,

$\xi_1 = 0.1$, $\xi_2 = 0.2$, $\alpha = 0.4$, and $\phi = 1.1$.

Table 2.10: Variation in system performance measures vs. N.

N	3	4	5	6	7	8
L_s	1.73287	2.21507	2.69421	3.16073	3.60819	4.03161
J_s	1.97401	2.17590	2.32287	2.43554	2.52532	2.59887
B_r	1.02598	0.82409	0.67712	0.56445	0.47467	0.40112
R_{ren1}	0.00201	0.00446	0.00532	0.00543	0.00524	0.00496
R_{ren2}	0.00850	0.03470	0.06644	0.10020	0.13399	0.16668
R_{ret1}	0.00301	0.00669	0.00798	0.00814	0.00786	0.00744
R_{ret2}	0.01275	0.05205	0.09966	0.15030	0.20098	0.25002
W_s	0.15645	0.54544	0.96360	1.38828	1.80595	2.20759
P_{idle}	0.29531	0.23036	0.18785	0.15868	0.13791	0.12272
P_w	0.28102	0.23321	0.19472	0.16616	0.14516	0.12955
P_b	0.42367	0.53643	0.61743	0.67516	0.71693	0.74773

From Tables 2.10 , we remark that along the increasing of N, the average balking rate B_r decreases due to the large capacity of the system. Then, the means system size L_s, and the mean waiting time W_s increase, consequently, the probability that the server is in normal busy period P_b increases, wherein, P_w and P_{idle} decrease, this implies an increase in the mean number of customers served with N. Moreover, the average reneging and retention rates R_{ren2} and R_{ret2} increase due to the significant number of customers in the system. While the behaviour of R_{ren1} and R_{ret1} is not monotone, it increases, then, decreases when N is above a certain threshold.

2.6.2 Economic analysis

In this part we present the variation in total expected cost, total expected revenue and total expected profit with the change in diverse parameters of the system. For the whole numerical study we fix the costs at $C_1 = 4$, $C_2 = 2$, $C_3 = 2$, $C_4 = 3$, $C_5 = 3$, $C_6 = 4$, $C_7 = 4$, $C_8 = 2$, $C_9 = 2$, $C_{10} = 3$, $C_{11} = 3$, $C_{12} = 2$, $C_{13} = 5$, $R = 25$. And consider the following Tables

- Table 11: $\lambda = 1.4 : 0.8 : 4.8$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,
- Table 12: $\lambda = 2.5$, $\mu_1 = 2.1 : 0.4 : 3.7$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,
- Table 13: $\lambda = 2.5$, $\mu_1 = 3.0$, $\mu_2 = 1.7 : 0.2 : 2.7$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,
- Table 14: $\lambda = 2.5$, $\mu_1 = 3.0$, $\mu_2 = 2.5$, $\nu = 1.3 : 0.2 : 2.3$, $\phi = 1.2$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,

- Table 15: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.3 : 0.2 : 1.3$, $\xi_2 = 0.1$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,

- Table 16: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.1$, $\xi_2 = 0.3 : 0.2 : 1.3$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,

- Table 17: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 0.3 : 0.2 : 1.3$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\beta = 0.6$, $\alpha = 0.4$, $N = 10$,

- Table 18: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\beta = 0.1 : 0.2 : 1$, $\alpha = 0.6$, $N = 10$,

- Table 19: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.2$, $\xi_1 = 0.6$, $\xi_2 = 0.4$, $\beta = 0.6$, $\alpha = 0.1 : 0.2 : 1$, $N = 10$,

- Table 20: $\lambda = 3.0$, $\mu_1 = 2.5$, $\mu_2 = 2.1$, $\nu = 1.7$, $\phi = 1.1$, $\xi_1 = 0.1$, $\xi_2 = 0.2$, $\beta = 0.6$, $\alpha = 0.4$, $N = 3 : 2 : 11$.

The numerical results are presented in following Tables and Graphes.

Table 2.11: Γ, Δ and Θ vs. λ.

λ	1.4	2.2	3	3.8	4.2	4.8
Γ	43.41467	47.16563	53.81363	61.46361	64.93732	69.43411
Δ	62.44631	87.83599	103.58542	110.98226	112.68849	113.98970
Θ	19.03164	40.67037	49.77179	49.51866	47.75117	44.55559

Table 2.12: Γ, Δ and Θ vs. μ_1.

μ_1	2.1	2.5	2.9	3.3	3.5	3.7
Γ	49.04408	48.72019	48.82543	49.31085	49.67482	50.10801
Δ	89.84638	95.38482	100.32531	104.82918	106.95748	109.01970
Θ	40.80231	46.66463	51.49987	55.51833	57.28266	58.91169

Table 2.13: Γ, Δ and Θ vs. μ_2.

μ_2	1.7	1.9	2.1	2.3	2.5	2.7
Γ	48.64273	48.91370	49.2815	49.73212	50.25305	50.83345
Δ	100.27834	101.48669	102.4811	103.30047	103.97722	104.53800
Θ	51.63561	52.57298	53.1996	53.56835	53.72417	53.70455

Table 2.14: Γ, Δ and Θ vs. ν.

ν	1.3	1.5	1.7	1.9	2.1	2.3
Γ	60.006248	60.42808	60.83935	61.24349	61.64399	62.04501
Δ	112.06111	113.81396	115.51876	117.17090	118.76361	120.29314
Θ	52.05486	53.38588	54.67941	55.92741	57.11961	58.24813

Table 2.15: Γ, Δ and Θ vs. ξ_1.

ξ_1	0.3	0.5	0.7	0.9	1.1	1.3
Γ	55.60031	55.55241	55.50744	55.46520	55.42550	55.38809
Δ	106.73059	106.62743	106.52766	106.43137	106.33855	106.24913
Θ	51.13028	51.07502	51.02023	50.96617	50.91305	50.86104

Table 2.16: Γ, Δ and Θ vs. ξ_2.

ξ_2	0.3	0.5	0.7	0.9	1.1	1.3
Γ	54.42735	54.18721	53.50838	52.98109	52.05005	52.20927
Δ	104.77963	104.26948	102.84655	101.58610	99.60370	99.46025
Θ	50.35228	50.08227	49.33816	48.60501	47.55365	47.25098

Table 2.17: Γ, Δ and Θ vs. ϕ.

ϕ	0.1	0.5	0.9	1.3	1.7	2.1
Γ	55.52628	55.08756	55.01420	54.98712	54.97424	54.96726
Δ	103.76382	105.37013	105.66036	105.78134	105.84754	105.88922
Θ	48.23755	50.28258	50.64616	50.79423	50.87330	50.92196

Table 2.18: Γ, Δ and Θ vs. β.

β	0.1	0.3	0.5	0.7	0.9	1
Γ	82.79219	72.35272	59.53657	49.17583	42.63061	40.23601
Δ	114.99820	114.53007	109.25291	96.80668	83.22630	77.19281
Θ	32.20601	42.17735	49.71635	47.63085	40.59570	36.95680

Table 2.19: Γ, Δ and Θ vs. α

α	0.1	0.3	0.5	0.7	0.9	1
Γ	58.62700	55.12727	52.71747	51.01291	49.76362	49.25887
Δ	106.81003	104.56888	102.68686	101.11251	99.78247	99.19185
Θ	48.18303	49.44161	49.96940	50.09960	50.01886	49.93298

Table 2.20: Γ, Δ and Θ vs. N.

N	3	5	6	7	9	11
Γ	46.70298	48.86349	50.13514	51.42076	53.87616	56.02955
Δ	88.66901	98.09453	100.70350	102.56714	104.95503	106.32773
Θ	41.96604	49.23104	50.56836	51.14637	51.07887	50.29818

General comments

– According to Table 2.11 and Figure 2.1, we remark that the increases of λ generates an increase in Γ and Δ, this is quite obvious. While the behavior of Θ is not monotonic, it increases, then decreases when λ is above a certain threshold, this can be explicable by the fact that a large number of incoming customers engenders a large number of customers served, and consequently the total expected profit increases, but when λ is

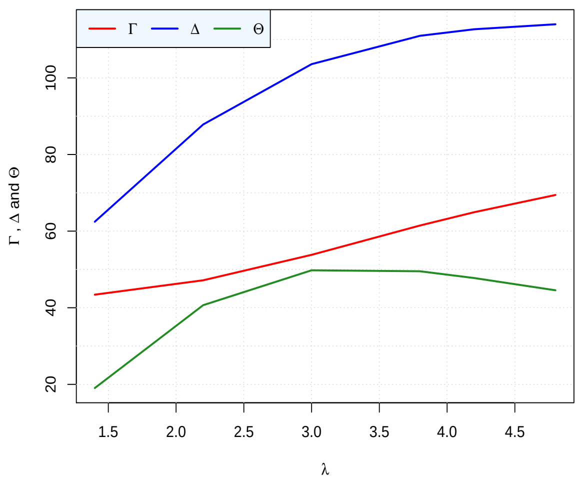

Figure 2.1: Γ, Δ and Θ vs. λ.

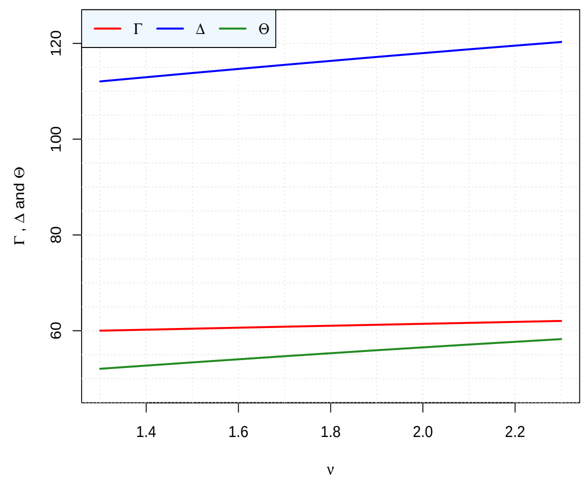

Figure 2.2: Γ, Δ and Θ vs. ν

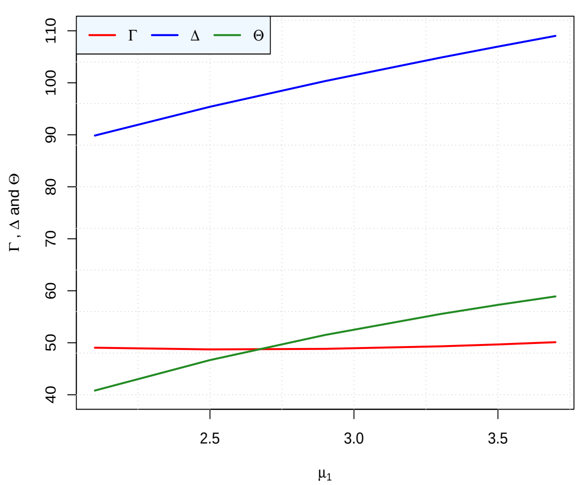

Figure 2.3: Γ, Δ and Θ vs. μ_1.

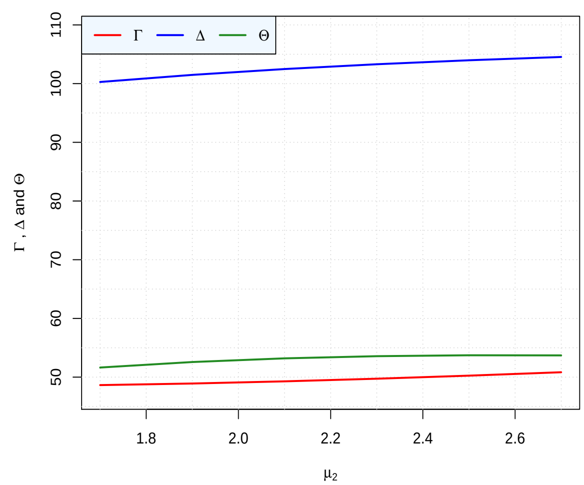

Figure 2.4: Γ, Δ and Θ vs. μ_2.

large enough, the customers in the system may renege due to the long queue length, this implies a loss customers.

– From Tables 2.12-2.14 and Figures 2.2-2.4, we remark that Γ, Δ and Θ all increase with the increasing of μ_1, μ_2, and ν. We can explain this by the fact that with the increasing of the service rates, the average balking rate B_r decreases. The customers are served faster, this leads to a decrease in the mean number of customers in the system L_s, in the probability that the system is idle P_{idle}, in the mean waiting time W_s, in

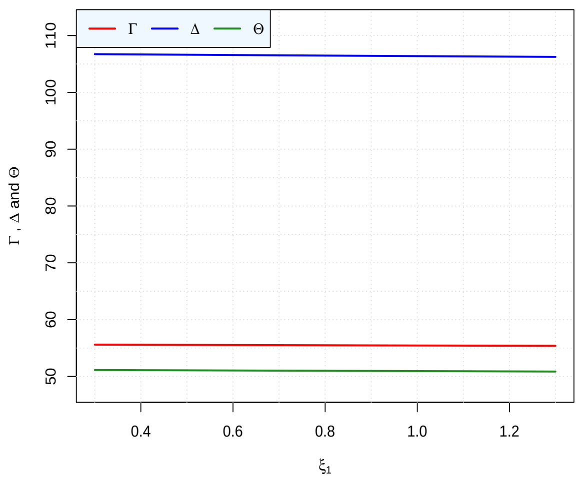

Figure 2.5: Γ, Δ and Θ vs. ξ_1.

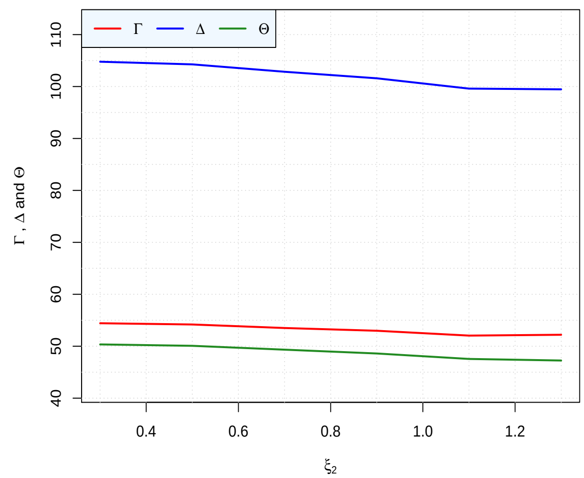

Figure 2.6: Γ, Δ and Θ vs. ξ_2.

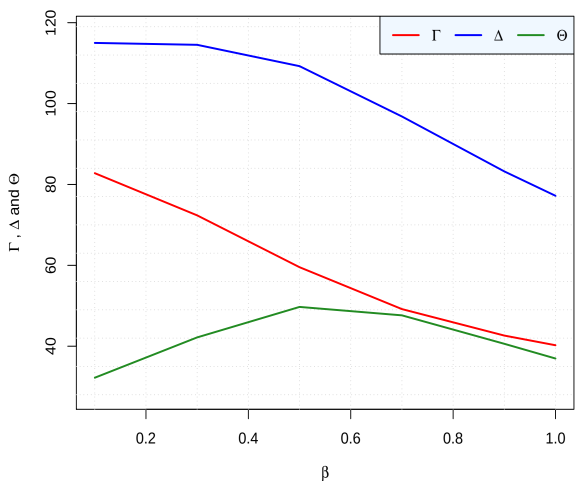

Figure 2.7: Γ, Δ and Θ vs. β.

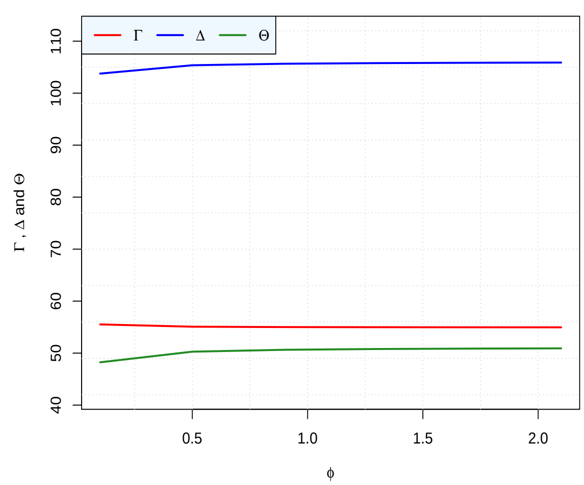

Figure 2.8: Γ, Δ and Θ vs. ϕ.

average reneging rates R_{ren1}, and R_{ren1}. Therefore, the expected total profit increases.

– From Tables 2.15-2.16 and Figures 2.5-2.6, we remark that Δ, and Θ decrease along the increasing of impatience rates ξ_1 and ξ_2. This is due to the fact that the mean waiting time of impatient customers decreases with the increasing of ξ_1 and ξ_2. Therefore, the average rate of loss customers increases, while the mean number of customers waiting for service and the busy period probability decrease which results in an increasing of the total expect cost Γ. Consequently this later generates a decrease in

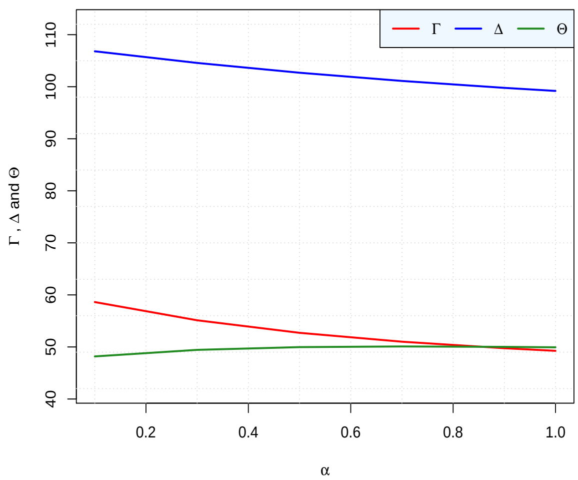

Figure 2.9: Γ, Δ and Θ vs. α.

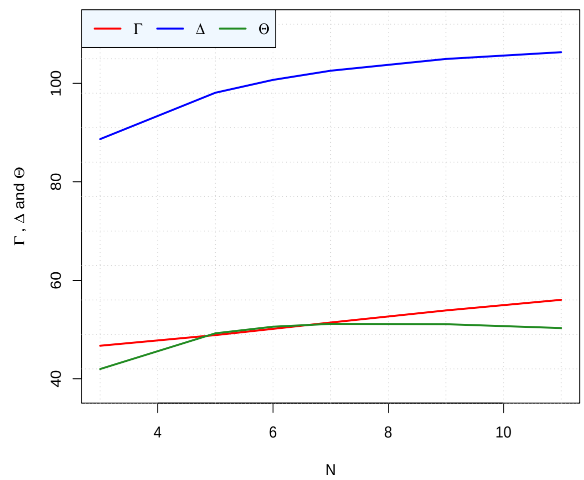

Figure 2.10: Γ, Δ and Θ vs. N.

the total cost profit Θ. Thus, it is quite clear that impatient phenomenon has a negative impact in the economy.

– From Table 2.17 and Figure 2.8, we see that Γ decreases with ϕ, while Δ and Θ increase along the increasing of the vacation rate ϕ. Obviously, the decrease in the mean vacation time implies a diminution in probability of loss customers, this leads to a high rate of customers served. Therefore, the total expected profit becomes significant.

– From Table 2.18 and Figure 2.7, we remark that along the increasing of non-feedback probability β, total expected cost Γ and total expected revenue Δ decrease. While, the total expected profit Θ is not monotonic with β, it first increases, then, decreases significantly. Therefore, one can deduce easily the negative impact of this probability on the cost profits.

– Through Table 2.19 and Figure 2.9, we observe that the increasing of non-retention probability α generates a decrease in Γ and Δ. While, the behavior of the total expected profit Θ is not monotone with α, it increases, then, when α is above a certain threshold, it decreases. This can be explained by the fact that when the non-retention probability α increases, the size of the system and the mean waiting time decrease, while the average reneging rate increases. This implies also that the probability of normal busy period P_b decreases. Therefore, the mean number of customers served is reduced. Moreover, the increase of Θ can be due to the choice of $\xi_1 = 0.6$ and $\xi_2 = 0.4$ So, it is quite evident that retention probability has a positive effect on the revenue generation and on the total expected profit of the system.

– From Table 2.20 and Figure 2.10, we remark that along the increasing N, total expected cost Γ, total expected revenue Δ increase. While, total expected profit Θ is not monotonic, it increases, then decreases when N is above a certain threshold. Obviously, the larger the size of the system, the smaller the average rate of balking, this generates a large number of customers served which engenders a positive impact on the total costs of the system and consequently on the economy of any firm. Note that the non-monotonicity of Θ can be due to the choice of the impatience rates ξ_1 and ξ_2.

2.7 Conclusion

In this paper we present a study of heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, balking, reneging and retention of reneged customers. It is supposed that impatience timers of customers in the system depend on the state of the server. The equations of the steady state probabilities are developed. The most important performance measures of the system are given. Then, based on the performance analysis, we formulate a cost model to determine the effect of different system parameters on the different characteristics as well as on total expected cost, total expected revenue, and total expected profit of the system.

In this study, the positive impact of retention probability on both characteristics and costs of the system under consideration has been shown. The present analysis has a large application in many real world systems as telecommunication networks, call centers and production-inventory systems. For further work, it will be interesting to consider a multiserver queueing system with heterogeneous service times, multiple working vacations, and impatient customers depending on the state of the servers. Moreover, one can develop a similar model wherein the servers are subject to sudden halt.

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Chapter 3

Sensitivity analysis of multiple vacation feedback queueing system with differentiated vacations, vacation interruptions and impatient customers

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Sensitivity analysis of multiple vacation feedback queueing system with differentiated vacations, vacation interruptions and impatient customers

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Abstract. The purpose of this paper is to study an infinite-buffer single-server queueing system with Bernoulli feedback, multiple vacations, differentiated vacations, vacation interruptions and impatient customers (balking and reneging). Two types of vacation are considered; type 1 vacation taken after the busy period and type 2 vacation taken when the server returns from a vacation and finds the system empty. Each vacation type may be interrupted when the number of customers in the system reaches a predefined threshold (each type of vacation has a different threshold). Via certain mechanism, reneged customers may be retained in the system. Using the recursive method, we obtain explicit expressions of the steady-state probabilities of the queueing system. Further, we present important performance measures and formulate a cost model. Finally, we carry out a sensitivity analysis through numerical experiments.

Keywords: Multiple vacations. Differentiated vacations. Vacation interruptions. Impatient customers. Bernoulli feedback. Cost model.
2010 Mathematics Subject Classification: 60K25, 68M20, 90B22.

3.1 Introduction

Vacation queues have been intensively considered because of their wide applications in a variety of congestion problems including computer systems, call centers, web services and communication networks, manufacturing systems, etc. Vacations queueing models have been the subject of big interest of many researchers. Excellent results on vacation models are found in the survey paper of Doshi (1986) and the monographs of Takagi (1991), Tian and Zhang (2006). For related literature, interested readers may refer to Chakravarthy (2009), Wang et al. (2011), Sandari and Srinivasan (2013), Choudhury and Deka (2016), Jain and Jain (2017) and references therein.

Vacation queueing models with impatient customers play a powerful role in our day-to-day life as well as various congestion situations; communication networks, call centers, systems operating in machining environment, manufacturing systems, transportation systems, etc. So, over recent years, a vast number of research papers has been done on this subject. Arumuganathan and Jeyakumar (2005) presented the steady state analysis of a bulk queue with multiple vacations, set up times with N-policy and close down times. Zhang et al. (2005) studied a single server queueing model with finite waiting space, impatient customers and server vacations.

Altman and Yechiali (2006) dealt with customers' impatience in single and multiple vacation models of diverse types of queues ($M/M/1$ and $M/G/1$ and $M/M/c$ queues). Further, Padmavathy et al. (2011) considered a vacation queues with impatient customers and a waiting server. The balking behavior of customers in the single-server queue with generally distributed service and vacation times has been examined by Economou et al. (2011).

Later, in Panda et al. (2016), authors considered a single server renewal input queueing model with balking, bernoulli-schedule controlled vacation and vacation interruption. Customers' equilibrium and socially optimal balking strategies in single-server Markovian queues with multiple vacations and N-policy was provided by Sun et al. (2016). Vijaya Laxmi and Jyothsna (2016) considered a discrete-time $M/M/1/N$ queue with impatient customers and Bernoulli-schedule vacation interruption under the early arrival system. Further, a study of single server Markovian queueing model with single and multiple vacations and impatience timers which depend of the states of the server was presented in Yue et al. (2016). Recently, customers' impatience in unreliable server queueing system under N-policy and vacation interruption was analyzed by Sharma (2017). Then, Bouchentouf and Yahiaoui (2017) examined a queueing system with feedback, reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption.

The research for the present paper is initially motivated by the desire to include Bernoulli feedback queue, impatience behavior (reneging and balking), and retention of reneged customers in Isijola-Adakeja and Ibe (2014), where authors have presented an $M/M/1/DV$ multiple vacation queueing system with differentiated vacations and vacation interruption considering two types of vacations. The model considered in this paper can be employed to model many real life congestion situations, like certain hospital emergency situations, call centers, post offices, etc. Further, from an economic point of view, it is beneficial to convince the reneged customers to do not leave the system to get their services.

This paper is structured as follows. The system description is presented in Section 2. In Section 3, the steady-state probabilities of the queueing system are obtained. In Section 4, we obtain explicit expressions of useful measures of effectiveness and develop a cost model. Further, in Section 5, numerical examples are performed. Finally, in Section 6 we conclude the paper.

3.2 System description

Consider an $M/M/1$ queueing system with Bernoulli feedback, multiple vacations, differentiated vacations, vacation interruptions, balking, reneging and retention of reneged customers. Customers arrive into the system according to a Poisson process with arrival rate λ, the service time is assumed to be exponentially distributed with rate μ. The service discipline is FCFS and there is an infinite space for customers to wait. In this work, two types of vacation are considered: type 1 vacation that may be taken after a busy period where at least one customer is served, and type 2 vacation which is taken when the server comes back from any vacation (either a type 1 vacation or a type 2 vacation) and finds the system empty. The period of a type 1 (resp. type 2) vacation is assumed to be exponentially distributed with rate ϕ_1 (resp. ϕ_2).
Further, we assume that the server's vacation can be interrupted when the number of customers in the system reaches n_1 (resp. n_2) when the server is on type 1 (resp. type 2) vacation. Moreover, we suppose that $n_1 > n_2$, as we want that the server will be interrupted earlier when he takes a vacation after zero busy period than when he takes a vacation after having a non-zero busy period.

Whenever a customer arrives at the system and finds the server busy, he activates an impatience timer T, exponentially distributed with parameter ξ, if the customer's service has not been completed before the customer's timer expires, the customer may leave the system. We suppose that the customers timers are independent and identi-

cally distributed random variables, independent of the size of the queue at that time. Further, using a certain mechanism, a reneged customer may abandon the system without getting service with probability α and can be retained in the system with probability α', where $\alpha + \alpha' = 1$.
A customer who on arrival finds at least one customer in the system, either decides to join the queue with probability θ or balk with probability θ', where $\theta + \theta' = 1$.

If the service is incomplete or unsatisfactory, the customer can either leave the system definitively with probability β or rejoin the end of the queue with probability β', where $\beta + \beta' = 1$.

3.3 Analysis of the model

In this part of paper we carry out the steady-state analysis of the system. Let $L(t)$ be the number of customers in the system at time t, and $J(t)$ denotes the state of the server at time t such that

$$J(t) = \begin{cases} 0, & \text{if the server is on busy period;} \\ 1, & \text{if the server is on type 1 vacation;} \\ 2, & \text{if the server is on type 2 vacation.} \end{cases}$$

Clearly, the process $\{(L(t), J(t)), t \geq 0\}$ is a continuous-time Markov process with state space $\Omega = \{(n,0) : n \geq 1\} \cup \{(n,j) : n \geq 0, j = 1,2\}$.

Let $P_{n,j} = \lim_{t\to\infty} P\{L(t) = n, J(t) = j, \ n \geq 0, j = \overline{0,2}\}$ denote the steady-state probabilities of the system.

3.3.1 Balance equations

Using Markov theory, the set of balance equations of our model is given as follows

$$(\lambda + \phi_1)P_{0,1} = (\beta\mu + \alpha\xi)P_{1,0}, \tag{3.1}$$

$$(\theta\lambda + \phi_1)P_{1,1} = \lambda P_{0,1}, \tag{3.2}$$

$$(\theta\lambda + \phi_1)P_{k,1} = \theta\lambda P_{k-1,1}, \ k = \overline{2, n_1 - 1}, \tag{3.3}$$

$$\lambda P_{0,2} = \phi_1 P_{0,1}, \tag{3.4}$$

$$(\theta\lambda + \phi_2)P_{1,2} = \lambda P_{0,2}, \tag{3.5}$$

$$(\theta\lambda + \phi_2)P_{k,2} = \theta\lambda P_{k-1,2}, \;\; k = \overline{2, n_2 - 1}, \tag{3.6}$$

$$\theta\lambda(P_{k,0} + P_{k,1} + P_{k,2}) = (\beta\mu + (k+1)\alpha\xi)P_{k+1,0}, \;\; k = \overline{1, n_2 - 1}, \tag{3.7}$$

$$(\beta\mu + n_2\alpha\xi)P_{n_2,0} = \theta\lambda(P_{n_2-1,0} + P_{n_2-1,1} + P_{n_2-1,2}), \tag{3.8}$$

$$\theta\lambda(P_{k,1} + P_{k,0}) = (\beta\mu + (k+1)\alpha\xi)P_{k+1,0}, \;\; k = \overline{n_2, n_1 - 1}, \tag{3.9}$$

$$\theta\lambda P_{k,0} = (\beta\mu + (k+1)\alpha\xi)P_{k+1,0}, \;\; k = n_1, n_1 + 1, ... \tag{3.10}$$

3.3.2 Steady-state solution

Applying the recursive method, we solve the above balance equations and obtain the steady state probabilities.

Using (3.1)-(3.3), we get

$$P_{0,1} = \omega_1 P_{1,0}, \qquad \text{where} \qquad \omega_1 = \left(\frac{\beta\mu + \alpha\xi}{\lambda + \phi_1}\right),$$

and

$$\begin{cases} P_{1,1} = \left(\dfrac{\lambda}{\theta\lambda + \phi_1}\right) P_{0,1}, & \text{k=1;} \\ P_{k,1} = \left(\dfrac{\theta\lambda}{\theta\lambda + \phi_1}\right) P_{k-1,1}, & k = \overline{2, n_1 - 1}. \end{cases}$$

Then, recursively we obtain

$$P_{k,1} = \delta_1 \beta_1^k P_{1,0}, \;\; k = \overline{1, n_1 - 1}, \qquad \text{where} \qquad \delta_1 = \frac{\omega_1}{\theta}, \text{ and } \beta_1 = \frac{\theta\lambda}{\theta\lambda + \phi_1}. \tag{3.11}$$

Via (3.4), we have

$$P_{0,2} = \omega_2 P_{1,0}, \qquad \text{where} \qquad \omega_2 = \frac{\phi_1}{\lambda}\omega_1.$$

Using (3.5) and (3.6), we get

$$\begin{cases} P_{1,2} = \left(\dfrac{\lambda}{\theta\lambda + \phi_2}\right) P_{0,2}, & \text{k=1;} \\ P_{k,2} = \left(\dfrac{\theta\lambda}{\theta\lambda + \phi_2}\right) P_{k-1,2}, & k = \overline{2, n_2 - 1}. \end{cases}$$

Thus,

$$P_{k,2} = \delta_2 \beta_2^k P_{1,0}, \ k = \overline{1, n_2 - 1}, \qquad \text{where} \qquad \delta_2 = \frac{\omega_2}{\theta}, \ \beta_2 = \frac{\theta\lambda}{\theta\lambda + \phi_2}. \tag{3.12}$$

Next, from (3.7), it yields

$$P_{k+1,0} = \frac{\theta\lambda}{\beta\mu + (k+1)\alpha\xi}\Big(P_{k,1} + P_{k,2} + P_{k,0}\Big), \qquad k = \overline{1, n_2 - 1}.$$

Thus, from this we obtain the following equation

$$P_{k,0} = \frac{\theta\lambda}{\beta\mu + k\alpha\xi}\Bigg\{(\theta\lambda)^{k-2}\prod_{i=2}^{k-1}\Big(\frac{1}{\beta\mu + i\alpha\xi}\Big) + \delta_1\beta_1^{k-1} + \delta_2\beta_2^{k-1} + \delta_1\sum_{i=1}^{k-2}\frac{(\theta\lambda)^{k-(i+1)}}{\prod_{j=i+1}^{k-1}\Big(\beta\mu + j\alpha\xi\Big)}\beta_1^i$$
$$+\delta_2\sum_{i=1}^{k-2}\frac{(\theta\lambda)^{k-(i+1)}}{\prod_{j=i+1}^{k-1}\Big(\beta\mu + j\alpha\xi\Big)}\beta_2^i\Bigg\}P_{1,0}, \qquad k = \overline{2, n_2 - 1}. \tag{3.13}$$

In the same manner as above, from (3.8), we get

$$P_{n_2,0} = \frac{\theta\lambda}{\beta\mu + n_2\alpha\xi}\Big(P_{n_2-1,0} + P_{n_2-1,1} + P_{n_2-1,2}\Big) = A(n_2, 0)P_{1,0},$$

where

$$A(n_2,0) = \Big(\frac{\theta\lambda}{\beta\mu + n_2\alpha\xi}\Big)\Bigg\{\frac{\theta\lambda}{\beta\mu + (n_2-1)\alpha\xi}\Bigg\{(\theta\lambda)^{n_2-3}\prod_{i=2}^{n_2-2}\Big(\frac{1}{\beta\mu + i\alpha\xi}\Big) + \delta_1\beta_1^{n_2-2} + \delta_2\beta_2^{n_2-2}$$
$$+\delta_1\sum_{i=1}^{n_2-3}\frac{(\theta\lambda)^{n_2-(i+1)}}{\prod_{j=i+1}^{n_2-2}\Big(\beta\mu + j\alpha\xi\Big)}\beta_1^i + \delta_2\sum_{i=1}^{n_2-3}\frac{(\theta\lambda)^{n_2-(i+1)}}{\prod_{j=i+1}^{n_2-2}\Big(\beta\mu + j\alpha\xi\Big)}\beta_2^i\Bigg\} + \delta_1\beta_1^{n_2-1} + \delta_2\beta_2^{n_2-1}\Bigg\}.$$

Now, from (3.9), we have

$$P_{k+1,0} = \frac{\theta\lambda}{\beta\mu + (k+1)\alpha\xi}\Big(P_{k,0} + P_{k,1}\Big), \qquad k = \overline{n_2, n_1 - 1},$$

this gives

$$P_{n_2+l,0} = \left\{ \frac{(\theta\lambda)^l}{\prod_{i=1}^{l}\Big(\beta\mu + (n_2+i)\alpha\xi\Big)} A(n_2,0) + \delta_1 \sum_{j=0}^{l} \frac{(\theta\lambda)^{l-j}}{\prod_{i=j+1}^{l}\Big(\beta\mu + (n_2+i)\alpha\xi\Big)} \beta_1^{n_2+j} \right\} P_{1,0}.$$

Therefore, we obtain

$$P_{k,0} = \left\{ \frac{(\theta\lambda)^{k-n_2}}{\prod_{i=1}^{k-n_2}\Big(\beta\mu + (n_2+i)\alpha\xi\Big)} A(n_2,0) + \delta_1 \sum_{j=0}^{k-n_2} \frac{(\theta\lambda)^{k-n_2-j}}{\prod_{i=j+1}^{k-n_2}\Big(\beta\mu + (n_2+i)\alpha\xi\Big)} \beta_1^{n_2+j} \right\} P_{1,0}, \quad k = \overline{n_2, n_1-1}. \tag{3.14}$$

Next, by using (3.10), we get

$$P_{k+1,0} = \left(\frac{\theta\lambda}{\beta\mu + (k+1)\alpha\xi} \right) P_{k,0}, \;\; k = n_1, n_1+1 \ldots.$$

Recursively, we find

$$P_{n_1+l,0} = \frac{(\theta\lambda)^l}{\prod_{j=1}^{l}\Big(\beta\mu + (n_1+j)\alpha\xi\Big)} P_{n_1,0}.$$

Thus,

$$P_{k,0} = \Phi P_{n_1,0}, \text{ with } \Phi = \frac{(\theta\lambda)^{k-n_1}}{\prod_{j=1}^{k-n_1}\Big(\beta\mu + (n_1+j)\alpha\xi\Big)}, \;\; k = n_1, n_1+1, \ldots. \tag{3.15}$$

Again, via (3.9), we obtain

$$P_{n_1,0} = \left(\frac{\theta\lambda}{\beta\mu + n_1\alpha\xi} \right) \Big(P_{n_1-1,1} + P_{n_1-1,0} \Big),$$

and from (3.14), we have

$$P_{n_1,0} = A(n_1,0) P_{1,0},$$

where

$$A(n_1,0)=\left(\frac{\theta\lambda}{\beta\mu+n_1\alpha\xi}\right)\left\{\frac{(\theta\lambda)^{n_1-n_2-1}}{\prod_{i=1}^{n_1-n_2-1}\Big(\beta\mu+(n_2+i)\alpha\xi\Big)}A(n_2,0)\right.$$

$$\left.+\delta_2\sum_{j=0}^{n_1-n_2-1}\frac{(\theta\lambda)^{n_1-n_2-j-1}}{\prod_{i=j+1}^{n_1-n_2-1}\Big(\beta\mu+(n_2+i)\alpha\xi\Big)}\beta_1^{n_2+j}+\delta_1\beta_1^{n_1-1}\right\}.$$

Therefore,

$$P_{k,0}=\Phi A(n_1,0)P_{1,0},\qquad k=n_1,n_1+1,.... \tag{3.16}$$

Finally, using the normalization condition, we get

$$P_{1,0}+\sum_{k=2}^{n_2-1}P_{k,0}+\sum_{k=n_2}^{n_1-1}P_{k,0}+\sum_{k=n_1}^{+\infty}P_{k,0}+P_{0,1}+\sum_{k=1}^{n_1-1}P_{k,1}+P_{0,2}+\sum_{k=1}^{n_2-1}P_{k,2}=1.$$

Then, using (3.11)-(3.14) and (3.16), we obtain

$$P_{1,0}=\frac{1}{1+B_1+B_2+B_3+B_4+B_5},$$

where

$$B_1=\sum_{k=2}^{n_2-1}\frac{\theta\lambda}{\beta\mu+k\alpha\xi}\left\{(\theta\lambda)^{k-2}\prod_{i=2}^{k-1}\left(\frac{1}{\beta\mu+i\alpha\xi}\right)+\delta_1\beta_1^{k-1}+\delta_2\beta_2^{k-1}\right.$$

$$\left.+\delta_1\sum_{i=1}^{k-2}\frac{(\theta\lambda)^{k-(i+1)}}{\prod_{j=i+1}^{k-1}\Big(\beta\mu+j\alpha\xi\Big)}\beta_1^{i}+\delta_2\sum_{i=1}^{k-2}\frac{(\theta\lambda)^{k-(i+1)}}{\prod_{j=i+1}^{k-1}\Big(\beta\mu+j\alpha\xi\Big)}\beta_2^{i}\right\},$$

$$B_2=\sum_{k=n_2}^{n_1-1}\left\{\frac{(\theta\lambda)^{k-n_2}}{\prod_{i=1}^{k-n_2}\Big(\beta\mu+(n_2+i)\alpha\xi\Big)}A(n_2,0)+\delta_1\sum_{j=0}^{k-n_2}\frac{(\theta\lambda)^{k-n_2-j}}{\prod_{i=j+1}^{k-n_2}\Big(\beta\mu+(n_2+i)\alpha\xi\Big)}\beta_1^{n_2+j}\right\},$$

$$B_3 = \sum_{k=n_1}^{+\infty} \frac{(\theta\lambda)^{k-n_1}}{\prod_{j=1}^{k-n_1}\Big(\beta\mu + (n_1+j)\alpha\xi\Big)} A(n_1,0),\ B_4 = \omega_1 + \delta_1 \sum_{k=1}^{n_1-1} \beta_1^k, \text{ and } B_5 = \omega_2 + \delta_2 \sum_{k=1}^{n_2-1} \beta_2^k.$$

3.4 Performance Measures and Cost Model

In this part of paper, we present different performance measures of the system, and based on these performance indices, we develop a cost model.

3.4.1 Performance measures

Using the steady-state probabilities presented previously, we can obtain useful performance measures of the system that are of general interest.

- The probability that the server is in busy period (P_B).

$$P_B = P_{1,0} + \sum_{k=2}^{n_2-1} P_{k,0} + \sum_{k=n_2}^{n_1-1} P_{k,0} + \sum_{k=n_1}^{+\infty} P_{k,0}.$$

- The probability that the server is on vacation (P_V).

$$P_V = P_{V_1} + P_{V_2} = \sum_{k=0}^{n_1-1} P_{k,1} + \sum_{k=0}^{n_2-1} P_{k,2} = 1 - P_B,$$

where P_{V_1} and P_{V_2} are the probabilities that the server is on type 1 vacation and type 2 vacation, respectively.

- The average number of customers in the system (L_s).

$$L_s = P_{1,0} + \sum_{k=2}^{n_2-1} kP_{k,0} + \sum_{k=n_2}^{n_1-1} kP_{k,0} + \sum_{k=n_1}^{+\infty} kP_{k,0} + \sum_{k=0}^{n_1-1} kP_{k,1} + \sum_{k=0}^{n_2-1} kP_{k,2}.$$

- The average number of customers in the queue (L_q).

$$L_q = \sum_{k=2}^{n_2-1} (k-1)P_{k,0} + \sum_{k=n_2}^{n_1-1} (k-1)P_{k,0} + \sum_{k=n_1}^{+\infty} (k-1)P_{k,0} + \sum_{k=1}^{n_1-1} kP_{k,1} + \sum_{k=1}^{n_2-1} kP_{k,2}.$$

- The average reneging rate (R_{ren}).

$$R_{ren} = \alpha\xi\left(P_{1,0} + \sum_{k=2}^{n_2-1} kP_{k,0} + \sum_{k=n_2}^{n_1-1} kP_{k,0} + \sum_{k=n_1}^{+\infty} kP_{k,0}\right).$$

- The average retention rate (R_{ret}).

$$R_{ret} = \alpha'\xi\left(P_{1,0} + \sum_{k=2}^{n_2-1} kP_{k,0} + \sum_{k=n_2}^{n_1-1} kP_{k,0} + \sum_{k=n_1}^{+\infty} kP_{k,0}\right).$$

- The average balking rate (B_r).

$$B_r = \theta'\lambda\left(P_{1,0} + \sum_{k=2}^{n_2-1} P_{k,0} + \sum_{k=n_2}^{n_1-1} P_{k,0} + \sum_{k=n_1}^{+\infty} P_{k,0} + \sum_{k=1}^{n_1-1} P_{k,1} + \sum_{k=1}^{n_2-1} P_{k,2}\right).$$

- The expected number of customers served per unit of time (E_{cs}).

$$E_{cs} = \beta\mu\left(P_{1,0} + \sum_{k=2}^{n_2-1} kP_{k,0} + \sum_{k=n_2}^{n_1-1} kP_{k,0} + \sum_{k=n_1}^{+\infty} kP_{k,0}\right).$$

3.4.2 Cost Model

In this subpart, we develop a model for the costs incurred in the queueing system, to this end let us define

- C_b : Cost per unit time when the server is busy.
- C_v : Cost per unit time when the server is on type 1 /type 2 vacation.
- C_q : Cost per unit time when a customer joins the queue and waits for service.
- C_s : Cost per service per unit time.
- C_{ren} : Cost per unit time when a customer reneges.
- C_{ret} : Cost per unit time when a customer is retained.
- C_{s-f} : Cost per unit time when a customer returns to the system as a feedback customer.

- C_{br} : Cost per unit time when a customer balks.

Let
• R be the revenue earned by providing service to a customer.

• Γ be the total expected cost per unit time of the system.

$$\Gamma = C_b P_B + C_v P_V + C_q L_q + C_{br} B_r + C_{ren} R_{ren} + C_{ret} R_{ret} + \mu(C_s + \beta' C_{s-f}).$$

• Δ be the total expected revenue per unit time of the system.

$$\Delta = R\mu P_B.$$

• Θ be the total expected profit per unit time of the system.

$$\Theta = \Delta - \Gamma.$$

3.5 Numerical analysis

In this section, we perform a sensitive numerical analysis to illustrate the impact of the system parameters on different performance measures and the total expected profit. The computations are given by developing a program in R software. For the whole numerical study we fix the costs at $C_b = 5$, $C_v = 2$, $C_q = 3$, $C_s = 4$, $C_{ren} = 3$, $C_{ret} = 5$, $C_{br} = 3$, $C_{s-f} = 2$, and $R = 25$.

Impact of arrival and service rates (λ) and (μ)

To examine the impact of λ and μ, we consider the following cases.

- Tables 3.1 and 3.3: $\lambda = 1.00 : 0.40 : 3.40$, $\mu = 5.00$, $\xi = 0.50$, $\phi_1 = 0.60$, $\phi_2 = 0.90$, $n_1 = 8$, $n_2 = 4$, $\theta = 0.80$, $\alpha = 0.50$, and $\beta = 0.60$.

- Tables 3.2 and 3.4: $\lambda = 3.00$, $\mu = 3.00 : 0.50 : 6.00$, $\xi = 1.00$, $\phi_1 = 1.00$, $\phi_2 = 0.80$, $n_1 = 8$, $n_2 = 4$, $\theta = 0.9$, $\alpha = 0.5$, and $\beta = 0.60$.

From Tables 3.1-3.4 and Figures 3.1-3.2 we remark that along the increases of the arrival (resp. service) rate λ (resp. μ), L_s and L_q increase (resp. decrease), this leads to an increase (resp. a decrease) in P_B, B_r, R_{ren} and R_{ret} , as it should be. Further, increases in λ and μ implies an increase in E_{cs} which results in the increasing of Γ, Δ and Θ, respectively, as intuitively expected.

Table 3.1: System performance measures vs. λ.

λ	P_B	L_s	L_q	B_r	R_{ren}	R_{ret}	E_{cs}
1.00	0.230323	1.372887	1.142564	0.135705	0.092503	0.026772	1.406790
1.40	0.301415	1.795022	1.493607	0.213454	0.155046	0.062617	2.167688
1.80	0.368932	2.155418	1.786486	0.293900	0.226351	0.111582	3.021292
2.20	0.434706	2.479265	2.044559	0.375951	0.306408	0.173462	3.972506
2.60	0.499547	2.790536	2.290989	0.459213	0.396352	0.249569	5.036779
3.00	0.563671	3.111412	2.547742	0.543514	0.498175	0.342365	6.238812
3.40	0.626857	3.464063	2.837207	0.628736	0.614725	0.455287	7.613304

Table 3.2: System performance measures vs. μ.

μ	P_B	L_s	L_q	B_r	R_{ren}	R_{ret}	E_{cs}
3.00	0.710071	3.004134	2.294063	0.275214	1.175551	0.736414	4.425961
3.50	0.661448	2.833762	2.172313	0.271057	1.042867	0.623889	4.613809
4.00	0.617257	2.701564	2.084307	0.267279	0.934458	0.537627	4.756190
4.50	0.577443	2.597701	2.020258	0.263876	0.845058	0.470577	4.868113
5.00	0.541695	2.514913	1.973218	0.260820	0.770520	0.417614	4.958953
5.50	0.509605	2.447932	1.938327	0.258076	0.707664	0.375074	5.034655
6.00	0.480755	2.392949	1.912194	0.255610	0.654072	0.340346	5.099086

Table 3.3: Γ, Δ and Θ for different values of λ.

λ	1.00	1.40	1.80	2.20	2.60	3.00	3.60
Γ	27.509454	28.322831	29.22546	30.21851	31.31318	32.52791	33.88739
Δ	28.79041	37.67687	46.11648	54.33831	62.44340	70.45883	78.35708
Θ	1.280961	9.354035	16.89102	24.11980	31.13022	37.93092	44.46970

Table 3.4: Γ, Δ and Θ for different values of μ.

μ	3.00	3.50	4.00	4.50	5.00	5.50	6.00
Γ	26.56458	27.84556	29.34512	31.01202	32.80717	34.70141	36.67304
Δ	53.25530	57.87673	61.72574	64.96237	67.71184	70.07072	72.11331
Θ	26.69072	30.03116	32.38062	33.95035	34.90466	35.36931	35.44027

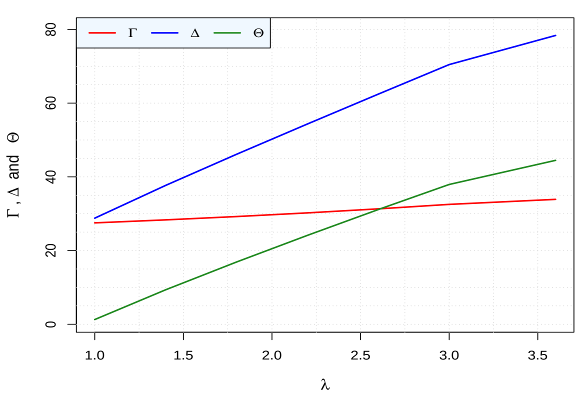

Figure 3.1: Γ, Δ and Θ vs. λ.

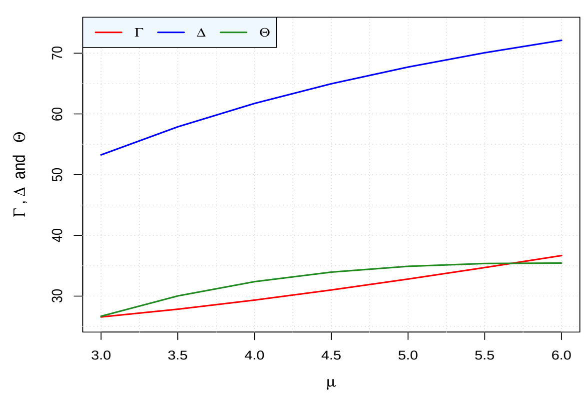

Figure 3.2: Γ, Δ and Θ vs. μ.

Impact of impatience rate (ξ)

To study the impact of ξ, we put $\lambda = 3.00$, $\mu = 5.00$, $\xi = 0.50 : 0.50 : 3.50$, $\phi_1 = 1.00$, $\phi_2 = 0.80$, $n_1 = 8$, $n_2 = 5$, $\theta = 0.90$, $\alpha = 0.40$, and $\beta = 0.60$.

Table 3.5: System performance measures vs. ξ

ξ	P_B	L_s	L_q	B_r	R_{ren}	R_{ret}	E_{cs}
0.50	0.669285	3.257864	2.588578	0.272498	0.506168	0.405302	7.850343
1.00	0.583702	2.678804	2.095103	0.265381	0.708000	0.387606	5.615459
1.50	0.525868	2.444018	1.918150	0.260572	0.849241	0.358569	4.574773
2.00	0.481555	2.313908	1.832353	0.256887	0.956366	0.336779	3.926740
2.50	0.445609	2.230707	1.785098	0.253898	1.040777	0.322067	3.468099
3.00	0.415479	2.172923	1.757444	0.251392	1.108931	0.312406	3.119526
3.50	0.389669	2.130578	1.740910	0.249246	1.164959	0.306131	2.842394

Table 3.6: Γ, Δ and Θ for different values of ξ.

ξ	0.50	1.00	1.50	2.00	2.50	3.00	3.50
Γ	32.37036	32.60928	32.69989	32.76832	32.83119	32.88943	32.94227
Δ	83.66068	72.96272	65.73355	60.19434	55.70117	51.93488	48.70858
Θ	51.29032	40.35345	33.03366	27.42602	22.86998	19.04544	15.76630

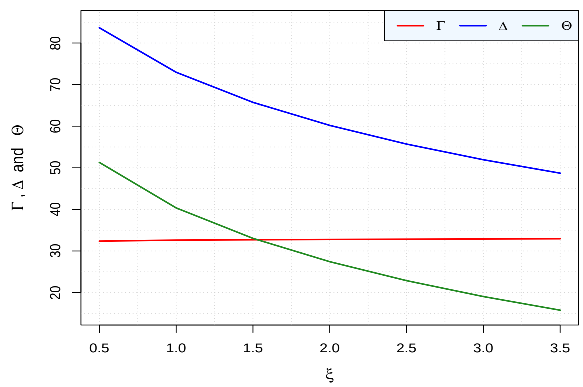

Figure 3.3: Γ, Δ and Θ vs. ξ.

Tables 3.5-3.6 and Figure 3.3 depict the impact of the impatience rate ξ, it is well shown that L_s, L_q, P_B, B_r and R_{ret} all decrease with ξ, whereas R_{ren} increases with the parameter ξ, which is quite reasonable; the number of reneged customers increases with the parameter ξ, this implies a decrease in the number of customers in the system, therefore the probability of busy period, the average rates of balking and retention decrease also. Further, as intuitively expected, the mean number of customers served E_{cs} decreases as ξ increases. Moreover, a decreasing trend is observed in Δ and Θ with the increase of ξ, while Γ increases with ξ. This is because the number of customers in the system decreases with ξ, which results in the decreasing of the number of customers served.

Impact of type 1 and type 2 vacation rates (ϕ_1) and (ϕ_2) as well as type 1 and type 2 vacation interruption thresholds (n_1) and (n_2)

To examine the effect of ϕ_1 and n_1 (resp. ϕ_2 and n_2), we consider the following cases.

- Tables 3.7 and 3.9: $\lambda = 3.00$, $\mu = 5.00$, $\xi = 1.00$, $\phi_1 = 0.50 : 0.50 : 2.50$, $\phi_2 = 2.00$, $n_1 = 10 : 5 : 20$, $n_2 = 8$, $\theta = 0.70$, $\alpha = 0.50$, and $\beta = 0.60$.
- Tables 3.8 and 3.10: $\lambda = 3.00$, $\mu = 5.00$, $\xi = 1.00$, $\phi_1 = 2.00$, $\phi_2 = 0.50 : 0.50 : 2.50$, $n_1 = 50$, $n_2 = 5 : 5 : 15$, $\theta = 0.70$, $\alpha = 0.50$, and $\beta = 0.60$.

Table 3.7: System performance measures vs. (ϕ_1, n_1).

ϕ_1	n_1	P_B	L_s	L_q	B_r	R_{ren}	R_{ret}	E_{cs}
	10	0.422135	3.056662	2.634527	0.807175	0.762541	0.041587	4.070553
0.50	15	0.411867	3.606383	3.194516	0.813177	0.830683	0.117194	4.394589
	20	0.407638	3.866849	3.459211	0.814921	0.854130	0.141596	4.506980
	10	0.472532	2.311160	1.838628	0.766424	0.698528	0.014798	3.863686
1.00	15	0.470614	2.422226	1.951613	0.768089	0.716322	0.032281	3.948029
	20	0.470247	2.444240	1.973993	0.768299	0.718895	0.034705	3.960313
	10	0.492016	2.058173	1.566157	0.745009	0.665373	0.009182	3.757397
1.50	15	0.491049	2.117313	1.630392	0.745913	0.675273	0.018766	3.804383
	20	0.490863	2.128470	1.637607	0.746020	0.676585	0.019998	3.810648
	10	0.511499	1.805185	1.293686	0.723594	0.632218	0.003574	3.651108
2.00	15	0.511483	1.812400	1.300917	0.723737	0.634223	0.005250	3.660736
	20	0.511479	1.812700	1.301221	0.723741	0.634275	0.005293	3.660982
	10	0.518436	1.728032	1.209596	0.713948	0.618543	0.002651	3.609525
2.50	15	0.518477	1.730849	1.212372	0.714008	0.619633	0.003511	3.610423
	20	0.518477	1.730899	1.212422	0.714009	0.619643	0.003519	3.614856

Table 3.8: System performance measures vs. (ϕ_2, n_2).

ϕ_2	n_2	P_B	L_s	L_q	B_r	R_{ren}	R_{ret}	E_{cs}
	5	0.470013	1.950241	1.480227	0.754515	0.611339	0.064832	3.460818
0.50	10	0.459874	2.637000	2.177127	0.776899	0.747348	0.005696	4.078685
	15	0.451622	3.000162	2.548541	0.781909	0.789144	0.001152	4.273749
	5	0.486758	1.828364	1.341606	0.737770	0.599065	0.057652	3.445963
1.00	10	0.489732	2.141068	1.651336	0.750115	0.688883	0.002192	3.860765
	15	0.488463	2.209231	1.720767	0.751243	0.699727	0.000151	3.911848
	5	0.495101	1.770043	1.274942	0.728004	0.591389	0.054956	3.434713
1.50	10	0.500803	1.984356	1.483553	0.737144	0.664613	0.001660	3.775444
	15	0.500196	2.021283	1.521091	0.737766	0.670098	0.000090	3.805642
	5	0.503443	1.711721	1.208277	0.718237	0.583713	0.052259	3.423462
2.00	10	0.511874	1.827644	1.315770	0.724172	0.640343	0.001100	3.690123
	15	0.511920	1.833334	1.321414	0.724288	0.642269	0.000029	3.699435
	5	0.507877	1.682327	1.174450	0.712258	0.578870	0.051283	3.415855
2.50	10	0.516690	1.771559	1.254869	0.717204	0.629459	0.001039	3.655063
	15	0.516777	1.774486	1.257708	0.717268	0.630837	0.000027	3.661772

Tables 3.7-3.10 and Figures 3.4-3.5 illustrate the effects of type 1 and type 2 vacation rates (ϕ_1) and (ϕ_2) as well as type 1 and type 2 vacation interruption thresholds (n_1) and (n_2), respectively. From the results obtained we have

Table 3.9: Γ, Δ and Θ for different values of (ϕ_1, n_1).

	$n_1 \backslash \phi_1$	0.50	1.00	1.50	2.00	2.50
Γ	10	32.18349	31.88644	31.76423	31.61980	31.56603
	15	32.75315	32.02648	31.84164	31.63458	31.57391
	20	32.93805	32.04585	31.86150	31.63495	31.57398
Δ	10	52.76687	59.06653	61.51205	63.93735	64.80445
	15	51.48339	58.82670	61.39214	63.93536	64.80967
	20	50.95472	58.78082	61.36894	63.93485	64.80960
Θ	10	20.58338	27.18009	29.65993	32.31755	33.23842
	15	18.73024	26.80022	29.56162	32.30078	33.23576
	20	18.01667	26.73497	29.52854	32.29990	33.23562

Table 3.10: Γ, Δ and Θ for different values of (ϕ_2, n_2).

	$n_2 \backslash \phi_2$	0.50	1.00	1.50	2.00	2.50
Γ	5	31.83176	31.75904	31.72637	31.67748	31.65343
	10	31.98084	31.79715	31.72997	31.63467	31.59526
	15	32.07378	31.81905	31.73842	31.63558	31.59478
Δ	5	58.75166	60.84476	61.89759	62.93042	63.48457
	10	57.48420	61.21652	62.70046	63.98421	64.58630
	15	56.45270	61.05793	62.62398	63.99004	64.59716
Θ	5	26.91990	29.08572	30.26844	31.25294	31.83114
	10	25.50336	29.41937	30.89556	32.34954	32.99105
	15	24.37892	29.23887	30.80778	32.35447	33.00238

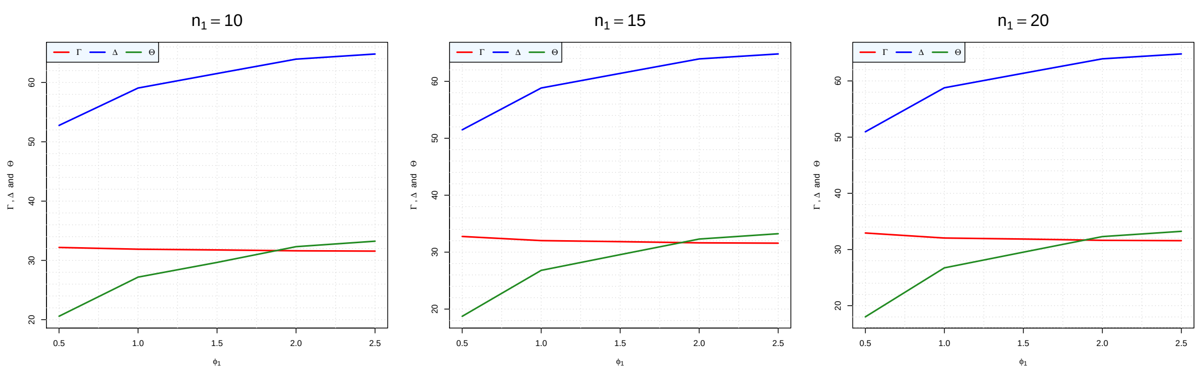

Figure 3.4: Γ, Δ and Θ vs. (ϕ_1, n_1).

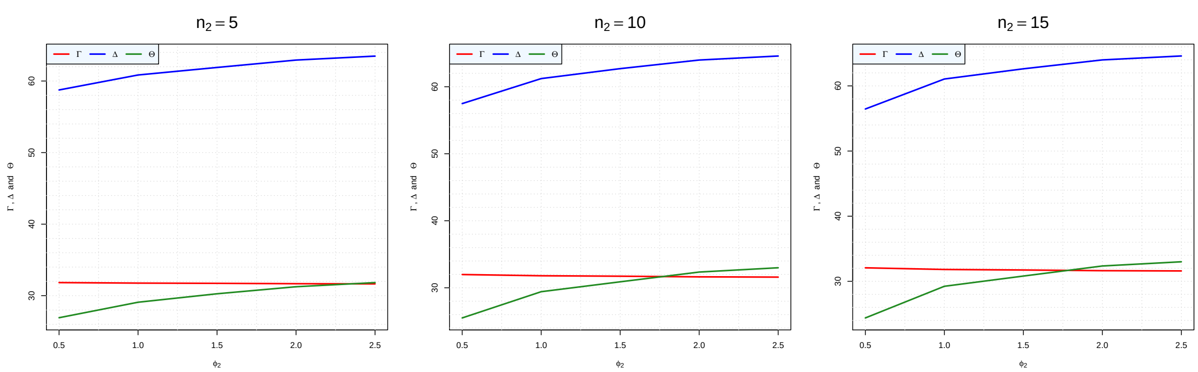

Figure 3.5: Γ, Δ and Θ vs. (ϕ_2, n_2).

- For fixed $n_i,\ i = 1,2$, along the increases of $\phi_i,\ i = 1,2$, a decreasing trend is observed in L_s, L_q, B_r, R_{ren} and R_{ret}, as intuitively expected. Whereas P_B increases with $\phi_i,\ i = 1,2$, this makes a perfect sense; the smaller the mean vacation time, the bigger the busy period. Moreover, the number of customers in the system decreases with vacation rates whatever the type of vacation which leads to the increase in the probability of busy period. Therefore, the average rates of balking, reneging and retention decrease.

- A decreasing trend is seen in E_{cs} with $\phi_i, i = 1,2$, this can be explained by the fact that as $\phi_i,\ i = 1,2$, increases, the vacation period decreases and the server switches to busy period during which customers may be impatient and leave the system, taking into account that the interruption thresholds $n_i,\ i = 1,2$, are taken slightly large. But, for appropriate values of different costs, this slight lost in the mean number of customers served does not affect negatively the total expected profit.

- As intuitively expected, for suitable values of the costs, Γ decreases as $\phi_i,\ i = 1,2$, increases, while Δ and Θ increase significantly with the increasing values of $\phi_i,\ i = 1,2$.

- For fixed $\phi_i, i = 1,2$, the performance measures L_s, L_q, B_r, R_{ren} and E_{cs} increase with $n_i, i = 1,2$. While R_{ret} increases with n_1 and decreases with n_2. This is because of the values chosen of vacation rates and vacation interruption thresholds.

- P_B decreases with n_1 when $\phi_1 = 0.5, 1.0, 1.5, 2.0$ and increases when $\phi_1 = 2.5$, while it decreases with n_2 when $\phi_2 = 0.5$, it is not monotone when $\phi_2 = 1.0, 1.5$ and increases when $\phi_2 = 2.0, 2.5$. This is due to the choice of the vacation rates and vacation interruption thresholds. Moreover, the higher the vacation rates, the greater the probability of busy period.

- For fixed $\phi_i,\ i = 1,2$, along the increasing of n_1, Γ increases, while it increases with n_2 when $\phi_2 = 0.5, 1.0, 1.5$, it is not monotone when $\phi_2 = 2$ and decreases in the case where $\phi_2 = 2.5$. Further, Δ decreases with n_1 when $\phi_1 = 0.5, 1.0, 1.5, 2.0$, and not monotone when $\phi_1 = 2.5$. Whereas it decreases with n_2 when $\phi_2 = 0.5$, not monotone when $\phi_1 = 1.0, 1.5$ and increases when $\phi_2 = 2.0, 2.5$. As a consequence, Θ monotonically decreases with n_1. While it decreases with the increasing values of n_2, when $\phi_2 = 0.50$, it is not monotone in the case where

$\phi_2 = 1.0, 1.5$, and increases when $\phi_2 = 2.0, 2.5$. Thus, as it is seen from the results obtained, early interruption of type 1 vacation has a significant impact on the total expected profit. Whereas, Θ is not monotone with type 2 vacation interruption threshold, this is due to the impatience phenomenon during busy period and the choice of type 1 and type 2 of vacation rates as well as type 1 and type 2 vacation interruption thresholds. However, theoretically, for our queueing model, it is not easy to obtain vacation rates and vacation interruption thresholds associated that affect positively the total expected profit.

Impact of non-balking probability (θ)

In order to check the effect of θ, we put $\lambda = 3.00$, $\mu = 5.00$, $\xi = 0.50$, $\phi_1 = 1.00$, $\phi_2 = 0.80$, $n_1 = 8$, $n_2 = 4$, $\theta = 0.10 : 0.10 : 0.90$, $\alpha = 0.50$, and $\beta = 0.60$.

Table 3.11: System performance measures vs. θ.

θ	P_B	L_s	L_q	B_r	R_{ren}	R_{ret}	E_{cs}
0.10	0.235634	1.075323	0.839689	2.204897	0.037174	0.001374	0.953884
0.20	0.287360	1.316758	1.029399	1.983997	0.078158	0.009631	1.417905
0.30	0.337355	1.543214	1.205859	1.755245	0.124356	0.027722	1.946898
0.40	0.386823	1.763335	1.376512	1.520767	0.176892	0.056402	2.552293
0.50	0.436370	1.984218	1.547848	1.281226	0.236544	0.096141	3.242417
0.60	0.486216	2.213723	1.727507	1.036649	0.304142	0.147718	4.026670
0.70	0.536334	2.460722	1.924388	0.786733	0.380768	0.212365	4.917727
0.80	0.586532	2.735448	2.148916	0.531003	0.467924	0.291853	5.933539
0.90	0.636495	3.050194	2.413699	0.268924	0.567734	0.388606	7.099662

Table 3.12: Γ, Δ and Θ for different values of θ.

θ	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
Γ	33.4399	33.0967	32.7894	32.5354	32.3431	32.2196	32.1733	32.1656	32.1614
Δ	29.4542	35.9199	42.1693	48.3529	54.5462	60.7769	67.0417	73.3164	79.5618
Θ	-3.9857	2.8232	9.3798	15.8174	22.2031	28.5573	34.8684	41.1008	47.1993

Clearly, from Tables 3.11-3.12 and Figure 3.6, we see that along the increasing of non-balking probability θ, the performance measures L_s, L_q, P_B, R_{ren} and R_{ret} all increase, while B_r decreases with θ, as it should be. Further, it is consistent with our intuition that the larger the probability of non-balking, the higher the number of customers served E_{cs}. This leads to a decrease in Γ and a significant increase in Δ and Θ. This trend matches absolutely with the realistic situation.

Impact of non-retention and non-feedback probabilities (α) and (β)

To examine the impact of α and of β, we consider the following cases.

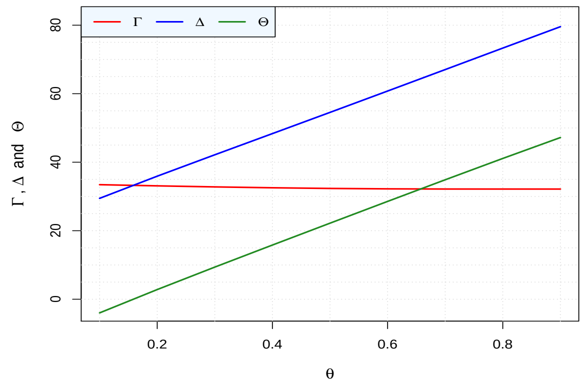

Figure 3.6: Γ, Δ and Θ vs. θ.

- Tables 3.13 and 3.14: $\lambda = 3.00$, $\mu = 5.00$, $\xi = 0.50$, $\phi_1 = 1.00$, $\phi_2 = 0.80$, $n_1 = 8$, $n_2 = 4$, $\theta = 0.70$, $\alpha = 0.10 : 0.10 : 0.90$, and $\beta = 0.60$.
- Tables 3.15 and 3.16: $\lambda = 3.00$, $\mu = 5.00$, $\xi = 0.50$, $\phi_1 = 1.00$, $\phi_2 = 0.80$, $n_1 = 8$, $n_2 = 5$, $\theta = 0.70$, $\alpha = 0.60$, and $\beta = 0.10 : 0.10 : 0.90$.

Table 3.13: System performance measures vs. α.

α	P_B	L_s	L_q	B_r	R_{ren}	R_{ret}	E_{cs}
0.10	0.639608	3.193360	2.553752	0.811962	0.122932	0.811869	7.664556
0.20	0.605978	2.890069	2.284090	0.803746	0.208848	0.565704	6.575937
0.30	0.578958	2.698767	2.119809	0.797146	0.276595	0.407173	5.858417
0.40	0.556147	2.563201	2.007054	0.791573	0.332776	0.295334	5.330477
0.50	0.536334	2.460722	1.924388	0.786733	0.380768	0.212365	4.917727
0.60	0.518794	2.380015	1.861221	0.782448	0.422616	0.148776	4.582377
0.70	0.503048	2.314623	1.811575	0.778602	0.459682	0.098878	4.302513
0.80	0.488761	2.260510	1.771750	0.775112	0.492922	0.058994	4.064236
0.90	0.475682	2.214987	1.739305	0.771917	0.523035	0.026623	3.858153

Table 3.14: Γ, Δ and Θ for different values of α.

α	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
Γ	34.7828	33.6842	32.9939	32.5181	32.1733	31.9154	31.7183	31.5653	31.44501
Δ	79.9510	75.7473	72.3698	69.5184	67.0417	64.8491	62.8810	61.0950	59.46025
Θ	45.1681	42.0630	39.3758	37.0002	34.8684	32.9337	31.1626	29.5297	28.01524

From Tables 3.13-3.16 and Figures 3.7-3.8, it is easy to observe that along the increases of α and β, the performance measures L_s, L_q, P_B, B_r and R_{ret} all decrease, as it should be. Further, an increase in the probability of non-feedback β leads to a decrease in R_{ren}, this implies an increase in E_{cs}. However, the increase in the probability of non-retention α implies an increase in R_{ren}, which implies a decrease in E_{cs}. Consequently, the increases in α and β generates a decrease in Γ, Δ and Θ, respectively. The negative impact of these two probabilities is quite clear in the economy.

Table 3.15: System performance measures vs. β.

β	P_B	L_s	L_q	B_r	R_{ren}	R_{ret}	E_{cs}
0.10	0.973630	5.870646	4.897016	0.893725	1.738876	0.969416	2.911200
0.20	0.895094	4.389540	3.494445	0.875038	1.240081	0.559856	4.197609
0.30	0.786456	3.426369	2.639913	0.849189	0.882475	0.310923	4.553516
0.40	0.683629	2.893932	2.210303	0.824722	0.661805	0.186544	4.630228
0.50	0.598166	2.597724	1.999559	0.804386	0.524616	0.124672	4.656364
0.60	0.529217	2.420934	1.891718	0.787980	0.433937	0.091435	4.678825
0.70	0.473423	2.306964	1.833540	0.774705	0.370096	0.071817	4.702464
0.80	0.427720	2.228476	1.800756	0.763830	0.322769	0.059193	4.725824
0.90	0.389754	2.171503	1.781748	0.754796	0.286256	0.050465	4.747600

Table 3.16: Γ, Δ and Θ for different values of β.

β	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
Γ	46.6657	41.8299	38.1089	35.4431	33.4048	31.7105	30.2137	28.8389	27.5447
Δ	121.703	111.886	98.3069	85.4536	74.7707	66.1520	59.1779	53.4649	48.7192
Θ	75.0379	70.0568	60.1980	50.0104	41.3658	34.4415	28.9641	24.6260	21.1745

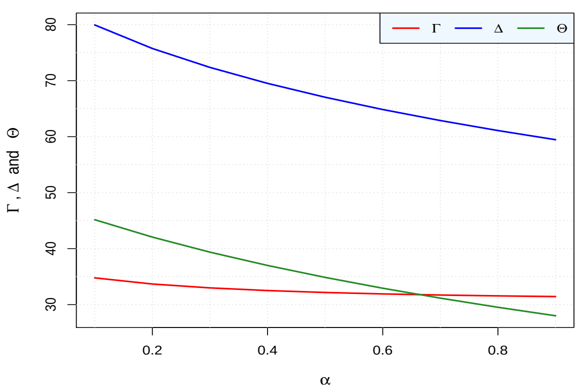

Figure 3.7: Γ, Δ and Θ vs. α.

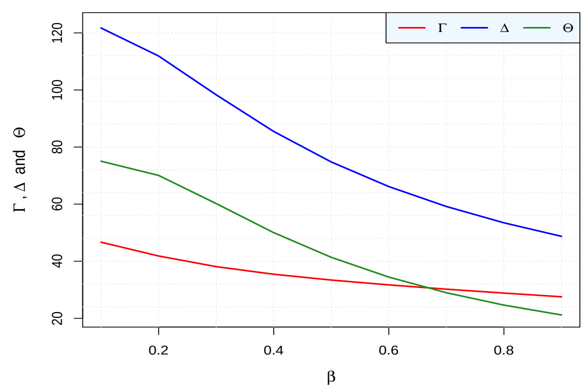

Figure 3.8: Γ, Δ and Θ vs. β.

3.6 Conclusion

In this paper we presented a sensitivity study of an infinite-buffer single server queueing system with Bernoulli feedback, multiple vacations, differentiated vacations, vacation interruptions, balking and reneging. Via certain mechanism, reneged customers may be retained in the system. Using the recursive approach, the exact expressions of the steady-state probabilities are obtained. Moreover, explicit expressions of useful performance measures are derived, and a cost model is developed. Finally, a variety of numerical results has been discussed.

For further work, it would be interesting to extend our results to a non-Markovian models which takes the system to more realistic environment. Furthermore, the model considered in the present investigation can be extended to a more general case with multiple servers, batch arrivals and servers subject to breakdowns with repairs.

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Chapter 4

Performance and economic analysis of Markovian Bernoulli feedback queueing system with vacations, waiting server and impatient customers

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Performance and economic analysis of Markovian Bernoulli feedback queueing system with vacations, waiting server and impatient customers

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Abstract. This paper concerns the analysis of a Markovian queueing system with Bernoulli feedback, single vacation, waiting server and impatient customers. We suppose that whenever the system is empty the sever waits for a random amount of time before he leaves for a vacation. Moreover, the customer's impatience timer depends on the states of the server. If the customer's service has not been completed before the impatience timer expires, the customer leaves the system, and via certain mechanism, impatient customer may be retained in the system. We obtain explicit expressions for the steady-state probabilities of the queueing model, using the probability generating function (PGF). Further, we obtain some important performance measures of the system and formulate a cost model. Finally, an extensive numerical study is illustrated.

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4.1 Introduction

Queueing models with vacations have a great impact in many real life situations, such models occur naturally in different fields such as computer and communication systems, flexible manufacturing systems, telephone services, production line systems, machine operating systems, post offices, etc. Over the past few decades, vacation queueing systems have paid attention of many researchers, excellent surveys on queueing systems with vacations can be found in Doshi (1986) and Takagi (1991) and in the monographs of Tian (2006) and Ke (2010). In recent years, there has been gowning interest in the study of queueing systems with impatient customers (balking and reneging). For related literature, interested readers may refer to Shin and Choo (2009), El-Paoumy and Nabwey (2011), Kumar et al. (2014), Kumar and Sharma (2014), Bouchentouf et al. (2014), Baek et al. (2017), Bouchentouf and Messabihi (2018) and references therein.

The studies of queueing models with impatient customers were ranked depending on the causes of the impatience behavior. In queueing literature, models where customers may be impatiens because of server vacations have been extensively analyzed. Yue et al. (2006) presented the optimal performance analysis of an $M/M/1/N$ queueing system with balking, reneging and server vacation. Altman and Yechiali (2006) gave the analysis of some queueing models such as $M/M/1$, $M/G/1$ and $M/M/c$ queues with server vacations and customer impatience, both single and multiple vacation cases were studied. Further, Altman and Yechiali (2008) investigated the infinite server queue with vacations and impatient customers. They obtained the probability generating function of the number of customers in the model and derived the performance measures of the system. Queueing systems with vacations and synchronized reneging have been done by Adan et al. (2009). Wu and Ke (2010) presented computational algorithm and parameter optimization for a multi-server system with unreliable servers and impatient customers. Later, the model given in Altman and Yechiali (2006) were extended by Yue et al. (2014) by considering a variant of the multiple vacation policy which includes both single vacation and multiple vacations. In Padmavathy et al. (2011), authors studied the steady state behavior of the vacation queues with impatient customers and a waiting server. Further, the transient solution of a $M/M/1$ multiple vacation queueing model with impatient customers has been investigated by Ammar (2015). Then, a study of single server Markovian queueing system with vacations and impatience timers which depend of the state of the server was presented in Yue et al. (2016). Recently, in Ammar (2017), author established the transient solution of an $M/M/1$ vacation queue with a waiting server and impatient customers.

The main objective of this article is to study an $M/M/1$ vacation queueing system with Bernoulli feedback, waiting server, reneging, and retention of reneged customers. It is supposed that whenever the busy period ended the server waits a random duration of time before beginning on a vacation. Moreover, we assume that the impatience timers of customers depend on the server's states. We obtain the steady-state solution of the queueing model, using the probability generating function (PGF). Further, we give explicit expressions of useful measures of effectiveness and formulate a cost model. Then, we present a sensitive numerical experiments to illuminate the interests of our theoretical results and to show the impact of the diverse parameters on the behavior of the system. Finally, an appropriate economic analysis is carried out numerically.

The model analyzed in this paper has a number of applications in practice. In most studies cited earlier, authors considered that the server leaves the system once the system is empty, but in many practical life situations the server waits a certain period of time before he leaves the system even if there is no customers, especially when we deal with a human behavior, examples can be found in post offices, banks, hospitals, etc.

Further, our study has another great scope, in most studies mentioned in the above literature, the basis of the research is the supposition that customers may be impatient because of server vacations. However, there are many situations where the customer can become impatient due to the long wait in the queue even if the server is present in the system, another example when the customer may leave the system during busy period is when he cannot see the server state, these situations can be found in telecommunication systems, call centers and production inventory systems.

The rest of the paper is organized in the following manner. In Section 2, we describe the model. In Section 3, we present the stationary analysis for the queueing model. In Section 4, we obtain different performance measures and formulate a cost model. Section 5 presents numerical results in the form of Tables and Figures. Finally, in Section 6 we conclude the paper.

4.2 System model

Consider a $M/M/1$ vacation queueing model with Bernoulli feedback, waiting server, reneging and retention of reneged customers. The model studied in this paper is based on following assumptions:

$*$ Customers arrive into the system according to a Poisson process with arrival rate λ, and service time is assumed to be exponentially distributed with parameter μ. The service discipline is FCFS and there is infinite space for customers to wait.

* When the busy period is finished the server waits a random duration of time before beginning on a vacation. This waiting duration is exponentially distributed with parameter η.

* If the server comes back from a vacation to an empty system he waits passively the first arrival, then he begins service. Otherwise, if there are customers waiting in the queue at the end of a vacation, the server starts immediately a busy period. That is single vacation policy. The period of vacation has an exponential distribution with parameter γ.

* Whenever a customer arrives at the system and finds the server on vacation (respectively. busy), he activates an impatience timer T_0 (respectively. T_1), which is exponentially distributed with parameter ξ_0 (respectively. ξ_1). If the customer's service has not been completed before the impatience timer expires, the customer may abandon the queue. We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers.

* Each reneged customer may leave the system without getting service with probability α and may be retained in the system with probability $\alpha' = (1-\alpha)$.

* After completion of each service, the customer can either leave the system definitively with probability β or return to the system and join the end of the queue with probability β', where $\beta + \beta' = 1$.

4.3 Stationary analysis

In this section, we use the probability generating function (PGF) to obtain the steady-state solution of the queueing system.

Let $L(t)$ be the number of customers in the system at time t, and $J(t)$ denotes the state of the server at time t such that

$$J(t) = \begin{cases} 1, & \text{when the server is in a busy period;} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the process $\{(L(t); J(t)); t \geq 0\}$ is a continuous-time Markov process with state space

$$\Omega = \{(j,n) : j = 0,1, \ n = 0,1,...\}.$$

Let $P_{j,n} = \lim_{t\to\infty} P\{J(t) = j, L(t) = n\}, \ j = 0,1, n = 0,1,..., \ (j,n) \in \Omega$, denote the system state probabilities.

Then, the steady-state balance equations of our model are given as follows:

$$(\lambda+\gamma)P_{0,0} = \alpha\xi_0 P_{0,1} + \eta P_{1,0}, \tag{4.1}$$

$$(\lambda+\gamma+n\alpha\xi_0)P_{0,n} = \lambda P_{0,n-1} + (n+1)\alpha\xi_0 P_{0,n+1}, \;\; n\geq 1, \tag{4.2}$$

$$(\lambda+\eta)P_{1,0} = \gamma P_{0,0} + (\beta\mu+\alpha\xi_1)P_{1,1}, \tag{4.3}$$

$$(\lambda+\beta\mu+n\alpha\xi_1)P_{1,n} = \lambda P_{1,n-1} + \gamma P_{0,n} + (\beta\mu+(n+1)\alpha\xi_1)P_{1,n+1}, \quad n\geq 1, \tag{4.4}$$

Theorem 4.3.1. *If we have a single server Bernoulli feedback queueing system with single vacation, waiting server, server's states-dependent reneging and retention of reneged customers, then*

1. *The steady-state probability $P_{0,.}$ is given by*

$$P_{0,.} = \left(\frac{\gamma\alpha\xi_0 + \delta_1 K_0(1)(1-\gamma)}{\gamma K_0(1)}\right)P_{0,0}. \tag{4.5}$$

2. *The steady-state probability $P_{1,.}$ is given by*

$$\begin{aligned} P_{1,.} = e^{\frac{\lambda}{\alpha\xi_1}}\Bigg(&\frac{\gamma}{\lambda+\eta}\left(\frac{\beta\mu}{\alpha\xi_1}K_1(1)+\frac{\eta}{\alpha\xi_1}K_2(1)\right)-\frac{\gamma}{\alpha\xi_1}K_3(1) \\ &+\frac{\beta\mu+\alpha\xi_1}{\lambda+\eta}\left(\frac{\beta\mu}{\alpha\xi_1}K_1(1)+\frac{\eta}{\alpha\xi_1}K_2(1)\right)\left(\frac{\alpha\xi_0-\delta_1K_0(1)}{\delta_2K_0(1)}\right)\Bigg)P_{0,0}, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} P_{0,0} = \Bigg\{&\frac{\delta_1\delta_2K_0(1)+\delta_2(\alpha\xi_0-\delta_1K_0(1))}{\gamma\delta_2K_0(1)} + e^{\frac{\lambda}{\alpha\xi_1}}\Bigg[\left(\frac{\beta\mu}{\alpha\xi_1}K_1(1)+\frac{\eta}{\alpha\xi_1}K_2(1)\right) \\ &\left(\frac{\gamma}{\lambda+\eta}+\left(\frac{\beta\mu+\alpha\xi_1}{\lambda+\eta}\left(\frac{\alpha\xi_0-\delta_1K_0(1)}{\delta_2K_0(1)}\right)\right)\right)-\frac{\gamma}{\alpha\xi_1}K_3(1)\Bigg]\Bigg\}^{-1}, \end{aligned} \tag{4.7}$$

$$K_0(z) = \int_0^z (1-s)^{\frac{\gamma}{\alpha\xi_0}-1} e^{-\frac{\lambda}{\alpha\xi_0}s}\,ds,$$

$$K_1(z) = \int_0^z s^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds, \ \ K_2(z) = \int_0^z (1-s)^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds,$$

and

$$K_3(z) = \int_0^z \left(1 - \frac{K_0(s)}{K_0(1)}\right) s^{\frac{\beta\mu}{\alpha\xi_1}} (1-s)^{-\left(\frac{\gamma}{\alpha\xi_0}+1\right)} e^{\left(\frac{\lambda}{\alpha\xi_0}-\frac{\lambda}{\alpha\xi_1}\right)s} ds.$$

Proof. Let

$$G_j(z) = \sum_{n=0}^{\infty} P_{j,n} z^n, \ \ j = 0, 1.$$

Then, multiplying Equation (4.2) by z^n, using Equations (4.1) and (4.3) and summing all possible values of n, we get

$$\alpha\xi_0(1-z)G_0'(z) - (\lambda(1-z)+\gamma)G_0(z) = -\{\delta_1 P_{00} + \delta_2 P_{11}\}, \tag{4.8}$$

with

$$\delta_1 = \left(\frac{\gamma\eta}{\lambda+\eta}\right) \text{ and } \delta_2 = \left(\frac{\eta(\beta\mu+\alpha\xi_1)}{\lambda+\eta}\right),$$

where $G_0'(z) = \frac{d}{dz}G_0(z)$.

In the same manner, from Equations (4.3) and (4.4) we obtain

$$\alpha\xi_1 z(1-z)G_1'(z) - (\lambda z - \beta\mu)(1-z)G_1(z) = -\gamma z G_0(z) + (\beta\mu(1-z)+\eta z)P_{1,0}. \tag{4.9}$$

Next, let $\Gamma = \delta_1 P_{00} + \delta_2 P_{11}$. Then, for $z \neq 1$, Equation (4.8) can be rewritten as follows

$$G_0'(z) - \left(\frac{\lambda}{\alpha\xi_0} + \frac{\gamma}{\alpha\xi_0(1-z)}\right)G_0(z) = -\frac{\Gamma}{\alpha\xi_0(1-z)}. \tag{4.10}$$

Multiplying both sides of (4.10) by the integrating factor $e^{\frac{-\lambda}{\alpha\xi_0}}(1-z)^{\frac{\gamma}{\alpha\xi_0}}$, then integrating from 0 to z we get

$$G_0(z) = e^{\frac{\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\gamma}{\alpha\xi_0}} \left\{G_0(0) - \frac{\Gamma}{\alpha\xi_0}K_0(z)\right\}, \tag{4.11}$$

with

$$K_0(z) = \int_0^z (1-s)^{\frac{\gamma}{\alpha\xi_0}-1} e^{-\frac{\lambda}{\alpha\xi_0}s} ds. \tag{4.12}$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{0,n} > 0$ and $z = 1$ is the root of the denominator of the right hand side of Equation (4.11), so $z = 1$ must be the root of the numerator of the right hand side of Equation (4.11).

Thus, we get

$$P_{0,0} = G_0(0) = \frac{\Gamma}{\alpha\xi_0} K_0(1). \tag{4.13}$$

This implies

$$P_{0,0} = \frac{\delta_2 K_0(1)}{\alpha\xi_0 - \delta_1 K_0(1)} P_{1,1}. \tag{4.14}$$

Consequently

$$P_{1,1} = \frac{\alpha\xi_0 - \delta_1 K_0(1)}{\delta_2 K_0(1)} P_{0,0}. \tag{4.15}$$

Next, substituting Equation (4.13) into (4.11), we obtain

$$G_0(z) = e^{\frac{\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\gamma}{\alpha\xi_0}} \left\{1 - \frac{K_0(z)}{K_0(1)}\right\} P_{0,0}. \tag{4.16}$$

For $z \neq 1$ and $z \neq 0$, Equation (4.9) can be rewritten as follows

$$G_1'(z) - \left(\frac{\lambda}{\alpha\xi_1} - \frac{\beta\mu}{\alpha\xi_1 z}\right) G_1(z) = \left(\frac{\beta\mu}{\alpha\xi_1 z} + \frac{\eta}{\alpha\xi_1(1-z)}\right) P_{1,0} - \frac{\gamma}{\alpha\xi_1(1-z)} G_0(z). \tag{4.17}$$

Then, we multiply both sides of Equation (4.17) by $e^{-\frac{\lambda}{\alpha\xi_1}z} z^{\frac{\beta\mu}{\alpha\xi_1}}$, we get

$$\frac{d}{dz}\left(e^{-\frac{\lambda}{\alpha\xi_1}z} z^{\frac{\beta\mu}{\alpha\xi_1}} G_1(z)\right) = \left\{\left(\frac{\beta\mu}{\alpha\xi_1 z} + \frac{\eta}{\alpha\xi_1(1-z)}\right) P_{1,0} - \frac{\gamma}{\alpha\xi_1(1-z)} G_0(z)\right\} e^{-\frac{\lambda}{\alpha\xi_1}z} z^{\frac{\beta\mu}{\alpha\xi_1}}. \tag{4.18}$$

Integrating from 0 to z, we have

$$G_1(z) = e^{\frac{\lambda}{\alpha\xi_1}z} z^{-\frac{\beta\mu}{\alpha\xi_1}} \left\{ \left(\frac{\beta\mu}{\alpha\xi_1} K_1(z) + \frac{\eta}{\alpha\xi_1} K_2(z) \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1} \int_0^z (1-s)^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} G_0(s) ds \right\}. \tag{4.19}$$

Where

$$K_1(z) = \int_0^z s^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds, \;\; K_2(z) = \int_0^z (1-s)^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds. \tag{4.20}$$

Using Equation (4.14) and substituting Equation (4.16) into (4.19), we get

$$G_1(z) = e^{\frac{\lambda z}{\alpha\xi_1}} z^{-\frac{\beta\mu}{\alpha\xi_1}} \left\{ \left(\frac{\beta\mu}{\alpha\xi_1} K_1(z) + \frac{\eta}{\alpha\xi_1} K_2(z) \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1} K_3(z) P_{0,0} \right\}. \tag{4.21}$$

With

$$K_3(z) = \int_0^z \left(1 - \frac{K_0(s)}{K_0(1)} \right) s^{\frac{\beta\mu}{\alpha\xi_1}} (1-s)^{-\left(\frac{\gamma}{\alpha\xi_0}+1\right)} e^{\left(\frac{\lambda}{\alpha\xi_0} - \frac{\lambda}{\alpha\xi_1}\right)s} ds. \tag{4.22}$$

Next, putting $z = 1$ in Equation (4.8), we get the probability that the server is on vacation, ($P_{0,.} = G_0(1) = \sum_{n=0}^{\infty} P_{0,n}$),

$$P_{0,.} = \left(\frac{\delta_1 P_{0,0} + \delta_2 P_{1,1}}{\gamma} \right). \tag{4.23}$$

And, putting $z = 1$ in Equation (4.21), we find the probability that the server is in busy period, ($P_{1,.} = G_1(1) = \sum_{n=0}^{\infty} P_{1,n}$),

$$P_{1,.} = e^{\frac{\lambda}{\alpha\xi_1}} \left\{ \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1} K_3(1) P_{0,0} \right\}. \tag{4.24}$$

From Equation (4.3), it yields

$$P_{1,0} = \left(\frac{\gamma}{\lambda+\eta} \right) P_{0,0} + \left(\frac{\beta\mu + \alpha\xi_1}{\lambda+\eta} \right) P_{1,1}. \tag{4.25}$$

Substituting Equation (4.25) into (4.24), we have

$$P_{1,.} = e^{\frac{\lambda}{\alpha\xi_1}} \left\{ \left(\frac{\gamma}{\lambda+\eta} \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) - \frac{\gamma}{\alpha\xi_1} K_3(1) \right) P_{0,0} + \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) \left(\frac{\beta\mu + \alpha\xi_1}{\lambda+\eta} \right) P_{1,1} \right\}. \tag{4.26}$$

Next, substituting Equation (4.15) into (4.23), we get (4.5). Then, substituting Equation (4.15) into (4.26), we obtain (4.6).

Finally, using the normalizing condition

$$\sum_{n=0}^{\infty} P_{0,n} + \sum_{n=0}^{\infty} P_{1,n} = 1,$$

which is equivalent to

$$P_{0,.} + P_{1,.} = 1. \tag{4.27}$$

And substituting Equations (4.15), (4.23) and (4.26) into (4.27), we obtain (4.7)

□

4.4 Performance measures and cost model

4.4.1 Performance measures

In this subpart useful performance measures are presented.

* The probability that the server is in a busy period (P_B).

$$\mathbb{P}(\text{Busy period}) = P_B = P_{1,.}.$$

* The probability that the server is on vacation (P_V).

$$\mathbb{P}(\text{Vacation period}) = P_V = 1 - \mathbb{P}(\text{Busy period}).$$

* The probability that the server is idle during busy period (P_I).

$$P_I = P_{1,0}.$$

* The average number of customers in the system when the server is taking vacation ($\mathbb{E}(L_0)$).

From Equation (4.8), using L'Hopital rule, we have

$$\mathbb{E}(L_0) = \lim_{z\to 1} G_0'(z) = \frac{-\lambda P_{0,.} + \gamma \mathbb{E}(L_0)}{-\alpha\xi_0}.$$

This implies

$$\mathbb{E}(L_0) = \left(\frac{\lambda}{\gamma + \alpha\xi_0}\right) P_{0,.}.$$

* The average number of customers in the system when the server is in busy period ($\mathbb{E}(L_1)$).

From Equation (4.9), using L'Hopital rule, we get

$$\begin{aligned}\mathbb{E}(L_1) &= \lim_{z\to 1} G_1'(z)\\ &= \left(\frac{\lambda - \beta\mu}{\alpha\xi_1}\right) P_{1,.} + \frac{\gamma}{\alpha\xi_1}\mathbb{E}(L_0) + \frac{\beta\mu}{\alpha\xi_1(\lambda+\eta)}\left(\gamma + \frac{(\beta\mu + \alpha\xi_1)(\alpha\xi_0 - \delta_1 K_0(1))}{\delta_2 K_0(1)}\right) P_{0,0}.\end{aligned}$$

* The average number of customers in the system ($\mathbb{E}(L)$).

$$\mathbb{E}(L) = \mathbb{E}(L_0) + \mathbb{E}(L_1).$$

* The average number of customers in the queue ($\mathbb{E}(L_q)$).

$$\begin{aligned}\mathbb{E}(L_q) &= \sum_{n=0}^{+\infty} nP_{0n} + \sum_{n=1}^{+\infty}(n-1)P_{1n}\\ &= \mathbb{E}(L) - (P_{1,.} - P_{1,0}).\end{aligned}$$

* The mean waiting time of a customer in the system (W_s).

$$W_s = \frac{\mathbb{E}(L_0) + \mathbb{E}(L_1)}{\lambda} = \frac{\mathbb{E}(L)}{\lambda}.$$

* The expected number of customers served per unit of time (E_{cs}).

$$E_{cs} = \beta\mu(P_{1,.} - P_{1,0}).$$

* The average rate of reneging (resp. retention) of impatient customers during vacation period.

$$R_{ren_0} = \alpha\xi_0\mathbb{E}(L_0),\ \ R_{ret_0} = (1-\alpha)\xi_0\mathbb{E}(L_0).$$

* The average rate of reneging (resp. retention) of impatient customers during busy period.

$$R_{ren_1} = \alpha\xi_1\mathbb{E}(L_1), \;\; R_{ret_1} = (1-\alpha)\xi_1\mathbb{E}(L_1).$$

Thus,
* The average rate of abandonment of a customer due to impatience (R_{ren}).

$$R_{ren} = R_{ren_0} + R_{ren_1}.$$

* The average rate of retention of impatient customers (R_{ret}).

$$R_{ret} = R_{ret_0} + R_{ret_1}.$$

4.4.2 Cost model

This subpart is devoted to develop a model for the costs incurred in the queueing system using the following symbols:

- C_1 : Cost per unit time when the server is working during busy period.
- C_2 : Cost per unit time when the server is idle during busy period.
- C_3 : Cost per unit time when the server is on vacation.
- C_4 : Cost per unit time when a customer joins the queue and waits for service.
- C_5 : Cost per service per unit time.
- C_6 : Cost per unit time when a customer reneges.
- C_7 : Cost per unit time when a customer is retained.
- C_8 : Cost per unit time when a customer returns to the system as a feedback customer.

Let
* R be the revenue earned by providing service to a customer.
* Γ be the total expected cost per unit time of the system.

$$\Gamma = C_1P_B + C_2P_I + C_3P_V + C_4\mathbb{E}(L_q) + C_6R_{ren} + C_7R_{ret} + \mu(C_5 + \beta' C_8).$$

* Δ be the total expected revenue per unit time of the system.

$$\Delta = R\mu(1 - P_V - P_{1,0}).$$

* Θ be the total expected profit per unit time of the system.

$$\Theta = \Delta - \Gamma.$$

4.5 Numerical analysis

4.5.1 Impact of system parameters on performance measures

Different performance measures of interest computed under different scenarios are given. These measures are obtained by using a MATLAB program coded by the authors. To illustrate the system numerically, the values for default parameters are considered using the following cases

- Table 4.1: $\lambda = 0.05 : 1.00 : 1.45$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 4.2 : $\lambda = 1.50$, $\mu = 0.40 : 2.00 : 5.60$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 4.3 : $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.05 : 0.50 : 0.95$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 4.4 : $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.05 : 0.85 : 1.30$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 4.5 : $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.05 : 0.10 : 0.55$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 4.6 : $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.05 : 0.10 : 0.55$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 4.7 : $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.10 : 0.10 : 1.00$, and $\alpha = 0.50$.
- Table 4.8 : $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.10 : 0.10 : 1.00$.

Table 4.1: Performance measures vs. λ.

λ	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
1.00	0.0272	0.7720	0.2280	0.6840	0.7883	1.4022	0.5060	0.5060	0.5440
1.05	0.0248	0.7795	0.2205	0.6931	0.8654	1.4169	0.5411	0.5411	0.5589
1.10	0.0227	0.7869	0.2131	0.7002	0.9439	1.4296	0.5762	0.5762	0.5738
1.15	0.0208	0.7943	0.2057	0.7052	1.0237	1.4407	0.6114	0.6114	0.5886
1.20	0.0191	0.8017	0.1983	0.7083	1.1049	1.4505	0.6466	0.6466	0.6034
1.25	0.0176	0.8090	0.1910	0.7094	1.1874	1.4591	0.6820	0.6820	0.6180
1.30	0.0161	0.8163	0.1837	0.7087	1.2713	1.4667	0.7175	0.7175	0.6325
1.35	0.0148	0.8234	0.1766	0.7063	1.3566	1.4735	0.7531	0.7531	0.6469
1.40	0.0137	0.8305	0.1695	0.7021	1.4434	1.4797	0.7890	0.7890	0.6610
1.45	0.0126	0.8375	0.1625	0.6964	1.5315	1.4853	0.8250	0.8250	0.6750

Table 4.2: Performance measures vs. μ.

μ	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
2.00	0.0144	0.8143	0.1857	0.7959	1.2864	1.3882	0.7457	0.7457	0.7543
2.40	0.0160	0.7938	0.2062	0.8839	1.0741	1.3053	0.6775	0.6775	0.8225
2.80	0.0174	0.7757	0.2243	0.9614	0.8883	1.2331	0.6179	0.6179	0.8821
3.20	0.0186	0.7597	0.2403	1.0300	0.7240	1.1694	0.5652	0.5652	0.9348
3.60	0.0197	0.7455	0.2545	1.0909	0.5775	1.1123	0.5182	0.5182	0.9818
4.00	0.0207	0.7328	0.2672	1.1453	0.4459	1.0607	0.4758	0.4758	1.0242
4.40	0.0216	0.7214	0.2786	1.1941	0.3268	1.0140	0.4374	0.4374	1.0626
4.80	0.0224	0.7111	0.2889	1.2383	0.2187	0.9713	0.4025	0.4025	1.0975
5.20	0.0231	0.7017	0.2983	1.2786	0.1201	0.9325	0.3707	0.3707	1.1293
5.60	0.0238	0.6931	0.3069	1.3154	0.0300	0.8970	0.3416	0.3416	1.1584

Table 4.3: Performance measures vs. ξ_0.

ξ_0	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.50	0.0130	0.8374	0.1626	0.6506	1.5209	1.4477	0.8253	0.8253	0.6747
0.55	0.0134	0.8372	0.1628	0.6106	1.5117	1.4148	0.8256	0.8256	0.6744
0.60	0.0139	0.8370	0.1630	0.5752	1.5036	1.3859	0.8260	0.8260	0.6740
0.65	0.0143	0.8369	0.1631	0.5438	1.4964	1.3601	0.8263	0.8263	0.6737
0.70	0.0148	0.8367	0.1633	0.5157	1.4899	1.3371	0.8266	0.8266	0.6734
0.75	0.0153	0.8365	0.1635	0.4904	1.4842	1.3164	0.8269	0.8269	0.6731
0.80	0.0158	0.8364	0.1636	0.4675	1.4790	1.2976	0.8272	0.8272	0.6728
0.85	0.0163	0.8362	0.1638	0.4466	1.4742	1.2806	0.8275	0.8275	0.6725
0.90	0.0167	0.8361	0.1639	0.4276	1.4699	1.2650	0.8278	0.8278	0.6722
0.95	0.0172	0.8360	0.1640	0.4101	1.4659	1.2507	0.8281	0.8281	0.6719

Table 4.4: Performance measures vs. ξ_1.

ξ_1	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.85	0.0131	0.8310	0.1690	0.7242	1.4598	1.4560	0.8380	0.8380	0.6620
0.90	0.0136	0.8248	0.1752	0.7508	1.3951	1.4306	0.8504	0.8504	0.6496
0.95	0.0140	0.8189	0.1811	0.7763	1.3364	1.4084	0.8623	0.8623	0.6377
1.00	0.0145	0.8132	0.1868	0.8007	1.2828	1.3890	0.8737	0.8737	0.6263
1.05	0.0149	0.8077	0.1923	0.8241	1.2338	1.3719	0.8846	0.8846	0.6154
1.10	0.0153	0.8024	0.1976	0.8467	1.1886	1.3568	0.8951	0.8951	0.6049
1.15	0.0157	0.7974	0.2026	0.8683	1.1469	1.3435	0.9052	0.9052	0.5948
1.20	0.0161	0.7925	0.2075	0.8892	1.1083	1.3317	0.9150	0.9150	0.5850
1.25	0.0164	0.7878	0.2122	0.9093	1.0723	1.3211	0.9244	0.9244	0.5756
1.30	0.0168	0.7833	0.2167	0.9288	1.0389	1.3117	0.9334	0.9334	0.5666

4.5.2 General Comments

* From Table 4.1 it is clearly seen that with the increases of the arrival rate λ, $P_{0,0}$ and P_V decrease, while P_B increases. Thus, the mean number of customers in the system during the busy period $\mathbb{E}(L_1)$ increases significatively which leads to an increase in the

Table 4.5: Performance measures vs. γ.

γ	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0284	0.7420	0.2580	0.9674	2.1932	2.1071	1.1740	1.1740	0.6646
0.15	0.0290	0.7933	0.2067	0.6890	1.9385	1.7517	0.9961	0.9961	0.7106
0.20	0.0290	0.8276	0.1724	0.5172	1.7849	1.5348	0.8879	0.8879	0.7414
0.25	0.0286	0.8522	0.1478	0.4032	1.6856	1.3925	0.8172	0.8172	0.7635
0.30	0.0281	0.8706	0.1294	0.3234	1.6179	1.2942	0.7685	0.7685	0.7801
0.35	0.0276	0.8850	0.1150	0.2654	1.5699	1.2235	0.7336	0.7336	0.7930
0.40	0.0270	0.8965	0.1035	0.2218	1.5348	1.1711	0.7077	0.7077	0.8033
0.45	0.0264	0.9059	0.0941	0.1882	1.5085	1.1311	0.6882	0.6882	0.8118
0.50	0.0258	0.9138	0.0862	0.1617	1.4884	1.1001	0.6730	0.6730	0.8189
0.55	0.0252	0.9204	0.0796	0.1404	1.4728	1.0755	0.6610	0.6610	0.8249

Table 4.6: Performance measures vs. η.

η	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0161	0.7919	0.2081	0.8919	1.5729	1.6432	0.8914	0.8914	0.6532
0.15	0.0187	0.7579	0.2421	1.0375	1.6647	1.8015	0.9669	0.9669	0.6369
0.20	0.0208	0.7316	0.2684	1.1502	1.7899	1.9601	1.0483	1.0483	0.6243
0.25	0.0224	0.7107	0.2893	1.2400	1.9383	2.1189	1.1338	1.1338	0.6142
0.30	0.0237	0.6936	0.3064	1.3132	2.1034	2.2778	1.2223	1.2223	0.6060
0.35	0.0248	0.6794	0.3206	1.3741	2.2811	2.4368	1.3130	1.3130	0.5992
0.40	0.0258	0.6674	0.3326	1.4254	2.4684	2.5959	1.4054	1.4054	0.5935
0.45	0.0266	0.6571	0.3429	1.4694	2.6632	2.7550	1.4992	1.4992	0.5886
0.50	0.0272	0.6483	0.3517	1.5074	2.8639	2.9142	1.5940	1.5940	0.5843
0.55	0.0278	0.6405	0.3595	1.5407	3.0696	3.0735	1.6897	1.6897	0.5806

Table 4.7: Performance measures vs. β.

β	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0020	0.9741	0.0259	0.1109	4.3719	2.9885	1.1207	1.1207	0.3793
0.20	0.0038	0.9503	0.0497	0.2128	3.6255	2.5589	0.9596	0.9596	0.5404
0.30	0.0060	0.9221	0.0779	0.3336	2.9646	2.1988	0.8246	0.8246	0.6754
0.40	0.0083	0.8932	0.1068	0.4578	2.3968	1.9031	0.7137	0.7137	0.7863
0.50	0.0104	0.8658	0.1342	0.5752	1.9133	1.6590	0.6221	0.6221	0.8779
0.60	0.0123	0.8410	0.1590	0.6815	1.4995	1.4541	0.5453	0.5453	0.9547
0.70	0.0140	0.8189	0.1811	0.7760	1.1416	1.2784	0.4794	0.4794	1.0206
0.80	0.0155	0.7995	0.2005	0.8594	0.8282	1.1251	0.4219	0.4219	1.0781
0.90	0.0169	0.7822	0.2178	0.9333	0.5510	0.9895	0.3711	0.3711	1.1289
1.00	0.0181	0.7669	0.2331	0.9991	0.3038	0.8686	0.3257	0.3257	1.1743

Table 4.8: Performance measures vs. α.

α	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0019	0.9710	0.0290	0.2900	6.3941	4.4560	0.5580	5.0220	0.9420
0.20	0.0049	0.9273	0.0727	0.5454	3.4759	2.6808	0.6454	2.5817	0.8546
0.30	0.0076	0.8916	0.1084	0.6502	2.4282	2.0523	0.7167	1.6724	0.7833
0.40	0.0101	0.8623	0.1377	0.6886	1.8756	1.7095	0.7754	1.1631	0.7246
0.50	0.0126	0.8375	0.1625	0.6964	1.5315	1.4853	0.8250	0.8250	0.6750
0.60	0.0151	0.8162	0.1838	0.6892	1.2957	1.3233	0.8676	0.5784	0.6324
0.70	0.0178	0.7976	0.2024	0.6746	1.1238	1.1989	0.9048	0.3878	0.5952
0.80	0.0205	0.7813	0.2187	0.6562	0.9926	1.0992	0.9375	0.2344	0.5625
0.90	0.0231	0.7667	0.2333	0.6362	0.8892	1.0170	0.9666	0.1074	0.5334
1.00	0.0258	0.7537	0.2463	0.6157	0.8056	0.9475	0.9926	0.0000	0.5074

number of customers served E_{cs}. Moreover, $\mathbb{E}(L_0)$ is not monotone with λ, while W_s increases as the arrival rate increases, this implies an increases in the average reneging and retention rates R_{ren} and R_{ret}.

∗ According to Table 4.2 we see that along the increases of the service rate μ, $P_{0,0}$, P_V,

$\mathbb{E}(L_0)$ and E_{cs} increase, whereas P_B and $\mathbb{E}(L_1)$ both decrease, as it should be expected. Moreover, with the increase in μ, the mean waiting time of a customer in the system W_s deceases, this leads to a decrease in R_{ren} and R_{ret}. Obviously, the higher the service rate, the smaller the average rate of abandonment and the larger the number of customers served.

$*$ From Table 4.3 we remark that when the reneging rate during vacation period ξ_0 increases, P_B, W_s, $\mathbb{E}(L_0)$ and $\mathbb{E}(L_1)$ decrease, while $P_{0,0}$, P_V, R_{ren} and R_{ret} increase. Consequently, E_{cs} decreases. As intuitively expected, the bigger the rate of reneging, the smaller the number of customers served.

$*$ According to Table 4.4, we observe that along the increases of the reneging rate during busy period ξ_1, P_B, $\mathbb{E}(L_1)$ and W_s decrease, this leads to a decrease in E_{cs}. Further, as expected, the increasing of ξ_1 implies an increase in $P_{0,0}$, P_V, $\mathbb{E}(L_0)$, R_{ren} and R_{ret}.

$*$ Table 4.5 illustrates that P_B increase with increasing values of the vacation rate γ, while $P_{0,0}$ is not monotonic with γ. Further, P_V, W_s, $\mathbb{E}(L_0)$ and $\mathbb{E}(L_1)$ decrease, this implies an increase in E_{cs}. On the other hand, R_{ren} and R_{ret} decrease significantly, which agrees with the intuitive expectation; the higher the rate of vacation, the bigger the probability of busy period and the greater the number of customers served.

$*$ According to Table 4.6, it is clearly observed that with the increase in the waiting server rate η, the probability of busy period P_B decreases which leads to a decrease in the mean number of customers served E_{cs}; this is because W_s, P_V and $\mathbb{E}(L_0)$ increase with η, which implies an increase in R_{ren}, R_{ret} and $P_{0,0}$. On the other hand the number of customers in the system during busy period $\mathbb{E}(L_1)$ increases; the reason is that the size of the system during vacation period becomes large with η.

$*$ The effect of the probability of non-feedback β is presented in Table 4.7, we see that P_B and W_s both decrease with increasing values of β. Further, as expected, $P_{0,0}$, P_V and $\mathbb{E}(L_0)$ increase as β increases, whereas $\mathbb{E}(L_1)$ decreases with increasing values of β; this is because the mean system size during vacation period increases with β. Further, it is well shown that R_{ren} and R_{ret} both decrease along the increasing of non-feedback probability β, which results in the increasing of E_{cs}.

$*$ The impact of non-retention probability α is shown in Table 4.8. As intuitively expected, along the increase of α, P_B and $\mathbb{E}(L_1)$ decrease, while P_V as α increases. Further, $\mathbb{E}(L_0)$ is not monotonic with the probability of non-retention. Moreover, W_s and R_{ret} both decrease with increasing of α whereas R_{ren} increases with the probability α, this leads to a decease of E_{cs}. This is quite reasonable; the smaller the probability of retaining impatient customers, the larger the average rate of reneged customers and the smaller the number of customers served.

4.5.3 Economic analysis

In this subpart, a sensitive economic analysis of the model is performed numerically and the results are discussed appropriately. We present the variation in total expected cost, total expected revenue and total expected profit with the change in different parameters of the system. For the whole numerical study we fix the costs at $C_1 = 5$, $C_2 = 3$, $C_3 = 5$, $C_4 = 3$, $C_5 = 4$, $C_6 = 3$, $C_7 = 2$, $C_8 = 2$, and $R = 50$.

Impact of arrival rate λ

We examine the impact of λ by keeping all other variables fixed, to this end we take $\lambda = 1.00 : 0.05 : 1.45$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$. Results of the analysis are summarized in Table 4.9 and Figure 4.1.

Table 4.9: Γ, Δ and Θ for different values of λ.

λ	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45
Γ	22.63	23.04	23.45	23.86	24.26	24.67	25.07	25.48	25.88	26.29
Δ	54.39	55.89	57.38	58.86	60.33	61.80	63.25	64.68	66.10	67.50
Θ	31.76	32.84	33.92	35.00	36.06	37.12	38.17	39.20	40.21	41.20

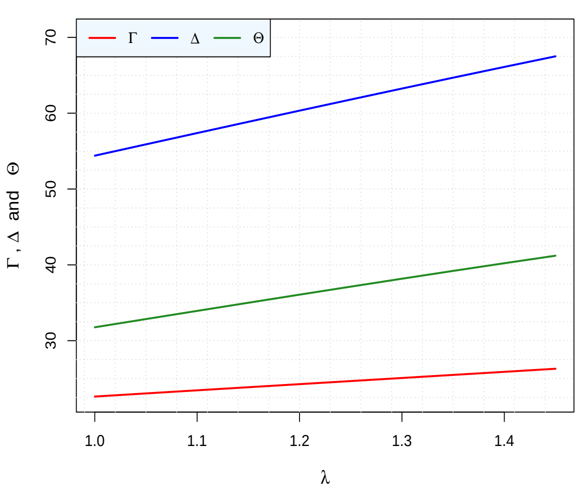

Figure 4.1: Γ, Δ and Θ vs. λ.

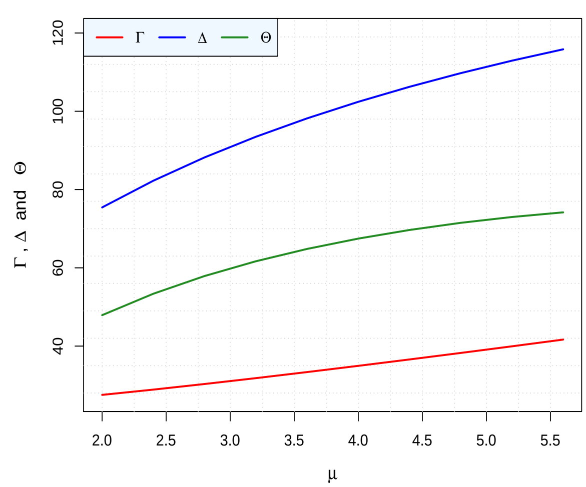

Figure 4.2: Γ, Δ and Θ vs. μ.

Following the obtained results we observe that Γ, Δ, and Θ all increase with the increasing of the arrival rate λ. This result agrees with our intuition; the number of the customers in the system increases with the increasing of λ, therefore a large number of customers is served. Consequently, the total expected profit increases.

Impact of service rate μ

To check the impact of service rate μ, the values of the parameters are chosen as follows: $\lambda = 1.50$, $\mu = 2.00 : 0.40 : 5.60$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.

Table 4.10: Γ, Δ and Θ for different values of μ.

μ	2.00	2.40	2.80	3.20	3.60	4.00	4.40	4.80	5.20	5.60
Γ	27.53	28.88	30.31	31.80	33.35	34.95	36.58	38.25	39.94	41.66
Δ	75.43	82.25	88.21	93.48	98.18	102.4	106.2	109.7	112.9	115.8
Θ	47.89	53.37	57.90	61.67	64.82	67.46	69.67	71.49	72.98	74.17

According to Table 4.10 and Figure 4.2 we see that Γ and Δ increase with increasing values of μ, this generates an increase in Θ. This result makes perfect sense, the higher the service rate, the greater the total expected profit of the system.

Impact of reneging rates ξ_0 and ξ_1

Let's study the effect of reneging rates in vacation and busy periods ξ_0 and ξ_1, To this end we consider the following cases

- Table 4.11: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 1.00$, $\xi_0 = 2.00 : 0.50 : 6.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 4.12: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85 : 0.05 : 1.30$, $\beta = 0.50$, and $\alpha = 0.50$.

Table 4.11: Γ, Δ and Θ for different values of ξ_0.

ξ_0	2.00	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00	6.50
Γ	23.97	23.99	24.01	24.02	24.03	24.04	24.05	24.05	24.05	24.06
Δ	78.59	78.47	78.37	78.28	78.21	78.15	78.10	78.06	78.03	78.01
Θ	54.62	54.48	54.36	54.26	54.18	54.11	54.06	54.01	53.97	53.95

From Tables 4.11 and 4.12 and Figures 4.3 and 4.4 we observe that

$\ast$ As expected, along the increasing of ξ_0, Γ increases while Θ and Δ decrease with ξ_0, this is because the average rate of reneged customers increases with ξ_0. Therefore

Table 4.12: Γ, Δ and Θ for different values of ξ_1.

ξ_1	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20	1.25	1.30
Γ	26.24	26.21	26.19	26.17	26.17	26.17	26.18	26.19	26.20	26.22
Δ	66.20	64.96	63.77	62.63	61.54	60.48	59.47	58.50	57.56	56.65
Θ	39.95	38.74	37.58	36.45	35.36	34.31	33.29	32.31	31.36	30.43

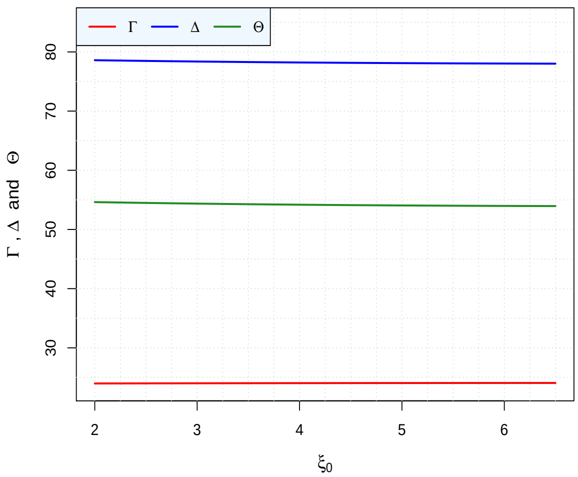

Figure 4.3: Γ, Δ and Θ vs. ξ_0.

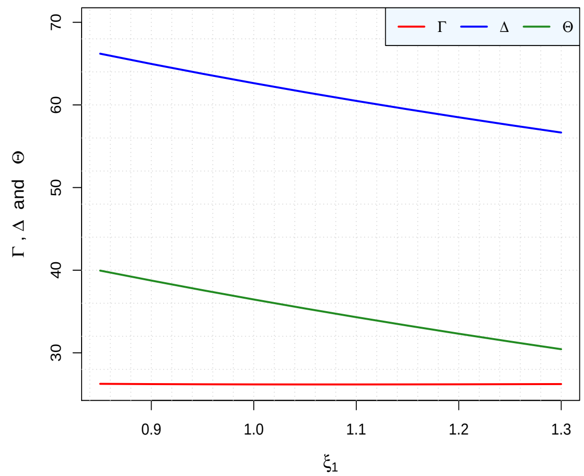

Figure 4.4: Γ, Δ and Θ vs. ξ_1.

the number of customers served decreases, which results in the decreasing of the total expected profit.

∗ With the increase of ξ_1, Δ decreases, while Γ is not monotonic with the parameter ξ_1. Further, Θ decreases with the increasing of the impatience rate, this is because the number of customers in the system decreases with ξ_1, this implies a decrease in P_B which results in the decreasing of E_{cs}.

Impact of vacation rate γ

To examine the impact of the vacation rate γ on the total expected profit, we take $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10 : 0.05 : 0.55$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.

Table 4.13: Γ, Δ and Θ for different values of γ.

γ	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
Γ	30.58	28.11	26.61	25.61	24.93	24.45	24.09	23.81	23.60	23.43
Δ	66.46	71.06	74.13	76.34	78.00	79.29	80.33	81.18	81.89	82.49
Θ	35.87	42.94	47.53	50.72	53.06	54.85	56.24	57.37	58.29	59.06

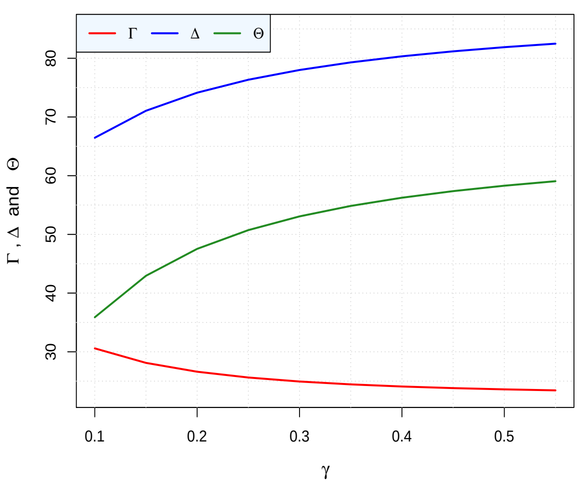

Figure 4.5: Γ, Δ and Θ vs. γ.

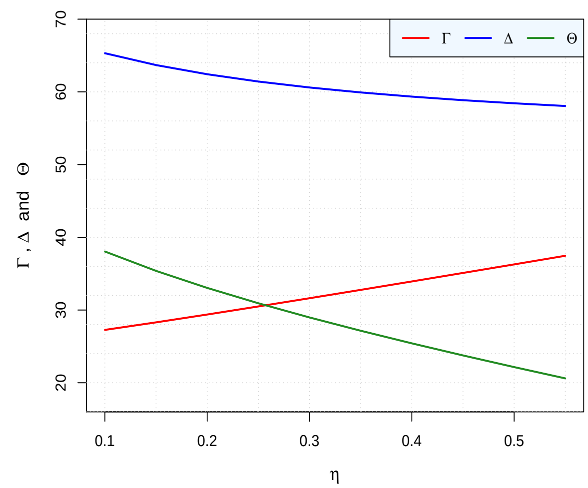

Figure 4.6: Γ, Δ and Θ vs. η.

From Table 4.13 and Figure 4.5 it is easily seen that the increases of the vacation rate γ implies a decrease in Γ and a considerable increase in Δ and Θ. This is quite explicable; as γ increases the vacation duration decreases and the server switches to busy period during which customers are served. Thus, this leads to a significant increase in the total expected profit.

Impact of waiting rate of a server η

Here, we examine the sensitivity of the total expected profit versus the waiting server rate η. For this case, we put $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10 : 0.05 : 0.55$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$. The numerical results are presented in Table 4.14 and Figure 4.6.

Table 4.14: Γ, Δ and Θ for different values of η.

η	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
Γ	27.26	28.30	29.38	30.49	31.62	32.77	33.93	35.09	36.27	37.45
Δ	65.31	63.68	62.42	61.42	60.60	59.92	59.34	58.85	58.43	58.06
Θ	38.04	35.38	33.04	30.92	28.98	27.15	25.42	23.75	22.15	20.60

From the obtained results we remark that with the increase in η, total expected cost Γ increases, while Δ and Θ monotonically decease. This is due to the fact that the probability of busy period during which service is provided decreases with the parameter η. Therefore, the total expected profit decreases considerably.

Impact of probability of non-retention α

To study the impact of α on the total expect profit, we choose the parameters values as follows: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.10 : 0.10 : 1.00$.

Table 4.15: Γ, Δ and Θ for different values of β.

α	0.10	0.20	0.3	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Γ	46.8570	34.38	30.05	27.75	26.29	25.26	24.49	23.88	23.39	22.98
Δ	94.2005	85.45	78.32	72.45	67.50	63.24	59.52	56.25	53.34	50.74
Θ	47.3435	51.07	48.27	44.69	41.20	37.97	35.03	32.36	29.95	27.76

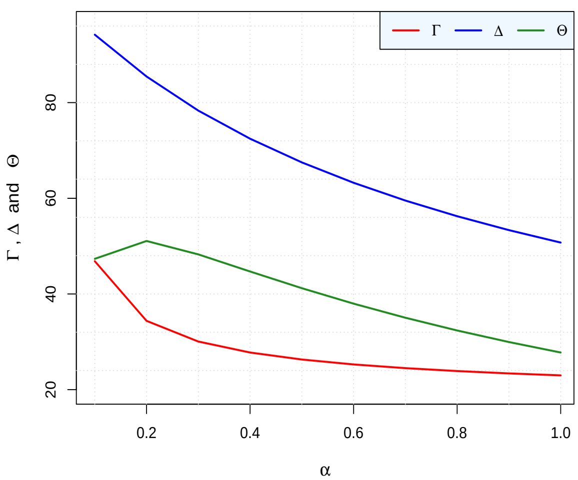

Figure 4.7: Γ, Δ and Θ vs. α.

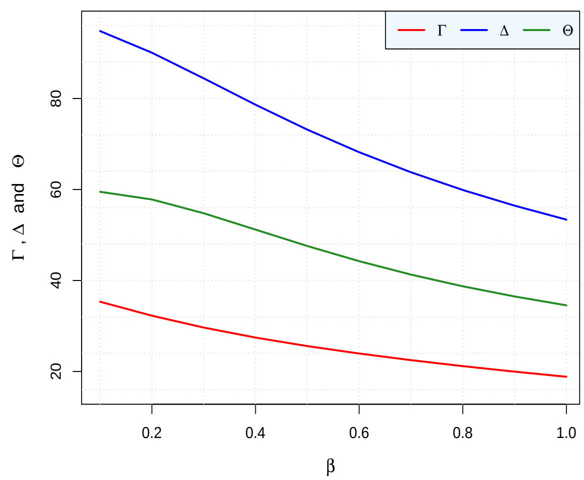

Figure 4.8: Γ, Δ and Θ vs. β.

According to Table 4.15 and Figure 4.7 we observe that the increases of non-retention probability α implies a decrease in Γ, Δ and Θ. A slight increase is observed in Θ when the parameter α is below a certain value, ($\alpha = 0.20$). Therefore, we can see that the probability of retaining reneged customers α' has a noticeable effect on the revenue generation and on the total expected profit of the system. This is because the number of customers served increases with the parameter α'. Thus, it is quite clear that the probability of retention has a positive impact in the economy.

Impact of probability of non-feedback β

Here, we put $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.10 : 0.10 : 1.00$, and $\alpha = 0.50$. The numerical results obtained for this situation is given in Table

4.16 and Figure 4.8

Table 4.16: Γ, Δ and Θ for different values of β.

β	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Γ	35.32	32.26	29.65	27.45	25.57	23.94	22.49	21.17	19.96	18.83
Δ	94.81	90.07	84.42	78.63	73.15	68.19	63.78	59.89	56.44	53.37
Θ	59.49	57.80	54.77	51.18	47.57	44.24	41.29	38.72	36.48	34.54

From the obtained results, it is clearly shown that Γ, Δ and Θ monotonically decrease as non-feedback probability β increases. The reason is that the number of the customers in the system decreases with the increasing of β, which leads to a decrease in the total expected profit.

4.6 Conclusion and future work

In this paper we studied an $M/M/1$ Bernoulli feedback queueing system with single exponential vacation, waiting server, reneging and retention of reneged customers, wherein the impatience timers of customers depend on the states of the server. The explicit expressions of the steady-state probabilities are obtained, using probability generating functions (PGFs). Useful measures of effectiveness of the queueing system are presented and a cost model is developed. Finally, an extensive numerical study is presented. Our system can be considered as a generalized version of the existing queueing models given by Yue et al.(2016) and Ammar (2017) associated with several practical situations.

The model considered in this paper can be extended to multiserver queueing system with delayed state-dependent service times, breakdowns and repairs.

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Chapter 5

Single server batch arrival Bernoulli feedback queueing system with waiting server, K-variant vacations and impatient customers

This chapter is submitted.

Single server batch arrival Bernoulli feedback queueing system with waiting server, K-variant vacations and impatient customers

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Abstract. This paper deals with an infinite capacity batch arrival single server Markovian Bernoulli feedback queueing system with waiting server, K-variant vacations, impatient customers and retention of reneged customers. The server activates a waiting timer whenever he completes the service of the last customer. If an arrival enters the system before the completion of the waiting timer, the server immediately begins the service. Otherwise, he decides to go on vacation after the expiration of the waiting timer. When a vacation period ends, the server switches to busy period, if there are customers in the queue. Otherwise, he takes another vacation and continues to do so till K consecutive vacations have been taken. After the end of the Kth vacation, the server switches to busy period and remains idle or busy depending on the availability of the customers in the system. During vacation period the customer may renege due to impatience. Using certain customer retention mechanism, the impatient customer can be retained in the system. Further, after getting incomplete or dissatisfying service, a customer may comeback to the system as a Bernoulli feedback customer to receive another regular service. The model is analyzed using probability generating function (PGF) technique. Various performance measures of the queueing system are derived. Then, by setting the appropriate parameters, some special cases are discussed. Moreover, a cost model for the queueing system is developed. The parameter optimization is illustrated numerically using quadratic fit search method (QFSM) and particle swarm optimization (PSO). Finally, numerical results are provided to explore the impact of system parameters on performance measures and costs of the queueing system.

Keywords: Variant multiple vacations. Impatient customers. Bernoulli feedback. Probability generating function. optimization. QSFM and PSO algorithm.
2010 Mathematics Subject Classification: 60K25, 68M20, 90B22.

5.1 Introduction

Vacation queues have broad applications in many real life situation systems, such as manufacturing and production systems, distribution and service systems, transportation systems, telecommunication industry, computer and communication systems, and so on. Queueing systems with vacation policies have been extensively studied in the past decades. For various results on different vacation models, the readers may refer to the survey paper of Doshi (1986), monographs of Takagi (1991), and Tian and Zhang (2006).

The queueing models discussed in the above literature suppose that the customers arrive one at a time. There are various situations wherein customers arrive in groups. Such queues, called batch arrival queueing models are prevalent in many practical situations for instance, in digital communication systems. Lee et al. (1994) analyzed a $M^X/G/1$ queue with bulk arrival, N-policy and multiple vacations. Madan and AI-Rawwash (2005) presented a study on the $M^X/G/1$ queue with feedback and optional server vacations based on a single vacation policy. The maximum entropy of the $M^X/M/1$ queueing system with multiple vacations and server breakdowns was provided in Wang et al. (2007). Then, Haridass and Arrumuganathan (2008) treated a $M^{[X]}/G/1$ queueing model with an unreliable server and with single vacation. Chang and Ke (2009) studied a batch retrial queueing model with J vacations. Aissani (2011) dealt with an $M^X/G/1$ energetic retrial queue with vacations and control. Later, a batch arrival single server retrial queue with modified vacations under N-policy was considered in Haridass and Arumuganathan (2015). Recently, Inoue et al. (2018) studied the impatient behavior of customers in an $M^{[X]}/G/1$ queueing model under steady state.

Vacation queueing systems with impatience play important roles in the analysis of many telephone switching systems, communication/telecommunication networks, computer systems and manufacturing systems. Zhang et al. (2005) considered a $M/M/1/N$ queue with balking, reneging and server vacations. Then, a study of customers' impatience in queues with server vacations has been given in Altman and Yechiali (2006). Performance analysis of an $M/M/c/N$ queueing system with balking, reneging and synchronous vacations of partial servers was carried out by Yue et al. (2006). Altman and Yechiali (2008) analyzed the infinite-server queueing model with system's additional tasks and impatient customers. Adan et al. (2009) treated queueing models with server vacations and synchronized abandonments. Later, Ammar (2015) gave the transient analysis of an $M/M/1$ queueing model with impatience and multiple vacations.

In real-life situations, the server does not go on vacation just as the system does when it gets empty. By considering a human behavior, we see that the server waits a certain amount of time before taking a vacation, even if the system does not have customers. Padmavathy et al. (2011) dealt with single server queueing models with impatient customers, server vacations and a waiting server, where the service times, customer impatience times, waiting times of the server in the empty system and the duration of the server vacations are all exponentially distributed. Ammar (2017) derived the transient solution of an $M/M/1$ vacation queue with a waiting server and impatient customers. Recently, a Markovian vacation queueing system with a waiting server and geometric abandonments has been developed in Deepa and Kalidass (2018). Recently, a Markovian queueing system with Bernoulli feedback, single vacation, waiting server and impatient customers has been discussed in Bouchentouf et al. (2018). Then, Bouchentouf and Guendouzi (2019) presented a cost optimization analysis for an $M^X/M/c$ queueing system with Bernoulli feedback, waiting servers and impatient customers under both single and multiple synchronous vacations.

Variant of multiple vacation schemes is relatively recent where it is permitted to the server to take a certain fixed number of consecutive vacations if the system remains empty at the end of a vacation. Such vacation policy was treated by Zhang and Tian (2001), the authors treated a $Geo/G/1$ queueing system with multiple adaptive vacations. Ke (2007) studied an $M^X/G/1$ queueing system with balking under a variant vacation. Literature related to variant multiple working vacations are found in Wang et al. (2011) and Yue et al. (2014). Recently, Laxmi and Rajesh (2016) studied a variant working vacations queue with customer impatience. Further, some performance measures of variant working vacations on batch arrival queue with reneging have been presented in Laxmi and Rajesh (2017).

The optimization of anufacturing/production, telecommunication and computer systems using queueing theory has been the subject of many studies in recent decades. Interesting papers in this area include the research works of Whitt (1984) in open and closed queueing networks, Dallery and Gershwin (1992), which describe the main queueing models and the results of the literature on the production lines, Cruz et al. (2018), which present the optimization of the performance of general finite single-server acyclic queueing networks, and Martins et al. (2019), which present performance analysis and optimization of buffers and servers in finite queueing networks.

In earlier literature, as it was mentioned, very few authors studied the comparable work on the variant vacations for queueing models with impatience at which the server may take a sequence of finite vacations in his idle time. But as far as the best of our knowledge, there is no considerable amount of research work on Bernoulli feedback queueing system with batch arrival, waiting server, variant vacations, impatient customers and retention of reneged customers. This motivates us to develop such a model and carry out its cost model. The system considered in this paper is motivated by questions regarding the performance modeling of queueing systems including call centers, customized manufacturing, traffic modeling, business and industries, computer communication, health sectors, post of office, medical sciences and many other areas.

The rest of the paper is arranged as follows, the mathematical model has been constructed in Section 2. The probability generating function of the steady state of the system is obtained in Section 3. In Section 4, various performance measures are derived. In Section 5, we give some special cases of our model. In Section 6, a cost model for the queueing system is developed in order to determine the optimal values of service rate, simultaneously, to minimize the total expected cost per unit time. For this purpose, we adopt QFSM and PSO algorithm to implement the optimization tasks. Section 7 is consecrated to numerical illustrations. In Section 8, we present the managerial insights. Finally, we conclude the paper in Section 9.

5.2 Model description

We consider an infinite-buffer Markovian queueing system at which customers arrive in batches according to a Poisson process with rate λ. Let X denote the batch size random variable of the arrival with probability mass function $P(X = l) = b_l$, $l = 1, 2, \ldots$.

The service is provided by a single server, and service time is assumed to be exponentially distributed with parameter μ. The customers are served on FCFS discipline. When the busy period is ended, the server waits a random period before taking a vacation, this waiting time is assumed to be exponentially distributed with parameter η.

When duration of the waiting server expires, the server leaves for vacation. Then, at a vacation period termination if he finds a customer at the vacation completion instant, he comes back to the busy period, otherwise, he takes a finite number, say K, of successive vacations. When the K consecutive vacations are complete, the server returns to busy period and depending on the arriving batch of customers, he stays idle or busy. The period of a vacation follows a exponentially distribution with parameter ϕ.

During vacation period, each incoming customer starts up an impatience timer independently of the other customers in the system, which is assumed to be exponentially distributed with parameter ξ. The impatient customers may leave the system (renege) with probability α. Using certain mechanism, they can be retained in the system with probability $\alpha' = 1 - \alpha$.

After completion of each service, this later may be incomplete or unsatisfactory, at this situation the customer may decide either to leave the system with probability β or return and join the tail of the queue with probability β', where $\beta + \beta' = 1$. Note that, there is no distinction between regular arrival and feedback ones, that is, the newly arrived and those that are fed back are served in order in which they join the tail of the primary queue.

The inter-arrival times, batch sizes, waiting server times, vacation times, service time and impatience times are independent of each other.

Practical applications of the model

– The proposed queueing model has prominent applications in diverse practical systems dealing with human behavior including post of office, banks, private healthcare, and private business firms at which customers may arrive in batches. At end of busy periods, the server waits for a while before proceeding for a vacation. Once the vacation period is over, the server switches to the busy period if there are customers in the queue; otherwise he may take a fixed consecutive vacations, at the end of the successive vacations, the server switches to busy period and stays idle or busy depending on the availability of the customers in the system. During the vacation period, a customer may quit the system whenever his waiting time is longer than his patience time. Further, customers may be dissatisfied with the quality of the service. In this case, they can rejoin the system as feedback customers to complete their service. Such systems can be modeled by our model developed in this paper.

– Another practical application of the proposed model arises in communication systems: Due to recent theoretical analyses, it is now broadly recognized that the impatience phenomenon is one of the determining factors for the performance of call centers. The asymptotic study of call center models has proven to provide useful managerial insights. From a business point of view, a call center is an entity that combines voice and data communication technologies, enabling a company to implement critical business strategies in order to reduce costs and increase revenues. It is typically set up for sales, marketing, technical support and customer service purposes.

Once the calls (arrival stream of customers in batches) are connected to the system, they can be filtered and forwarded through a proactive support service. The filter may be a software or a live representative who assesses the customer's problem and then transfers the calls to a designated representative. Once the calls are forwarded to the suitable representatives, the customer service representatives will work on resolving the customers's problems (service). In addition, in call center, arrival stream of in batches called outbound call, in the form of e-mail sent to the call center with a request to be called back will be processed in the center in the order of their arrival when there is no incoming call. Once the customers are serviced and no call are connected (empty system), the agents stay active and look for a new calls (waiting server) for a certain period of time. After that, they go on vacation. At the end of the vacation period, the agents come back to the busy period, if there are a new calls, they start working on them; otherwise they may take a fixed consecutive vacations. When the number of fixed vacation in taken, the agents return to the busy period where they start working if they find customers waiting in the queue; otherwise they stay idle. When the system is on vacation, the flow of new requests (customers) continues, but each customer activates his own impatience timer, such that, if the system is still on vacation when the time expires, the customers leave the system.

– Customer's impatience represents a threat to businesses, it leads to the loss of potential customers. For any firm, it's not just about losing a customer, it's about harming the company's brand image by giving negative feedback about the firm's quality of service. Therefore, the reneging has a very bad effect on the business and the goodwill of a company. Thus, to avert this serious problem, a firm has to use certain retention strategies to convince the impatient customer to remain in the system. So, the concept of customer retention is of great importance for the management of the firm. This could be either by increasing the rate of the service, introducing an extra service channel or presenting more advantageous offers to customers.

5.3 The equilibrium state distribution

In this section, we study the steady-state distribution of the system. Let $L(t)$ be the number of customers in the system and $S(t)$ denote the status of the server at time t, such that

$$S(t) = \begin{cases} j, & \text{when the server is taking the } (j+1)th \text{ vacation at time } t, \\ & j = \overline{0, K-1}; \\ K, & \text{the server is in busy period at time } t. \end{cases}$$

The bi-variate $\{(L(t); S(t)); t \geq 0\}$ represents two dimensional infinite state continuous-time Markov chain with state space $\Omega = \{(n, j) : n \geq 0, j = \overline{0, K}\}$.

Let $P_{n,j} = \lim_{t\to\infty} P\{L(t) = n, S(t) = j\}$, $n \geq 0, j = \overline{0, K}$ denote the system state probabilities of the process $\{(L(t), S(t)), t \geq 0\}$.

The state transition diagram corresponding to our system is depicted in Figure 5.1.

The steady-state balance equations that govern our model are deduced as

$$(\lambda + \phi)P_{0,0} = \alpha\xi P_{1,0} + \eta P_{0,K}, \tag{5.1}$$

$$(\lambda + \phi + \alpha\xi)P_{1,0} = \lambda b_1 P_{0,0} + 2\alpha\xi P_{2,0}, \;\; n = 1, \tag{5.2}$$

$$(\lambda + \phi + n\alpha\xi)P_{n,0} = \lambda \sum_{m=1}^{n} b_m P_{n-m,0} + (n+1)\alpha\xi P_{n+1,0}, \;\; n \geq 2, \tag{5.3}$$

$$(\lambda + \phi)P_{0,j} = \alpha\xi P_{1,j} + \phi P_{0,j-1}, \;\; 1 \leq j \leq K-1, \tag{5.4}$$

$$(\lambda + \phi + n\alpha\xi)P_{n,j} = \lambda \sum_{m=1}^{n} b_m P_{n-m,j} + (n+1)\alpha\xi P_{n+1,j}, n \geq 1, \;\; 1 \leq j \leq K-1, \tag{5.5}$$

$$(\lambda + \eta)P_{0,K} = \phi P_{0,K-1} + \beta\mu P_{1,K}, \tag{5.6}$$

$$(\lambda + \beta\mu)P_{n,K} = \lambda \sum_{m=1}^{n} b_m P_{n-m,K} + \beta\mu P_{n+1,K} + \phi \sum_{j=0}^{K-1} P_{n,j}, \;\; n \geq 1. \tag{5.7}$$

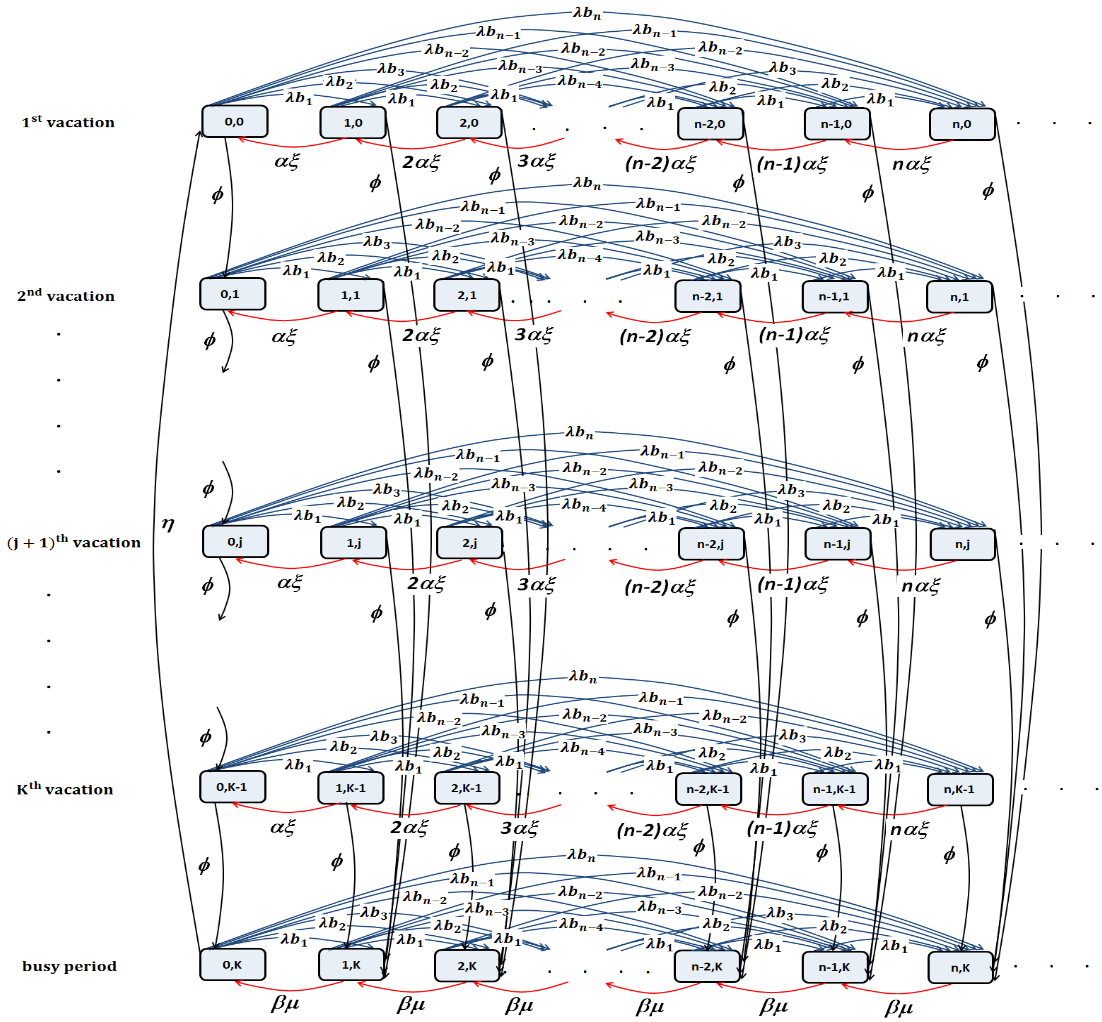

Figure 5.1: State-transition-rate diagram of the model.

Theorem 5.3.1. *If $\lambda E(X) < \beta\mu$, then the steady-state-probabilities $P_{n,j}$ are given as*

$$P_{.,j} = \sum_{n=0}^{\infty} P_{n,j} = A^{j-1} P_{0,0}, \qquad j = \overline{0, K-1}, \tag{5.8}$$

and

$$P_{.,K} = \sum_{n=0}^{\infty} P_{n,K} = \frac{1}{\beta\mu - \lambda B'(1)} \left\{ \frac{\phi\lambda B'(1)}{\alpha\xi + \phi} \frac{1 - A^K}{A(1-A)} + \frac{\beta\mu\alpha\xi}{\eta C} \right\} P_{0,0}, \tag{5.9}$$

where

$$P_{0,0} = \left\{ \frac{\beta\mu\alpha\xi}{\eta C(\beta\mu - \lambda B'(1))} + \frac{1 - A^K}{A(1-A)} \left(\frac{\phi\lambda B'(1)}{(\beta\mu - \lambda B'(1))(\alpha\xi + \phi)} + 1 \right) \right\}^{-1},$$

such that

$$A = \frac{\phi C}{\alpha\xi},$$

with

$$C = \int_0^1 e^{\frac{\lambda}{\alpha\xi}H(x)}(1-x)^{\frac{\phi}{\alpha\xi}-1}dx, \quad \textit{and} \quad H(z) = \int_0^z \frac{B(x)-1}{1-x}dx,$$

where $B(x)$ is the probability generating function of the batch arrival size X, and $B'(1) = E(X)$ is the first moment of random variable X.

Proof. The state probabilities are obtained by solving equations (5.1)-(5.7) using probability generating functions (PGFs).
Let us define the PGFs of $P_{n,j}$ as

$$G_j(z) = \sum_{n=0}^{\infty} P_{n,j}z^n, \ |z| \leq 1, \ j = \overline{0,K},$$

and the PGF of the batch arrival size X as

$$B(z) = \sum_{n=1}^{\infty} b_n z^n, \ |z| \leq 1.$$

Multiplying equations (5.1)-(5.3) by z^n and summing all possible values of n, then re-arranging all the terms, we obtain

$$(1-z)\alpha\xi G_0'(z) - [\lambda(B(z)-1)-\phi]G_0(z) = -\eta P_{0,K}, \tag{5.10}$$

In the same way, using equations (5.4)-(5.5) and (5.6)-(5.7), we respectively get

$$(1-z)\alpha\xi G_j'(z) - [\lambda(B(z)-1)-\phi]G_j(z) = -\phi P_{0,j-1}, j = \overline{1,K-1}, \tag{5.11}$$

and

$$[\lambda z(B(z)-1)+\beta\mu(1-z)]G_K(z) + z\phi\sum_{j=0}^{K-1}G_j(z) = z\phi\sum_{j=0}^{K-2}P_{0,j} + [\beta\mu(1-z)+z\eta]P_{0,K}. \tag{5.12}$$

By taking $z=1$ in equations (5.10) and (5.11), we respectively find

$$\phi G_0(1) = \eta P_{0,K}, \tag{5.13}$$

and

$$G_j(1) = P_{0,j-1}, j = \overline{1, K-1}. \tag{5.14}$$

Now, we can write equations (5.10) and (5.11) for $z \neq 1$ as

$$G_0'(z) + \left[\frac{\lambda}{\alpha\xi}H'(z) - \frac{\phi}{\alpha\xi(1-z)}\right]G_0(z) = -\frac{\eta}{\alpha\xi(1-z)}P_{0,K}, \tag{5.15}$$

and

$$G_j'(z) + \left[\frac{\lambda}{\alpha\xi}H'(z) - \frac{\phi}{\alpha\xi(1-z)}\right]G_j(z) = -\frac{\phi}{\alpha\xi(1-z)}P_{0,j-1}, \ j = \overline{1, K-1}, \tag{5.16}$$

where

$$H'(z) = \frac{B(z)-1}{1-z}.$$

Next, by multiplying (5.15) and (5.16) by $e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}}$, we obtain

$$\frac{d}{dz}\left(e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}}G_0(z)\right) = -\frac{\eta}{\alpha\xi}e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}-1}P_{0,K}, \tag{5.17}$$

and

$$\frac{d}{dz}\left(e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}}G_j(z)\right) = -\frac{\phi}{\alpha\xi}e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}-1}P_{0,j-1}, j = \overline{1, K-1}. \tag{5.18}$$

Then, integrating form 0 to z, we get

$$G_0(z) = e^{-\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{-\frac{\phi}{\alpha\xi}}\left\{G_0(0) - \frac{\eta}{\alpha\xi}C(z)P_{0,K}\right\}, \tag{5.19}$$

and

$$G_j(z) = e^{-\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{-\frac{\phi}{\alpha\xi}}\left\{G_j(0) - \frac{\phi}{\alpha\xi}C(z)P_{0,j-1}\right\}, \ j = \overline{1, K-1}, \tag{5.20}$$

where

$$C(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi}H(x)}(1-x)^{\frac{\phi}{\alpha\xi}-1}dx.$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{n,0} > 0$ and $z = 1$ is the root of the denominator of the right hand side of equation (5.19), so $z = 1$ must be the root of the numerator of the right hand side of equation (5.19). Thus, we get

$$P_{0,0} = G_0(0) = \frac{\eta}{\alpha\xi}CP_{0,K}, \tag{5.21}$$

where

$$C =: C(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi}H(x)}(1-x)^{\frac{\phi}{\alpha\xi}-1}dx.$$

This implies

$$P_{0,K} = \frac{\alpha\xi}{\eta C}P_{0,0}. \tag{5.22}$$

Similarly, as $G_j(1) = \sum_{n=0}^{\infty} P_{n,j} > 0,\ \ j = \overline{1,K-1}$, and $z = 1$ is the root of the denominator of the right hand side of equation (5.20), so $z = 1$ must be the root of the numerator of the right hand side of equation (5.20). Thus

$$P_{0,j} = G_j(0) = \frac{\phi}{\alpha\xi}CP_{0,j-1}, \quad j = \overline{1,K-1}. \tag{5.23}$$

Now, using equation (5.23) repeatedly, we obtain

$$P_{0,j} = A^j P_{0,0},\ \ j = \overline{1,K-1},\ \text{ where } A = \frac{\phi C}{\alpha\xi}. \tag{5.24}$$

Next, substituting equations (5.22) and (5.24) into (5.19) and (5.20) respectively, we get

$$G_0(z) = e^{-\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{-\frac{\phi}{\alpha\xi}}\left\{1 - \frac{C(z)}{C}\right\}P_{0,0}, \tag{5.25}$$

and

$$G_j(z) = e^{-\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{-\frac{\phi}{\alpha\xi}}\left\{1-\frac{C(z)}{C}\right\}A^j P_{0,0},\ \ j=\overline{1,K-1}. \tag{5.26}$$

From equations (5.25)-(5.26), we find the expression of the probability generating function $G_j(z)$ for $j=\overline{0,K-1}$ in terms of $P_{0,0}$, and from equation (5.22) we have $P_{0,K}$ in terms of $P_{0,0}$, while $P_{0,j}$ in terms of $P_{0,0}$ is given equation (5.24). Thus, substituting equations (5.22) and (5.24)-(5.26) in equation (5.12), we obtain the expression of the probability generating function $G_K(z)$ in term of $P_{0,0}$.

Next, substituting equations (5.13)-(5.14) into (5.12), we get

$$G_K(z) = \frac{\beta\mu(1-z)P_{0,K} - z\phi\sum_{j=0}^{K-1}(G_j(z)-G_j(1))}{\lambda z(B(z)-1)+\beta\mu(1-z)}. \tag{5.27}$$

Then, applying L'Hospital's rule, we find

$$G_K(1) = \frac{\beta\mu P_{0,K} + \phi\sum_{j=0}^{K-1}G_j'(1)}{\beta\mu - \lambda B'(1)}. \tag{5.28}$$

Next, form equations (5.10) and (5.13), using L'Hospital's rule, we get

$$G_0'(1) = \frac{\lambda B'(1)}{\alpha\xi+\phi}G_0(1). \tag{5.29}$$

Similarly, form equations (5.11) and (5.14), we have

$$G_j'(1) = \frac{\lambda B'(1)}{\alpha\xi+\phi}G_j(1),\ \ j=\overline{1,K-1}. \tag{5.30}$$

Then, equations (5.29) and (5.30) imply

$$G_j'(1) = \frac{\lambda B'(1)}{\alpha\xi+\phi}G_j(1),\ \ j=\overline{0,K-1}. \tag{5.31}$$

Now, via equations (5.25) and (5.26), using L'Hospital's rule, we obtain

$$G_j(1) = A^{j-1}P_{0,0}, j=\overline{0,K-1}. \tag{5.32}$$

This implies

$$\sum_{j=0}^{K-1} G_j'(1) = \frac{\lambda B'(1)}{\alpha\xi + \phi} \frac{1 - A^K}{A(1-A)} P_{0,0}. \tag{5.33}$$

Next, substituting equations (5.22) and (5.33) into (5.28), we get

$$G_K(1) = \frac{1}{\beta\mu - \lambda B'(1)} \left\{ \frac{\phi\lambda B'(1)}{\alpha\xi + \phi} \frac{1 - A^K}{A(1-A)} + \frac{\beta\mu\alpha\xi}{\eta C} \right\} P_{0,0}. \tag{5.34}$$

Finally, in order to obtain $P_{0,0}$ we use the normalization condition given as

$$\sum_{n=0}^{\infty} \sum_{j=0}^{K-1} P_{n,j} + \sum_{n=0}^{\infty} P_{n,K} = 1 \Longleftrightarrow \sum_{j=0}^{K-1} G_j(1) + G_K(1) = 1. \tag{5.35}$$

So, substituting equations (5.32) and (5.34) into (5.35), we get

$$P_{0,0} = \left\{ \frac{\beta\mu\alpha\xi}{\eta C(\beta\mu - \lambda B'(1))} + \frac{1 - A^K}{A(1-A)} \left(\frac{\phi\lambda B'(1)}{(\beta\mu - \lambda B'(1))(\alpha\xi + \phi)} + 1 \right) \right\}^{-1}.$$

□

5.4 System Performance measures

The indices that are of general interest for the evaluation of the performances of our system include:

– The probability that the server is idle during busy period.

$$P_{0,K} = \frac{\alpha\xi}{\eta C} P_{0,0}.$$

– The probability that the server is in vacation period.

$$P_v = \frac{1 - A^K}{A(1-A)} P_{0,0}.$$

– The probability that the server is serving customers during busy period.

$$P_b = 1 - P_v - P_{0,K}.$$

– The mean system size when the server is on vacation.

$$E[L_V] = \sum_{j=0}^{K-1} \lim_{z\to 1} G'_j(z).$$

Form equation (5.33), we have

$$E[L_V] = \frac{\lambda B'(1)}{\alpha\xi+\phi}\frac{1-A^K}{A(1-A)}P_{0,0}.$$

– The mean system size when the server is in busy period.

$$E[L_K] = \lim_{z\to 1} G'_K(z).$$

Differentiating equation (5.27) and using L'Hospital's rule, we obtain

$$E[L_K] = \frac{\phi}{2(\beta\mu-\lambda B'(1))}\sum_{j=0}^{K-1} G''_j(1) + \frac{\phi(2\beta\mu+\lambda B''(1))}{2(\beta\mu-\lambda B'(1))^2}\sum_{j=0}^{K-1} G'_j(1) + \frac{\beta\mu\lambda(2B'(1)+B''(1))}{2(\beta\mu-\lambda B'(1))^2}P_{0,K}, \tag{5.36}$$

where $G''_j(1)$ is obtained by differentiating twice $G_j(z)$ at $z=1$ for $j=\overline{0,K}$. So, differentiating twice equations (5.10) and (5.11) and taking $z=1$, we find

$$\sum_{j=0}^{K-1} G''_j(1) = \frac{2\lambda B'(1)}{2\alpha\xi+\phi}E[L_V]. \tag{5.37}$$

Next, substituting equation (5.37) into (5.36), we find

$$E[L_K] = \left[\frac{\phi\lambda B'(1)}{(2\alpha\xi+\phi)(\beta\mu-\lambda B'(1))} + \frac{\phi(2\beta\mu+\lambda B''(1))}{2(\beta\mu-\lambda B'(1))^2}\right]E[L_V] + \frac{\beta\mu\lambda(2B'(1)+B''(1))}{2(\beta\mu-\lambda B'(1))^2}P_{0,K}.$$

– The mean system size.

$$E[L] = E[L_V] + E[L_K].$$

– The mean queue length.

$$E[L_q] = \sum_{j=0}^{K}\sum_{n=1}^{\infty}(n-1)P_{n,j} = E[L] - \left[1 - \sum_{j=0}^{K} P_{0,j}\right].$$

– The mean number of customers served per unit time.

$$N_s = \beta\mu \sum_{n=1}^{\infty} P_{n,K} = \beta\mu P_b.$$

– The average rate of abandonment of a customer due to impatience.

$$R_a = \alpha\xi \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} nP_{n,j} = \alpha\xi E[L_V].$$

– The average rate of retention of impatient customers.

$$R_e = (1-\alpha)\xi \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} nP_{n,j} = (1-\alpha)\xi E[L_V].$$

5.5 Particular cases

In this section, we present some special cases of our queueing model which are consistent with the existing literature. These latter can be deduced by setting appropriate parameters as follows:

Case 1: No variant vacations, no batch arrival, no retention, and no feedback

When $K = 1$, $b_1 = 1$, $\alpha = 1$, and $\beta = 1$, that is, if the server comes back from vacation to an empty system, he remains idle waiting for the first arrival, then he starts a busy periods. Customers arrive to the system one by one, they are persistent and never returns to the system as a feedback customer. In this case, the equations (5.1)–(5.7) can be abstracted as follow:

$$(\lambda + \phi)P_{0,0} = \xi P_{1,0} + \eta P_{0,1}, \tag{5.38}$$

$$(\lambda + \phi + n\xi)P_{n,0} = \lambda P_{n-1,0} + (n+1)\xi P_{n+1,0}, \;\; n \geq 1, \tag{5.39}$$

$$(\lambda + \eta)P_{0,1} = \phi P_{0,0} + \mu P_{1,1}, \tag{5.40}$$

$$(\lambda + \mu)P_{n,1} = \lambda P_{n-1,1} + \mu P_{n+1,1} + \phi P_{n,0}, \; n \geq 1. \tag{5.41}$$

Then, the steady-state probabilities $P_{.,0}$ and $P_{.,1}$ from equations (5.38)–(5.41) are as:

$$P_{.,0} = \frac{(\phi+\xi)(\mu(\eta C - \xi P_{0,0}) - \lambda\eta C)}{\phi C(\phi\mu + \xi(\mu-\lambda))},$$

and

$$P_{.,1} = \frac{\lambda\phi\eta C + \xi\mu(\phi+\xi)P_{0,0}}{\eta C(\mu\phi + \xi(\mu-\lambda))},$$

where

$$P_{0,0} = \frac{\phi\eta C(\phi+\xi)(\mu-\lambda)}{(\mu\phi\eta + \mu\eta\xi - \lambda\eta\xi + \mu\phi^2 + \mu\phi\xi)\xi},$$

and

$$C = \int_0^1 e^{\frac{-\lambda}{\xi}x}(1-x)^{\frac{\phi}{\xi}-1}dx.$$

which match with the results of $M/M/1$ queueing model with single vacation, waiting server, and impatient customers, given in Padmavathy et al. (2011).

Case 2: No waiting server, no batch arrival, no retention, and no feedback

When $\eta \to +\infty$, $b_1 = 1$, $\alpha = 1$, and $\beta = 1$, that is, Whenever a system becomes empty, the server goes on vacation. Customers arrive to the system one by one, they are persistent and never returns to the system as a feedback customer. In this case, the equations (5.1)–(5.7) can be abstracted as follow:

$$(\lambda+\phi)P_{0,0} = \xi P_{1,0} + \mu P_{1,K}, \tag{5.42}$$

$$(\lambda+\phi+n\xi)P_{n,0} = \lambda P_{n-1,0} + (n+1)\xi P_{n+1,0},\ \ n \geq 1, \tag{5.43}$$

$$(\lambda+\phi)P_{0,j} = \xi P_{1,j} + \phi P_{0,j-1},\ \ 1 \leq j \leq K-1, \tag{5.44}$$

$$(\lambda+\phi+n\xi)P_{n,j} = \lambda P_{n-1,j} + (n+1)\xi P_{n+1,j},\ \ n \geq 1,\ \ 1 \leq j \leq K-1, \tag{5.45}$$

$$\lambda P_{0,K} = \phi P_{0,K-1}, \tag{5.46}$$

$$(\lambda+\mu)P_{n,K} = \lambda P_{n-1,K} + \mu P_{n+1,K} + \phi \sum_{j=0}^{K-1} P_{n,j},\ n \geq 1. \tag{5.47}$$

From the latest equations, the steady-state-probabilities of the number of customers in the system have the following from:

$$P_{.,j} = A^{j-1}P_{0,0}, \qquad j = \overline{0, K-1},$$

and

$$P_{.,K} = \frac{\phi}{\mu-\lambda}\left(\frac{\lambda(1-A^K)}{(\phi+\xi)A(1-A)} + \frac{\mu}{\lambda}A^{K-1}\right)P_{0,0},$$

where

$$P_{0,0} = \left\{\frac{(\mu\phi + (\mu-\lambda)\xi)(1-A^K)}{(\mu-\lambda)(\phi+\xi)A(1-A)} + \frac{\mu\phi A^{K-1}}{\lambda(\mu-\lambda)}\right\}^{-1},$$

with

$$A = \frac{\phi C}{\xi},$$

such that

$$C = \int_0^1 e^{\frac{-\lambda}{\xi}x}(1-x)^{\frac{\phi}{\xi}-1}dx.$$

The obtained results match with that of an $M/M/1$ queueing model with impatient customers and a variant of multiple vacation policy presented by Yue et al. (2014).

5.6 Cost model

This part is devoted to develop a model for the costs incurred in the queueing system using the following elements:

- C_1 : Cost per unit time when the server is working during busy period.
- C_2 : Cost per unit time when the server is idle during busy period.
- C_3 : Cost per unit time when the server is on vacation.

- C_4 : Cost per unit time when customers join the queue and wait for service.
- C_5 : Cost per service per unit time.
- C_6 : Cost per unit time of serving a feedback customer.
- C_7 : Cost per unit time when a customer reneges.
- C_8 : Cost per unit time when a customer is retained.

The costs for this queueing model can be presented as follows:

– The busy period cost (C_b). When there are customers in the system, the server works on them. Thus, busy period cost is needed and is given by $C_b = C_1 P_b$.

– The idle period cost (C_{id}). When the server finishes serving all customers present in the system, he waits a random period of time for a new arrival before he goes on vacation. A cost for this period is required, it is given by $C_{id} = C_2 P_{0,K}$.

– The vacation period cost (C_v). Once the normal busy period is over, the server switches to the vacation period. A cost for this period is necessary, it is done by $C_v = C_3 P_v$.

– The cost for the number of customers waiting for service (C_q). A cost is needed for customers in the queue waiting to be serviced, it is given as $C_q = C_4 E[L_q]$.

– The service cost (C_s). The service cost per customer should be included in the total cost of the queueing system. It is given as $C_s = \mu C_5$.

– The service cost for feedback customers (C_f). After getting incomplete or unsatisfactory service, the customers may comeback to the system as feedback customers in order to get another service. Thus, a cost is required for these customers. The service cost per feedback customer is done as $C_f = \beta' \mu C_6$.

– The cost due to lost abandonment (C_a). During vacation period, customers may lose their patience and leave the system. Thus, the reneging cost per customer is required and it is given as $C_a = C_7 R_a$.

– The retention cost for reneged customers (C_e). Due to reneging of impatient customers, the retention cost must be incorporated into the total cost of the system. Therefore, the retention cost per customer is $C_e = C_8 R_e$.

Finally, using the cost parameters listed above, the total expected cost per unit time of the system is presented as

$$\mathcal{T}_c = C_1 P_b + C_2 P_{0,K} + C_3 P_v + C_4 E[L_q] + \mu(C_5 + \beta' C_6) + C_7 R_a + C_8 R_e.$$

The total expected profit per unit time of the system is given by

$$\mathcal{T}_p = \mathcal{T}_r - \mathcal{T}_c,$$

where $\mathcal{T}_r$ is the total expected revenue per unit time of the system,

$$\mathcal{T}_r = Rev \times \mu \times P_b,$$

where Rev denotes the revenue earned by providing service to a customer.

The analytic study of the optimization behavior of the expected cost function is a non-trivial task to undertake, since the decision variables appear in an expression which is a highly complex. Thus, we consider in this paper the cost optimization problem under a given cost structure via quadratic fit search method (QFSM) and particle swarm optimization (PSO). We employ the two methods to solve the optimization problem, then we compare the two algorithms. For a detailed algorithm of QFSM and PSO, one may refer to Rao (2009) and Radin (1997), respectively.

We focus on the optimization of the service rate μ in different cases in order to minimize the cost function F. Therefore, a total expected cost function must be developed in order to determine an optimum regular service rate μ^* and the optimum expected cost $F(\mu^*)$. Consequently, the optimization problem can be illustrated mathematically as:

$$\text{Minimize: } F(\mu) = \mathcal{T}_c.$$

5.7 Numerical results

To study the behavior of system characteristics with respect to the changes of its parameters, we execute a numerical experiment by coding computer program in R software. For computational convenience, we arbitrarily choose the values of different system parameters and costs. We suppose that the arrival batch size X follows a geometric distribution with parameter p, that is

$$P(X = l) = b_l = (1-p)^{l-1}p,\ 0 < p < 1\ (l = 1, 2, \ldots).$$

Then, we easily have

$$B(z) = \frac{pz}{1-(1-p)z}, \quad E(X) = B'(1) = \frac{1}{p}, \quad \text{and} \quad E(X^2) = B''(1) = \frac{2(1-p)}{p^2}.$$

Next, for the whole analysis, we fixe $C_1 = 20$, $C_2 = 10$, $C_3 = 8$, $C_4 = 20$, $C_5 = 30$, $C_6 = 15$, $C_7 = 20$, $C_8 = 25$, and $Rev = 200$.

5.7.1 Optimization of service rate μ

The main goal in this part of paper is to determine the optimal value of service rate μ for different cases in order to minimize the cost function F. The tolerance of QFSM is $\epsilon = 10^{-6}$, swarm size, maximum number of iterations and learning factors of PSO are taken as 20, 100 and $c_1 = c_2 = 2$, respectively.

The total expected cost incurred on the system $\mathcal{T}_c$ can be minimized with respect to the decision parameter service μ using QFSM and PSO. To make the study more useful from the cost-benefit view point, the total cost function is presented in Tables 5.1-5.4 and plotted (using QFSM) in Figures 5.2-5.5 by varying values of λ, K, η, and ϕ, respectively.

- Table 5.1 (resp. Figure 5.2) presents the minimum values of μ along with $F(\mu^*)$ (resp. $\mathcal{T}_c$) for various λ. The other parameters are chosen as $K = 7$, $p = 0.75$, $\beta = 0.7$, $\eta = 3$, $\phi = 1.1$, $\alpha = 0.6$, and $\xi = 1.5$.

- Table 5.2 (resp. Figure 5.3) illustrates the optimum values of μ along with $F(\mu^*)$ (resp. $\mathcal{T}_c$) for various values of K. The other parameters are taken as $\lambda = 1$, $p = 0.75$, $\beta = 0.7$, $\eta = 3$, $\phi = 1.1$, $\alpha = 0.6$, and $\xi = 1.5$.

- Table 5.3 (resp. Figure 5.4) displays the optimal values of μ along with $F(\mu^*)$ (resp. $\mathcal{T}_c$) for different values of η. The other parameters are fixed as $K = 7$, $p = 0.75$, $\beta = 0.7$, $\lambda = 1$, $\phi = 1.1$, $\alpha = 0.6$, and $\xi = 1.5$.

- Table 5.4 (resp. Figure 5.5) illustrates the minimum values of μ along with $F(\mu^*)$ (resp. $\mathcal{T}_c$) for various values of ϕ. The other parameters are taken as $K = 7$, $p = 0.75$, $\beta = 0.7$, $\lambda = 1$, $\eta = 3$, $\alpha = 0.6$, and $\xi = 1.5$.

Table 5.1: The optimal values μ^* and $F(\mu^*)$ for various values of λ.

	QFSM		PSO	
λ	μ^*	$F(\mu^*)$	μ^*	$F(\mu^*)$
0.55	1.818215	102.873936	1.8182	102.8739
0.65	2.082011	115.988351	2.0820	115.9884
0.75	2.340349	128.746239	2.3404	128.7462
0.85	2.594235	141.207636	2.5942	141.2076
0.95	2.844401	153.416642	2.8444	153.4166

Table 5.2: The optimal values μ^* and $F(\mu^*)$ for various values of K.

	QFSM		PSO	
K	μ^*	$F(\mu^*)$	μ^*	$F(\mu^*)$
1	2.989518	158.255259	2.9895	158.2553
3	2.972585	159.195156	2.9726	159.1952
5	2.969294	159.379618	2.9693	159.3796
7	2.968266	159.437363	2.9683	159.4374
9	2.967902	159.457829	2.9679	159.4578

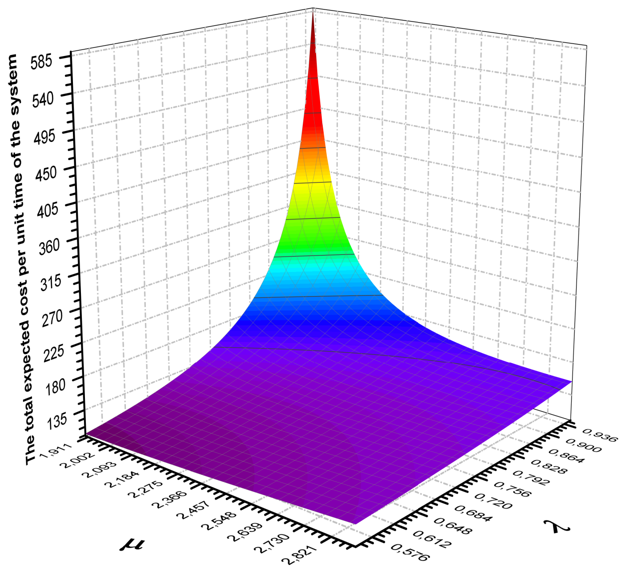

Figure 5.2: $\mathcal{T}_c$ vs. λ and μ.

Table 5.3: The optimal values μ^* and $F(\mu^*)$ for various values of η.

	QFSM		PSO	
η	μ^*	$F(\mu^*)$	μ^*	$F(\mu^*)$
0.7	3.006193	157.345151	3.0062	157.3452
1.4	2.985068	158.50078	2.9851	158.5008
2.1	2.975202	159.048926	2.9752	159.0489
2.8	2.969484	159.368998	2.9695	159.369
3.5	2.965751	159.578848	2.9658	159.5788

Table 5.4: The optimal values μ^* and $F(\mu^*)$ for various values of ϕ.

	QFSM		PSO	
ϕ	μ^*	$F(\mu^*)$	μ^*	$F(\mu^*)$
0.6	2.904913	162.926958	2.9049	162.927
1.2	2.975776	158.931596	2.9758	158.9316
1.8	3.005696	156.699431	3.0057	156.6994
2.4	3.022255	155.312738	3.0226	155.3127
3.0	3.032777	154.382524	3.0328	154.3825

From Figures 5.3-5.5, it is clearly seen the convexity of the curves for different values of K, η, and ϕ which shows that there exists a certain value of the service rate μ

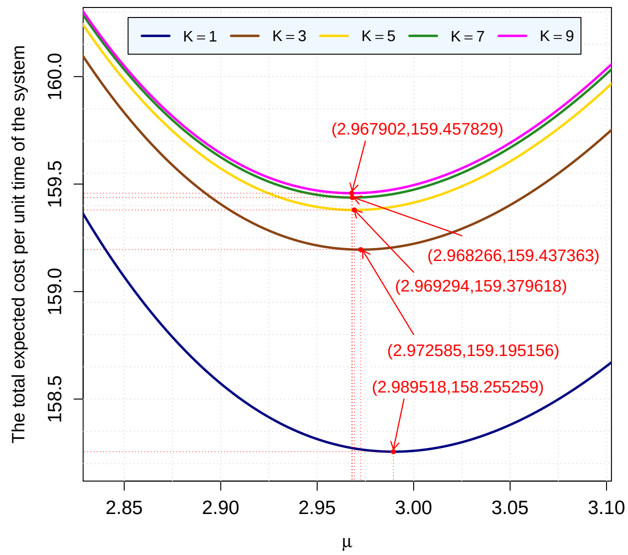

Figure 5.3: $\mathcal{T}_c$ vs. K and μ.

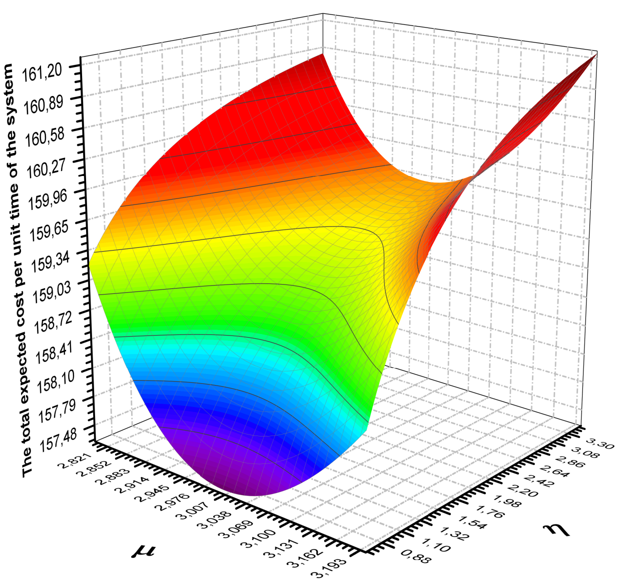

Figure 5.4: $\mathcal{T}_c$ vs. η and μ.

that minimizes the total expected cost function for the chosen set of model parameters. However, the convexity is not clear in Figure 5.2, this is due to the choice of the parameters, especially λ and μ. We had to choose the values for the two parameters in such a way that the stability condition $\lambda E(X) < \beta\mu$ is verified.

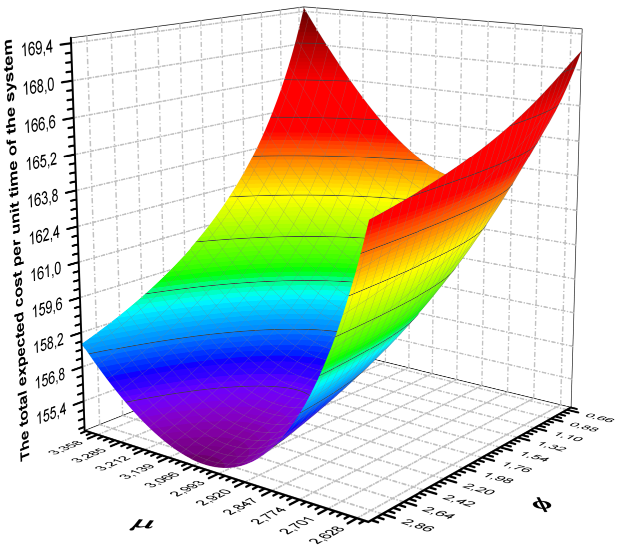

Figure 5.5: $\mathcal{T}_c$ vs. ϕ and μ.

Further, it is worth noting that the fastest optimization algorithms look only for a local solution, a point where the objective function is smaller than any other possible points nearby. They don't always find the best of all such minima, namely the global solution. Global solutions are needed (or highly desirable) in some applications, but they are generally difficult to identify and even more difficult to locate. An important particular case is convex programming, where all local solutions are also global solutions. Linear programming problems fall into the category of convex programming. Nevertheless, nonlinear general problems, both constrained and unconstrained, may have local solutions that are not global solutions. From Figures 5.3-5.5, we clearly observe that the optimal value is the global solution, as when F is convex. Then, any local minimizer μ^* is a global minimizer of F, see Theorem 2.5 in Nocedal and Wright (2006).

Tables 5.1-5.4 present the optimum values of μ along with the minimum expected cost for various values of λ, K, η, and ϕ, respectively. A decreasing (resp. increasing) trend is seen in μ^* with the increase of η and K (resp. λ and ϕ). Further, the optimal expected cost $F(\mu^*)$ increases with the increase of λ, η and K. This is because the mean number of the customers in the system as well as the average rate of lost customers increase with the increasing of λ, η and K which results in the increasing of the optimal expected cost. In addition, $F(\mu^*)$ decreases with ϕ which is quite reasonable; the mean queue length and the average rate of reneging decrease which leads to the decreasing in the optimal expected cost.

Via the results presented in the Tables, we observe that both the two adopted methods give identical results, but the convergence is faster in PSO algorithm. Then, as it has been pointed out in Laxmi et al. (2013), the advantage of using the PSO algorithm lies in the ease with which it may be adjusted and implemented utilizing only a velocity operator in order to drive the search in the search space, while the QFSM depends on the proper choice of the initial 3-point approximation.

Furthermore, the results obtained are quite interesting and can be applied to many real-time machining systems for upgrading the system by suitable choice of service rate μ.

5.7.2 Impact of λ, p, and β

We check the effect of λ, p, and β on different performance measures and costs, the values of these parameters are presented in Table 5.5 and Figures 5.6-5.7. The other parameters of the model as taken as $K = 5$, $\eta = 3$, $\phi = 1.1$, $\alpha = 0.6$, $\xi = 0.9$, and $\mu = 3.5$.

Table 5.5: Impact of λ, p, and β.

λ	p	β	$E[L_V]$	$E[L_K]$	R_a	P_b	P_v	$P_{0,K}$	$\mathcal{T}_c$	$\mathcal{T}_r$
		0.7	0.3612	0.8683	0.1950	0.3286	0.5923	0.0791	154.4013	229.9915
	0.60	0.8	0.3858	0.6840	0.2083	0.2827	0.6327	0.0846	146.5769	197.9150
		0.9	0.4044	0.5629	0.2184	0.2481	0.6633	0.0886	139.7439	173.6907
		0.7	0.3209	0.5137	0.1733	0.2558	0.6578	0.0864	146.0160	179.0679
0.6	0.75	0.8	0.3359	0.4186	0.1814	0.2209	0.6886	0.0905	139.5682	154.6544
		0.9	0.3473	0.3530	0.1876	0.1944	0.7120	0.0936	133.5388	136.0991
		0.7	0.2846	0.3538	0.1537	0.2094	0.7000	0.0906	141.6145	146.5738
	0.9	0.8	0.2947	0.2939	0.1591	0.1813	0.7249	0.0938	135.6575	126.8854
		0.9	0.3024	0.2513	0.1633	0.1598	0.7439	0.0963	129.9293	111.8600
		0.7	0.3779	1.2075	0.2041	0.3929	0.5312	0.0759	161.3242	275.0287
	0.60	0.8	0.4126	0.9266	0.2228	0.3371	0.5802	0.0829	151.9645	235.9600
		0.9	0.4387	0.7493	0.2369	0.2952	0.6167	0.0881	144.3036	206.6104
		0.7	0.3471	0.6796	0.1874	0.3045	0.6099	0.0856	149.8402	213.1156
0.7	0.75	0.8	0.3681	0.5452	0.1988	0.2624	0.6468	0.0908	142.8413	183.6394
		0.9	0.3840	0.4547	0.2074	0.2304	0.6748	0.0948	136.4914	161.3262
		0.7	0.3131	0.4550	0.1691	0.2485	0.6602	0.0913	144.2319	173.9113
	0.9	0.8	0.3272	0.3740	0.1767	0.2147	0.6899	0.0954	138.0070	150.2730
		0.9	0.3379	0.3173	0.1825	0.1890	0.7125	0.0985	132.1152	132.2917
		0.7	0.3810	1.6623	0.2058	0.4602	0.4687	0.0711	170.1157	322.1654
	0.60	0.8	0.4280	1.2348	0.2311	0.3936	0.5265	0.0799	158.3052	275.5482
		0.9	0.4632	0.9783	0.2501	0.3439	0.5697	0.0864	149.4117	240.7166
		0.7	0.3651	0.8834	0.1972	0.3549	0.5614	0.0837	154.1822	248.4298
0.8	0.75	0.8	0.3933	0.6961	0.2124	0.3051	0.6047	0.0902	146.4034	213.5712
		0.9	0.4145	0.5735	0.2239	0.2676	0.6374	0.0950	139.6203	187.2912
		0.7	0.3362	0.5737	0.1815	0.2887	0.6202	0.0911	147.0579	202.1041
	0.9	0.8	0.3549	0.4660	0.1917	0.2490	0.6548	0.0962	140.4808	174.3076
		0.9	0.3692	0.3921	0.1993	0.2189	0.6810	0.1001	134.3817	153.2326

– For fixed p and β, along the increases of λ, an increasing trend is observed in P_b,

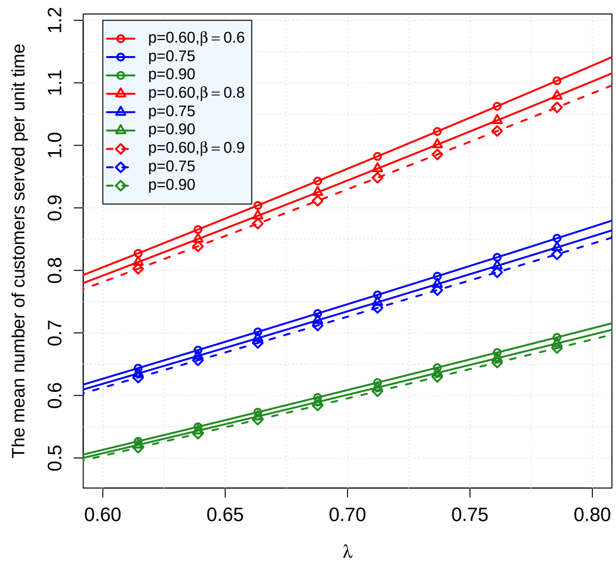

Figure 5.6: Effect of λ, p, and β on N_s.

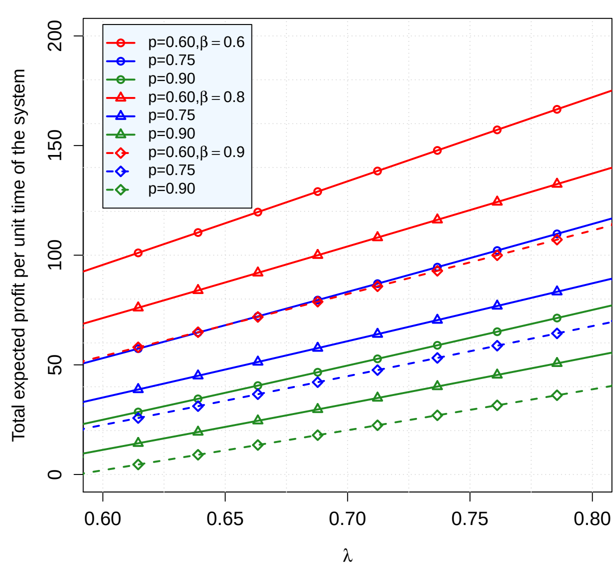

Figure 5.7: Effect of λ, p, and β on $\mathcal{T}_p$.

$E[L_V]$, $E[L_K]$, and R_a, while a decreasing trend is seen in $P_{0,K}$ and P_v with λ. This implies an increasing in $\mathcal{T}_c$, $\mathcal{T}_r$ and $\mathcal{T}_p$. The obtained results agree absolutely with our intuition; the mean number of the customers in the system increases with the increasing of λ. Thus, the larger the mean number of customers in the system, the higher the mean number of customers served.

– For fixed λ and β, with the increases of p, an increasing trend is remarked in P_v and $P_{0,K}$. Further, a decreasing trend is observed in P_b, $E[L_V]$, $E[L_K]$, and R_a with p. This implies a diminution in $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_p$. This is because the mean number of customers in the system decreases with p. Thus, the mean number of customers served is reduced.

– For fixed λ and p, along the increases of β, we observe an increasing trend in $P_{0,K}$, P_v, $E[L_V]$, and R_a. In addition, a decreasing trend is seen in P_b and $E[L_K]$ with β. This leads to a decrease in $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_p$.

– For fixed p and β, along the increase of λ, N_s increases monotonically, as it should be. Moreover, one may also see that for higher values of β and smaller values p, N_s is reduced. Therefore, $\mathcal{T}_p$ decreases.

5.7.3 Impact of K, ξ, and α

We vary K, ξ, and α, their values are given in the respective Table 5.6 and Figures 5.8-5.9. The other default parameters are chosen as $p = 0.75$, $\beta = 0.7$, $\lambda = 1$, $\eta = 3$, $\phi = 1.1$, and $\mu = 3$.

Table 5.6: Impact of K, ξ, and α.

K	ξ	α	$E[L_K]$	$E[L_V]$	N_s	R_a	R_e	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_p$
		0.3	2.4102	0.2899	1.2638	0.0696	0.1624	163.7101	451.3396	287.6295
	0.8	0.6	2.2180	0.2625	1.2073	0.1260	0.0840	158.6905	431.1953	272.5049
		0.9	2.0845	0.2398	1.1607	0.1726	0.0192	155.0977	414.5361	259.4384
		0.3	2.2180	0.2625	1.2073	0.1260	0.2940	163.9399	431.1953	267.2554
1	1.6	0.6	1.9847	0.2207	1.1215	0.2119	0.1412	156.7723	400.5296	243.7573
		0.9	1.8426	0.1904	1.0592	0.2741	0.0305	152.2130	378.2885	226.0755
		0.3	2.0845	0.2398	1.1607	0.1726	0.4028	164.6884	414.5361	249.8477
	2.4	0.6	1.8426	0.1904	1.0592	0.2741	0.1828	156.0203	378.2885	222.2682
		0.9	1.7061	0.1578	0.9924	0.3409	0.0379	150.8832	354.4259	203.5427
		0.3	2.4259	0.3399	1.2518	0.0816	0.1903	165.4641	447.0590	281.5949
	0.8	0.6	2.1984	0.3134	1.1829	0.1504	0.1003	159.6924	422.4637	262.7714
		0.9	2.0342	0.2913	1.1236	0.2097	0.0233	155.4353	401.2820	245.8467
		0.3	2.1984	0.3134	1.1829	0.1504	0.3510	165.9605	422.4637	256.5032
4	1.6	0.6	1.9062	0.2725	1.0717	0.2616	0.1744	157.5341	382.7672	225.2331
		0.9	1.7138	0.2418	0.9851	0.3482	0.0387	151.8575	351.8160	199.9585
		0.3	2.0342	0.2913	1.1236	0.2097	0.4894	167.0877	401.2820	234.1943
	2.4	0.6	1.7138	0.2418	0.9851	0.3482	0.2322	156.6943	351.8160	195.1217
		0.9	1.5136	0.2074	0.8853	0.4481	0.0498	150.0523	316.1646	166.1123
		0.3	2.4273	0.3443	1.2507	0.0826	0.1928	165.6206	446.6771	281.0565
	0.8	0.6	2.1964	0.3184	1.1805	0.1529	0.1019	159.7915	421.5995	261.8079
		0.9	2.0287	0.2969	1.1195	0.2138	0.0238	155.4721	399.8358	244.3637
		0.3	2.1964	0.3184	1.1805	0.1529	0.3567	166.1605	421.5995	255.4390
7	1.6	0.6	1.8969	0.2787	1.0658	0.2675	0.1783	157.6248	380.6526	223.0278
		0.9	1.6961	0.2489	0.9749	0.3584	0.0398	151.8088	348.1880	196.3792
		0.3	2.0287	0.2969	1.1195	0.2138	0.4989	167.3495	399.8358	232.4863
	2.4	0.6	1.6961	0.2489	0.9749	0.3584	0.2389	156.7867	348.1880	191.4013
		0.9	1.4831	0.2153	0.8683	0.4650	0.0517	149.9207	310.1067	160.1860

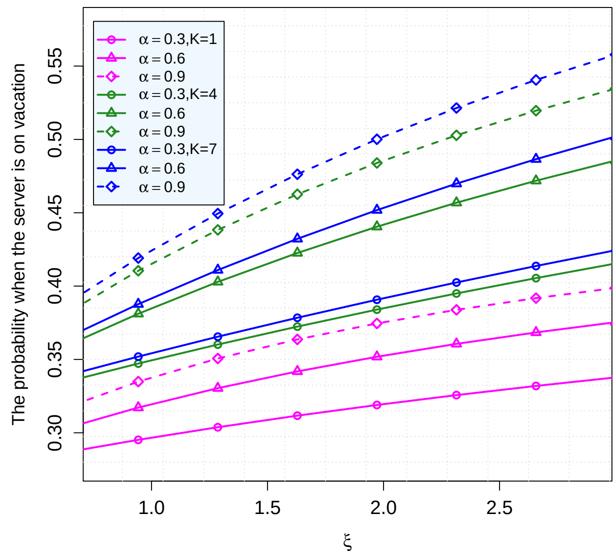

Figure 5.8: Effect of K, ξ, and α on P_v.

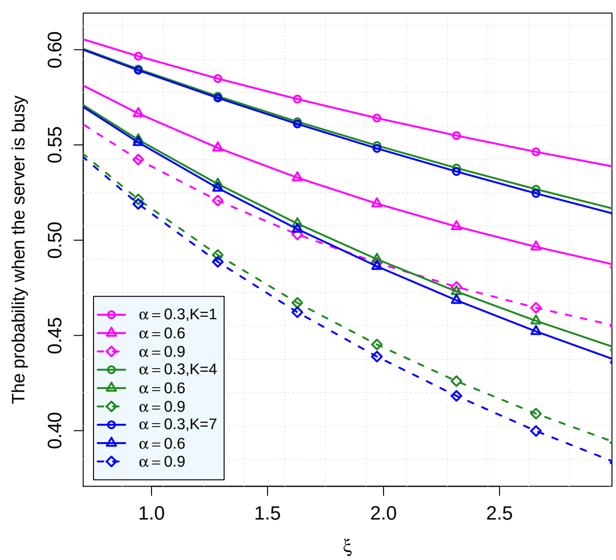

Figure 5.9: Effect of K, ξ, and α on P_b.

– For fixed α, increases in K and ξ implies an increase in R_a, R_e, and $\mathcal{T}_c$, while $E[L_V]$ increases with K and decreases with ξ. The other performance measures and costs, $E[L_K]$, N_s, $\mathcal{T}_r$, and $\mathcal{T}_p$ decrease with the increasing of K and ξ.

– For fixed K and ξ, except R_a, all the other performance measures and costs decrease with the increase of α. Moreover, as intuitively expected, P_v (resp. P_b) increases (resp. decreases) with ξ. In addition, the higher α and K, the smaller the probability that the server is working during the busy period P_b and the bigger the probability that the server is on vacation P_v. This is due to the fact that, the number of customers in the system decreases with α and ξ, and increase with K. Thus, the probability of busy period decreases which implies a decrease in the mean number of customers served. Further, one can conclude that the retention probability α' has a nice impact on the economy of the queueing system.

5.7.4 Impact of μ, ϕ, and η

We examine the impact of μ, ϕ, and η, their values are mentioned in Table 5.7 and Figure 5.10. The other model parameters are arbitrarily selected as $\lambda = 3$, $\beta = 0.8$, $p = 0.8$, $K = 3$, $\xi = 0.1$, $\alpha = 0.4$, $\eta = 3$, and $\phi = 2$.

Table 5.7: Impact of μ, ϕ, and η.

μ	ϕ	η	P_b	P_v	$P_{0,K}$	$E[L]$	$E[L_q]$	$\mathcal{T}_p$
5.0	2.0	2.0	0.9368	0.0385	0.0247	19.6869	18.7270	377.7967
		2.5	0.9368	0.0418	0.0214	19.7672	18.8054	376.1619
		3.0	0.9367	0.0444	0.0189	19.8288	18.8656	374.9070
	2.5	2.0	0.9370	0.0360	0.0270	19.4320	18.4755	383.0391
		2.5	0.9369	0.0394	0.0237	19.4966	18.5383	381.7286
		3.0	0.9369	0.0420	0.0211	19.5470	18.5872	380.7077
	3.0	2.0	0.9371	0.0339	0.0290	19.2690	18.3150	386.3820
		2.5	0.9370	0.0374	0.0256	19.3221	18.3665	385.3081
		3.0	0.9370	0.0401	0.0229	19.3640	18.4070	384.4610
5.4	2.0	2.0	0.8667	0.0813	0.0520	9.1744	8.2590	573.7768
		2.5	0.8665	0.0883	0.0452	9.2558	8.3364	572.0877
		3.0	0.8665	0.0936	0.0399	9.3182	8.3957	570.7913
	2.5	2.0	0.8670	0.0759	0.0571	8.9177	8.0094	579.2146
		2.5	0.8669	0.0831	0.0500	8.9834	8.0713	577.8618
		3.0	0.8668	0.0887	0.0445	9.0346	8.1195	576.8080
	3.0	2.0	0.8672	0.0716	0.0612	8.7536	7.8506	582.6724
		2.5	0.8671	0.0789	0.0540	8.8078	7.8506	581.5640
		3.0	0.8671	0.0846	0.0483	8.8505	7.9412	580.6899
5.8	2.0	2.0	0.8063	0.1181	0.0756	6.2294	5.3524	618.5547
		2.5	0.8062	0.1282	0.0656	6.3117	5.4288	616.8190
		3.0	0.8060	0.1360	0.0580	6.3749	5.4875	615.4870
	2.5	2.0	0.8068	0.1103	0.0829	5.9713	5.1044	624.1603
		2.5	0.8067	0.1207	0.0726	6.0379	5.1655	622.7712
		3.0	0.8065	0.1289	0.0646	6.0898	5.2131	621.6892
	3.0	2.0	0.8070	0.1040	0.0890	5.8061	4.9472	627.7167
		2.5	0.8070	0.1146	0.0784	5.8612	4.9971	626.5789
		3.0	0.8069	0.1230	0.0701	5.9047	5.0365	625.6816

– As we expect, for fixed ϕ and η, the performance measures P_v, and $P_{0,K}$ increase with the increase in μ, while P_b, $E[L_K]$, $E[L]$ and $E[L_q]$ decrease monotonically. Therefore, $\mathcal{T}_p$ increases because of the number of customers served, which is quite reasonable.

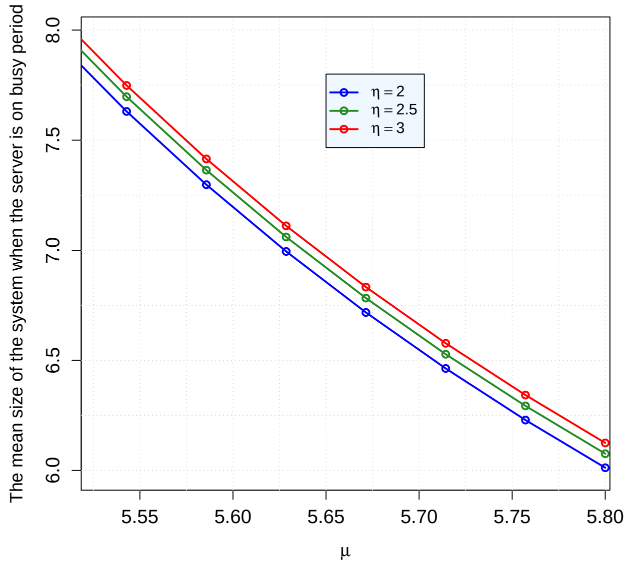

Figure 5.10: $E[L_K]$ vs. μ and η.

– For fixed μ and η, it is depicted that P_b and $P_{0,K}$ increase with ϕ, while P_v, $E[L]$, and $E[L_q]$ decrease with the increasing values of ϕ. Thus, $\mathcal{T}_p$ increases significantly. This trend matches with the realistic situation.

– For fixed μ and ϕ, along the increasing of η, an increasing trend is observed in P_v, $E[L]$, and $E[L_q]$ and a decreasing trend is seen in P_b, $E[L_K]$, $P_{0,K}$, and $\mathcal{T}_p$. This is because the number of customers during the vacation period increases with η. Hence, as the impatience occurs during this period, the number of customers reneged augments which results in the increasing of the total expected profit.

5.8 Managerial insights

Businesses are increasingly dependent on Internet services (call centers) as the basis of their revenues. The downtime (vacation and/or breakdowns) can therefore be directly translated into a loss of customers and consequently of income. The issue addressed in this work concerns the impatient behavior of customers in infinite buffer batch arrival single server Markovian Bernoulli feedback queueing system with waiting server, and K-variant vacations. A managerial implication of fundamental importance lies precisely in the retention strategy which would take, from this point of view, a strongly positive connotation. This study focuses on this concept, offering different companies interesting management perspectives for impatient customer. After presenting the theoretical analysis of the considered queueing model, it was proposed to examine the relationship between the different system parameters and performance measures as well as the cost model of the queueing system. The final outcome carried out in this investigation, theoretical and numerical analysis of the queueing system, marked a transition in the theory of queueing systems with vacation and impatient customers. It is worth noting that the analysis has been more difficult because of diverse parameters governing the system.

5.9 Conclusion

This study focused on the analysis of an infinite capacity batch arrival single server Markovian Bernoulli feedback queueing system subject to functioning K-variant vacations by including the assumption of waiting server, customer's impatience and retention of reneged customers. The steady-state study of the system has been carried out using the PGFs method to evaluate various system metrics in terms of steady-state probabilities. We also considered a cost optimization problem using particle swarm optimization (PSO) and quadratic fit search method (QFSM). The results presented show that the two methods adopted give identical results, but the convergence is faster in the PSO algorithm. Further, we investigated the effect of different parameters on the performance measures and the cost functions of the system through numerical experiments. Our queueing system may be considered as a generalized version of many existing queueing models given by Padmavathy et al. (2011) and Yue et al. (2014) equipped with various features and associated with diverse practical situations. Our model can be further extended to a more general case with general type service times and lead times. Furthermore, the realistic feature of bulk failure can be included.

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Chapter 6

The $M^X/M/c$ Bernoulli feedback queue with variant multiple working vacations and impatient customers: Performance and economic analysis

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The $M^X/M/c$ Bernoulli feedback queue with variant multiple working vacations and impatient customers: Performance and economic analysis

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Abstract. The present paper deals with an $M^X/M/c$ Bernoulli feedback queueing system with variant multiple working vacations and impatience timers which depend on the states of the servers. Whenever a customer arrives at the system, he activates an random impatience timer. If his service has not been completed before his impatience timer expires, the customer may abandon the system. Using certain customer retention mechanism, the impatient customer can be retained in the system. After getting incomplete or unsatisfactory service, with some probability, each customer may comeback to the system as a Bernoulli feedback. Using the probability generating functions (PGFs), we derive the steady-state solution of the model. Then, we obtain useful performance measures. Moreover, we carry out an economic analysis. Finally, numerical study is performed to explore the effects of the model parameters on the behavior of the system.

Keywords: Queueing models. batch arrival. variant working vacations. impatient customers. Bernoulli feedback.
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6.1 Introduction

Vacation queueing models with impatient customers are very helpful in providing basic framework for efficient design and study of diverse practical situations including telephone switchboard, inventory problems with perishable goods, computer and communication network, data/voice transmission, manufacturing system, etc.

In recent past, vacation queueing models have been widely studied. Doshi (1986), Takagi (1991) and Tian and Zhang (2006) are excellent survey works on the subject. An extensive amount of the literature is available on queueing models with server vacation and arrival batch and can be found in Madan and AI-Rawwash (2005), Wang et al. (2007), Haridass and Arrumuganathan (2008), etc.

Working vacation queues with customer impatience have attracted the interest of many researchers. Altman and Yechiali (2008) treated the infinite-server queueing model with system's additional tasks and impatient customers. Perel and Yechiali (2010) considered a 2-phase Markovian random environment with impatient customers. working vacation queueing model with customer impatience has been analyzed by Yue et al. (2012). Then, Zhang et al. (2013) presented an equilibrium balking strategies in Markovian queues with working vacations. Sun et al. (2014) gave the equilibrium and optimal behavior of customers in Markovian queues with multiple working vacations. Goswami (2014) analyzed a queueing model with impatient customers with Bernoulli schedule working vacations and vacation interruption. Laxmi and Jyothsna (2015) dealt with balking and reneging multiple working vacations queue with heterogeneous servers. Later, Tian et al. (2016) presented equilibrium and optimal strategies in $M/M/1$ queueing model with working vacations and vacation interruptions. Recently, in Bouchentouf and Yahiaoui (2017), a study on feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption has been done. For more literature on customer impatience in working vacation queueing models, the authors may be referred to Selvaraju and Goswami (2013) and Laxmi and Jyothsna (2013,2014).

Variant of multiple vacation policy is relatively a recent one where it is permitted to the server to take a certain fixed number of consecutive vacations, if the system remains empty at the end of a vacation. This sort of vacation schedule was carried out by Zhang and Tian (2001). In their paper, a $Geo/G/1$ queueing model with multiple adaptive vacations has been analyzed. Literature related to variant multiple working vacations can be found in Ke (2007), Ke et al. (2010), Wang et al. (2011) and Yue et al. (2014). Recently, Laxmi and Rajesh (2016) studied a variant working vacations queue with customer impatience. Furthermore, the performance measures of batch arrival queue with variant working vacations and reneging have been presented in Laxmi and

Rajesh (2017).

In the present investigation, we carry out an analysis of an $M^X/M/c$ Bernoulli feedback queueing model with variant of multiple working vacations, reneging which depend on the states of the servers and retention of reneged customers. The queueing model presented in this paper has many practical situations. Moreover, as the impatience has strongly bas effect on the economy of any firm, a great idea of retention of impatient customers is incorporated in this work. Besides, to the best of our knowledge, modeling of multi-server queueing system with Bernoulli feedback, variant of working vacations, impatience timers which depend on the states of the servers and retention of reneged customers has not been attempted in literature. This paper makes a contribution in this sense.

The paper is arranged as follows. We describe the model in Section 2. The theoretical analysis of the system is presented in Section 3. Useful measures of effectiveness and the cost of our model are given in Section 4. To validate the analytical results and to facilitate the sensitivity analysis, we present some numerical results for system performance and cost model in Section 5. Some concluding remarks and notable features of investigation done are highlighted in Section 6.

6.2 The model formulation

Consider Markov model for an infinite buffer multi-server queueing system with batch arrival, variant of working vacations, Bernoulli feedback, impatient customers which depend on the states of the servers and retention of reneged customers. For the mathematical formulation of the queueing model, the following notations and assumptions are given:

Customers arrive in batches according to a Poisson process with rate λ. The arrival batch size X is a random variable with probability mass function $P(X = l) = b_l;\ l = 1,2,...$. The service times during normal busy period is assumed to be exponentially distributed with mean $1/\mu$. And during the vacation time the service is provided according to an exponential distribution with parameter η, with $\mu \geq \eta$. The queueing system consists of c servers such that all the servers go for working vacation and vacation time synchronously once the system becomes empty, and they also return to the system as one at the same time. If the servers return from working vacation and vacation period to find an empty queue, they immediately leave all together for another vacation and working; otherwise, they return to serve the queue. Vacation and working vacation periods are assumed to be exponentially distributed with mean $1/\phi$.

If the servers find customer at a working vacation completion instant, they all comeback to regular busy period; otherwise, they take all together K vacations sequentially. When the K consecutive working vacations are complete, all servers switch to busy period and depending on the arriving batch of customers, they stay idle or busy. So, in variant multiple vacation policy, if the system remains empty at the end of a vacation, the servers are permitted to take a finite number, say K, of consecutive vacations.

Whenever a customer arrives at the system and finds the servers on vacation or working vacation (respectively. busy) period, he activates an impatience timer T_1 (respectively. T_2), which is exponentially distributed with parameter ξ_1 (respectively. ξ_2). If the customer's service has not been completed before the customer's timer expires, this later may leave the system. We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers. Each impatient customer may abandon the system with probability α and can be retained in the queue with complementary probability $\alpha' = (1-\alpha)$. If the service is uncomplete or unsatisfactory, the customer can either leave the system definitively with probability β or rejoin the end of the queue of the system for another service with probability β', where $\beta + \beta' = 1$. Note that, both customers, the newly arrived and those that are fed back are served in order in which they join the tail of the primary queue.
The inter-arrival times, working vacation and vacation periods, normal busy period are mutually independent.

6.3 Theoretical analysis of the model

Let $N(t)$ denote the number of customers in the system at time t, and let $\kappa(t)$ be the status of the servers at time t. For the mathematical representation of the present model at an instant t, we consider the following states of the system based on the server status,

$$\kappa(t) = \begin{cases} j, & \text{the servers are taking the } (j+1)^{th} \text{ vacation at time } t,\ j = \overline{0, 1, K-1}, \\ K, & \text{the servers are idle or busy at time } t. \end{cases}$$

Figure 6.1 depicts the state transition diagram.

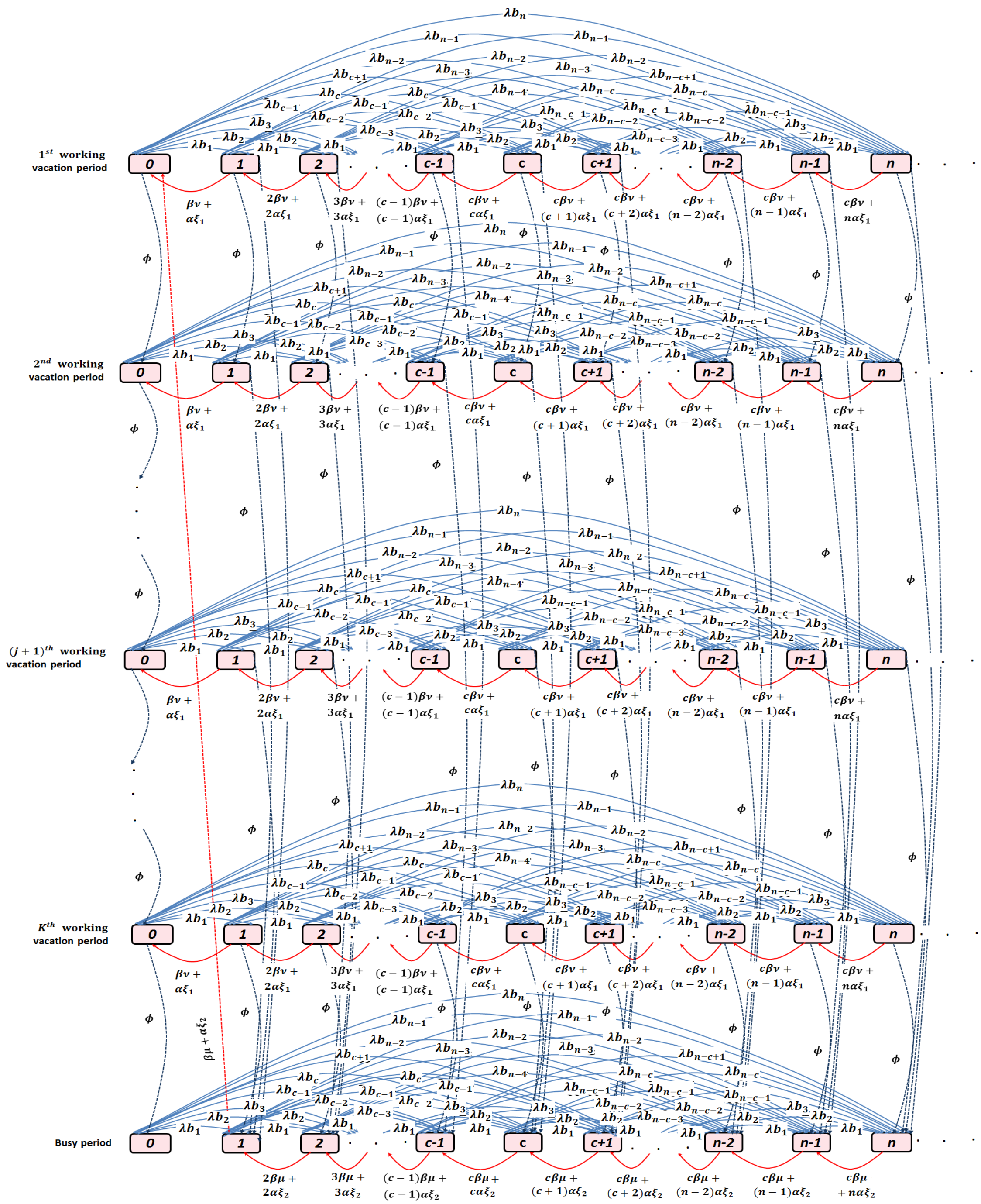

Figure 6.1: State-transition-rate diagram.

The bi-variate process $\{(N(t), \kappa(t)),\ \ t \geq 0\}$ represents two dimensional infinite state Markov chain in continuous time with state space

$$\Omega = \{(n, j):\ \ n \geq 0;\ \ j = \overline{0, K}\}.$$

Let $P_{n,j} = \lim_{t\to\infty} P(N(t) = n; \kappa(t) = j),\ n \geq 0;\ j = \overline{0,K}$ be the steady-state probabilities of the process $\{(N(t); \kappa(t)); t \geq 0\}$.

6.3.1 PGFs and Balance equations

Define the probability generating functions as

$$G_j(z) = \sum_{n=0}^{\infty} P_{n,j} z^n,\ |z| \leq 1,\ j = \overline{0,K}$$

$$G_j'(z) = \frac{d}{dz} G_j(z) = \sum_{n=1}^{\infty} n P_{n,j} z^{n-1},\ j = \overline{0,K}$$

$$B(z) = \sum_{n=1}^{\infty} b_n z^n,\ \text{with}\ B(1) = \sum_{n=1}^{\infty} b_n = 1.$$

In order to develop the model, the steady state Chapman-Kolmogrov equations for the system states is constructed as follows:

$$(\lambda + \phi) P_{0,0} = (\beta\nu + \alpha\xi_1) P_{1,0} + (\beta\mu + \alpha\xi_2) P_{1,K}, \tag{6.1}$$

$$(\lambda + \phi + \beta\nu + \alpha\xi_1) P_{1,0} = \lambda b_1 P_{0,0} + 2(\beta\nu + \alpha\xi_1) P_{2,0}, \tag{6.2}$$

$$(\lambda + \phi + n(\beta\nu + \alpha\xi_1)) P_{n,0} = \lambda \sum_{m=1}^{n} b_m P_{n-m,0} + (n+1)(\beta\nu + \alpha\xi_1) P_{n+1,0},\ 2 \leq n \leq c-1, \tag{6.3}$$

$$(\lambda + \phi + c\beta\nu + n\alpha\xi_1) P_{n,0} = \lambda \sum_{m=1}^{n} b_m P_{n-m,0} + (c\beta\nu + (n+1)\alpha\xi_1) P_{n+1,0},\ n \geq c, \tag{6.4}$$

$$(\lambda + \phi) P_{0,j} = (\beta\nu + \alpha\xi_1) P_{1,j} + \phi P_{0,j-1},\ 1 \leq j \leq K, \tag{6.5}$$

$$(\lambda + \phi + \beta\nu + \alpha\xi_1) P_{1,j} = \lambda b_1 P_{0,j} + 2(\beta\nu + \alpha\xi_1) P_{2,j},\ 1 \leq j \leq K-1, \tag{6.6}$$

$$(\lambda+\phi+n(\beta\nu+\alpha\xi_1))P_{n,j}=\lambda\sum_{m=1}^{n}b_mP_{n-m,j}+(n+1)(\beta\nu+\alpha\xi_1)P_{n+1,j},\quad 2\le n\le c-1,\ 1\le j\le K-1, \tag{6.7}$$

$$(\lambda+\phi+c\beta\nu+n\alpha\xi_1)P_{n,j}=\lambda\sum_{m=1}^{n}b_mP_{n-m,j}+(c\beta\nu+(n+1)\alpha\xi_1)P_{n+1,j},\quad n\ge c,\ 1\le j\le K-1, \tag{6.8}$$

$$\lambda P_{0,K}=\phi P_{0,K-1}, \tag{6.9}$$

$$(\lambda+\beta\mu+\alpha\xi_2)P_{1,K}=\lambda b_1P_{0,K}+2(\beta\mu+\alpha\xi_2)P_{2,K}+\phi\sum_{j=0}^{K-1}P_{1,j}, \tag{6.10}$$

$$(\lambda+n(\beta\mu+\alpha\xi_2))P_{n,K}=\lambda\sum_{m=1}^{n}b_mP_{n-m,K}+(n+1)(\beta\mu+\alpha\xi_2)P_{n+1,K}+\phi\sum_{j=0}^{K-1}P_{n,j},\ 2\le n\le c-1, \tag{6.11}$$

$$(\lambda+c\beta\mu+n\alpha\xi_2)P_{n,K}=\lambda\sum_{m=1}^{n}b_mP_{n-m,K}+(c\beta\mu+(n+1)\alpha\xi_2)P_{n+1,K}+\phi\sum_{j=0}^{K-1}P_{n,j},\ n\ge c. \tag{6.12}$$

The normalizing condition is given as

$$\sum_{n=0}^{\infty}\sum_{j=0}^{K}P_{nj}=1. \tag{6.13}$$

Multiplying Eqs.(6.1)-(6.4) by z^n, and summing all possible values of n, we get

$$(1-z)z\alpha\xi_1 G_0'(z) + [\lambda z(B(z)-1) - z(\phi + c\beta\nu) + c\beta\nu]G_0(z) = \beta\nu(1-z)\sum_{n=0}^{c-1}(c-n)P_{n,0}z^n - (\alpha\xi_2 + \beta\mu)zP_{1,K}. \tag{6.14}$$

In a similar manner, we get from Eqs.(6.5)-(6.8)

$$(1-z)z\alpha\xi_1 G_j'(z) + [\lambda z(B(z)-1) - z(\phi + c\beta\nu) + c\beta\nu]G_j(z) = \beta\nu(1-z)\sum_{n=0}^{c-1}(c-n)P_{n,j}z^n - \phi z P_{0,j-1},\ 1 \le j \le K-1. \tag{6.15}$$

In the same way, from Eqs.(6.9)-(6.12)

$$(1-z)z\alpha\xi_2 G_K'(z) + [\lambda z(B(z)-1) + c\beta\mu(1-z)]G_K(z) = -z\phi\sum_{j=0}^{K-1}G_j(z)$$
$$\beta\mu(1-z)\sum_{n=0}^{c-1}(c-n)P_{n,K}z^n + z(\beta\mu + \alpha\xi_2)P_{1,K} + z\phi\sum_{j=0}^{K-2}P_{0,j}. \tag{6.16}$$

Next, using the recursive method, we get

$$\begin{cases} P_{n,0} &= \gamma_n P_{0,0} + \varphi_n P_{1,K}, \\ P_{n,j} &= \gamma_n P_{0,j} + \omega_n P_{0,j-1}, \end{cases}$$

where

$$\gamma_n = \begin{cases} 1, & \text{if } n = 0; \\ \dfrac{\lambda + \phi}{\beta\nu + \alpha\xi_1}, & \text{if } n = 1. \\ \psi_{n-1}\gamma_{n-1} - \dfrac{A}{n}\displaystyle\sum_{i=1}^{n-1} b_i\gamma_{n-1-i} & \text{if } 2 \le n \le c-1, \end{cases}$$

$$\varphi_n = \begin{cases} 0, & \text{if } n = 0; \\ -\dfrac{\beta\mu + \alpha\xi_2}{\beta\nu + \alpha\xi_1}, & \text{if } n = 1. \\ \psi_{n-1}\varphi_{n-1} - \dfrac{A}{n}\displaystyle\sum_{i=1}^{n-1} b_i \varphi_{n-1-i} & \text{if } 2 \le n \le c-1, \end{cases}$$

$$\omega_n = \begin{cases} 0, & \text{if } n = 0; \\ -\dfrac{\phi}{\beta\nu + \alpha\xi_1}, & \text{if } n = 1. \\ \psi_{n-1}\omega_{n-1} - \dfrac{A}{n}\displaystyle\sum_{i=1}^{n-1} b_i \omega_{n-1-i} & \text{if } 2 \le n \le c-1, \end{cases}$$

and

$$A = \frac{\lambda}{\beta\nu + \alpha\xi_1}, \quad \psi_n = \frac{\lambda + \phi + n(\beta\nu + \alpha\xi_1)}{(n+1)(\beta\nu + \alpha\xi_1)}.$$

6.3.2 Solutions of the Differential Equations

For $z \neq 1$ and $z \neq 0$, Eqs. (6.14) and (6.15) can be written respectively as

$$G_0'(z) + \left(\frac{\lambda}{\alpha\xi_1}H'(z) - \frac{(\phi + c\beta\nu)}{(1-z)\alpha\xi_1} + \frac{c\beta\nu}{(1-z)z\alpha\xi_1}\right)G_0(z) = \frac{\beta\nu}{z\alpha\xi_1}Q_0(z)P_{0,0} + \left(\frac{\beta\nu}{\alpha\xi_1 z}Q_1(z) - \frac{\alpha\xi_2 + \beta\mu}{(1-z)\alpha\xi_1}\right)P_{1,K}, \tag{6.17}$$

for $j = \overline{1, K-1}$.

$$G_j'(z) + \left(\frac{\lambda}{\alpha\xi_1}H'(z) - \frac{(\phi + c\beta\nu)}{(1-z)\alpha\xi_1} + \frac{c\beta\nu}{(1-z)z\alpha\xi_1}\right)G_j(z) = \frac{\beta\nu}{z\alpha\xi_1}Q_0(z)P_{0,j} + \left(\frac{\beta\nu}{\alpha\xi_1 z}Q_2(z) - \frac{\phi}{(1-z)\alpha\xi_1}\right)P_{0,j-1}, \tag{6.18}$$

such that

$$Q_0(z) = \sum_{n=0}^{c-1}(c-n)\gamma_n z^n, Q_1(z) = \sum_{n=0}^{c-1}(c-n)\varphi_n z^n, Q_2(z) = \sum_{n=0}^{c-1}(c-n)\omega_n z^n,$$

and

$$H(z) = \int_0^z \frac{B(x)-1}{1-x}dx, \text{ and } H'(z) = \frac{B(z)-1}{1-z}.$$

Now, by taking $z = 1$ in Eqs.(6.14) and (6.15), we have respectively

$$\phi G_0(1) = (\alpha\xi_2 + \beta\mu)P_{1,K}, \tag{6.19}$$

and

$$G_j(1) = P_{0,j-1} \tag{6.20}$$

Next, to solve the linear differential equations (6.17) and (6.18), we multiply both sides of the above equations by $e^{\frac{\lambda}{\alpha\xi_1}H(z)}(1-z)^{\frac{\phi}{\alpha\xi_1}}z^{\frac{c\beta\nu}{\alpha\xi_1}}$, then integrating form 0 to z, we obtain

$$G_0(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(z)}}{(1-z)^{\frac{\phi}{\alpha\xi_1}}z^{\frac{c\beta\nu}{\alpha\xi_1}}}\left\{\frac{\beta\nu}{\alpha\xi_1}K_0(z)P_{0,0} + \left(\frac{\beta\nu}{\alpha\xi_1}K_1(z) - \frac{\alpha\xi_2+\beta\mu}{\alpha\xi_1}K_2(z)\right)P_{1,K}\right\}, \tag{6.21}$$

for $j = \overline{1, K-1}$.

$$G_j(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(z)}}{(1-z)^{\frac{\phi}{\alpha\xi_1}}z^{\frac{c\beta\nu}{\alpha\xi_1}}}\left\{\frac{\beta\nu}{\alpha\xi_1}K_0(z)P_{0,j} + \left(\frac{\beta\nu}{\alpha\xi_1}K_3(z) - \frac{\phi}{\alpha\xi_1}K_2(z)\right)P_{0,j-1}\right\}, \tag{6.22}$$

where

$$K_0(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_1}H(x)}(1-x)^{\frac{\phi}{\alpha\xi_1}}x^{\frac{c\beta\nu}{\alpha\xi_1}-1}Q_0(x)dx,$$

$$K_1(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_1}H(x)}(1-x)^{\frac{\phi}{\alpha\xi_1}}x^{\frac{c\beta\nu}{\alpha\xi_1}-1}Q_1(x)dx,$$

$$K_2(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_1}H(x)}(1-x)^{\frac{\phi}{\alpha\xi_1}-1}x^{\frac{c\beta\nu}{\alpha\xi_1}}dx,$$

and

$$K_3(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_1}H(x)}(1-x)^{\frac{\phi}{\alpha\xi_1}}x^{\frac{c\beta\nu}{\alpha\xi_1}-1}Q_2(x)dx.$$

Next, $z = 0$ and $z = 1$ are the roots of the numerator of the right hand side of (6.21) and (6.22). Then, taking $z = 1$ in (6.21) and (6.22) respectively, we get

$$P_{1,K} = \theta_1 P_{0,0}, \text{ where } \theta_1 = \frac{\beta\nu K_0(1)}{(\beta\mu+\alpha\xi_2)K_2(1)-\beta\nu K_1(1)}, \tag{6.23}$$

and

$$P_{0,j} = C^j P_{0,0}, 1 \leq j \leq K-1 \quad \text{where} \quad C = \frac{\phi K_2(1)-\beta\nu K_3(1)}{\beta\nu K_0(1)}. \tag{6.24}$$

Via Eqs.(6.9) and (6.24)

$$P_{0,K} = \theta_0 P_{0,0}, \tag{6.25}$$

where

$$\theta_0 = \frac{\phi}{\lambda}C^{K-1}.$$

Substituting Eqs.(6.23) and (6.24) in Eqs.(6.21) and (6.22) respectively, we get

$$G_0(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(z)}}{(1-z)^{\frac{\phi}{\alpha\xi_1}}z^{\frac{c\beta\nu}{\alpha\xi_1}}}\left\{\frac{\beta\nu K_0(z)+(\beta\nu K_1(z)-(\beta\mu+\alpha\xi_2)K_2(z))\theta_1}{\alpha\xi_1}\right\}P_{0,0}, \tag{6.26}$$

and for $j = \overline{1, K-1}$

$$G_j(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(z)}}{(1-z)^{\frac{\phi}{\alpha\xi_1}}z^{\frac{c\beta\nu}{\alpha\xi_1}}}\left\{\beta\nu K_0(z)+\frac{\beta\nu K_3(z)-\phi K_2(z)}{C}\right\}\frac{C^j}{\alpha\xi_1}P_{0,0}. \tag{6.27}$$

Thus,

$$\sum_{j=0}^{K-1} G_j(z) = \Psi(z)P_{0,0}, \; j = \overline{0, K-1}, \tag{6.28}$$

with

$$\Psi(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(z)}}{(1-z)^{\frac{\phi}{\alpha\xi_1}} z^{\frac{c\beta\nu}{\alpha\xi_1}}} \left\{ \frac{\beta\nu K_0(z) + (\beta\nu K_1(z) - (\beta\mu + \alpha\xi_2)K_2(z))\theta_1}{\alpha\xi_1} \right.$$

$$\left. + \frac{C}{\alpha\xi_1}\left(\beta\nu K_0(z) + \frac{\beta\nu K_3(z) - \phi K_2(z)}{C}\right)\left(\frac{1-C^{K-1}}{1-C}\right) \right\}.$$

By taking $z = 1$ in Eq. (6.16), we find

$$\phi \sum_{j=0}^{K-1} G_j(1) = (\beta\mu + \alpha\xi_2)P_{1,K} + \phi \sum_{j=0}^{K-2} P_{0,j}. \tag{6.29}$$

Consequently, we have

$$\sum_{j=0}^{K-1} G_j(1) = \left\{ \frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1-C^{K-1}}{1-C} \right\} P_{0,0}. \tag{6.30}$$

Next, we have to solve the differential equation (6.16). So, we must express recursively the quantity $P_{n,K}$ in terms of $P_{0,0}$. In the same manner as previously, it yields

$$P_{n,K} = \theta_n P_{0,0},$$

with

$$\theta_n = \begin{cases} \theta_0, & \text{if } n = 0; \\ \theta_1, & \text{if } n = 1. \\ \sigma_{n-1}\theta_{n-1} - \dfrac{B}{n}\displaystyle\sum_{i=1}^{n-1} b_i\theta_{n-1-i} - \dfrac{E}{n}(\gamma_{n-1}H(K) + \omega_{n-1}h(K)) & \text{if } 2 \le n \le c-1, \end{cases}$$

such that

$$\sigma_n = \frac{\lambda + n(\beta\mu + \alpha\xi_2)}{(n+1)(\beta\mu + \alpha\xi_2)},$$

$$B = \frac{\lambda}{\beta\mu + \alpha\xi_2}, \quad E = \frac{\phi}{\beta\mu + \alpha\xi_2},$$

and

$$H(K)=\sum_{j=0}^{K-1}C^{j}=\frac{1-C^{K}}{1-C},\ \ h(K)=\sum_{j=0}^{K-1}C^{j-1}=\frac{1-C^{K}}{C(1-C)}.$$

By substituting Eq. (6.29) in Eq. (6.16), we have

$$G_K'(z)+\left(\frac{\lambda}{\alpha\xi_2}H(z)'+\frac{c\beta\mu}{z\alpha\xi_2}\right)G_K(z)=\frac{\beta\mu}{z\alpha\xi_2}Q_3(z)P_{0,0}-\frac{\phi\sum_{j=0}^{K-1}[G_j(z)-G_j(1)]}{(1-z)\alpha\xi_2}, \tag{6.31}$$

where

$$Q_3(z)=\sum_{n=0}^{c-1}(c-n)\theta_n z^n.$$

Multiplying Eq. (6.31) by $e^{\frac{\lambda}{\alpha\xi_2}H(z)}z^{\frac{c\beta\mu}{\alpha\xi_2}}$ and integrating from 0 to z, then using Eqs. (6.28) and (6.30), we get

$$G_K(z)=\frac{e^{-\frac{\lambda}{\alpha\xi_2}H(z)}}{z^{\frac{c\beta\mu}{\alpha\xi_2}}}\left\{-\frac{\phi}{\alpha\xi_2}\left(K_4(z)-\left(\frac{\beta\mu+\alpha\xi_2}{\phi}\theta_1+\frac{1-C^{K-1}}{1-C}\right)K_5(z)\right)+\frac{\beta\mu}{\alpha\xi_2}K_6(z)\right\}P_{0,0}, \tag{6.32}$$

where

$$K_4(z)=\int_0^z e^{\frac{\lambda}{\alpha\xi_2}H(x)}x^{\frac{c\beta\mu}{\alpha\xi_2}}(1-x)^{-1}\Psi(x)dx,$$

$$K_5(z)=\int_0^z e^{\frac{\lambda}{\alpha\xi_2}H(x)}x^{\frac{c\beta\mu}{\alpha\xi_2}}(1-x)^{-1}dx,$$

and

$$K_6(z)=\int_0^z e^{\frac{\lambda}{\alpha\xi_2}H(x)}x^{\frac{c\beta\mu}{\alpha\xi_2}-1}Q_3(x)dx.$$

Now, Taking $z=1$ in Eq. (6.32), and using the normalization condition

$$\sum_{j=0}^{K-1}G_j(1)+G_K(1)=1,$$

we obtain

$$P_{0,0}=\left\{e^{-\frac{\lambda}{\alpha\xi_2}H(1)}\left\{-\frac{\phi}{\alpha\xi_2}\left(K_4(1)-\left(\frac{\beta\mu+\alpha\xi_2}{\phi}\theta_1+\frac{1-C^{K-1}}{1-C}\right)K_5(1)\right)\right.\right.$$
$$\left.\left.+\frac{\beta\mu}{\alpha\xi_2}K_6(1)\right\}+\left(\frac{\beta\mu+\alpha\xi_2}{\phi}\theta_1+\frac{1-C^{K-1}}{1-C}\right)\right\}^{-1} \tag{6.33}$$

6.4 Performance measures and cost model

6.4.1 Measures of effectiveness

Performance measures are significant features of queueing systems as they reflect the effectiveness of the considered queueing system. The queueing model developed may be of great importance by using some useful characteristics which can be in the future employed for the prediction, development and improvement of the concerned real world queueing system. In this section, we formulate some important system performance measures in terms of steady state probabilities.

* The average number of customers in the system $E(L)$.

$$E(L)=E(L_{WV})+E(L_K).$$

Differentiating Eq. (6.14), taking $z=1$ and using Eq. (6.19), we get

$$(\alpha\xi_1+\phi)G_0'(1)=(\lambda B'(1)-c\beta\nu)G_0(1)+\beta\nu(Q_0(1)P_{0,0}+Q_1(1)P_{1,K}). \tag{6.34}$$

In the same manner, for $j=\overline{1,K-1}$, Differentiating Eq.(6.15), taking $z=1$ and using Eq. (6.20), we get

$$(\alpha\xi_1+\phi)G_j'(1)=(\lambda B'(1)-c\beta\nu)G_j(1)+\beta\nu(Q_0(1)P_{0,j}-Q_2(1)P_{0,j-1}), \tag{6.35}$$

where

$$Q_0(1)=\sum_{n=0}^{c-1}(c-n)\gamma_n,\ Q_1(1)=\sum_{n=0}^{c-1}(c-n)\varphi_n,\ Q_2(1)=\sum_{n=0}^{c-1}(c-n)\omega_n.$$

From Eq.(6.34), we obtain

$$G_0'(1)=\left\{\frac{\lambda B'(1)-c\beta\nu}{\alpha\xi_1+\phi}\left(\frac{\beta\mu+\alpha\xi_2}{\phi}\right)\theta_1+\frac{\beta\nu(Q_0(1)+\theta_1Q_1(1))}{\alpha\xi_1+\phi}\right\}P_{0,0}. \tag{6.36}$$

From Eq. (6.35), summing over all possible values of j, $j = \overline{1, K-1}$, we obtain

$$\sum_{j=1}^{K-1} G_j'(1) = \left\{ \left(\frac{1 - C^{K-1}}{C(1-C)} \right) \frac{\lambda B'(1) + \beta\nu(Q_0(1)C - Q_2(1) - c)}{\alpha\xi_1 + \phi} \right\} P_{0,0}. \tag{6.37}$$

Furthermore, the mean system size when the servers are on working vacation, denoted by $E(L_{WV})$, is obtained as follows:

$$E(L_{WV}) = G_0'(1) + \sum_{j=1}^{K-1} G_j'(1). \tag{6.38}$$

Substituting Eqs. (6.36) and (6.37) in (6.38), we get

$$\begin{aligned} E(L_{WV}) = &\left\{ \frac{\lambda B'(1) - c\beta\nu}{\alpha\xi_1 + \phi} \left(\frac{\beta\mu + \alpha\xi_2}{\phi} \right) \theta_1 + \frac{\beta\nu(Q_0(1) + \theta_1 Q_1(1))}{\alpha\xi_1 + \phi} \right. \\ &\left. + \left(\left(\frac{1 - C^{K-1}}{C(1-C)} \right) \frac{\lambda B'(1) + \beta\nu(Q_0(1)C - Q_2(1) - c)}{\alpha\xi_1 + \phi} \right) \right\} P_{0,0}. \end{aligned} \tag{6.39}$$

Next, from Eq.(6.16), and using L'Hospital rule, we find

$$\begin{aligned} E(L_K) = \lim_{z\to 1} G_K'(z) = \lim_{z\to 1} &\left\{ \frac{-(\lambda z(B(z) - 1) + c\beta\mu(1-z))}{(1-z)\alpha\xi_2} G_K(z) \right. \\ &\left. - \frac{\phi \sum_{j=0}^{K-1} [G_j(z) - G_j(1)]}{(1-z)\alpha\xi_2} + \frac{\beta\mu}{z\alpha\xi_2} Q_3(z) P_{0,0} \right\}. \end{aligned}$$

This implies

$$E(L_K) = \frac{\lambda B'(1) - \beta\mu}{\alpha\xi_2} G_K(1) + \frac{\phi}{\alpha\xi} \sum_{j=1}^{K-1} G_j'(1) + \frac{\beta\mu}{\alpha\xi_2} Q_3(1) P_{0,0}, \tag{6.40}$$

where

$$Q_3(1) = \sum_{n=0}^{c-1} (c-n)\theta_n.$$

* The mean of the queue length L_q.

$$L_q = \sum_{j=0}^{K} \sum_{n=c+1}^{\infty} (n-c)P_{n,j}$$
$$= E(L) - c + \left\{ \left(Q_0(1) + \frac{Q_2(1)}{C} \right) \left(\frac{1-C^K}{1-C} \right) + Q_3(1) \right\} P_{0,0}.$$

* The mean expected number of customers served per unit time N_s.

$$N_s = \beta\mu \sum_{n=1}^{c-1} nP_{n,K} + c\beta\mu \sum_{n=c}^{\infty} P_{n,K} + \beta\nu \sum_{j=0}^{K} \sum_{n=1}^{c-1} nP_{n,j} + c\beta\nu \sum_{j=0}^{K} \sum_{n=c}^{\infty} P_{n,j}$$

$$= c\beta\left(\mu(P_b + P_{0,K}) + \nu P_{wv}\right) + \beta\left(\mu Q_3(1) + \nu(Q_0(1)H(K) + Q_2(1)h(K))\right) P_{0,0}.$$

* The probability that the servers are in working vacation period P_{WV}.

$$P_{WV} = \sum_{j=0}^{K-1} G_j(1) = \left\{ \frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1-C^{K-1}}{1-C} \right\} P_{0,0}.$$

* The probability that the servers are idle in working vacation P_{idle}.

$$P_{idle} = \sum_{j=0}^{K-1} P_{0,j} = \frac{1-C^K}{1-C} P_{0,0}.$$

* The probability that the servers are busy P_{busy}.

$$P_{busy} = 1 - P_{0,K} - P_{WV}.$$

* The average rate of abandonment of a customer due to impatience R_a.

$$R_a = \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} n\alpha\xi_1 P_{n,j} + \sum_{n=0}^{\infty} n\alpha\xi_2 P_{n,K}$$

$$= \alpha\xi_1 E(L_{WV}) + \alpha\xi_2 E(L_K).$$

* The average rate of retention of impatient customers R_e.

$$R_e = \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} n(1-\alpha)\xi_1 P_{n,j} + \sum_{n=0}^{\infty} n(1-\alpha)\xi_2 P_{n,K}$$

$$= (1-\alpha)\xi_1 E(L_{WV}) + (1-\alpha)\xi_2 E(L_K).$$

6.4.2 Economic model

To construct the cost model, we consider the following cost (in unit) elements associated with different events:

- C_1 : Cost per unit time when the servers are busy.
- C_2 : Cost per unit time when the servers are idle during busy period.
- C_3 : Cost per unit time when the servers are idle during working vacation period.
- C_4 : Cost per unit time when the servers are on working vacation period.
- C_5 : Cost per unit time when a customer joins the queue and waits for service.
- C_6 : Cost per unit time when a customer reneges.
- C_7 : Cost per unit time when a customer is retained.
- C_8 : Cost per service per unit time when the servers are in busy period.
- C_9 : Cost per service per unit time when the servers are in working vacation period.
- C_{10} : Cost per unit time when a customer returns to the system as a feedback customer.
- C_{11} : Fixed server purchase cost per unit.

Let
R be the revenue earned by providing service to a customer.

Γ be the total expected cost per unit time of the system.

$$\Gamma = C_1 P_b + C_2 P_{0,K} + C_3 P_e + C_4 P_{WV} + C_5 L_q + C_6 R_a + C_7 R_e$$

$$+ c\mu C_8 + c\nu C_9 + c\beta'(\mu+\nu) C_{10} + cC_{11}.$$

Δ be the total expected revenue per unit time of the system.

$$\Delta = R \times N_s.$$

Θ be the total expected profit per unit time of the system.

$$\Theta = \Delta - \Gamma.$$

6.5 Numerical analysis

To analyze the parameter impact on the system performance, numerical calculus are carried out and some ones are presented in the form of Graphs and Tables. The characteristics and different costs are obtained by using R program coded by the authors. First of all let us assume that the arrival batch size X follows a geometric distribution with parameter q, that is $P(X = l) = (1-q)^{l-1}q$, with $0 < q < 1$, and $l = 1, 2, \ldots$. Consequently, $B(z) = \dfrac{qz}{1-(1-q)z}$.

To illustrate the system numerically, the values for default parameters are considered as:

First, we consider the following cases:

- Table 6.1 : $\lambda = 2.9 : 0.1 : 3.3$, $K = (1,5,9)$, $c = 3$, $q = 0.8$, $\mu = 4$, $\nu = 3.8$, $\phi = 0.1$, $\beta = 0.8$, $\alpha = 0.8$, $\xi_1 = 0.5$, $\xi_2 = 0.8$.
- Table 6.2 : $\lambda = 3$, $K = 3$, $c = 3$, $q = (0.5, 0.7, 0.9)$, $\mu = 4$, $\nu = 3.8$, $\phi = 0.07 : 0.02 : 0.15$, $\beta = 0.8$, $\alpha = 0.8$, $\xi_1 = 0.5$, $\xi_2 = 0.8$.
- Table 6.3 : $\lambda = 3$, $K = 3$, $c = 3$, $q = 0.8$, $\mu = 4.6 : 0.4 : 6.2$, $\nu = 3.8$, $\phi = 0.1$, $\beta = 0.8$, $\alpha = 0.8$, $\xi_1 = 0.5$, $\xi_2 = (0.8, 0.94, 1.04)$.
- Table 6.4 : $\lambda = 3.4$, $K = 3$, $c = 3$, $q = 0.8$, $\mu = 4.0$, $\nu = 0.35 : 0.2 : 1.15$, $\phi = 0.1$, $\beta = 0.8$, $\alpha = (0.5, 0.7, 0.9)$, $\xi_1 = 0.5$, $\xi_2 = 0.8$.
- Table 6.5 : $\lambda = 2.9$, $K = 3$, $c = 3$, $q = 0.8$, $\mu = 4.0$, $\nu = 3.8$, $\phi = 0.1$, $\beta = (0.5, 0.7, 0.9)$, $\alpha = 0.8$, $\xi_2 = 0.79 : 0.02 : 0.87$, $\xi_1 = 0.5$.
- Table 6.6 : $\lambda = 3.4$, $K = 3$, $c = (1,2,3)$, $q = 0.8$, $\mu = 4.0$, $\nu = 3.8$, $\phi = 0.1$, $\beta = 0.8$, $\alpha = 0.8$, $\xi_1 = 4.5 : 0.5 : 6.5$, $\xi_2 = 0.8$.

Second, for economic cost results, we consider the following situations: $C_1 = 5$, $C_2 = 3$, $C_3 = 4$, $C_4 = 5$, $C_5 = 5$, $C_6 = 5$, $C_7 = 5$, $C_8 = 4$, $C_9 = 4$, $C_{10} = 5$, $C_{11} = 4$, and $R = 50$. Numerical results are presented in the following Tables and Figures.

6.5.1 Discussion on the results

- From Table 6.1 and Figures 6.2-6.3, we see that for different values of variant vacation K, along the increase of the arrival rate λ, the probability that the system

Table 6.1: Total costs vs. λ.

K	1			5			9		
λ	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
2.9	120.4041	531.6658	411.2618	195.2688	568.5601	373.2913	195.3084	568.5796	373.2712
3.0	122.1671	534.7137	412.5467	192.7104	569.8081	377.0976	192.7494	569.8274	377.0781
3.1	123.9192	537.6298	413.7105	190.3974	571.0168	380.6194	190.4356	571.0360	380.6003
3.2	125.6620	540.4197	414.7576	188.3150	572.1867	383.8717	188.3524	572.2056	383.8533
3.3	126.7224	545.4875	418.7651	182.1972	573.0985	390.8506	182.2479	573.0985	390.5806

Table 6.2: Total costs vs. ϕ.

q	0.5			0.7			0.9		
ϕ	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
0.07	160.2763	595.4973	435.2210	172.8560	572.8560	400.0426	216.3829	545.9906	329.6077
0.09	160.7400	596.2907	435.5507	170.2858	578.0479	407.7620	219.4602	554.4544	334.9942
0.11	160.8474	596.8863	436.0389	166.8535	581.9042	415.0506	213.1461	565.2968	352.1507
0.13	160.7960	597.3476	436.5516	161.7721	584.5145	422.7425	205.0141	571.0750	366.0609
0.15	160.6748	597.7125	437.0377	158.9608	586.6848	427.7240	197.1398	575.5170	378.3772

Table 6.3: Total costs vs. μ.

ξ_2	0.80			0.94			1.04		
μ	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
4.60	183.9987	644.5488	460.5501	191.9106	635.7698	443.8591	197.0989	630.0409	432.9421
5.00	187.7598	691.1525	503.3926	196.2569	679.9235	483.6665	198.8866	676.0608	477.1742
5.40	191.4837	737.2152	545.7281	200.5359	723.5063	522.9703	203.3135	718.9175	515.6040
5.80	195.1493	782.8152	587.6659	204.7636	766.5681	561.8045	207.7011	761.2520	553.5509
6.20	198.7750	827.9931	629.2181	208.9442	809.1618	600.2176	212.0479	803.1188	591.0709

Table 6.4: Total costs vs. ν.

α	0.5			0.7			0.9		
ν	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
0.35	98.7895	548.2676	449.4781	95.5000	498.4809	402.9810	92.9471	458.5866	365.6395
0.55	100.1176	554.3015	454.1939	96.4025	508.2268	411.8243	93.7156	472.5236	378.8080
0.75	101.9522	558.9834	457.0311	97.8672	516.6151	418.7479	95.1487	484.0831	388.9344
0.95	104.2273	562.3944	458.1671	99.8645	523.7082	423.8435	97.1571	494.2884	397.1314
1.15	106.8859	565.1780	458.2921	102.3833	529.8369	427.4536	99.7588	502.9528	403.1940

Table 6.5: Total costs vs. ξ_2.

β	0.5			0.7			0.9		
ξ_2	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
0.79	195.2539	580.2338	384.9799	190.4259	572.1482	381.7223	199.3447	564.7339	365.3892
0.81	195.4810	579.6329	384.1518	191.5068	571.3736	379.8668	201.4988	563.8195	362.3207
0.83	195.7182	579.0377	383.3195	190.1446	572.0249	381.8803	203.6107	562.9214	359.3108
0.85	195.9633	578.4492	382.4859	191.1776	571.2889	380.1113	205.6878	562.0369	356.3491
0.87	196.2156	577.8671	381.6515	192.1779	570.5641	378.3661	207.7247	561.1681	353.4434

Table 6.6: Total costs vs. ξ_1.

c	1			2			3		
ξ_1	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
4.50	58.1448	198.6960	140.5512	97.5554	397.3824	299.8269	138.0011	596.6616	458.6605
5.00	58.1369	198.6854	140.5485	97.5312	397.2814	299.7502	137.9293	596.3670	458.4377
5.50	58.1297	198.6729	140.5431	97.5028	397.1844	299.6816	137.8431	596.0944	458.2513
6.00	58.1230	198.6588	140.5358	97.4712	397.0918	299.6207	137.7470	595.8445	458.0976
6.50	57.2817	193.4057	136.1240	91.7429	385.8117	294.0688	126.7755	578.6476	451.8720

becomes empty P_{00} decreases. Thus, the mean number of customers served in-

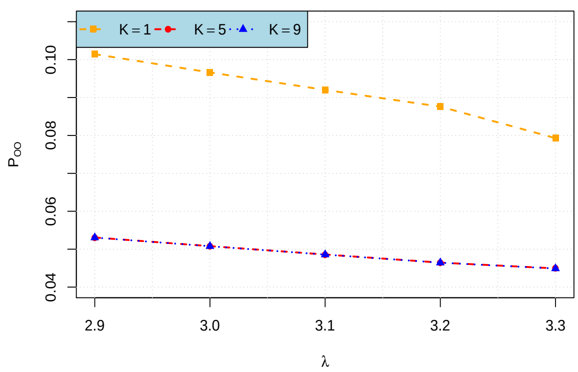

Figure 6.2: Impact of λ on P_{00}.

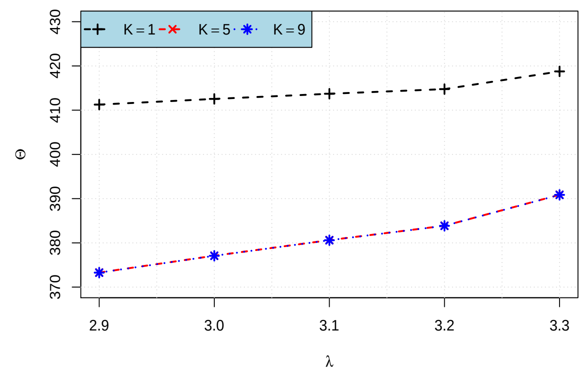

Figure 6.3: Impact of λ on Θ.

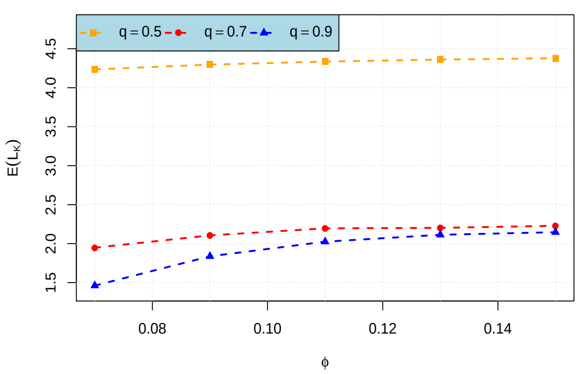

Figure 6.4: Impact of ϕ on $E(L_K)$

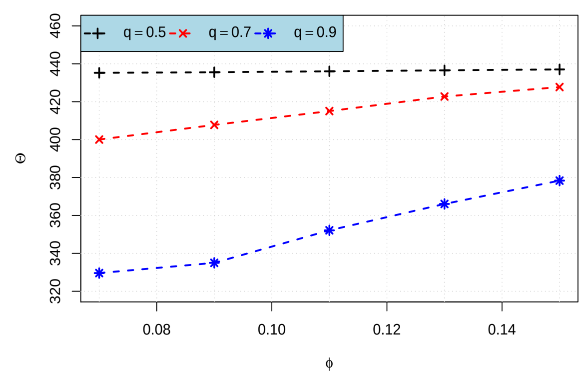

Figure 6.5: Impact of ϕ on Θ.

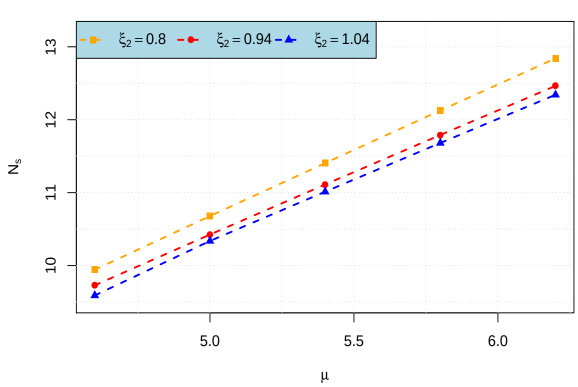

Figure 6.6: Impact of μ on N_s

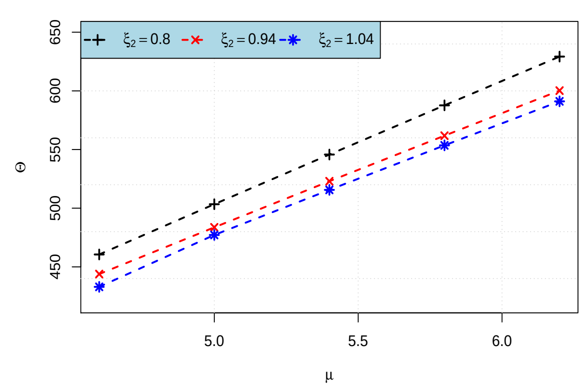

Figure 6.7: Impact of μ on Θ.

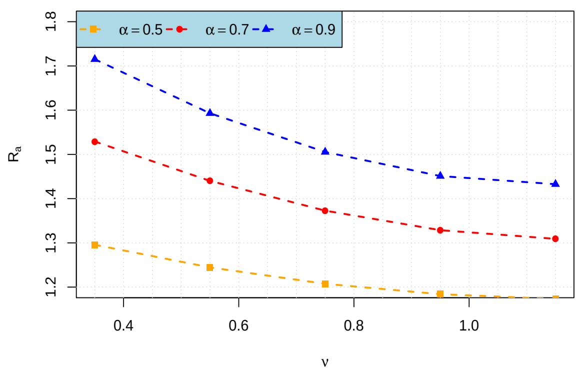

Figure 6.8: Impact of ν on R_a

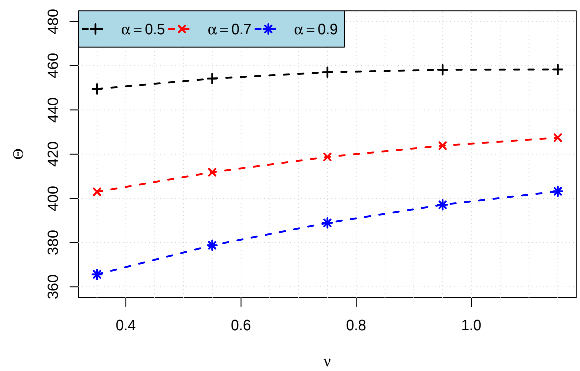

Figure 6.9: Impact of ν on Θ.

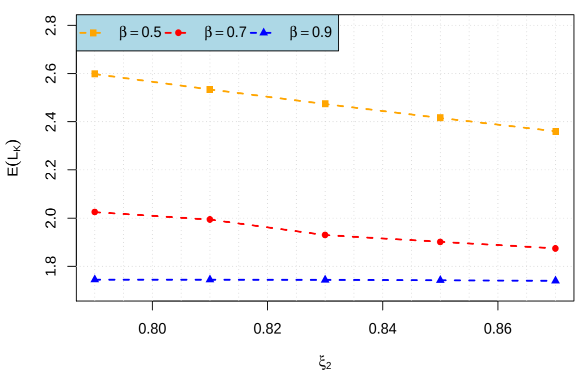

Figure 6.10: Impact of ξ_2 on $E(L_K)$

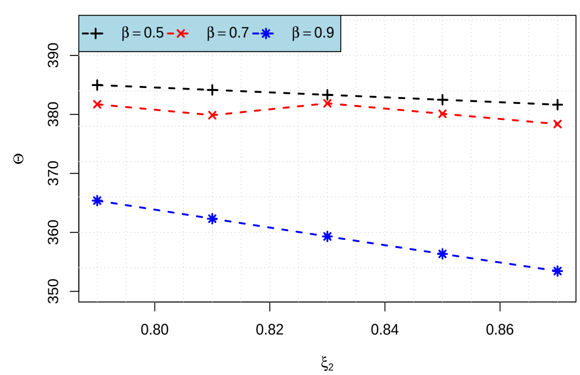

Figure 6.11: Impact of ξ_2 on Θ.

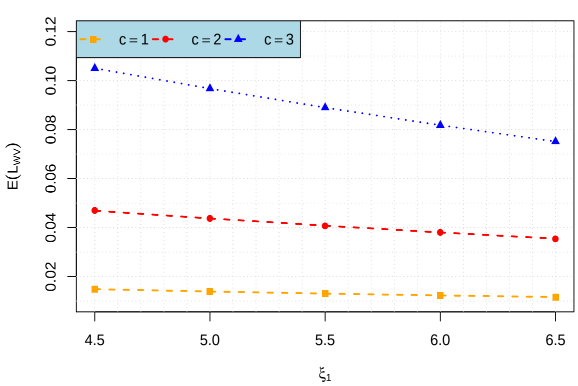

Figure 6.12: Impact of ξ_1 on $E(L_{WV})$

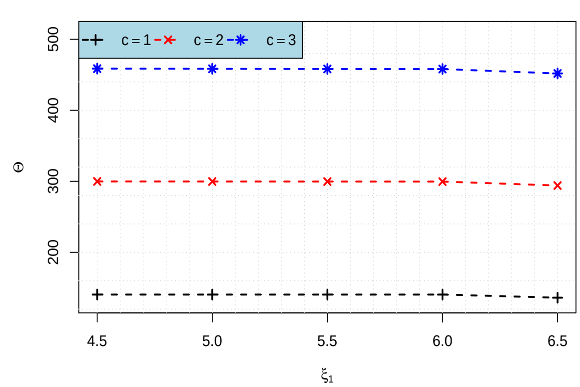

Figure 6.13: Impact of ξ_1 on Θ

creases. This implies an increase in the total expected profit Θ. Further, it is well observed that the increase of the number of variant vacation has a bad effect on the system.

- The impact of vacation rate ϕ is depicted in Table 6.2 and Figures 6.4-6.5 for different mean batch sizes $1/q$. It can be observed that for fixed q, as ϕ increases, the mean size of the system when the servers are in normal busy period $E(L_K)$ increases, as intuitively expected. On the other hand, for fixed ϕ, $E(L_K)$ increases with $1/q$, as it should be. Thus, it is clearly obvious that the total expected profit Θ increases with the increasing of ϕ, while the augmentation of q implies a lost in Θ.

- In Table 6.3 and Figures 6.6-6.7, we illustrate the effect of service rate during busy period μ, for various impatience rate during busy period ξ_2. It is quite clear that with the increase in the service rate μ, the mean number of customers served augments. Thus, the total expected profit Θ increases. Obviously, the number of customers served decreases when ξ_2 increases. Thus, we have a significant total expected profit Θ for large values of μ and small values of ξ_2.

- According to the results presented in Table 6.4 and Figures 6.8-6.9, we see that the average rate of abandonment R_a decreases with the increases in the service rate during vacation period ν. This is because the mean number of customers served augments with ν. Consequently, the average rate of abandonment is reduced. Further, the increase in the probability of non-retention α implies an increase in R_a. Finally, it is well observed that the increases in the service rate during vacation period ν, and in the retention probability α' have a nice impact on the total expected profit Θ.

- The impact of the impatience rate during busy period ξ_2 for different values of non-feedback probabilities β is illustrated in Table 6.5 and Figures 6.10-6.11. It is clearly shown that with the increase in impatience rate during normal busy period ξ_2, the mean size on the system when the servers are in normal busy period $E(L_K)$ decreases, this implies a diminution in the mean number of customers served. Consequently, the total expected profit Θ decreases. Furthermore, from the above presentations it is well seen that the feedback probability β' has a nice effect on the economy of the system.

- Figures 6.12 and 6.13 plot the impatience rate during working vacation period ξ_1 for different values of number of servers c. It is well observed that when the impatience rate ξ_1 is large, the mean size of the system when the servers are on working vacation period decreases. Therefore, the mean number of customers served is reduced. This leads to a decrease in Θ. On the other hand, from Table 6.6, we observe that when the number of servers becomes large, the total expected profit is significant. This is due to the fact that the mean number of customers served increases with c, while the average rate of abandonment decreases with the increasing of the number of the servers.

6.6 Conclusions and future scope

In the present study, we explored reneging behaviour in multi-server Bernoulli feedback queueing system with batch arrival, variant of multiple working vacations and retention of the reneged customers. For the analysis purpose, we investigated various system characteristics in terms of steady state probabilities using the probability generating functions (PGFs). Reneging and retention probabilities incorporated in our model may play an important role in the economy of the concerned system. Numerical experiments performed can be useful and benefic to explore the impacts of system parameters on the performance measures in different contexts. The model developed may provide lucrative perspicacity to the production managers, system engineers etc. To make the system modelling more closer to the real world problems, an extension of our results for a non-Markovian models is a pointer to future research. Moreover, we can extend this study in future by incorporating the bulk failure.

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Chapter 7

Cost optimization analysis for an $M^X/M/c$ vacation queueing system with waiting servers and impatient customers

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Cost optimization analysis for an $M^X/M/c$ vacation queueing system with waiting servers and impatient customers

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Abstract. This paper deals with the study of an $M^X/M/c$ Bernoulli feedback queueing system with waiting servers and two different policies of synchronous vacations (single and multiple vacation policies). During vacation period, the customers may leave the system (reneging), and using certain customer retention mechanism, the reneged customers may be retained in the system. The probability generating function (PGF) has been used to obtain the steady state probabilities of the model. Various performances measures of the system are derived. Then, a cost model is developed. Further, a cost optimization problem is considered using quadratic fit search method. Finally, a variety of numerical illustrations are discussed for the applicability of the model.

Keywords: Multi-server queueing systems. Single vacation. Multiple vacation. Impatient customers. Bernoulli feedback. Probability generating function. Optimization.
2010 Mathematics Subject Classification: 60K25, 68M20, 90B22.

7.1 Introduction

Performance of modeling vacation queueing systems has attracted many researchers owing to their large applications in many real life congestion problems including computer and communication systems, manufacturing and production systems along with other queueing systems having industrial importance. A detailed surveys of the literature devoted to such systems are found in Doshi (1986), Takagi (1991), Tian and Zhang (2006) and references therein.

Modeling vacation queueing models with impatient customers is very important in order to obtain novel managerial insights. The lost revenues due to impatience in several industries may be enormous. Literature analysis has shown an extensive studies of these models. Altman and Yechiali (2006) dealt with customers' impatience in queues with server vacation. Zhang et al. (2005) gave the analysis of an $M/M/1/N$ queueing model with balking, reneging and server vacations. Later, Ammar (2015) carried out the transient analysis of an $M/M/1$ queue with impatient behavior and multiple vacations. Panda and Goswami (2016) studied the equilibrium balking strategies for a $GI/M/1$ queue with Bernoulli-schedule vacation and vacation interruption. Recently, in Ammar (2017), the transient solution of an $M/M/1$ vacation queue with a waiting server and impatient customers has been established. For more literature on customer's impatience in vacation queues, the authors can be referred to Yue et al. (2006), Padmavathy et al. (2011), Misra and Goswami (2015), Yue et al. (2016), Sun et al. (2016), Bouchentouf and Yahiaoui (2017), and references therein.

Queueing systems with batch arrival represent the case where arrivals enter the system in batches rather than one by one. Few examples of arrivals in batches to a system are customers in elevators, supermarkets, banks, etc. Considerable works on vacation models with batch arrivals were conducted by many researchers. Lee et al. (1996) analyzed the fixed bulk service queueing system with single and multiple vacations. Later, Jau-Chuan Ke (2007) dealt with a $M^X/G/1$ queueing model with balking and variant vacation policy, Wang et al. (2007) presented the maximum entropy analysis of the $M^X/M/1$ queueing system with multiple vacations and server breakdowns. Then, Baruah et al. (2012) treated the balking and the re-service in a vacation queueing model with batch arrival and two types of heterogeneous service. Baruah et al. (2013) dealt with a batch arrival queue with second optional service and reneging during vacation periods. Recently, Sasikala et al. (2017) presented the steady state behaviour of $M^X/G^K/1$ queueing model with control policy on request for re-service, N-policy, balking and multiple vacations.

The study of multi-server vacation systems with impatient customers is far more complex compared to impatience in single server vacation models, consequently, a limited literature is available. The $M/M/c/N$ queuing system with balking, reneging and synchronous vacations of some partial servers together was presented by Yue et al. (2006). Altman and Yechiali (2008) treated the infinite-server queues with system's additional tasks and impatient customers. Then, a computational algorithm and parameter optimization for a multi-server queue with unreliable server and impatient customers have been discussed by Chia and Jau-Chaun (2010). Later, Yue et al. (2014) dealt with an $M/M/c$ queueing system with impatient customers and synchronous vacation, where impatience is due to the servers' vacation. Recently, the analysis of a $M/M/c$ queueing model with single and multiple synchronous working vacations was presented in Majid and Manoharan (2017).

This paper deals with an infinite buffer multi-server vacation queueing system with batch arrival, Bernoulli feedback and waiting servers wherein customers may renege during vacation period, and they can be retained in the system, via certain strategy (convinced to stay in order to be serviced). The steady-state probabilities of the queueing system are obtained through probability generating functions (PGFs). Useful performance measures of the queueing system are derived. The cost profit analysis of the model is carried out. The optimization of the model is performed using quadratic fit search method (QFSM) in order to minimize the total expected cost of the system with respect to the service rate. A numerical study is presented to illustrate the impact of various system parameters on different performance measures and total expected profit of the system. The analysis carried out in this paper is very important and useful to any insurance firm. Among the advantages of the obtained results is to show the positive impact of waiting server and customer retention strategy.

The rest of paper is arranged as follows. Section 2 provides a general description of the model with multiple and single vacation policies. In Section 3, we develop the queueing model under multiple vacation policy (MVP) and carry out the steady-state analysis of the system, then we derive the explicit expressions of the various performance measures of the queueing system. The model under single vacation policy (SVP) is analyzed in Section 4 following the same methodology presented in the previous section. In Section 5, we develop a model for the costs incurred and perform the appropriate optimization using a quadratic fit search method (QFSM). Section 6 presents numerical examples in order to demonstrate the applicability of the theoretical investigation, and finally we conclude the paper in Section 7.

7.2 Model Description

We consider an $M^X/M/c$ Bernoulli feedback queueing system under single and multiple vacations wherein customers may leave the system due to impatience during the absence of the servers. Using certain retention customer mechanism, the reneged customers may be retained in the system. The model considered in this work is based on following assumptions:

(i) Customers arrive in batches according to a Poisson process with rate λ. The sizes of successive arriving batches are i.i.d. random variables X_1, X_2,...distributed with probability mass function $P(X = l) = b_l;\ l = 1, 2, 3, ...$

(ii) The customers are served on a First-Come First-Served (FCFS) queue discipline. The service times are assumed to follow exponential distribution with mean $1/\mu$.

(iii) When the busy period is finished the servers wait a random duration of time before beginning on a vacation. This waiting duration is exponentially distributed with mean $1/\eta$.

(iv) The queueing model consists of c servers. In synchronous vacation policy, all the servers leave for a vacation simultaneously, once the system becomes empty and they also comeback to the system as one at the same time.
In multiple vacation policy (MVP), the servers continue to take vacations until they find the system nonempty at a vacation completion instant. While, in single vacation policy (SVP), when the vacation ends and servers find the system empty, they remains idle until the first arrival occurs. Vacation periods are assumed to be exponentially distributed with mean $1/\phi$.

(v) Customers in batches are supposed to enter the queueing system, join the queue, if the servers are unavailable due to vacation, a batch of customers activates an independent impatience timer T, with exponentially distributed duration, with mean $1/\xi$. If T expires while the servers are still on vacation, the customers may abandon the system. Further, using certain mechanism, each impatient customer may abandon the system, with probability α, and can be retained in the queue, with complementary probability $(1 - \alpha)$. Moreover, If the service is uncomplete or unsatisfactory, the customers can either leave the system definitively, with probability β, or rejoin the end of the queue of the system for another service, with complementary probability $(1 - \beta)$. Note that $\rho = \frac{\lambda E(X)}{c\beta\mu} < 1$ is the stability condition of the system, where $E(X)$ is the mean of a batch of arrivals.

(vi) We assume that the inter-arrival times, batch sizes, server waiting times, vacation times, service times and impatience times are independent of each other.

Let $\{L(t); t \geq 0\}$ be the number of customers in the system at time t, and $S(t)$ be the state of servers at time t, where $S(t)$ is defined as follows:

$$S(t) = \begin{cases} 1, & \text{when the servers are in busy period at time } t; \\ 0, & \text{when the servers are in vacation period at time } t. \end{cases}$$

Then, $\{(S(t), L(t)); t \geq 0\}$ defines a two-dimensional continuous Markov process with state space

$$\Omega = \{(s,n) : s = 0,1,\ n = 0,1,...\}.$$

Let

$$P_{s,n} = \lim_{t\to\infty} P\{S(t) = s, L(t) = n\}, s = 0,1, n = 0,1,...,$$

denote the system steady-state probabilities.

7.3 Analysis of the model under MVP

In this section, we study the model considered in Section 2 under multiple vacation policy, the state-transition-rate diagram is presented in Figure 7.1.

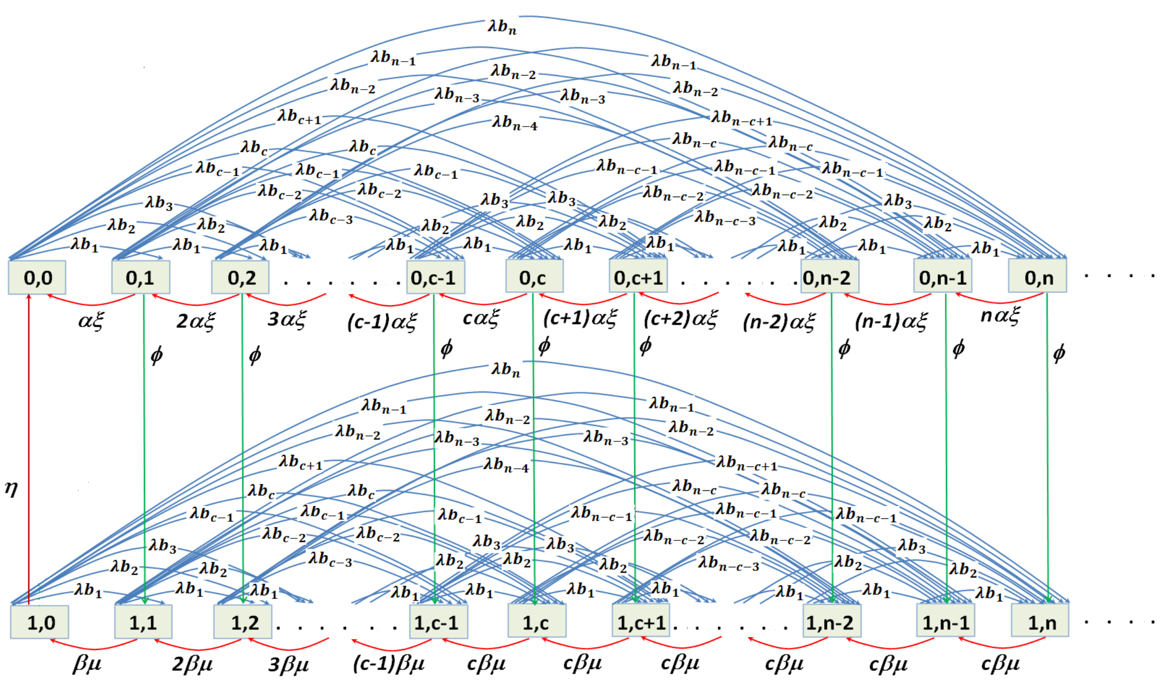

Figure 7.1: State-transition-rate diagram of the model under MVP.

7.3.1 Steady state solution of the model

Using the Markov theory, the set of steady-state equations are written as follows

$$\lambda P_{0,0} = \alpha\xi P_{0,1} + \eta P_{1,0}, \tag{7.1}$$

$$(\lambda + \phi + \alpha\xi)P_{0,1} = \lambda b_1 P_{0,0} + 2\alpha\xi P_{0,2}, \quad n = 1, \tag{7.2}$$

$$(\lambda + \phi + n\alpha\xi)P_{0,n} = \lambda \sum_{m=1}^{n} b_m P_{0,n-m} + (n+1)\alpha\xi P_{0,n+1}, \;\; n \geq 2, \tag{7.3}$$

$$(\lambda + \eta)P_{1,0} = \beta\mu P_{1,1}, \tag{7.4}$$

$$(\lambda + \beta\mu)P_{1,1} = \lambda b_1 P_{1,0} + 2\beta\mu P_{1,2} + \phi P_{0,1}, \quad n = 1, \tag{7.5}$$

$$(\lambda + n\beta\mu)P_{1,n} = \lambda \sum_{m=1}^{n} b_m P_{1,n-m} + (n+1)\beta\mu P_{1,n+1} + \phi P_{0,n}, \quad 2 \leq n \leq c-1, \tag{7.6}$$

$$(\lambda + c\beta\mu)P_{1,n} = \lambda \sum_{m=1}^{n} b_m P_{1,n-m} + c\beta\mu P_{1,n+1} + \phi P_{0,n}, \quad n \geq c. \tag{7.7}$$

And the normalizing condition is given as

$$\sum_{n=0}^{\infty} P_{0,n} + \sum_{n=0}^{\infty} P_{1,n} = 1. \tag{7.8}$$

The probability generating function (PGF) of $P_{s,n}$ is defined as

$$G_s(z) = \sum_{n=0}^{\infty} P_{s,n} z^n, \;\; s = 0, 1. \tag{7.9}$$

The probability generating function (PGF) of the batch size X is as

$$B(z) = \sum_{l=1}^{\infty} b_l z^l, |z| \leq 1, \;\text{with}\; B(1) = \sum_{l=1}^{\infty} b_l = 1. \tag{7.10}$$

The steady-state probabilities of the queueing system are obtained by solving the equations (7.1)-(7.7) using PGF.

By multiplying equations (7.1)-(7.3) by z^n, and summing over n, then re-arranging all the terms, we get

$$(1-z)\alpha\xi G_0'(z) + [\lambda(B(z)-1)-\phi]G_0(z) = -[\phi P_{0,0} + \eta P_{1,0}]. \tag{7.11}$$

In a similar manner, form equations (7.4)-(7.7), we have

$$\begin{aligned}[\lambda z(B(z)-1) + c\beta\mu(1-z)]G_1(z) + z\phi G_0(z) = \beta\mu(z-1)\sum_{n=0}^{c-1}(n-c)P_{1,n}z^n \\ +z[\eta P_{1,0} + \phi P_{0,0}].\end{aligned} \tag{7.12}$$

Then, by taking $z = 1$ in equation (7.11) or equation (7.12), we obtain

$$\phi G_0(1) = \eta P_{1,0} + \phi P_{0,0}. \tag{7.13}$$

We solve the differential equation (7.11) using the same method used in Altman and Yechiali (2006). Thus equation (7.11) can be written as

$$G_0'(z) + \left[\frac{\lambda}{\alpha\xi}H'(z) - \frac{\phi}{\alpha\xi(1-z)}\right]G_0(z) = -\left[\frac{\phi}{\alpha\xi(1-z)}P_{0,0} + \frac{\eta}{\alpha\xi(1-z)}P_{1,0}\right], \tag{7.14}$$

where

$$H(z) = \int_0^z \frac{B(x)-1}{1-x}dx \text{ and } H'(z) = \frac{B(z)-1}{1-z}.$$

Then, we multiply both sides of equation (7.14) by $e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}}$, we get

$$\frac{d}{dz}\left(e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}}G_0(z)\right) = -\left[\frac{\phi}{\alpha\xi}P_{0,0} + \frac{\eta}{\alpha\xi}P_{1,0}\right]e^{\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{\frac{\phi}{\alpha\xi}-1}. \tag{7.15}$$

Now, integrating equation (7.15) from 0 to z, we get

$$G_0(z) = e^{-\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{-\frac{\phi}{\alpha\xi}}\left\{G_0(0) - \frac{K(z)}{\alpha\xi}\left[\phi P_{0,0} + \eta P_{1,0}\right]\right\}, \tag{7.16}$$

where

$$K(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi}H(x)}(1-x)^{\frac{\phi}{\alpha\xi}-1}dx. \tag{7.17}$$

Since $G_0(1) = P_{0,.} = \sum_{n=0}^{\infty} P_{0,n} > 0$ and $z = 1$ is the root of the denominator of the right hand side of equation (7.16), so $z = 1$ must be the root of the numerator of the right hand side of equation (7.16).

Thus, we get

$$P_{0,0} = G_0(0) = \frac{K(1)}{\alpha\xi}\Big[\phi P_{0,0} + \eta P_{1,0}\Big], \tag{7.18}$$

with

$$K(1) = \int_0^1 e^{\frac{\lambda}{\alpha\xi}H(x)}(1-x)^{\frac{\phi}{\alpha\xi}-1}dx. \tag{7.19}$$

From equation (7.18), it yields

$$P_{1,0} = \theta_0 P_{0,0}, \text{ where } \theta_0 = \frac{\alpha\xi - \phi K(1)}{\eta K(1)}. \tag{7.20}$$

By substituting equation (7.20) into equation (7.16), we obtain

$$G_0(z) = e^{-\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{-\frac{\phi}{\alpha\xi}}\left\{1 - \frac{K(z)}{K(1)}\right\}P_{0,0}. \tag{7.21}$$

And by substituting equation (7.20) into equation (7.13), we find the probability that the servers are in vacation period, $(G_0(1) = P_{0,.} = \sum_{n=0}^{\infty} P_{0,n})$,

$$G_0(1) = \frac{\alpha\xi}{\phi K(1)}P_{0,0}. \tag{7.22}$$

It is clearly seen that equation (7.12) expresses $G_1(z)$ in terms of $P_{0,0}, P_{1,0}, P_{1,n}$ and $G_0(z)$. From equation (7.21), we see that $G_0(z)$ is expressed in terms of $P_{0,0}$, then in equation (7.20), $P_{1,0}$ is given in terms of $P_{0,0}$. Thus, to define $G_1(z)$ in terms of $P_{0,0}$, we need to express $P_{1,n}$ in terms of $P_{0,0}$. To this end, we firstly have to write $P_{0,n}$ in terms of $P_{0,0}$.

From equation (7.1), using equation (7.20), we get

$$P_{0,1} = \omega_1 P_{0,0}, \tag{7.23}$$

where $\omega_1 = \frac{\lambda - \eta\theta_0}{\alpha\xi}$.

From equation (7.2), using equation (7.23), we find

$$P_{0,2} = \omega_2 P_{0,0}, \tag{7.24}$$

where $\omega_2 = \psi_1\omega_1 - \frac{\lambda}{2\alpha\xi} b_1\omega_0$, $\psi_1 = \frac{\lambda+\phi+\alpha\xi}{2\alpha\xi}$, and $\omega_0 = 1$.

Then, from equation (7.3), for $n = 2$, using equations (7.23) and (7.24), we obtain

$$P_{0,3} = \omega_3 P_{0,0}, \tag{7.25}$$

where $\omega_3 = \psi_2\omega_2 - \frac{\lambda}{3\alpha\xi}\left(b_1\omega_1 + b_2\omega_0\right)$, and $\psi_2 = \frac{\lambda+\phi+2\alpha\xi}{3\alpha\xi}$.

Then, recursively, it yields

$$P_{0,n} = \omega_n P_{0,0}, \tag{7.26}$$

where

$$\omega_n = \begin{cases} 1, & \text{if } n = 0, \\ \frac{\lambda-\eta\theta_0}{\alpha\xi}, & \text{if } n = 1, \\ \psi_{n-1}\omega_{n-1} - \dfrac{\lambda}{n\alpha\xi}\displaystyle\sum_{i=1}^{n-1} b_i\omega_{n-1-i}, & \text{if } 2 \le n \le c-1, \end{cases}$$

with

$$\psi_{n-1} = \frac{\lambda + \phi + (n-1)\alpha\xi}{n\alpha\xi}.$$

Next, we need to write $P_{1,n}$ in terms of $P_{0,0}$. Via equation (7.4), using equation (7.20), we get

$$P_{1,1} = \theta_1 P_{0,0}, \tag{7.27}$$

where $\theta_1 = \frac{\lambda+\eta}{\beta\mu}\theta_0$.

From equation (7.5), using equations (7.20), (7.23) and (7.27), we obtain

$$P_{1,2} = \theta_2 P_{0,0}, \tag{7.28}$$

where $\theta_2 = \rho_1\theta_1 - \frac{\phi}{2\beta\mu}\omega_1 - \frac{\lambda}{2\beta\mu}b_1\theta_0$, and $\rho_1 = \frac{\lambda+\beta\mu}{2\beta\mu}$.

Then, from equation (7.6), for $n = 2$, using equations (7.20), (7.24), (7.27) and (7.28), we find

$$P_{1,3} = \theta_3 P_{0,0}, \tag{7.29}$$

where $\theta_3 = \rho_2\theta_2 - \frac{\phi}{3\beta\mu}\omega_2 - \frac{\lambda}{3\beta\mu}\left(b_1\theta_1 + b_2\theta_0\right)$, and $\rho_2 = \frac{\lambda+2\beta\mu}{3\beta\mu}$.

Then, recursively, we get

$$P_{1,n} = \theta_n P_{0,0}, \tag{7.30}$$

where

$$\theta_n = \begin{cases} \theta_0, & \text{if } n = 0, \\ \frac{\lambda+\eta}{\beta\mu}\theta_0, & \text{if } n = 1, \\ \rho_{n-1}\theta_{n-1} - \dfrac{\phi}{n\beta\mu}\omega_{n-1} - \dfrac{\lambda}{n\beta\mu}\displaystyle\sum_{i=1}^{n-1} b_i\theta_{n-1-i}, & \text{if } 2 \le n \le c-1, \end{cases}$$

with

$$\rho_{n-1} = \frac{\lambda + (n-1)\beta\mu}{n\beta\mu}.$$

Finally, using equations (7.12),(7.20),(7.21), and (7.30), $G_0(z)$ and $G_1(z)$ are expressed in terms of $P_{0,0}$. So, it remains to determine this quantity. From equation (7.14), applying L'Hopital rule, we find

$$\lim_{z\to 1} G_0'(z) = G_0'(1) = \frac{\lambda B'(1)}{\alpha\xi + \phi} G_0(1). \tag{7.31}$$

Next, substituting equation (7.22) in equation (7.31), we obtain

$$G_0'(1) = \frac{\alpha\xi\lambda B'(1)}{(\alpha\xi + \phi)\phi K(1)} P_{0,0}. \tag{7.32}$$

Then, substituting equation (7.13) in equation (7.12), we get

$$G_1(z) = \frac{\beta\mu(1-z)R(z)P_{0,0} - z\phi(G_0(z) - G_0(1))}{\lambda z(B(z)-1) + c\beta\mu(1-z)}, \tag{7.33}$$

where

$$R(z) = \sum_{n=0}^{c-1}(c-n)\theta_n z^n.$$

From equation (7.33), applying L'Hopital rule, we obtain the probability that the servers are in busy period, $(G_1(1) = P_{1,.} = \sum_{n=0}^{\infty} P_{1,n})$,

$$\lim_{z\to 1} G_1(z) = G_1(1) = \frac{\phi G_0'(1) + \beta\mu R(1)P_{0,0}}{c\beta\mu - \lambda B'(1)}, \tag{7.34}$$

with

$$R(1) = \sum_{n=0}^{c-1}(c-n)\theta_n.$$

Finally, by substituting equations (7.22), (7.32) and (7.34) in equation (7.8), we find

$$P_{0,0} = \left\{\frac{\beta\mu}{c\beta\mu - \lambda B'(1)}R(1) + \frac{\alpha\xi}{\phi K(1)}\left(\frac{\phi\lambda B'(1)}{(c\beta\mu - \lambda B'(1))(\alpha\xi + \phi)} + 1\right)\right\}^{-1}.$$

7.3.2 Performance measures

Once the steady-state probabilities are obtained, one can evaluate different performance measures of the considered model.

• The mean system size ($E[L]$). Let L denote the number of customers in the system. The mean system size is given as

$$E[L] = E[L_0] + E[L_1].$$

∗ Let L_0 be the system size when the servers are in vacation period. Then, the mean system size when the servers are in vacation period ($E[L_0]$) is given as

$$E[L_0] = \lim_{z\to 1} G_0'(z) = G_0'(1),$$

which is a direct consequence of equation (7.32).

∗ Let L_1 be the system size when the servers are in busy period. Then, the mean system size when the servers are in busy period ($E[L_1]$) is given as

$$E[L_1] = \lim_{z\to 1} G_1'(z) = G_1'(1).$$

Via equation (7.33), using L'Hopital rule, we obtain

$$\begin{aligned} E[L_1] = G_1'(1) = &\frac{\phi}{2(c\beta\mu - \lambda B'(1))} G_0''(1) + \frac{\phi(2c\beta\mu + \lambda B''(1))}{2(c\beta\mu - \lambda B'(1))^2} G_0'(1) \\ &+ \left(\frac{\beta\mu\lambda(2B'(1) + B''(1))}{2(c\beta\mu - \lambda B'(1))^2} R(1) + \frac{\beta\mu}{c\beta\mu - \lambda B'(1)} R'(1) \right) P_{0,0}, \end{aligned} \tag{7.35}$$

where $G_0''(1)$ is obtained by differentiating twice $G_0(z)$ at $z = 1$. Thus, using equation (7.11), we find

$$G_0''(1) = \frac{2\lambda B'(1)}{2\alpha\xi + \phi} G_0'(1) + \frac{\lambda B''(1)}{2\alpha\xi + \phi} G_0(1). \tag{7.36}$$

Further

$$R'(1) = \sum_{n=1}^{c-1} n(c-n)\theta_n.$$

Then, substituting equation (7.36) in equation (7.35), we get

$$\begin{aligned} E[L_1] = &\left(\frac{\phi(2c\beta\mu + \lambda B''(1))}{2(c\beta\mu - \lambda B'(1))^2} + \frac{\lambda\phi B'(1)}{(c\beta\mu - \lambda B'(1))(2\alpha\xi + \phi)} \right) E[L_0] \\ &+ \left(\frac{\beta\mu\lambda(2B'(1) + B''(1))}{2(c\beta\mu - \lambda B'(1))^2} R(1) + \frac{\beta\mu}{c\beta\mu - \lambda B'(1)} R'(1) \right) P_{0,0} \\ &+ \frac{\lambda\phi B''(1)}{2(c\beta\mu - \lambda B'(1))(2\alpha\xi + \phi)} G_0(1). \end{aligned}$$

• The mean number of customers in the queue ($E[L_q]$).

$$E[L_q] = \sum_{n=0}^{\infty} n P_{0,n} + \sum_{n=c+1}^{\infty} (n-c) P_{1,n} = E[L] - c(1 - P_v) + R(1) P_{0,0}.$$

• The probability that the servers are in vacation period (P_v). From equation (7.22), we obtain

$$P_v = G_0(1) = \frac{\alpha\xi}{\phi K(1)} P_{0,0}.$$

• The probability that the servers are idle during busy period (P_e). From equation (7.20), we get

$$P_e = \frac{\alpha\xi - \phi K(1)}{\eta K(1)} P_{0,0}.$$

• The probability that the servers are working (serving customers) during busy period (P_b).

$$P_b = 1 - P_v - P_e.$$

• The mean number of customers served per unit time (N_s).

$$N_s = \beta\mu \sum_{n=1}^{c-1} nP_{1,n} + c\beta\mu \sum_{n=c}^{\infty} P_{1,n} = \beta\mu\big(c(P_b + P_e) + R(1)P_{0,0}\big).$$

• The average rate of abandonment of customers due to impatience (R_a).

$$R_a = \alpha\xi \sum_{n=0}^{\infty} nP_{0,n} = \alpha\xi E[L_0].$$

• The average retention rate of impatient customers (R_e).

$$R_e = (1-\alpha)\xi \sum_{n=0}^{\infty} nP_{0,n} = (1-\alpha)\xi E[L_0].$$

7.4 Analysis of the model under SVP

This section is devoted to the study of the system under single vacation policy. The transition- rate diagram depicting the state of the system is shown in Figure 7.2.

7.4.1 Steady state solution of the model

Via the Markov theory, the set of steady-state equations are as follows

$$(\lambda + \phi)P_{0,0} = \alpha\xi P_{0,1} + \eta P_{1,0}, \tag{7.37}$$

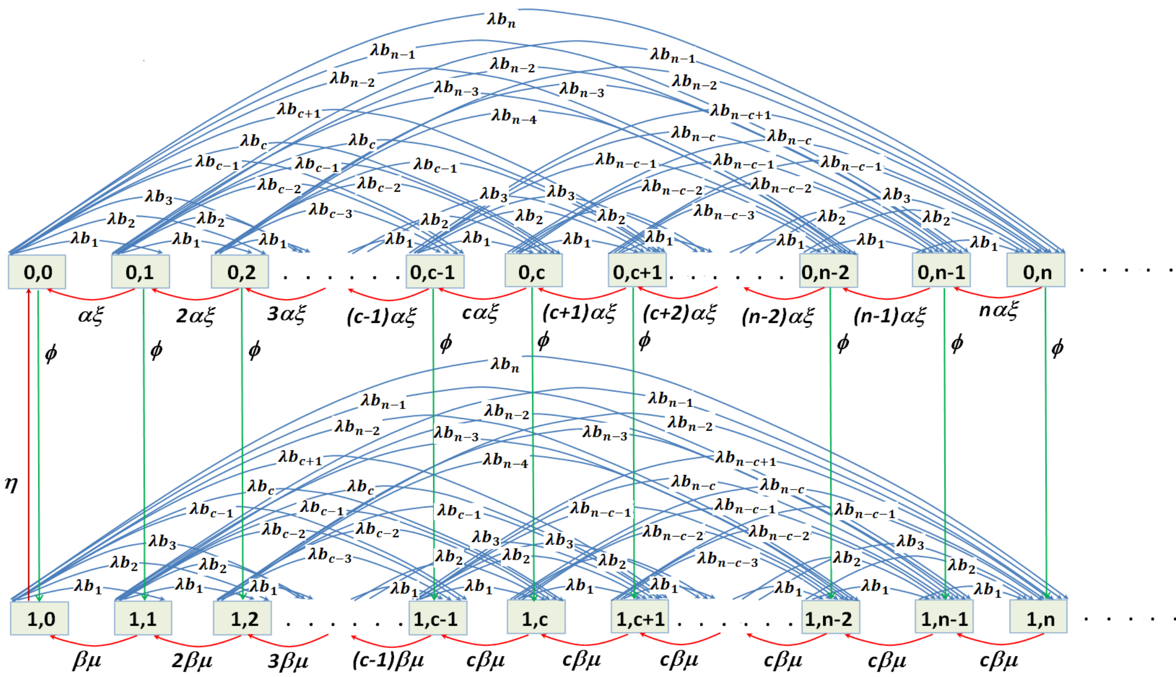

Figure 7.2: State-transition-rate diagram of the model under SVP.

$$(\lambda+\phi+\alpha\xi)P_{0,1}=\lambda b_1 P_{0,0}+2\alpha\xi P_{0,2}, \quad n=1, \tag{7.38}$$

$$(\lambda+\phi+n\alpha\xi)P_{0,n}=\lambda\sum_{m=1}^{n} b_m P_{0,n-m}+(n+1)\alpha\xi P_{0,n+1}, \quad n\geq 2, \tag{7.39}$$

$$(\lambda+\eta)P_{1,0}=\phi P_{0,0}+\beta\mu P_{1,1}, \tag{7.40}$$

$$(\lambda+\beta\mu)P_{1,1}=\lambda b_1 P_{1,0}+2\beta\mu P_{1,2}+\phi P_{0,1}, \quad n=1, \tag{7.41}$$

$$(\lambda+n\beta\mu)P_{1,n}=\lambda\sum_{m=1}^{n} b_m P_{1,n-m}+(n+1)\beta\mu P_{1,n+1}+\phi P_{0,n}, \quad 2\leq n\leq c-1, \tag{7.42}$$

$$(\lambda+c\beta\mu)P_{1,n}=\lambda\sum_{m=1}^{n} b_m P_{1,n-m}+c\beta\mu P_{1,n+1}+\phi P_{0,n}, \quad n\geq c, \tag{7.43}$$

The normalizing condition is given in equation (7.8).

The PGF of $P_{s,n}$ is given in equation (7.9), and that of the batch size X has already been done in (7.10).

The state probabilities are obtained by solving the equations (7.37)-(7.43) using PGF.

Now, multiplying equation (7.37)-(7.39) by z^n, and summing n, then re-arranging all the terms, we have

$$(1-z)\alpha\xi G_0'(z) + [\lambda(B(z)-1)-\phi]G_0(z) = -\eta P_{1,0}. \tag{7.44}$$

In a similar manner, form equations (7.40)-(7.43), it yields

$$[\lambda z(B(z)-1)+c\beta\mu(1-z)]G_1(z) + z\phi G_0(z) = \beta\mu(z-1)\sum_{n=0}^{c-1}(n-c)P_{1,n}z^n + \eta z P_{1,0}. \tag{7.45}$$

By taking $z=1$ in equation (7.44) or equation (7.45), we obtain

$$\phi G_0(1) = \eta P_{1,0}. \tag{7.46}$$

We solve equation (7.44) by following the method presented in Altman and Yechiali (2006).

Using equation (7.40), we get

$$P_{1,0} = \frac{\phi}{\lambda+\eta}P_{0,0} + \frac{\beta\mu}{\lambda+\eta}P_{1,1}. \tag{7.47}$$

Equation (7.44) can be written as

$$G_0'(z) + \left[\frac{\lambda}{\alpha\xi}H'(z) - \frac{\phi}{\alpha\xi(1-z)}\right]G_0(z) = -\left[\frac{\delta_1}{\alpha\xi(1-z)}P_{0,0} + \frac{\delta_2}{\alpha\xi(1-z)}P_{1,1}\right], \tag{7.48}$$

where

$$\delta_1 = \frac{\eta\phi}{\lambda+\eta}, \quad \delta_2 = \frac{\eta\beta\mu}{\lambda+\eta}.$$

The solution of the equation (7.44) is computed as before and given as follows

$$G_0(z) = e^{-\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{-\frac{\phi}{\alpha\xi}}\left\{G_0(0) - \frac{K(z)}{\alpha\xi}\left[\delta_1 P_{0,0} + \delta_2 P_{1,1}\right]\right\}. \tag{7.49}$$

Since $G_0(1) = P_{0,.} = \sum_{n=0}^{\infty} P_{0,n} > 0$ and $z = 1$ is the root of the denominator of the right hand side of equation (7.49), thus $z = 1$ must be the root of the numerator of the right hand side of equation (7.49).

Consequently,

$$P_{0,0} = G_0(0) = \frac{\delta_1 P_{0,0} + \delta_2 P_{1,1}}{\alpha\xi} K(1). \tag{7.50}$$

This implies

$$P_{1,1} = M_1 P_{0,0}, \text{ where } M_1 = \frac{\alpha\xi(\lambda+\eta)}{\eta\beta\mu K(1)} - \frac{\phi}{\beta\mu}. \tag{7.51}$$

Consequently,

$$G_0(z) = e^{-\frac{\lambda}{\alpha\xi}H(z)}(1-z)^{-\frac{\phi}{\alpha\xi}}\left\{1 - \frac{K(z)}{K(1)}\right\}P_{0,0}. \tag{7.52}$$

Next, substituting equation (7.51) into (7.40), and equation (7.53) into (7.46), we get respectively

$$P_{1,0} = M_0 P_{0,0}, \text{ where } M_0 = \frac{\alpha\xi}{\eta K(1)}, \tag{7.53}$$

and

$$G_0(1) = \frac{\alpha\xi}{\phi K(1)} P_{0,0}. \tag{7.54}$$

Now, equation (7.52) shows that $G_0(z)$ can be expressed in terms of $P_{0,0}$ and equation (7.53) expresses $P_{1,0}$ in terms of $P_{0,0}$. So, to get $P_{1,n}$ in terms of $P_{0,0}$ for $n = 0,...,c-1$, at first we have to express $P_{0,n}$ in terms of $P_{0,0}$ for $n = 0,...,c-1$.

Using equations (7.37)-(7.39), recursively, we get

$$P_{0,n} = \gamma_n P_{0,0}, \tag{7.55}$$

with

$$\gamma_n = \begin{cases} 1, & \text{if } n = 0, \\ \frac{\lambda+\phi-\eta M_0}{\alpha\xi}, & \text{if } n = 1, \\ \psi_{n-1}\gamma_{n-1} - \dfrac{\lambda}{n\alpha\xi}\displaystyle\sum_{i=1}^{n-1} b_i\gamma_{n-1-i}, & \text{if } 2 \leq n \leq c-1. \end{cases}$$

Next, via equations (7.40)-(7.42), using recursive method, we obtain

$$P_{1,n} = M_n P_{0,0}, \tag{7.56}$$

where

$$M_n = \begin{cases} M_0, & \text{if } n = 0, \\ M_1, & \text{if } n = 1, \\ \rho_{n-1}M_{n-1} - \dfrac{\phi}{n\beta\mu}\gamma_{n-1} - \dfrac{\lambda}{n\beta\mu}\displaystyle\sum_{i=1}^{n-1} b_i M_{n-1-i}, & \text{if } 2 \leq n \leq c-1. \end{cases}$$

Thus, $G_0(z)$ and $G_1(z)$ can be easily deduced in terms of $P_{0,0}$.
From equation (7.44), using equation (7.46), and applying L'Hopital rule, we have

$$\lim_{z\to 1} G_0'(z) = G_0'(1) = \frac{\lambda B'(1)}{\alpha\xi + \phi} G_0(1). \tag{7.57}$$

Substituting equation (7.54) into equation (7.57), we obtain

$$G_0'(1) = \frac{\alpha\xi\lambda B'(1)}{\phi K(1)(\alpha\xi + \phi)} P_{0,0}. \tag{7.58}$$

Next, substituting equation (7.46) into equation (7.45), we have

$$G_1(z) = \frac{\beta\mu(1-z)Q(z)P_{0,0} - z\phi(G_0(z) - G_0(1))}{\lambda z(B(z)-1) + c\beta\mu(1-z)}, \tag{7.59}$$

where

$$Q(z) = \sum_{n=0}^{c-1} (c-n) M_n z^n.$$

From equation (7.59), applying L'Hopital rule, it yields

$$G_1(1) = \frac{\phi G_0'(1) + \beta\mu Q(1) P_{0,0}}{c\beta\mu - \lambda B'(1)}, \tag{7.60}$$

with

$$Q(1) = \sum_{n=0}^{c-1} (c-n)M_n.$$

Next, substituting equation (7.58) into equation (7.60), we get

$$G_1(1) = \left\{ \frac{\alpha\xi\lambda B'(1)}{K(1)(\alpha\xi+\phi)(c\beta\mu-\lambda B'(1))} + \frac{\beta\mu}{c\beta\mu-\lambda B'(1)} Q(1) \right\} P_{0,0}. \tag{7.61}$$

Finally, by substituting equations (7.54) and (7.61) into equation (7.8), we get

$$P_{0,0} = \left\{ \frac{\alpha\xi}{\phi K(1)} \left(1 + \frac{\phi\lambda B'(1)}{(c\beta\mu-\lambda B'(1))(\alpha\xi+\phi)} \right) + \frac{\beta\mu}{c\beta\mu-\lambda B'(1)} Q(1) \right\}^{-1}.$$

7.4.2 Performance measures

• The mean system size ($E[L]$). L is the number of customers in the system.

$$E[L] = E[L_0] + E[L_1].$$

∗ Let L_0 be the system size when the servers are in vacation period, the mean system size when the servers are on vacation ($E[L_0]$) has been already given in equation (7.58).

∗ Let L_1 be the system size when the servers are in busy period, ($E[L_1]$) be the mean system size when the servers are on busy period. From equation (7.59), taking $z = 1$ and using L'Hopital rule, we obtain

$$\begin{aligned} E[L_1] = G_1'(1) = {} & \frac{\phi}{2(c\beta\mu-\lambda B'(1))} G_0''(1) + \frac{\phi(2c\beta\mu+\lambda B''(1))}{2(c\beta\mu-\lambda B'(1))^2} G_0'(1) \\ & + \left(\frac{\beta\mu\lambda(2B'(1)+B''(1))}{2(c\beta\mu-\lambda B'(1))^2} Q(1) + \frac{\beta\mu}{c\beta\mu-\lambda B'(1)} Q'(1) \right) P_{0,0}, \end{aligned} \tag{7.62}$$

where $G_0''(1)$ is obtained by differentiating twice $G_0(z)$ at $z = 1$, therefore, using equation (7.44), we get

$$G_0''(1) = \frac{2\lambda B'(1)}{2\alpha\xi+\phi} G_0'(1) + \frac{\lambda B''(1)}{2\alpha\xi+\phi} G_0(1), \tag{7.63}$$

and

$$Q'(1) = \sum_{n=1}^{c-1} n(c-n)M_n.$$

Now, substituting equation (7.63) into (7.62), we get

$$E[L_1] = \left(\frac{\phi(2c\beta\mu + \lambda B''(1))}{2(c\beta\mu - \lambda B'(1))^2} + \frac{\lambda\phi B'(1)}{(c\beta\mu - \lambda B'(1))(2\alpha\xi + \phi)}\right)E[L_0]$$
$$+\left(\frac{\beta\mu\lambda(2B'(1) + B''(1))}{2(c\beta\mu - \lambda B'(1))^2}Q(1) + \frac{\beta\mu}{c\beta\mu - \lambda B'(1)}Q'(1)\right)P_{0,0}$$
$$+\frac{\lambda\phi B''(1)}{2(c\beta\mu - \lambda B'(1))(2\alpha\xi + \phi)}G_0(1).$$

• The mean number of customers in the queue ($E[L_q]$).

$$E[L_q] = \sum_{n=0}^{\infty} nP_{0,n} + \sum_{n=c+1}^{\infty}(n-c)P_{1,n} = E[L] - c(1 - P_v) + Q(1)P_{0,0}.$$

• The probability that the servers are in vacation period (P_v). From equation (7.54), we obtain

$$P_v = G_0(1) = \frac{\alpha\xi}{\phi K(1)}P_{0,0}.$$

• The probability that the servers are idle during busy period (P_e). From equation (7.53), we get

$$P_e = \frac{\alpha\xi}{\eta K(1)}P_{0,0}.$$

• The probability that the servers are working (serving customers) during busy period (P_b).

$$P_b = 1 - P_v - P_e.$$

• The mean number of customers served per unit time (N_s).

$$N_s = \beta\mu\sum_{n=1}^{c-1} nP_{1,n} + c\beta\mu\sum_{n=c}^{\infty}P_{1,n} = \beta\mu\big(c(P_b + P_e) + Q(1)P_{0,0}\big).$$

- The average rate of abandonment of customers due to impatience (R_a).

$$R_a = \alpha\xi \sum_{n=0}^{\infty} nP_{0,n} = \alpha\xi E[L_0].$$

- The average retention rate of impatient customers (R_e).

$$R_e = (1-\alpha)\xi \sum_{n=0}^{\infty} nP_{0,n} = (1-\alpha)\xi E[L_0].$$

7.5 Cost model

Practically, queueing managers are interested in minimizing operating cost of unit time. In this part of paper, we first formulate a steady-state expected cost function per unit time, where the service rate μ is the decision variable. Our main goal is to determine the optimum value of μ in order to minimize the expected cost function. To this end, we have to define the following cost elements:

- C_1 : Cost per unit time when the servers are working during busy period.
- C_2 : Cost per unit time when the servers are idle during busy period.
- C_3 : Cost per unit time when the servers are in vacation period.
- C_4 : Cost per unit time when customers join the queue and wait for service.
- C_5 : Cost per service per unit time.
- C_6 : Cost per unit time of serving a feedback customer.
- C_7 : Cost per unit time when a customer reneges.
- C_8 : Cost per unit time when a customer is retained in the system.
- C_9 : Fixed server purchase cost per unit.
- R : The revenue earned by providing service to a customer.
  Let
- $\mathcal{T}_c$ be the total expected cost per unit time of the system:

$$\mathcal{T}_c = C_1 P_b + C_2 P_e + C_3 P_v + C_4 E[L_q] + c\mu(C_5 + \beta' C_6) + C_7 R_a + C_8 R_e + cC_9.$$

- $\mathcal{T}_r$ be the total expected revenue per unit time of the system:

$$\mathcal{T}_r = R \times N_s$$

- $\mathcal{T}_p$ be the total expected profit per unit time of the system:

$$\mathcal{T}_p = \mathcal{T}_r - \mathcal{T}_c.$$

Quadratic fit search method

This part considers the cost optimization problem under a given cost structure via quadratic fit search method (QFSM), this technique utilizes a 3-point pattern for fitting a quadratic function that has a unique optimum, see Rardin (1997). So, we focus on the optimization of the service rate μ in different cases in order to minimize the expected cost function $\mathcal{T}_c$ denoted in this part by F. Assume that all system parameters have fixed values, and the only controlled parameter is the service rate μ.

Thus, the optimization problem can be illustrated mathematically as:

$$\text{Minimize: } F(\mu) = C_1 P_b + C_2 P_e + C_3 P_v + C_4 E[L_q] + c\mu(C_5 + \beta' C_6) + C_7 R_a + C_8 R_e + cC_9.$$

As it has been mentioned in Laxmi et al. (2014), given a 3-point pattern, we may fit a quadratic function via corresponding functional values that has a unique minimum, x^q, for the given objective function $F(x)$. Quadratic fit utilizes this approximation to improve the current 3-point pattern by replacing one of its points with optimum x^q. The unique optimum x^q of the quadratic function agreeing with $F(x)$ at 3-point operation (x^l, x^m, x^u) is given as

$$x^q \cong \frac{1}{2}\left[\frac{F(x^l)((x^m)^2 - (x^u)^2) + F(x^m)((x^u)^2 - (x^l)^2) + F(x^u)((x^l)^2 - (x^m)^2)}{F(x^l)(x^m - x^u) + F(x^m)(x^u - x^l) + F(x^u)(x^l - x^m)}\right]$$

7.6 Numerical results

In this section, we illustrate the obtained resulting formulas numerically, we first carry out the optimization of the queueing system, using quadratic fit search method (QFSM) to minimize the expected cost function F with respect to the service rate, then we discuss the influence of different system parameters on the various performance measures of the queueing system as well as on total expected cost, total expected revenue and total expected profit. We assume that the batch size X follows a geometric distribution with parameter p, that is,

$$b_l = P(X = l) = (1-p)^{l-1}p,\ 0 < p < 1\ (l = 1, 2, \ldots).$$

Then, it is easy to observe that

$$B(z) = \frac{pz}{1-(1-p)z}, E(X) = B'(1) = \frac{1}{p}, \text{ and } E(X^2) = B''(1) = \frac{2(1-p)}{p^2}.$$

For the whole analysis in this numerical part, we fixe $C_1 = 40$, $C_2 = 25$, $C_3 = 20$, $C_4 = 30$, $C_5 = 50$, $C_6 = 20$, $C_7 = 20$, $C_8 = 30$, and $C_9 = 10$.

7.6.1 Optimization analysis

In order to carry out the numerical analysis on the parameter optimisation for the queueing system under consideration, we consider the values for default parameters as $c = 2$, $p = 0.70$, $\lambda = 1.00$, $\beta = 0.80$, $\eta = 3.00$, $\phi = 2.20$, $\alpha = 0.60$, and $\xi = 0.20$, and the tolerance of QFSM is $\epsilon = 10^{-6}$.

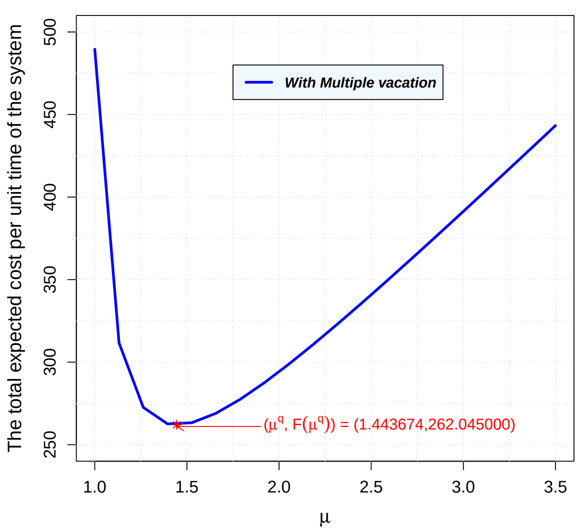

Figure 7.3: The optimum service rate μ^* under multiple vacation policy.

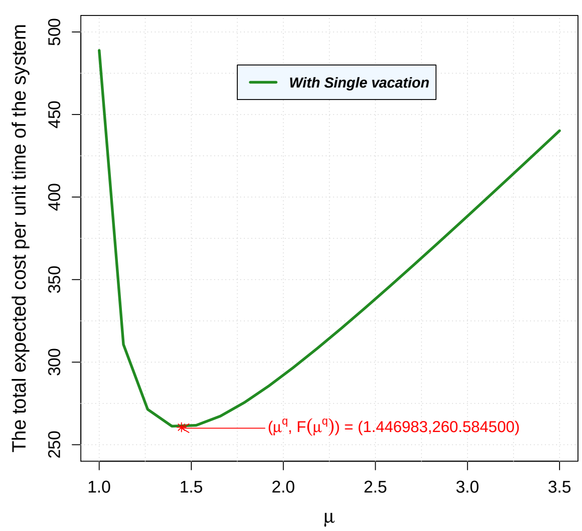

Figure 7.4: The optimum service rate μ^* under single vacation policy.

From Figures 7.3-7.4, we clearly see the convexity of the curves, which shows that there exists a certain value of the service rate μ that minimizes the total expected

Table 7.1: Search for optimum service rate μ^* under multiple vacation policy.

iter	μ^l	μ^m	μ^u	$F(\mu^l)$	$F(\mu^m)$	$F(\mu^u)$	μ^q	$F(\mu^q)$
01	1.050000	2.750000	3.500000	383.304600	365.811400	443.316700	2.010933	295.150300
07	1.050000	1.595699	1.634616	383.304600	265.898700	267.792700	1.561236	264.473400
14	1.050000	1.474157	1.481616	383.304600	262.234000	262.333800	1.468219	262.168900
21	1.050000	1.450477	1.452088	383.304600	262.045900	262.060100	1.449175	262.051500
28	1.050000	1.445212	1.445577	383.304600	262.045600	262.045800	1.444916	262.045400
35	1.050000	1.444012	1.444095	383.304600	262.045100	262.045100	1.443944	262.045100
42	1.050000	1.443738	1.443755	383.304600	262.045000	262.045000	1.443720	262.045000
49	1.050000	1.443670	1.443678	383.304600	262.045000	262.045000	1.443674	262.045000

Table 7.2: Search for optimum service rate μ^* under single vacation policy.

iter	μ^l	μ^m	μ^u	$F(\mu^l)$	$F(\mu^m)$	$F(\mu^u)$	μ^q	$F(\mu^q)$
01	1.050000	2.750000	3.500000	382.489400	363.101400	440.196600	2.022338	294.000500
07	1.050000	1.600834	1.640239	382.489400	264.464800	266.371900	1.566160	263.037800
14	1.050000	1.478071	1.485637	382.489400	260.777800	260.879400	1.472040	260.711500
21	1.050000	1.453966	1.455613	382.489400	260.594700	260.600100	1.452634	260.591200
28	1.050000	1.448941	1.448941	382.489400	260.585000	260.585300	1.448260	260.584800
35	1.050000	1.447324	1.447410	382.489400	260.584500	260.585300	1.447253	260.584500
42	1.050000	1.447037	1.447056	382.489400	260.584500	260.584500	1.447022	260.584500
49	1.050000	1.446982	1.446983	382.489400	260.584500	260.584500	1.446983	260.584500

cost function for the chosen set of model parameters. By adopting QFSM and choosing the initial 3-point pattern as $(\mu^l, \mu^m, \mu^u) = (1.05, 2.75, 3.5)$, in multiple vacation, and $(\mu^l, \mu^m, \mu^u) = (1.05, 2.75, 3.5)$, in single vacation, and after finite iterations, we observe that the minimum expected operating cost per unit time converges to the solution $F = 262.045100$ at $\mu^* = 1.443674$, under multiple vacation and converges to $F = 260.584500$ at $\mu^* = 1.446983$, under single vacation.

Further, from Tables 7.1-7.2, and Figures 7.3-7.4, we observe that the optimum service rate μ^* of multiple vacation model is smaller than that of single vacation model, while the minimum expected cost $F(\mu^*)$ of multiple vacation model is bigger than that of single vacation model.

Table 7.3: The optimal values μ^* and $F(\mu^*)$ for different values of λ.

	MVP		SVP	
λ	μ^*	$F(\mu^*)$	μ^*	$F(\mu^*)$
0.70	1.080571	210.4515	1.084180	209.6032
0.80	1.203464	228.0262	1.206953	226.9668
0.90	1.324392	245.2030	1.327779	243.9394
1.00	1.443674	262.0450	1.446983	260.5845
1.10	1.561503	278.5999	1.564713	276.9498

Using QFS technique, the optimal values of μ and the minimum expected cost $F(\mu^*)$ are shown in Tables 7.3, 7.4 and 7.5 for various values of λ, ϕ and η, respectively. We observe from Table 7.3 that for both single and multiple vacations, as the arrival rate λ increases, both the optimal service rate and the minimum expected cost increase,

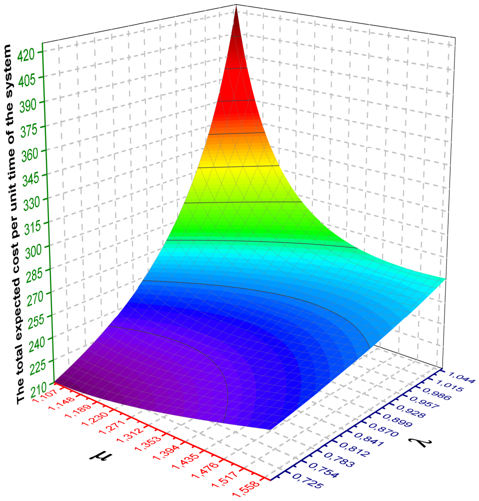

Figure 7.5: $\mathcal{T}_c$ versus λ and μ in MVP.

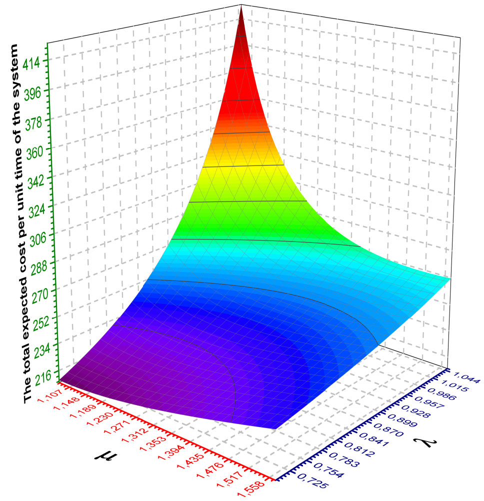

Figure 7.6: $\mathcal{T}_c$ versus λ and μ in SVP.

Table 7.4: The optimal values μ^* and $F(\mu^*)$ for different values of ϕ.

	MVP		SVP	
ϕ	μ^*	$F(\mu^*)$	μ^*	$F(\mu^*)$
0.80	1.425489	279.9405	1.427247	277.9489
1.20	1.434895	270.6155	1.437281	268.6739
1.60	1.439685	265.8825	1.442519	264.1108
2.00	1.442593	263.0613	1.445755	261.4950
2.40	1.444537	261.2039	1.447950	259.8484

Table 7.5: The optimal values μ^* and $F(\mu^*)$ for different values of η.

	MVP		SVP	
η	μ^*	$F(\mu^*)$	μ^*	$F(\mu^*)$
1.00	1.446642	260.7151	1.451625	258.7063
1.50	1.445304	261.2975	1.449833	259.4021
2.00	1.444523	261.6465	1.448581	259.9053
2.50	1.444013	261.8790	1.447658	260.2862
3.00	1.443674	262.0450	1.446949	260.5845

the increase in the optimal service rate with λ is as expected in view of the stability of the system. Moreover, it is quite clear from Figures 7.5 and 7.6 that for both MVP and SVP, the total expected cost increases with λ and μ, as intuitively expected. Then, from Table 7.4, we observe that for both single and multiple vacation policies, the optimal service rate increases with ϕ, while the minimum expected cost decreases as ϕ increases. On the other hand, Figures 7.7 and 7.8 show that for both MVP and SVP, the total expected cost decreases with ϕ, which agrees with our intuition, while it is not monotone with the parameter μ; it first decreases if the service rate μ is less than some threshold parameter, then it increases when μ is above this threshold value. Further,

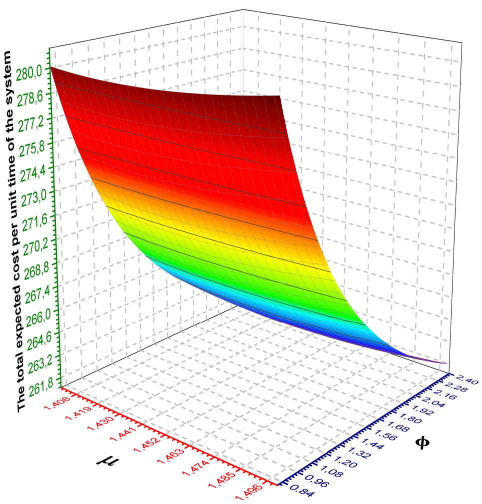

Figure 7.7: $\mathcal{T}_c$ versus ϕ and μ in MVP.

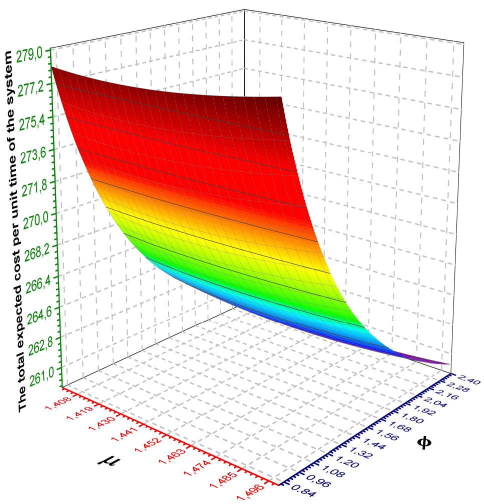

Figure 7.8: $\mathcal{T}_c$ versus ϕ and μ in SVP.

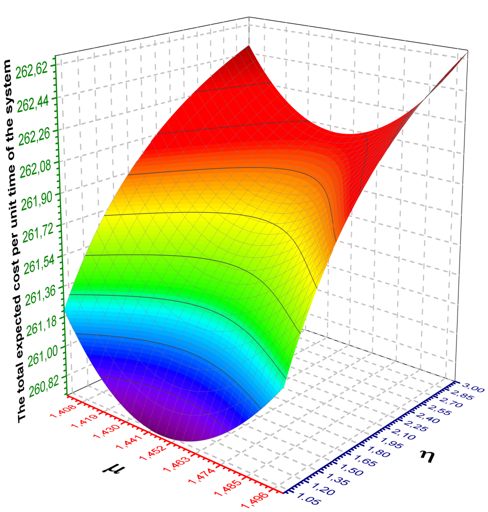

Figure 7.9: $\mathcal{T}_c$ versus η and μ in MVP.

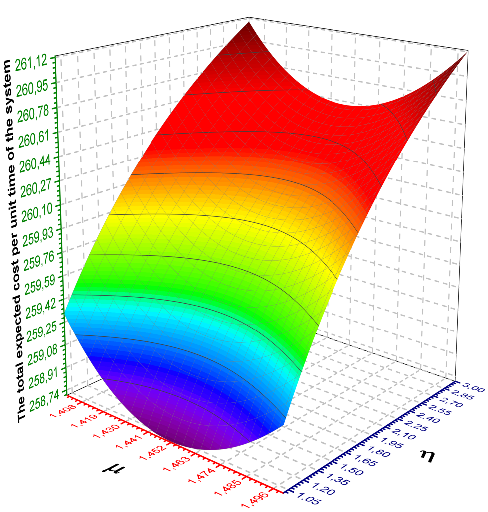

Figure 7.10: $\mathcal{T}_c$ versus η and μ in SVP.

from Table 7.5, it is clearly seen that the optimal service rate decreases with η, whereas, the minimum expected cost increases as η increases, this is quite obvious. Moreover, Figures 7.9 and 7.10 point out that for both MVP and SVP, the total expected cost increases with η, whereas it is not monotone with μ; it first decreases when the service rate μ is below a certain threshold value, then it increases when μ is greater than this threshold value. The non-monotonicity of the total expected cost with μ, displayed in

Figures 7.7- 7.10, can be due to the choice of the system parameters.

7.6.2 Performance and cost-profit analysis

In this subsection, we perform a sensitivity analysis to understand how different performance measures, total expected cost, total expected revenue, and total expected profit vary with different system parameters.

Impact of arrival rate (λ) **and batch size** (p).

Let the values for default parameters be fixed as $c = 2$, $\beta = 0.90$, $\eta = 3.00$, $\phi = 1.50$, $\alpha = 0.60$, $\xi = 3.50$, and $\mu = 1.50$.

Table 7.6: Impact of λ and p.

	λ	p	$E[L]$	N_s	P_e	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_p$
With multiple vacation	0.70	0.65	0.97286	0.97058	0.06191	152.77216	291.17513	138.40297
		0.75	0.74095	0.86469	0.06319	148.24494	259.40575	111.16081
		0.85	0.60110	0.78567	0.06357	145.37701	235.70059	90.323580
	0.80	0.65	1.21954	1.12674	0.06329	157.15329	338.02346	180.87017
		0.75	0.90281	1.00127	0.06577	151.26648	300.38062	149.11415
		0.85	0.71935	0.90839	0.06696	147.72496	272.51708	124.79212
	0.90	0.65	1.52497	1.28564	0.06300	162.35242	385.69260	223.34018
		0.75	1.09166	1.13953	0.06696	154.56575	341.85851	187.29276
		0.85	0.85197	1.03224	0.06913	150.15713	309.67332	159.51619
With single vacation	0.70	0.65	1.06753	0.74619	0.23936	150.50700	223.85729	73.350290
		0.75	0.81422	0.61197	0.25509	145.74809	183.59043	37.842330
		0.85	0.66234	0.51080	0.26679	142.81731	153.23999	10.422670
	0.80	0.65	1.32890	0.96625	0.21694	155.21125	289.87474	134.66349
		0.75	0.98592	0.81299	0.23539	148.96740	243.89841	94.931010
		0.85	0.78777	0.69823	0.24901	145.30197	209.46816	64.166190
	0.90	0.65	1.64967	1.17791	0.19455	160.79661	353.37170	192.57509
		0.75	1.18476	1.00559	0.21584	152.49444	301.67655	149.18211
		0.85	0.92746	0.87738	0.23146	147.88262	263.21447	115.33185

From Table 7.6, we observe that for both single and multiple vacation policies, for fixed p, with the increases of λ, the mean system size $E[L]$ increases, which results in the increasing of the mean number of customers served N_s. Further, along the increasing of λ, the probability that the servers are idle during busy period P_e decreases in the model with SVP, while it is not monotone in the model with MVP; it increases, then decreases, when $p = 0.65$, and increases in the case where $p = 0.75, 0.85$. This is due to the choice of the system parameters. In addition, $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_p$ all increase with λ. This is quite reasonable, the bigger the arrival rate, the larger the number of customers served and the greater the total expected cost, the total expected revenue and the total expected profit.

On the other hand, for both policies, for fixed λ, with the increasing of p, the probabil-

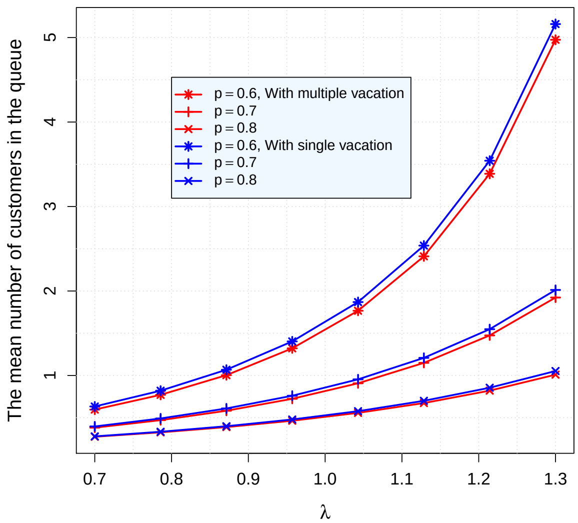

Figure 7.11: Impact of λ on $E[L_q]$ in MVP and SVP.

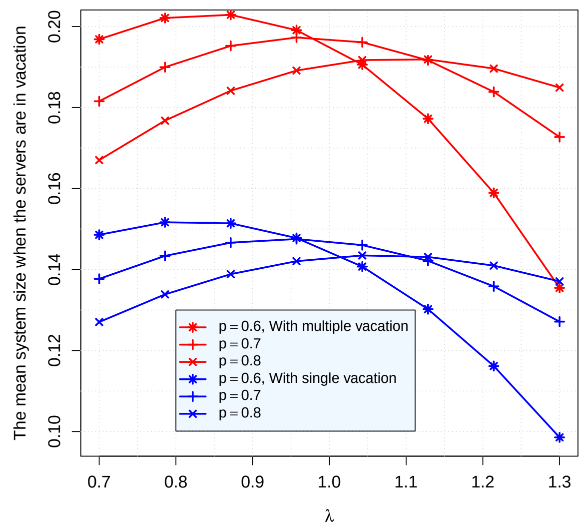

Figure 7.12: Impact of λ on $E[L_0]$ in MVP and SVP.

ity that the servers are idle P_e increases, while $E[L]$ and N_s decrease with the parameter p, this leads to a decrease in $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_p$, as intuitively expected.

Figures 7.11-7.12 show the effect of the arrival rate λ on the expected number of customers in the queue $E[L_q]$ and on the size of the system when the servers are on vacation $E[L_0]$, for different values of batch size p, under multiple and single vacation policies. It can be observed that for fixed p, with the increase of λ, $E[L_q]$ increases monotonically as it should be. While $E[L_0]$ first increases, then decreases in the case where $\lambda > 0.80$ & $p = 0.60$, $\lambda > 1.00$ & $p = 0.70$, and $\lambda > 1.10$ & $p = 0.80$. Obviously, $E[L_q]$ increases with $1/p$, while $E[L_0]$ decreases with the parameter $1/p$, which is coherent with the fact that increasing the arrival rates increase the queue length during the busy period and decreases the system size when the servers are in vacation.

Further, one may also observe that for higher values of p, $E[L_q]$ of multiple vacation model is smaller than that of single vacation model, while $E[L_0]$ of multiple vacation model is higher than that of single vacation model. This is due to the fact that in single vacation policy, whenever the busy period ended, the servers switch to the busy period and stay there until the first arriving customer enters the system, consequently the queue length $E[L_q]$ increases and $E[L_0]$ decreases. Contrariwise, in multiple vacation policy, once the vacation period is finished, the servers switch to the busy period, if at that moment no customer is observed in the queue, they immediately comeback to the vacation period, which results in the increasing of the size of the system during this period $E[L_0]$.

Impact of waiting rate of the severs (η) and vacation rate (ϕ).

In this subpart, we fixed the parameters as $c = 2$, $p = 0.70$, $\lambda = 0.90$, $\beta = 0.80$, $\eta = 3.00$, $\phi = 0.50$, $\alpha = 0.60$, $\xi = 3.20$, and $\mu = 2.20$.

Table 7.7: System performance measures vs. η.

	η	P_v	P_b	$E[L_0]$	$E[L_1]$	R_a	R_e	N_s
With multiple vacation	1.00	0.430930	0.488966	0.254386	2.334917	0.488421	0.325614	1.093950
	1.50	0.470251	0.471474	0.277598	2.217380	0.532988	0.355325	1.065748
	2.00	0.492731	0.461473	0.290868	2.150184	0.558467	0.372311	1.049624
	2.50	0.507281	0.455000	0.299458	2.106692	0.574959	0.383306	1.039188
	3.00	0.517468	0.450468	0.305471	2.076241	0.586505	0.391003	1.031882
With single vacation	1.00	0.302642	0.546037	0.178655	2.718388	0.343018	0.228679	1.185962
	1.50	0.359414	0.520781	0.212169	2.548688	0.407364	0.271576	1.145243
	2.00	0.396614	0.504232	0.234129	2.437491	0.449527	0.299685	1.118562
	2.50	0.422875	0.492550	0.249631	2.358992	0.479292	0.319528	1.099727
	3.00	0.442404	0.483862	0.261159	2.300618	0.501426	0.334284	1.085720

Table 7.8: System performance measures vs. ϕ.

	ϕ	P_v	P_b	$E[L_0]$	$E[L_1]$	R_a	R_e	N_s
With multiple vacation	0.50	0.517468	0.450468	0.305471	2.076241	0.586505	0.391003	1.031882
	0.80	0.428737	0.533169	0.225177	2.454730	0.432340	0.288227	1.221750
	1.10	0.376571	0.581728	0.178132	2.675744	0.342014	0.228009	1.333429
	1.40	0.342205	0.613676	0.147248	2.820324	0.282717	0.188478	1.407038
	1.70	0.317842	0.636296	0.125431	2.922099	0.240828	0.160552	1.459247
With single vacation	0.50	0.442404	0.483862	0.261159	2.300618	0.501426	0.334284	1.085720
	0.80	0.346108	0.561596	0.181779	2.679892	0.349017	0.232678	1.256686
	1.10	0.289268	0.604667	0.136835	2.895358	0.262722	0.175148	1.349564
	1.40	0.251263	0.631480	0.108117	3.033645	0.207584	0.138389	1.405972
	1.70	0.223768	0.649430	0.088306	3.129558	0.169548	0.113032	1.442606

Table 7.9: $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_c$ vs. η.

	MVP			SVP		
η	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_p$	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_p$
1.00	171.0821	328.1849	157.1029	172.1893	355.7885	183.5992
1.50	170.7427	319.7243	148.9816	171.6993	343.5729	171.8736
2.00	170.5486	314.8873	144.3386	171.3782	335.5686	164.1903
2.50	170.4231	311.7565	141.3335	171.1516	329.9180	158.7664
3.00	170.3351	309.5646	139.2294	170.9830	325.7160	154.7330

The impact of waiting rate of the servers η and vacation rate ϕ in single and multiple vacations are shown in Tables 7.7-7.10. It is clearly seen that for both multiple and single vacation policies, P_v, $E[L_0]$, R_a and R_e all increase with η and decrease with ϕ. While P_b, $E[L_1]$, and N_s decrease with η and increase with ϕ. Therefore, for both policies, $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_p$ decrease with η and increase with ϕ. These results are consistent

Table 7.10: $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_c$ vs. ϕ.

	MVP			SVP		
ϕ	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_p$	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_p$
0.50	170.3351	309.5646	139.2294	170.9830	325.7160	154.7330
0.80	171.3911	366.5250	195.1338	172.0484	377.0059	204.9575
1.10	171.9970	400.0286	228.0316	172.6472	404.8691	232.2219
1.40	172.3860	422.1114	249.7253	173.0279	421.7915	248.7635
1.70	172.6546	437.7740	265.1194	173.2900	432.7819	259.4919

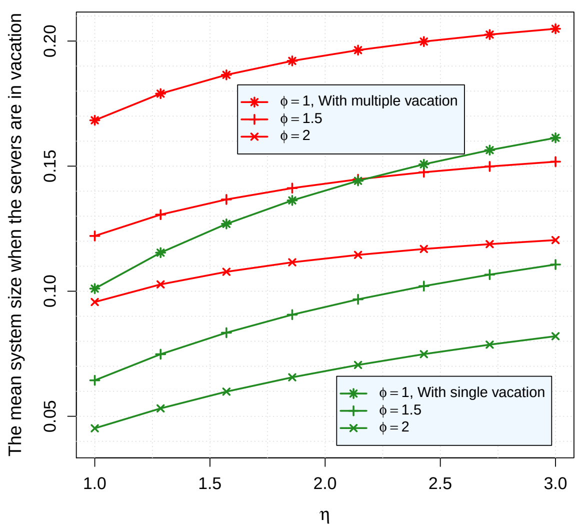

Figure 7.13: Impact of η on $E[L_0]$ in MVP and SVP.

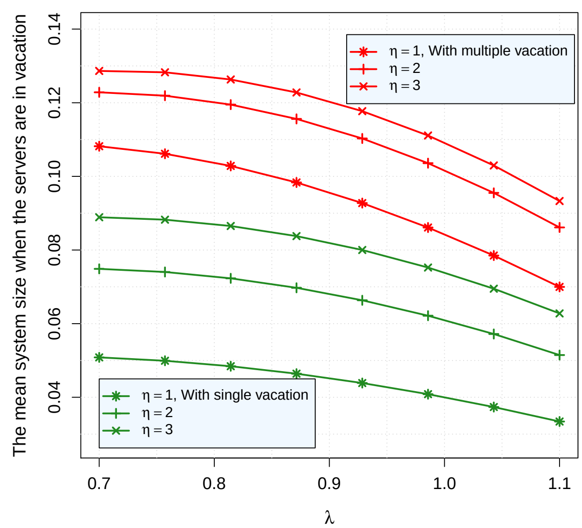

Figure 7.14: Impact of λ on $E[L_0]$ in MVP and SVP.

with our intuition; the probability of busy period increases with ϕ (resp. decreases with η), thus the mean number of customers served increases with ϕ (resp. decreases with η), therefore, the total expected profit increases with ϕ (resp. decreases with η). On the other hand, the probability of vacation period decreases with the parameter ϕ (resp. increases with the parameter η). Consequently, the average rate of reneging decreases with ϕ (resp. increases with η). Consequently, the total expected profit increases with increasing values of ϕ and decreases along the increasing of η.

From Figures 7.13-7.14 we see that for both single and multiple vacations, $E[L_0]$ increases with η and decreases with λ and ϕ, as it should be expected. Then, evidently for lower values of ϕ, $E[L_0]$ of multiple vacation model is higher than that of single vacation model. On the other hand, for higher values of η, $E[L_0]$ of multiple vacation model is greater than that of single vacation model. Consequently, we can conclude that the model with waiting servers outperforms the model without this policy.

Impact of impatience rate (ξ) and non-retention probability (α).

In this subpart, we choose the default parameters as $c = 2$, $p = 0.70$, $\lambda = 0.90$, $\beta = 0.80$, $\eta = 2.00$, $\phi = 1.50$, $\mu = 2.20$.

Table 7.11: Impact of ξ and α.

	ξ	α	$E[L]$	N_s	R_a	R_e	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_p$
With multiple vacation	0.20	0.25	1.57260	1.56348	0.01940	0.05819	209.2591	469.0433	259.7843
		0.50	1.53353	1.54527	0.03799	0.03799	208.3662	463.5815	255.2153
		0.75	1.49770	1.52764	0.05584	0.01861	207.5425	458.2923	250.7498
	0.80	0.25	1.46464	1.51054	0.07299	0.21897	211.1577	453.1625	242.0048
		0.50	1.35354	1.44673	0.13562	0.13562	208.2473	434.0202	225.7729
		0.75	1.26600	1.38898	0.19040	0.06347	205.9043	416.6931	210.7888
	1.40	0.25	1.37866	1.46206	0.12078	0.36233	213.0528	438.6178	225.5650
		0.50	1.22834	1.36198	0.21542	0.21542	208.5768	408.5930	200.0162
		0.75	1.11845	1.27563	0.29312	0.09771	205.2185	382.6889	177.4704
With single vacation	0.20	0.25	1.48691	1.20539	0.01537	0.04610	206.2113	361.6174	155.4061
		0.50	1.45496	1.18637	0.02999	0.02999	205.4669	355.9117	150.4448
		0.75	1.42577	1.16795	0.04393	0.01464	204.7821	350.3842	145.6021
	0.80	0.25	1.39896	1.15007	0.05723	0.17169	207.5825	345.0215	137.4390
		0.50	1.30984	1.08337	0.10493	0.10493	205.1588	325.0103	119.8515
		0.75	1.24105	1.02314	0.14540	0.04847	203.2324	306.9424	103.7099
	1.40	0.25	1.32984	1.09938	0.09376	0.28127	208.9241	329.8147	120.8906
		0.50	1.21196	0.99511	0.16346	0.16346	205.2052	298.5327	93.32747
		0.75	1.12925	0.90638	0.21768	0.07256	202.4852	271.9147	69.42946

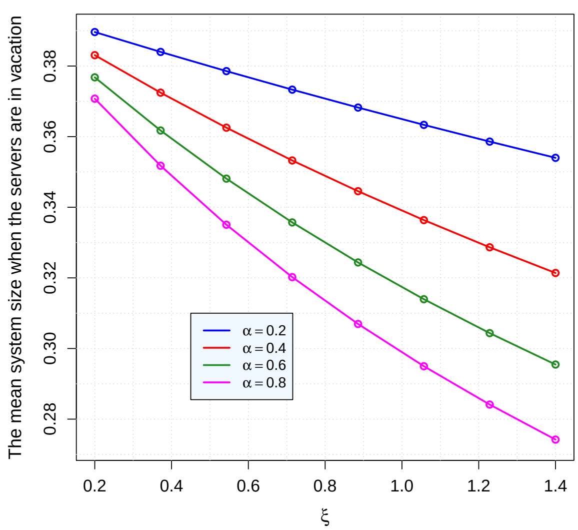

Figure 7.15: Impact of ξ on $E[L_0]$ in MVP.

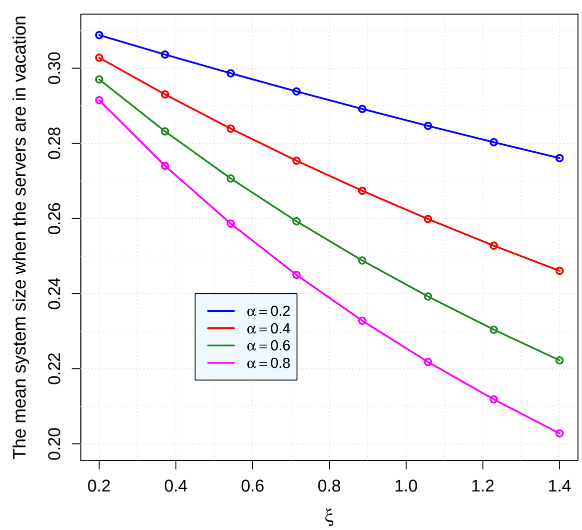

Figure 7.16: Impact of ξ on $E[L_0]$ in SVP.

Table 7.11 illustrates the impact of ξ and α, for both single and multiple vacation policies. As expected, for both MVP and SVP, increases in ξ and α implies a decrease in $E[L]$ and N_s. This is because the size of the system decreases with the increasing of ξ and α. Thus, the mean number of customers served decreases as the two parameters

ξ and α increase. Further, R_a increases with ξ and α, whereas, R_e increases with ξ and decreases with α, as it should be. Therefore, $\mathcal{T}_c$, $\mathcal{T}_r$, and $\mathcal{T}_p$ monotonically decrease with α, $\mathcal{T}_c$ is not monotone with ξ, while $\mathcal{T}_r$ and $\mathcal{T}_p$ decrease significantly with the increasing values of ξ, this is because of the significant number of lost customers. From this, it is clearly obvious that the retention probability has a positive impact on the economy of the system, this probability is very useful for any firm operating in the field of finance, supply chain, manufacturing, and so on.

Figures 7.15-7.16 depict the effect of ξ for different values of α in both single and multiple vacation policies. From the figures, it can be seen that as the impatience rate ξ increases, the mean system size when the servers are on vacation period $E[L_0]$ monotonically decreases for any α, as intuitively expected. Moreover, from both figures, we observe that $E[L_0]$ is high when the non-retention probability α is small. Further, as it should be expected, $E[L_0]$ of multiple vacation model is greater than that of single vacation model.

Impact of non-feedback probability (β) **and number of the severs** (c).

In this part, we take $p = 0.70$, $\lambda = 0.90$, $\eta = 2.00$, $\phi = 1.50$, $\alpha = 0.60$, $\xi = 1.00$, and $\mu = 2.20$.

Table 7.12: Impact of c and β.

		MVP			SVP		
c	β	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_p$	$\mathcal{T}_c$	$\mathcal{T}_r$	$\mathcal{T}_p$
2	0.70	214.1259	429.2409	215.1150	211.5811	350.1083	138.5272
	0.80	207.1737	416.6931	209.5195	204.2018	306.9425	102.7406
	0.90	201.1740	400.7218	199.5478	197.8585	258.1941	60.33554
3	0.70	281.6618	837.3160	555.6542	293.5541	810.1472	516.5932
	0.80	288.7010	838.0231	549.3221	285.9941	797.1694	511.1753
	0.90	295.9462	830.8510	534.9048	278.6898	773.2725	494.5827

From Table 7.12 and Figures 7.17-7.18, we see that for both single and multiple vacations, N_s increases with μ, c, and β, respectively. Further, for both MVP and SVP, for fixed β, the total expected cost, the total expected revenue, and the total expected profit increase significantly with the increasing of c. This is quite reasonable, the greater the number of servers in the system, the larger the number of customers served and the higher the total expected profit. In addition, in both MVP and SVP, for fixed c, the total expected revenue and the total expected profit decrease when β increases. While in the model with SVP, $\mathcal{T}_c$ decreases with the parameter β, and in the model with MVP, it decreases with β, when $c = 2$, and increases along the increasing of β, when $c = 3$.

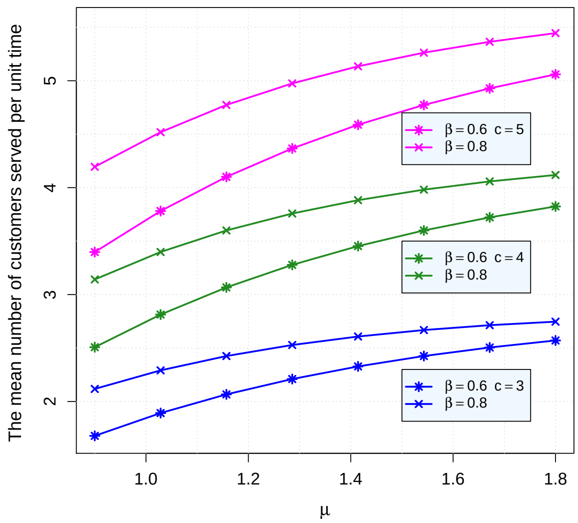

Figure 7.17: Impact of μ on N_s in MVP.

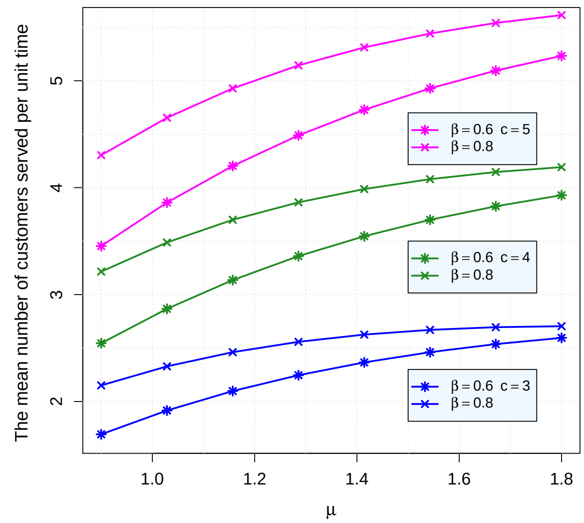

Figure 7.18: Impact of μ on N_s in SVP.

Thus, we can say that a feedback probability has a nice effect on the economy of the system. Moreover, as intuitively expected, N_s of single vacation model is higher than that of multiple vacation model.

7.7 Conclusion and future scope

In this paper, we carried out a study of a infinite-buffer multi-server Bernoulli feedback queueing system with batch arrivals, waiting servers, impatient customers and retention of reneged customer, under single and multiple vacation policies. We obtained the closed-form expressions for the steady-state probabilities of the queueing model, using the probability generating function (PGF). Various performance measures of the system are evaluated. We also performed a cost model and considered a cost optimization problem using quadratic fit search method (QFSM) in order to obtain the optimum values of the service rate for different values of arrival rate, waiting rate of the servers and vacation rate. Important numerical results have been illustrated, which may be useful to explore the impact of system parameters on different performance measures and total expected cost, total expected revenue and total expected profit, respectively. The obtained results have potential applications in modeling computer and telecommunication systems, computer networks, manufacturing, and so on. For further works, it will be interesting to apply the technique used in this paper in order to study more complex models such as $Geo^X/Geo/c$ and $M^X/M/c$ with breakdowns, impatient customers and asynchronous multiple and single vacations. Furthermore, the model under investigation can be analyzed under the provision of time dependent arrival and service rates which leads the system to more realistic environment.

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Chapter 8

Conclusions and future work

In this thesis, we examined different queueing systems with impatient customers. Although extensive research has been done on this topic in the last decades, we believe that combining different features including reneging during busy and vacation periods, waiting server, feedback and retention of reneged customers together for diverse systems with vacation and working vacations, then studying their impact on the performance measures as well as the cost-profit of the systems has not been addressed, yet.

In what follows, we first reiterate the main conclusions of this thesis, then we propose some possible extensions that can contribute to the literature on the queueing systems with impatient customers.

8.1 Results

In Chapter 1, a comprehensive review of the recent literature on impatience in queueing systems, feedback queues, vacation and working vacation queues was provided. In this chapter, we represented the exposition of our research within the body of existing literature.

In Chapter 2, we studied an heterogeneous two-server queueing system with Bernoulli feedback, multiple working vacations, balking, reneging and retention of reneged customers, we assumed that impatience timers of customers in the system depend on the state of the server. In this work, we extended the problem in Laxmi and Jyothsna (2015) by considering Bernoulli feedback, reneging in either normal busy and working vacation periods, wherein the reneged customers may be retained via ceratin mechanism. We developed the equations of the steady state probabilities via the supplementary variable and recursive techniques, then we derived useful performance measures of the queueing system. Further, we formulated a cost model in order to determine the impact of various system parameters on the different characteristics as well as on total expected cost, total expected revenue, and total expected profit of the system. In this study, we showed via a numerical analysis the impact of different system parameters on both characteristics and costs of the considered queueing system.

In Chapter 3, we considered an infinite-buffer single server queueing system with Bernoulli feedback, multiple vacations, differentiated vacations, vacation interruptions, balking and reneging such that reneged customers may be retained in the system. Via the recursive method, we obtained the exact expressions of the steady-state probabilities. Then, we presented explicit expressions of important performance measures, and developed a cost model. In addition, a variety of numerical results has been discussed.

In Chapter 4, we dealt with a $M/M/1$ Bernoulli feedback queueing system with single exponential vacation, waiting server, reneging and retention of reneged customers, wherein the impatience timers of customers depend on the states of the server. We obtained the explicit expressions of the steady-state probabilities using probability generating functions (PGFs). In addition, we derived important measures of effectiveness of the queueing system and formulated a cost model. Further, via an extensive numerical study, we showed the impact of different system parameters on the performance measures and cost-profit of the queueing model. The considered queueing system can be considered as a generalized version of the queueing models which exist in the literature presented in Yue et al. (2016) and Ammar (2017) associated with many practical situations.

In Chapter 5, we analyzed an infinite capacity batch arrival single server Markovian Bernoulli feedback queueing system subject to functioning K-variant vacation policy with waiting server, impatient customers and retention of reneged customers. We established the steady-state study of the queueing system using the PGFs method and evaluated diverse system metrics in terms of steady-state probabilities. In addition, we considered a cost optimization problem using particle swarm optimization (PSO) and quadratic fit search method (QFSM). Using a comprehensive numerical experiment we showed that the two algorithms performed quite well. Further, we investigated the effect of different parameters on the performance measures and the cost functions of the system through numerical experiments.

Chapter 6 explored the impatience behavior in multi-server Bernoulli feedback queueing system with batch arrival, variant of multiple working vacations, and retention of the reneged customers. We investigated diverse system characteristics in terms of steady state probabilities via the probability generating function method. We carried out an economic analysis and showed the effect of the different system parameters on the system characteristics as well as on total expected cost, total expected revenue, and total expected profit of the system.

In Chapter 7, we treated a infinite-buffer multi-server Bernoulli feedback queueing model with batch arrivals, waiting servers, impatient customers and retention of reneged customer, under single and multiple vacations. We derived the closed-form expressions for the steady-state probabilities of the queueing system using the probability generating function (PGF). Then, we evaluated useful performance measures of the system. We also performed a cost model and considered a cost optimization problem using quadratic fit search method (QFSM). We aimed to find the optimum values of the service rate for different values of arrival rate, waiting rate of the servers and vacation rate such that the average total cost is minimized and proved the convexity of the cost function in each decision variable. In addition, we explored the impact of system parameters on different performance measures and total expected cost, total expected revenue and total expected profit, respectively.

– The obtained results in this thesis have potential applications in many real world systems including computer and telecommunication systems, computer networks, manufacturing, call centers and production-inventory systems and so on.

8.2 Further research

The framework developed in this thesis allows us to examine the impact of different system parameters including service rates during busy and working vacation periods, vacation rate, waiting rate of server, reneging, retention of reneged customers, Bernoulli feedback, and others in enhancing performance (i.e, operational costs reduction) and profitability of the queueing systems in different settings. The models developed in this thesis provide lucrative perspicacity to the production managers and system engineers. Our analyses can be used as a building block to compose more complex queueing models. For instance, providing a rigorous proof for transient solution analysis for multiserver queueing systems with asynchronous multiple and single vacation policies, vacation interruption and impatient customers as an interesting extension to what we have carried out for the queueing models given in Chapters 4-7. Further, to make the system modelling more closer to the real world problems, these models can be further extended to a more general cases with general type service times and lead times.

Another interesting case for future research can be the realistic feature of synchronous and asynchronous breakdowns and repairs for the multiserver queueing systems given in this thesis.

In such models, analysing the effect of the system parameters on the system characteristics and profitability will help the system manager to enhance the cost efficiency and responsiveness of the systems. It should be noted that each of the above mentioned extensions adds sufficient complexity and makes the problem less tractable.

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ملخص. نقوم في هذه الأطروحة بدراسة سلوك نفاذ صبر الزبائن في مختلف أنظمة طوابير الانتظار لعدة أسباب من بينها حالة الخادم، نوعية الخدمة، طول وقت الانتظار، إلخ. أولا تحصلنا على احتمالات في حالة الاستقرار لنظام طابور الانتظار $M/M/2/N$ مع خادمين غير متجانسين، تغذية راجعة، إجازة الخوادم، تقديم الخدمة خلال فترة إجازة، العزوف، تنصل الزبائن المتعلق بحالة الخادم و الاحتفاظ بالزبائن المتنصلين، باستخدام طريقة المتغير الإضافي و الطريقة التراجعية. ثانيا استخدمنا طريقة تراجعية من أجل الحصول على الحل لنظام طابور الانتظار ذو قدرة استيعاب غير منتهية مع اختلاف الإجازات، إعاقة الإجازات، العزوف، التنصل والاحتفاظ بالزبائن المتنصلين خلال فترة الاشتغال. ثالثا، نحلل سلوك نفاذ صبر الزبائن (التنصل)، مدى تأثير الاحتفاظ و التغذية الراجعة في أنظمة طوابير الانتظار $M/M/1$ مع إجازة وحيدة، $M^X/M/1$ مع K إجازة متتالية، $M^X/M/c$ مع K إجازة متتالية و تقديم الخدمة خلال إجازات، $M^X/M/c$ مع إجازة وحيدة و مع إجازة متعددة، حيث أن الخوادم في النظام الأول و الثاني و الرابع غير مسموح أن يأخذوا إجازات عند شغور النظام إلا بعد فترة انتظار عشوائية. نقوم بدراسة و تحليل حالة المراوحة من أجل هذه الأنظمة باستخدام وظيفة توليد الاحتمال. إضافة إلى ذلك، نقوم باشتقاق مختلف مقاييس الأداء و تقديم التحليل الاقتصادي لمختلف النماذج المدروسة في هذه الأطروحة. إضافة إلى ذلك، نقوم بدراسة تحسينية لنظام طابور الانتظار الرابع والسادس باستخدام طريقتي PSO و QFSM.

Abstract. In this thesis, we analyze the impatient behaviour in different queueing systems due to different factors including server state, quality of service, waiting time, etc. Firstly, we obtain the steady-state probabilities for an $M/M/2/N$ queueing system with two heterogeneous servers, feedback, vacation, working vacation, balking, reneging which depends on server state and retention of reneged customers, using supplementary variable and recursive techniques. Secondly, we use the recursive method to establish the solution of an infinite capacity queueing system with differentiated vacations, vacation interruption, balking, reneging during the busy period and retention of reneged customers. Thirdly, we analyze the impatient behavior (reneging), the impact of retention of reneged customers and feedback in an $M/M/1$ queueing system with single vacation and waiting server, $M^X/M/1$ with waiting server and K-variant vacations, $M^X/M/c$ with K-variant working vacations as well as $M^X/M/c$ with waiting servers and both single multiple vacation policies. We establish the stationary analysis for these queueing systems using the probability generating function. In addition, we derive useful performance measures and present the economic analysis of the different models presented in this thesis. In addition, we study the optimization of the fourth and sixth queueing systems using the PSO and QFSM methods.

Résumé. Dans cette thèse nous analysons le comportement d'impatience dans différents systèmes de files d'attente, due aux différents facteurs notamment l'état de serveur, qualité de service, le temps d'attente, etc. Dans un premier lieu, nous obtenons les probabilités d'état stable pour un système de file d'attente $M/M/2/N$ avec deux serveurs hétérogènes, feedback, vacances, service pendant les vacances, dérobade, abandon qui dépendent de l'état du serveur et rétention des clients abandonnés, en utilisant la méthode de variable supplémentaire et la récursivité. En second lieu, nous utilisons la méthode récursive afin d'établir la solution d'un système de file d'attente de capacité infinie avec des vacances différentiées, interruption de vacances, dérobade, abandon pendant la période d'occupation et rétention des clients abandonnés. En troisième lieu, nous analysons le comportement d'impatience (abandon), l'impact des rétentions et du feedback dans les systèmes de files d'attente $M/M/1$ avec vacance unique, $M^X/M/1$ avec K vacances consécutives, $M^X/M/c$ avec K vacances consécutives et services pendant les vacances, $M^X/M/c$ avec vacances uniques et multiples, où les serveurs dans le premier, deuxième et quatrième systèmes sont autorisés à prendre des vacances chaque fois que le système est vide après une période d'attente aléatoire . Nous établissons l'analyse stationnaire pour ces systèmes de files d'attente en utilisant la fonction génératrice des probabilités. En outre, Nous dérivons importantes mesures de performance et présentons l'analyse économique des différents modèles présentés dans cette thèse. En outre, nous étudions dans cette thèse l'optimisation du quatrième, et sixième système de files d'attente en utilisant les méthodes PSO et QFSM.