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Stochastic Analysis Of Some General Fractional Stochastic Processes

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"***I have always sought knowledge and truth, and believed that in order to approach God, it is no better here than to seek knowledge and truth ..***" [Ibn al-Haytham]

" ***The study of mathematics, like the Nile, begins in minuteness but ends in magnificence.***"[Charles Caleb Colton]

"***Go down deep enough into anything and you will find mathematics ..***" [Dean Schlicter]

Dedication

*All praise to **Allah**, today we fold the day's tiredness and the errand summing up between the cover of this humble work.*

I dedicate my work to:

*My great teacher and messenger, **Mohammed-peace and grace from Allah be upon him**,* who taught us the purpose of life, My homeland Palestine

***My parents**, who have been our source of inspiration and gave us strength when we thought of giving up, who continually provide their moral, spiritual, emotional, and financial. God save them.*

My grandmother, God rest her soul.

*Whose are a freind to the spirit and a gift to the heart **Akram, Naima**.*

*My aunt **Aicha**, her hasband **Mohammed** and her children **Assia** and **Sami**.*

My uncles and Cousins.

*To my close friend **Ikram Aissani**, **Esma Ouis** and **Bouchra Bendjebara** to my teacher and freind **Ikhlass Belabbas** and their famillies.*

*Last but not least I am dedicating this to my chikha **Amina Lytim**, gone forever away from our loving eyes and who left a void never to be filled in our lives. Though your life was short, I will make sure your memory lives on as long as I shall live.*

All those if my pen forget them, my heart will not forgotten them.

All those who are looking glory and pride in Islam and nothing else .

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Contents

Abstract

In the present master thesis, we seek to introduce the mean properties of the grey Brownian motion and to study the stochastic differential equations driven by generalized grey noise.

First, we give some preliminary background of stochastic calculus from the notion of stochastic processes and stochastic integration to stochastic differential equations. Then we give an overview of the theory of fractional calculus. We wanted to study a parametric class of stochastic processes to model both fast and slow anomalous diffusion. This class called generalized grey Brownian motion (ggBm) which is defined canonically in the so called grey noise space, then we show that it is possible to define it in an unspecified probability space. The ggBm is made up of self-similar with stationary increments processes (H-sssi) and depends on two real parameters $\alpha \in (0, 2)$ and $\beta \in (0, 1]$. It includes fractional Brownian motion when $\alpha \in (0, 2)$ and $\beta = 1$, and standart Brownian motion when $\alpha = \beta = 1$. Then we establish a substitution formula for stochastic differential equation driven by generalized grey noise.

Key words: Standard Brownian motion. Fractional Brownian motion. Grey noise. Grey Brownian motion. Stochastic differential equations. Fractional calculus.

Résumé

Dans ce travail, nous cherchons à présenter les propriétés du mouvement Brownien gris, et à étudier les équations différentielles stochastiques dirigées par le bruit gris généralisé.

Premièrement, nous donnons quelques notions préliminaires sur le calcul stochastique partons de la notion de processus stochastiques et l'intégration stochastique afin de résoudre des équations différentielles stochastique, puis nous donnons un aperçu sur la théorie du calcul fractionnaire. Nous voulions étudier une classe paramétrique de processus stochastiques afin de modéliser une diffusion anormale rapide et lente. Cette classe appelée mouvement Brownien gris généralisé (mBgg) qui est définie canoniquement dans l'espace de bruit gris, alors nous montrons qu'il est possible de le définir dans un espace de probabilité non spécifié. Le mBgg est constitué de processus auto-similaires avec des incréments stationnaires (H-asas) et dépend de deux paramètres réels $\alpha \in (0,2)$ et $\beta \in (0,1]$. Il comprend un mouvement Brownien fractionnaire lorsque $\alpha \in (0,2)$ et $\beta = 1$ et le mouvement Brownien standart lorsque $\alpha = \beta = 1$. Ensuite, nous établissons une formule de substitution pour les équations différentielles stochastiques dirigées par le bruit gris généralisé.

Mots clés: Mouvement Brownien standard. Mouvement Brownien fractionnaire. Bruit gris. Mouvement Brownien gris. Equation différentielle stochastique. Calcul fractionnaire.

Introduction

In recent years fractional Brownian motion and processes related to fractional dynamics have become an object of intensive study. Mathematically these processes in general lack both the Markov and the semi-martingale property, so that many of the classical methods from stochastic analysis do not apply, making their analysis more challenging. These processes are capable of modeling systems with long-range self interaction and memory effects by using fractional differential equations.

In 1992 Schneider introduced the notion of grey Brownian motion [31] in order to solve Caputo fractional differential equation. In the 90's Mainardi and al [11] started a systematic study of fractional differential equations, and introduced the generalized grey Brownian motion (ggBm for short). Mura, Mainardi [23] studied a class of self-similar stochastic processes with stationary increments to model anomalous diffusion in physics and they investigate the class of grey Brownian motion [6]. In 2018, Da Silva and Erraoui [8] studied the singularity of generalized grey Brownian motions with different parameters.

On the other hand, the field of stochastic differential equations and its applications has gained a lot of importance during the past three decades, mainly because it has become a powerful tool in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering [26]. Recently, there has been a significant development in the existence and uniqueness of solution of stochastic differential equations. For more details on this topic, see for example the book of [26].

In contrast to the much attention attracted by stochastic differential equations and stochastic differential equations driven by fractional Brownian motion, stochastic differential equations driven by generalized grey Brownian motion is an unknown new subject in the literature and there are only a few papers published in this field. Da Silva and Erraoui [7] studied the existence and the upper bound for the density of solutions of stochastic differential equations driven by generalized grey noise. Bock and Da Silva [3] derived solution of a linear Wick-type stochastic differential equation (SDE) driven by grey Brownian motion in a suitable distribution space.

This Master thesis aims on the one hand, to answer the mentioned above questions and to construct a comprehensive picture of what grey Brownian motion is? and secondly, the study of a class of stochastic differential equations deriven by grey noise.

This master thesis consists of three chapters.

The first one is devoted to stochastic calculus. After giving a background on stochastic calculus starting from the notion of stochastic processes. We move to the definition of stationary processes, self-similarity, $H-$sssi processes. As an example we give the mathematical definition of Brownian motion and its properties. After that we introduce Itô calculus and we finish the chapter by a brief overview of stochastic differential equations.

The Seconde chapter is devoted to a brief summary of the theory of fractional calculus. We wanted to make an overview of the concepts, basic properties, in the same time we give a general review on the fractional Brownian motion, we start by mathematical definition then the main properties of this stochastic processes and we finish this chapter by the main representation of fractional Brownian motion.

The Third chapter is the essence of this master thesis its deal to the stochastic differential equation via generalized grey Brownian motion, at first we present the generalized grey Brownian motion it's definition, properties then we present a global result about a substituation formula for a class of stochastic differential equation.

Finally, we give a conclusion. In witch we summarize the main results of this work.

Chapter 1

Preliminary Background

This chapter provides theoretical basis for this work. Definitions, properties of stochastic processes are discussed and some of concepts are clarified, such as self-similarity, H-sssi. There exists a vast literature that treats stochastic process. For more detail, we refer the reader to [23, 14, 28, 15, 10, 33].

1.1 Basic definitions

In this section we introduce some basic concepts concerning continuous time stochastic processes used freely later on. Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and recall that a map $Z : \Omega \to \mathbb{R}$ is called a ***random variable*** if Z is measurable as a map from $(\Omega, \mathcal{F})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$.

What is a continuous time stochastic process? For us it is simply a family of random variables:

Definition 1.1.1. *Let $I = [0, T]$ for some $T \in (0, \infty)$ or $I = [0, \infty)$. A family of random variables $X = (X_t)_{t\in I}$ with $X_t : \Omega \longrightarrow \mathbb{R}$ is called* ***stochastic process*** *with index set I.*

The definition of a stochastic process can be given more generally by allowing more general I and other state spaces than $\mathbb{R}$. In our case there are two different views on the stochastic process X:

- The family $X = (X_t)_{t\in I}$ describes random functions by $\omega \mapsto f(\omega) = (X_t(\omega))_{t\in I}$. The function $t \mapsto X_t(\omega)$ is called ***path*** or ***trajectory*** of X.

- The family $X = (X_t)_{t\in I}$ describes a process, which is, with respect to the time variable t, an ordered family of random variables $t \mapsto X_t$.

The two approaches differ by the roles of ω and t.

Definition 1.1.2. *Let $X = (X_t)_{t\in I}$ and $Y = (Y_t)_{t\in I}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. The processes X and Y are **versions** or **(modifications)** of each other provided that*

$$\mathbb{P}(X_t = Y_t) = 1, \quad for\ all\ t \in I.$$

Definition 1.1.3. *Let $X = (X_t)_{t\in I}$ and $Y = (Y_t)_{t\in I}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. The processes X and Y are **indistinguishable** if and only if*

$$\mathbb{P}(X_t = Y_t, \quad for\ all\ t \in I) = 1.$$

Definition 1.1.4. ***The finite-dimensional distributions*** *of the real valued stochastic process $X_t = \{X_t\}_{t\in I}$ are the measures $\mu_{t_1,\dots,t_k}$, defined on $\mathbb{R}^k$, such that*

$$\mu_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = \mathbb{P}(\{X_{t_1} \in A_1, \cdots, X_{t_k} \in A_k\}), \tag{1.1}$$

where $k \in \mathbb{N}$ and $t_i \in I, i = 1, ..., k$ a time sequence, $\{A_1, \dots, A_k\}$ are Borel sets on $\mathbb{R}$.

Theorem 1.1.1. *Let $X = (X_t, t \in I)$ and $Y = (Y_t, t \in I)$ be two real valued processes. Then X and Y have the same distribution (or law) if and only if their finite-dimensional distributions agree.*

Proof: See (Watkins, [33]).

Theorem 1.1.2. *([26]) For all $\{t_i\}_{i=1,\dots,k} \subset I$, $k \in \mathbb{N}$ let $\nu_{t_1,\dots,t_k}$ be probability measures on $\mathbb{R}^k$, such that :*

1. *for all permutations π on $\{1, 2, \dots, k\}$,*

$$\nu_{t_{\pi(1)},\dots,t_{\pi(k)}}(A_1 \times ... \times A_k) = \nu_{t_1,\dots,t_k}(A_{\pi^{-1}(1)} \times \cdots \times A_{\pi^{-1}(k)}),$$

2. *for any $m \in \mathbb{N}$,*

$$\nu_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = \nu_{t_1,\dots,t_k,t_{k+1},\dots,t_{k+m}}(A_1 \times \cdots \times A_k \times \mathbb{R} \times \cdots \times \mathbb{R}).$$

Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a real valued stochastic process X defined on it, such that:

$$\nu_{t_1,\ldots,t_k}(A_1 \times \ldots \times A_k) = \mathbb{P}(\{X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k\}),$$

for any $\{t_i\}_{i=1,\ldots,k} \subset I$, $k \in \mathbb{N}$ and $A_i \in \mathcal{B}$.

Definition 1.1.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family of σ-algebras $\mathcal{F}_t = (\mathcal{F}_t)_{t\in I}$ is called **filtration** if $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $0 \leq s \leq t \in I$.*

The quadruple $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in I})$ is called stochastic basis.

Definition 1.1.6. *Let $X = (X_t)_{t\in I}, X_t : \Omega \to \mathbb{R}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{t\in I}$ be a filtration.*

- *The process X is called **measurable** provided that the function $(\omega, t) \to X_t(\omega)$ considered as map between $\Omega \times I$ and $\mathbb{R}$ is measurable with respect to $\mathcal{F} \otimes \mathcal{B}(I)$ and $\mathcal{B}(\mathbb{R})$.*
- ***Adapted** to the filtration $(\mathcal{F}_t)_{t\in I}$ if for each $t \in I$ one has that $X(t)$ is $\mathcal{F}_t$-measurable.*

Let us recall the notion of a martingale.

Definition 1.1.7. *Let $X = (X_t)_{t\in I}$ be $(\mathcal{F}_t)_{t\in I}$-adapted and such that $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$, X is called **martingale** provided that for all $0 \leq s \leq t \in I$, one has*

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s, \qquad a.s.$$

Definition 1.1.8. *Let $X = (X_t)_{t\in I}$ be a stochastic process. The process X is **continuous** provided that $t \mapsto X_t(\omega)$ is continuous for all $\omega \in \Omega$.*

Gaussian processes form a class of stochastic processes used in several branches in pure and applied mathematics.

Definition 1.1.9. *A real-valued stochastic process is called **Gaussian** of all its finite-dimensional distributions are Gaussian, in other words, if they are multivariate normal distributions.*

1.1.1 Stationary processes

Stationarity is a rather intuitive concept, it means that the statistical properties of the process do not change over time.

Definition 1.1.10. *A stochastic process $\{X_t\}_{t\geq 0}$ is said a* ***stationary process*** *if any collection $\{X_{t_1}, X_{t_2}, \dots, X_{t_n}\}$ has the same distribution of $\{X_{t_1+\tau}, X_{t_2+\tau}, \dots, X_{t_n+\tau}\}$ for all $t \geq 0$ and each $\tau \geq 0$. That is,*

$$\{X_{t_1}, X_{t_2}, \dots, X_{t_n}\} \stackrel{d}{=} \{X_{t_1+\tau}, X_{t_2+\tau}, \dots, X_{t_n+\tau}\}.$$

Definition 1.1.11. *A stochastic process $\{X_t\}_{t\geq 0}$ is said a* ***stationary increment process****, shortly* ***si****, if for all $t \geq 0$ and for any $h \geq 0$:*

$$\{X_{t+h} - X_h\}_{t\geq 0} \stackrel{d}{=} \{X_t - X_0\}_{t\geq 0}. \tag{1.2}$$

1.1.2 Self-similar processes

Self-similar(shortly **ss**) processes, introduced by Lamperti [15, 10], are the ones that are invariant under suitable translations of time and scale. In the last few years there has been an explosive growth in the study of self-similar processes.

Definition 1.1.12. ***(Self-similar processes)*** *A real valued stochastic process $X = \{X_t\}_{t\geq 0}$ is said self-similar with index $H \geq 0$, shortly* ***H-ss****, if for all $t \geq 0$ and for any $a > 0$*

$$\{X_{at}\}_{t\geq 0} \stackrel{d}{=} \{a^H X_t\}_{t\geq 0}.$$

Remark 1.1.1. *Observe that, if X is an* ***H-ss*** *process, then all the finite-dimensional distributions of X in $[0, \infty[$ are completely determined by the distribution in any finite real interval.*

Corollary 1.1.1. *For $H > 0$,* ***H-ss*** *process starts at 0 a.s.*

Proof: We have for all a that $X_0 = X_{a0} \stackrel{d}{\sim} a^H X_0$. Then, letting $a \to 0$, we obtain the result.

Proposition 1.1.1. *Let $X = \{X_t\}_{t\geq 0}$ be a non-degenerate[1] stationary process, then X can not be an* ***H-ss*** *process.*

[1]A process $\{X_t\}_{t\geq 0}$ is said to be degenerate if for any $t \geq 0$, $X_t = 0$ almost surely.

Proof: Indeed, for any $a > 0$:

$$X_t \stackrel{d}{=} X_{at} \stackrel{d}{=} a^H X_t,$$

by stationarity and self-similarity of the process X. Let $a \longrightarrow \infty$. Then the family of random variables on the right diverge with positive probability, whereas the random variable on the left is finite with probability one, leading to a contradiction.

Nevertheless, there is an important connection between self-similar and stationary processes.

Proposition 1.1.2. *Let $\{X_t\}_{t\geq 0}$ be an **H-ss** process; then the process*

$$Y(t) = e^{-tH} X(e^t), \quad t \in \mathbb{R} \tag{1.3}$$

is stationary. We have the converse, in the sense that if $(Y_t)_{t\in\mathbb{R}}$ is stationary, then

$$X_t = t^H Y(ln(t)), \quad t \geq 0 \tag{1.4}$$

is $H-ss$.

Proof: Let $\theta_1, ...\theta_d$ be real numbers. If $\{X(t), 0 < t < \infty\}$ is **H-ss**, then for any $t_1, ..., t_d \in \mathbb{R}^1$ and $h > 0$,

$$\begin{aligned}
\sum_{j=1}^{d} \theta_j Y(t_j + h) &= \sum_{j=1}^{d} \theta_j e^{-t_j H} e^{-hH} X(e^h e^{t_j}) \\
&\stackrel{d}{=} \sum_{j=1}^{d} \theta_j e^{-t_j H} X(e^{t_j}) \\
&= \sum_{j=1}^{d} \theta_j Y(t_j),
\end{aligned}$$

proving that $\{Y(t), t \in \mathbb{R}\}$ is stationary.

Conversely, if $\{Y(t), t \in \mathbb{R}\}$ is stationary, then for $t_1, ..., t_d > 0$ and $a > 0$

$$\begin{aligned}\sum_{j=1}^{d}\theta_j X(at_j) &= \sum_{j=1}^{d}\theta_j a^H t_j^H Y(ln(a)+ln(t_j)) \\ &\stackrel{d}{=} \sum_{j=1}^{d}\theta_j a^H t_j^H Y(ln(t_j)) \\ &= \sum_{j=1}^{d}\theta_j a^H X(t_j),\end{aligned}$$

proving that $\{X(t), t > 0\}$ is **H-ss**.

The transformation defined by (1.3) is called the Lamperti transformation.

1.1.3 H-sssi processes

Definition 1.1.13. *A stochastic process $X = \{X_t\}_{t\in I}$, $\mathcal{F}$-adapted, which is **H-ss** with stationary increments, is said **H-sssi** process with exponent H.*

In the following we always suppose that $\mathbb{E}(X_t^2) < \infty, t \in I$. let $X = \{X_t\}_{t\in I}$, $\mathcal{F}$-adapted, be an **H-sssi** process with finite variance [2], the following properties hold:

1. $X_0 = 0$ almost surely.

2. If $H \neq 1$, then for any $t \geq 0$, $\mathbb{E}(X_t) = 0$.

3. One has:
$$X(-t) \stackrel{d}{=} -X(t),$$
it follows from the first property and the stationarity of the increments:
$$X(-t) \stackrel{a.s.}{=} X(-t) - X(0) \stackrel{d}{=} X(0) - X(t) \stackrel{a.s.}{=} -X(t).$$
The above property allows us to extend the definition of an **H-sssi** process to the whole real line (i.e $\{X_t\}_{t\in\mathbb{R}}$).

4. Let $\sigma^2 = \mathbb{E}(X_1^2)$. Then,
$$\mathbb{E}(X_t^2) = |t|^{2H}\sigma^2. \qquad (1.5)$$

[2] We always consider finite variance **H-sssi** process because it have many interesting properties.

Indeed, from the third property and the self-similarity:

$$\mathbb{E}X(t)^2 = \mathbb{E}X^2(|t|sign(t)) = |t|^{2H}\mathbb{E}X^2(sign(t)) = |t|^{2H}\mathbb{E}(X_1^2) = |t|^{2H}\sigma^2.$$

5. The autocovariance function of an **H-sssi** process [3] X, with $\mathbb{E}(X_1^2) = \sigma^2$, turns out to be:

$$\gamma_{s,t}^H = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \tag{1.6}$$

   It follows from the fourth property and the stationarity of the increments

$$\mathbb{E}(X_sX_t) = \frac{1}{2}(\mathbb{E}X_s^2 + \mathbb{E}X_t^2 - \mathbb{E}(X_t - X_s)^2).$$

6. If $X = \{X_t\}_{t\in I}$ is an **H-sssi** process, then one must have $H \leq 1$.

   The constraint of the scaling exponent follows directly from the stationarity of the increments:

$$2^H\mathbb{E}|X_1| = \mathbb{E}|X_2| = \mathbb{E}|X_2 - X_1 + X_1| \leq \mathbb{E}|X_2 - X_1| + \mathbb{E}|X_1| = 2\mathbb{E}|X_1|,$$

   therefore, $2^H \leq 2 \Longleftrightarrow H \leq 1$.

Remark 1.1.2. *The case $H = 1$ corresponds a.s. to $X_t = tX_1$. Indeed, on the $L^2(\Omega, \mathbb{P})$ norm:*

$$\mathbb{E}(X_t - tX_1)^2 = \mathbb{E}(X_t^2 + t^2X_1^2 - 2tX_tX_1) = \sigma^2(2t^2 - 2t^2) = 0.$$

1.1.4 The basic examples of stochastic processes, The Brownian motion

The most important stochastic process is the Brownian motion. It was first discussed by Louis Bachelier in 1900, and independently by Einstein in his 1905 paper. The modern mathematical treatment of Brownian motion (abbreviated to BM), also called the Wiener process is due to Wiener in 1923, who proved that there exists a version of BM with continuous paths. Note that BM is a Gaussian process, a Markov process and a martingale. Hence its importance in the theory of stochastic process. It serves as a basic building block for many more complicated processes. For further history of Brownian motion and related processes we cite Meyer [19], Klebaner [14] and Pitman [28].

[3] Sometimes, we refer to the **H-sssi** process $\{X_t\}_{t\in I}$ with the word standard if $\sigma^2 = 1$

1.1.4.1 Definition of Brownian Motion

We now start to define and study Brownian motion (Wiener process).

Definition 1.1.14. ***(Brownian motion)*** *A stochastic process $\{B(t), t \geq 0\}$ is said to be a Brownian motion with variance parameter $\sigma^2 > 0$ if:*

(i) $B(0) = 0$.

(ii) ***(Independent increments.)*** *For each $0 \leq t_1 < t_2 < \ldots < t_m$,*

$$B(t_1), B(t_2) - B(t_1), \ldots, B(t_m) - B(t_{m-1}),$$

are independent r.v.'s.

(iii) ***(Stationary increments.)*** *For each $0 \leq s < t, B(t) - B(s)$ has a normal distribution with mean zero and variance $\sigma^2(t - s)$.*

(iv) ***(Continuity of paths.)*** *$\{B(t)\}_{t \geq 0}$ are continuous functions of t.*

Remark 1.1.3.
- *Notice that the natural filtration of the Brownian motion is $\mathcal{F}_t^B = \sigma(B_s, s \leq t)$.*
- *If $\sigma^2 = 1$, we said that $\{B(t) : t \geq 0\}$ is a standard Brownian motion.*

1.1.4.2 Properties of Brownian motion

1- Martingale property

A martingale is a very special type of stochastic process.

Lemma 1.1.1. *An $\mathcal{F}_t$-Wiener process B_t is an $\mathcal{F}_t$-martingale.*

Proof: We need to prove that $\mathbb{E}(B_t|\mathcal{F}_s) = B_s$ for any $t > s$. But as B_s is $\mathcal{F}_s$-measurable (by adaptedness) this is equivalent to $\mathbb{E}(B_t - B_s|\mathcal{F}_s) = 0$, and this is clearly true by the definition of the Wiener process (as $B_t - B_s$ has zero mean and is independent of $\mathcal{F}_s$).

2- Markov property

The reason why Markov processes are so important comes from the fact that they are fundamental class of stochastic processes, with many applications in real life problems outside mathematics.

Definition 1.1.15. *An $\mathcal{F}_t$ adapted process X_t is called an $\mathcal{F}_t$-Markov process if we have $\mathbb{E}(f(X_t)|\mathcal{F}_s) = \mathbb{E}(f(X_t)|X_s)$ for all $t \geq s$ and all bounded measurable functions f. When the filtration is not specified, the natural filtration $\mathcal{F}_t^X$ is implied.*

Lemma 1.1.2. *An $\mathcal{F}_t$-Wiener process B_t is an $\mathcal{F}_t$-Markov process.*

Proof: We refer the reader to $(Klebaner, [14])$.

3- Self-similarity

Theorem 1.1.3. *B is an **H-ss** process with $H = 1/2$.*

Proof: It is enough to show that for every $a > 0, \{a^{1/2}B(t)\}$ is also Brownian motion. Conditions (i), (ii) and (iv) follow from the same conditions for $\{B(t)\}$. As to (iii), Gaussianity and mean-zero property also follow from the properties of $\{B(t)\}$.
As to the variance, $\mathbb{E}\left[\left(a^{1/2}B(t)^2\right)\right] = t$. And for all $t_1, t_2 \in \mathbb{R}$, the autocovariance function $\mathbb{E}\left[(B(at_1)B(at_2))\right] = \min(at_1, at_2) = a\min(t_1, t_2) = \mathbb{E}\left[\left(a^{1/2}B(t_1)a^{1/2}B(t_2)\right)\right]$. Thus $\{a^{1/2}B(t)\}$ is a Brownian motion.

4- Non-differentiability

Theorem 1.1.4. *For any t almost all trajectories of Brownian motion are not differentiable at t.*

Proof: We refer the reader to (Klebaner, [14]).

5- Hölder continuity

Proposition 1.1.3. *Brownian motion paths are a.s locally γ-Hölder continuous for $\gamma \in [0, 1/2)$.*

Proof: We refer the reader to (Klebaner, [14]).

6- Quadratic variation

Definition 1.1.16. *The quadratic variation of Brownian motion $B(t)$ is defined as*

$$[B,B](t) = [B,B]([0,t]) = \lim_{n\to\infty} \sum_{i=1}^{n} \left| B_{t_i^n} - B_{t_{i-1}^n} \right|^2,$$

where for each n, $\{t_i^n, 0 \leq i \leq n\}$ is a partition of $[0,t]$, and the limit is taken over all partitions with $\delta_n = \max_i(t_{i+1}^n - t_i^n) \to 0$ as $n \to \infty$, and in the sense of convergence in probability.

Theorem 1.1.5. *(Klebaner, [14]). Quadratic variation of a Brownian motion over $[0,t]$ is t.*

1.2 Introduction to stochastic integration

Let us consider the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of the Bm $B(t)$, $t \geq 0$.

Definition 1.2.1. *Let $\mathcal{V}(S,T)$ be the class of real **measurable** functions $f(t,\omega)$, defined on $[0,\infty) \times \Omega$, such that:*

1. *$f(t,\omega)$ is $\mathcal{F}_t$-adapted.*
2. *$\mathbb{E}\left(\int_S^T f(t,\cdot)^2 dt\right) < \infty$.*

1.2.1 Itô integral

1.2.1.1 Itô integral definition

Let $f \in \mathcal{V}(S,T)$. We want to define the Itô integral of f in the interval $[S,T)$. Namely:

$$\mathcal{I}(f)(\omega) = \int_S^T f(t,\omega) dB_t(\omega), \tag{1.7}$$

where B_t is a standard $(\mathbb{E}(B(1)^2) = 1)$ one dimensional Brownian motion. We begin defining the integral for a special class of functions:

Definition 1.2.2. *(**Simple functions**) A function $\phi \in \mathcal{V}(S,T)$ is called **simple function** (or elementary), if it can be expressed as a superposition of characteristic functions.*

$$\phi(t,\omega) = \sum_{k\geq 0} e_k(\omega) 1_{[t_k,t_{k+1})}(t). \tag{1.8}$$

Definition 1.2.3. *Let $\phi \in \mathcal{V}(S,T)$ be a simple function of the form of (1.8) , then we define the stochastic integral:*

$$\int_S^T \phi(t,\omega) dB_t = \sum_{k\geq 0} e_k(\omega)(B_{t_{k+1}} - B_{t_k})(\omega). \tag{1.9}$$

Lemma 1.2.1. *(**Ito isometry**, [23]) Let $\phi \in \mathcal{V}(S,T)$ be a simple function, then:*

$$\mathbb{E}\left(\left(\int_S^T \phi(t,\cdot)dB_t\right)^2\right) = \mathbb{E}\left(\int_S^T \phi(t,\cdot)^2 dt\right). \tag{1.10}$$

Remark 1.2.1. *Observe that (1.10) is indeed an isometry. In fact, it can been written as equality of norms in L^2 spaces:*

$$\left\|\int_S^T \phi(t,\cdot)dB_t\right\|_{L^2(\Omega,\mathbb{P})} = \|\phi\|_{L^2([S,T]\times\Omega)}.$$

We have the following important proposition.

Proposition 1.2.1. *Let $f \in \mathcal{V}$, then there exists a sequence of simple functions $\phi_n \in \mathcal{V}$, $n \in \mathbb{N}$, which converges to f in the L^2-norm. Namely,*

$$\lim_{n\longrightarrow\infty} \int_S^T \mathbb{E}\left((f(t,\cdot) - \phi_n(t,\cdot))^2\right) dt = \lim_{n\longrightarrow\infty} ||f - \phi_n||^2_{L^2([S,T]\times\Omega)} = 0. \tag{1.11}$$

Proof: See (Mura, [23]).

Definition 1.2.4. *(**Itô integral**) Let $f \in \mathcal{V}(S,T)$ the Itô integral from S to T of f is defined as the $L^2(\Omega,\mathbb{P})$ limit:*

$$\mathcal{I}(f) = \int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\longrightarrow\infty} \int_S^T \phi_n(t,\omega)dB_t(\omega), \tag{1.12}$$

where $\phi_n \in \mathcal{V}$, $n \in \mathbb{N}$, is a sequence of simple functions which converges to $f \in L^2([S,T]\times\Omega)$.

Remark 1.2.2. *Observe, in view of (1.11), that the definition above does not depend on the actual choice of $\{\phi_n, n \in \mathbb{N}\}$.*

By definition, we have that Itô isometry holds for Itô integrals:

Corollary 1.2.1. *(**Itô isometry for Ito integrals**, [23]) Let $f \in \mathcal{V}(S,T)$, then:*

$$\mathbb{E}\left(\left(\int_S^T f(t,\cdot)dB_t\right)^2\right) = \mathbb{E}\left(\int_S^T f(t,\cdot)\, dt\right). \tag{1.13}$$

Corollary 1.2.2. *(Mura, [23]) If $f_n(t,\omega) \in \mathcal{V}(S,T)$ converges to $f(t,\omega) \in \mathcal{V}(S,T)$ as $n \longrightarrow \infty$ in the $L^2([S,T]\times\Omega)$-norm, then:*

$$\int_S^T f_n(t,\cdot)dB_t \longrightarrow \int_S^T f(t,\cdot)dB_t, \tag{1.14}$$

in the $L^2(\Omega,\mathbb{P})$-norm.

1.2.1.2 Properties of the Itô integral

Proposition 1.2.2. *(Mura, [23]) Let $f, g \in \mathcal{V}(0,T)$ and let $0 \leq S < U < T$. Then:*

1. $\displaystyle\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t.$

2. *For some constant* $a \in \mathbb{R}$, $\displaystyle\int_S^T (af + g) dB_t = a\int_S^T f dB_t + \int_S^T g dB_t.$

3. $\displaystyle\mathbb{E}\left[\int_S^T f dB_t\right] = 0.$

4. $\displaystyle\int_S^T f dB_t$ *is* $\mathcal{F}_T$*-measurable.*

5. *The process* $M_t(\omega) = \displaystyle\int_0^T f(t,\omega) dB_s(\omega)$ *where* $f \in \mathcal{V}(0,T)$ *for any* $t > 0$, *is a martingale with respect to* $\mathcal{F}_t$.

1.2.2 Extensiens of Itô integral

The construction of the Itô Integral can be extended to a class of function $f(t,\omega)$ which satisfies a weak integration condition. This generalization is indeed necessary because it is not difficult to find functions which do not belong to $\mathcal{V}$. Therefore, we introduce the following class of functions

Definition 1.2.5. *Let $\mathcal{W}(S,T)$ be the class of real measurable functions $f(t,\omega)$, defined on $[0,\infty) \times \Omega$, such that*

1. $f(t,\omega)$ *is* $\mathcal{F}_t$*-adapted.*

2. $\displaystyle\mathbb{P}\left(\int_S^T f(t,\cdot)^2 dt < \infty\right) = 1.$

In the construction of stochastic integrals for the class of functions belonging to Ω we can no longer use the L^2 notion of convergence, but rather we have to use convergence in probability. In fact, for any $f \in \mathcal{B}$, one can show that there exists a sequence of simple functions $\phi_n \in \mathcal{W}$ such that

$$\int_S^T |\phi_n(t,\cdot) - f(t,\cdot)|^2 dt \longrightarrow 0, \tag{1.15}$$

in probability. For such a sequence one has that the sequence $\left\{\displaystyle\int_S^T \phi_n(t,\omega) dB_t(\omega), n \in \mathbb{N}\right\}$ converges in probability to some random variable. Moreover, the limit does not depends on the approximating sequence ϕ_n.

Definition 1.2.6. ***(Itô integral II)*** *Let $f \in \mathcal{W}(S,T)$. The Itô integral from S to T of f is defined as the limit in probability:*

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n \longrightarrow \infty} \int_S^T \phi_n(t,\omega)dB_t(\omega), \tag{1.16}$$

where $\phi_n \in \mathcal{W}, n \in \mathbb{N}$, is a sequence of simple functions which converges to f in probability.

Remark 1.2.3. *Note that this integral is not in general a martingale. However, it is a local martingale.*

1.3 Stochastic Differential Equations

We call stochastic differential equation (SDE) an equation of the form

$$dX_t = b(t, X(t))dt + \sigma(t, X(t))dB(t), \qquad X|_{t=0} = x, \tag{1.17}$$

where $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}), x$ is $\mathcal{F}_0$-measurable, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0;T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ have some regularity specified case by case and the solution $(X_t)_{t\geq 0}$ is a d-dimensional continuous adapted process. The meaning of the equation (1.17) is identic to

$$X_t = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s). \tag{1.18}$$

If there exists a stochastic process X_t that satisfies this equation, we say that it *solves* the stochastic differential equation.

The main goal of this section is to find conditions on the coefficients b an σ that guarantee the existence and uniqueness of solutions.

However, there are a number of subtle points involved:

- First, the existence of the integrals in (1.18) requires some degree of regularity on X_t and the functions b and σ; in particular, it must be the case that for all $t \geq 0$, with probability one, $\int_0^t |b(s, X(s))|ds < \infty$ and $\int_0^t \sigma^2(s, X(s))ds < \infty$.

- Second, the definition requires that the process X_t live on the same probability space as the given Wiener process B_t, and that it be adapted to the given filtration. It turns out that for certain coefficient functions b and σ, solutions to the stochastic integral equation (1.18) may exist for some Wiener processes and some admissible filtration but not for others.
  - The solution is a *strong solution* if it is valid for each given Wiener process (and initial value), that is it is sample pathwise unique.
  - A solution is a *weak solution* if it is valid for given coefficients, but unspecified Wiener process, that is its probability law is unique.

More precisely.

Definition 1.3.1. *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an admissible filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$. A* ***strong solution*** *of the stochastic differential equation (1.18) with initial condition $x \in \mathbb{R}$ is an adapted process X_t with continuous paths such that for all $t \geq 0$,*

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s).$$

Definition 1.3.2. *A* ***weak solution*** *of the stochastic differential equation (1.18) with initial condition x is a continuous stochastic process X_t defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for some Wiener process B_t and some admissible filtration $\mathcal{F}_t$ the process $X(t)$ is adapted and satisfies the stochastic integral equation (1.18).*

Let us come to uniqueness. Similarly to existence, there are two concepts.

Definition 1.3.3. ***(pathwise uniqueness)*** *We say that pathwise uniqueness holds for equation (1.18) if, given any filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with a Brownian motion $(B_t)_{t\geq 0}$ given any deterministic initial condition $X_0 = x$, if $(X_t^{(1)})_{t\geq 0}$ and $(X_t^{(2)})_{t\geq 0}$ are two continuous $\mathcal{F}_t$-adapted process which fulfill (1.18), then they are indistinguishable.*

Definition 1.3.4. ***(uniqueness in law)*** *We say that there is uniqueness in law for equation (1.17) if a given two weak solutions on any pair of spaces, their laws coincide.*

Theorem 1.3.1. *If the coefficients b and σ satisfy the following conditions:*

1. *A Lipschitz condition in x and y.* $\exists K, \forall x \in \mathbb{R}^n, \forall y \in \mathbb{R}^n, \forall t \geq 0:$

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq K\|x-y\|.$$

2. *A linear growth condition:* $\exists K, \forall x \in \mathbb{R}^n, \forall t \geq 0:$

$$\|b(t,x))\| + \|\sigma(t,x)\| \leq K(1 + \|x\|).$$

Then there exists a unique strong solution X to the stochastic differential equation (1.17) with continuous trajectories and there exists a constant C such that

$$\mathbb{E}[\|X_t\|^2] \leq Ce^{Ct}(1 + \|x\|^2).$$

Proof: See (Øksendal, [26]).

Chapter 2

Fractional Calculus

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last three decades fractional calculus has been applied to almost every field of mathematics, science, biology, engineering and technology. Our aim in this chapter is to introduce some very elementary facts about fractional calculus then we discuss the definition and the well-known properties of the fractional Brownian motion. In this section, we have taken definitions and used notions from [1, 5, 9, 15, 18, 21, 20, 27, 29].

2.1 Useful Mathematical Functions

Before looking at the definition of fractional integrals or derivatives, we will first discuss some useful mathematical definitions that are inherently tied to fractional calculus and will commonly be encountered.

2.1.1 The Gamma Function

The Gamma function, denoted by $\Gamma(x)$, is a generalization of the factorial function $n!$ and defined as.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x \in \mathbb{R}. \tag{2.1}$$

Some of the basic properties of Γ function, are:

$$\begin{cases} \Gamma(1) = \Gamma(2) = 1 \\ \Gamma(x+1) = x\Gamma(x) \qquad x \in \mathbb{R}^+ \\ \Gamma(n) = (n-1)! \qquad n \in N^* \end{cases}$$

From the above we can get:

$$\begin{cases} \Gamma(\frac{1}{2}) = \sqrt{\pi} \\ \Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi} \\ \Gamma(\frac{-3}{2}) = \dfrac{\Gamma(\frac{-3}{2}+1)}{\frac{-3}{2}} = \dfrac{\Gamma(\frac{-1}{2})}{\frac{-3}{2}} = \dfrac{\Gamma(\frac{1}{2})}{\frac{-3}{2}\frac{-1}{2}} = \dfrac{4}{3}\sqrt{\pi} \end{cases}$$

2.1.2 The Mittag-Lefler Function

The Mittag-Leffler function is named after a Swedish mathematician who defined and studied it in 1903. The function is a direct generalization of the exponential function, it plays a major role in fractional calculus. Firstly, we introduce one parameter function by using series, namely

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \tag{2.2}$$

Then, we define the Mittag-Leffler function with two parameters, as:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. \tag{2.3}$$

Note that $E_{\alpha,\beta}(0) = 1$. Also, for some specific values of α, and β, the Mittag-Leffler function reduces to some familiar functions. Namely:

$$\begin{cases} E_{1,1}(x) = \sum_{k=0}^{\infty} \dfrac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = \exp(x), \\ E_{1,2}(x) = \sum_{k=0}^{\infty} \dfrac{x^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \dfrac{x^k}{(k+1)!} = \dfrac{1}{x}\sum_{k=0}^{\infty} \dfrac{x^{k+1}}{(k+1)!} = \dfrac{\exp(x)-1}{x} \\ E_{1,3}(x) = \sum_{k=0}^{\infty} \dfrac{x^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \dfrac{x^k}{(k+2)!} = \dfrac{1}{x^2}\sum_{k=0}^{\infty} \dfrac{x^{k+2}}{(k+2)!} = \dfrac{\exp(x)-1-x}{x^2} \\ E_{1,m}(x) = \frac{1}{x^{m-1}}[\exp(x) - \sum_{k=0}^{m-2} \dfrac{x^k}{k!}] \\ E_{2,1}(-x^2) = \sum_{k=0}^{\infty} \dfrac{(-x^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k}}{(2k)!} = \cos(x) \\ E_{2,2}(-x^2) = \sum_{k=0}^{\infty} \dfrac{(-x^2)^k}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+1}}{x(2k+1)!} = \dfrac{\sin(x)}{x} \end{cases}$$

2.2 Fractional Derivatives and Integrals

This section is devoted to review three important definitions of fractional derivatives and integrals.

2.2.1 Gr$\ddot{u}$nwald-Letnikove, 1867-1868

Grünwald-Letnikov derivative is a basic extension of the natural derivative to fractional one. It was introduced by A. Grünwald in 1867, and then by A. Letnikov in 1868. Hence, it is written as

Definition 2.2.1. *Let $\alpha \in (0,1)$ be fixed and let $f : \mathbb{R} \to \mathbb{R}$ be a given function. The R-Grünwald-Letnikov derivative of order α of f is defined, respectively as:*

$$D_{+}^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t - jh).$$

We recall that the binomial coefficients can be defined as: $\begin{pmatrix} \alpha \\ n \end{pmatrix} = \frac{\alpha!}{n!(\alpha-n)!}$.

2.2.2 Riemann-Liouville definition

The Riemann-Liouville Operator is still the most frequently used when fractional integration is performed. which is considered as a direct generalization of Cauchy's formula for an n-times integral :

$$\int_a^x dx_1 \int_a^{x_1} dx_2 ... \int_a^{x_{n-1}} f(x_n) dx_n = \frac{1}{(n-1)!} \int_a^x \frac{f(t)}{(x-t)^{1-n}} dt. \tag{2.4}$$

Example 2.2.1. *let $f(x) = x$, $n = 3$ and $a = 0$ then (2.4) becomes*

$$\int_0^x \int_0^{x_1} \int_0^{x_2} x_3 dx_3 x_2 x_1 = \frac{1}{2!} \int_0^x \frac{t}{(x-t)^{-2}} dt, \tag{2.5}$$

and by integration one gets

$$\frac{1}{2!} \int_0^x \frac{t}{(x-t)^{-2}} dt = \frac{x^4}{4!}. \tag{2.6}$$

Since $(n-1)! = \Gamma(n)$, Riemann realized that (2.4) might have meaning even when n takes non-integer values. Thus perhaps it was natural to define fractional integration as follows.

Definition 2.2.2. *Let $f \in L_1([a,b])$ and $a \leq x \leq b$ then*

$$I^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt,$$

$$I^{\alpha}_{b-}f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt,$$

*are called the **Riemann-Liouville fractional integral** of order $\alpha > 0$.*

Lemma 2.2.1. *(Miller, [20]) Assuming arbitrary function f and $m, n \geq 0$ the following equations hold .*

1. *Semi-group property:*

$$I^m_a I^n_a f = I^{m+n}_a f.$$

2. *Commutative property:*

$$I^m_a I^n_a f(x) = I^n_a I^m_a f(x).$$

Definition 2.2.3. *Let $f \in L_1([a,b])$ and $a \leq x \leq b$ then*

$$D^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}} dt,$$

$$D^{\alpha}_{b-}f(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(t)}{(t-x)^{\alpha}} dt,$$

*which is called the **Riemann-Liouville fractional derivative** of order $0 < \alpha < 1$.*

To find a profound understanding of Riemann-Liouville derivative, some of the most crucial properties of this operator are mentioned in the following:

Lemma 2.2.2. *(Miller, [20]) Let f_1 and f_2 are two functions on $[a,b]$ as well as $c_1, c_2 \in \mathbb{R}, n > 0$, and $m > n$. Regarding these, the following equations hold*

1. *Linearity rules:*

$${}^{RL}D^n_a(f_1 + f_2) ={}^{RL} D^n_a f_1 +{}^{RL} D^n_a f_2, \qquad {}^{RL}D^n_a(c_1 f_1) = c_1^{RL} D^n_a(f_1).$$

2. *Zero rule:*

$$D^0 f = f.$$

3. *Product rule:*

$$^{RL}D_t^q(fg) = \sum_{j=0}^{\infty} \begin{pmatrix} q \\ j \end{pmatrix} \quad ^{RL}D_t^j(f)^{RL}D_t^{q-j}(g).$$

4. *In the general, semi-group property does not hold for Riemann-Liouville fractional derivative. Indeed, the following equation is not always true.*

$$^{RL}D^a \quad ^{RL}D^b f =^{RL} D^{a+b} f.$$

2.2.3 Caputo definition

Since Riemann-Liouville fractional derivatives failed in the description and modeling of some complex phenomena, Caputo derivative was introduced in 1967.

Definition 2.2.4. *The Caputo derivative of fractional order a $(n-1 \leq \alpha < n)$ of a function f is defined as*

$$^C_aD_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \qquad (n-1 \leq \alpha < n)$$

$$^C_bD_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_t^b \frac{f^{(n)}(\tau)d\tau}{(\tau-t)^{\alpha+1-n}}, \qquad (n-1 \leq \alpha < n)$$

Let f is a enough differentiable function, c_1, $c_2 \in \mathbb{R}$, and $m > n \geq 0$. The following proprieties are satisfies

1. Caputo derivative is the left inverse of Riemann-Liouville integral.

$$^CD_a^n I_a^n f = f.$$

2.

$$I_a^n \, ^CD_a^n f(x) = f(x) - \sum_{k=0}^{m-1} \frac{D^k f(a)}{k!}(x-a)^k.$$

3. Linearity:

$$^CD_a^n(c_1 f_1 + c_2 f_2) = c_1(^CD_a^n f_1) + c_2(^CD_a^n f_2).$$

4. Leibniz equation

$$^CD_a^n[fg](x) = \frac{(x-a)^{-n}}{\Gamma(1-n)} g(a)(f(x)-f(a)) + (^CD_a^n g(x))f(x)$$

$$+ \sum_{k=1}^{\infty} \begin{pmatrix} n \\ k \end{pmatrix} (I_a^{k-n} g(x))^C D_a^k f(x).$$

5. The semi-group property, the following equation holds,

$$^{C}D_{a}^{\alpha} \quad ^{C}D_{a}^{\beta} f =^{C} D^{\alpha+\beta} f.$$

2.3 Fractional Brownian motion

Among all the fractional stochastic processes applied to modeling natural and manmade systems, fractional Brownian motion (fBm) can be regarded as the most widely used. This process was first introduced by Kolmogorov in 1940, and studied by Mandelbrot and Van Ness in 1968. The fBm is a generalization of Brownian motion where a stochastic integral representation in terms of a standard Brownian motion was established.

Definition 2.3.1. *A Gaussian Process* $B_t^H = \{B^H(t), t \geq 0\}$ *is called a fractional Brownian motion (fBm) of Hurst index* $H \in (0,1)$ *if it has mean zero with covariance function*

$$R_H(t,s) = \mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{2.7}$$

In the next paragraph we list several properties of the fractional Brownian motion that are of our main interest.

- For $H = 1$, we set $B_t^H = B_t^1 = t\xi$, where ξ is a standard normal random variable.
- For $H = \frac{1}{2}$, B_t^H is the standard Wiener process.
- For $H > 1/2$, increments $B_t^H - B_s^H$ for any $0 \leq s < t$ are positively correlated and conversely for $H < 1/2$ the increments are negatively correlated.

Hence the following three properties are obtained through $R_H(t,s)$ in Definition 2.3.1.

1. ***Self-similarity:*** $\{a^{-H}B^H(at), t \geq 0\}$ has the same law as $\{B^H(t), t \geq 0\}$.
2. ***Stationary increments:*** $B^H(t+s) - B^H(t)$ has the same law as $B^H(t)$ for $s, t \geq 0$.
3. ***Variance:*** $\mathbb{E}[B^H(t)^2] = t^{2H}$, for all $t \geq 0$.

Proof: By definition $\mathbb{E}[B^H(t)] = 0$ and hence $\mathbb{E}[a^{-H}B^H(at)] = 0$ also.

1. Thus to show that both processes have the same probability distribution, it is sufficient enough to show that they both have the same covariance. Let $a > 0$ and $s, t \geq 0$.

$$\begin{aligned}
\mathbb{E}[a^{-H}B^H(at)a^{-H}B^H(as)] &= a^{-2H}\mathbb{E}[B^H(at)B^H(as)] \\
&= \frac{1}{2}a^{-2H}[(at)^{2H} + (as)^{2H} - \mid at - as \mid^{2H}] \\
&= \frac{1}{2}a^{-2H}a^{2H}[(t)^{2H} + (s)^{2H} - \mid t - s \mid^{2H}] \\
&= \frac{1}{2}[(t)^{2H} + (s)^{2H} - \mid t - s \mid^{2H}] \\
&= \mathbb{E}[B^H(t)B^H(s)].
\end{aligned}$$

And thus fBm is self-similar.

2. Again, since $\mathbb{E}[B^H(t+s) - B^H(t)]$ is clearly zero, it suffices to show that both processes have equal covariance. Let $r, s, t \geq 0$, then

$$\begin{aligned}
\mathbb{E}[(B^H(t+s) - B^H(s))(B^H(r+s) - B^H(s))] &= \mathbb{E}[B^H(t+s)B^H(r+s)] \\
&- \mathbb{E}[B^H(t+s)B^H(s)] \\
&- \mathbb{E}[B^H(s)B^H(r+s)] + \mathbb{E}[B^H(s)B^H(s)] \\
&= \frac{1}{2}((t+s)^{2H} + (r+s)^{2H} - \mid t - r \mid^{2H}) \\
&- \frac{1}{2}([(t+s)^{2H} + s^{2H} - t^{2H}]) \\
&- \frac{1}{2}([s^{2H} + (r+s)^{2H} - r^{2H}] + [s^{2H} + s^{2H}]) \\
&= \frac{1}{2}(t^{2H} + r^{2H} - \mid t - r \mid^{2H}) \\
&= \mathbb{E}[B^H(t)B^H(r)].
\end{aligned}$$

Hence fBm has stationary increments.

3. $\mathbb{E}[B^H(t)^2] = \mathbb{E}[B^H(t)B^H(t)] = \frac{1}{2}(t^{2H} + t^{2H} - |t-t|^{2H}) = t^{2H}$.

2.3.1 Long range dependency

Define $X(n) = B^H(n+1) - B^H(n), n \geq 1$. Then clearly $X(n)$ is a Gaussian stationary sequence with unit variance. Moreover the covariance function of $X(t)$ is

$$r^H(n) = E[X(0)X(n)] = 1/2((n+1)^{2H} - 2n^{2H} + (n-1)^{2H}).$$

If $H = 1/2$ then we get that $r(n) = 0$ implying that the increments of $X(n)$ are uncorrelated.

But, if $H \neq 1/2$, we get that as n tends to infinity $r^H(n) \backsim H(2H-1)n^{2H-2}$. Thus we get

- If $0 < H < 1/2$ then $\sum_{n=0}^{\infty} |r^H(n)| < \infty$.
- If $1/2 < H < 1$ then $\sum_{n=0}^{\infty} |r^H(n)| = \infty$, in this case the process B^H is a long memory process.

2.3.2 Hölder continuity

We recall that according to the Kolmogorov criterion (Klebaner, [14]), a process $X = (X_t)_{t\in\mathbb{R}}$ admits a continuous modification if there exist constants $\alpha \geq 1$, $\beta > 0$, and $k > 0$ such that

$$\mathbb{E}\big[|X(t) - X(s)|^{\alpha}\big] \leq k|t-s|^{1+\beta},$$

for all $s, t \in \mathbb{R}$.

Theorem 2.3.1. *Let $H \in (0,1)$. The fbm B^H admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*

Proof: We recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Hölder continuous of order α, $0 < \alpha \leq 1$ and write $f \in \mathcal{C}^{\alpha}(\mathbb{R})$, if there exists $M > 0$ such that

$$|f(t) - f(s)| \leq M|t-s|^{\alpha},$$

for every $s, t \in \mathbb{R}$. For any $\alpha > 0$ we have

$$\mathbb{E}\big[|B^H(t) - B^H(s)|^{\alpha}\big] = \mathbb{E}\big[|B^H(1)|^{\alpha}\big]|t-s|^{\alpha H}.$$

2.3.3 Path differentiability

By (Maslowski, [17]) we also obtain that the process B^H is not mean square differentiable and it does not have differentiable sample paths.

Proposition 2.3.1. *Let $H \in (0,1)$. The fBm sample path $B^H(.)$ is not differentiable. In fact, for every $t_0 \in [0,\infty)$*

$$\lim_{t\to t_0}\sup\left|\frac{B^H(t)-B^H(t_0)}{t-t_0}\right| = \infty,$$

with probability one.

Proof: Here we recall the proof of (Maslowski, [17]). Note that we assume $B^H(0)=0$. The result is proved by exploiting the self-similarity of B^H. Consider the random variable

$$\mathcal{R}_{t,t_0} := \frac{B^H(t)-B^H(t_0)}{t-t_0},$$

that represents the incremental ratio of B^H. Since B^H is self-similar see(Biagini, [1]), we have that the law of $\mathcal{R}_{t,t_0}$ is the same of $(t-t_0)^{H-1}B^H(1)$. If one considers the event

$$A(t,w) := \left\{\sup_{0\le s\le t}\left|\frac{B^H(s)}{s}\right| > d\right\},$$

then for any sequence $(t_n)_{n\in\mathbb{N}}$ decreasing to 0, we have

$$A(t_n,w) \supseteq A(t_{n+1},w),$$

and

$$A\left(t_n,w\right) \supseteq \left(\left|\frac{B^H(t_n)}{t_n}\right| > d\right) = \left(|B^H(1)| > t_n^{1-H}d\right).$$

But,

$$\lim_{n\longrightarrow\infty}\left(|B^H(1)| > t_n^{1-H}d\right).$$

Since this is true for any d, it must be the case that the derivative does not exist at any point along any sample path of $B^H(t)$.

2.3.4 Non semi-martingale property

The definition of the Itô integral is a direct consequence of the martingale property of Brownian motion. But fBm does not exhibit this property, in fact, fBm is not even a semi-martingale. There are many different proofs revealing this fact. We state the theorem and present a simple proof here. But first, we need to find the p-variation of B^H.

Definition 2.3.2. *The p-variation of a stochastic process $(X(t))_{t\in[0,T]}$is defined as*

$$\mathcal{V}_p(X,[0,T]) := sup_\pi \sum_{i=1}^{n} |X(t_i)-X(t_{i-1})|^p, \tag{2.8}$$

where π is a finite partition of $[0,T]$. The index of p-variation of a process is defined to be

$$I(X,[0,T]) := \inf\{p>0; \mathcal{V}_p(X,[0,T])<\infty\}. \tag{2.9}$$

Lemma 2.3.1. $I(B^H,[0,T]) = \frac{1}{H}$ *Moreover,* $\mathcal{V}_p(B^H(t),[0,T]) = 0$ *when* $pH > 1$ *and* $\mathcal{V}_p(B^H(t),[0,T]) = \infty$ *when* $pH < 1$.

Proof: A proof can be found in see(Biagini, [1]).

This can be seen when we take into consideration that

$$\mathbb{E}[|B^H(t_i) - B^H(t_{i-1})|^p] = \mathbb{E}[|B^H(1)|^p]|t_i - t_{i-1}|^{pH},$$

and plugging this into (2.8) and applying (2.9).

Theorem 2.1. $\{B^H(t) : t \geq 0\}$, *for* $H \neq 1/2$, *is not semimartingale.*

Proof: A process $\{X(t), t \geq 0\}$ is called a semimartingale if it admits the Doob-Meyer decomposition $X(t) = X(0) + M(t) + A(t)$, where $M(t)$ is an $\mathcal{F}_t$ local martingale with $M(0) = 0$, $A(t)$ is a càdlàg adapted process of locally bounded variation and $X(0)$ is $\mathcal{F}_0$-measurable. Moreover, any semimartingale has locally bounded quadratic variation (Biagini, [1]). Now, let $X(t) = B^H(t)$. If $H \in (0,1/2)$, then $B^H(t)$ cannot even be a martingale since it has infinite quadratic variation, hence, it is not a semimartingale.

If $H \in (1/2,1)$ then the quadratic variation of $B^H(t)$ is zero. So, let's suppose that it is a semimartingale. Then, $M(t) = B^H(t) - A(t)$ has quadratic variation equal to zero. So, from [1], $M(t) = 0$ for all t a.s. Then that would mean that $B^H(t) = A(t)$, but this can't be the case since $B^H(t)$ has unbounded variation. Hence $B^H(t)$ is not a semimartingale for any $H \neq 1/2$.

2.3.5 Integral representation of fractional Brownian motion

Now we show that the fractional Brownian motion can be represented as a stochastic integral.

The standard fBm as introduced by Mandelbrot and Van Ness is defined by the following moving average representation:

$$B^H(t) = \frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0}[(t-u)^{H-1/2} - (-u)^{H-1/2}dB(u) + \int_0^t (t-u)^{H-1/2}dB(u)\right\} \tag{2.10}$$

where $B(t)$ is the standard Brownian motion, Γ is the gamma function. Equation(2.10) can be written more compactly as

$$B^H(t) = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{\infty} [(t-u)_+^{H-1/2} - (-u_+)^{H-1/2}] dB(u). \tag{2.11}$$

Chapter 3

Stochastic Differential Equation Via Generalized Grey Brownian Motion

Grey Brownian motion was introduced by Schneider in [31, 32] as a stochastic model for slow-anomalous diffusion described by the time fractional diffusion equation. Later Minardi, Mura and Pagnini [21],[22], extended this class, to the so called "generalized" grey Brownian motion which includes stochastic models for slow and fast-anomalous diffusion, i.e., the time evolution of the marginal density function is described by differential equations of fractional type. In this chapter we will not reproduce the all construction of the ggBm, but we will refer to the mention of the latter and some of its properties. Then we establish a global result on stochastic differential equation driven by generalized grey noise, the interested reader is referred to [7] and references therein.

3.1 Preliminary notions

Definition 3.1.1. ***(Schwartz space)*** *The space $\mathcal{S}(\mathbb{R}^n)$ is the space of all the functions $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, such that for any multi-indices $j = (j_1, j_2, \ldots, j_n)$ and $k = (k_1, k_2, ..., k_n)$*

$$\sup_{x \in \mathbb{R}^n} |x^j D^k f(x)| < \infty. \tag{3.1}$$

Definition 3.1.2. ***(Tempred distribution)*** *The space of all tempered distributions on $\mathbb{R}$, denoted $\mathcal{S}'(\mathbb{R}^n)$, is the dual space of $\mathcal{S}(\mathbb{R}^n)$. That is, it is the set of all functions that are linear and continuous.*

Definition 3.1.3. ***(Completely monotonic function)*** *A function f with domain* $(0, \infty)$ *is said to be completely monotonic, if it possesses derivatives* $f^{(n)}(x)$ *for all* $n = 0, 1, 2, ...$ *and if* $(-1)^n f^{(n)}(x) \geq 0$ *for all* $x > 0$.

Definition 3.1.4. *A continuous map* $\Phi : X \longrightarrow \mathbb{C}$ *is called a* ***characteristic functional*** *on X if it is:*

1. *Normalized:* $\Phi(0) = 1$,
2. *Positive defined:* $\sum_{i,j=1}^{m} \overline{c_i} \Phi(\xi_i - \xi_j) c_j \geq 0, \; m \in \mathbb{Z}, \; \{c_i\}_{i=1,....,m} \in \mathbb{C}, \; \{\xi_i\}_{i=1,...,m} \in X$.

Proposition 3.1.1. *Let F be a completely monotonic function defined on the positive real line. Therefore, there exists a unique characteristic functional, defined on a real separable Hilbert space H, such that:*

$$\Phi(\xi) = F(\|\xi\|^2), \quad \xi \in H.$$

Definition 3.1.5. ***(Nuclear space***, [23]*) A topological vector space X, with the topology defined by a family of Hilbert norms, is said a nuclear space if for any Hilbert norm* $\| \cdot \|_p$ *there exists a larger norm* $\| \cdot \|_q$ *such that the inclusion map* $X_q \hookrightarrow X_p$ *is an Hilbert−Schmidt operator.*

Remark 3.1.1. *Nuclear spaces have many of the good properties of the finite-dimensional Euclidean spaces* $\mathbb{R}^d$. *For example, a subset of a nuclear space is compact if and only if is bounded and closed. Moreover, spaces whose elements are '**smooth**' in some sense tend to be nuclear spaces.*

Theorem 3.1.1. ***(Minlos theorem***, [23]*) Let X be a nuclear space. For any characteristic functional* Φ *defined on X there exists a unique probability measure* μ *defined on the measurable space* $(X', \mathcal{B})$, *where* $\mathcal{B}$ *is regarded as the Borel* σ*-algebra generated by the weak topology on* X', *such that:*

$$\int_{X'} e^{i\langle w, \xi \rangle} d\mu(w) = \Phi(\xi), \quad \xi \in X. \tag{3.2}$$

3.2 Grey Brownian Motion

3.2.1 Grey noise

We denote by $L^2(\mathbb{R}) := L^2(\mathbb{R}, dx)$ the Hilbert space of real-valued square integrable measurable functions w.r.t. the Lebesgue measure. The inner product in $L^2(\mathbb{R})$ is denoted by $(.,.)$ and the corresponding norm by $|\ .\ |$. As a densely imbedded nuclear space in $L^2(\mathbb{R})$ we choose the Schwartz test function space $\mathcal{S}(\mathbb{R})$ equipped with the scalar product.

$$(\xi, \eta)_\alpha := \mathcal{C}(\alpha) \int_{\mathbb{R}} \overline{\widetilde{\xi}(x)} \widetilde{\eta}(x) |x|^{1-\alpha} dx, \quad \xi, \eta \in \mathcal{S}(\mathbb{R}),\ 0 < \alpha < 2, \tag{3.3}$$

with $\mathcal{C}(\alpha) = \Gamma(\alpha + 1) \sin(\frac{\pi\alpha}{2})$. Together with the dual space $\mathcal{S}'(\mathbb{R})$ we obtain the basic nuclear triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}).$$

The canonical dual pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ is denoted by $\langle .,. \rangle$ and given as an extension of the scalar product in $L^2(\mathbb{R})$ by

$$\langle f, \varphi \rangle = (f, \varphi), \quad f \in L^2(\mathbb{R}), \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

On $\mathcal{S}'(\mathbb{R})$ we choose the Borel σ-algebra $\mathcal{B}$ generated by the cylinder sets. Thus, we have a measurable space $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$.

In order to define the Mittag-Leffler measure which is a family of probability measures on $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$ whose characteristic functions are given by the Mittag-Leffler functions. using proposition 3.1.1 which states that starting from a completely monotonic function F, we can define characteristic functionals on $\mathcal{S}(\mathbb{R})$ by setting $\Phi(\xi) = F(\|\xi\|_\alpha^2)$. Then, by Minlos Theorem 3.1.1, the following definition makes sense.

Definition 3.2.1. *For any $\beta \in (0, 1]$ the Mittag-Leffler measure is defined as the unique probability measure μ_β on $\mathcal{S}'(\mathbb{R})$ by fixing its characteristic functional*

$$\int_{\mathcal{S}'} e^{i\langle w, \varphi \rangle} d\mu_\beta(w) = E_\beta(-\frac{1}{2}\langle \varphi, \varphi \rangle), \quad \varphi \in \mathcal{S}(\mathbb{R}) \tag{3.4}$$

Remark 3.2.1. *1. The measure μ_β is also called grey noise (reference) measure, (Mura [23], Bingham [2], Bondesson [4]).*

2. The range $0 < \beta \leq 1$ ensures the complete monotonicity of $E_\beta(-x)$.

In other words, this is sufficient to show that

$$\mathcal{S}(\mathbb{R}) \ni \varphi \mapsto E_\beta(-\frac{1}{2}\langle\varphi,\varphi\rangle) \in \mathbb{R}$$

is a characteristic function in $\mathcal{S}(\mathbb{R})$*.*

It follows from (3.4) that all moments of μ_β exists and we have

Lemma 3.2.1. *For any* $\varphi \in \mathcal{S}(\mathbb{R})$ *and* $n \in \mathbb{N}_0$ *we have*

$$\int_{\mathcal{S}'(\mathbb{R})} \langle w,\varphi\rangle^{2n+1} d\mu_\beta(w) = 0.$$

$$\int_{\mathcal{S}'(\mathbb{R})} \langle w,\varphi\rangle^{2n} d\mu_\beta(w) = \frac{(2n)!}{2^n\Gamma(\beta n+1)} \mid \varphi \mid^{2n}.$$

Remark 3.2.2.

- *In the approach of (Mura, [21]) the grey noise measure is defined via the characteristic function* $E_\beta(-(.,.)_\alpha)$ *and denoted by* $\mu_{\alpha,\beta}$*. This means that first the parameters* $0 < \alpha < 2$ *and* $0 < \beta < 1$ *are fixed and then generalized grey Brownian motion* $B_t^{\alpha,\beta}$ *is constructed in* $L^2(\mu_{\alpha,\beta})$*. The measure* μ_β *as defined above is named general grey noise measure since for fixed* $0 < \beta < 1$ *all generalized grey Brownian motions* $B_t^{\alpha,\beta}$ *for* $0 < \alpha < 2$ *can be constructed in the single space* $L^2(\mu_\beta)$*.*
- *In the case* $\alpha = \beta$ *we write* $B^\beta(t)$ *instead of* $B_t^{\beta,\beta}$*.* $B^\beta(t)$ *is called grey Brownian motion.*

Definition 3.2.2. *We consider the generalized stochastic process* $X_{\alpha,\beta}$ *defined canonically on the generalized grey noise space* $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_{\alpha,\beta})$*, called grey noise by*
$X_{\alpha,\beta}(\varphi) : \mathcal{S}'(\mathbb{R}) \to \mathbb{R}, w \mapsto X_{\alpha,\beta}(\varphi)(w) := \langle w,\varphi\rangle$*.*

Properties

1. Characteristic function:

$$\mathbb{E}(e^{i\lambda X_{\alpha,\beta}(\varphi)}) := E_\beta(-\lambda^2\|\varphi\|_\alpha^2). \tag{3.5}$$

2. Moments:

$$\mathbb{E}(X_{\alpha,\beta}(\varphi)^k) = \begin{cases} 0, & k = 2n+1 \\ \frac{(2n)!}{\Gamma(\beta n+1)}\|\varphi\|_\alpha^{2n}, & k = 2n. \end{cases}$$

3. For any $f \in \mathcal{S}(\mathbb{R})$, we have $X_{\alpha,\beta}(f) \in L^2(\mu_{\alpha,\beta})$ and

$$\|X_{\alpha,\beta}(f)\|^2_{L^2(\mu_{\alpha,\beta})} = \frac{2}{\Gamma(\beta+1)}\|f\|^2_\alpha.$$

3.2.2 Generalized grey Brownian motion.

In this subsection we briefly introduce the mathematical definition and the main of proprieties of generalized grey Brownian motion.

Definition 3.2.3. *The stochastic process*

$$\{B_{\alpha,\beta}(t)\}_{t\geq 0} = \{X_{\alpha,\beta}(\mathbf{1}_{[0,t)})\}_{t\geq 0}. \tag{3.6}$$

*is called **'generalized' (standard) grey Brownian motion**.*

The "generalized" grey Brownian motion $\{B_{\alpha,\beta}\}$ has the following properties that come directly from the grey noise properties.

1. $B_{\alpha,\beta}(0) = 0$ a.s. Moreover, for each $t \geq 0$, $\mathbb{E}(B_{\alpha,\beta}(t)) = 0$ and

$$\mathbb{E}(B_{\alpha,\beta}(t)^2) = \frac{2}{\Gamma(\beta+1)}t^\alpha. \tag{3.7}$$

2. The autocovariance function is:

$$\mathbb{E}(B_{\alpha,\beta}(t)B_{\alpha,\beta}(s)) = \gamma_{\alpha,\beta}(t,s) = \frac{1}{\Gamma(\beta+1)}(t^\alpha + s^\alpha - |t-s|^\alpha). \tag{3.8}$$

3. For any $t, s \geq 0$, the characteristic function of the increments is:

$$\mathbb{E}(e^{iy(B_{\alpha,\beta}(t)-B_{\alpha,\beta}(s))}) = E_\beta(-y^2|t-s|^\alpha), \quad y \in \mathbb{R}. \tag{3.9}$$

The third property follows from the linearity of the grey noise definition, we suppose $0 \leq s < t$, we have $y(B_{\alpha,\beta}(t) - B_{\alpha,\beta}(s)) = yX_{\alpha,\beta}(\mathbf{1}_{[0,t)} - \mathbf{1}_{[0,s)}) = X_{\alpha,\beta}(y\mathbf{1}_{[s,t)})$, and $\|y\mathbf{1}_{[s,t)}\|^2_\alpha = y^2(t-s)^\alpha$.

Proposition 3.2.1. *For any $0 < \alpha < 2$ and $0 < \beta \leq 1$, the process $B_{\alpha,\beta}(t), t \geq 0$, is a self-similar with stationary increments process (**H-sssi**), with $H = \alpha/2$.*

Proof: This result is actually a consequence of the linearity of the noise definition. Given a sequence of real numbers $\{\theta_j\}_{j=1,\dots,n}$ we have to show that for any $0 < t_1 < t_2 < \dots < t_n$ and $a > 0$:

$$\mathbb{E}\left(\exp\left(i\sum_j \theta_j B_{\alpha,\beta}(at_j)\right)\right) = \mathbb{E}\left(\exp\left(i\sum \theta_j a^{\alpha/2} B_{\alpha,\beta}(t_j)\right)\right).$$

The linearity of the grey noise definition allows to write the above equality as

$$\mathbb{E}\left[\exp\left(iX_{\alpha,\beta}\left(\sum_j \theta_j \mathbf{1}_{[0,at_j)}\right)\right)\right] = \mathbb{E}\left[\exp\left(iX_{\alpha,\beta}\left(a^{\alpha/2}\sum_j \theta_j \mathbf{1}_{[0,t_j)}\right)\right)\right].$$

Using (3.5) we have

$$F_\beta\left(\|\sum_j \theta_j \mathbf{1}_{[0,at_j)}\|_\alpha^2\right) = F_\beta\left(\|a^{\alpha/2}\sum_j \theta_j \mathbf{1}_{[0,t_j)}\|_\alpha^2\right),$$

which, because of the complete monotonicity, reduces to

$$\|\sum_j \theta_j \mathbf{1}_{[0,at_j)}\|_\alpha^2 = a^\alpha \|\sum_j \theta_j \mathbf{1}_{[0,t_j)}\|_\alpha^2.$$

In view of (3.3) and the fact that $\tilde{\mathbf{1}}_{[0,t)}(x) = \frac{1}{\sqrt{2\pi}}\frac{e^{ixt}-1}{ix}$, the above equality is checked after a simple change of variable in the integration. In the same way we can prove the stationarity of the increments.We have to show that for any $h \in \mathbb{R}$:

$$\mathbb{E}\left[\exp\left(i\sum_j \theta_j (B_{\alpha,\beta}(t_j+h) - B_{\alpha,\beta}(h))\right)\right] = \mathbb{E}\left[\exp\left(i\sum_j \theta_j (B_{\alpha,\beta}(t_j))\right)\right].$$

We use the linearity property to write

$$\mathbb{E}\left[\exp\left(iX_{\alpha,\beta}\left(\sum_j \theta_j \mathbf{1}_{[h,t_j+h)}\right)\right)\right] = \mathbb{E}\left[\exp\left(iX_{\alpha,\beta}\left(\sum_j \theta_j \mathbf{1}_{[0,t_j)}\right)\right)\right].$$

By using the definition and the complete monotonicity, we have

$$\|\sum_j \theta_j \mathbf{1}_{[h,t_j+h)}\|_\alpha^2 = \|\sum_j \theta_j \mathbf{1}_{[0,t_j)}\|_\alpha^2.$$

Remark 3.2.3. *In view of Proposition* 3.2.1, $\{B_{\alpha,\beta}(t)\}$ *forms a class of **H-sssi** stochastic processes indexed by two parameters* $0 < \alpha < 2$ *and* $0 < \beta \le 1$. *This class includes fractional Brownian motion* ($\beta = 1$)*, grey Brownian motion* ($\alpha = \beta$) *and Brownian motion* ($\alpha = \beta = 1$).

3.2.3 Characterization of the ggBm

We want now to characterize the ggBm through its finite dimensional structure. We know that all the ggBm finite dimensional probability density functions are defined only by their autocovariance matrix. The following proposition holds

Proposition 3.2.2. *Let $B_{\alpha,\beta}$ be a ggBm, then for any collection $\{B_{\alpha,\beta}(t_1), ..., B_{\alpha,\beta}(t_n)\}$, the joint probability density function is given by:*

$$f_{\alpha,\beta}(x_1, x_2, \dots, x_n; \gamma_{\alpha,\beta}) = \frac{(2\pi)^{-\frac{n-1}{2}}}{\sqrt{2\Gamma(1+\beta)^n \det \gamma_{\alpha,\beta}}} \int_0^\infty \frac{1}{\tau^{n/2}} M_{1/2}\left(\frac{\zeta}{\tau^{1/2}}\right) M_\beta(\tau) d\tau. \quad (3.10)$$

with:

$\zeta = \left(2\Gamma(1+\beta)^{-1} \sum_{i,j=1}^{n} x_i \gamma_{\alpha,\beta}^{-1}(t_i, t_j) x_j\right)^{1/2}, \gamma_{\alpha,\beta}(t_i, t_j) = \frac{1}{\Gamma(1+\beta)}(t_i^\alpha + t_j^\alpha - |t_i - t_j|^\alpha), \quad i, j = 1, ..., n.$

and M is the M-Wright function.

Proof: in order to show (3.10), we calculate its n-dimensional Fourier transform and we find that it is equal to

$$\mathbb{E}\left(\exp(i \sum_{j=1}^{n} \theta_j B_{\alpha,\beta}(t_j))\right) = E_\beta\left(-\Gamma(1+\beta)\frac{1}{2}\sum_{i,j=1}^{n} \theta_i \theta_j \gamma_{\alpha,\beta}(t_i, t_j)\right).$$

We have

$$\int_{\mathbb{R}^n} exp\left(i \sum_{j=1}^{n} \theta_j x_j\right) f_{\alpha,\beta}(x_1, ..., x_n; \gamma_{\alpha,\beta}) d^n x =$$

$$\frac{(2\pi)^{-\frac{n-1}{2}}}{\sqrt{2\Gamma(1+\beta)^n \det \gamma_{\alpha,\beta}}} \int_0^\infty \frac{1}{\tau^{n/2}} M_\beta(\tau) \int_{\mathbb{R}^n} exp\left(i \sum_{j=1}^{n} \theta_j x_j\right) M_{1/2}(\frac{\xi}{\tau^{1/2}}) d^n x d\tau.$$

We remember that $M_{1/2}(r) = \frac{1}{\sqrt{\pi}} e^{r^2/4}$, thus we get

$$\int_0^\infty \frac{1}{\tau^{n/2}} M_\beta(\tau) \int_{\mathbb{R}^n} exp\left(i \sum_{j=1}^{n} \theta_j x_j\right) \times$$

$$\frac{(2\pi)^{-\frac{n}{2}}}{\sqrt{\Gamma(1+\beta)^n \det \gamma_{\alpha,\beta}}} exp\left(-\Gamma(1+\beta)^{-1} \sum_{i,j=1}^{n} x_i \gamma_{\alpha,\beta}^{-1}(t_i, t_j) x_j / 2\tau\right) d^n x d\tau.$$

We make the change of variables $x = \Gamma(1+\beta)^{1/2}\tau^{1/2}y$, with $x, y \in \mathbb{R}^n$, and we get

$$\int_0^\infty M_\beta(\tau) \int_{\mathbb{R}^n} exp\left(i\Gamma(1+\beta)^{1/2}\tau^{1/2}\sum_{j=1}^n \theta_j y_j \right) \times$$
$$\frac{(2\pi)^{-\frac{n}{2}}}{\sqrt{\det \gamma_{\alpha,\beta}}} exp\left(-\sum_{i,j=1}^n \frac{y_i \gamma_{\alpha,\beta}^{-1}(t_i,t_j) y_j}{2} \right) d^n y d\tau =$$

$$\int_0^\infty M_\beta(\tau) exp\left(-\Gamma(1+\beta)\tau \sum_{i,j=1}^n \frac{\theta_i \gamma_\alpha(t_i,t_j)\theta_j}{2} \right) d\tau = \int_0^\infty e^{-\tau s} M_\beta(\tau) d\tau = E_\beta(s),$$

where $s = \Gamma(1+\beta)\sum_{i,j=1}^n \theta_i\theta_j\gamma_{\alpha,\beta}(t_i,t^j)/2$.

Applying the Kolmogorov extension theorem (see Theorem 1.1.2), the above proposition allows us to define the ggBm in an unspecified probability space. In fact, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the following proposition characterizes the ggBm:

Proposition 3.2.3. *(Mura, [23]) Let* $X(t), t \geq 0$, *be a stochastic process, defined in a certain probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *such that*

1. $X(t)$ *has covariance matrix indicated by* $\gamma_{\alpha,\beta}$ *and finite-dimensional distributions defined by (eq. 3.10).*
2. $\mathbb{E}[X^2(t)] = \frac{2}{\Gamma(1+\beta)}t^\alpha$ *for* $0 < \beta \leq 1$ *and* $0 < \alpha < 2$.
3. $X(t)$ *has stationary increments,*

then $X(t)$, $t \geq 0$, *is a generalized grey Brownian motion.*

In fact condition 2) together with condition 3) imply that $\gamma_{\alpha,\beta}$ must be the ggBm autocovariance matrix (3.8).

Corollary 3.2.1. *(Mura, [23]) Let* $X(t)$, $t \geq 0$, *be a stochastic process defined in a certain probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *Let* $H = \alpha/2$ *with* $0 < \alpha < 2$ *and suppose that* $\mathbb{E}[X(1)^2] = 2/\Gamma(1+\beta)$. *The following statements are equivalent:*

i) X *is H-**sssi** with finite-dimensional distribution defined by (3.10),*

ii) X is a generalized grey Brownian motion with scaling exponent $\alpha/2$ and "fractional order" parameter β,
iii) X has zero mean, covariance function $\gamma_{\alpha,\beta}(t,s)$, $t,s \geq 0$, defined by (3.8) and finite dimensional distribution defined by (3.10).

3.2.4 Representations of ggBm

In this subsection we will show that gBm $B_{\alpha,\beta}$ admits different representations which will be useful in proving certain properties on the next sections. This is related to the fact that these representations involves certain known processes, such as fractional Brownian motion (fBm).

3.2.4.1 Normal variance mixture

Proposition 3.2.4. *[23] Let $B_{\alpha,\beta}(t), t \geq 0$, be a ggBm, then*

$$\{B_{\alpha,\beta}(t), t \geq 0\} \stackrel{d}{=} \{\sqrt{Y_\beta} X_\alpha(t), t \geq 0\}, \tag{3.11}$$

where $X_\alpha(t)$ is a standard fBm, Y_β is an independent nonnegative random variable with probability density function $M_\beta(\tau), \tau \geq 0$.

The representation (3.11) is particularly interesting. Since, many question, in particularly those related to the distribution properties of $B_{\alpha,\beta}(t)$, can be reduced to question concerning the fBm $X_\alpha(t)$, which are easier since $X_\alpha(t)$ is a Gaussian process.

3.2.4.2 One dimensional representation

Here we obtain two representations of ggBm as subordinations, valid for one-dimensional distributions. First we show that ggBm may be represented as a subordination of Brownian motion by a β-stable subordinator, (Schneider, [32]).

1- Subordination of Brownian motion

Let $S = S(t), t \in [0,1]$ be a β-stable subordinator and define the inverse process of S by

$$E(x) := \inf\{t : S(t) > x, \quad x \in \mathbb{R}^+\}.$$

$E(x)$ is $1/\beta$-self-similar process with no independent/stationary increments. Bingham [2] and Bondesson and al. [4] showed that $E(x)$ has a Mittag-Leffler distribution

$$\mathbb{E}(e^{-sE(x)}) = E_\beta(-sx^\beta).$$

It follows that

$$\mathbb{E}(e^{-sE(x^{\alpha/\beta})}) = E_\beta(-sx^\alpha).$$

On the other hand, we have the equality in law $E(x) = (S(1)/x)^{-\beta}$ which implies

$$\mathbb{E}(e^{-sS^{-\beta}(1)}) = E_\beta(-s).$$

and

$$\mathbb{E}(e^{-sS^{-\beta}(t^{-\alpha})}) = E_\beta(-st^\alpha).$$

As a consequence, we obtain the following representation for the ggBm:

$$B_{\alpha,\beta}(t) = B(E(t^{\alpha/\beta})) = B(S^{-\beta}(t^{-\alpha})),$$

where B is a standard Brownian motion independent of S and the equalities are valid only for one-dimensional distributions.

2- Subordination of fBm

The second representation of ggBm as a subordination of fBm uses as subordinator a process with one-dimensional distribution related to the M-Wright function.
Let $D_\beta = \{D_\beta(t), t \geq 0\}$ be the process with one-dimensional distribution given by

$$f_{D_\beta(t)}(x) = t^{-\beta} M_\beta(xt^{-\beta}), \quad x, t \geq 0.$$

The ggBm is represented as

$$B_{\alpha,\beta}(t) = B_H\left(D_\beta^{1/\alpha}(t^{\alpha/\beta})\right),$$

where the fBm B_H and D_β are independent. The equality is valid only for one-dimensional distributions. Density $f_{D_\beta(t)}$ is the fundamental solution of the time-fractional drift equation

$$\mathcal{D}_t^\beta f_{D_\beta(t)}(x) = -\frac{\partial}{\partial x} f_{D_\beta(t)}(x),$$

where $\mathcal{D}_t^\beta$ denotes the Caputo derivative.

3.2.5 Other properties of the ggBm

This subsection is devoted to the study of some other properties such as the $p-$variation of ggBm, Hölder continuity...

3.2.5.1 The p-variation of generalized grey Brownian motion

The approach taken is inspired from the one used for the fBm.

Proposition 3.2.5. *We have the following limit in probability*

$$\lim_{n\longrightarrow+\infty} n^{p\frac{\alpha}{2}-1}\sum_{j=1}^{n}\left|B_{\alpha,\beta}\left(\frac{j}{n}\right)-B_{\alpha,\beta}\left(\frac{j-1}{n}\right)\right|^{p}=\mathbb{E}(|B_{\alpha,\beta}(1)|^{p}).$$

Proof: See (Silva,[8]).

Proposition 3.2.6. *(Silva,[8]) We have the following limit in probability*

$$V_{p,n}:=\sum_{j=1}^{n}\left|B_{\alpha,\beta}\left(\frac{j}{n}\right)-B_{\alpha,\beta}\left(\frac{j-1}{n}\right)\right|^{p}\quad\overrightarrow{n\longrightarrow+\infty}\quad\begin{cases}0 & a.s.\quad if\ \ p\alpha/2>1\\ \infty & a.s.\quad if\ p\alpha/2<1\\ \mathbb{E}(|B_{\alpha,\beta}(1)|^{p})\ a.s.\ if\ p=\frac{2}{\alpha}.\end{cases}$$

Remark 3.2.4. *The ggBm is not a semimartingale. In addition, $B_{\alpha,\beta}$ cannot be of finite variation on $[0,1]$ and by scaling and stationarity of the increment on any interval.*

Proof: Indeed there is a subsequence such that $V_{p,n}$ converge almost surely to ∞ for $p=1$ and $\alpha\in(0,2)$. If $\alpha\in(1,2)$ we can choose $p\in(2/\alpha,2)$ such that $V_{p,n}$ converge to 0 for some subsequence. This implies that the quadratic variation of $B_{\alpha,\beta}$ is zero. If $\alpha\in(0,1)$ we can choose $p>2$ such that $2p/\alpha<1$ and the p-variation of $B_{\alpha,\beta}$ must be infinite. So, in any case $B_{\alpha,\beta}$ can not be a semimartingale.

3.2.5.2 Hölder continuity

Proposition 3.2.7. *(Grothaus, [12]) Let $0<\alpha<2$ and $0<\beta<1$. Then for all $p\in N$ there exists $K<\infty$ such that $\mathbb{E}_{\mu\beta}(\mid B_t^{\alpha,\beta}-B_s^{\alpha,\beta}\mid^{2p})=K\mid t-s\mid^{\alpha p},t,s\geq 0$.*

The last proposition ensures that generalized grey Brownian motion has a continuous version. Indeed, choose $p\in\mathbb{N}$ such that $ap>1$ then the previous proposition provides the estimate $\mathbb{E}_{\mu\beta}((B_t^{\alpha,\beta}-B_s^{\alpha,\beta})^{2p})\leq k\mid t-s\mid^{1+p}$ with $q=ap-1>0$. This estimate is sufficient to apply Kolmogorov's continuity theorem.

3.2.5.3 Long-range dependency

Remark 3.2.5. *(Mura, [22]) Because of the stationarity of the increments, the anomalous diffusion appears deeply related to the long-range dependence characterization of* $B_{\alpha,\beta}(t)$*. We remember that an H-sssi process has long-range dependence (or long memory) if* $1/2 < H < 1$*. This means that the discrete time process of its increments exhibits long-range correlation. That is, the increments' autocorrelation function* $r(k)$ *tends to zero with a power law as* k *goes to infinity. Therefore, when* $0 < \alpha < 1$ *the diffusion is slow and the process has short memory. While when* $1 < \alpha < 2$ *the diffusion is fast and the process has long memory.*

3.3 Stochastic differential equations driven by generalized grey noise

In this section we establish a substitution formula for stochastic differential equation driven by generalized grey noise.

We consider the following stochastic differential equation (SDE) on $\mathbb{R}^n$

$$X_t = x_0 + \sum_{j=1}^{d} \int_0^t V_j(X_s) dB^j_{\alpha,\beta}(s) + \int_0^t V_0(X_s) ds, \qquad t \in [0,T]. \tag{3.12}$$

where $x_0 \in \mathbb{R}^n$, $T > 0$ is a fixed time, $B_{\alpha,\beta} = (B^1_{\alpha,\beta}, ..., B^d_{\alpha,\beta})$ is a d-dimensional ggBm, $\alpha \in (1,2), \beta \in (0,1]$ and $\{V_j, 0 \leq j \leq d\}$ is a collection of vector fields of $\mathbb{R}^n$.

The stochastic integral appearing in (3.12) is a pathwise Riemann-Stieltjes, see (Young, [34]), under suitable assumptions on $V = (V_1, ..., V_d)$, the equation (3.12) has a unique solution which is $(\frac{\alpha}{2})$-*Hölder* continuous for all $\varepsilon > 0$. This result was obtained in (Lyons, [16]). Nualart and Rãşcanu [24] have established the existence of a unique solution for a class of general differential equations that includes (3.12) using the fractional integration by parts formula obtained by Zähle for Young integral, see [35]. The representation in law of $B_{\alpha,\beta}$ see (3.11), allows us to consider, instead of the equation (3.12), the following equation:

$$X^H_t = x_0 + \sum_{j=1}^{d} \int_0^t V_j(X^H_s) d(\sqrt{Y_\beta} B^j_H)(s) + \int_0^t V_0(X^H_s) ds, \qquad t \in [0,T]. \tag{3.13}$$

This is due to the fact that the solutions of the SDEs (3.12) and(3.13) induces the same distribution on the space of continuous functions $\mathcal{C}([0,T],\mathbb{R}^n)$. Furthermore, since the stochastic integral in (3.13) is a pathwise Riemann-Stieltjes integral, then the SDE (3.13) can be written as

$$X_t^H = x_0 + \sqrt{Y_\beta}\sum_{j=1}^{d}\int_0^t V_j(X_s^H)dB_H^j(s) + \int_0^t V_0(X_s^H)ds, \qquad t \in [0,T]. \tag{3.14}$$

The main purpose of this section is to establish a substitution formula (SF) for equation (3.14). Let us now describe this approach. For each $y > 0$, we consider the following equation

$$X_t^H(y) = x_0 + \sqrt{y}\sum_{j=1}^{d}\int_0^t V_j(X_s^H(y))dB_H^j(s) + \int_0^t V_0(X_s^H(y))ds \tag{3.15}$$

It is well known that, under suitable assumptions, see e.g. Nualart and Rãşcanu [13], that the SDE (3.15) has a strong $(1-\lambda)$- *Hölder* continuous solution with $1-H < \lambda < \frac{1}{2}$. To establish a SF, the natural idea is to replace y in (3.15) by the random variable (Y_β) and prove that $X_\cdot^H(Y_\beta)$ satisfies the SDE (3.14). For more details on the SF we refer to Nualart [25]. To handle this problem, the key is to prove, for each $t \in [0,T]$ the following equalities

$$\int_0^t V_j(X_s^H(y))dB_H^j(s)|_{y=Y_\beta} = \int_0^t V_j(X_s^H(Y_\beta))dB_H^j(s), \quad j = 1,...,d, \tag{3.16}$$

and

$$\int_0^t V_0(X_s^H(y))ds|_{y=Y_\beta} = \int_0^t V_0(X_s^H(Y_\beta))ds. \tag{3.17}$$

To this end we need to study the regularity of the solution $X_t^H(y)$ of the SDE (3.15) with respect to y.

3.3.1 Preliminary

According to Mura and Pagnini [22], the ggBm $B_{\alpha,\beta}$ is a stochastic process defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that for any collection $0 \le t_1 < t_2 < \ldots < t_n < \infty$ the joint probability density function of $(B_{\alpha,\beta}(t_1),\ldots,B_{\alpha,\beta}(t_n))$ is given by (3.11).

Using the fact that the process $B_{\alpha,\beta}$ has $(\frac{\alpha}{2}-\varepsilon)-$ *Hölder* continuous trajectories for all $\varepsilon>0$. So, we can use the integral introduced by Young [34] with respect to $B_{\alpha,\beta}$. That is, for any *Hölder* continuous function f of order such that $\gamma+(\alpha/2)>1$ and every subdivision $(t_i^n)_{i=0,\ldots,T}$ of $[0,T]$, whose mesh tends to 0, as n goes to ∞, the Riemann sums

$$\sum_{i=0}^{n-1} f(t_i^n)(B_{\alpha,\beta}(t_{i+1}^n)-B_{\alpha,\beta}(t_i^n)),$$

converge to a limit which is independent of the subdivision $(t_i^n)_{i=0,\ldots,T}$. We denote this limit by

$$\int_0^T f(t)dB_{\alpha,\beta}(t).$$

For $0<\lambda<1$ we denote by $\mathcal{C}^\lambda([0,T],\mathbb{R}^d)$ the space of all $\lambda-$ *Hölder* continuous functions $f:[0,T]\to\mathbb{R}^d$, equipped with the norm

$$\| f \|_\lambda = \| f \|_{[0,T],\infty} + \| f \|_{[0,T],\lambda},$$

where

$$\| f \|_{[0,T],\infty} = \sup_{0\le t\le T} | f(t) |, \qquad \| f \|_{[0,T],\lambda} = \sup_{0\le s<t\le T} \frac{| f(t)-f(s) |}{| t-s |^\lambda}.$$

For $k,n,m\in\mathbb{N}$ we denote by $\mathcal{C}_b^k := \mathcal{C}_b^k(\mathbb{R}^n,\mathbb{R}^m)$ the space of all bounded functions on $\mathbb{R}^n$ which are k times continuously differentiable in Fréchet sense with bounded derivative up to the kth order, equipped with the norm

$$\| f \|_{\mathcal{C}_b^k} = \| f \|_\infty + \| \mathcal{D}f \|_\infty + \ldots + \| \mathcal{D}^k f \|_\infty < \infty.$$

We also denote by $\mathcal{C}_b^\infty := \mathcal{C}_b^\infty(\mathbb{R}^n,\mathbb{R}^m)$ the class of all infinitely differentiable (in Fréchet sense) bounded functions on $\mathbb{R}^n$ with bounded derivatives of all orders.

3.3.2 Substitution theorem

Throughout this subsection we assume that the coefficients V_0 and V satisfy the following hypothesis

$$\textbf{(H.1)} \qquad V_0\in\mathcal{C}_b^1, \qquad V\in\mathcal{C}_b^2.$$

First we give the regularity of the solution $X_t^H(y)$ of the SDE (3.15) with respect to y. This result will be proved using the following Fernique-type lemma.

Lemma 3.3.1. *(Saussereau, [30])*

- *(i). Let $T > 0$ and $1/2 < \delta < H < 1$ be given. Then, for any $\tau < 1/(128(2T)^{2(H-\delta)})$, we have*

$$\mathbb{E}(exp(\tau \parallel B_H \parallel^2_{[0,T],\delta})) \leq (1 - 128\tau(2T)^{2(H-\delta)})^{-1/2}.$$

- *(ii). For any integer $k \geq 1$ we have*

$$\mathbb{E}(\parallel B_H \parallel^{2k}_{[0,T],\delta}) \leq 32^k(2T)^{2k(H-\delta)}(2k)!.$$

Remark 3.3.1. *For any $\tau < 1/128(2T)^{2(H-\delta)})$ we have the following tail norm estimate for B_H:*

$$\mathbb{P}[\parallel B_H \parallel_{[0,T],\delta} > r] \leq Mexp(-\tau r^2), \tag{3.18}$$

where $M = \left(1 - 128\tau(2T)^{2(H-\delta)}\right)^{-1/2}$.

The following estimate is crucial in the proof of our main result, Theorem 3.3.1 below. It is worth to notice that such estimate was obtained and improved by Hu and Nualart [13].

Proposition 3.3.1. *(Nualart, [13]) Let $T > 0$ and $1/2 < \delta < H < 1$ be given. Under Hypothesis (**H.1**) there exist a positive constant C_n depending on $T, \delta, H, \parallel V_0 \parallel_{\mathcal{C}^1_b}$ and $\parallel V \parallel_{\mathcal{C}^2_b}$ such that*

$$\begin{aligned} \parallel X^H(y) - X^H(\widetilde{y}) \parallel_\delta \leq & C_n \mid \sqrt{y} - \sqrt{\widetilde{y}} \mid \parallel V \parallel_{\mathcal{C}^1_b} \parallel B_H \parallel_{[0,T],\delta} \\ & \times \left(1 + \parallel B_H \parallel_{[0,T],\delta}\right)^{2/\delta} exp\left(C_n \parallel B_H \parallel_{[0,T],\delta}\right)^{1/\delta}, \end{aligned}$$

for all $\mid y \mid, \mid \widetilde{y} \mid \leq n$.

Now we are ready to state the regularity of the solution $X^H(y)$ of the SDE (3.15) with respect to y.

Proposition 3.3.2. *Let $T > 0$ and $1/2 < \delta < H < 1$ be given. Under Hypothesis (**H.1**) there exist a positive $\widetilde{C}_n > 0$ depending on $T, \delta, H, \parallel V_0 \parallel_{\mathcal{C}^1_b}$ and $\parallel V \parallel_{\mathcal{C}^2_b}$ such that*

$$\mathbb{E}\left(\sup_{s \leq t} \mid X^H(y) - X^H(\widetilde{y}) \mid^4\right) \leq \widetilde{C}_n \mid y - \widetilde{y} \mid^2, \quad t \in [0,T],$$

for all $\mid y \mid, \mid \widetilde{y} \mid \leq n$.

Proof: Let $t \in [0,T]$ and $|y|, |\widetilde{y}| \leq n$ be fixed. Using the estimate in Proposition 3.3.1 we obtain

$$\mathbb{E}\left(\sup_{s\leq t} \mid X^H(y) - X^H(\widetilde{y}) \mid^4\right) \leq C_n^4 \mid \sqrt{y} - \sqrt{\widetilde{y}} \mid^4 \parallel V \parallel_{\mathcal{C}_b^1}$$

$$\times \mathbb{E}\left(\parallel B_H \parallel_{[0,T],\delta}^4 \left(1 + \parallel B_H \parallel_{[0,T],\delta}\right)^{8/\delta} exp\left(4C_n \parallel B_H \parallel_{[0,T],\delta}\right)^{1/\delta}\right)\right)$$

It follows from the assertions (i) and (ii) of Lemma 3.3.1 and the following Young inequality

$$4C_n \parallel B_H \parallel_{[0,T],\delta}^{1/\delta} \leq \frac{2\delta - 1}{2\delta}\left(\frac{4C_n}{\varepsilon}\right)^{2\delta/(2\delta-1)} + \varepsilon^{2\delta} \parallel B_H \parallel_{[0,T],\delta}^2$$

that, for small enough ε, there exist a constant $\widetilde{C}_n > 0$ depending on $T, \delta, H, \parallel V_0 \parallel_{\mathcal{C}_b^1}$ and$\parallel V \parallel_{\mathcal{C}_b^2}$ such that

$$\mathbb{E}(\sup_{s\leq t} \mid X_s^H(y) - X_s^H(\widetilde{y} \mid^4) \leq \widetilde{C}_n \mid y - \widetilde{y} \mid^2$$

The following proposition provides the substitution formulas (3.16) and (3.17).

Proposition 3.3.3. *Under Hypothesis (**H.1**) the equalities* (3.16) *and* (3.17) *are satisfied.*

Proof: First let's recall that, for $j = 1, ..., d$ and any $y > 0$, the Young integrals

$$\int_0^T V_j(X_s^H(y))dB_H^j(s), \tag{3.19}$$

and

$$\int_0^T V_j(X_s^H(Y_\beta))dB_H^j(s), \tag{3.20}$$

exist. Indeed, if $1 - H < \lambda < \frac{1}{2}$, then for each $y > 0$, the SDE (3.15) has a strong $(1-\lambda)$ -*Hölder* continuous solution $X^H(y)$. Therefore, the process $X_s^H(y)$ has $(1-\lambda) - Hölder$ continuous paths. Then the existence of the preceding integrals follows from the Lipschitz condition of V_j and the *Hölder* continuity of the paths of B_H^j. As a consequence, for any subdivision $(t_k^n)_{k=0,\ldots,n-1}$ of $[0,T]$, whose mesh tends to 0 as n goes to ∞, and each $y \geq 0$, the Riemann sums

$$S_n^j(y) = \sum_{k=0}^{n-1} V_j\left(X_{t_k^n}^H(y)\right)\left(B_H^j(t_{k+1}^n) - B_H^j(t_k^n)\right),$$

and

$$R_n^j = \sum_{k=0}^{n-1} V_j\left(X^H_{t^n_k}(Y_\beta)\right)\left(B^j_H(t^n_{k+1}) - B^j_H(t^n_k)\right),$$

converge to (3.19) and (3.20), respectively. Now to prove (3.16), it suffices to show that $S_n^j(Y_\beta) = R_n^j$, converge, as n goes to ∞, to

$$\int_0^T V_j(X_s^H(y))dB^j_H(s) \mid_{y=Y_\beta} .$$

Taking into account that the fBm with Hurst parameter H has locally bounded $p-$variation for $p > 1/H$ and the regularity of the solution $X_t^H(y)$ with respect to y, cf. Proposition 3.3.2, then the above mentioned convergence follows from Nualart [25] and the following estimate,

$$\mathbb{E} \mid S_n^j(y) - S_n^j(\widetilde{y}) \mid^4 = \mathbb{E} \mid \sum_{k=0}^{n-1}\left(V_j(X^H_{t^n_k}(y)) - V_j(X^H_{t^n_k}(\widetilde{y}))\right)\left(B^j_H(t^n_{k+1}) - B^j_H(t^n_k)\right) \mid^4,$$

$$\leq C \mid y - \widetilde{y} \mid^2,$$

for all $\mid y \mid, \mid \widetilde{y} \mid \leq n$.

The equality (3.17) is easy to prove.

The main result of this section is the following theorem.

Theorem 3.3.1. *The process* $\{X_t^H(Y_\beta), t \in [0,T]\}$ *satisfies the SDE* (3.13).

Proof: It follows from the classical Kolmogorov criterion that, for each $t \in [0,T]$ there exists a modification of the process $\{X_t^H(y), y \geq 0\}$ that is a continuous process whose paths are γ- *Hölder* for every $\gamma \in [0, \frac{1}{4})$. Now using the equalities (3.16) and (3.17) we obtain that the process $\{X_t^H(Y_\beta), t \in [0,T]\}$ satisfies the SDE (3.13) by substituting $y = Y_\beta(w)$ in the SDE (3.15). This completes the proof.

Conclusion

In this work we investigated the class of grey Brownian motion $B_{\alpha,\beta}(0 < \alpha < 2, 0 < \beta \leq 1)$ which is a class of self-similar stochastic processes with stationary increments.

First we have giving a background on stochastic calculus starting from the notions of processes and filtration, definition of stationary processes, self-similarity, H-sssi processes and we took the Brownian motion as an example.

Secondly we have discussed some useful mathematical definitions that are inherently tied to fractional calculus then some definitions of fractional integrals and derivatives are given, after that we have introduced the fractional Brownian motion.

Finally we showed that grey Brownian motion is in general a non-Gaussian process, is not a semimartingale and is of course Non-Markovian. This class of processes includes, fractional Brownian motion, Brownian motion and other $\frac{\alpha}{2}$-sssi process as special cases. GgBm admits different representations in terms of certain known processes such as Brownian motion subordinator, fractional Brownian motion subordinator and normal variance mixture $B_{\alpha,\beta}(t) = \sqrt{Y_\beta} X_\alpha(t), t \geq 0$, were $X_\alpha(t)$ is a standard fBm. This last representation is very interesting since many question related to gBm $B_{\alpha,\beta}$ may be reduced to questions concerning the fBm which is easier since it is Gaussian. Then we studied the problem of stochastic differential equation driven by gBm, we have established a substitution formula for stochastic differential equation driven by generalized grey noise.

Appendix

I. Riemann-Stieltjes integral

The Riemann-Stieltjes integral of a deterministic function $f : [a, b] \to \mathbb{R}$ (integrand) with respect to another deterministic function $g : [a, b] \to \mathbb{R}$ (integrator) is defined as the limit of sums

$$\sum_{i=1}^{n} f(s_i)g(t_i) - g(t_{i-1})$$

where $s_i \in [t_{i-1}; t_i], \pi = a = t_0 < t_1 < \ldots < t_n = b$, as the mesh of the partition π,$mesh(\pi) := max_{i=1,2,\ldots,n}|t_i - t_{i-1}|$, goes to 0. The Riemann-Stieltjes integral, denoted by

$$(R - S)\int_{[a,b]} f dg.$$

II. Hilbert-Schmidt Operators

An Hilbert-Schmidt operator is a bounded operator A, defined on an Hilbert space H, such that there exists an orthonormal basis $\{e_i\}_{i\in\mathcal{I}}$ of H with the property $\sum_{i\in\mathcal{I}} ||Ae_i||^2 < \infty$.

III. Anomalous diffusion

Anomalous diffusion is characterized by the asymptotic time power-law behaviour of the variance for large times: $\sigma^2(t) \sim t^\gamma$. Namely, the diffusion is slow if the exponent γ is lesser than one, normal if it is equal to one and fast if it is greater than one.

IV. The Wright function $W_{\lambda,\mu}(z)$

The Wright function, denoted by $W_{\lambda,\mu}(z)$, is named in honour of the British mathematician Edward Maitland Wright, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the theory of partitions.

Definition

The Wright function is defined by the series representation, convergent in the whole complex plane,

$$W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in \mathbb{C}.$$

The Auxiliary Functions of the Wright Type

Mainardi, in his first analysis of the time-fractional diffusion equation [11], introduced the two (Wright-type) entire auxiliary functions,

$$F_\nu(z) := W_{-\nu,0}(-z), \quad 0 < \nu < 1.$$

$$M_\nu(z) := W_{-\nu,1-\nu}(-z), \quad 0 < \nu < 1.$$

As a matter of fact, functions $F_\nu(z)$ and $M_\nu(z)$ are particular cases of the Wright function by setting $\lambda = -\nu$ and $\mu = 0$ or $\mu = 1$, respectively.

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