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Fractional Stochastic Differential Equations

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Dedication

*All praise to **Allah**, today we fold the day's tiredness and the errand summing up between the cover of this humble work.*

I dedicate my work to:

*My great teacher and messenger, **Mohammed-peace and grace from Allah be upon him***, who taught us the purpose of life.

*My parents **Boualem Bendjebara** and **Khaoula Fesraoui**, for their generous support they provided me throughout my entire life and particularly through the process of pursuing the master degree because of their unconditional love and prayers, so I have the chance to complete this thesis. God save them.*

My grandmother. God rest her soul.

*A gift to the heart, my aunt **Radia**.*

*Whose love flows in my veins, and my heart always remembers them, to my brother **Mohamed** and my cousin **Kenza**.*

*Whom I have truly tasted the taste of life, my close freinds **Imene Slimani**, **Ghania Benalioua** and **Amel Debbas** and their famillies.*

Last but not least I am dedicating this to all the people in my life who touch my heart.

All those if my pen forget them, my heart will not forgotten them.

All those who are looking glory and pride in Islam and nothing else .

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Contents

Abstract

This master thesis focused on the study of the existence and the uniqueness of solution for a class of fractional stochastic differential equation.
First, we brought the reader through the fundamental notions of stochastic processes and stochastic integration as well as stochastic differential equations. Then we gave a global view on the fractional calculus, after a short introduction and some preliminaries, we explored the Gr*ü*unwald-Letnikov, Riemann-Liouville and Caputo approaches for defining a fractional derivative. Then, we proved some basic properties of fractional derivatives, such as linearity, the Leibniz rule and composition. Thereafter, we applied the definitions of the fractional derivatives to a few examples. As application of fractional derivatives we gave a commonly used economic model.
This master thesis ends with investigating a global result on the existence and uniqueness of solutions for Caputo fractional stochastic differential equations of order $\alpha \in (1/2, 1)$ whose coefficients satisfy a standard Lipschitz condition, and using a temporally weighted norm.

Key words: Fractional stochastic differential equations. Existence and uniqueness of solutions. Temporally weighted norm. Fractional calculus. Caputo fractional derivative.

Résumé

Ce mémoire de master a pour but, de faire une étude sur l'existence et l'unicité de la solution d'une équation differentielle stochastique fractionnaire.

Tout d'abord, nous avons donné au lecteur les notions fondamentales des processus stochastiques, l'intégration stochastique et les équations differentielles stochastiques, qui nous seront utiles tout au long du présent mémoire. Nous avons introduit ensuite une vue globale sur le calcul fractionnaire, après une courte introduction et quelques notions préliminaires, nous nous sommes concentrés sur les approches de Gr*ü*unwald-Letnikov, Riemann-Liouville et Caputo pour la définition d'une dérivée fractionnaire. De plus, on a introduit et étudié quelques propriétés de base des dérivés fractionnaires, telles que la linéarité, la composition de ces opérateurs et la règle de Leibniz. D'autre part, ces définitions sont appliquées à quelques exemples et une application d'un modèle économique est éxaminée.

Ce mémoire de master se termine par un résultat global sur l'existence et l'unicité des solutions pour les équations différentielles stochastiques fractionnaires de Caputo d'ordre $\alpha \in (1/2, 1)$ dont les coefficients satisfont la condition standard de Lipschitz, en utilisant une norme temporelle pondérée.

Mots clés: Equations différentielles stochastiques fractionnaires. Existence et unicité des solutions. Norme temporelle pondérée. Calcul fractionnaire. Dérivée fractionnaire de Caputo.

Introduction

Our understanding of nature relies on calculus, which in turn relies on the intuitive concept of the derivative. It's descriptive power comes from the fact that it analyses the behavior at scales small enough that its properties change linearly, so avoiding complexities that arise at larger ones. Fractional calculus generalizes this concept from integer to non-integer order.

Fractional calculus is the branch of calculus that generalizes the derivative of a function to non-integer order, allowing calculations such as deriving a function to 1/2 order, the name "fractional" is used for denoting this kind of derivative. Most authors on this topic will cite a particular date as the birthday of so called "fractional calculus". In a letter dated September 30th, 1695. L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function $f(x) = x$. L'Hopital's posed the question to Leibniz, what would the result be if $n = 1/2$. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn". In these words fractional calculus was born.

Notable contributions work in the broad area of fractional calculus were done in a slow progress up to 1900. Leibniz, Euler(1738), Fourier(1822), Abel(1826), Liouville-Riemann (1832-1847), Grünwald-Letnikov(1867-1868), Hadamard(1892). After 1900, the fractional calculus attend a fast development and in an attempt to formulate particular problems, other definitions were proposed. We mention some of them, Weyl(1917), Marchaud(1927), Erdelyi-Kober(1940), Caputo(1967). For more in-depth information, applications and related topics it is possible to consult some of the classical references, Oldham-Spanier [12], Samko and al [19], Miller and Ross [14].

On the other hand, the increasing interest in applications of fractional calculus has motivated the development and the investigation of numerical methods specifically devised to solve fractional differential equations(FDEs). Fractional differential equations are now receiving an increasing attention due to their applications in a variety of disciplines such as mechanics, physics, chemistry, biology, electrical engineering, control theory, finance. For more details, we refer the interested reader to the monographs [8], and references therein.

In contrast to the huge number of publications in deterministic fractional differential equations, there have been only a few papers dealing with stochastic fractional differential equations involving with a Caputo fractional time derivative and most of these publications have attempted to establish a result on the existence and uniqueness of mild solutions. We refer the reader to [18, 1] for results on the existence and uniqueness of this type of solutions. Pedjeu and Ladde [16] studied the existence and uniqueness of some class of stochastic fractional differential equation. Kamrani [7] discussed the numerical solution of stochastic fractional differential equation.

The present master thesis aims on the one hand, to introduce the concept of fractional calculus, the branch of mathematics which explores fractional integrals and derivatives. And secondly, the study of the existence and the uniqueness of solution for a class of fractional stochastic differential equation.

This Master thesis, is divided into three chapters, it's organized as follows:
In **Chapter 1** we gather some preliminary results. In particular, we introduce some basic concepts concerning continuous time stochastic processes. First we recall what is a continuous time stochastic process, and when do two stochastic processes coincide then, different types of measurability, the notion of a martingale. After that we introduce Itô calculus. At last we give a brief overview on stochastic differential equations.

The **Chapter 2** focuses on the theory of fractional integrals and derivatives, we first give some basic functions, such as the Gamma function, the Beta function and the Mittag-

Leffler function, the definitions of the Grünwald-Letnikov, Riemann-Liouville and the Caputo integral and derivatives, then introduce some more approaches to define a differintegral also frequently used such as Hadamard and Hilfer. Then, we study many of its properties. Finally, we briefly discuss some examples and an application of fractional calculus and examine a commonly used economic model.

The core of this master thesis is the **Chapter 3** in this chapter we establish a result on the global existence and uniqueness of solutions for Caputo fractional stochastic differential equations of order $\alpha \in (1/2, 1)$. We first introduce briefly some notations and primely facts, after that we state the main result then show proof of the result on the global existence and uniqueness of solution using a temporally weighted norm.

Finally, we give a conclusion. In witch we summarize the main results of this work.

Chapter 1

Preliminary Background

In this chapter we give some basic notions on stochastic processes, after that we introduce Itô calculus. At last we give a brief overview on stochastic differential equations. There exists a vast literature that treats stochastic process. For more detail, we refer the reader to [2, 4, 5, 9, 13, 17, 20].

1.1 Basic definitions

In this section we introduce some basic concepts concerning continuous time stochastic processes used freely later on. Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and recall that a map $Z : \Omega \to \mathbb{R}$ is called a ***random variable*** if Z is measurable as a map from $(\Omega, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$.

What is a continuous time stochastic process? For us it is simply a family of random variables.

Definition 1.1.1. *Let $I = [0, T]$ for some $T \in (0, \infty)$ or $I = [0, \infty)$. A family of random variables $X = (X_t)_{t \in I}$ with $X_t : \Omega \longrightarrow \mathbb{R}$ is called* ***stochastic process*** *with index set I.*

The definition of a stochastic process can be given more generally by allowing more general I and other state spaces than $\mathbb{R}$. In our case there are two different views on the stochastic process X.

- The family $X = (X_t)_{t \in I}$ describes random functions by $\omega \mapsto f(\omega) = (X_t(\omega))_{t \in I}$.

The function $t \mapsto X_t(\omega)$ is called ***path*** or ***trajectory*** of X.

- The family $X = (X_t)_{t\in I}$ describes a process, which is, with respect to the time variable t, an ordered family of random variables $t \mapsto X_t$.

The two approaches differ by the roles of ω and t.

Definition 1.1.2. *Let $X = (X_t)_{t\in I}$ and $Y = (Y_t)_{t\in I}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. The processes X and Y are **versions** or **(modifications)** of each other provided that*

$$\mathbb{P}(X_t = Y_t) = 1, \ for \ all \ t \in I.$$

Definition 1.1.3. *Let $X = (X_t)_{t\in I}$ and $Y = (Y_t)_{t\in I}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. The processes X and Y are **indistinguishable** if and only if*

$$\mathbb{P}(X_t = Y_t, \ for \ all \ t \in I) = 1.$$

Definition 1.1.4. ***The finite-dimensional distributions*** *of the real valued stochastic process $X_t = \{X_t\}_{t\in I}$ are the measures $\mu_{t_1,\dots,t_k}$, defined on $\mathbb{R}^k$, such that*

$$\mu_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = \mathbb{P}(\{X_{t_1} \in A_1, \cdots, X_{t_k} \in A_k\}), \tag{1.1}$$

where $k \in \mathbb{N}$ and $\{t_i\}_{i=1,\dots,k} \subset I$ a time sequence, $\{A_1, \dots, A_k\}$ are Borel sets on $\mathbb{R}$.

Theorem 1.1.1. *Let $X = (X_t, t \in I)$ and $Y = (Y_t, t \in I)$ be two real valued processes. Then X and Y have the same distribution (or law) if and only if their finite-dimensional distributions agree.*

Proof: See, (Watkins, [20]).

Theorem 1.1.2. *([11]) For all $\{t_i\}_{i=1,\dots,k} \subset I$, $k \in \mathbb{N}$ let $\nu_{t_1,\dots,t_k}$ be probability measures on $\mathbb{R}^k$, such that*

1. *For all permutations π on $\{1, 2, \dots, k\}$,*

$$\nu_{t_{\pi(1)},\dots,t_{\pi(k)}}(A_1 \times ... \times A_k) = \nu_{t_1,\dots,t_k}(A_{\pi^{-1}(1)} \times \cdots \times A_{\pi^{-1}(k)}).$$

2. *For any $m \in \mathbb{N}$,*

$$\nu_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = \nu_{t_1,\dots,t_k,t_{k+1},\dots,t_{k+m}}(A_1 \times \cdots \times A_k \times \mathbb{R} \times \cdots \times \mathbb{R}).$$

Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a real valued stochastic process X defined on it, such that

$$\nu_{t_1,\dots,t_k}(A_1 \times \dots \times A_k) = \mathbb{P}(\{X_{t_1} \in A_1, \dots, X_{t_k} \in A_k\}),$$

for any $t_i \in I$, $k \in \mathbb{N}$ and $A_i \in \mathcal{B}$.

Definition 1.1.5. *A stochastic process $\{X_t\}_{t\geq 0}$ is said a* ***stationary process*** *if any collection $\{X_{t_1}, X_{t_2}, \dots, X_{t_n}\}$ has the same distribution of $\{X_{t_1+\tau}, X_{t_2+\tau}, \dots, X_{t_n+\tau}\}$ for all $t \geq 0$ and each $\tau \geq 0$. That is*

$$\{X_{t_1}, X_{t_2}, \dots, X_{t_n}\} \stackrel{d}{=} \{X_{t_1+\tau}, X_{t_2+\tau}, \dots, X_{t_n+\tau}\}.$$

Definition 1.1.6. *A stochastic process $\{X_t\}_{t\geq 0}$ is said a* ***stationary increment process****, shortly* ***si****, if for any $h \geq 0$, for all $t \geq 0$,*

$$\{X_{t+h} - X_h\}_{t\geq 0} \stackrel{d}{=} \{X_t - X_0\}_{t\geq 0}. \tag{1.2}$$

Definition 1.1.7. *A real valued stochastic process $X = \{X_t\}_{t\geq 0}$ is said* ***self-similar*** *with index $H \geq 0$, shortly* ***H-ss****, if for any $t \geq 0$, $a > 0$,*

$$\{X_{at}\}_{t\geq 0} \stackrel{d}{=} \{a^H X_t\}_{t\geq 0}.$$

Definition 1.1.8. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family of σ-algebras $\mathcal{F}_t = (\mathcal{F}_t)_{t\in I}$ is called* ***filtration*** *if $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $0 \leq s \leq t \in I$.*

The quadruple $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ is called stochastic basis.

Definition 1.1.9. *Let $X = (X_t)_{t\in I}, X_t : \Omega \to \mathbb{R}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{t\in I}$ be a filtration.*

- *The process X is called* ***measurable*** *provided that the function $(\omega, t) \mapsto X_t(\omega)$ considered as map between $\Omega \times I$ and $\mathbb{R}$ is measurable with respect to $\mathcal{F} \otimes \mathcal{B}(I)$ and $\mathcal{B}(\mathbb{R})$.*

- ***Adapted*** *to the filtration* $(\mathcal{F}_t)_{t\in I}$ *if for each* $t \in I$ *one has that* $X(t)$ *is* $\mathcal{F}_t$*-measurable.*

Let us recall the notion of a martingale.

Definition 1.1.10. *Let* $X = (X_t)_{t\in I}$ *be* $(\mathcal{F}_t)_{t\in I}$*-adapted and such that* $\mathbb{E}|X_t| < \infty$, *for all* $t \geq 0$, *X is called* ***martingale*** *provided that for all* $0 \leq s \leq t \in I$, *one has*

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s. \qquad a.s$$

Definition 1.1.11. *Let* $X = (X_t)_{t\in I}$ *be a stochastic process. The process X is* ***continuous*** *provided that* $t \mapsto X_t(\omega)$ *is continuous for all* $\omega \in \Omega$.

Gaussian processes form a class of stochastic processes used in several branches in pure and applied mathematics.

Definition 1.1.12. *A real-valued stochastic process is called* ***Gaussian*** *of all its finite-dimensional distributions are Gaussian, in other words, if they are multivariate normal distributions.*

1.1.1 The basic examples of stochastic processes, The Brownian motion

The most important stochastic process is the Brownian motion. It was first discussed by Louis Bachelier (1900), and independently by Einstein in his 1905 paper. The modern mathematical treatment of Brownian motion (abbreviated to BM), also called the Wiener process is due to Wiener in 1923, who proved that there exists a version of BM with continuous paths. Note that BM is a Gaussian process, a Markov process, and a martingale. Hence its importance in the theory of stochastic process. It serves as a basic building block for many more complicated processes. For further history of Brownian motion and related processes we cite Meyer [13], Klebaner [9] and Pitman [17].

1.1.1.1 Definition of Brownian Motion

We now start to define and study Brownian motion (Wiener process).

Definition 1.1.13. ***(Brownian motion)*** *A stochastic process* $\{B(t), t \geq 0\}$ *is said to be a Brownian motion with variance parameter* $\sigma^2 > 0$ *if* *(i)* $B(0) = 0$.

(ii) ***(Independent increments.)*** *For each* $0 \leq t_1 < t_2 < \ldots < t_m$,

$$B(t_1), B(t_2) - B(t_1), \ldots, B(t_m) - B(t_{m-1}),$$

are independent r.v.'s.

(iii) ***(Stationary increments.)*** *For each* $0 \leq s < t, B(t) - B(s)$ *has a normal distribution with mean zero and variance* $\sigma^2(t-s)$.

(iv) ***(Continuity of paths.)*** $\{B(t)\}_{t \geq 0}$ *are continuous functions of* t.

Remark 1.1.1.

- *Notice that the natural filtration of the Brownian motion is* $\mathcal{F}_t^B = \sigma(B_s, s \leq t)$.
- *If* $\sigma^2 = 1$, *we said that* $\{B(t) : t \geq 0\}$ *is a standard Brownian motion.*

1.1.1.2 Properties of Brownian motion

1- Martingale property

A martingale is a very special type of stochastic process.

Lemma 1.1.1. *An* $\mathcal{F}_t$*-Wiener process* B_t *is an* $\mathcal{F}_t$*-martingale.*

Proof: We need to prove that $\mathbb{E}(B_t|\mathcal{F}_s) = B_s$ for any $t > s$. But as B_s is $\mathcal{F}_s$-measurable (by adaptedness) this is equivalent to $\mathbb{E}(B_t - B_s|\mathcal{F}_s) = 0$, and this is clearly true by the definition of the Wiener process (as $B_t - B_s$ has zero mean and is independent of $\mathcal{F}_s$).

2- Markov property

The reason why Markov processes are so important comes from the fact that they are fundamental class of stochastic processes, with many applications in real life problems outside mathematics.

Definition 1.1.14. *An* $\mathcal{F}_t$ *adapted process* X_t *is called an* $\mathcal{F}_t$*-Markov process if we have* $\mathbb{E}(f(X_t)|\mathcal{F}_s) = \mathbb{E}(f(X_t)|X_s)$ *for all* $t \geq s$ *and all bounded measurable functions* f. *When the filtration is not specified, the natural filtration* $\mathcal{F}_t^X$ *is implied.*

Lemma 1.1.2. *An $\mathcal{F}_t$-Wiener process B_t is an $\mathcal{F}_t$-Markov process.*

Proof: We refer the reader to (Klebaner, [9]).

3- Self-similarity

Theorem 1.1.3. *B is an H-ss process with $H = 1/2$.*

Proof: It is enough to show that for every $a > 0, \{a^{1/2}B(t)\}$ is also Brownian motion. Conditions (i), (ii) and (iv) follow from the same conditions for $\{B(t)\}$. As to (iii), Gaussianity and mean-zero property also follow from the properties of $\{B(t)\}$.
As to the variance, $\mathbb{E}\left[\left(a^{1/2}B(t)^2\right)\right] = t$. And for all $t_1, t_2 \in \infty$, the autocovariance function $\mathbb{E}\left[(B(at_1)B(at_2))\right] = \min(at_1, at_2) = a\min(t_1, t_2) = \mathbb{E}\left[\left(a^{1/2}B(t_1)a^{1/2}B(t_2)\right)\right]$. Thus $\{a^{1/2}B(t)\}$ is a Brownian motion.

4- Non-differentiability

Theorem 1.1.4. *For any t almost all trajectories of Brownian motion are not differentiable at t.*

Proof: We refer the reader to (Klebaner, [9]).

5- Hölder continuity

Proposition 1.1.1. *Brownian motion paths are a.s. locally γ-Hölder continuous for $\gamma \in [0, 1/2)$.*

Proof: We refer the reader to (Klebaner, [9]).

6- Quadratic variation

Definition 1.1.15. *The quadratic variation of Brownian motion $B(t)$ is defined as*

$$[B, B](t) = [B, B]([0, t]) = \lim_{n\to\infty} \sum_{i=1}^{n} \left| B_{t_i^n} - B_{t_{i-1}^n} \right|^2,$$

where for each n, $\{t_i^n,\ 0 \le i \le n\}$ is a partition of $[0, t]$, and the limit is taken over all partitions with $\delta_n = \max_i(t_{i+1}^n - t_i^n) \to 0$ as $n \to \infty$, and in the sense of convergence in probability.

Theorem 1.1.5. *(Klebaner, [9]) Quadratic variation of a Brownian motion over* $[0, t]$ *is* t.

1.2 Introduction to stochastic integration

Let us consider the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, B_t is a standard Brownian motion.

Definition 1.2.1. *Let* $\mathcal{V}(S, T)$ *be the class of real* ***measurable*** *functions* $f(t, \omega)$*, defined on* $[0, \infty) \times \Omega$*, such that*

1. $f(t, \omega)$ *is* $\mathcal{F}_t$*-adapted.*
2. $\mathbb{E}\left(\int_S^T f(t, \cdot)^2 dt\right) < \infty$.

1.2.1 Itô integral

1.2.1.1 Itô integral definition

Let $f \in \mathcal{V}(S, T)$. We want to define the Itô integral of f in the interval $[S, T)$. Namely

$$\mathcal{I}(f)(\omega) = \int_S^T f(t, \omega) dB_t(\omega). \tag{1.3}$$

The idea is natural. First we define $\mathcal{I}(f)$ for a simple class of functions f. Then define $\int_S^T f(t, \omega) dB_t(\omega)$, as the limit of as $\int_S^T \phi dB_t(\omega)$, as the limit $f \to \phi$.

We begin defining the integral for a special class of functions.

Definition 1.2.2. *(**Simple functions**) A function* $\phi \in \mathcal{V}(S, T)$ *is called* ***simple function*** *(or elementary), if it can be expressed as a superposition of characteristic functions.*

$$\phi(t, \omega) = \sum_{k \geq 0} e_k(\omega) \mathbf{1}_{[t_k, t_{k+1})}(t). \tag{1.4}$$

Definition 1.2.3. *Let* $\phi \in \mathcal{V}(S, T)$ *be a simple function of the form of (1.4), then we define the stochastic integral as*

$$\int_S^T \phi(t, \omega) dB_t = \sum_{k \geq 0} e_k(\omega)(B_{t_{k+1}} - B_{t_k})(\omega). \tag{1.5}$$

Lemma 1.2.1. *(**Ito isometry**, [15]) Let $\phi \in \mathcal{V}(S,T)$ be a simple function, then*

$$\mathbb{E}\left(\left(\int_S^T \phi(t,\cdot)dB_t\right)^2\right) = \mathbb{E}\left(\int_S^T \phi(t,\cdot)^2 dt\right). \tag{1.6}$$

Remark 1.2.1. *Observe that (1.6) is indeed an isometry. In fact, it can been written as equality of norms in L^2 spaces*

$$\left\|\int_S^T \phi(t,\cdot)dB_t\right\|_{L^2(\Omega,\mathbb{P})} = \|\phi\|_{L^2([S,T]\times\Omega)}.$$

We have the following important proposition.

Proposition 1.2.1. *Let $f \in \mathcal{V}$, then there exists a sequence of simple functions $\phi_n \in \mathcal{V}, n \in \mathbb{N}$, which converges to f in the L^2-norm. Namely*

$$\lim_{n\longrightarrow\infty} \int_S^T \mathbb{E}\left((f(t,\cdot) - \phi_n(t,\cdot))^2\right) dt = \lim_{n\longrightarrow\infty} ||f - \phi_n||^2_{L^2([S,T]\times\Omega)} = 0. \tag{1.7}$$

Proof: See (Mura, [15]).

Definition 1.2.4. *(**Itô integral I**) Let $f \in \mathcal{V}(S,T)$ the Itô integral from S to T of f is defined as the $L^2(\Omega,\mathbb{P})$ limit*

$$\mathcal{I}(f) = \int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\longrightarrow\infty} \int_S^T \phi_n(t,\omega)dB_t(\omega), \tag{1.8}$$

where $\phi_n \in \mathcal{V}$, $n \in \mathbb{N}$, is a sequence of simple functions which converges to $f \in L^2([S,T]\times\Omega)$.

Remark 1.2.2. *Observe, in view of (1.7), that the definition above does not depend on the actual choice of $\{\phi_n, n \in \mathbb{N}\}$.*

By definition, we have that Itô isometry holds for Itô integrals.

Corollary 1.2.1. *(**Itô isometry for Itô integrals**, [15]) Let $f \in \mathcal{V}(S,T)$, then*

$$\mathbb{E}\left(\left(\int_S^T f(t,\cdot)dB_t\right)^2\right) = \mathbb{E}\left(\int_S^T f(t,\cdot)dt\right). \tag{1.9}$$

Corollary 1.2.2. *[15] If $f_n(t,\omega) \in \mathcal{V}(S,T)$ converges to $f(t,\omega) \in \mathcal{V}(S,T)$ as $n \longrightarrow \infty$ in the $L^2([S,T]\times\Omega)$-norm, then*

$$\int_S^T f_n(t,\cdot)dB_t \longrightarrow \int_S^T f(t,\cdot)dB_t, \tag{1.10}$$

in the $L^2(\Omega,\mathbb{P})$-norm.

1.2.1.2 Properties of the Itô integral

Proposition 1.2.2. *[15] Let $f, g \in \mathcal{V}(0,T)$ and let $0 \leq S < U < T$. Then*

1. $\displaystyle\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t.$
2. *For some constant $a \in \mathbb{R}$,* $\displaystyle\int_S^T (af+g) dB_t = a\int_S^T f dB_t + \int_S^T g dB_t.$
3. $\displaystyle\mathbb{E}\left[\int_S^T f dB_t\right] = 0.$
4. $\displaystyle\int_S^T f dB_t$ *is $\mathcal{F}_T$-measurable.*
5. *The process $M_t(\omega) = \displaystyle\int_0^T f(t,\omega) dB_s(\omega)$ where $f \in \mathcal{V}(0,T)$ for any $t > 0$, is a martingale with respect to $\mathcal{F}_t$.*

1.2.2 Extensiens of Itô integral

The construction of the Itô Integral can be extended to a class of function $f(t,\omega)$ which satisfies a weak integration condition. This generalization is indeed necessary because it is not difficult to find functions which do not belong to $\mathcal{V}$. Therefore, we introduce the following class of functions.

Definition 1.2.5. *Let $\mathcal{W}(S,T)$ be the class of real measurable functions $f(t,\omega)$, defined on $[0,\infty) \times \Omega$, such that*

1. *$f(t,\omega)$ is $\mathcal{F}_t$-adapted.*
2. $\displaystyle\mathbb{P}\left(\int_S^T f(t,\cdot)^2 dt < \infty\right) = 1.$

For any $f \in \mathcal{W}$, one can show that there exists a sequence of simple functions $\phi_n \in \mathcal{W}$ such that

$$\int_S^T |\phi_n(t,\cdot) - f(t,\cdot)|^2 dt \longrightarrow 0, \tag{1.11}$$

in probability. For such a sequence one has that the sequence $\left\{\displaystyle\int_S^T \phi_n(t,\omega) dB_t(\omega), n \in \mathbb{N}\right\}$ converges in probability to some random variable. Moreover, the limit does not depends on the approximating sequence ϕ_n. Thus, we may define

Definition 1.2.6. *(**Itô integral II**) Let $f \in \mathcal{W}(S,T)$. The Itô integral from S to T of f is defined as the limit in probability*

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n \longrightarrow \infty} \int_S^T \phi_n(t,\omega)dB_t(\omega), \tag{1.12}$$

where $\phi_n \in \mathcal{W}, n \in \mathbb{N}$, is a sequence of simple functions which converges to f in probability.

1.2.3 Stochastic Differential Equations

We call stochastic differential equation (SDE) an equation of the form

$$dX_t = b(t, X(t))dt + \sigma(t, X(t))dB(t), \qquad X|_{t=0} = x, \tag{1.13}$$

where $(B_t)_{t \geq 0}$ is a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, x is $\mathcal{F}_0$-measurable, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ have some regularity specified case by case and the solution $(X_t)_{t \geq 0}$ is a d-dimensional continuous adapted process. The meaning of the equation (1.13) is identic to

$$X_t = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s). \tag{1.14}$$

If there exists a stochastic process X_t that satisfies this equation, we say that it *solves* the stochastic differential equation.

The main goal of this section is to find conditions on the coefficients b and σ that guarantee the existence and uniqueness of solutions.

However, there are a number of subtle points involved

- First, the existence of the integrals in (1.14) requires some degree of regularity on X_t and the functions b and σ; in particular, it must be the case that for all $t \geq 0$, with probability one, $\int_0^t |b(s, X(s))|ds < \infty$ and $\int_0^t \sigma^2(s, X(s))ds < \infty$.

- Second, the definition requires that the process X_t live on the same probability space as the given Wiener process B_t, and that it be adapted to the given filtration. It turns out that for certain coefficient functions b and σ, solutions to the stochastic integral equation (1.14) may exist for some Wiener processes and some admissible filtration but not for others.

- The solution is a *strong solution* if it is valid for each given Wiener process (and initial value), that is it is sample pathwise unique.
- A solution is a *weak solution* if it is valid for given coefficients, but unspecified Wiener process, that is its probability law is unique.

More precisely.

Definition 1.2.7. *Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an admissible filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$. A* ***strong solution*** *of the stochastic differential equation (1.14) with initial condition $x \in \mathbb{R}$ is an adapted process X_t with continuous paths such that for all $t \geq 0$,*

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s).$$

Definition 1.2.8. *A* ***weak solution*** *of the stochastic differential equation (1.14) with initial condition x is a continuous stochastic process X_t defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for some Wiener process B_t and some admissible filtration $\mathcal{F}_t$ the process $X(t)$ is adapted and satisfies the stochastic integral equation (1.14).*

Let us come to uniqueness. Similarly to existence, there are two concepts.

Definition 1.2.9. ***(pathwise uniqueness)*** *We say that pathwise uniqueness holds for equation (1.14) if, given any filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with a Brownian motion $(B_t)_{t\geq 0}$ given any deterministic initial condition $X_0 = x$, if $(X_t^{(1)})_{t\geq 0}$ and $(X_t^{(2)})_{t\geq 0}$ are two continuous $\mathcal{F}_t$-adapted process which fulfill (1.14), then they are indistinguishable.*

Definition 1.2.10. ***(uniqueness in law)*** *We say that there is uniqueness in law for equation (1.13) if a given two weak solutions on any pair of spaces, their laws coincide.*

Theorem 1.2.1. *If the coefficients b and σ satisfy the following conditions*

1. *A Lipschitz condition in x and y. $\exists K, \forall x \in \mathbb{R}^n, \forall y \in \mathbb{R}^n, \forall t \geq 0,$*

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|.$$

2. *A linear growth condition,* $\exists K, \forall x \in \mathbb{R}^n, \forall t \geq 0,$

$$\|b(t,x))\| + \|\sigma(t,x)\| \leq K(1 + \|x\|).$$

Then there exists a unique strong solution X *to the stochastic differential equation (1.13) with continuous trajectories and there exists a constant* C *such that*

$$\mathbb{E}[\|X_t\|^2] \leq Ce^{Ct}(1 + \|x\|^2).$$

Proof: See (Oksendal, 36+[11]).

Chapter 2

Fractional Calculus

As it's mentioned in the introduction, the fractional calculus is the mathematical field in which the differentiation operator is applied to a function a nonintegral number of times. This subject is as old as the differential calculus, and goes back to times when Leibnitz and Newton invented differential calculus. In this chapter we aim to introduce fractional calculus we start with the Grunwald-Letnikov's one to move then to the precise definitions of the left and right Riemann-Liouville (R-L) fractional integrals and derivatives and the definition of the fractional derivatives in the Caputo sense. Other fractional derivatives are given such as Hadamard and Hilfer. Then, we study the main of their properties. Finally, we briefly discussed some examples and as an application of fractional calulus we examined a commonly used economic model. Further details on the material discussed here can be found in [6, 10, 12] and references therein.

2.1 Useful Mathematical Functions

Before looking at the definition of fractional integrals or derivatives, we will first discuss some useful mathematical definitions that are inherently tied to fractional calculus and will commonly be encountered. These include the Gamma function, the Beta function and the Mittag-Leffler function.

2.1.1 The Gamma Function

The Gamma function, denoted by $\Gamma(x)$, is a generalization of the factorial function $n!$ and defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x \in \mathbb{R}. \tag{2.1}$$

Some of the basic properties of Γ function, are

$$\begin{cases} \Gamma(1) = \Gamma(2) = 1. \\ \Gamma(x+1) = x\Gamma(x), \qquad x \in \mathbb{R}^+. \\ \Gamma(n) = (n-1)!, \qquad n \in \mathbb{N}^*. \end{cases}$$

From the above we can get

$$\begin{cases} \Gamma(\frac{1}{2}) = \sqrt{\pi}. \\ \Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}. \\ \Gamma(\frac{-3}{2}) = \dfrac{\Gamma(\frac{-3}{2}+1)}{\frac{-3}{2}} = \dfrac{\Gamma(\frac{-1}{2})}{\frac{-3}{2}} = \dfrac{\Gamma(\frac{1}{2})}{\frac{-3}{2}\frac{-1}{2}} = \dfrac{4}{3}\sqrt{\pi}. \end{cases}$$

2.1.2 The Beta Function

The Beta function is very important for the computation of the fractional derivatives of the power function. It is defined by the two parameter integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x, y \in \mathbb{R}. \tag{2.2}$$

It should be mention that the Beta function has also some properties, the key one is its relationship to the gamma function.

- $$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{2.3}$$
- $B(x,y) = B(y,x)$.
- $B(1/2, 1/2) = \pi$.

2.1.3 The Mittag-Lefler Function

The Mittag-Leffler function is named after a Swedish mathematician who defined and studied it in 1903. The function is a direct generalization of the exponential function, it plays a major role in fractional calculus. Firstly, we introduce one parameter function by using series, namely

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \tag{2.4}$$

Then, we define the Mittag-Leffler function with two parameters, as

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. \tag{2.5}$$

Note that $E_{\alpha,\beta}(0) = 1$. Also, for some specific values of α, and β, the Mittag-Leffler function reduces to some familiar functions, namely

$$\begin{cases} E_{1,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \exp(x). \\ E_{1,2}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} = \frac{\exp(x) - 1}{x}. \\ E_{1,3}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{x^k}{(k+2)!} = \frac{1}{x^2} \sum_{k=0}^{\infty} \frac{x^{k+2}}{(k+2)!} = \frac{\exp(x) - 1 - x}{x^2}. \\ E_{1,m}(x) = \frac{1}{x^{m-1}}[\exp(x) - \sum_{k=0}^{m-2} \frac{x^k}{k!}]. \\ E_{2,1}(-x^2) = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \cos(x). \\ E_{2,2}(-x^2) = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{x(2k+1)!} = \frac{\sin(x)}{x}. \end{cases}$$

2.2 Fractional Derivatives and Integrals

This section is devoted to review three important definition of fractional derivatives and Integrals.

2.2.1 Grünwald-Letnikove, 1867-1868

Grünwald-Letnikov derivative is a basic extension of the natural derivative to fractional one. It was introduced by A. Grünwald in 1867, and then by A. Letnikov in 1868. Hence, it is written as

Definition 2.2.1. *Let $\alpha \in (0,1)$ be fixed and let $f : \mathbb{R} \to \mathbb{R}$ be a given function. The R-Grünwald-Letnikov derivative of order α of f is defined, respectively as*

$$D_{+}^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t - jh).$$

We recall that the binomial coefficients can be defined as $\begin{pmatrix} \alpha \\ n \end{pmatrix} = \frac{\alpha!}{n!(\alpha-n)!}$.

2.2.2 Riemann-Liouville, 1832-1847

The Riemann-Liouville Operator is still the most frequently used when fractional integration is performed. Which is considered as a direct generalization of Cauchy's formula for an n-times integral

$$\int_a^x dx_1 \int_a^{x_1} dx_2 ... \int_a^{x_{n-1}} f(x_n) dx_n = \frac{1}{(n-1)!} \int_a^x \frac{f(t)}{(x-t)^{1-n}} dt. \tag{2.6}$$

Example 2.2.1. *As an example let $f(x) = x$, $n = 3$ and $a = 0$ then (2.6) becomes*

$$\int_0^x \int_0^{x_1} \int_0^{x_2} x_3 dx_3 x_2 x_1 = \frac{1}{2!} \int_0^x \frac{t}{(x-t)^{-2}} dt, \tag{2.7}$$

and by integration one gets

$$\frac{1}{2!} \int_0^x \frac{t}{(x-t)^{-2}} dt = \frac{x^4}{4!}. \tag{2.8}$$

Since $(n-1)! = \Gamma(n)$, Riemann realized that (2.6) might have meaning even when n takes non-integer values. Thus perhaps it was natural to define fractional integration as follows.

Definition 2.2.2. *Let $f \in L_1([a,b])$ and $a \leq x \leq b$ then*

$$I_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt,$$

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt,$$

*are called the **Riemann-Liouville fractional integral** of order $\alpha > 0$.*

Definition 2.2.3. *Let $f \in L_1([a,b])$ and $a \leq x \leq b$ then*

$$D_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}} dt,$$

$$D_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{b} \frac{f(t)}{(t-x)^{\alpha}} dt,$$

which is called the ***Riemann-Liouville fractional derivative*** *of order* $0 < \alpha < 1$.

2.2.3 Caputo, 1969

Since Riemann-Liouville fractional derivatives failed in the description and modeling of some complex phenomena, Caputo derivative was introduced in 1967.

Definition 2.2.4. *The Caputo derivative of fractional order* $(n-1 \leq \alpha < n)$ *of a function* f *is defined as*

$${}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_{a}^{t} \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, \qquad (n-1 \leq \alpha < n).$$

$${}_{b}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_{t}^{b} \frac{f^{(n)}(\tau) d\tau}{(\tau - t)^{\alpha+1-n}}, \qquad (n-1 \leq \alpha < n).$$

Remark 2.2.1.

- *Such kind of equations arise in the mathematical modeling of various physical phenomena, such as heat conduction in materials with memory.*
- *Initial conditions for the Caputo derivatives are expressed in terms of initial values of integer order derivatives.*
- *For this reason, many authors either resort to Caputo derivatives, or use the Riemann-Liouville derivatives but avoid the problem of initial values of fractional derivatives by treating only the case of zero initial conditions.*
- *It is known that for zero initial conditions the Riemann-Liouville, Grünwald-Letnikov and Caputo fractional derivatives coincide.*

2.3 Other Fractional Derivatives

As previously mentioned, different definitions for fractional derivative with the different properties can be proposed, which all of them are valid and mathematically acceptable. However, the main question is "which relation should be applied in modeling of a specific phenomenon? In other words, which definition would be more appropriate for a specific problem?". As a rule of thumb, since they tend to interpret natural phenomena, the definition which is more consistent with the experimental results have more privilege than the other fractional definitions.

2.3.1 Marchaud fractional derivative

In his doctoral thesis, Marchaud defined the following fractional differentiation.

Definition 2.3.1. ***(Marchaud derivative: 1927)*** *For a function defined $\mathbb{R}$ and for every $\alpha \in (0,1)$ distinguishing two types of derivative, respectively from the right and from the left one*

$$D_+^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} + \int_0^\infty \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt,$$

and

$$D_-^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} + \int_{-\infty}^0 \frac{f(x) - f(x+t)}{t^{1+\alpha}} dt.$$

These fractional derivatives are well defined when f is a bounded, locally Hölder continuous function in $\mathbb{R}$.

Remark 2.3.1. *If we compare the Marchaud derivative with respect to the Riemann-Liouville one, we immediately realize that, in the latter one, the classical derivative operator appears, while, in the first one, it does not. This is one of the key points that Marchaud's definition makes evident. That is, Marchaud derivative avoids applying the classical derivative after an integration in order to define the fractional operator.*

2.3.2 Hilfer fractional derivative

In the recent years new alternative definitions of fractional operators have been introduced in the literature. An interesting example is the so-called Hilfer derivative. The idea behind the introduction of this derivative is to interpolate between the Riemann-Liouville and the Caputo derivatives. As it is clear from the definition below, the Hilfer derivative depends on the parameter $\nu \in [0,1]$ that balances the individual contributions of the two fractional derivatives.

Definition 2.3.2. ***(Hilfer derivative: 2000)*** *Let $\mu \in (0,1)$, $\nu \in [0,1]$, $f \in L^1[a,b], a < t < b$, the Hilfer derivative is defined as*

$$(D_{a+}^{\mu,\nu} f)(t) = \left(I_{a+}^{\nu(1-\mu)} \frac{d}{dt} \left(I_{a+}^{(1-\nu)(1-\mu)} f \right) \right)(t).$$

$$(D_{b-}^{\mu,\nu} f)(t) = \left(I_{b-}^{\nu(1-\mu)} \frac{d}{dt} \left(I_{b-}^{(1-\nu)(1-\mu)} f \right) \right)(t).$$

Remark 2.3.2. *Hilfer derivatives coincide with Riemann-Liouville derivatives for $\nu = 0$ and with Caputo derivatives for $\nu = 1$.*

2.3.3 Canavati fractional derivative

There is another defenition of fractional derivatives that is useful in deriving inequalities. This is the Canavati fractional derivative. It is "between" the Riemann-Liouville derivative and the Caputo derivative.

Definition 2.3.3. ***(Canavati derivative: 2009)*** *Let $n-1 < \alpha < n, f \in \mathcal{C}^\alpha([a,b])$. Then, the Canavati derivative of order α is defined as*

$${}_a^{Can}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} + \frac{d}{dt} \int_a^t \frac{f^{(n-1)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau.$$

2.4 Basic Properties of Fractional Derivatives

2.4.1 Semigroup Properties of Fractional Integral Operators

Theorem 2.4.1. *For any $f \in C([a,b])$ the Riemann-Liouville fractional integral satisfies*

$$I_{a+}^\alpha I_{a+}^\beta f(x) = I_{a+}^{\alpha+\beta} f(x), \tag{2.9}$$

for $\alpha > 0, \beta > 0$.

Proof: The proof is rather direct, we have by definition

$$I_{a+}^\alpha I_{a+}^\beta f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{dt}{(x-t)^{1-\alpha}} \int_a^t \frac{f(u)}{(t-u)^{1-\beta}} du, \tag{2.10}$$

and since $f(x) \in C([a,b])$ we can by Fubini's theorem interchange order of integration and by setting $t = u + s(x-u)$ we obtain

$$I_{a+}^\alpha I_{a+}^\beta f(x) = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{f(u)}{(x-u)^{1-\alpha-\beta}} du = I_{a+}^{\alpha+\beta} f(x). \tag{2.11}$$

2.4.2 Linearity

Let f and g are functions for which the given derivatives or integrals operator are defined and $\lambda, \mu \in \mathbb{R}$ are real constants.

$$ {}_aD_t^p(\lambda f(t) + \mu g(t)) = \lambda {}_aD_t^p f(t) + \mu {}_aD_t^p g(t). $$

Proof:

- For Grüunwald-Letnikov fractional derivative we have

$$ \begin{aligned} {}_aD_t^p(\lambda f(t) + \mu g(t)) &= \lim_{h\to 0} h^{-p} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} p \\ r \end{pmatrix} (\lambda f(t-rh) + \mu g(t-rh)) \\ &= \lambda \lim_{h\to 0} h^{-p} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} p \\ r \end{pmatrix} f(t-rh) \\ &+ \mu \lim_{h\to 0} h^{-p} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} p \\ r \end{pmatrix} g(t-rh) \\ &= \lambda {}_aD_t^p f(t) + \mu {}_aD_t^p g(t). \end{aligned} $$

- For Riemann-Liouville differintegral will also be given

$$ \begin{aligned} {}_aD_t^{-p}(\lambda f(t) + \mu g(t)) &= \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} (\lambda f(t) + \mu g(t)) d\tau \\ &= \lambda \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau + \mu \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} g(\tau) d\tau \\ &= \lambda {}_aD_t^{-p} f(t) + \mu {}_aD_t^{-p} g(t). \end{aligned} $$

2.4.3 Zero Rule

It can be proved that if f is continuous for $t \geq a$ then we have

$$ \lim_{p\to 0} {}_aD_t^{-p} f(t) = f(t). $$

Proof:

The proof can be found in (Oldham, [12]). Hence, we define

$$ {}_aD_t^0 f(t) = f(t). $$

2.4.4 Product Rule & Leibniz's Rule

If f and g are functions we know the derivative of their product is given by the product rule

$$(f.g)' = f'.g + f.g'.$$

This can be generalized to

$$(fg)^{(n)} = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} f^{(k)} g^{(n-k)},$$

which is also known as the Leibniz rule. In the last expression f and g are n-times differentiable functions. If f and g and their derivatives are continuous in [a,t] it can be proved that the Leibniz rule for fractional derivatives is given by the following expression

$$_aD_t^p(f(t)g(t)) = \sum_{k=0}^{m} \begin{pmatrix} p \\ k \end{pmatrix} {}_aD_t^k f(t)\, {}_aD_t^{p-k} g(t). \tag{2.12}$$

A similar proof can be given for the fractional integral.

2.4.5 Composition

2.4.5.1 Fractional integration of a fractional integral

The Riemann-Liouville fractional integral has the following important property

$$_aD_t^{-p}({}_aD_t^{-q}f(t)) =_a D_t^{-q}({}_aD_t^{-p}f(t)) =_a D_t^{-p-q}f(t), \tag{2.13}$$

which is called the composition rule for the Riemann-Liouville fractional integrals. Using the definition, the proof is quite straightforward

$$\begin{aligned} {}_aD_t^{-p}\left({}_aD_t^{-q}f(t)\right) &= \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}({}_aD_t^{-q}f(\tau))d\tau \\ &= \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\left(\frac{1}{\Gamma(q)}\int_a^\tau (\tau-\xi)^{q-1}f(\xi)d\xi\right)d\tau \\ &= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^\tau (t-\tau)^{p-1}(\tau-\xi)^{q-1}f(\xi)d\xi d\tau. \end{aligned}$$

Changing the order of integration we obtain

$$_aD_t^{-p}\left({}_aD_t^{-q}f(t)\right) = \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\int_a^\tau (t-\tau)^{p-1}(\tau-\xi)^{q-1}d\tau d\xi.$$

We make the substitution $\frac{\tau-\xi}{t-\xi}=\zeta$ from which it follows that $d\tau=(t-\xi)d\zeta$ and the new interval of integration is $[0,1]$. Now we are able to rewrite the last expression as

$$\begin{aligned} {}_aD_t^{-p}\left({}_aD_t^{-q}f(t)\right) &= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\left((t-\xi)^{p+q-1}\int_0^1(1-\zeta)^{p-1}\zeta^{q-1}d\zeta\right)d\xi \\ &= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\left((t-\xi)^{p+q-1}B(p,q)\right)d\xi. \end{aligned}$$

Using identity (2.3) to express the Beta function in terms of the Gamma function we obtain

$$\begin{aligned} {}_aD_t^{-p}\left({}_aD_t^{-q}f(t)\right) &= \frac{1}{\Gamma(p)\Gamma(q)}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\int_a^t f(\xi)(t-\xi)^{p+q-1}d\xi \\ &= \frac{1}{\Gamma(p+q)}\int_a^t (t-\xi)^{p+q-1}f(\xi)d\xi \\ &= {}_aD_t^{-p-q}f(t). \end{aligned}$$

2.4.5.2 Fractional differentiation of a fractional integral

An important property of the Riemann-Liouville fractional derivative is

$$ {}_aD_t^{p}\left({}_aD_t^{-q}f(t)\right) ={}_a D_t^{p-q}f(t), \tag{2.14}$$

where f has to be continuous and if $p \geq q \geq 0$, the derivative ${}_aD_t^{p-q}f$ exists. This property is called the composition rule for the Riemann-Liouville fractional derivatives. We shall prove this property, but first we need another property which actually is a special case of the previous one with $q=p$

$$ {}_aD_t^{p}\left({}_aD_t^{-p}f(t)\right) = f(t), \tag{2.15}$$

where $p>0$ and $t>a$. This implies that the Riemann-Liouville fractional differentiation operator is the left inverse of the Riemann-Liouville fractional integration of the same order p. We prove this in the following way

- First we consider the case $p=n\in\mathbb{N}^*$, then we have

$$\begin{aligned} {}_aD_t^{n}\left({}_aD_t^{-n}f(t)\right) &= \frac{d^n}{dt^n}\frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1}f(\tau)d\tau \\ &= \frac{d}{dt}\int_a^t f(\tau)d\tau = f(t). \end{aligned}$$

- For the non-integer case we take $k-1 \leq p < k$ and use (2.13) to write

$$ {}_aD_t^{-k} f(t) = {}_a D_t^{-(k-p)} \left({}_aD_t^{-p} f(t) \right). $$

Now using the definition of the Riemann-Liouville differintegral we obtain

$$ \begin{aligned} {}_aD_t^{p} \left({}_aD_t^{-p} f(t) \right) &= \frac{d^k}{dt^k} \left[{}_aD_t^{-(k-p)} \left({}_aD_t^{-p} f(t) \right) \right] \\ &= \frac{d^k}{dt^k} \left[{}_aD_t^{-k} f(t) \right] = f(t). \end{aligned} $$

- Now we are able to prove (2.14). We consider two cases. First we'll deal with $q \geq p \geq 0$. Then we have

$$ {}_aD_t^{p} \left({}_aD_t^{-q} f(t) \right) = {}_a D_t^{p} \left[{}_aD_t^{-p} \left({}_aD_t^{-(q-p)} f(t) \right) \right] = {}_a D_t^{p-q} f(t). $$

This follows directly from (2.13) and (2.15). Now we will consider the second case in which we have $p > q \geq 0$. Using (2.13) we see that

$$ \begin{aligned} {}_aD_t^{p} \left({}_aD_t^{-q} f(t) \right) &= \frac{d^k}{dt^k} \left[{}_aD_t^{-(k-p)} \left({}_aD_t^{-q} f(t) \right) \right] \\ &= \frac{d^k}{dt^k} \left({}_aD_t^{p-q-k} f(t) \right) = \frac{d^k}{dt^k} \left({}_aD_t^{-(k-(p-q))} f(t) \right) \\ &= {}_aD_t^{p-q} f(t). \end{aligned} $$

So in both cases we proved equation (2.14).

Remark 2.4.1. *The converse of (2.15) is not true, so ${}_aD_t^{-p} \left({}_aD_t^{p} f(t) \right) \neq f(t)$. The proof for this can be found in (Koning, [10]).*

2.5 Examples

This section deals with some examples of fractional derivatives. First we will take a look at the power function and thereafter explore the exponential function and trigonometric functions.

2.5.1 The Power Function

The Power function is very important in mathematics since many functions can be derived from an infinite power series. First we will use the Riemann- Liouville fractional integral to compute the integral of order $p \in \mathbb{R}_+^*$ of the power function $(t-a)^\beta$. Plugging this into the equation gives

$$ {}_aD_t^{-p}(t-a)^\beta = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1}(\tau-a)^\beta d\tau. $$

If we make the substitution $\frac{\tau-a}{t-a} = \xi$ from which it follows that $d\tau = (t-a)d\xi$ and the new interval of integration is $[0,1]$, we can rewrite the last expression as

$$ \begin{aligned} {}_aD_t^{-p}(t-a)^\beta &= \frac{(t-a)^{\beta+p}}{\Gamma(p)} \int_0^1 (1-\xi)^{p-1}\xi^\beta d\xi \\ &= \frac{(t-a)^{\beta+p}}{\Gamma(p)} B(p, \beta+1) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+p+1)}(t-a)^{\beta+p}, \end{aligned} \tag{2.16} $$

where in the last equation we made use of (2.3). It follows that $\beta > -1$.

Next we will compute the derivative of order $r \in \mathbb{R}_+^*$ of the same power function $(t-a)^\beta$ using the Riemann-Liouville fractional derivative.

Again filling in $f(t) = (t-a)^\beta$ gives

$$ {}_aD_t^r(t-a)^\beta = \frac{d^k}{dt^k}\left({}_aD_t^{-(k-r)}(t-a)^\beta\right). $$

Now we are able to use the integral of the power function we have just computed in (2.16). If we replace the order p by $k-r > 0$ we can rewrite the last expression as

$$ \begin{aligned} {}_aD_t^r(t-a)^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+k-r+1)} \frac{d^k}{dt^k}(t-a)^{\beta+k-r} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-r+1)}(t-a)^{\beta-r}, \end{aligned} \tag{2.17} $$

with $\beta > -1$.

The following two examples can clarify this using concrete numbers.

Example 2.5.1. *The half-derivative of $f(x) = x$, so in the last expression we set $t = x$, $a = 0$, $\beta = 1$ and $r = \frac{1}{2}$. Then we obtain*

$$ \begin{aligned} {}_aD_t^{\frac{1}{2}}(x-0)^1 &= \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}(x-0)^{1-\frac{1}{2}} \\ {}_aD_t^{\frac{1}{2}}x &= \frac{\Gamma(2)}{\Gamma(3)}x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}. \end{aligned} $$

Example 2.5.2. *The derivative of order $\frac{3}{4}$ of $f(x) = x^2$, so again in formula (2.17) we set $t = x$, $a = 0$, but now $\beta = 2$ and $r = \frac{3}{4}$. This gives us*

$$\begin{aligned}
{}_aD_t^{\frac{3}{4}}(x-0)^2 &= \frac{\Gamma(2+1)}{\Gamma(2-\frac{3}{4}+1)}(x-0)^{2-\frac{3}{4}} \\
{}_aD_t^{\frac{3}{4}}x^2 &= \frac{\Gamma(3)}{\Gamma(2\frac{1}{4})}x^{\frac{1}{4}} \approx 1.76522x^{\frac{1}{4}}.
\end{aligned}$$

2.5.2 The Exponential Function

Another frequently used function in mathematics is the exponential function. We shall use the Weyl fractional integral, which is formally equal to the Riemann- Liouville fractional integral [10], to compute the integral of order $p \in \mathbb{R}_+^*$ of the function $f(t) = \exp(\lambda t)$, where $\lambda \in \mathbb{C}$, and setting a equal to $-\infty$ gives us

$${}_{-\infty}D_t^{-p}\exp(\lambda t) = \frac{1}{\Gamma(p)}\int_{-\infty}^{t}(t-\tau)^{p-1}\exp(\lambda\tau)d\tau.$$

This expression can be rewritten as

$${}_{-\infty}D_t^{-p}\exp(\lambda t) = \lambda^{1-p}\frac{1}{\Gamma(p)}\int_{-\infty}^{t}(\lambda(t-\tau))^{p-1}\exp(\lambda\tau)d\tau.$$

If we make the substitution $\xi = \lambda(t-\tau)$ it follows that ξ goes from $\infty \to 0$ and $-\lambda d\tau = d\xi$ so $d\tau = -\lambda^{-1}d\xi$. Now we can rewrite the last expression as

$$\begin{aligned}
{}_{-\infty}D_t^{-p}\exp(\lambda t) &= -\lambda^{1-p}\frac{1}{\Gamma(p)}\int_{-\infty}^{0}\xi^{p-1}\exp(\lambda t-\xi)\lambda^{-1}d\xi \\
&= \lambda^{1-p}\frac{1}{\Gamma(p)}\int_{0}^{+\infty}\xi^{p-1}\exp(\lambda t-\xi)\lambda^{-1}d\xi \\
&= \lambda^{-p}\frac{\exp(\lambda t)}{\Gamma(p)}\int_{0}^{+\infty}\xi^{p-1}\exp(-\xi)d\xi.
\end{aligned}$$

Using the Gamma function, we get

$${}_{-\infty}D_t^{-p}\exp(\lambda t) = \lambda^{-p}\frac{\exp(\lambda t)}{\Gamma(p)}\Gamma(p) = \lambda^{-p}\exp(\lambda t).$$

The fractional derivative of order p can be obtained in the same way and is given by

$${}_{-\infty}D_t^{p}\exp(\lambda t) = \lambda^{p}\exp(\lambda t), \qquad p \in \mathbb{R}. \tag{2.18}$$

Example 2.5.3. *Compute the fractional derivative of order $\alpha = 1/3$ for function $f(t) = e^{5t}$.*

$${}_{-\infty}D_t^{1/3}(e^{5t}) = 5^{1/3}e^{5t}.$$

2.5.3 The Trigonometric Functions

In this example we would like to explore the differintegral of the sine and cosine functions. We are able to use the last example since we can write the trigonometric functions in terms of the exponential function in the following way

$$\sin(t) = \frac{\exp(it) - \exp(-it)}{2i}. \qquad \cos(t) = \frac{\exp(it) + \exp(-it)}{2}.$$

First we will explore the Weyl differintegral of order $p \in \mathbb{R}$ of the sine function

$$_{-\infty}D_t^p \sin(t) =_{-\infty} D_t^p \left(\frac{\exp(it) - \exp(-it)}{2i} \right).$$

If we now use the linearity of the Weyl differintegral, which follows directly from the linearity of the Rieman-Liouville differintegral since they are formally equal, the last expression can be rewritten as

$$_{-\infty}D_t^p \sin(t) = \frac{1}{2i} \left(_{-\infty}D_t^p \exp(it) -_{-\infty} D_t^p \exp(-it) \right).$$

If we now use the expression for the differintegral of the exponential function (2.18) given in the last example we obtain

$$\begin{aligned} _{-\infty}D_t^p \sin(t) &= \frac{1}{2i} \left(i^p \exp(it) - (-i)^p \exp(-it) \right) = \frac{1}{2i} \Big(\exp(i\frac{\pi}{2}p) \exp(it) - \exp(-i\frac{\pi}{2}p) \exp(-it) \Big) \\ &= \frac{1}{2i} \Big(\exp(i(t + \frac{\pi}{2}p)) - \exp(-i(t + \frac{\pi}{2}p)) \Big) = \sin(t + \frac{\pi}{2}p). \end{aligned}$$

The differintegral for the cosine function can be obtained in the same way and is given by

$$_{-\infty}D_t^p \cos(t) = \cos(t + \frac{\pi}{2}p).$$

Example 2.5.4. *Compute the fractional derivative of order $\alpha = 1/2$ for function $f(t) = sin(3t)$. Taking $a = -\infty$,*

$$_{-\infty}D_t^{1/2}(sin(3t)) = \sqrt{3}(sin(3t + \frac{\pi}{4})).$$

2.6 Applications

We will treat one simple economic example to show how fractional calculus can be implemented in a commonly used model.

2.6.1 Economic example

Let's say a customer buys a product for a price **b euro**. The customer does not pay instantly for the product, but chooses to pay off in y months. The interest rate of the seller is $r\%$ per month. The monthly payment the customer is charged is denoted by **m euro**. If we define $f(\tau)$ to be the remaining debt at the end of the τ^{th} month, it can be shown that we have

$$f(\tau) = b(1+r)^{\tau} - \frac{m}{r}[(1+r)^{\tau} - 1]. \tag{2.19}$$

At $\tau = y$ the customer should have payed off his product so then we must have $f(y) = 0$. Now we are able to solve (2.19) for m which gives

$$m = \frac{b(1+r)^{y} r}{(1+r)^{y} - 1}. \tag{2.20}$$

Usually this problem can be solved using the following differential equation

$$f'(\tau) - rf(\tau) = -m. \tag{2.21}$$

If we want to approximate this with a fractional differential equation we rewrite the last formula and consider

$$_0D_t^p f(\tau) - rf(\tau) = -m, \qquad with \ \ 0 < p \leq 1. \tag{2.22}$$

As in [10] we can solve this fractional differential equation by taking the Laplace transform on both sides, and the linearity of the Laplace transform we obtain

$$\begin{aligned} L\{_0D_t^p f(\tau)\} - L\{rf(\tau)\} &= -L\{m\} \\ s^p F(s) -_0 D_t^{p-1} f(0) - rF(s) &= -\frac{m}{s}. \end{aligned}$$

We assume $_0D_t^{p-1}f(0)$ exists and call it c. Now we are able to solve for $F(s)$ and obtain

$$F(s) = \frac{c}{s^p - r} - \frac{m}{s(s^p - r)}.$$

Using Table 1 ([10], p 23) we take the inverse Laplace transform on both sides and get

$$f_p(\tau) = c\tau^{p-1} E_{p,p}(r\tau^p) - m\tau^p E_{p,p+1}(r\tau^p). \tag{2.23}$$

Using the fact that [10]

$$\lim_{\tau \to 0^+} \tau^{p-1} E_{p,p}(r\tau^p) = 1,$$

and

$$\lim_{\tau \to 0^+} \tau^p E_{p,p+1}(r\tau^p) = 0.$$

Therefore, if we evaluate expression (2.23) in $\tau = 0$ we get

$$f_p(0) = c.$$

Since $f_p(\tau)$ denotes the remaining debt at the end of month τ, the last expression can be seen as the debt at the beginning, which is equal to the price of the product **b euro**. So we have $b = f_p(0) = c$ and we can rewrite (2.23) as

$$f_p(\tau) = b\tau^{p-1} E_{p,p}(r\tau^p) - m\tau^p E_{p,p+1}(r\tau^p). \tag{2.24}$$

Chapter 3

Stochastic Fractional Differential Equations

The use of fractional orders differential and integral operators in mathematical models has become increasingly widespread in recent year. Several forms of fractional stochastic differential equations have been proposed in standard models and there has been of significant interest in studying their solution. In this chapter we shall discuss the global existence and uniqueness of solution of a class of a Caputo fractional order $\alpha \in (\frac{1}{2}, 1)$ stochastic differential equations, using a temporally weighted norm and whose coefficients satisfy a standard Lipschitz condition. The main reference of this chapter is [1, 3, 18].

3.1 Preliminaries

Consider a Caputo fractional stochastic differential equation (for short Caputo FSDE) of order $\alpha \in (\frac{1}{2}, 1)$ of the following form

$$^{C}D^{\alpha}_{0+}X(t) = b(t, X(t)) + \sigma(t, X(t))\frac{dB_t}{dt}, \tag{3.1}$$

where $b, \sigma : [0, \infty)\times\mathbb{R}^d \to \mathbb{R}^d$, are measurable and $(B_t)_{t\in[0,\infty)}$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t := \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$. For each $t \in [0, \infty)$, let $\mathfrak{X}_t := \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ denote the space of all $\mathcal{F}_t-$ measurable, mean square integrable functions $f = (f_1, , ..., f_d)^T : \Omega \to \mathbb{R}^d$ with

$$\|f\|_{ms} := \sqrt{\sum_{i=1}^{d} \mathbb{E}(|f_i|^2)} = \sqrt{\mathbb{E}\|f\|^2},$$

where $\mathbb{R}^d$ is endowed with the standard Euclidean norm.

A process $X : [0,\infty) \to \mathbb{L}(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathcal{F}_t-$ adapted if $X(t) \in \mathfrak{X}_t$ for all $t \geq 0$.

Definition 3.1.1. *For each $\eta \in \mathfrak{X}_0$ a $\mathcal{F}_t-$adapted process X is called a solution of (3.1) with the initial condition $X(0) = \eta$ if the following equality holds for $t \in [0,\infty)$,*

$$X(t) = \eta + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-\tau)^{\alpha-1} b(\tau, X(\tau)) d\tau + \int_0^t (t-\tau)^{\alpha-1} \sigma(\tau, X(\tau)) dB_\tau \right), \quad (3.2)$$

where $\Gamma(\alpha)$ is the Gamma function.

3.2 The main result

In the remaining of this section, we assume that the coefficients b and σ satisfy the following standard conditions

- (**H.1**) There exists L > 0 such that for all $x, y \in \mathbb{R}^d$, $t \in [0,\infty)$,

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq L\|x-y\|.$$

- (**H.2**) $\sigma(.,0)$ is essentially bounded, i.e.

$$\|\sigma(.,0)\|_\infty := esssup_{\tau \in [0,\infty)} \|\sigma(\tau,0)\| < \infty,$$

and $b(.,0)$ is $\mathbb{L}^2$ integrable, i.e.

$$\int_0^\infty \|b(\tau,0)\|^2 d\tau < \infty.$$

The main result in this chapter is to show the global existence and uniqueness solutions of (3.1) when (**H.1**) and (**H.2**) hold. Furthermore, we also show the continuity dependence of solutions on the initial values.

Theorem 3.2.1. *Suppose that (**H.1**) and (**H.2**) hold. Then*

- *(i) for any $\eta \in \mathfrak{X}_0$, the initial value problem (3.1) with the initial condition $X(0) = \eta$ has a unique global solution on the whole interval $[0, \infty)$ denoted by $\varphi(., \eta)$.*
- *(ii) on any bounded time interval $[0, T]$, where $T > 0$, the solution $\varphi(., \eta)$ depends continuously on η, i.e.*

$$\lim_{\zeta \to \eta} \sup_{t \in [0,T]} \| \varphi(t, \zeta) - \varphi(t, \eta) \|_{ms} = 0.$$

3.3 Proof of the main result

Our aim in this section is to prove the result on global existence, uniqueness and continuity dependence on the initial values of solutions to the equation (3.1). In fact, in order to prove the Theorem 3.2.1 it is equivalent to show the existence and uniqueness solutions on an arbitrary interval $[0, T]$, where T > 0 is arbitrary. In what follows we choose and fix a $T > 0$ arbitrarily. Let $\mathbb{H}^2([0, T])$ be the space of all $\mathcal{F}_t$-adapted and measurable processes X such that

$$\| X \|_{\mathbb{H}^2} =: \sup_{0 \leq t \leq T} \| X(t) \|_{ms} < \infty.$$

Obviously, $\mathbb{H}^2([0, T], \| \, . \, \|_{\mathbb{H}^2})$, is a Banach space. For any $\eta \in \mathfrak{X}_0$, we define an operator $\tau_\eta : \mathbb{H}^2([0, T]) \to \mathbb{H}^2([0, T])$ by

$$\tau_\eta \xi(t) = \eta + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t - \tau)^{\alpha - 1} b(\tau, \xi(\tau)) d\tau + \int_0^t (t - \tau)^{\alpha - 1} \sigma(\tau, \xi(\tau)) dB_\tau \right). \tag{3.3}$$

The following lemma is devoted to showing that this operator is well-defined.

Lemma 3.3.1. *For any $\eta \in \mathfrak{X}_0$, the operator τ_η is well-defined.*

Proof: Let $\xi \in \mathbb{H}^2([0, T])$ be arbitrary. From the definition of $\tau_\eta \xi$ as in (3.3) and the inequality $\|x + y + z\|^2 \leq 3(\|x\|^2 + \|y\|^2 + \|z\|^2)$ for all $x, y, z \in \mathbb{R}^d$, we have for all

$t \in [0,T]$,

$$\begin{aligned} \|\tau_\eta \xi(t)\|^2_{ms} &\leq \|3\eta\|^2_{ms} + \frac{3}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\| \int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau \right\|^2\right) \\ &+ \frac{3}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\| \int_0^t (t-\tau)^{\alpha-1} \sigma(\tau,\xi(\tau))dB_\tau \right\|^2\right). \end{aligned} \tag{3.4}$$

By the Hölder inequality, we obtain

$$\begin{aligned} \mathbb{E}\left(\left\| \int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau \right\|^2\right) &\leq \int_0^t (t-\tau)^{2\alpha-2} d\tau \mathbb{E}\left(\int_0^t \|b(\tau,\xi(\tau))\|^2 d\tau\right) \\ &= \frac{t^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\int_0^t \|b(\tau,\xi(\tau))\|^2 d\tau\right). \end{aligned} \tag{3.5}$$

From (**H.1**), we derive

$$\begin{aligned} \|b(\tau,\xi(\tau))\|^2 &\leq 2(\|b(\tau,\xi(\tau)) - b(\tau,0)\|^2 + \|b(\tau,0)\|^2) \\ &\leq 2L^2\|\xi(\tau)\|^2 + 2\|b(\tau,0)\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}\left(\int_0^t \|b(\tau,\xi(\tau))\|^2 d\tau\right) &\leq 2L^2\mathbb{E}\left(\int_0^t \|\xi(\tau)\|^2 d\tau\right) + 2\int_0^t \|b(\tau,0)\|^2 d\tau \\ &\leq 2L^2 T \sup_{t\in[0,T]} \mathbb{E}(\|\xi(t)\|^2) + 2\int_0^T \|b(\tau,0)\|^2 d\tau, \end{aligned}$$

which together with (3.5) implies that

$$\mathbb{E}\left(\left\| \int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau \right\|^2\right) \leq \frac{2L^2T^{2\alpha}}{2\alpha-1}\|\xi\|^2_{\mathbb{H}^2} + \frac{2T^{2\alpha-1}}{2\alpha-1}\int_0^T \|b(\tau,0)\|^2 d\tau. \tag{3.6}$$

Now, using Itô's isometry, we obtain

$$\begin{aligned} \mathbb{E}\left(\left\| \int_0^t (t-\tau)^{\alpha-1} \sigma(\tau,\xi(\tau))dB_\tau \right\|^2\right) &= \sum_{1\leq i\leq d} \mathbb{E}\left(\int_0^t (t-\tau)^{\alpha-1}\sigma_i(\tau,\xi(\tau))dB_\tau\right)^2 \\ &= \sum_{1\leq i\leq d} \mathbb{E}\left(\int_0^t (t-\tau)^{2\alpha-2} \mid \sigma_i(\tau,\xi(\tau)) \mid^2 d\tau\right) \\ &= \mathbb{E}\int_0^t (t-\tau)^{2\alpha-2}\|\sigma(\tau,\xi(\tau))\|^2 d\tau. \end{aligned}$$

From (**H.1**), we also have

$$\|\sigma(\tau,\xi(\tau))\|^2 \leq 2L^2\|\xi(\tau)\|^2 + 2\|\sigma(\tau,0)\|^2 \leq 2L^2\|\xi(\tau)\|^2 + 2\|\sigma(.,0)\|^2_\infty.$$

Therefore, for all $t \in [0, T]$ we have

$$\begin{aligned}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,\xi(\tau))dB_\tau\right\|^2\right) &\le 2L^2\mathbb{E}\int_0^t (t-\tau)^{2\alpha-2}\|\xi(\tau)\|^2 d\tau \\ &+ 2\|\sigma(.,0)\|_\infty^2 \int_0^t (t-\tau)^{2\alpha-2} d\tau \\ &\le 2L^2\frac{T^{2\alpha-1}}{2\alpha-1}\|\xi\|^2_{\mathbb{H}^2} + \frac{2T^{2\alpha-1}}{2\alpha-1}\|\sigma(.,0)\|_\infty^2.\end{aligned}$$

This together with (3.4) and (3.6) implies that $\|\tau_\eta\xi\|_{\mathbb{H}^2} < \infty$. Hence, the map τ_η is well-defined.

To prove existence and uniqueness of solutions, we will show that the operator τ_η defined as above is contractive under a suitable temporally weighted norm (for the same method to prove the existence and uniqueness of solutions of stochastic differential equations). Here, the weight function is the Mittag-Leffler function $E_{2\alpha-1}(.)$ defined as

$$E_{2\alpha-1}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma((2\alpha-1)k+1)}, \quad for\ all \quad t \in \mathbb{R}.$$

Lemma 3.3.2. *For any $\alpha > \frac{1}{2}$ and $\gamma > 0$, the following inequality holds*

$$\frac{\gamma}{\Gamma(2\alpha-1)}\int_0^t (t-\tau)^{2\alpha-2}E_{2\alpha-1}(\gamma\tau^{2\alpha-1})d\tau \le E_{2\alpha-1}(\gamma t^{2\alpha-1}).$$

Proof: Let $\gamma > 0$ be arbitrary. Consider the corresponding linear Caputo fractional differential equation of the following form

$$^{c}D_{0+}^{2\alpha-1}x(t) = \gamma x(t). \tag{3.7}$$

The Mittag-Leffler function $E_{2\alpha-1}(\gamma t^{2\alpha-1})$ is a solution of (3.7). Hence, the following equality holds

$$E_{2\alpha-1}(\gamma t^{2\alpha-1}) = 1 + \frac{\gamma}{\Gamma(2\alpha-1)}\int_0^t (t-\tau)^{2\alpha-2}E_{2\alpha-1}(\gamma\tau^{2\alpha-1})d\tau,$$

which completes the proof.

Proof of The Theorem 3.2.1: Let $T > 0$ be arbitrary. Choose and fix a positive constant γ such that

$$\gamma > \frac{3L^2(T+1)\Gamma(2\alpha-1)}{\Gamma(\alpha)^2}. \tag{3.8}$$

On the space $\mathbb{H}^2([0,T])$, we define a weighted norm $\|.\|_\gamma$ as below

$$\|X\|_\gamma := \sup_{t\in[0,T]}\sqrt{\frac{\mathbb{E}(\|X(t)\|^2)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}}, \qquad for \quad all \quad X \in \mathbb{H}^2([0,T]). \tag{3.9}$$

Obviously, two norms $\|.\|_{\mathbb{H}^2}$ and $\|.\|_\gamma$ are equivalent. Thus, $(\mathbb{H}^2(0,T), \|.\|_\gamma)$ is also a Banach space.

- Choose and fix $\eta \in \mathfrak{X}_0$. By virtue of Lemma 3.3.1, the operator τ_η is well defined. We will prove that the map τ_η is contractive with respect to the norm $\|.\|_\gamma$.

  For this purpose, let $\xi, \hat{\xi} \in \mathbb{H}^2([0,T])$ be arbitrary. From (3.3) and the inequality $\|x+y\|^2 \le 2(\| x \|^2 + \| y \|^2)$ for all $x, y \in \mathbb{R}^d$, we derive the following inequalities for all $t \in [0,T]$,

$$\begin{aligned}\mathbb{E}\left(\left\|\tau_\eta\xi(t) - \tau_\eta\hat{\xi}(t)\right\|^2\right) &\le \frac{2}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(b(\tau,\xi(t)) - b(\tau,\hat{\xi}(t)))d\tau\right\|^2\right)\\ &+ \frac{2}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\xi(t)) - \sigma(\tau,\hat{\xi}(t)))dB_\tau\right\|^2\right).\end{aligned}$$

  Using the Hölder inequality and (**H.1**), we obtain

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(b(\tau,\xi(\tau)) - b(\tau,\hat{\xi}(\tau)))d\tau\right\|^2\right)\\ &\le L^2 t\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\xi(\tau)-\hat{\xi}(\tau)\|^2)d\tau.\end{aligned}$$

  On the other hand, by Itô's isometry and (H1), we have

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\xi(\tau)) - \sigma(\tau,\hat{\xi}(\tau)))dB_\tau\right\|^2\right)\\ &= \mathbb{E}\int_0^t (t-\tau)^{2\alpha-2}\|\sigma(\tau,\xi(\tau)) - \sigma(\tau,\hat{\xi}(\tau))\|^2 d\tau\\ &\le L^2\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\xi(\tau)-\hat{\xi}(\tau))\|^2)d\tau.\end{aligned}$$

  Thus, for all $t \in [0,T]$ we have

$$\mathbb{E}\left(\left\|\tau_\eta\xi(t) - \tau_\eta\hat{\xi}(t)\right\|^2\right) \le \frac{2L^2(t+1)}{\Gamma(\alpha)^2}\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\xi(\tau)-\hat{\xi}(\tau))\|^2)d\tau,$$

  which together with the definition of $\|.\|_\gamma$ as in (3.9) implies that

$$\frac{\mathbb{E}\left(\left\|\tau_\eta\xi(t) - \tau_\eta\hat{\xi}(t)\right\|^2\right)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \le \frac{2L^2(t+1)}{\Gamma(\alpha)^2}\frac{\int_0^t (t-\tau)^{2\alpha-2}E_{2\alpha-1}(\gamma t^{2\alpha-1})d\tau}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\|\xi-\hat{\xi}\|^2{}_\gamma.$$

In light of Lemma 3.3.2, we have for all $t \in [0,T]$,

$$\frac{\mathbb{E}\left(\left\|\tau_\eta\xi(t) - \tau_\eta\hat{\xi}(t)\right\|^2\right)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \leq \frac{2\Gamma(2\alpha-1)L^2(T+1)}{\Gamma(\alpha)^2\gamma}\|\xi - \hat{\xi}\|^2{}_\gamma.$$

Consequently,

$$\|\tau_\eta\xi - \tau_\eta\hat{\xi}\|_\gamma \leq \kappa\|\xi - \hat{\xi}\|_\gamma, \qquad \textit{where} \quad \kappa := \sqrt{\frac{2\Gamma(2\alpha-1)L^2(T+1)}{\Gamma(\alpha)^2\gamma}}.$$

By (3.8), we have $\kappa < 1$ and therefore the operator τ_η is a contractive map on $(\mathbb{H}^2([0,T]), \|.\|_\gamma)$. Using the Banach fixed point theorem, there exists a unique fixed point of this map in $(\mathbb{H}^2([0,T]))$. This fixed point is also the unique solution of (3.1) with the initial condition $X(0) = \eta$. The proof of this part is complete.

- Choose and fix $T > 0$ and $\eta, \zeta \in \mathfrak{X}_0$. Since $\varphi(., \eta)$ and $\varphi(., \zeta)$ are solutions of (3.1) it follows that

$$\begin{aligned}\varphi(t,\eta) - \varphi(t,\zeta) = \eta - \zeta \quad &+ \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(b(\tau,\varphi(\tau,\eta)) - b(\tau,\varphi(\tau,\zeta)))d\tau \\ &+ \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\varphi(\tau,\eta)) - \sigma(\tau,\varphi(\tau,\zeta)))dB_\tau.\end{aligned}$$

Hence, using the inequality $\|x+y+z\|^2 \leq 3(\|x\|^2 + \|y\|^2 + \|z\|^2)$ for all $x,y,z \in \mathbb{R}^d$ and using (**H.1**), the Hölder inequality and It$\hat{o}$'s isometry, we obtain

$$\begin{aligned}\mathbb{E}(\|\varphi(t,\eta) - \varphi(t,\zeta)\|^2) \leq \frac{3L^2(t+1)}{\Gamma(\alpha)^2}&\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\varphi(t,\eta) - \varphi(t,\zeta)\|^2)d\tau \\ &+3\mathbb{E}(\|\eta-\zeta\|^2).\end{aligned}$$

By definition of $\|.\|_\gamma$, we have

$$\begin{aligned}\frac{\mathbb{E}(\|\varphi(t,\eta) - \varphi(t,\zeta)\|^2)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \quad \leq \quad &\frac{3L^2(t+1)}{\Gamma(\alpha)^2}\frac{\int_0^t (t-\tau)^{2\alpha-2}E_{2\alpha-1}(\gamma\tau^{2\alpha-1})d\tau}{E_{2\alpha-1}(\gamma\tau^{2\alpha-1})} \times \\ &\|\varphi(.,\eta) - \varphi(.,\zeta)\|^2{}_\gamma + 3\mathbb{E}(\|\eta-\zeta\|^2).\end{aligned}$$

By virtue of Lemma 3.3.2, we have

$$\|\varphi(.,\eta) - \varphi(.,\zeta)\|^2{}_\gamma \leq \frac{3L^2(T+1)\Gamma(2\alpha-1)}{\gamma\Gamma(\alpha)^2}\|\varphi(.,\eta) - \varphi(.,\zeta)\|^2{}_\gamma + 3\|\eta-\zeta\|^2{}_{ms}.$$

Thus, by (3.8) we have

$$\left(1 - \frac{3L^2(T+1)\Gamma(2\alpha-1)}{\gamma\Gamma(\alpha)^2}\right)\|\varphi(.,\eta) - \varphi(.,\zeta)\|^2{}_\gamma \leq 3\|\eta-\zeta\|^2{}_{ms}.$$

Hence,

$$\lim_{\eta \to \zeta} \sup_{t \in [0,T]} \|\varphi(t,\eta) - \varphi(t,\zeta)\|_{ms} = 0.$$

The proof is complete.

Conclusion

Our main goal in this work is the study of the existence and the uniqueness of solution for a class of fractional stochastic differential equation.

First, we discussed some fundamental notions of stochastic processes and stochastic integration as well as stochastic differential equations. Next, we introduced the concept of fractional calculus; the branch of mathematics which explores fractional integrals and derivatives. Although the history of fractional calculus is three hundred years old, it is still receiving great interest and acceptance from the research community. In the recent years new alternative definitions of fractional operators have been introduced in the literature: Coimbra derivative (2003), Jumarie derivative (2006), Chen derivative (2010), local fractional Yang derivative (2012).

At last we investigated a global result on the existence and uniqueness of solutions for Caputo fractional stochastic differential equations of order $\alpha \in (1/2; 1)$ whose coefficients satisfy a standard Lipschitz condition, and using a temporally weighted norm called Bielecki norm. With respect to this norm, it was proved that the operator associated with the stochastic integral equation is globally contractive and its fixed point gives rise to the appropriate global solution of the system. Furthermore, we also show that the solutions depend continuously on the initial values.

Appendix

I. Banach space

A Banach space is a complete, normed, vector space.

II. Contraction mapping

A map $T : X \to X$ is a contraction if for some $k \in (0, 1)$,

$$\|Tx - Ty\| \leq k\|x - y\|, \qquad for\ all \quad x, y \in X.$$

The Contraction mapping theorem

Let X be a Banach space and T be a contraction mapping. Then has an unique fixed point.

III. Fixed point

If $T : X \to X$, then a point $x \in X$ such that

$$T(x) = x,$$

is called a fixed point of T.

IV. Essentially bounded function

The function f is said to be essentially bounded when the function $x \mapsto |f(x)|$ has an almost upper bound. We then note $\|f\|_\infty = \operatorname{ess\,sup} |f|$.

V. Almost upper bound

Let a measurable space $(X, \mathcal{A}, \mu)$ and f a function on X with real values. A real a is called an almost upper bound of f if $f(x) \leq a$ for almost every element x of X, in other words, if the set $\{x \in X \mid f(x) > a\}$ is negligible.

VI. Weighted Norm

A weighted norm is a finite norm that involves multiplication by a particular function referred to as the weight.

For example, in the space of functions from a set $U \subset \mathbb{R}$ to $\mathbb{R}$ under the norm $\|\cdot\|_U$ defined by: $\|f\|_U = \sup\limits_{x\in U} |f(x)|$, functions that have infinity as a limit point are excluded. However, the weighted norm $\|f\| = \sup\limits_{x\in U} \left| f(x)\frac{1}{1+x^2} \right|$ is finite for many more functions, so the associated space contains more functions.

VII. Hölder Inequality

Let $1 \leq p \leq \infty$, q be the conjugate exponent of p i.e $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p$ and $g \in L^q$ then $f.g \in L^1$ and $\|f.g\| \leq \|f\|_p \|g\|_q$.

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