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# STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-LIPSCHTZIAN COEFFICIENTS

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par

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# Contents

# Introduction

Stochastic differential equations (SDEs) were first initiated and developed by K. Itô (1942). Today they have become a very powerful tool applied to mathematics, physics, chemistry, biology, medical sciences, and almost all sciences. Let us explain why we need SDEs.

In nature, physics, society, engineering and so on we always meet two kinds of functions with respect to time: one is deterministic, and another is random. For example, in financial market we deposit money $\pi_t$ in a bank. This can be seen as our having bought some units $\eta_t^0$ of a bond, where the bond's price $P_t^0$ satisfies the following ordinary differential equation

$$d\,P_t^0 = P_t^0 r_t dt,\ P_0^0 = 1,\ t \in [0,T]$$

where $r_t$ is the rate of the bond, and the money that we deposit in the bank is $\pi_t = \eta_t^0 P_t^0 = \eta_t^0 \exp[\int_0^t r_s ds]$. Obviously, usually, $P_t^0 = \exp[\int_0^t r_s ds]$ is non-random, since the rate $r_t$ is usually deterministic. However, if we want to buy some stocks from the market, each stock's price is random. For simplicity let us assume that in the financial market there is only one stock, and its price is $P_t^1$. Obviously, it will satisfy a differential equation as follows:

$$dP_t^1 = P_t^1(b_t\, dt + d(\text{a stochastic perturbation})),\ P_0^1 = P_0^1,\ t \in [0,T]$$

where all of the above processes are 1-dimensional. Here the stochastic perturbation is very important, because it influences the price of the stock, which will cause us to earn or lose money if we buy the stock. One important problem aries naturally. How can we model this stochastic perturbation? Can we make calculations to get the solution of the stock's price $P_t^1$, as we do in the case of the bond's price $P_t^0$? The answer is positive, usually a continuous stochastic perturbation will be modeled by a stochastic integral $\int_0^t \sigma_s dw_s$, where $w_t$, $t \geq 0$ is the so-called Brownian motion process (BM), or the Wiener process. The 1-dimensional BM $w_t$, $t \geq 0$ has the following nice properties: 1) (independent increment property). It has an independent increment property, that is, for any $0 < t_1 < \cdots < t_n$ the system $\{w_0, w_{t_1} - w_0, w_{t_2} - w_{t_1}, \cdots, w_{t_n} - w_{t_{n-1}}\}$ is an independent system. Or say, the increments, which happen in disjoint time intervals, occurred independently. 2) (Normal distribution property). Each increments is Normally distributed. That is, for any $0 \leq s < t$ the increment $w_t - w_s$ on this time interval is a normal random variable with mean $m$, and variance $\sigma^2(t-s)$. We write this as $w_t - w_s \sim N(m, \sigma^2(t-s))$. 3) (Stationary distribution property). The probability distribution of each increment only depends on the length of the time interval, and it does not depend on the starting point of the time interval. That is, the $m$ and $\sigma^2$ appearing in property 3) are constants. 4) (Continuous trajectory property). Its trajectory is continuous. That is BM $w_t$, $t \geq 0$ is continuous in $t$.

Since the simplest or say, the most basic continuous stochastic perturbation, intuitively will have the above four properties, the modeling of the general continuous stochastic perturbation by a stochastic integral with respect to this basic BM $w_t$, $t \geq 0$ is quite natural. However, the 1-dimensional BM also has some strange property: Even though it is continuous in $t$, it is nowhere differentiable in $t$. So we cannot define the stochastic integral $\int_0^t \sigma_s(\omega) dw_s(\omega)$ for each given $\omega$. That is why K. Itô (1942) invented a completely new way to define this stochastic integral.

Our first task in this work is to introduce the Itô stochastic integral and to establish the Itô formula and discuss its applications: solving SDE. The second task is to introduce the concepts of solutions and to discuss their existence and uniqueness and the related important theory. Since, actually, in the realistic world we will always meet some jump type stochastic perturbation.

This memory is organized as follows:

An introduction where we place our work and its plan.

The first chapter is devoted to the theory of stochastic calculus.

In the second chapter, we will discuss kinds of stochastic differential equations (SDE)

with jumps.
In the last chapter discussing solutions for stochastic differential equations (SDEs) with jumps and with non-Lipschitzian coefficients, is necessary and useful from the practical point of view.

# Chapter 1

# Stochastic calculus

## 1.1 Martingale theory and stochastic integral for point processes

A stochastic integral is a kind of integral quite different from the usual deterministic integral. However, its theory has broad and important application in science, mathematics itself, economic, finance, and elsewhere. A stochastic integral can be completely characterized by martingale theory. In this chapter we will discuss the elementary martingale theory, which forms the foundation of stochastic analysis and stochastic integral. As a first step we also introduce the stochastic integral with respect to a point process.

### 1.1.1 Concept of martingale

In some sense the martingale conception can be explained by a fair game. Let us interpret it as follows: in a game suppose that a person at the present time $s$ has wealth $x_s$ for the game, and at the future time t he will have the wealth $x_t$. The expected money for this person at the future time $t$ is naturally expressed as $\mathbb{E}[x_t/\mathfrak{F}_s]$, where $\mathbb{E}[\cdot]$ means the expectation value of $\cdot$, $\mathfrak{F}_s$ means the information up to time $s$, which is known by the gambler, and $\mathbb{E}[\cdot/\mathfrak{F}_s]$ is the conditional expectation value of(.) under given $\mathfrak{F}_s$. Obviously, if the game is fair, then it should be

$$\mathbb{E}[x_t/\mathfrak{F}_s] = x_s, \forall t \geq s.$$

This is exactly the definition of a martingale for a random process $x_t, t \geq 0$. Let us make it more explicit for later development. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, $\{\mathfrak{F}_t\}_{t\geq 0}$ be an information family, which satisfies the so-called "usual conditions":
$(i)$ $\mathfrak{F}_s \subset \mathfrak{F}_t$, as $0 \leq s \leq t$;
$(ii)$ $\mathfrak{F}_{t+} = \cap_{h>0}\mathfrak{F}_{t+h}$.
Here condition $(i)$ means that the information increases with time, and condition (ii) that the information is right continuous, or say, $\mathfrak{F}_{t+h} \downarrow \mathfrak{F}_t$, as $h \downarrow 0$. In this case we call $\{\mathfrak{F}_t\}_{t\geqslant 0}$ a $\sigma$-field filtration .

**Definition 1.1.1.** *A real random process $\{x_t\}_{t\geq 0}$ is called a martingale (super-martingale, sub-martingale) with respect to $\{\mathfrak{F}_t\}_{t\geq 0}$,or $\{x_t, \mathfrak{F}_t\}_{t\geq 0}$ is a martingale (super-martingale, sub-martingale), if*
*$(i)$ $x_t$ is integrable for each $t \geq 0$; that is, $\mathbb{E}|x_t| < \infty$, $\forall t \geq 0$;*
*$(ii)$ $x_t$ is $\mathfrak{F}_t$-adapted; that is, for each $t \geq 0$, $x_t$ is $\mathfrak{F}_t$-measurable;*
*$(iii)$ $\mathbb{E}[x_t/\mathfrak{F}_s] = x_s$,(respectively, $\leq, \geq$), a.s. $\forall 0 \leq s \leq t$.*

For the random process $\{x_t\}_{t\in[0,T]}$ and the random process $\{x_n\}_{n=1}^{\infty}$ with discrete time similar definitions can be given.

### 1.1.2 Stopping times, predictable process

**Definition 1.1.2.** *A random variable $\tau(\omega) \in [0, \infty]$ is called a $\mathfrak{F}_t$-stopping time, or simply, a stoping time, if for any $t \geq 0$, $\{\tau(\omega) \leq t\} \in \mathfrak{F}_t$.*

The intuitive interpretation of a stopping time is as follows: if a gambler has a right to stop his gamble at any time $\tau(\omega)$, he would of course like to choose the best time to stop. Suppose he stops his game before time t, i.e. he likes to make $\tau(\omega) \leq t$, then the maximum information he can get about his decision is only the information up to $t$, i.e $\{\tau(\omega) \leq t\} \in \mathfrak{F}_t$. The trivial example for a stopping time is $\tau(\omega) \equiv t, \forall \omega \in \Omega$. That is to say, any constant time $t$ actually is a stopping time. For a discrete random variable $\tau(\omega) \in \{0, 1, 2, \cdots, \infty\}$ the definition can be reduced to that $\tau(\omega)$ is a stopping time, if for any $n \in N$, $\{\tau(\omega) = n\} \in \mathfrak{F}_n$, since $\{\tau(\omega) = n\} = \{\tau(\omega) \leq n\} - \{\tau(\omega) \leq n-1\}$, and $\{\tau(\omega) \leq n\} = \cup_{k=1}^{n}\{\tau(\omega) = k\}$. The following properties of general stopping times will be useful later.

**Lemma 1.1.1.** *[70] $\tau(\omega)$ is a stopping time, if and only if $\{\tau(\omega) < t\} \in \mathfrak{F}_t, \forall t$.*

**Lemma 1.1.2.** *[70] Let $\sigma, \tau, \sigma_n, n \in N^\star$ be stopping times. Then*
*$(i)\, \sigma \wedge \tau, \sigma \vee \tau$,*
*$(ii)\, \sigma = \lim\limits_{n\to\infty} \sigma_n$, when $\sigma_n \uparrow or \sigma_n \downarrow$,*
*are all stopping times.*

**Proposition 1.1.1.** *[70] Let $\sigma, \tau, \sigma_n, n = 1, 2, \cdots$ be stopping times.*

1. *If $\sigma(\omega) \leq \tau(\omega), \forall\omega$, then $\mathfrak{F}_\sigma \subset \mathfrak{F}_\tau$.*
2. *If $\sigma_n(\omega) \downarrow \sigma(\omega), \forall\omega$, then $\cap_{n=1}^{\infty}\mathfrak{F}_{\sigma n} = \mathfrak{F}_\sigma$.*
3. *$\sigma \in \mathfrak{F}_\sigma$.(we use $f \in \mathfrak{F}_\sigma$ to mean that $f$ is $\mathfrak{F}_\sigma$ -measurable).*

**Definition 1.1.3.** *An $\mathbb{R}^d$ -valued process $\{x_t\}_{t\geq 0}$ is called measurable (respectively, progressive measurable), if the mapping*

$$(t, \omega) \in [0, \infty) \times \sigma \to x_t(\omega) \in \mathbb{R}^d$$

*(respectively, for each $t \geq 0, (s, \omega) \in [0, t] \times \Omega \to x_t(\omega) \in \mathbb{R})$ is $\mathfrak{B}([0, \infty)) \times \mathfrak{F}/\mathfrak{B}(\mathbb{R})$-measurable); (respectively, $\mathfrak{B}([0, t]) \times \mathfrak{F}_t/\mathfrak{B}(\mathbb{R})$-measurable); that is, $\{(t, \omega) : x_t(\omega) \in B\} \in \mathfrak{B}([0, \infty)) \times \mathfrak{F}, \forall B \in \mathfrak{B}(\mathbb{R}^d)$; (respectively, $\{(s, \omega) : s \in [0, t], x_s(\omega) \in B\} \in \mathfrak{B}([0, t]) \times \mathfrak{F}_t, \forall B \in \mathfrak{B}(\mathbb{R}^d)$).*

Let us introduce two useful $\sigma$-algebras as follows: Denote by $\mathcal{P}$(respectively,$\mathcal{O}$ ) as the smallest $\sigma$-algebra on $[0, \infty) \times \Omega$ such that all left-continuous (respectively, right-continuous) $\mathfrak{F}_t$-adapted processes

$$y_t(\omega) : [0, \infty) \times \Omega \to y_t(\omega) \in \mathbb{R}^d$$

are measurable. $\mathcal{P}$ (respectively, $\mathcal{O}$) is called the predictable (respectively, optional) $\sigma$-algebra. Thus, the following definition is natural.

**Definition 1.1.4.** *A process $\{x_t\}_{t\geq 0}$ is called predictable (optional), if the mapping*

$$(t,\omega) \in [0,\infty) \times \Omega \to x_t(\omega) \in \mathbb{R}^d$$

*is $\mathcal{P}/\mathfrak{B}(\mathbb{R}^d)$-measurable (respectively $\mathcal{O}/\mathfrak{B}(\mathbb{R}^d)$-measurable).*

**Theorem 1.1.1.** *[70] If $\{x_t\}_{t\geq 0}$ is a $\mathbb{R}^d$-valued progressive measurable process, then for each stopping time $\sigma$, $Z_\sigma I_{\sigma<\infty}$ is $\mathfrak{F}_\sigma$-measurable.*

### 1.1.3 Martingales with discrete Time

**Theorem 1.1.2.** *[70] Let $\{x_n\}_{n\in N}$ be a martingale (super-martingale, sub-martingale), $\sigma \leq \tau$ be two bounded stopping times. Then $\{x_n\}_{n\in N}$ is a strong martingale (respectively, strong super-martingale, strong sub-martingale), i.e.*

$$\mathbb{E}[x_\tau/\mathfrak{F}_\sigma] = x_\sigma(respectivly, \leq, \geq),\ a.s.$$

**Theorem 1.1.3.** *[70] Let $\{x_n\}_{n\in N}$ be a sub-martingale. Then for every $\lambda > 0$ and natural number $N$*

$$\begin{aligned}
\lambda\mathbb{P}(\max\nolimits_{0\leq n\leq N} x_n \geq \lambda) &\leq \mathbb{E}(x_N \mathbb{1}_{\max 0\leq n\leq N} x_n \geq \lambda) &&\leq \mathbb{E}(x_N^+) \\
&\leq \mathbb{E}|x_N|, \\
\lambda\mathbb{P}(\min\nolimits_{0\leq n\leq N} x_n \leq -\lambda) &\leq -\mathbb{E}x_0 + \mathbb{E}(x_N I_{\min 0\leq n\leq N} x_n > -\lambda) &&\leq \mathbb{E}_{x_0}^- + \mathbf{E}(x_N^+) \\
&\leq \mathbf{E}|x_0| + \mathbf{E}|x_N|.
\end{aligned}$$

### 1.1.4 Uniform integrability

It is well known in the theory of real analysis that if a sequence of measurable functions is dominated by an integrable function, then one can take the limit under the integral sign for the function sequence. That is the famous Lebesgue's dominated convergence theorem. However, sometimes it is difficult to find such a dominated function. In this case the uniform integrability of that function sequence can be a great help. Actually, in many cases it is a powerful tool .

**Definition 1.1.5.** *family of functions $A \subset L^1(\Omega, \mathfrak{F}, \mathbb{P})$ is called uniformly integrable, if $\lim\limits_{\lambda\to\infty} \sup\limits_{f\in A} \mathbb{E}(f\mathbb{1}_{|f|>\lambda}) = 0$ , where $L^1(\Omega, \mathfrak{F}, \mathbb{P})$ is the totality of random variables $\xi$, (that is, all $\xi$ are $\mathfrak{F}$-measurable) such that $\mathbb{E}|\xi| < \infty$.*

**Lemma 1.1.3.** *[70] Suppose that* $\{x_n\}_{n=1}^{\infty} \subset L^1(\Omega, \mathfrak{F}, \mathbb{P})$ *is uniformly integrable and as* $n \to \infty$, $x_n \to x$, *in probability i.e.* $\forall \varepsilon > 0$, $\lim\limits_{n\to\infty} \mathbb{P}(|x_n - x| > \varepsilon) = 0$, *then*

$$\lim_{n\to\infty} \mathbb{E}|x_n - x| = 0. \text{ (i.e. } x_n \to x, \text{ in } L^1(\Omega, \mathfrak{F}, \mathbb{P})\text{ )}$$

*In particular,* $\lim\limits_{n\to\infty} \mathbb{E}(x_n) = \mathbb{E}(x)$

**Lemma 1.1.4.** *[70] Suppose that* $A \subset L^1(\Omega, \mathfrak{F}, \mathbb{P})$. *Any one of the following conditions makes* $A$ *uniformly integrable:*

1. *There exists an integrable* $g \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ *such that* $|x| \leq g, \forall x \in A$.
2. *There exists a* $p > 1$ *such that* $\sup\limits_{x\in A} \mathbb{E}|x(\omega)|^P < \infty$ .

**Theorem 1.1.4.** *[70] Suppose that* $\{x_n\}_{n=1}^{\infty} \subset L^1(\Omega, \mathfrak{F}, \mathbb{P})$. *Then the following two statements are equivalent:*

1. $\{x_n\}_{n=1}^{\infty}$ *is uniformly integrable.*
2. $\sup\limits_{n\geq 1} \mathbb{E}|x_n| < \infty$; *and* $\forall \varepsilon > 0, \exists \delta > 0$ *such that* $\forall B \in \mathfrak{F}$, *as* $\mathbb{P}(B) < \delta$, $\sup\limits_{n\geq 1} \mathbb{E}|x_n| \mathbb{1}_B < \varepsilon$.
3. *Furthermore, if there exists an* $x \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ *such that as* $n \to \infty$, $x_n \to x$, *in probability; then the following statement is also equivalent to (1).*
4. $x_n \to x$, *in* $L^1(\Omega, \mathfrak{F}, \mathbb{P})$.

### 1.1.5 Martingales with continuous time

**Theorem 1.1.5.** *[70] Let* $\{x_t\}_{t\geq 0}$ *be a real right- continuous martingale (super-martingale, sub-martingale) with respect to* $\{\mathfrak{F}_t\}$, *and* $\{\sigma_t\}_{t\in[0,\infty]}$ *be a family of bounded stopping times such that* $\mathbb{P}(\sigma_t \leq \sigma_s) = 1$, *if* $t < s$. *Then* $\{x_t\}_{t\geq 0}$ *is a strong martingale (respectively, strong super-martingale, strong sub-martingale), i.e. as t<s,*

$$\mathbb{E}[x_{\sigma_s}/\mathfrak{F}_{\sigma_t}] = x_{\sigma_t} \text{ (respectively, } \leq, \geq\text{), a.s.}$$

### 1.1.6 Doob-Meyer decomposition theorem

In the incomplete financial market to price some option will involve the problem connected to the Doob-Meyer decomposition of some sub-martingales or super-martingales. Besides, this decomposition theorem is also a fundamental tool in stochastic analysis and its applications.

**Theorem 1.1.6.** *[70] Let $\{x_n\}_{n\in N}$ be a sub-martingale. The there exists a unique decomposition such that*

$$x_n = M_n + A_n,\, n \in N,$$

*where $\{M_n\}_{n\in N}$ is a martingale, and $\{A_n\}_{n\in N}$ is an increasing process, both are $\{\mathfrak{F}_n\}_{n\in N}$ -adapted, and $\{A_n\}_{n\in N}$ is predictable, where predictable means that $A_n \in \mathfrak{F}_{n-1}, \forall n = 1, 2, \cdots$, and $A_0 = 0$.*

### 1.1.7 Poisson random measure

A dynamical system will always encounter some jump stochastic perturbations. The simplest type comes from a stochastic point process. To understand it properly requires some preparation. Let $(Z, \mathfrak{B}_z)$ be a measurable space.

**Definition 1.1.6.** *A map $\mu(B,\omega) : \mathfrak{B}_z \times \Omega \to \mathbb{R}_+ \cup \{\infty\}$ is called a random measure on $\mathfrak{B}_z \times \Omega$, if*

1. *for any fixed $B \in \mathfrak{B}_z, \mu(B,\cdot)$ is a random variable but with values in $R_+ \cup \{\infty\}$;*
2. *for any fixed $\omega \in \Omega$, $\mu(\cdot,\omega)$ is a $\sigma$-finite measure. (Here, $\sigma$-finite means that there exists $\{U_n\}_{n=1}^{\infty} \subset \mathfrak{B}_z$ such that $Z = \cup_{n=1}U_n$ and $\mu(U_n,\omega) < \infty, \forall n$).*

Here the definition of a random variable taking values in $\mathbb{R}_+\{\infty\}$, is the same as that taking values in $\mathbb{R}$. Let us introduce the Poisson random measure as follows:

**Definition 1.1.7.** *A random measure $\mu(B,\omega)$ is called a poisson random measure on $\mathfrak{B}_z \times \Omega$, if it is non-negative integer valued (possibly $\infty$) such that*

1. *for each $B \in \mathfrak{B}_z$, $\mu(B,\cdot)$ is poisson distributed; i.e.*

$$\mathbb{P}\left(\{\omega : \mu(B,\omega) = n\}\right) = e^{-\lambda(B)}\frac{\lambda(B)^n}{n}, n \in N;$$

*where* $\lambda(B) = \mathbb{E}\,\mu(B,\omega), \forall B \in \mathfrak{B}_z$, *is usually called the mean measure, or the intensity measure of* $\mu$;

2. *if* $\mathfrak{B}_z \supset \{B_j\}_{j=1}^m$ *are disjoint, then* $\{\mu(B_j,\cdot)\}_{j=1}^m$ *are independent.*

Here as in the real analysis we still define $0.\infty = 0$. Thus if $\lambda(B) = \infty$, then all $\mathbb{P}(\{\omega : \mu(B,\omega) = n\}) = 0, n \in N$, hence $\mu(B,\omega) = \infty, \mathbb{P}$ a.s. The existence of a Poisson random measure is given by the following theorem.

**Theorem 1.1.7.** *[70] For any* $\sigma$*-finite measure* $\lambda$ *on* $(Z, \mathfrak{B}_z)$ *there exists a Poisson random measure* $\mu$ *with* $\lambda(B) = \mathbb{E}\,\lambda(B), \forall B \in \mathfrak{B}_z$.

### 1.1.8 Poisson point process

Now let us introduce the concept of random point processes. Assume that $(Z, B_z)$ is a measurable space. Suppose that $D_p \subset (0,\infty)$ is a countable set, then a mapping $p$: $D_p \to Z$, is called a point function (valued) on $Z$. Endow $(0,\infty) \times Z$ with the product $\sigma$-field $\mathfrak{B}((o,\infty)) \times \mathfrak{B}_z$, and define a counting measure through $p$ as follows:

$$N_p\left((0,t] \times U\right) = \sharp\{s \in D_p : s \le t,\, p(s) \in U\}, \forall t > 0, U \in \mathfrak{B}_z)$$

where $\sharp$ means the numbers of $\cdot$ counting in the set $\{\cdot\}$. Now let us consider a function of two variables $p(t,\omega)$ such that for each $\omega \in \Omega, p(\cdot,\omega)$ is a point function on Z, i.e. $p(\cdot,\omega) : D_{p(\cdot,\omega)} \to Z$, where $D_{p(\cdot,\omega)} \subset (0,\infty)$ is a countable set. Naturally, its counting measure is defined by

$$N_p((0,t] \times U,\omega) = N_{p(\omega)}((0,t] \times U) = \sharp\{s \in D_p : s \le t,\, p(s,\omega) \in U\}, \forall t > 0, U \in \mathfrak{B}_z$$

and we introduce the definition as follows:

**Definition 1.1.8.** 1. *If* $N_p((0,t] \times U,\omega)$ *is a random measure on* $(\mathfrak{B}((0,\infty)) \times \mathfrak{B}_z) \times \Omega$, *then* $p$ *is called a (random) point process.*

2. *If* $N_p((0,t] \times U,\omega)$ *is a Poisson random measure on* $(\mathfrak{B}((0,\infty)) \times \mathfrak{B}_z) \times \Omega$, *then* $p$ *is called a Poisson point process.*

3. *For a Poisson point process $p$ if its intensity measure $n_p(dtdx) = \mathbb{E}(N_p((dtdx))$ satisfies that*

$$n_p(dtdx) = \pi(dx)dt$$

*where $\pi(dx)$ is some measure on $(Z, \mathfrak{B}_z)$, then $p$ is called a stationary Poisson point process. $\pi(dx)$ is called the characteristic measure of $p$.*

One sees that the concept of a Poisson point process is finer than a Poisson process, because it also considers where jumps occur, as well as the jumps themselves. Sometimes to such situations more attention should be paid. For example, in many cases to count how many times the degree of an earthquake exceeds some level (that is, the point process drops in some area), where the earthquake happened in some area, is more important than counting all of the times it has happened. Actually, the forecast of an earthquake is only that its power is stronger than some degree. When the earthquake is very very small, usually, it is not necessary to forecast it. So the point process is more realistic.

### 1.1.9 Stochastic integral for point process

In a dynamical system the stochastic jump perturbation usually can be modeled as a stochastic integral with respect to some point process (i.e. its counting measure), or its martingale measure. In this section we will discuss how to define such stochastic integral. The idea is first to define it in the simple case by Lebesgue-Stieltjes integral for each or almost all $\omega \in \Omega$ (is said to define it pathwise). Then consider it in the general case through some limits. For this now let us consider a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with an increasing $\sigma$-field family $\{\mathfrak{F}_t\}_{t\geq 0}$, which satisfies the usual condition explained in the first section of this chapter. From now on all random variables and random processes are defined on it if without further explanation.

**Definition 1.1.9.** *Suppose that $p$ is a point process on $Z$, and*

$$N_p(t, U) = \sum_{s\in D_p, s\leq t} \mathbb{1}_U(p(s))$$

*is its counting measure.*

1. *$p$ is called $\mathfrak{F}_t$-adapted, if its counting measure is $\mathfrak{F}_t$-measurable for each $t \geq 0$ and each $U \in \mathfrak{B}_z$.*

2. *$p$ is called $\sigma$-finite, if $\exists \{U_n\}_{n=1}^{\infty} \subset \mathfrak{B}_z$, such that $\mathbb{E}\, N_p(t, U_n) < \infty, \forall t > 0, \forall n$, and $Z = \cup_{n=1}^{\infty} U_n$.*

From now on we only discuss the $\mathfrak{F}_t$-adapted and $\sigma$-finite point process $p$. Denote $\Gamma_p = \{U \in \mathfrak{B}_z : \mathbb{E}\, N_p(t, U) < \infty, \forall t > 0\}$ . Obviously, for any $U \in \Gamma_p, N_p(t, U)$ is a sub-martingale, since it is non- negative and increasing in $t$. Hence by Doob-Meyer's decomposition Theorem 1.1.6 there exists a unique $\mathfrak{F}_t$-adapted martingale $\widetilde{N_p}(t, U)$ and a unique $\mathfrak{F}_t$-adapted natural increasing process $\widehat{N}_p(t, U)$ such that

$$N_p(t, U) = \widetilde{N_p}(t, U) + \widehat{N}_p(t, U) \tag{1.1}$$

Notice that the equality only holds true $\mathbb{P}-$ a.s. for the given $U$. Hence $\widehat{N}_p(t, U)$ may not be a measure for $U \in \mathfrak{B}_z$, a.s. Moreover, it also may not be continuous in $t$. However, in most practical case we need $\widehat{N}_p(t, U)$ to have such properties.

**Definition 1.1.10.** *A point process $p$ is said to be of class $(QL)$ (meaning Quasi Left-continuous) if in the D-M decomposition expression 1.1:*

*(i) $\widehat{N}_p(t, U)$ is continuous in $t$ for any $U \in \Gamma_p$;*

*(ii) $\widehat{N}_p(t, U)$ is a $\sigma$-finite measure on $(Z, \mathfrak{B}_z)$ for any given $t \geq 0$, $\mathbb{P}-$ a.s.*

We will call $\widehat{N}_p(t, U)$ the compensator of $N_p(t, U)$ (or $p$). We now introduce the following definition for the $\mathfrak{F}_t$-Poisson point process.

**Definition 1.1.11.** *A point process $p$ is called an $\mathfrak{F}_t$-Poisson point process, if it is a Poisson point process,$\mathfrak{F}_t$-adapted, and $\sigma$-finite, such that $N_p(t + h, U) - N_p(t, U)$ is independent of $\mathfrak{F}_t$ for each $h > 0$ and each $U \in \Gamma_p$.*

Notice that $\widetilde{\mathfrak{F}}_t = \sigma[N_p((0, s] \times U); s \leq t, U \in \mathfrak{B}_z] \subset \mathfrak{F}_t$, and in general these may not equal to each other. This is why we have to assume that for a Poisson point process $N_p(t + h, U) - N_p(t, U)$ is independent of $\mathfrak{F}_t$. From now on we only discuss point processes which belong to class $(QL)$. By definition one can consider that the following proposition holds true.

**Proposition 1.1.2.** *[70] A $(\mathfrak{F}_t-)$ point processes $p$ is a stationary $(\mathfrak{F}_t)$-Poisson point process of class $(QL)$, if and only if its compensator has the form:*
$\widehat{N}_p(t,U) = t\pi(U), \forall t > 0, U \in \Gamma_p$, *where $\pi(\cdot)$ is a $\sigma$-finite measure on $\mathfrak{B}_z$.*

Now let us discuss the integral with respect to the point process. In the simple case it can be defined by the Lebesgue-Stieltjse integral. First, we have the following Lemma.

**Lemma 1.1.5.** *[70] For any given $U \in \mathfrak{B}_z$ and any bounded $\mathfrak{F}_t$-predictable process $f(t,\omega)$ let*

$$x_t(\omega) = \int_0^t f(s,\omega) d\widetilde{N}_p(s,U) = \int_0^t f(s,\omega) dN_p(s,U) - \int_0^t f(s,w) d\widehat{N}_p(s,U) =$$
$$\sum_{s \le t, s \in D_{p(\omega)}} f(s,\omega) \mathbb{1}_U(p(s,\omega)) - \int_0^t f(s,\omega) d\widehat{N}_p(s,U).$$

*Then $x_t$ is a $\mathfrak{F}_t$-martingale.*

The integral defined in the above lemma motivates us to define the stochastic integrals with respect to the counting measure and martingale measure generated by a point process of the class $(QL)$ for some class of stochastic processes as the integrands thru Lebesgue-Stieltjes integral.

## 1.2 Brownian motion, stochastic integral and Itô's formula

For a dynamic system the simplest continuous stochastic perturbation is naturally considered to be a Brownian motion (BM), since it is a Normal process (or say, a Guassian process) with independent increments which are also normally distributed. In general, a continuous stochastic perturbation will be modeled as some stochastic integral with respect to the BM. However, the BM has the strange property that even though its trajectory is continuous in t, it is not differentiable for all t. So for a stochastic integral with respect to BM one has to use a different approach - the martingale approach is used to define it.

### 1.2.1 Brownian motion and its nowhere differentiability

**Definition 1.2.1.** *A $d$-dimensional random process $\{x_t\}_{t\geq 0}$ is called a Brownian Motion $(BM)$ or a Weiner process, if*

1. *its initial probability law is given by some probability measure $\mu$, i.e. $\forall\Gamma \in \mathfrak{B}(\mathbb{R}^d), \mathbb{P}(x_0 \in \Gamma) = \mu(\Gamma)$;*
2. *it has independent increments, i.e. $\forall 0 = t_0 < t_1 < \cdots < t_m$, the increments $x_{t_0}, x_{t_1} - x_{t_0}, x_{t_2} - x_{t_1}, \cdots, x_{t_m} - x_{t_{m-1}}$ are independent;*
3. *$\forall 0 \leq s < t, x_t^i - x_s^i \sim N(0,(t-s)), i = 1,2,\cdots,d$, i.e. each real component increment $x_t^i - x_s^i$ is Normally distributed with the mean $\mathbb{E}(x_t - x_s) = 0$ and the variance $V(x_t - x_s) = (t-s)$; where $x_t = (x_t^1, x_t^2, \cdots, x_t^d)$, and $\{x_t^i\}_{i=1}^d$ is an independent random variable family for each $t > 0$;*
4. *it is continuous in $t$, a.s, that is, for almost all $\omega \in \Omega$ the trajectory $x_t(\omega)$ is continuous in $t$.*

*Now for $t > 0, x \in \mathbb{R}^d$ let $p(t,x) = (2\pi t)^{d/2} \exp[-|x|^2/2t]$.*

Now let us set $W^d =$ the set of all continuous $d$-dimensional functions $w(t)$ defined for $t \geq 0$. $\mathfrak{B}(W^d) =$ the smallest $\sigma$-field including all Borel cylinder sets in $W^d$, where a Borel cylinder set means a set $B \subset W^d$ of the following form

$$B = \{w : (w(t_1), \cdots, w(t_n)) \in A\},$$

for some finite sequence $0 \leq t_1 < t_2 < \cdots < t_n$ and $A \in \mathfrak{B}(\mathbb{R}^{nd})$. From above one sees that given a Brownian motion $\{x_t\}_{t\geq 0}$, this will lead to the generation of a probability measure $\mathbb{P}$ defined on $\mathfrak{B}(W^d)$. Such a probability measure is called a Wiener measure with the initial measure (or say, the initial law) $\mu$. Conversely, if we have a Wiener measure $\mathbb{P}$ with initial measure $\mu$ on $\mathfrak{B}(W^d)$, let $(\Omega, \mathfrak{F}, \mathbb{P}) = (W^d, \mathfrak{B}(W^d), \mathbb{P}), x(t,w) = w(t), \forall t \geq 0, w \in W^d$, then we obtain a $BM$ $\{x_t\}_{t\geq 0}$ defined on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. So the $BM$ is in one to one correspondence with the Wiener measure. Now a natural question arises: does the $BM$, that is, the

Wiener measure exist? The existence of the Brownian motion is established by the following theorem.

**Theorem 1.2.1.** *[70] For any probability measure $\mu$ on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ the Wiener measure $P_\mu$ with the initial law $\mu$ exists uniquely.*

**Definition 1.2.2.** *If a d-dimensional BM $\{x_t\}_{t\geq 0}$ is such that $\mathbb{P}(x_0 = 0) = 1$, that is, $\mu = \delta_0$ the probability measure concentrated at the single point $\{0\}$, then it is called the standard BM and denoted by $\{w_t\}_{t\geq 0}$.*

From now on we always discuss the standard Brownian motion, and it is simply denoted by $BM$. Brownian motion has some nice properties. For example, its trajectory is continuous, i.e. $\{x_t\}_{t\geq 0}$ is continuous. Moreover, it can be a square integrable martingale.

**Corollary 1.2.1.** *[70] If $\{x_t\}_{t\geq 0}$ is a d-dimensional $\mathfrak{F}_t$-BM and $\mathbb{E}|x_0|^2 < \infty$, then*

1. *$\{x_t\}_{t\geq 0}$ is a square integrable $\mathfrak{F}_t$-martingale;*

2. *$x_t^i x_t^j - \delta_{ij} t$ is a $\mathfrak{F}_t$-martingale.*

However, a $BM$ also has the following strange properties.

**Theorem 1.2.2.** *[70] Suppose that $\{x_t\}_{t\geq 0}$ is a 1-dimensional BM, then $\mathbb{P}-$ a.s, for any given $\alpha > \frac{1}{2}$, $\{x_t\}_{t\geq 0}$ is not Hölder-continuous with index $\alpha$ for each $t \geq 0$.*

**Definition 1.2.3.** *We say that $\{x_t\}_{t\geq 0}$ is Hölder-continuous with index $\alpha$ at $t_0 > 0$, if $\forall \varepsilon > 0, \exists \delta > 0$ such that as $|t - t_0| < \delta, |x_t - x_{t_0}| < \varepsilon |t - t_0|^\alpha$.*

theorem 1.2.2 actually tells us that the trajectory of $BM$ is not Lipschitzian continuous at each point $t$, so it is nowhere differentiable for $t \geq 0$. Hence it is also not finite variational on any finite interval of $t$, since each finite variational function of $t$ should be almost everywhere differentiable for $t$. Thus we arrive at the following corollary.

**Corollary 1.2.2.** *[70]*

1. *The trajectory of a BM is nowhere differentiable for $t \geq 0$, $\mathbb{P}$-a.s.*

2. *The trajectory of a BM is not finite variational on any finite interval of $t$, $\mathbb{P}$-a.s.*

### 1.2.2 Spaces $\mathcal{L}^0$ and $\mathcal{L}^2$

To discuss Itô's integral we first need to consider its integrand processes.

**Definition 1.2.4.** *Let*

$$
\begin{aligned}
\mathcal{L}^2 &= \left\{ \begin{array}{c} \{f(t,\omega)\}_{t\geq 0} : it\, is\, \mathfrak{F}_t - adapted,\, real\, such\, that \forall\, T>0 \\ \| f \|_{2,T}^2 = \mathbb{E}\displaystyle\int_0^T f^2(t,\omega)dt < \infty. \end{array} \right\}, \\
\mathcal{L}^0 &= \left\{ \begin{array}{l} \{f(t,\omega)\}_{t\geq 0} : it\, is\, \mathfrak{F}_t - adapted,\, real\, and \\ \exists : 0 = t_0 < t_1 < \cdots < t_n < \cdots \to \infty,\, and \\ \exists \varphi_i(\omega) \in \mathfrak{F}_{t_i}, \sup_i \| \varphi_i \|_\infty \, such\, that \\ f(t,\omega) = \varphi_0(\omega)\mathbb{1}_{t=0}(t) + \displaystyle\sum_{i=0}^{\infty} \varphi_i(\omega)\mathbb{1}_{[t_i,t_{i+1}]}(t). \end{array} \right\} \\
\mathcal{L}_T^2 &= \left\{ \{f(t,\omega)\}_{t\in[0,T]} : \{f(t,\omega)\}_{t\geq 0} \in \mathcal{L}^2 \right\} \\
\mathcal{L}_T^0 &= \left\{ \{f(t,\omega)\}_{t\in[0,T]} : \{f(t,\omega)\}_{t\geq 0} \in \mathcal{L}^0 \right\}
\end{aligned}
$$

Here $\| \varphi_i \|_\infty = ess\sup |\varphi_i(t,\omega)|$. Now let us discuss the relationship between $\mathcal{L}^2$ and $\mathcal{L}^0$.

**Lemma 1.2.1.** *[70] For $f = \{f(t,\omega)\}_{t\geq 0} \in \mathcal{L}^2$ let*

$$\| f \|_2 = \sum_{n=0}^{\infty} \frac{1}{2^n}(\| f \|_{2,n} \wedge 1)$$

*Then*

1. *$\| \cdot \|_2$ is a metric, and $\mathcal{L}^2$ is complete under this metric, if we make the identification $f = f', \forall f, f' \in \mathcal{L}^2$, as $\| f - f' \|_{2,n} = 0, \forall n$.*

2. *$\mathcal{L}^0$ is dense in $\mathcal{L}^2$ with respect to the metric $\| \cdot \|_2$ .*

### 1.2.3 Ito's integrals on $\mathcal{L}^2$

First, we will define the Itô integral for $\mathcal{L}^0$. Suppose that a $\mathfrak{F}_t$-Brownian motion $\{w_t\}_{t\geq 0}$ (Wiener process) is given on $(\Omega, \mathfrak{F}, \mathbb{P})$.

**Definition 1.2.5.** *For every* $f = \{f(t,\omega)\}_{t\geq 0} \in \mathcal{L}^0$*:*

$$f(t,\omega) = \varphi_0(\omega)\mathbb{1}_{t=0}(t) + \sum_{i=0}^{\infty} \varphi_i(\omega)\mathbb{1}_{[t_i,t_{i+1}]}(t)$$

*define for* $t_n \leq t < t_{n+1}, n \in N$,

$$I(f)(t,\omega) = \int_0^t f(s,\omega)dw(s,\omega) = \sum_{i=0}^{n} \varphi_i(\omega)(w(t_{i+1},\omega) - w(t_i,\omega)).$$

Firstly, it is easily seen that the stochastic integral also has an expression, which is actually a finite sum for each $0 < t < \infty$,

$$I(f)(t) = \sum_{i=0}^{\infty} \varphi_i(w(t_{i+1}\wedge t) - w(t_i \wedge t)),$$

moreover, $I(f)(t)$ is continuous in $t$. Secondly, it has the following property.

**Proposition 1.2.1.** *[70]*

1. *I(f)(0)=0, a.s. and for any* $\alpha,\beta \in \mathbb{R}; f,g \in \mathcal{L}^0$

$$I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$$

2. $\forall \in \mathcal{L}^0$*, and for each* $t > s \geq 0$,

$$\mathbb{E}[(I(f)(t) - I(f)(s))^2/\mathfrak{F}_s] = \mathbb{E}\left[\int_s^t f^2(u,\omega)du/\mathfrak{F}_s\right].$$

**Definition 1.2.6.** $I(f)$ *defined above is called the stochastic integral or the Itô integral of* $f \in \mathcal{L}^2$ *with respect to a BM* $\{w(t)\}_{t\geq 0}$*, and it is denoted by*

$$I(f)(t) = \int_0^t f(s)dw(s) = \int_0^t f(s,\omega)dw(s,\omega).$$

Beware of the fact that the integral is not defined pathwise. So, actually,

$$I(f)(t)(\omega_0) = \left(\int_0^t f(s)dw(s)\right)(\omega_0) = \left(\int_0^t f(s,\omega)dw(s,\omega)\right)(\omega_0), \mathbb{P}-\text{ a.s.}$$

### 1.2.4 Itô's integrals on $\mathcal{L}^{2,loc}$

First, let us introduce the concept of local martingales.

**Definition 1.2.7.** *1. An $\mathfrak{F}_t$-adapted real random process $\{x_t\}_{t\geq 0}$ is called a local $\mathfrak{F}_t$-martingale and denoted by $\{x_t\}_{t\geq 0} \in \mathcal{M}^{loc}$, if $\exists\, \sigma_n \uparrow \infty, \sigma_n < \infty$ is a $\mathfrak{F}_t$-stopping time for each $n$, such that $\{x_{t\wedge\sigma}\}_{t\geq 0}$ is a $\mathfrak{F}_t$-martingale for each $n$.*

*2. In addition, if $\{x_{t\wedge\sigma_n}\}_{t\geq 0}$ is a square integrable $\mathfrak{F}_t$-martingale for each $n$, then $\{x_t\}_{t\geq 0}$ is called a locally square integrable $\mathfrak{F}_t$-martingale, and it is denoted by $\{x_t\}_{t\geq 0} \in \mathcal{M}^{2,loc}$.*

*3. Write*

$$\mathcal{M}^{2,loc,c} = \left\{ \; \{x_t\}_{t\geq 0} \in \mathcal{M}^{2,loc} : \{x_t\}_{t\geq 0} \quad \textit{is cotinuous in } t \textit{ with}\, x_0 = 0 \; \right\}.$$

Now consider the definition of Itô's integral on $\mathcal{L}^{2,loc}$. For each $f \in \mathcal{L}^{2,loc}$, that is $\int_0^t |f(s,\omega)|^2 ds < \infty$, a.s., let $\sigma_n = \inf\{t \geq 0 : \int_0^t |f(s,\omega)|^2 ds > n\} \wedge n$. Then $\sigma_n \uparrow \infty, \sigma_n < \infty$ is a stopping time for each $n$. Obviously, $\{f(t,\omega)\mathbb{1}_{t\leq\sigma_n}\}_{t\geq 0} \in \mathcal{L}^2$ for each $n$, since

$$\mathbb{E}\int_0^T |f(t,\omega)\mathbb{1}_{t\leq\sigma_n}|^2 dt = \mathbb{E}\int_0^{T\wedge\sigma_n} f^2(t,\omega)dt \leq n < \infty, \forall T > 0.$$

Define in a natural way

$$I(f)(t\wedge\sigma_n) = \int_0^{t\wedge\sigma_n} f(s,\omega)dw_s = \int_0^t f(s,\omega)\mathbb{1}_{s\leq\sigma_n} dw_s, \forall n \in N^*.$$

This stochastic integral is well defined, since for $m > n$,

$$\int_0^{t\wedge\sigma_n} f(s,\omega)\mathbb{1}_{s\leq\sigma_m} dw_s = \int_0^t f(s,\omega)\mathbb{1}_{s\leq\sigma_m}\mathbb{1}_{s\leq\sigma_n} dw_s = \int_0^t f(s,\omega)\mathbb{1}_{s\leq\sigma_n}$$

Moreover, by definition $\{I(f)(t)\}_{t\geq 0} \in \mathcal{M}^{2,loc,c}$.

**Definition 1.2.8.** *$\forall f \in \mathcal{L}^{2,loc}$, define $\{I(f)(t)\}_{t\geq 0} \in \mathcal{M}^{2,loc,c}$ as above, then it is called the stochastic integral or the Itô integral of $f$ with respect to the BM $\{w(t)\}_{t\geq 0}$, and it is always denoted by*

$$I(f)(t) = \int_0^t f(s,\omega)dw_s(\omega) = \int_0^t f(s)dw_s.$$

*All of these integrals are called stochastic integrals.*

Finally, let us consider an $r$-dimensional $\mathfrak{F}_t$ $BM$

$$\{w(t)\}_{t\geq 0} = \{w^1(t), \cdots, w^r(t)\}_{t\geq 0}$$

Suppose that $f_i \in \mathcal{L}^{2,loc}, i = 1, \cdots, r$. Then the stochastic integral

$$\left\{ \int_0^t f_i(s,\omega)dw_s^i \right\}_{t\geq 0} \in \mathcal{M}^{2,loc,c}.$$

is defined for each $i = 1, \cdots, r$. We have the following proposition.

**Proposition 1.2.2.** *[70] There exist $\sigma_n \uparrow \infty$, $\sigma_n < \infty$ is a $\mathfrak{F}_t$-stopping time such that for each $n$ and $\forall t > s \geq 0$*

$$\mathbb{E}\left[ \int_{s\wedge\sigma_n}^{t\wedge\sigma_n} f_i(u,\omega)dw_u^i \int_{s\wedge\sigma_n}^{t\wedge\sigma_n} f_j(u,\omega)dw_u^j/\mathfrak{F}_s \right] = \delta_{ij}\mathbb{E}\left[ \int_{s\wedge\sigma_n}^{t\wedge\sigma_n} (f_if_j)(u,\omega)ds/\mathfrak{F}_s \right].$$

### 1.2.5 Stochastic integrals with respect to martingales

In this section we are going to discuss the stochastic integral $\int_0^t f(s,\omega)dM_s$, where $\{M_t\}_{t\geq 0} \in \mathcal{M}^2$. Recall that the procedure for defining Itô's integral $\int_0^t f(s,\omega)dw_s$ is as follows: First we define it for $f \in \mathcal{L}^0$ which is a simple process, and find that $\int_0^t f(s,\omega)dw_s \in \mathcal{M}^{2,c}$. Then, after establishing the one to one correspondence between space $\mathcal{L}^2$ and $\mathcal{M}^{2,c}$ with the same metric, for each $f \in \mathcal{L}^2$, we can take a sequence of $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^0$ which tends to $f$ in $\mathcal{L}^2$. So the corresponding sequence of integrals

$$\left\{ \int_0^t f_n(s,\omega)dw_s \right\}_{n=1}^{\infty} \in \mathcal{M}^{2,c}$$

will also tend to a limit in $\mathcal{M}^{2,c}$, which we denote by $\int_0^t f(s,\omega)dw_s$, and define it to be the stochastic integral for $f$. Note that for a $BM$ $\{w_t\}_{t\geq 0}$ we have that

$$w_t^2 = \text{a martingale} + t, \forall t \geq 0,$$

and we establish a one to one correspondence as follows: for each $T > 0$,

$$f \in \mathcal{L}_T^2 \Longleftrightarrow \left\{ \int_0^t f(s,\omega)dw_s \right\}_{t\in[0,T]} \in \mathcal{M}_T^{2,c},$$

with the same norm $\| f \|_{2,T} = \left[ \int_0^t f^2(s,\omega)dw_s \right]^{1/2}$. Now for a $\{M_t\}_{t\geq 0} \in \mathcal{M}^2$ we want to do the same thing. So first we need a $D-M$ decomposition for its square. For simplicity we discuss the one-dimensional processes.

**Proposition 1.2.3.** *[70]*

1. *If $\{M_t\}_{t\geq 0} \in \mathcal{M}^2$, then $\{M_t^2\}_{t\geq 0}$ has a unique $D-M$ decomposition as follows:*

$$M_t^2 = a\ martingale + \langle M \rangle_t,$$

*where $\langle M \rangle_t$ is a natural (predictable) integrable increasing process, and it is called the (predictable) characteristic process for $M_t$*

2. *If $\{M_t\}_{t\geq 0}, \{N_t\}_{t\geq 0} \in \mathcal{M}^2$, then $\{M_t N_t\}_{t\geq 0}$ has a unique decomposition (it may be still called the $D-M$ decomposition) as follows:*

$$M_t N_t = a\ martingale + \langle M, N \rangle_t,$$

*where $\langle M, N \rangle_t$ is a natural (predictable) integrable finite variational process, i.e. it is the difference of two natural (predictable) integrable increasing processes, and it is called the cross (predictable) characteristic process (or (predictable) quadratic variational $\mathfrak{F}_t$-adapted process) for $M_t$ and $N_t$.*

3. *If* $\{M_t\}_{t\geq 0}, \{N_t\}_{t\geq 0} \in \mathcal{M}^{2,loc}$, *then*
*(i)* $\exists\sigma_n \uparrow \infty, \sigma_n < \infty$ *is a stopping time for each* $n$ *such that* $\{M_{t\wedge\sigma_n}\}_{t\geq 0}, \{N_{t\wedge\sigma_n}\}_{t\geq 0} \in \mathcal{M}^2$ *for each* $n$*;*
*(ii) there exist a unique predictable process* $\{\langle M, N\rangle_t\}_{t\geq 0}$ *such that*

$$\langle M, N\rangle_{t\wedge\sigma_n} = \langle M^{\sigma_n} N^{\sigma_n}\rangle_t, \forall n \text{ and } \forall t > 0,$$

*where we write* $M_t^{\sigma_n} = M_{t\wedge\sigma_n}$, *and* $N_t^{\sigma_n} = N_{t\wedge\sigma_n}$

For the continuity of $\langle M, N\rangle_t$ we have the following proposition.

**Proposition 1.2.4.** *[70] Any one of the following conditions makes* $\langle M, N\rangle_t$ *continuous in* $t$*:*
*(i)* $\{\mathfrak{F}_t\}_{t\geq 0}$ *is continuous in time, i.e. if* $\sigma_n \uparrow \sigma$ *and they are all stopping times, then* $\mathfrak{F}_\sigma = \vee_n \mathfrak{F}_{\sigma_n}$; *(ii)* $M, N \in \mathcal{M}^{2,c}$.

For stochastic integrals with respect to the martingale $\{M_t\}_{t\geq 0} \in \mathcal{M}^2$, we need to introduce the space of integrand processes as in the case with respect to $BM$ $\{w_t\}_{t\geq 0} \in \mathcal{M}^{2,c}$.

**Definition 1.2.9.** *1. Write*

$$\mathcal{L}_M^2 = \left\{ \begin{array}{l} \{f(t,\omega)\}_{t\geq 0} : \textit{it is} \mathfrak{F}_t - \textit{predictable such that} \forall T > 0 \\ (\| f \|_{2,T}^M)^2 = \mathbb{E}\displaystyle\int_0^T f^2(t,\omega) d\langle M\rangle_t < \infty. \end{array} \right\}.$$

*For* $f = \{f(t,\omega)\}_{t\geq 0} \in \mathcal{L}_M^2$ *set*
$\| f \|_2^M = \displaystyle\sum_{n=1}^\infty \frac{1}{2^n}(\| f \|_{2,n}^M \wedge 1).$

2. $$\mathcal{L}_M^{2,loc} = \left\{ \begin{array}{l} \{f(t,\omega)\}_{t\geq 0} : \textit{it is} \mathfrak{F}_t - \textit{predictable such that if} \exists \sigma_n \uparrow \infty, \sigma_n \\ \textit{is a} \mathfrak{F}_t - \textit{stopping time for each } n, \textit{and} \\ \mathbb{E}\displaystyle\int_0^{T\wedge\sigma_n} f^2(t,\omega) d\langle M\rangle_t < \infty, \forall T > 0, \forall n \end{array} \right\}.$$

*3. $\mathcal{L}^0$ is defined the same as in Definition 1.2.4.*

Note that if $\mathbb{E}\int_0^{T\wedge\sigma_n} f^2(t,\omega)d\langle M\rangle_t < \infty, \forall T > 0, \forall n,$ then $\mathbb{P}-$ a.s. $\int_0^{N\wedge\sigma_n} f^2(t,\omega)d\langle M\rangle_t < \infty, \forall n, \forall N = 1,2,\cdots$. Therefore, $\int_0^{T} f^2(t,\omega)d\langle M\rangle_t < \infty, \forall T > 0, \mathbb{P}$-a.s. In general, the inverse is not necessary true. However, if $\langle M\rangle_t$ is continuous in $t$, then the inverse is also true. Reasoning almost completely in the same way as in lemma 1.2.1, one arrives at the following lemma.

**Lemma 1.2.2.** *[70] $\mathcal{L}^0$ is dense in $\mathcal{L}^2_M$ with respect to the metric $\|\cdot\|_2^M$.*

Now we can define the stochastic integral $\int_0^t f(s,\omega)dM_s$ with respect to $\{M_t\}_{t\geq 0}$ , first for $f \in \mathcal{L}^0$, and then for $f \in \mathcal{L}^2_M$, and finally for $f \in \mathcal{L}^{2,loc}_M$ in completely the same way as when defining $\int_0^t f(s,\omega)dw_s$. However, we would like to define it in another way, even if it is more abstract and different, because it is then faster and easier to show all of its properties.

**Definition 1.2.10.** *For $M \in \mathcal{M}^{2,loc}$ and $f \in \mathcal{L}^{2,loc}_M$ (or $M \in \mathcal{M}^2$ and $f \in \mathcal{L}^2_M$) if $X = \{x_t\}_{t\geq 0} \in \mathcal{M}^{2,loc}(X = \{x_t\}_{t\geq 0} \in \mathcal{M}^2)$ satisfies that*

$$\langle M, N\rangle(t) = \int_0^t f(u)d\langle M, N\rangle(u), \tag{1.2}$$

*$\forall N \in \mathcal{M}^{2,loc}(N \in \mathcal{M}^2), \forall t \geq 0$, then set $x_t = I^M(f)(t)$, and call it the stochastic integral of $f$ with respect to martingale $M$.*

In the rest of this section we always assume that $M \in \mathcal{M}^{2,loc}$. First let us show the uniqueness of $X \in \mathcal{M}^{2,loc}$ in Definition 1.2.10. In fact, if there is another $X' \in \mathcal{M}^{2,loc}$ such that 1.2 holds, then $\langle X - X', N\rangle = 0, \forall N \in \mathcal{M}^{2,loc}$. Hence by taking $N = X - X'$ one finds that $X = X'$. Secondly, we need to show that such a definition is equivalent to the usual one, which was explained before this definition.

**Proposition 1.2.5.** *[70] If $f \in \mathcal{L}^2_M$ is a stochastic step function, i.e. $f \in \mathcal{L}^2_M$, and $\exists\, \sigma_n \uparrow, \sigma_0 = 0$, $\sigma_0$ an is a $\mathfrak{F}_t$-stopping time for each $n$ such that*

$$f(t,\omega) = f_0(\omega)\mathbb{1}_{t=0} + \sum_{n=0}^{\infty} f_n(\omega)\mathbb{1}_{[\sigma_n,\sigma_{n+1}]}(t),$$

*where $f_n \in \mathfrak{F}_{\sigma_n}$, then*

$$I^M(f)(t) = \sum_{n=0}^{\infty} f_n(\omega)(M_{t\wedge\sigma_{n+1}} - M_{t\wedge\sigma_n}) = \int_0^t f(u)dM(u) \tag{1.3}$$

**Lemma 1.2.3.** *[70] If $M, N \in \mathcal{M}^2, f \in \mathcal{L}^2_M, g \in \mathcal{L}^2_N$, then*

$$\int_0^t |f \cdot g|(u)d|\langle M,N\rangle|_u \le |\int_0^t f^2(u)d\langle M\rangle_u|^{1/2}|\int_0^t g^2(u)d\langle N\rangle_u\,|^{1/2},$$

*where $|\langle M\rangle|_u$ is the total variation of $M_t, t \in [0,u]$. Note that we always write $d|\langle M,N\rangle|_u = |d\langle M,N\rangle|_u$.*

### 1.2.6 Itô's formula for continuous semi-Martingales

In calculus if $f(x), x(t) \in C^1$ and both are non-random, then

$$df(x(t)) = f'(x(t))dx(t).$$

However, this formula for a random process $x(t,\omega)$ is not necessary true.
For example, one can show that

$$|w(t)|^2 = 2\int_0^t w(s)dw(s) + t, \tag{1.4}$$

or symbolically, we can denote it by

$$d|w(t)|^2 = 2w(t)dw(t) + dt,$$

where $w(t)$ is the 1-dimensional standard $BM$. Actually, this is a special case of the famous Itô's formula. So one can understand how important and useful Itô's formula is in stochastic analysis and stochastic calculus.

**Proposition 1.2.6.** *[70] If $f(x) \in C^2(R)$, and $M \in \mathcal{M}^{2,loc,c}$, then*

$$f(M_t) - f(M_0) = \int_0^t f'(M_s)dM_s + \frac{1}{2}\int_0^t f''(M_s)d\langle M\rangle_s. \tag{1.5}$$

Now let us consider Itô's formula for the continuous semi-martingale. Suppose that

$$x_t = x_0 + A_t + M_t,$$

where $x_0 \in \mathfrak{F}_0, \{A_t\}_{t\geq 0}$ is a continuous finite variational ($\mathfrak{F}_t$-adapted) process with $A_0 = 0, \{M_t\}_{t\geq 0} \in \mathcal{M}^{2,loc,c}$. We will call such an $x_t$ a continuous semi-martingale. The same proof will show the following result.

**Theorem 1.2.3.** *[70] If $f(x) \in C^2(\mathbb{R})$, then*

$$f(x_t) - f(x_0) = \int_0^t f'(x_s)dA_s + \int_0^t f'(x_s)dM_s + \frac{1}{2}\int_0^t f''(x_s)d\langle M\rangle_s \tag{1.6}$$

### 1.2.7 Itô's formula for semi-Martingales with jumps

In the practical case we will always encounter some stochastic perturbation with jumps. So we need an Itô's formula for a semi-martingale with jumps. Consider

$$x_t = x_0 + A_t + M_t + \int_0^{t+}\int_Z (s,z,\omega)N_p(ds,dz) + \int_0^{t+}\int_Z g(s,z,\omega)\widetilde{N_p}(ds,dz),$$

where $x_0, \{A_t\}_{t\geq 0}$ and $\{M_t\}_{t\geq 0}$ are the same as in theorem 1.2.3, and $p$ is a $\mathfrak{F}$-point process of the class $(QL), f \in \mathcal{F}_p, g \in \mathcal{F}_p^{2,loc}$ such that $f(s,z,\omega)g(s,z,\omega) = 0$. We will call such a $x_t$ a semi-martingale with jumps. Here the last two terms are called jump terms. All jumps of $x_t$ are caused by them, and the condition $f(s,z,\omega)g(s,z,\omega) = 0$ means that the last two terms do not have the same jump times. We have the following general Itô's formula:

**Theorem 1.2.4.** *[70] If $F(x) \in C^2(\mathbb{R})$, then*

$$F(x_t) - F(x_0) = \int_0^t F'(x_s)dA_s + \int_0^t F'(x_s)dM_s + \frac{1}{2}\int_0^t F''(x_s)d\langle M\rangle_s$$

$$+\int_0^{t+}\int_Z [F(x_{s-}+f(s,z,\omega))-F(x_{s-})]N_p(ds,dz)$$

$$+\int_0^{t+}\int_Z [F(x_{s-}+g(s,z,\omega))-F(x_{s-})]\widetilde{N_p}(ds,dz)$$

$$+\int_0^{t+}\int_Z [F(x_s+g(s,z,\omega))-F(x_s)-F'(x_s)g(s,z,\omega)]\widehat{N}_p(ds,dz). \tag{1.7}$$

*holds. Moreover, 1.7 also can be rewritten as*

$$F(x_t)-F(x_0)=\int_0^t F'(x_s)dA_s+\int_0^t F'(x_s)dM_s+\frac{1}{2}\int_0^t F''(x_s)d\langle M\rangle_s$$

$$+\int_0^{t+}\int_Z F'(x_{s-})g(s,z,\omega)\widetilde{N_p}(ds,dz)+\int_0^{t+}\int_Z F'(x_{s-})f(s,z,\omega)N_p(ds,dz)$$

$$+\int_0^{t+}\int_Z [F(x_{s-}+f(s,z,\omega))-F(x_{s-})-F'(x_{s-})f(s,z,\omega)]N_p(ds,dz)$$

$$+\int_0^{t+}\int_Z [F(x_{s-}+g(s,z,\omega))-F(x_{s-})-F'(x_{s-})g(s,z,\omega)]N_p(ds,dz)$$

*or, more simply,*

$$F(x_t)-F(x_0)=\int_0^t F'(x_{s-})dx_s+\frac{1}{2}\int_0^t F''(x_s)d\langle M\rangle_s$$

$$+\int_0^{t+}\int_Z [F(x_{s-}+f(s,z,\omega))-F(x_{s-})-F'(x_{s-})f(s,z,\omega)]N_p(ds,dz)$$

$$+\int_0^{t+}\int_Z [F(x_{s-}+g(s,z,\omega))-F(x_{s-})-F'(x_{s-})g(s,z,\omega)]N_p(ds,dz) \tag{1.8}$$

### 1.2.8 Itô's formula for $d$-dimensional semi-martingales and integration by parts

The above Itô's formula for one-dimensional semi-martingales with jumps is easily generalized to that for the $n$-dimensional case. Consider a $d$-dimensional semi-martingales with jumps as follows: $x_t = (x_t^1, \cdots, x_t^d)$, where for $i = 1, 2, \cdots, d$

$$x_t^i = x_0^i + A_t^i + M_t^i$$

$$+ \int_0^{t+} \int_Z f^i(s, z, \omega) N_p(ds, dz) + \int_0^{t+} \int_Z g^i(s, z, \omega) \widetilde{N_p}(ds, dz)$$

where $x_0 \in \mathfrak{F}_0, \{A_t\}_{t \geq 0}$ is a finite variational ($\mathfrak{F}_t$-adapted) process, and $\{M_t\}_{t \geq 0} \in \mathcal{M}^{2,loc,c}$ , all are $d$-dimensional, and $p$ is a $\mathfrak{F}_t$-point process of the class $(QL), f \in \mathcal{F}_p, g \in \mathcal{F}_p^{2,loc}$ such that $f^i(s, z, \omega) g^j(s, z, \omega) = 0, \forall i, j = 1, 2, \cdots, d$. Then we have the following theorem.

**Theorem 1.2.5.** *[70] If a real function $F(x) \in C^2(\mathbb{R}^d)$, then*

$$F(x_t) - F(x_0) = \sum_{i=1}^d \int_0^t F'_{x^i}(x_{s-}) dx_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t F''_{x^i x^j}(x_s) d\langle M^i, M^j \rangle_s$$

$$+ \int_0^{t+} \int_Z [F(x_{s-} + f(s, z, \omega)) - F(x_{s-}) - \sum_{i=1}^d F'_{x^i}(x_{s-}) f^i(s, z, \omega)] N_p(ds, dz)$$

$$+ \int_0^{t+} \int_Z [F(x_{s-} + g(s, z, \omega)) - F(x_{s-}) - \sum_{i=1}^d F'_{x^i}(x_{s-}) g^i(s, z, \omega)] N_p(ds, dz), \quad (1.9)$$

*or,*

$$F(x_t) - F(x_0) =$$

$$\sum_{i=1}^d \int_0^t F'_{x^i}(x_{s-}) dA_s^i + \sum_{i=1}^d \int_0^t F'_{x^i}(x_{s-}) dM_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t F''_{x^i x^j}(x_s) d\langle M^i, M^j \rangle_s$$

$$+ \int_0^{t+} \int_Z [F(x_{s-} + g(s, z, \omega)) - F(x_{s-})] \widetilde{N_p}(ds, dz)$$

$$+\int_0^{t+}\int_Z [F(x_{s-}+f(s,z,\omega))-F(x_{s-})]N_p(ds,dz),$$

$$+\int_0^{t+}\int_Z [F(x_{s-}+g(s,z,\omega))-F(x_{s-})-\sum_{i=1}^d F'_{x^i}(x_{s-})g^i(s,z,\omega)]\widehat{N}_p(ds,dz),$$

**Remark 1.2.1.** *1. If we denote*

$$[x^i,x^j]_t=\langle x^{ic},x^{jc}\rangle_t+\sum_{s\le t}(\Delta x^i\Delta x^j)=\langle M^i,M^j\rangle_t+\sum_{s\le t}(\Delta x^i\Delta x^j),$$

*which is called the cross quadratic variational process (cross characteristics) of semi-martingales* $\{x_t^i\}_{t\ge 0}$ *and* $\{x_t^j\}_{t>\ge 0}$*, then 1.9 can be rewritten as*

$$F(x_t)-F(x_0)=\sum_{i=1}^d\int_0^t F'_{x^i}(x_{s-})dx_s^i+\frac{1}{2}\sum_{i,j=1}^d\int_0^t F''_{x^ix^j}(x_s)d[x^i,x^j]_s$$

$$+\int_0^{t+}\int_Z [F(x_{s-}+f(s,z,\omega))-F(x_{s-})-\sum_{i=1}^d F'_{x^i}(x_{s-})f^i(s,z,\omega)$$

$$-\frac{1}{2}\sum_{i,j=1}^d\int_0^t F''_{x^ix^j}(x_{s-})f^i(s,z,\omega)f^j(s,z,\omega)]N_p(ds,dz)$$

$$+\int_0^{t+}\int_Z [F(x_{s-}+g(s,z,\omega))-F(x_{s-})-\sum_{i=1}^d F'_{x^i}(x_{s-})g^i(s,z,\omega)$$

$$-\frac{1}{2}\sum_{i,j=1}^d\int_0^t F''_{x^ix^j}(x_{s-})g^i(s,z,\omega)g^j(s,z,\omega)]N_p(ds,dz)$$

*2. Ito's formula 1.9 can also be written symbolically in differential form as*

$$dF(x_t) = \sum_{i=1}^{d} F'_{x^i}(x_{t-})dx_t^i + \frac{1}{2}\sum_{i,j=1}^{d} F''_{x^ix^j}(x_t)d\langle M^iM^j\rangle_t$$

$$\int_Z [F(x_{t-} + f(s,z,\omega)) - F(x_{t-}) - \sum_{i=1}^{d} F'_{x^i}(x_{t-})f^i(s,z,\omega)]N_p(dt,dz),$$

$$\int_Z [F(x_{t-} + g(s,z,\omega)) - F(x_{t-}) - \sum_{i=1}^{d} F'_{x^i}(x_{t-})g^i(s,z,\omega)]N_p(dt,dz),$$

*or,*

$$dF(x_t) = \sum_{i=1}^{d} F'_{x^i}(x_{t-})dx_t^i + \frac{1}{2}\sum_{i,j=1}^{d} F''_{x^ix^j}(x_{t-})d[x^i,x^j]_t$$

$$+ d\eta_t(F,g) + d\eta_t(F,f) \tag{1.10}$$

*where*

$$d\eta_t(F,g) = \int_Z [F(x_{t-} + g(t,z,\omega)) - F(x_{t-}) - \sum_{i=1}^{d} F'_{x^i}(x_{t-})g^i(t,z,\omega)]$$

$$-\frac{1}{2}\sum_{i,j=1}^{d} F''_{x^ix^j}(x_{t-})g^i(t,z,\omega)g^j(t,z,\omega)]N_p(dt,dz),$$

*and $d\eta_t(F,f)$ is similarly defined.*

By Itô's formula one easily derives the formula of integration by parts for semi-martingales with jumps. Suppose the semi-martingale $\{x_t^i\}_{t\geq 0}$ are given as above, $i = 1, 2, \cdots, d$. Then we have the following theorem (integration by parts).

**Theorem 1.2.6.** *[70] We have:*

$$dx_t^ix_t^j = x_t^idx_t^j + x_t^jdx_t^i + d[x^i,x^j]_t,$$

*or equivalently,*

$$x_t^ix_t^j - x_0^ix_0^j = \int_0^t x_s^idx_s^j + \int_0^t x_s^jdx_s^i + [x^i,x^j]_t.$$

### 1.2.9 Independence of BM and poisson point processes

As an application of Itô's formula we can prove the independence of BM and Poisson point processes, which is very important in stochastic analysis. For simplicity let us first discuss the independence of a 1-dimensional Brownian motion and a Poisson point process.

**Theorem 1.2.7.** *[70] Assume that $\{x_t\}_{t\geq 0}$ is a 1-dimensional $\mathfrak{F}_t$-semi-martingale, and $p$ is a $\mathfrak{F}_t$-point process of class $(QL)$. If*

1. *$M_t = x_t - x_0 \in \mathcal{M}^{2,loc,c}$, $\langle M\rangle_t = t$;*

2. *The compensator $\widehat{N}_p(dt, dz)$ of $p$ is a non-random $\sigma$-finite measure on $[0,\infty)\times Z$; then $\{x_t\}_{t\geq 0}$ is a 1-dimensional $\mathfrak{F}_t$-BM, and $p$ is a $\mathfrak{F}_t$-Poisson point process such that they are independent.*

*The above theorem is easily generalized to the d-dimensional case.*

**Theorem 1.2.8.** *[70] Assume that $\{x_t\}_{t\geq 0}$ is a d-dimensional $\mathfrak{F}_t$-semi-martingale, where $x_t = (x_t^1, \cdots, x_t^d)$, and $p_i, i = 1, 2, \cdots, n$, are $\mathfrak{F}_t$-point processes of class $(QL)$ on state spaces $Z_i, i = 1, 2, \cdots, n$, respectively. If*

1. *$M_t^i = x_t^i - x_0^i \in \mathcal{M}^{2,loc,c}$, $\langle M^i M^j\rangle_t = \delta_{ij}t$; $i, j = 1, 2, \cdots, d$,*

2. *the compensator $\widehat{N}_{p_i}(dt, dz)$ of $p_i$ is a non-random $\sigma$-finite measure on $[0,\infty)\times Z, i = 1, 2, \cdots, n$; and the domains $D_{p_i(\omega)}, i = 1, 2, \cdots, n$, are mutually disjoint, a.s. Then $\{x_t\}_{t\geq 0}$ is a d-dimensional $\mathfrak{F}_t$-BM, and $p_i(i = 1, 2, \cdots, n)$ is a $\mathfrak{F}_t$-Poisson point process such that they are mutually independent.*

### 1.2.10 Strong Markov property of BM and poisson point processes

The martingale characterization of BM and Poisson point processes (theorem 1.2.8) can be used to show that a BM is still a BM if it starts again from any stopping time, and this property is also true for a stationary Poisson point process.

**Theorem 1.2.9.** *[70] If $x_t = (x_t^1, \cdots, x_t^d)$ is a d-dimensional $\mathfrak{F}_t$-BM, and $\sigma$ is a $\mathfrak{F}_t$-stopping time with $\mathbb{P}(\sigma < \infty) = 1$, then $\{x_t^*\}_{t\geq 0} = \{x_{t+\sigma}\}_{t\geq 0}$ is a d-dimensional $\mathfrak{F}_t^* = \mathfrak{F}_{t+\sigma} - BM$. In particular, $\{w_t^*\}_{t\geq 0} = \{x_{t+\sigma} - x_\sigma\}_{t\geq 0}$ is a standard BM independent of $\mathfrak{F}_0^* = \mathfrak{F}_\sigma$.*

**Theorem 1.2.10.** *[70] If $p$ is a stationary $\mathfrak{F}_t$-Poisson point process on some space $Z$ with the characterictic measure $\pi(dz)$,and $\sigma$ is a $\{\mathfrak{F}_t\}_{t\geq 0}$-stopping time with $\mathbb{P}(\sigma < \infty) = 1$, then $p^* = \{p^*(t)\}_{t\in D_{p^*}} = \{p(t+\sigma)\}_{t+\sigma\in D_p}$ is a stationary $\{\mathfrak{F}_t^*\}_{t\geq 0} = \{\mathfrak{F}_{t+\sigma}\}_{t\geq 0}$-Poisson point process with the same characterictic measure $\pi(dz)$.*

By the previous two theorems one immediately sees that a standard BM and a stationary Poisson point processes are both stationary strong Markov processes. That is to say, if $\{w_t\}_{t\geq 0}$ is a $d$-dimensional $\mathfrak{F}_t$-standard BM, then for any $\mathfrak{F}_t$-stopping time $\sigma$ with $\mathbb{P}(\sigma < \infty) = 1$, it satisfies $\forall A \in \mathfrak{B}(\mathbb{R}^d), \forall t > 0,$

$$\mathbb{P}(w_{t+\sigma} \in A/\mathfrak{F}_\sigma), \text{ a.s.}$$

In fact, by Theorem 1.2.9 one has that it is equivalent to

$$\mathbb{P}(w_t^* \in A/\mathfrak{F}_0^*) = \mathbb{P}(w_t \in A/\mathfrak{F}_0) \Leftrightarrow \mathbb{P}(w_t^* \in A) = \mathbb{P}(w_t \in A)$$

where $\{w_t^*\}_{t\geq 0}$ is a $\mathfrak{F}_t^*$-BM. The last equality is obviously true. It is natural to define the strong Markov property of a stationary $\mathfrak{F}_t$-point process $p$ as follows: If $p$ satisfies that $\forall t > 0, \forall k = 1, 2, \cdots,$

$$\mathbb{P}(N_p((0, t+\sigma] \times U) = k/\mathfrak{F}_\sigma) = \mathbb{P}(N_p((0, t] \times U) = k/\mathfrak{F}_0) \text{ a.s.},$$

then $p$ is called a strong Markov $\mathfrak{F}_t$-point process. Thus we arrive at the following corollary.

**Corollary 1.2.3.** *[70] The $\mathfrak{F}_t$-standard BM and $\mathfrak{F}_t$-stationary Poisson point processes are both stationary strong Markov processes.*

### 1.2.11 Martingale representation theorem

The martingale representation theorem is very useful in the mathematical financial market and in the filtering problems. More precisely, we have the following theorem.

**Theorem 1.2.11.** *[70] Let $m(t)$ be a square integrable $\mathbb{R}^d$-valued $\mathfrak{F}_t^{w,k}$-martingale, where $\mathfrak{F}_t^{w,k}$ is the $\sigma$-algebra generated (and completed) by $\{w_s, k_s, s \leq t\}$, and $\{w_t\}_{t\geq 0}$ is a $d_1$-dimensional BM, $\{k_t\}_{t\geq 0}$ is a stationary $d_2$-dimensional poisson point process of the class $(QL)$ such that the components $\{k_t^1\}_{t\geq 0}, \cdots, \{k_t^{d_2}\}_{t\geq 0}$ have disjoint domains and disjoint ranges. Then there exists a unique $(q_t, p_t) \in L^2_{\mathfrak{F}^{w,k}}(R^{d\otimes d_1}) \times F^2_{\mathfrak{F}^{w,k}}(R^{d\otimes d^2})$ such that*

$$m(t) = m(0) + \int_0^t q_s dw_s + \int_0^t \int_Z p_s(z)\widetilde{N}_k(ds, dz).$$

*Here we write*

$$L^2_{\mathfrak{F}}(\mathbb{R}^{d\otimes d_1}) = \{f(t,\omega) : f(t,\omega) \; is \; \mathfrak{F}_t - adapted, \mathbb{R}^{d\otimes d_1} - valued \;\; such \;\; that$$

$$\mathbb{E}\int_0^T |f(t,\omega)|^2 dt < \infty, for \, any \, T < \infty\}$$

*and*

$$F^2_{\mathfrak{F}}(\mathbb{R}^{d\otimes d_2}) = \{f(t,z,\omega) : f(t,z,\omega)\} \; is \; \mathbb{R}^{d\otimes d_2} - valued, -\mathfrak{F}_t \, predictable \;\; such \;\; that$$

$$\mathbb{E}\int_0^T \int_Z |f(t,z,\omega)|^2 \pi(dz) dt < \infty, \forall T < \infty\}$$

# Chapter 2

# Stochastic differential equations

In the practical case a dynamical system will always be disturbed by some stochastic perturbation, one type of which is continuous, and it can be modeled by some stochastic integral with respect to the BM, and the other is of the jump type, which is usually modeled by some stochastic integral with respect to the martingale measure generated by a point process. In this chapter we will discuss such kinds of stochastic differential equations (SDE) with jumps.

## 2.1 Strong solutions to SDE with jumps

### 2.1.1 Notation

Suppose we are given a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with a $\sigma$-field filtration $\{\mathfrak{F}_t\}_{t\geq 0}$. Consider the following SDE with jumps in $d$-dimensional space :

$$x_t = x_0 + \int_0^t b(s, x_s, \omega)ds + \int_0^t \sigma(s, x_s, \omega)dw_s$$

$$+ \int_0^t \int_Z c(s, x_{s-}, z, \omega)\widetilde{N}_k(ds, dz), t \geq 0 \tag{2.1}$$

where $w_t^T = (w_t^1, \cdots, w_t^{d_1}), 0 \leq t$, is a $d_1$-dimensional $\mathfrak{F}_t$-adapted standard Brownian motion $(BM), w_t^T$ is the transpose of $w_t^T; k^T = (k_1, \cdots, k_{d_2})$ is a $d_2$-dimensional

$\mathfrak{F}_t$-adapted stationary Poisson point process with independent components, and $\widetilde{N}_{k_i}(ds, dz)$ is the Poisson martingale measure generated by $k_i$ satisfying

$$\widetilde{N}_{k_i}(ds, dz) = N_{k_i}(ds, dz) - \pi(dz)ds, i = 1, \cdots, d_2.$$

Here $\pi(\cdot)$ is a $\sigma$-finite measure on a measurable space $(Z, \mathfrak{B}(Z))$, and $N_{k_i}(ds, dz)$ is the Poisson counting measure generated by $k_i$. From now on, for simplicity, we will always denote the integral $\int_0^t = \int_{(0,t]} = \int_0^{t+}$.

**Definition 2.1.1.** *$\{x_t\}_{t\geq 0}$ (or, simply, $x_t$) is said to be a $(\mathfrak{F}_t)$-solution of 2.1, if $\{x_t\}_{t\geq 0}$ satisfies 2.1. In the case that $x_t \in \mathfrak{F}_t^{w,k}, \forall t \geq 0$, where $\mathfrak{F}_t^{w,k}$ is the $\sigma$-algebra generated (and completed) by $w_s, k_s, s \leq t$, and then it is called a strong solution.*

From Definition 2.1.1 it is seen that for discussing the solution of 2.1 we always need to assume that the coefficients satisfy the following assumption:

**$(A)_1$:** $b$ and $\sigma : [0, \infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d, c : [0, \infty) \times \mathbb{R}^d \times Z \times \Omega \to \mathbb{R}^d$ are jointly measurable and $\mathfrak{F}_t$-adapted where, furthermore, $c$ is $\mathfrak{F}_t$-predictable. Moreover, to simplify the discussion of , we will also suppose that all $N_{k_i}(ds, Z), 1 \leq i \leq d_2$, have no common jump time; i.e. we always make the following assumption

**$(A)_2$:** $N_{k_i}(\{t\}, U)N_{k_j}(\{t\}, U) = 0$, as $i \neq j$, for all $U \in \mathfrak{B}(Z)$.

**Definition 2.1.2.** *We say that the pathwise uniqueness of solutions to 2.1 holds, if for any two solutions $\{x_t^i\}_{t\geq 0}, i = 1, 2$ satisfying 2.1 on the same probability space with the same Brownian motion $\{w_t\}_{t\geq 0}$ and Poisson martingale measure $\widetilde{N}_k(dt, dz), \mathbb{P}\left(\sup_{t\geq 0} |x_t^1 - x_t^2| = 0\right) = 1$.*

### 2.1.2 A priori estimate and uniqueness of solutions

Now let us introduce some notation which is useful later.

$$S_{\mathfrak{F}}^{2,loc}(\mathbb{R}^d) = \left\{ \begin{array}{c} f(t, \omega) : f(t, \omega) \text{ is } \mathfrak{F}_t - \text{adapted}, \mathbb{R}^d - \text{valued such that} \\ \mathbb{E} \sup_{t\in[0,T)} |f(t, \omega)|^2 < \infty, \forall T < \infty, \end{array} \right\}.$$

**Lemma 2.1.1.** *Assume that $x_t$ is a solution of 2.1, and assume that*

1. $\mathbb{E}|x_0|^2 < \infty$, *and*
2. $\langle x \cdot b(t,x,\omega)\rangle \leq c(t)(l + |x|^2)$,

$$|\sigma(t,x,\omega)|^2 + \int_Z |c(t,x,z,\omega)|^2 \pi(dz) \leq c(t)(1+|x|^2),$$

*where $0 \leq c(t)$ is non-random, such that $C_T = \int_0^T c(t)dt < \infty$, for any $T < \infty$. Then*

$$\mathbb{E}(\sup_{t\in[0,T)} |x_t|^2) \leq k_T < \infty,$$

*where $k_T \geq 0$ is a constant only depending on $C_t$ and $\mathbb{E}|x_0|^2$ . Hence one has that under the assumption of this lemma the solution of 2.1 always satisfies $\{x_t\}_{t\geq 0} \in S^{2,loc}_{\mathfrak{F}}(\mathbb{R}^d)$.*

*Proof.* Let $\tau_N = \inf\{t \geq 0 : |x_t| > N\}$. By Itô' formula

$$|x_{t\wedge\tau_N}|^2 = |x_0|^2 + 2\int_0^{t\wedge\tau_N} x_s \cdot b(s,x_s,\omega)ds + 2\int_0^{t\wedge\tau_N} x_s \cdot \sigma(s,x_s,\omega)dw_s$$

$$+ \int_0^{t\wedge\tau_N} |\sigma(s,x_s,\omega)|^2 ds + 2\int_0^{t\wedge\tau_N}\int_Z x_s \cdot c(s,x_{s-},z,\omega)\widetilde{N}_k(ds,dz)$$

$$+ \int_0^{t\wedge\tau_N}\int_Z |c(s,x_{s-},z,\omega)|^2 N_k(ds,dz). \tag{2.2}$$

Since for any $T < \infty$, as $t \in [0,T]$,

$$\mathbb{E}\int_0^{t\wedge\tau_N} |x_s \cdot \sigma(s,x_s,\omega)|^2 ds = \mathbb{E}\int_0^{t\wedge\tau_{N-}} |x_s \cdot \sigma(s,x_s,\omega)|^2 ds$$

$$\leq N^2 \int_0^t c(s)(1+|N|^2)ds < \infty.$$

Hence $\left\{ \int_0^{t\wedge\tau_N} x_s\cdot\sigma(s,x_s,\omega)dw_s \right\}_{t\in[0,T]}$ is a martingale. A similar conclusion holds for $\int_0^{t\wedge\tau_N}\int_Z x_s\cdot c(s,x_{s-},z,\omega)\widetilde{N}_k(ds,dz)$. Hence by the martingale inequality

$$\mathbb{E}\|x\|_{t\wedge\tau_N}^2 \leq \mathbb{E}|x_0|^2 + k_0\mathbb{E}\int_0^{t\wedge\tau_N} c(s)(1+\|x\|_s^2)ds$$

$$+\frac{1}{2}\mathbb{E}\|x\|_{t\wedge\tau_N}^2 \leq \mathbb{E}|x_0|^2 + \frac{1}{2}\mathbb{E}\|x\|_{t\wedge\tau_N}^2 + k_0\int_0^t c(s)(1+\mathbb{E}\|x_{s\wedge\tau_N}\|^2)ds, \tag{2.3}$$

where we write $\|x\|_t^2 = \sup_{s\leq t}|x_s|^2$, and we have used the fact that

$$\begin{aligned}
2\mathbb{E}\sup_{s\leq t}\left| \int_0^{s\wedge\tau_N} x_s\cdot\sigma(s,x_s,\omega)dw_s \right| &\leq k_0'\mathbb{E}\sqrt{\int_0^{t\wedge\tau_N}|x_s\cdot\sigma(s,x_s,\omega)|^2ds} \\
&\leq \frac{1}{4}\mathbb{E}\|x\|_{t\wedge\tau_N}^2 + k_0'^2\mathbb{E}\int_0^{t\wedge\tau_N}|\sigma(s,x_s,\omega)|^2ds \\
&\leq \frac{1}{4}\mathbb{E}\|x\|_{t\wedge\tau_N}^2 + k_0'^2\mathbb{E}\int_0^{t\wedge\tau_N} c(s)(1+\|x\|_s^2)ds,
\end{aligned}$$

and a similar inequality also holds for

$$2\mathbb{E}\sup_{s\leq t}\left|\int_0^{t\wedge\tau_N}\int_Z x_s\cdot c(s,x_{s-},z,\omega)\widetilde{N}_k(ds,dz)\right|.$$

Therefor

$$\frac{1}{2}\mathbb{E}\|x\|_{t\wedge\tau_N}^2 \leq \mathbb{E}|x_0|^2 + k_0\int_0^t c(s)(1+\mathbb{E}\|x_{s\wedge\tau_N}\|^2)ds.$$

By Gronwall's inequality one finds that when $t\in[0,T]$

$$\mathbb{E}\|x\|_{t\wedge\tau_N}^2 \leq k_T'\exp\left(2k_0\int_0^T c(s)ds\right) = k_T < \infty, \tag{2.4}$$

where $k_T' = 2\mathbb{E}|x_0|^2 + 2\int_0^T c(s)ds$. Letting $N\uparrow\infty$, by Fatou's lemma one finds that

$$\mathbb{E}\|x\|_T^2 \leq k_T,$$

where $k_T = \left(2\mathbb{E}|x_0|^2 + 2\int_0^T c(s)ds\right)\exp\left(2k_0\int_0^T c(s)ds\right)$ depends on $\mathbb{E}|x_0|^2$ and $\int_0^T c(s)ds$ only. $\square$

**Lemma 2.1.2.** *Assume that $b(t,x,\omega)$ and $\sigma(t,x,\omega)$ are uniformly locally bounded in $x$, that is, for each $0 < r < \infty$,*

$$|b(t,x,\omega)| + |\sigma(t,x,\omega)| \leq k_r, \text{ as } |x| \leq r,$$

*where $k_r \leq 0$ is a constant depending only on $r$; and assume that for each $N = 1, 2, \cdots, T < \infty$ there exist non-random functions $c_T^N(t)$ and $\rho_T^N(u)$ such that as $|x_1|, |x_2| \leq N$; and $t \in [0,T]$*

$$2(x_1 - x_2)\cdot(b(t,x_1,\omega) - b(t,x_2,\omega)) + |\sigma(t,x_1,\omega) - \sigma(t,x_2,\omega)|^2$$

$$+ \int_Z |c(t,x_1,z,\omega) - c(t,x_2,z,\omega)|^2 \pi(dz) \leq c_T^N(t)\rho_T^N(|x_1 - x_2|^2),$$

*where $c_T^N(t)$ is non-negative such that $\int_0^T c_T^N(t)dt < \infty$, and $\rho_T^N(u)$ defined on $u \geq 0$, is non-negative, increasing, continuous and concave such that*

$$\int_{0+} du/\rho_T^N(u) = \infty.$$

*Then the solution of 2.1 is pathwise unique.*

*Proof.* Assume $\{x_t^i\}_{t\geq 0}, i = 1, 2$ are two solutions of 2.1 with the same BM $\{w_t\}_{t\geq 0}$ and Poisson martingale measure $\widetilde{N}_k(dt,dz)$. Let

$$X_t = x_t^1 - x_t^2,\ \widehat{b}(s, x_s^1, x_s^2, \omega) = b(s, x_s^1, \omega) - b(t, x_s^2, \omega),$$

and

$$\tau_N = \inf\{t \geq 0 : |x_t^1| + |x_t^2| > N\}\ .$$

Then by Itô's formula as in 2.2 one sees that

$$Z_{t\wedge\tau_N} = \mathbb{E}|X_{t\wedge\tau_N}|^2] = \mathbb{E}\int_0^{t\wedge\tau_N} \Big[2X_s \cdot \widehat{b}(s, x_s^1, x_s^2, \omega) + |\widehat{\sigma}(s, x_s^1, x_s^2, \omega)|^2$$

$$+ \int_Z |\widehat{c}(s, x_s^1, x_s^2, z, \omega)|^2 \pi(dz)\Big]\, ds \leq \mathbb{E}\int_0^{t\wedge\tau_N -} c_T^N(s)\rho_T^N(|X_s|^2)ds$$

$$\leq \int_0^t c_T^N(s)\rho_T^N(Z_{s\wedge\tau_N})ds, \text{ as } t \in [0, T].$$

Hence by the following Lemma 2.1.3 for any $T < \infty, \mathbb{P}-$ a.s. $Z_{t\wedge\tau_N} = 0, \forall t \in [0, T]$. Letting $N \to \infty$ one finds that $\mathbb{P}-$ a.s.

$$Z_t = 0, \forall t \in [0, T].$$

By the *RCLL* (right continuous with left limit) property of $\{x_t^i\}_{t\geq 0}, i = 1, 2$ the conclusion now follows.

□

**Lemma 2.1.3.** *If* $\forall t \geq 0$ *a real non-random function* $y_t$ *satisfies*

$$0 \leq y_t \leq \int_0^t \rho(y_s)ds < \infty,$$

*where* $\rho(u)$ *defined on* $u \geq 0$, *is non-negative, increasing such that* $\rho(0) = 0, \rho(u) > 0$, *as* $u > 0$; *and*

$$\int_{0+} du/\rho(u) = \infty,$$

*then*

$$y_t = 0, \forall t \geq 0.$$

*Proof.* Let

$$z_t = \int_0^t \rho(y_s)ds.$$

Obviously, one only needs to show that $\forall t \geq 0$, $z_t = 0$. Indeed, $z_t$ is absolutely continuous, increasing and,

$$\dot{z}_t = \rho(y_t) \leq \rho(z_t). \tag{2.5}$$

Set

$$t_0 = \sup\{t \geq 0 : z_s = 0, \forall s \in [0,t]\}.$$

If $t_0 < \infty$, then $z_t > 0$, as $t > t_0$. Hence by assumption and from 2.5 for any $\delta > 0$

$$\infty = \int_{(0,z(t_0+\delta))} du/\rho(u) = \int_{(t_0,t_0+\delta)} dz_t/\rho(z_t) \leq \int_{(t_0,t_0+\delta)} dt \leq \delta.$$

This is a contradiction. Therefore $t_0 = \infty$. □

### 2.1.3 Existence of solutions for the Lipschitzian case

In this section we are going to discuss the existence and uniqueness of solution to SDE 2.1. First, we introduce a notation which will be used later.

$$L^2_{\mathfrak{F}}(\mathbb{R}^d) = \left\{ \begin{array}{c} f(t,\omega) : f(t,\omega) \,\text{is}\, \mathfrak{F}_t - \text{adapted}, \mathbb{R}^d - \text{valued} \\ \text{such that}\, \mathbb{E} \displaystyle\int_0^\tau |f(t,\omega)|^2 dt < \infty \end{array} \right\}$$

**Theorem 2.1.1.** *Assume that*

1. *$b$ and $\sigma : [0,\infty) \times \mathbb{R}_d \times \Omega \to \mathbb{R}^d$,*

$$c : [0,\infty) \times R_d \times Z \times \Omega \to \mathbb{R}^d$$

*are jointly measurable and $\mathfrak{F}_t$-adapted, where furthermore, $c$ is $\mathfrak{F}_t$-predictable such that $\mathbb{P}-$ a.s.*

$$|b(t,x,\omega)| \leq c(t)(1+|x|),$$

$$|\sigma(t,x,w)|^2 + \int_Z |c(t,x,z,\omega)|^2 \pi(dz) \leq c(t)(1+|x|^2),$$

*where $c(t)$ is non-negative and non-random such that*

$$\int_0^T c(t)dt < \infty;$$

2. $|b(t, x_1, \omega) - b(t, x_2, \omega)| \leq c(t)|x_1 - x_2|,$

$$|\sigma(t, x_1, \omega) - \sigma(t, x_2, \omega)|^2 + \int_Z |c(t, x_1, z, \omega) - c(t, x_2, z, \omega)|^2 \pi(dz)$$

$$\leq c(t)|x_1 - x_2|^2,$$

*where $c(t)$ satisfies the same conditions as in* (1)*;*

3. $x_0 \in \mathfrak{F}_0, \mathbb{E}|x_0|^2 < \infty.$

*Then 2.1 has a pathwise unique $\mathfrak{F}_t$-adapted solution $\{x_t\}_{t\geq 0} \in S^{2,loc}_{\mathfrak{F}}(\mathbb{R}^d)$. In the case that $b(t, x, \omega)$ and $\sigma(t, x, \omega)$ are $\mathfrak{F}_t^{w,\widetilde{N}_k}$-adapted, and $c(t, x, z, \omega)$ is $\mathfrak{F}_t^{w,\widetilde{N}_k}$-predictable, then the solution is also $\mathfrak{F}_t^{w,\widetilde{N}_k}$-adapted, i.e. it is a strong solution.*

*Proof.* Let us use the contraction mapping principle to prove this result. Introduce a new norm as follows: for any given $T < \infty$ and for $(x.) \in \tilde{B} = L^2_{\mathfrak{F}}(\mathbb{R}^d)$ let

$$\|(x.)\|_M^2 = \sup_{t\in[0,T]} e^{-b_0 A(t)} \mathbb{E}\|x\|_t^2,$$

where $\|x\|_t = \sup_{s\leq t} |x_s|$, and $b_0 \geq 0$ is a constant, which will be determined later, and

$$A(t) = \int_0^t c(s)ds$$

write

$$H = \{(x.) \in \tilde{B} : \|(x.)\|_M < \infty\}.$$

Then $H$ is a Banach space. For any $(\bar{x}^i.) \in H, i = 1, 2$, denote $(x^i.), i = 1, 2$, by the following SDE:

$$x_t^i = x_0 + \int_0^t b(s, \bar{x}_s^i, \omega)ds + \int_0^t \sigma(s, \bar{x}_s^i, \omega)dw_s$$

$$+ \int_0^t \int_z c(s, \tilde{x}_{s-}^i, z, \omega)\widetilde{N}_k(ds, dz), 0 \le t, i = 1, 2.$$

By assumption (1) one easily sees that $(x^i.) \in H, i = 1, 2$. Let

$$X_t = x_t^1 - x_t^2.$$

Similarly, define $\bar{X}_{t.}$ Then by Itô's formula as in 2.2, and discussing similarly as in 2.3, one has that

$$\mathbb{E}[\|X\|_t^2] \le \gamma^{-1} \int_0^t k_0 c(s)\mathbb{E}\|\bar{X}\|_s^2 ds + \gamma \int_0^t k_0 c(s)\mathbb{E}\|X\|_s^2 ds$$

$$= \gamma^{-1} I_t^1 + \gamma \int_0^t k_0 c(s)\mathbb{E}\|\bar{X}\|_s^2 ds$$

$$\le \gamma^{-1} I_t^1 + \int_0^t \exp(\gamma k_0(A(t) - A(s)))k_0 c(s) I_s^1 ds$$

where $k_0 \ge 1$ is a fixed constant, and we have applied Lemma 2.1.4 below. Note that $0 \le A(t)$ is increasing, so that

$$\begin{aligned}
e^{-b_0 A(t)} I_t^1 &= e^{-b_0 A(t)} \int_0^t k_0 c(s)\mathbb{E}\|\bar{X}\|_s^2 ds \\
&\le \sup_{s \le t} e^{-b_0 A(s)} \mathbb{E}\|\bar{X}\|_s^2 \int_0^t e^{-b_0(A(t)-A(s))} k_0 c(s) ds \\
&\le \sup_{s \le t} e^{-b_0 A(s)} \mathbb{E}\|\bar{X}\|_s^2 \cdot k_0' \int_0^t c(s) ds \\
&\le k_T'' \sup_{s \le T} e^{-b_0 A(s)} \mathbb{E}\|\bar{X}\|_s^2.
\end{aligned}$$

where $k_T'' > 0$ is a constant depending on $\int_0^T c(s)ds$ only. Thus, if we write $u(s) = \mathbb{E}|\bar{X}_s|^2$, then

$$e^{-b_0 A(t)} \int_0^t \exp(\gamma k_0((A(t) - A(s))))k_0 c(s) I_s^1 ds$$

$$\leq \sup_{s\leq t} e^{-b_0 A(s)} I_s^1 \int_0^t e^{-k_0(\frac{b_0}{k_0}-\gamma)(A(t)-A(s))} k_0 c(s) ds$$

$$\leq k_T'' \sup_{s\leq t} e^{-b_0 A(s)} u(s)(\frac{b_0}{k_0}-\gamma)^{-1} \leq k_T'' \sup_{s\leq T} e^{-b_0 A(s)} u(s)(\frac{b_0}{k_0}-\gamma)^{-1}.$$

Hence

$$\|(X.)\|_M^2 \leq \max(k_T''\gamma^{-1}, k_T''(\frac{b_0}{k_0}-\gamma)^{-1})\|(\bar{X}.)\|_M^2.$$

After appropriately choosing $\gamma$ and $b_0$ to make

$$\max(k_T''\gamma^{-1}, k_T''(\frac{b_0}{k_0}-\gamma)^{-1}) < l$$

by the contraction mapping principle one finds that there exist a unique solution $\{\bar{x}_t\}_{t\geq 0} \in L^2_{\mathfrak{F}}(\mathbb{R}^d)$ satisfying 2.1. Let us show the following result: there exists a version $\{x_t\}_{t\geq 0}$ of $\{\bar{x}_t\}_{t\geq 0}$, that is, for each $t \in [0,T]$ $\mathbb{P}(x_t \neq \bar{x}_t) = 0$, such that $\{x_t\}_{t\geq 0}$ is RCLL (right continuous and with left limit) and $\{x_t\}_{t\geq 0}$ is a solution of 2.1. In fact, write

$$x_t = x_0 + \int_0^t b(s, \bar{x}_s, \omega) ds + \int_0^t \sigma(s, \bar{x}_s, \omega) dw_s$$

$$+ \int_0^t \int_z c(s, \bar{x}_{s-}, z, \omega) \widetilde{N}_k(ds, dz), t \in [0,T];$$

$$y_t = x_0 + \int_0^t b(s, x_s, \omega) ds + \int_0^t \sigma(s, x_s, \omega) dw_s$$

$$+ \int_0^t \int_z c(s, x_{s-}, z, \omega) \widetilde{N}_k(ds, dz), t \in [0,T].$$

Then $\mathbb{E}\int_0^T |x_t - \bar{x}_t|^2 dt = 0$. So, there exists a set $\Lambda_1 \times \Lambda_2 \in ([0,T]) \times \mathfrak{F}$ such that $\mathbb{E}\int_0^T \mathbb{1}_{\Lambda_1\times\Lambda_2}(t,\omega) dt = 0$, and $x_t(\omega) = \bar{x}_t(\omega)$, as $(t,\omega)$ does not belong $\Lambda_1 \times \Lambda_2$. Hence, for each $t \in [0,T]$,

$$\mathbb{E}|x_t - y_t|^2 \le 3\left[\mathbb{E}\left|\int_0^t |b(s,\bar{x}_s) - b(s,x_s)|ds\right|^2 + \mathbb{E}\int_0^t |\sigma(s,\bar{x}_s) - \sigma(s,x_s)|^2 ds\right.$$

$$\left. + \mathbb{E}\int_0^t \int_Z |c(s,\bar{x}_{s-},z) - c(s,x_{s-},z)|^2 \pi(dz)ds\right] = 0.$$

So, the above fact holds true. Now, by Lemma 2.1.1 and 2.1.2 the solution is also pathwise unique such that $\mathbb{E}(\sup_{t\in[0,T]} |x_t|^2) \le k_T < \infty$, for each $T < \infty$, where $k_T$ is a constant depending on $T$ and $\int_0^T c(s)ds$ only. So we have show that there is a unique solution $\{x_t\}_{t\in[0,T]}$ for each given $T < \infty$. By the uniqueness of the solution we immediately obtain a solution $\{x_t\}_{t\ge 0}$ , which is also unique. When all coefficients are $\mathfrak{F}_t^{w,\widetilde{N}_k}$-adapted, then by construction one easily sees that $\{x_t\}_{t\ge 0}$ is also $\mathfrak{F}_t^{w,\widetilde{N}_k}$-adapted, i.e. it is a strong solution.

$\square$

**Lemma 2.1.4.** *[70] (Gronwall's inequality)*

*If* $0 \le y_t \le \gamma v_t + \int_0^t c(s)y_s ds, \forall t \ge 0,$ *where* $\gamma > 0$ *is a constant, and* $c(s) \ge 0$*, then* $\forall t \ge 0$

$$y_t \le \gamma v_t + \int_0^t \exp\left(\int_s^t c(r)dr\right) c(s)v_s ds.$$

## 2.2 Examples of weak solutions

For the existence and weak uniqueness of a weak solution of SDE with jumps we have the following examples.

**Example 2.2.1.** *Assume that*

$$|b(t,x)| \le k_0,$$

$$|\sigma(t,x)|^2 + \int_Z |c(t,x,z)|^2 \pi(dz) \le k_0(1 + |x|^2),$$

*and for each $N = 1, 2, \cdots$ there exist a non-random function $c^N(t)$ such that as $|x_1|$ and $|x_2| \leq N$,*

$$|\sigma(t, x_1, \omega) - \sigma(t, x_2, \omega)|^2 + \int_Z |c(t, x_1, z, \omega) - c(t, x_2, z, \omega)|^2 \pi(dz) \leq c^N(t)|x_1 - x_2|^2,$$

*where $c^N(t) \geq 0$ satisfies that $\int_0^T c^N(t)dt < \infty$ for each $T < \infty$; and $\sigma^{-1}(t, x)$ exists and is bounded $|\sigma^{-1}(t, x)| \leq k_0$, where $x, b, c \in \mathbb{R}^d, \sigma \in R^{d\otimes d}$. Then the following SDE with jumps in d-dimensional space on $t \in [0, T]$:*

$$x_t = x_0 + \int_0^t b(s, x_s)ds + \int_0^t \sigma(s, x_s)dw_s + \int_0^t \int_Z c(s, x_{s-}, z)\widetilde{N}_k(ds, dz), \tag{2.6}$$

*where $x_0 \in \mathbb{R}^d$ is a constant vector, has a weak unique weak solution.*

In the case that the SDE 2.6 has no jump term, that is, $c = 0$, if $\sigma = \sigma(x)$ does not depend on $t$, then we can weaken the condition on $\sigma$ to get a weak solution. However, in this case the weak uniqueness is not necessarily true.

**Example 2.2.2.** *Assume that $\sigma = \sigma(x)$ does not depend on $t$, and*

$$|b(t, x)| \leq k_0,$$

$$|\sigma(x)|^2 \leq k_0(1 + |x|^2),$$

*and $\sigma^{-1}(x)$ exists and is bounded $|\sigma^{-1}(x)| \leq k_0$, where $x, b \in \mathbb{R}^d, \sigma \in R^{d\otimes d}$*
*Then for any $T < \infty$*

$$x_t = x_0 + \int_0^T b(s, x_s)ds + \int_0^t \sigma(x_s)dw_s, t \in [0, T],$$

*has a weak solution, where $x_0 \in \mathbb{R}^d$ is a constant vector.*

# Chapter 3

# Stochastic differential equations with non-Lipschitzian coefficients

In this chapter we will use the smoothness method and the Skorokhod weak convergence technic to discuss the existence and uniqueness of strong solutions and weak solutions for SDE with jumps and with non- Lipschitzian coefficients.

## 3.1 Strong solutions, continuous Coefficients with $\rho$-conditions

In this section we will use the smoothness method to obtain the existence and uniqueness of a strong solution for a SDE with continuous coefficients, which satisfy some so-called $\rho$-condition. Consider the following SDE with jumps:

$$x_t = x_0 + \int_0^t b(s, x_s, \omega)ds + \int_0^t \sigma(s, x_s, \omega)dw_s$$

$$+ \int_0^t \int_z c(s, x_{s-}, z, \omega)\widetilde{N}_k(ds, dz), \forall t \geq 0 \tag{3.1}$$

where $\{w_t\}_{t\geq 0}$ is a d-dimensional BM, $\widetilde{N}_k(ds, dz)$ is the Poisson martingale measure generated by a Poisson point process $k(\cdot)$ such that $\widetilde{N}_k(ds, dz) = N_k(ds, dz) - \pi(dz)dt$, where $N_k(ds, dz)$ is the counting measure with the compensator $\pi(dz)dt$ generated by $k(\cdot), \pi(\cdot)$ is a $\sigma$-finite measure on some measurable space $(Z, \mathfrak{B}_z)$, and $b \in \mathbb{R}^d, \sigma \in R^{d\otimes d}, c \in \mathbb{R}^d$. In 3.1 if $c = 0$, we get a continuous SDE. Furthermore, if $\sigma = 0$, then 3.15 will be reduced to a continuous ODE for each fixed $\omega$. So we will call the case $\sigma = 0$ a degenerate case, no matter if $c = 0$ or not. We will use the smoothness technic to show the results.

**Theorem 3.1.1.** *[70]*

1. *Assume that*

$$\begin{aligned} b &= b(t, x, \omega) : [0, \infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d, \\ \sigma &= \sigma(t, x, \omega) : [0, \infty) \times \mathbb{R}^d \times \Omega \to R^{d\otimes d}, \\ c &= c(t, x, \omega) : [0, \infty) \times \mathbb{R}^d \times Z \times \Omega \to \mathbb{R}^d, \end{aligned}$$

*are $\mathfrak{F}_t^{w,\widetilde{N}_k}$- adapted and measurable processes such that $\mathbb{P}$- a.s.*

$$|b(t, x, \omega)| \leq c_1(t)(1 + |x|),$$

$$|\sigma(t, x, \omega)|^2 + \int_Z c(t, x, z, \omega)|^2 \pi(dz) \leq c_1(t)(1 + |x|^2),$$

*where $\mathfrak{F}_t^{w,\widetilde{N}_k}$ is the $\sigma$-field generated by $w$ and $\widetilde{N}_k$ up to time $t$, that is, $\mathfrak{F}_t^{w,\widetilde{N}_k} = \sigma(w_s, \widetilde{N}_k((0, s], U), \forall U \in \mathfrak{B}_z, s \leq t)$, and $c_1(t)$ is non-negative and non-random such that for each $T < \infty$*

$$\int_0^T c_1(t)dt < \infty;$$

2. *$b(t, x, \omega)$ and $\sigma(t, x, \omega)$ are continuous in $x$; and*

$$\lim_{h\to 0} \int_Z |c(t, x + h, z, \omega) - c(t, x, z, \omega)|^2 \pi(dz) = 0;$$

*3. for each $N = 1, 2, \cdots$ , and each $T < \infty$,*

$$2\langle (x_1 - x_2), (b(t, x_1, \omega) - b(t, x_2, \omega))\rangle$$

$$+|\sigma(t, x_1, \omega) - \sigma(t, x_2, \omega)|^2 + \int_z c(t, x_1, z, \omega) - c(t, x_2, z, \omega)|^2 \pi(dz)$$

$$\leq c_T^N(t)\rho_T^N(|x_1 - x_2|^2),$$

*as $|x_i| \leq N, i = 1, 2, t \in [0, T]$; where $\int_0^T c_T^N(t)dt < \infty$; and $\rho_T^N(u) \geq 0$,as $u \geq 0$, is non-random, strictly increasing, continuous and concave such that*

$$\int_{0+} du/\rho_T^N(u) = \infty.$$

*Then for any given constant $x_0 \in \mathbb{R}^d$ 3.1 has a pathwise unique strong solution.*

First, let us give an example of the existence of a solution to an SDE in the case that $\forall T < \infty, \int_0^T c_1(t)dt < \infty$, $c_1(t)$ is unbounded and, moreover, $b_1$ is also unbounded and non-Lipschitzian continuous in $x$.

**Example 3.1.1.** *Let $b(t, x) = -\mathbb{1}_{t \neq 0}\mathbb{1}_{x \neq 0} t^{-\alpha_1} x|x|^{-\beta}$, where $\alpha_1 < 1, 0 < \beta < 1$, and suppose that $\sigma$ and $c$ satisfy (1) and (2) in Theorem 3.1.1, and satisfy the condition (3) in Theorem 3.1.1 with $b = 0$. Then 3.1 has a pathwise unique strong solution. Obviously, $c(t) = \mathbb{1}_{s \neq 0} s^{-\alpha_1}$, is unbounded in $t$, and $b$ is also unbounded in $t$ and $x$, and is non-Lipschitz continuous in $x$.*

*Proof.* Notice that $\forall x, x' \in \mathbb{R}^d$

$$\langle x - x', -x|x|^{-\beta} + x'|x'|^{-\beta}\rangle = -|x|^{2-\beta} - |x'|^{2-\beta}$$

$$+|x'|^{-\beta}\langle x, x'\rangle + |x|^{-\beta}\langle x, x'\rangle \leq -|x|^{2-\beta} - |x'|^{2-\beta}$$

$$+|x'|^{-\beta+1}|x| + |x|^{-\beta+1}|x'| = (|x| - |x'|)(|x'|^{1-\beta} - |x|^{1-\beta}) \leq 0.$$

Hence Theorem 3.1.1 applies. □

Before we prove Theorem 3.1.1, let us first establish a lemma.

**Lemma 3.1.1.** *Under assumptions (1) and (2) in Theorem 3.1.1 there exist $b^n$, $\sigma_n$ and $c^n, n = 1, 2, \cdots$ , satisfying the following conditions:*

1. $|b^n(t, x, \omega)| \leq 2c_1(t)(l + |x|)$, *as* $n \geq N_0$;

$$|\sigma^n(t, x, \omega)|^2 + \int_Z c^n(t, x, z, \omega)|^2\pi(dz) \leq 8c_1(t)(1 + |x|^2),$$

   *where* $N_0 > 0$ *is a constant;*

2. *as* $x, x' \in \mathbb{R}^d$

$$|b^n(t, x, \omega) - b^n(t, x', \omega)| \leq k_n c_1(t)|x - x'|,$$

$$|\sigma^n(t, x, \omega) - \sigma^n(t, x', \omega)|^2 + \int_Z |c_n(t, x, z, \omega) - c^n(t, x', z, \omega)|^2\pi(dz)$$

$$\leq k_n c_1(t)|x - x'|,$$

   *where* $k_n \geq 0$ *is a constant only depending on* $n$,

3. *for any* $N > 0$ *and for each* $t \geq 0, \omega \in \Omega$, *as* $n \to \infty$

$$\sup_{|x| \leq N} |b^n(t, x, \omega) - b(t, x, \omega)| \to 0,$$

$$\sup_{|x| \leq N} |\sigma^n(t, x, \omega) - \sigma(t, x, \omega)|^2$$

$$+ \sup_{|x| \leq N} \int_Z |c^n(t, x, z, \omega) - c(t, x, z, \omega)|^2\pi(dz) \to 0.$$

*Proof.* Let us smooth out $b$ only with respect to $x$ to get $b^n$, i.e. define

$$b^n(t,x,\omega) = \int_{\mathbb{R}^d} b(t, x - n^{-1}\bar{x}, \omega) J(\bar{x}) d\bar{x},$$

where for all $u \in \mathbb{R}^d$

$$J_d(u) = \begin{cases} c_d \exp(-(1 - |u|^2)^{-1}), \text{ for } |u| < 1, \\ 0, \text{otherwise}, \end{cases}$$

and the constant $c_d$, satisfies $\int_{\mathbb{R}^d} (u) du = 1$. Then

$$|b^n(t,x,\omega)| \leq \int_{\mathbb{R}^d} |b(t, x - n^{-1}\bar{x}, \omega)| J(\bar{x}) d\bar{x}$$

$$\leq c_1(t) \int_{\mathbb{R}^d} (1 + |x - n^{-1}\bar{x}|) J(\bar{x}) d\bar{x}$$

$$\leq c_1(t)(1 + |x| + n^{-1} \int_{\mathbb{R}^d} |\bar{x}| J(\bar{x}) d\bar{x}) = c_1(t)(1 + |x| + n^{-1}k_0)$$

$$\leq (1 + n^{-1}k_0) c_1(t)(1 + |x|) \leq 2c_1(t)(1 + |x|), \text{ as } n > k_0.$$

So $b^n$ satisfies (1). On the other hand,

$$|b^n(t,x,\omega) - b^n(t,x',\omega)| = |n^d \int_{\mathbb{R}^d} b(t,\bar{x},\omega) J(n(x - \bar{x})) d\bar{x}$$

$$-n^d \int_{\mathbb{R}^d} b(t,\bar{x},\omega) J(n(x' - \bar{x})) d\bar{x}| \leq n^d \int_{\mathbb{R}^d} |b(t,\bar{x},\omega)| |J(n(x - \bar{x}))$$

$$-J(n(x' - \bar{x}))| d\bar{x} \leq n^d c_1(t) |x - \bar{x}|$$

$$\times \int_{\mathbb{R}^d} \int_0^1 (1 + |\bar{x}|) \text{grad} \left[ J(n(x - \bar{x} + \theta(x' - x))) \right] d\theta d\bar{x} \leq k_n c_1(t) |x - x'|.$$

So (2) is established for $b^n$. Now by Heine-Borel's finite covering theorem for any $N > 0$ and any given $\tilde{\epsilon} > 0$ one can find a $\delta > 0, \delta$ may depend on $t$ and $\omega$ such that $\frac{1}{n} < \delta, |b(t, x - n^{-1}, \omega) - b(t,x,\omega)| < \tilde{\epsilon}, \forall |x| \leq N$; because $b$ is continuous in $x$. Hence, as $n \geq \frac{1}{\delta}$,

$$\sup_{|x|\le N}|b^n(t,x,\omega)-b(t,x,\omega)|$$

$$\le |\int_{\mathbb{R}^d}\sup_{|x|\le N}|b(t,x-n^{-1}\bar{x},\omega)-b(t,x,\omega)|J(\bar{x})d\bar{x}|$$

$$=|\int_{|\bar{x}\le 1|}\sup_{|x|\le N}|b(t,x-n^{-1}\bar{x},\omega)-b(t,x,\omega)|J(\bar{x})d\bar{x}<\tilde{\epsilon}.$$

Thus (3) is also true for $b^n$. Now, defining $\sigma^n$ and $c^n$ similarly, it is easily seen that $\sigma_n, n=1,2,\cdots$, also satisfy (1), (2) and (3). For $c^n$ the proof is also similar. In fact,

$$\begin{aligned}\int_Z |c^n(t,x,z,\omega)|^2\pi(dz) &= \int_Z |\int_{\mathbb{R}^d} c(t,x-n^{-1}\bar{x},z,\omega)J(\bar{x})d\bar{x}|^2\pi(dz)\\ &\le \int_Z\int_{\mathbb{R}^d}|c(t,x-n^{-1}\bar{x},z,\omega)|^2J(\bar{x})d\bar{x}|^2\pi(dz)\\ &\le c_1(t)\int_{\mathbb{R}^d}(1+|x-n^{-1}\bar{x}|^2)J(\bar{x})d\bar{x}\le 2c_1(t)(1+2|x|^2\\ &+2n^{-1}\int_{\mathbb{R}^d}|\bar{x}|^2J(\bar{x})d\bar{x}\le 2c_1(t)(1+2|x|^{2+2n^{-1}\tilde{k}_0})\\ &\le 4c_1(t)(1+|x|^2),\, as\, n>2\tilde{k}_0\end{aligned}$$

So (1) is proved for $c^n$. On the other hand,

$$\begin{aligned}&\int_Z |c^n(t,x,z,\omega)-c^n(t,x',z,\omega)|^2\pi(dz)\\ &\le n^d\int_{\mathbb{R}^d}|c(t,\bar{x},z,\omega)|^2\pi(dz)|J(n(x-\bar{x})-J(n(x'-\bar{x}))|^2d\bar{x}\\ &\le n^dc_1(t)|x-x'|^2\int_{\mathbb{R}^d}\int_0^1(1+|\bar{x}|^2)|grad[J(n(x-\bar{x}+\theta(x'-x)))]^2|d\theta d\bar{x}\\ &\le k_nc_1(t)|x-\bar{x}|.\end{aligned}$$

So for $c^n$ (2) is also established. Finally, by a similar proof as in $b^n$ one easily derives that (3) is also true for $c^n$. □

Now let us prove Theorem 3.1.1.

*Proof.* For $b^n$, $\sigma^n$ and $c^n$ obtained from the above lemma by theorem, there exists a pathwise unique strong solution $(x_t^n)$ satisfying the following SDE

$$x_t^n = x_0 + \int_0^t b^n(s, x_s^n, \omega)ds + \int_0^t \sigma^n(s, x_s^n, \omega)dw_s$$

$$+ \int_0^t \int_Z c^n(s, x_{s-}^n, z, \omega)\widetilde{N}_k(ds, dz). \tag{3.2}$$

By Ito's formula

$$\mathbb{E}[|x_t^m - x_t^n|^2] = 2\mathbb{E}[\int_0^t (x_s^m - x_s^n) \cdot (b^m(s, x_s^m, \omega) - b^n(s, x_s^n, \omega))ds$$

$$+ \int_0^t |\sigma^m(s, x_s^m, \omega) - \sigma^n(s, x_s^n, \omega)|^2 ds$$

$$+ \int_0^t \int_Z |c^m(s, x_s^m, z, \omega) - c^n(s, x_s^n, z, \omega)|^2 \pi(dz)ds]$$

$$= \mathbb{E}[\int_0^t \int_{\mathbb{R}^d} [2(x_s^m - x_s^n) \times (b(s, x_s^m - m^{-1}\bar{x}, \omega)$$

$$-b(s, x_s^n - n^{-1}\bar{x}, \omega)) + |\sigma(s, x_s^m - m^{-1}\bar{x}, \omega) - \sigma(s, x_s^n - n^{-1}\bar{x}, \omega)|^2$$

$$+ \int_Z |c(s, x_s^m - m^{-1}\bar{x}, z, \omega) - c(s, x_s^n - n^{-1}\bar{x}, z, \omega)|^2 \pi(dz)].J(\bar{x})d\bar{x}]$$

$$\leq \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \{c_1(s)\rho(|x_s^m - x_s^n - (m^{-1} - n^{-1})\bar{x}|^2)$$

$$+20c_1(s)|(m^{-1} - n^{-1})\bar{x}|\} J(\bar{x})d\bar{x}ds.$$

Hence as $t \in [0, T]$

$$\mathbb{E}[|x_t^m - x_t^n|^2 \leq k_T'(m^{-1} + n^{-1})$$

$$+ k_0' \int_0^t \{c_1(s) \int_{\mathbb{R}^d} \rho(\mathbb{E}|x_s^m - x_s^n - (m^{-1} - n^{-1})\bar{x}|^2)J(\bar{x})d\bar{x}\}ds \tag{3.3}$$

So for all n and $\forall T < \infty$

$$\mathbb{E}(\sup_{t \leq T} |x_t^n|^2)ds \leq k_T < \infty, \tag{3.4}$$

Hence, by Fatou's lemma, it is easily seen that

$$\overline{\lim}_{m,n\to\infty}\mathbb{E}[|x_t^m - x_t^n|^2] \le \tilde{k}_T \int_0^t c_1(s)\rho_1\left(\overline{\lim}_{m,n\to\infty}\mathbb{E}|x_s^m - x_s^n|^2\right) ds,$$

where $\rho_1(u) = \rho(u) + u$. Therefore, $\overline{\lim}_{m,n\to\infty}\mathbb{E}|x_t^m - x_t^n|^2 = 0$. By 3.3 one also finds that for each $T < \infty \overline{\lim}_{m,n\to\infty}\mathbb{E}\int_0^T |x_t^m - x_t^n|^2 dt = 0$. So there exists an $(x_t) \in L^2_{\mathfrak{F}}(\mathbb{R}^d)$ such that for each $T < \infty$

$$\overline{\lim}_{n\to\infty}\mathbb{E}\int_0^T |x_t^n - x_t|^2 dt = 0.$$

On the other hand, by the above result one also has that for each $t \ge 0$

$$\overline{\lim}_{n\to\infty}\mathbb{E}|x_t^n - x_t|^2 = 0.$$

So $x_t^n \to x_t$, in probability for each $t$, and one can choose a subsequence $\{n_k\}$ of $\{n\}$ , denoted by $\{n\}$ again, such that $\mathbb{P}-$ a.s. as $n \to \infty$,

$$x_t^n \to x_t^0, \forall t = r_k, k = 1, 2, \cdots;$$

where $\{r_k\}_{k=1}^\infty \subset [0, T]$ is the totality of rational numbers in $[0, T]$. Hence by Fatou's lemma

$$\mathbb{E}\sup_{t\le T}|x_t^0| \le \mathbb{E}[\sup_k \lim_{n\to\infty}|x_{r_k}^n|^2] \le \underline{\lim}_{n\to\infty}\mathbb{E}[\sup_{t\le T}|x_t^n|^2] \le k_T. \tag{3.5}$$

So, we have

$$\int_0^t\int_Z c^n(s, x_{s-}^n, z, \omega)\widetilde{N}_k(ds, dz) \to \int_0^t\int_Z c(s, x_{s-}, z, \omega)\widetilde{N}_k(ds, dz), \text{in } \mathbb{P} \tag{3.6}$$

one may assume that $\sup_{t\le T}|x_t^n| \le k_0, \forall n$ and $\sup_{t\le T}|x_t| \le k_0$. However, in this case, as $t \in [0, T]$, for any $\epsilon > 0$

$$\mathbb{P}\left(\left|\int_0^t\int_Z c^n(s, x_{s-}^n, z, \omega)\widetilde{N}_k(ds, dz) - \int_0^t\int_Z c(s, x_{s-}, z, \omega)\widetilde{N}_k(ds, dz)\right| > \epsilon\right)$$

$$\le \frac{1}{4\epsilon^2}\mathbb{E}\int_0^T\int_Z |c^n(s, x_s^n, z, \omega) - c(s, x_s^n, z, \omega)|^2 \mathbb{1}_{|x_t^n|\le k_0}\mathbb{1}_{|x_t|\le k_0}\pi(dz)ds$$

$$+\frac{1}{4\epsilon^2}\mathbb{E}\int_0^T\int_Z |c(s,x_{s-}^n,z,\omega)-c(s,x_s,z,\omega)|^2 1\!\!1_{|x_t^n|\le k_0}1\!\!1_{|x_t|\le k_0}\pi(dz)ds$$

$$\le \frac{1}{\epsilon^2}\int_0^T \sup_{|x|\le k_0}\int_Z |c^n(s,x,z,\omega)-c(s,x,z,\omega)|^2\pi(dz)ds$$

$$+\frac{1}{4\epsilon^2}\mathbb{E}\int_0^T\int_Z |c(s,x_s^n,z,\omega)-c(s,x_s,z,\omega)|^2 1\!\!1_{|x_t^n|\le k_0}1\!\!1_{|x_t|\le k_0}\pi(dz)ds$$

$$= I^{1,n}+I^{2,n}.$$

Notice from Lemma 3.1.1 one finds that

$$\sup_{|x|\le k_0}\int_Z |c^n(t,x,z)|^2\pi(dz)\le 8c_1(t)(1+k_0^2), \text{ and}$$

$$\lim_{n\to\infty}\sup_{|x|\le k_0}\int_Z |c^n(s,x,z)-c(s,x,z)|^2\pi(dz)=0.$$

Thus one can apply Lebesgue's dominated convergence theorem to get

$$\lim_{n\to\infty} I^{1,n}=0 \tag{3.7}$$

Moreover, one also finds that as $n\to\infty$,

$$\mathbb{E}\int_0^T\int_Z |c(s,x_s^n,z,\omega)-c(s,x_s,z,\omega)|^2 1\!\!1_{|x_t^n|\le k_0}1_{|x_t|\le k_0}\pi(dz)ds\to 0. \tag{3.8}$$

In fact,

$$\mathbb{P}\left(\int_Z |c(s,x_s^n,z,\omega)-c(s,x_s,z,\omega)|^2 1\!\!1_{|x_t^n|\le k_0}1\!\!1_{|x_t|\le k_0}\pi(dz)>\epsilon\right)$$

$$\le \mathbb{P}(|x_s^n-x_s^0|>\delta)$$

$$+\mathbb{P}(1\!\!1_{|x_s^n-x_s^0|\le\delta,|x_s^n|\le k_0,|x_s^0|\le k_0}\int_Z |c(s,x_s^n,z)-c(s,x_s^0,z)|^2\pi(dz)>\epsilon)$$

$$= J_1^{n,\delta}+J_2^{n,\delta}.$$

Now since

$$\lim_{h\to 0}\sup_{|x|\leq k_0}\int_Z |c(s,x+h,z)-c(s,x,z)|^2\pi(dz)=0,$$

one can take a small enough $\delta>0$ such that

$$\sup_{|x|\leq k_0,|h|\leq\delta}\int_Z |c(s,x+h,z)-c(s,x,z)|^2\pi(dz)<\epsilon.$$

Hence for this $\delta>0$, $J_2^{n,\delta}=0$. Furthermore, for arbitrary given $\tilde{\epsilon}>0$ there exists a $\widetilde{N}$ such that as $n\geq\widetilde{N}$, $J_1^{n,\delta}<\tilde{\epsilon}$. Thus, for each $s$, as $n\to\infty$,

$$\textstyle\int_Z |c(s,x_s^n,z,\omega)-c(s,x_s,z,\omega)|^2 1\!\!1_{|x_t^n|\leq k_0}1\!\!1_{|x_t|\leq k_0}\pi(dz)\to 0, \text{ in } \mathbb{P}.$$

Hence, Lebesgue's dominated convergence theorem applies, and 3.8 holds. Thus 3.6 follows. By the same token one easily shows that as $n\to\infty$,

$$\int_0^t b^n(s,x_s^n,\omega)ds\to\int_0^t b(s,x_s,\omega)ds, \text{ in } \mathbb{P};$$

and

$$\int_0^t \sigma^n(s,x_s^n,\omega)ds\to\int_0^t \sigma(s,x_s,\omega)dw_s, \text{ in } \mathbb{P}.$$

Therefore, $(x_t)$ is a solution of 3.1. □

## 3.2 The Skorokhod weak convergence technic

To discuss the existence of a weak solution for a SDE under some weak conditions the following Skorokhod weak convergence technic is very useful and we will use it frequently in this chapter. Let us establish a lemma, which is very useful in the discussion of the existence of a weak solution to an SDE. In the rest of this Chapter let us assume that

$$Z=\mathbb{R}^d-\{0\}, \text{ and } \int_Z\frac{|x|^2}{1+|x|^2}\pi(dz)<\infty.$$

**Lemma 3.2.1.** *Suppose that*

$$|b^n(t,x,\omega)| \leq c_1(t)(1+|x|),\ as\ n \geq N_0;$$

$$|\sigma^n(t,x,\omega)|^2 + \int_Z |c^n(t,x,z,\omega)|^2 \pi(dz) \leq c_1(t)(1+|x|^2),$$

*where $N_0 > 0$ is a constant. Assume that for each $n = 1, 2, \cdots$, $x_t^n$ is the solution of the following SDE:*

$$x_t^n = x_0 + \int_0^t b^n(s, x_s^n)ds + \int_0^t \sigma^n(s, x_s^n)dw_s$$

$$+ \int_0^t \int_Z c^n(s, x_{s-}^n, z)q(ds, dz),$$

*where we denote $q(dt, dz) = \widetilde{N}_k(dt, dz)$ the Poisson martingale measure with the compensator $\pi(dz)dt$ such that $q(dt, dz) = p(dt, dz) - \pi(dz)dt$ and*

$$p(dt, dz) = N_k(dt, dz).$$

*Then the following fact holds, this fact we may call "the result of SDE from the Skorokhod weak convergence technic": There exists a probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{\mathbb{P}})$ (actually, $\widetilde{\Omega} = [0,1], \widetilde{\mathfrak{F}} = \mathfrak{B}([0,1])$) and a sequence of RCLL processes $(\tilde{x}_t^n, \tilde{w}_t^n, \tilde{\zeta}_t^n), n = 0, 1, 2, \cdots$, defined on it such that $(\tilde{x}_t^n, \tilde{w}_t^n, \tilde{\zeta}_t^n), n = 1, 2, \cdots$ have the same finite probability distributions as those of $(x_t^n, w_t, \zeta_t), n = 1, 2, \cdots$, where*

$$\zeta_t = \int_0^t \int_{|z| \leq 1} z\widetilde{N}_k(ds, dz) + \int_0^t \int_{|z|>1} zN_k(ds, dz)$$

*and as $n \to \infty, \forall t \geq 0$,*

$$\tilde{\eta}_t^n \to \tilde{\eta}_t^0,\ in\ probability,\ as\ \tilde{\eta}_t^n = \tilde{x}_t^n, \tilde{w}_t^n, \tilde{\zeta}_t^n, n = 0, 1, 2, \cdots$$

*Write*

$$\tilde{p}^n(dt, dz) = \sum_{s \in dt} \mathbb{1}_{(0 \neq \Delta\tilde{\zeta}_s^n \in dz)}(s), \tilde{q}^n(dt, dz) = \tilde{p}^n(dt, dz) - \pi(dz)dt,$$

$$\forall n = 0, 1, 2, \cdots$$

*Then $\tilde{p}^n(dt, dz)$ is a Poisson random counting measure with the compensator $\pi(dz)dt$ for each $n = 0, 1, 2, \cdots$, and it satisfies the condition*

$$\tilde{\zeta}_t^n = \int_0^t \int_{|z|\leq 1} z\tilde{q}^n(ds, dz) + \int_0^t \int_{|z|>1} z\tilde{p}^n(ds, dz), n = 0, 1, 2, \cdots$$

*Moreover, $\tilde{w}_s^n$ and $\tilde{w}_t^0$ are BMs on the probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$ and, $\tilde{P}^n(dt, dz)$ and $\tilde{p}^0(dt, dz)$ are Poisson martingale measures with the same compensator $\pi(dz)dt$. Furthermore, $(\tilde{x}_t^n)$ satisfies the following SDE with $\tilde{w}_t^n$ and $\tilde{q}^n(dt, dz)$ on $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$.*

$$\tilde{x}_t^n = x_0 + \int_0^t b^n(s, \tilde{x}_s^n)ds + \int_0^t \sigma^n(s, \tilde{x}_s^n)d\tilde{w}_s^n$$

$$+ \int_0^t \int_Z c^n(s, \tilde{x}_{s-}^n, z)\tilde{q}^n(ds, dz).$$

*Proof.* By the properties of $b^n, \sigma^n$, and $c^n$, we have

$$\sup_n \mathbb{E}(\sup_{t\leq T} |x_t^n|^2) \leq k_T.$$

Moreover, as $r \leq t \leq T$;

$$\mathbb{E}|x_t^n - x_r^n|^2 \leq 3\mathbb{E}|\int_r^t b^n(s, x_s^n)ds|^2 + 3\mathbb{E}\int_r^t |\sigma^n(s, x_s^n)|^2 ds$$

$$+3\mathbb{E}\int_r^t \int_Z |c^n(s, x_s^n, z)|^2 \pi(dz)ds \leq 6(t - r)\int_r^t \mathbb{E}(1 + |x_s^n|)^2 c_1(s)ds$$

$$+24\int_r^t \mathbb{E}(1 + |x_s^n|)^2 c_1(s)ds \leq k_T'(t - r).$$

So

$$\sup_n \sup_{t_1,t_2\leq T;|t_1-t_2|\leq h} \mathbb{E}(|x_{t_1}^n - x_{t_2}^n|^2) \leq k_T' h.$$

Thus for each $T \geq 0, \epsilon > 0$

$$\lim_{N\to\infty}\sup_n\sup_{t\le T}\mathbb{P}\{|x_t^n|>N\}\le\lim_{N\to\infty}\frac{k_T}{N^2}=0,$$

$$\lim_{h\downarrow 0}\sup_n\sup_{t_1,t_2\le T,|t_1-t_2|\le h}\mathbb{P}\{|x_{t_1}^n-x_{t_2^n}|>\epsilon\}\le\lim_{h\downarrow 0}k'_T h=0,$$

Therefore,$\forall n\ge 1$

$$\begin{cases}\lim\limits_{N\to\infty}\sup\limits_n\sup\limits_{t\le T}\mathbb{P}(|x_t^n|>N)=0,\\ \lim\limits_{h\downarrow 0}\sup\limits_n\sup\limits_{t_1,t_2\le T,|t_1-t_2|\le h}\mathbb{P}(|x_{t_1}^n-x_{t_2^n}|>\epsilon)=0.\end{cases}\tag{3.9}$$

Now write

$$\zeta_t=\int_0^t\int_{|z|\le 1}z\widetilde{N}_k(ds,dz)+\int_0^t\int_{|z|>1}zN_k(ds,dz)=\zeta_t^1+\zeta_t^2.$$

Let us show that $\zeta_t$ also satisfies 3.9. In fact, by the martingale inequality

$$\mathbb{E}\sup_{t\le T}|\zeta_t^1|^2\le\int_0^T\int_{|z|\le 1}|z|^2\pi(dz)ds\le 2T\int_{|z|\le 1}\frac{|z|^2}{1+|z|^2}\pi(dz)<\infty.$$

Hence $\lim\limits_{N\to\infty}\sup\limits_{t\le T}\mathbb{P}(|\zeta_t^1|>N)\to 0.$

Write $\bar{I}_t^2=\int_0^t\int_{|z|>1}|z|N_k(ds,dz).$

Since $\bar{I}_t^2$ is RCLL, $\{0<s\le T:\Delta\bar{I}_s^2>1\}$ is a finite set, so

$$\sum_{0<s\le T}\Delta\bar{I}_s^2 I_{(\Delta\bar{I}_s^2>1)}=\sum_{k=1}^{n(\omega)}|z_k(\omega)|\mathbb{1}_{|z_k(\omega)|>1}<\infty.$$

Hence $\mathbb{P}(\sup\limits_{t\le T}|I_t^2|<\infty)=1$. In particular, $\lim\limits_{N\to\infty}\sup\limits_{t\le T}\mathbb{P}(|\zeta_t^2|>N)=0$. Now for arbitrary $\epsilon>0$

$$\mathbb{P}(|\zeta_t-\zeta_s|>\epsilon)\le\mathbb{P}(\left|\int_s^t\int_{|z|\le 1}z\widetilde{N}_k(ds,dz)\right|>\epsilon/2)$$

$$+\mathbb{P}(\left|\int_s^t\int_{|z|>1}z\widetilde{N}_k(ds,dz)\right|>\epsilon/2)=J_1+J_2.$$

It is evident that as $|t-s| \leq h \to 0$

$$J_1 \leq (2/\epsilon)^2 \mathbb{E}\left|\int_s^t \int_{|z|\leq 1} z\widetilde{N}_k(ds,dz)\right|^2 \leq (2/\epsilon)^2 \int_{|z|\leq 1} |z|^2 \pi(dz)|t-s|$$

$$\leq 2(2/\epsilon)^2 \int_{|z|\leq 1} \frac{|z|^2}{1+|z|^2}\pi(dz)|t-s| \to 0.$$

Notice that $N_k(dt,dz)$ is a Poisson random measure with the compensator

$$\pi(dz)dt, as|t-s| \leq h \to 0$$

$$J_2 \leq \mathbb{P}(N_k((s,t],|z|>1)>0) = 1-\exp(-\int_s^t \int_{|z|>1} \pi(dz)dr)$$

$$\leq 1-\exp(-\pi(|z|>1)h) \to 0,$$

where $\pi(|z|>1) = \int_{|z|>1} \pi(dz) = 2\int_{|z|>1} \frac{|z|^2}{1+|z|^2}\pi(dz) < \infty$. Hence $\zeta_t$ satisfies 3.9, that is,

$$\lim_{N\to\infty} \sup_{t\leq T} \mathbb{P}(|\zeta_t| > N) = 0, \text{ and}$$

$$\lim_{h\downarrow 0} \sup_{t_1,t_2\leq T, |t_1-t_2|\leq h} \mathbb{P}(|\zeta_{t_1}-\zeta_{t_2}| > \epsilon) = 0.$$

Since $\mathbb{E}|w_t - w_s|^2 = |t-s|$. One also easily shows that 3.9 holds for $w_t$. Hence Skorokhod's theorem applies to $\{x_t^n, \zeta_t, w_t\}$. □

**Remark 3.2.1.** *By this lemma one sees that if "the result of SDE from the Skorokhod weak convergence technic " holds, and we can prove that*

$$\left|\int_0^t (b^n(s,\tilde{x}_s^n) - b(s,\tilde{x}_s^0))ds\right| \to 0, \textit{ in probability } \widetilde{\mathbb{P}},$$

$$\int_0^t \sigma^n(s,\tilde{x}_s^n)d\widetilde{w}_s^n \to \int_0^t \sigma(s,\tilde{x}_s^0)d\widetilde{w}_s^0, \textit{ in } \widetilde{\mathbb{P}},$$

$$\int_0^t \int_Z c^n(s, \tilde{x}^n_{s-}, z)\tilde{q}^n(ds, dz) \to \int_0^t \int_Z c^n(s, \tilde{x}^0_{s-}, z)\tilde{q}^0(ds, dz), \, in \, \widetilde{\mathbb{P}}, \tag{3.10}$$

*then* $(\tilde{\Omega}, \tilde{\mathfrak{F}}, (\tilde{\mathfrak{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}}; \{\tilde{w}^0_t\}_{t\geq 0}, \tilde{q}^0(dt, dz), \{\tilde{x}^0_t\}_{t\geq 0})$*, or say* $\tilde{x}^0_t$*, is a weak solution of 3.11 in the next section.*

## 3.3 Weak solutions, Continuous coefficients

The technic used in proving Theorem 3.1.1 motivates us to obtain an existence theorem for weak solutions of SDE with jumps and with $\sigma$, which can be degenerate. Consider the following SDE with non-random coefficients: $\forall t \geq 0$,

$$x_t = x_0 + \int_0^t b(s, x_s)ds + \int_0^t \sigma(s, x_s)dw_s + \int_0^t \int_Z c(s, x_{s-}, z)\widetilde{N}_k(ds, dz). \tag{3.11}$$

**Theorem 3.3.1.** *Assume that*

1.
$$\begin{aligned} b &= b(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d, \\ \sigma &= \sigma(t, x) : [0, \infty) \times \mathbb{R}^d \to R^{d\otimes d}, \\ c &= c(t, x, z) : [0, \infty) \times \mathbb{R}^d \times Z \to \mathbb{R}^d, \end{aligned}$$

   *are jointly Borel measurable such that* $\mathbb{P}-$ *a.s.*

$$\int_Z |c(t, x, z)|^2 \pi(dz) \leq c_1(t)(1 + |x|^2),$$

   *where* $c_1(t)$ *is non-negative such that for each* $T < \infty$

$$\int_0^T c_1(t)dt < \infty;$$

2. $|b(t, x)|^2 + |\sigma(t, x)| \leq c_1(t)(1 + |x|^2)$*, where* $c_1(t)$ *has the same property as in* (1)*;*

3. $b(t, x)$ *is continuous in* $x$ *and* $\sigma(t, x)$ *is jointly continuous in* $(t, x)$*; and*

$$\lim_{h,h'\to 0}\int_Z |c(t+h',x+h,z)-c(t,x,z)|^2\pi(dz)=0;$$

*4.* $Z=\mathbb{R}^d-\{0\}$, $\int_Z \frac{|z|^2}{1+|z|^2}\pi(dz)<\infty$. *Then for any given constant* $x_0\in\mathbb{R}^d$ *3.11 has a weak solution.*

*Proof.* We can smooth out $b,\sigma$ and $c$ only with respect to $x$ to get $b^n$, $\sigma^n$, and $c^n$, respectively. Then we have a pathwise unique strong solution $x_t^n$ satisfying a SDE similar to 3.2, but here all coefficients $b^n$, $\sigma^n$, and $c^n$ do not directly depend on $\omega$. Now applying Lemma 3.2.1 "the result of SDE from the Skorokhod weak convergence technic" holds. So we only need to show 3.10 in Remark 3.2.1 holds. However, since $\forall t\geq 0$,

$$\tilde{x}_t^0\to\tilde{x}_t^0,\text{ in probability }\widetilde{\mathbb{P}},\text{ as }n\to\infty,$$

as in the proof of Theorem 3.1.1 one finds that 3.4 and 3.5 hold. So, we may assume that all $\{\tilde{x}_t^n, t\in[0,T]\}_{n=0}^\infty$ are uniformly bounded, that is, $|\tilde{x}_t^n|\leq k_0, \forall t\in[0,T], \forall n=0,1,2,\cdots$ in all following discussion on the convergence in probability. Now for an arbitrary given $\epsilon>0$

$$\begin{aligned}
&\widetilde{\mathbb{P}}\left(\left|\int_0^t\int_Z c^n(s,\tilde{x}_{s-}^n,z)\tilde{q}^n(ds,dz)-\int_0^t\int_Z c(s,\tilde{x}_{s-}^0,z)\tilde{q}^0(ds,dz)\right|>\epsilon\right)\\
\leq\ &\widetilde{\mathbb{P}}\left(\left|\int_0^t\int_Z (c^n(s,\tilde{x}_{s-}^n,z)-c(s,\tilde{x}_{s-}^n,z))\tilde{q}^n(ds,dz)\right|>\epsilon/3\right)\\
+\ &\widetilde{\mathbb{P}}\left(\left|\int_0^t\int_Z (c^n(s,\tilde{x}_{s-}^n,z)-c(s,\tilde{x}_{s-}^0,z))\tilde{q}^n(ds,dz)\right|>\epsilon/3\right)\\
+\ &\widetilde{\mathbb{P}}\left(\left|\int_0^t\int_Z c^n(s,\tilde{x}_{s-}^0,z)\tilde{q}^n(ds,dz)-\int_0^t\int_Z c(s,\tilde{x}_{s-}^0,z)\tilde{q}^0(ds,dz)\right|>\epsilon/3\right)\\
=\ &\sum_{i=1}^3 I_i^n.
\end{aligned}$$

Obviously,

$$\begin{aligned}
I_1^n &\leq \frac{9}{\epsilon^2}\mathbb{E}^{\widetilde{\mathbb{P}}}\int_0^t\sup_{|x|\leq k_0}\int_z |c^n(s,x,z)-c(s,x,z)|^2\pi(dz)ds=I_{11}^n,\text{and}\\
I_2^n &\leq \frac{9}{\epsilon^2}\mathbb{E}^{\widetilde{\mathbb{P}}}\int_0^t |c^n(s,\tilde{x}_s^n,z,\omega)-c(s,\tilde{x}_s^n,z,\omega)|^2\mathbb{1}_{|\tilde{x}_t^n|\leq k_0}\mathbb{1}_{|\tilde{x}_t|\leq k_0}\pi(dz)ds=I_{21}^n,
\end{aligned}$$

Now as the proof of 3.7 and 3.8 one finds that

$$\begin{aligned} \lim_{n\to\infty} I_1^n &\leq \lim_{n\to\infty} I_{11}^n &= 0, \text{and} \\ \lim_{n\to\infty} I_2^n &\leq \lim_{n\to\infty} I_{21}^n &= 0 \end{aligned}$$

Let us show that $\lim\limits_{n\to\infty} I_3^n = 0$. In fact, for any $0 < T < \infty$,

$$I_3^n \leq 2\left(\frac{12}{\epsilon}\right)^2 \mathbb{E}\int_0^T \int_{0<|z|<\delta} |c(s,\tilde{x}_s^0,z)|^2 \pi(dz)ds$$

$$+\widetilde{\mathbb{P}}\left(\left|\int_0^t \int_{|z|\geq\delta} 1\!\!1_{|\tilde{x}_s^0|\leq k_0} c(s,\tilde{x}_{s-}^0,z)\tilde{q}^n(ds,dz) - \int_0^t \int_{|z|\geq\delta} 1\!\!1_{|\tilde{x}_s^0|\leq k_0} c(s,\tilde{x}_{s-}^0,z)\tilde{q}^0(ds,dz)\right| > \frac{\epsilon}{6}\right)$$

$$= I_2^\delta + I_3^{n,\delta}.$$

Notice that as $\delta \downarrow 0$,

$$\mathbb{E}\int_0^T \int_{\{0<|z|<\delta\}} |c(s,\tilde{x}_s^0,z)|^2 \pi(dz)ds < \infty, \text{ and } \{0<|z|<\delta\} \downarrow \emptyset.$$

So one can take a small enough $\delta > 0$ such that $I_2^\delta < \tilde{\epsilon}/3$. Observe that

$$I_3^{n,\delta} \leq 2\left(\frac{18}{\epsilon}\right)^2 \mathbb{E}\int_0^T \int_{|z|\geq\delta} 1\!\!1_{\sup\limits_{s\leq T} |\tilde{x}_s^0| \leq k_0} |c(s,\tilde{x}_{s-}^0,z)|$$

$$- \sum_{i=0}^{2^m-1} |c(\frac{iT}{2^m},\tilde{x}_{\frac{iT}{2^m}}^0,z) 1\!\!1_{(\frac{iT}{2^m},\frac{(i+1T)}{2^m}]}(s)|^2 \pi(dz)ds$$

$$+\widetilde{\mathbb{P}}(\sum_{i=0}^{2^m-1} 1\!\!1_{\sup\limits_{s\leq T} |\tilde{x}_s^0| \leq k_0} |\int_{\frac{iT}{2^m}}^{\frac{(i+1)T}{2^m}} \int_{|z|\geq\delta} c\left(\frac{iT}{2^m},\tilde{x}_{\frac{iT}{2^m}}^0,z\right)\tilde{p}^n(ds,dz)$$

$$- \int_{\frac{iT}{2^m}}^{\frac{(i+1)T}{2^m}} \int_{|z|\geq\delta} c\left(\frac{iT}{2^m},\tilde{x}_{\frac{iT}{2^m}}^0,z\right)\tilde{p}^0(ds,dz) > \frac{\epsilon}{6}$$

$$= I_{31}^m + I_{32}^{m,\delta},$$

where $0 < \frac{T}{2^m} < \frac{2T}{2^m} < \cdots < \frac{iT}{2^m} < \cdots < T$ is a division on $[0, T]$. Since by conditions $(1) - (3)$ $\lim_{m\to\infty} I_{31}^m = 0$, one can choose a large enough $m$ such that $I_{31}^m < \tilde{\epsilon}/6$. For these given $m, \delta$ there exists a $\widetilde{N}$ such that as $n \geq \widetilde{N}$, $I_{32}^{m,\delta} < \tilde{\epsilon}/6$. So, we have proved that $\lim_{n\to\infty} I_3^n = 0$, and eventually we obtain that

$$\lim_{n\to\infty} \sum_{i=1}^{3} I_i^n = 0$$

That is, the third limit in 3.10 holds. The proofs of the remaining results are similar and even simpler. Thus $\tilde{x}_t^0$ is a weak solution.

□

For that the coefficient $b$ can be greater than linear growth we can establish the following thoerem.

**Theorem 3.3.2.** *Assume that*

$$\begin{aligned} |b(t,x)| &\leq c_1(t)(1 + |x|\Pi_{k=1}^m g_k(x)), \\ |b(t,x)| &\leq k_0(1 + |x|^2\Pi_{k=1}^m g_k(x)), \end{aligned}$$

*where $c_1(t) \geq 0$ and $g_k(x)$ is such that*

$$g_k(x) = 1 + \underbrace{\ln(1 + \ln(1 + \cdots \ln}_{k-times}(1 + |x|^{2n_0}))),$$

*($n_0$ is some natural number).*
*Then for any given constant $x_0 \in \mathbb{R}^d$ 3.11 has a weak solution on $t \geq 0$.*

*Proof.* For each $n = 1, 2, \cdots$ introduce a real smooth function $W^n(x), x \in \mathbb{R}^d$, such that $0 \leq W^n(x) \leq 1$ and $W^n(x) = 1$, as $|x| \leq n$; $W^n(x) = 0$, as $|x| \geq n + 1$. Write

$$b^n(t,x) = b(t,x)W^n(x), \sigma^n(t,x) = \sigma(t,x)W^n(x).$$

Then by Theorem 3.3.1 for each $n$ there exists a weak solution $x_t^n$ with a BM $x_t^n$ and a Poisson martingale measure $\widetilde{N}_{k^n}(dt, dz)$, which has the same compensator $\pi(dz)dt$, defined on some probability space $(\Omega^n, \mathfrak{F}^n, \{\mathfrak{F}_t^n\}, \mathbb{P}^n)$ such that $\mathbb{P}^n-$ a.s. $\forall t \geq 0$,

$$x_t^n = x_0 + \int_0^t b^n(s, x_s^n)ds + \int_0^t \sigma^n(s, x_s^n)dw_s^n + \int_0^t \int_Z c(s, x_{s-}^n, z)\widetilde{N}_{k^n}(ds, dz).$$

Construct a space $\Omega^n = D \times W_0 \times D$, where $D$ and $W_0$ are the totality of all RCLL real functions and all real continuous functions $f(t)$ with $f(0) = 0$, defined on $[0, \infty)$, respectively. Map $(x^n(., \omega), w^n(., \omega), \zeta^n(., \omega))$ into the $\Omega$, where

$$\zeta_t^n = \int_0^t \int_{|z|<1} z\widetilde{N}_{k^n}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN_{k^n}(ds, dz),$$

and

$$\begin{aligned} N_{k^n}((0,t], U) &= \sum_{0<s\leq t} \mathbb{1}_{0\neq\Delta\zeta_s^n\in U}, \text{for} t \geq 0, U \in \mathfrak{B}(Z), \\ \widetilde{N}_{k^n}(dt, dz) &= N_{k^n}(dt, dz) - \pi(dz)dt. \end{aligned}$$

From this map we get a probability law $\mathbb{P}_{x_0}^n$ on $\Omega^n$. Now let

$$\Omega = \Pi_{n=1}^{\infty}\Omega^n, \mathfrak{F} = \Pi_{n=1}^{\infty}\mathfrak{F}^n, \mathbb{P} = \Pi_{n=1}^{\infty}\mathbb{P}_{x_0}^n,$$

where $\mathfrak{F}^n = \mathfrak{B}_D \times \mathfrak{B}_{W_0} \times \mathfrak{B}_D$, and define $\forall\omega = (\omega^1, \cdots, \omega^n, \cdots) \in \Omega$,

$$\begin{aligned} \tilde{x}_t^1(\omega) &= \tilde{x}_t^1(\omega^1), \tilde{w}_t^1(\omega) &= \tilde{w}_t^1(\omega^1), \tilde{\zeta}_t^1(\omega) &= \zeta_t^1(\omega^1), \\ \cdots\cdots \\ \tilde{x}_t^n(\omega) &= \tilde{x}_t^n(\omega^n), \tilde{w}_t^n(\omega) &= \tilde{w}_t^n(\omega^n), \tilde{\zeta}_t^n(\omega) &= \zeta_t^n(\omega^n), \\ \cdots\cdots \end{aligned}$$

Then one finds that for each $n$, $\tilde{x}_t^n$ satisfies the following SDE: $\mathbb{P}-$ a.s.

$$\tilde{x}_t^n = x_0 + \int_0^t b^n(s, \tilde{x}_s^n)ds + \int_0^t \sigma_n(s, \tilde{x}_s^n d\widetilde{w}_s^n)$$

$$+ \int_0^t \int_Z c(s, \tilde{x}_{s-}^n, z)\widetilde{N}_{k'^n}(ds, dz), \forall t \geq 0,$$

where

$$\begin{aligned} N_{k'^n}((0,t], U) &= \sum_{0<s\leq t} I_{0\neq\Delta\tilde{\zeta}_s^n\in U}, \text{for}\, t \geq 0, U \in \mathfrak{B}(Z), \\ \widetilde{N}_{k'^n}(dt, dz) &= N_{k'^n}(dt, dz) - \pi(dz)dt, \end{aligned}$$

and

$$\tilde{\zeta}_t^n = \int_0^t \int_{|z|<1} z\widetilde{N}_{k'^n}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN_{k'^n}(ds, dz).$$

Let us show that the following facts hold for $\eta_t^n = \tilde{\zeta}_t^n, \widetilde{w}_t^n$, and $\tilde{x}_t^n$:

$$\lim_{N\to\infty} \sup_n \sup_{t\leq T} \mathbb{P}(|\eta_t^n| > N) = 0,$$

$$\lim_{h\downarrow 0} \sup_n \sup_{t_1,t_2\leq T, |t_1-t_2|\leq h} \mathbb{P}(|\eta_{t_1}^n - \eta_{t_2}^n| > \epsilon) = 0, \tag{3.12}$$

In fact, as the proof of Lemma 3.2.1 one easily sees that $\zeta_t^n$ satisfies the condition:

$$\lim_{N\to\infty} \sup_n \sup_{t\leq T} \mathbb{P}(|\tilde{\zeta}_t^n| > N)$$

$$= \lim_{N\to\infty} \sup_n \sup_{t\leq T} \mathbb{P}_{x_0}^n(|\zeta_t^n| > N)$$

$$= \lim_{N\to\infty} \sup_{t\leq T} \mathbb{P}_{x_0}^1(|\zeta_t^1| > N) = 0,$$

and

$$\lim_{h\downarrow 0} \sup_n \sup_{t_1,t_2\leq T, |t_1-t_2|\leq h} \mathbb{P}(|\tilde{\zeta}_{t_1}^n - \tilde{\zeta}_{t_2}^n| > \epsilon)$$

$$= \lim_{h\downarrow 0} \sup_n \sup_{t_1,t_2\leq T, |t_1-t_2|\leq h} \mathbb{P}_{x_0}^n(|\zeta_{t_1}^n - \zeta_{t_2}^n| > \epsilon)$$

$$= \lim_{h\downarrow 0} \sup_{t_1,t_2\leq T, |t_1-t_2|\leq h} \mathbb{P}_{x_0}^1(|\zeta_{t_1}^1 - \zeta_{t_2}^1| > \epsilon) = 0,$$

because all $\{\zeta_t^n\}_{t\geq 0}, n = 1, 2, \cdots$ have the same probability laws. So 3.12 holds for $\zeta_t^n$. Similarly, $\widetilde{w}_t^n$ also satisfies the 3.12. Now applying Itô's formula to $g_{m+1}(\tilde{x}_t^n)$, one finds that $\mathbb{P}-$ a.s.

$$\begin{aligned} g_{m+1}(\tilde{x}_t^n) = \;& g_{m+1}(x_0) + \int_0^t g'_{m+1}(\tilde{x}_s^n) b^n(s, \tilde{x}_s^n) ds \\ + \;& \int_0^t g'_{m+1}(\tilde{x}_{s-}^n) \sigma(s, \tilde{x}_s^n) dw_s + \frac{1}{2} \int_0^t \|g''_{m+1}(\tilde{x}_s^n) \sigma(s, \tilde{x}_s^n)\|^2 ds \\ + \;& \int_0^t \int_Z g'_{m+1}(\tilde{x}_{s-}^n) c(s, \tilde{x}_{s-}^n, z) \widetilde{N}_{k'^n}(ds, dz) \\ + \;& \int_0^t \int_Z [g_{m+1}(\tilde{x}_{s-}^n c(s, \tilde{x}_{s-}^n, z)) - g_{m+1}(\tilde{x}_{s-}^n) \\ - \;& g'_{m+1}(\tilde{x}_{s-}^n) c(s, \tilde{x}_{s-}^n, z)] N_{k'^n}(ds, dz), \end{aligned}$$

where we write $g'_{m+1}(x) = grad g_{m+1}(x)$, and $g''_{m+1}(x) = \left[\frac{\partial^2}{\partial x_i \partial x_j} g_{m+1}(x) =\right]_{i,j=1}^d$

By evaluation and from the assumption one sees that

$$|g'_{m+1}(x) b^n(s,x)| \leq \Pi_{k=1}^m g_k^{-1}(x) \frac{2n_0 |x|^{2n_0 - 2}}{1 + |x|^{2n_0}} |x \cdot b^n(s,x)| \leq k_0 c_1(t),$$

$$\|g''_{m+1}(x) \sigma(s,x)\|^2$$

$$\leq \sum_{i,j,l=1}^d \{\Pi_{k=1}^m g_k^{-1}(x) \left[ \frac{2n_0 \delta_{ij} |x|^{2n_0-2} + 4n_0(n_0-1) x_i x_j |x|^{2n_0-4}}{1+|x|^{2n}} - \frac{4n_0^2 x_i x_j |x|^{4n_0-4}}{(1+|x|^{2n_0})^2} \right]$$

$$-\Pi_{k=1}^m g_k^{-1}(x) \frac{4n_0^2 x_i x_j |x|^{4n_0-4}}{(1+|x|^{2n_0})^2} \sum_{k=0}^m \Pi_{l=1}^m g_l^{-1}(x)\} (\sigma_{il}\sigma_{jl})(t,x) \leq k_0,$$

$$\|g''_{m+1}(x)\|^2 \leq k_0,$$

where $k_0 > 0$ is a constant, and we write $g_0(x) = 1$. Hence using the fact that

$$\sup_{t \leq T} \ln(1 + |x_t|) = \ln(1 + \sup_{t \leq T} |x_t|)$$

one finds that as $T < \infty$, $\forall n$, when $N \to \infty$,

$$\mathbb{P}(\sup_{t \leq T} |\tilde{x}_t^n| > N) \leq \frac{1}{g_{m+1}(N)} \mathbb{E} g_{m+1}(N) (\sup_{t \leq T} |\tilde{x}_t^n|)$$

$$= \frac{1}{g_{m+1}(N)} \mathbb{E} \sup_{t \leq T} g_{m+1}(N)(|\tilde{x}_t^n|) \leq k'_0 (1 + \int_0^T c_1(t) dt) / g_{m+1}(N) \to 0$$

This means that

$$\lim_{N\to\infty} \sup_{n=1,2,\cdots} (\mathbb{P} \sup_{t\leq T} |\tilde{x}_t^n| < N) = 0. \tag{3.13}$$

Furthermore, by Itô's formula to $g_{m+1}(\tilde{x}_s^n - \tilde{x}_r^n), s \in (r,t]$, one finds that $\mathbb{P}-$ a.s.

$$\begin{aligned} g_{m+1}(\tilde{x}_s^n - \tilde{x}_r^n) = {} & 1 + \int_r^t g'_{m+1}(\tilde{x}_s^n - \tilde{x}_r^n) b^n(s, \tilde{x}_s^n) ds \\ + {} & \int_r^t g'_{m+1}(\tilde{x}_s^n - \tilde{x}_r^n)\sigma(s, \tilde{x}_s^n) dw_s + \frac{1}{2}\int_r^t \|g''_{m+1}(\tilde{x}_s^n - \tilde{x}_r^n)\sigma(s, \tilde{x}_s^n)\|^2 ds \\ + {} & \int_r^t \int_Z g'_{m+1}(\tilde{x}_{s-}^n - \tilde{x}_r^n) c(s, \tilde{x}_{s-}^n, z) \widetilde{N}_{k'^n}(ds, dz) \\ + {} & \int_r^t \int_Z [g'_{m+1}(\tilde{x}_{s-}^n - \tilde{x}_r^n c(s, \tilde{x}_{s-}^n, z)) g'_{m+1}(\tilde{x}_{s-}^n - \tilde{x}_r^n) \\ - {} & g'_{m+1}(\tilde{x}_{s-}^n - \tilde{x}_r^n) c(s, \tilde{x}_{s-}^n, z) N_{k'^n}(ds, dz). \end{aligned}$$

Thus one similarly has that $\forall \epsilon > 0, 0 \leq t - r \leq h, t \leq T, \forall n,$

$$\mathbb{P}(|\tilde{x}_t^n - \tilde{x}_r^n| > \epsilon) \leq \sup_n \mathbb{P}(\sup_{t\leq T} |\tilde{x}_t^n| > N)$$

$$+ \frac{1}{g_{m+1}(\epsilon) - 1} \mathbb{E}(g_{m+1}(\tilde{x}_t^n - \tilde{x}_r^n) - 1) 1\!\!1_{\sup_{t\leq T} |\tilde{x}_t^n| \leq N}$$

$$\leq k'_0 \left[\int_r^t c_1(t) dt + (t-r)\right] / (1 - g_{m+1}(\epsilon)) \to 0, \text{when}\, h \to 0.$$

Therefore, 3.12 holds for $\eta_t^n = \tilde{x}_t^n$. Hence Skorokhod's theorem applies. By this and by "the result of SDE from the Skorokhod weak convergence technic" holds. (See Lemma 3.2.1 and Remark 3.2.1). So we only need to show that 3.10 in Remark 3.2.1 holds. For this let us first show that $\lim_{N\to\infty} \mathbb{P}(\sup_{t\leq T} |\tilde{x}_t^0| > N) = 0$. In fact,

$$\begin{aligned}
&\mathbb{P}(\sup_{t\leq T}|\tilde{x}_t^0| > N) = \mathbb{P}(\sup_{k=1,2,\cdots}|\tilde{x}_{r_k}^0| > N) \\
\leq\ & \mathbb{P}(\sup_{k=1,2,\cdots}|\tilde{x}_{r_k}^0 - \tilde{x}_{r_k}^{n_k}| > \frac{N}{2}) + \mathbb{P}(\sup_{k=1,2,\cdots}|\tilde{x}_{r_k}^{n_k}| > \frac{N}{2}) \\
\leq\ & \mathbb{P}(\sup_{k=1,2,\cdots}|\tilde{x}_{r_k}^0 - \tilde{x}_{r_k}^{n_k}| > \frac{1}{2}) + \mathbb{P}(\sup_{k=1,2,\cdots}|\tilde{x}_{r_k}^{n_k}| > \frac{N}{2}) = I_1 + I_2^N
\end{aligned}$$

where $\{r_k\}_{k=1}^{\infty}$ is the set of all rational numbers in $[0,T]$. However, for arbitrary given $\tilde{\epsilon} > 0$ and for each $r_k$ we may take an $n_k$ large enough such that $\mathbb{P}(|\tilde{x}_{r_k}^0 - \tilde{x}_{r_k}^{n_k}| > \frac{1}{2}) < \frac{\tilde{\epsilon}}{2^{k+1}}, k = 1,2,\cdots$. Hence $I_1 \leq \sum_{k=1}^{\infty}\frac{\tilde{\epsilon}}{2^{k+1}} = \frac{\tilde{\epsilon}}{2}$. On the other hand, by 3.13 there exists a $\widetilde{N}$ such that as $\forall N \geq \widetilde{N}, I_2^N < \frac{\tilde{\epsilon}}{2}$. Therefore, $\lim_{N\to\infty}\mathbb{P}(\sup_{t\leq T}|\tilde{x}_t^0| > N) = 0$ holds true.

Now let us prove the second limit in 3.10. Notice that from 3.13 and the result just proved we may assume that $|\tilde{x}_t^n| \leq k_0, \forall t \in [0,T], \forall n = 0,1,2,\cdots$. Now for any given $\epsilon > 0$

$$\begin{aligned}
&\widetilde{\mathbb{P}}(\left|\int_0^t \sigma^n(s,\tilde{x}_s^n)d\widetilde{w}_s^n - \int_0^t \sigma(s,\tilde{x}_s^0)d\widetilde{w}_s^0\right| > \epsilon) \\
\leq\ & \left(\frac{2}{\epsilon}\right)^2 \mathbb{E}\int_0^t \left|\sigma^n(s,\tilde{x}_s^n) - \sigma(s,\tilde{x}_s^0)\right|^2 1\!\!1_{|\tilde{x}_s^n|\leq k_0} 1\!\!1_{|\tilde{x}_s^0|\leq k_0} ds \\
+\ & \mathbb{P}(\left|\int_0^t 1\!\!1_{|\tilde{x}_s^0|\leq k_0}\sigma(s,\tilde{x}_s^n)d\widetilde{w}_s^n - \int_0^t 1\!\!1_{|\tilde{x}_s^0|\leq k_0}\sigma(s,\tilde{x}_s^0)d\widetilde{w}_s^0\right| > \frac{\epsilon}{2}) \\
=\ & I_3^n + I_4^n.
\end{aligned}$$

Notice that for any $\epsilon > 0$ as $n > k_0$,

$$\begin{aligned}
&\widetilde{\mathbb{P}}(|\sigma^n(s,\tilde{x}_s^n) - \sigma(s,\tilde{x}_s^0)|^2 1\!\!1_{|\tilde{x}_s^n|\leq k_0} 1\!\!1_{|\tilde{x}_s^0|\leq k_0} > \epsilon) \\
=\ & \widetilde{\mathbb{P}}(|\sigma(s,\tilde{x}_s^n) - \sigma(s,\tilde{x}_s^0)|^2 1\!\!1_{|\tilde{x}_s^0|\leq k_0} 1\!\!1_{|\tilde{x}_s^0|\leq k_0} > \epsilon \\
\leq\ & \widetilde{\mathbb{P}}(|\tilde{x}_s^n - \tilde{x}_s^0| > \eta) \\
+\ & \widetilde{\mathbb{P}}(|\sigma(s,\tilde{x}_s^n) - \sigma(s,\tilde{x}_s^0)|^2 1\!\!1_{|\tilde{x}_s^0|\leq k_0} 1\!\!1_{|\tilde{x}_s^0|\leq k_0} 1\!\!1_{|\tilde{x}_s^n - \tilde{x}_s^0|>\eta} > \epsilon).
\end{aligned}$$

Since $\sigma(s,x)$ is continuous in $x$, so it is uniformly continuous in $|x| < k_0$ Hence one can choose a small enough $\eta > 0$ (which can depend on $s$) such that as $|x' - x"| \leq \eta$

and $|x'|, |x''| \leq k_0, |\sigma(s,x') - \sigma(s,x")| < \epsilon$. This means that we can have the result that as $n \to \infty$,

$$\widetilde{\mathbb{P}}(|\sigma^n(s,\tilde{x}_s^n) - \sigma(s,\tilde{x}_s^0)|^2 \mathbb{1}_{|\tilde{x}_s^n| \leq k_0} \mathbb{1}_{|\tilde{x}_s^0| \leq k_0} > \epsilon) \leq \widetilde{\mathbb{P}}(|\tilde{x}_s^n - \tilde{x}_s^0| > \eta) \to 0$$

So, by Lebesgue's dominated convergence theorem as $n \to \infty, I_3^n \to 0$. Now notice that $\sigma(t,x)$ is jointly continuous, so if we write $\sigma_m(t,x)$ as its smooth functions, then

$$\lim_{m \to \infty} |\sigma_m(t,x) - \sigma(t,x)|^2 = 0, \forall t, x;$$

and

$$|\sigma_m(t,x) - \sigma_m(s,y)| \leq k_m[|t-s| + |x-y|],$$

where $k_m \geq 0$ is a constant depending only on $m$. Observe that

$$I_4^n \leq 2\left(\frac{2}{\epsilon}\right)^2 \mathbb{E} \int_0^T \left|\sigma(s,\tilde{x}_s^0) - \sigma_m(s,\tilde{x}_s^0)\right|^2 \mathbb{1}_{|\tilde{x}_s^0| \leq k_0} ds$$

$$\mathbb{P}(\left|\int_0^t \mathbb{1}_{|\tilde{x}_s^0| \leq k_0} \sigma_m(s,\tilde{x}_s^n) d\widetilde{w}_s^n - \int_0^t \mathbb{1}_{|\tilde{x}_s^0| \leq k_0} \sigma_m(s,\tilde{x}_s^0) d\widetilde{w}_s^0\right| > \frac{\epsilon}{3})$$

$$= I_{41}^m + I_{42}^{m,n}$$

So for any given $\tilde{\epsilon} > 0$ by Lebesgue's dominated convergence theorem we can choose a large enough m such that $I_{41}^m < \tilde{\epsilon}/2$. Then we can have $\lim_{n \to \infty} I_{42}^{m,n} = 0$. Thus we obtain that $\lim_{n \to \infty} I_4^n = 0$, and the second limit in 3.10 is established. The proof for the remaining results are similar. □

## 3.4 Existence of strong solutions and applications to ODE

Applying the above results and using the Yamada-Watanabe type theorem, we immediately obtain the following theorems on the existence of a pathwise unique strong solution to SDE 3.15.

**Theorem 3.4.1.** *[70] Under the assumption of Theorem 3.3.2 if, in addition, the following condition for the pathwise uniqueness holds: (PWU1) for each* $N = 1, 2, \cdots$ *, and each* $T < \infty$,

$$2\langle (x_1 - x_2), (b(t, x_1) - b(t, x_2)\rangle$$

$$+|\sigma(t, x_1) - \sigma(t, x_2)|^2 + \int_Z |c(t, x_1, z) - c(t, x_2, z)|^2 \pi(dz)$$

$$\leq c_T^N(t)\rho_T^N(|x_1 - x_2|^2),$$

*as* $|x_i| \leq N, i = 1, 2, t \in [0, T]$*; where* $c_T^n(t) > 0$ *such that* $\int_0^T c_T^N(t)dt < \infty$*; and* $\rho_T^N(u) \geq 0$*, as* $\geq 0$*, is strictly increasing, continuous and concave such that*

$$\int_{0+} du/\rho_T^N(u) = \infty;$$

*then 3.11 has a pathwise unique strong solution.*

Furthermore, by using Theorem 3.3.2 and Theorem 3.4.1 we immediately obtain a result on the ODE.

**Theorem 3.4.2.**

1. *If* $b(t, x)$ *is jointly Borel measurable and continuous in* $x$ *such that*

$$|b(t, x)| \leq c_1(t)(1 + |x|\Pi_{k=1}^m g_k(x)),$$

   *where* $c_1(t)$ *and* $g_k(x)$ *have the same properties as in Theorem 3.3.2, then the ODE*

$$x_t = x_0 + \int_0^t b(s, x_s)ds, t \geq 0 \tag{3.14}$$

   *has a solution. (It is not necessary unique).*

2. *In addition, if* $b(t, x)$ *is such the (PWU1) condition only for* $b$ *in Theorem 3.4.1, then ODE 3.14 has a unique solution.*

*Proof.* (1) is obtained by Theorem 3.3.1 by setting $\sigma = 0, c = 0$; and (2) follows from Theorem 3.4.1 by letting $\sigma = 0, c = 0$. □

**Example 3.4.1.** *Let $b(t,x) = \mathbb{1}_{t\neq 0}\mathbb{1}_{x\neq 0}t^{-\alpha_1}x|x|^{-\beta}$, where $\alpha_1 < 1, 0 < \beta < 1$. Then ODE 3.14 has a solution. Let $b(t,x) = -\mathbb{1}_{t\neq 0}\mathbb{1}_{x\neq 0}t^{-\alpha_1}x|x|^{-\beta}$ where $\alpha_1 < 1, 0 < \beta < 1$. Then ODE 3.14 has a unique solution.*

## 3.5 Weak solutions, measurable coefficient case

In this section we will discuss the existence of weak solutions of SDEs with measurable coefficients. In this case we have to assume that the SDEs are non-degenerate. In this case one sees that the Krylov type estimate is a very powerful tool for establishing the existence theorem for weak solutions of SDE with jumps under very weak conditions. Consider a $d$-dimensional SDE with jumps as follows: $t \geq 0$,

$$x_t = x_0 + \int_0^t b(s,x_s)ds + \int_0^t |\sigma(s,x_s)dw_s + \int_0^t \int_Z c(s,x_{s-},z)\widetilde{N}_k(ds,dz), \qquad (3.15)$$

where $w_t$ and $\widetilde{N}_k(dt,dz)$ have the same meaning as in 3.1 and all coefficients $b, \sigma$ and $c$ are non-random.

**Theorem 3.5.1.** *[70] Assume that*

1. *$Z = \mathbb{R}^d - \{0\}$, and $\pi(dz) = dz/|z|^{d+1}$;*
2. *$|b(t,x)| + |\sigma(t,x)| + \int_Z |c(t,x,z)|^2\pi(dz) \leq k_0$, where $k_0 > 0$ is a constant, $b, \sigma$ and $c$ are Borel measurable functions;*
3. *there exists a constant $\delta_0 > 0$ such that for all $\mu \in \mathbb{R}^d$,*

$$\langle \sigma(t,x)\mu, \mu\rangle \geq |\mu|^2\delta_0.$$

*Then there exists a weak solution for 3.15.*

Before we prove this theorem let us establish the following lemma.

**Lemma 3.5.1.** *Under assumption of Theorem 3.5.1 there exist smooth functions:* $\forall n = 1, 2, \cdots$

$$b^n(t,x), \sigma^n(t,x), (t,x) \in [0,\infty) \times \mathbb{R}^d,$$

*which are the smoothness functions of* $b(t,x), \sigma(t,x)$, *on* $[0,\infty) \times \mathbb{R}^d$, *respectively; and there exist smooth functions:* $\forall n = 1, 2, \cdots$

$$\tilde{c}^n(t,x,z), (t,x,z) \in [0,\infty) \times \mathbb{R}^d \times \{\epsilon_n \le |z| \le \epsilon_n^{-1}\} = A_n,$$

*which are the smoothness functions of* $c(t,x,z)$, *on* $A_n$, *where en* $\epsilon_n \downarrow 0$, *such that set* $c^n(t,x,z) = \tilde{c}^n(t,x,z) 1_{\{\epsilon_n \le |z| \le \epsilon_n^{-1}\}}$, *then*

1. $|b^n(t,x)| \le k_0, |\sigma^n(t,x)| \le k_0,$
$$\int_Z |c^n(t,x,z)|^2 \pi(dz) \le 2k_0, \forall n = 1, 2, \cdots;$$
2. $\forall \mu \in \mathbb{R}^d, \forall (t,x) \in [0,\infty) \times \mathbb{R}^d,$
$$\langle \sigma^n(t,x)\mu, \mu \rangle \ge |\mu|^2 \delta_0;$$
3. $\forall n = 1, 2, \cdots, \forall t \ge 0, \forall x, x' \in \mathbb{R}^d,$
$$|b^n(t,x) - b^n(t',x')| + |\sigma^n(t,x) - \sigma^n(t',x')| \le k_n k_0 [|x - x'| + |t - t'|],$$
$$\int_{\{\epsilon_n \le |z| \le \epsilon_n^{-1}\}} |c^n(s,x,z) - c^n(s,x',z)|^2 \pi(dz) \le k_n k_0 [|x - x'|^2 + |t - t'|^2],$$
4. *for each* $T < \infty, \forall N = 1, 2, \cdots, \forall q \ge 1,$ *as* $n \to \infty$,
$$\|b^n - b\|_{q,[0,T] \times S_N} + \|\sigma^n - \sigma\|_{q,[0,T] \times S_N} \to 0. \text{ and}$$
$$\| \int_{\{\epsilon_n \le |z| \le \epsilon_n^{-1}\}} |c^n - c|^2 \pi(dz) \|_{q,[0,n] \times S_n} < \frac{1}{2^n},$$

*where* $S_n = \{x \in \mathbb{R}^d : |x| \leq n$

5. $\| \int_Z |c^n - c|^2 \pi(dz) \|_{q,[0,T]\times S_N} \to 0$ *as* $n \to \infty; \forall T, N < \infty.$

*Proof.* Let us smooth out $\sigma$ to get $\sigma_n$, i.e. let for all $u \in R^{d+1}$

$$J_{d+1}(u) = \begin{cases} c_{d+1} \exp(-(1-|u|^2)^{-1}), \mathrm{for}|u| < 1, \\ 0, \mathrm{otherwise}, \end{cases}$$

such that the constant $c_{d+1}$ satisfies the condition

$$\int_{\mathbb{R}^{d+1}} J_{d+1}(u) du = 1.$$

and write for $(t, x) \in [0, \infty) \times \mathbb{R}^d, n = 1, 2, \cdots$

$$\sigma^n(t, x) = \int_{\mathbb{R}^{d+1}} \sigma(t - n^{-1}\bar{t}, x - n^{-1}\bar{x}) J(\bar{t}, \bar{x}) d\bar{t} d\bar{x}$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^1} \sigma(t - n^{-1}\bar{t}, x - n^{-1}\bar{x}) J(\bar{t}, \bar{x}) d\bar{t} d\bar{x}$$

where we define $\sigma(t, x) = 0$, for $t < 0$. $\sigma^n$ are usually called the smoothness functions of $\sigma$ on $[0, \infty) \times \mathbb{R}^d$. For these $\sigma^n, n = 1, 2, \cdots$ let us show that they satisfy (1) – (4). In fact, $\forall \mu \in \mathbb{R}^d$

$$|\sigma^n(t, x)| \leq \int_{\mathbb{R}^{d+1}} \sigma(t - n^{-1}\bar{t}, x - n^{-1}\bar{x}) J(\bar{t}, \bar{x}) d\bar{t} d\bar{x} \leq k_0,$$

$$\langle \sigma^n \mu, \mu \rangle = \int_{\mathbb{R}^{d+1}} \langle \sigma(t - n^{-1}\bar{t}, x - n^{-1}\bar{x}) \mu, \mu \rangle J(\bar{t}, \bar{x}) d\bar{t} d\bar{x} \leq k_0 \geq |\mu|^2 \delta_0,$$

Moreover, because the $\sigma^n(t, x)$ are the smoothness functions of $\sigma(t, x)$, so $\sigma^n(t, x) \to \sigma(t, x)$, a.s. [7],[28]. Hence for any $q > 0$, for each $T < \infty$ and $N = 1, 2, \cdots$, as $n \to \infty$,

$$\|\sigma^n - \sigma\|_{q,[0.T]\times[-N,N]^{\otimes d}} \to 0. \tag{3.16}$$

Furthermore, one easily sees that for each $n = 1, 2, \cdots$ as $x, x' \in \mathbb{R}^d$

$$|\sigma^n(t,x)-\sigma^n(t,x')|\leq k_0k_0|x-x'|.$$

Thus $\sigma^n, n=1,2,\cdots$, satisfy $(1)-(4)$. In the same way one can construct $b^n(t,x), n=1,2,\cdots$ , such that they satisfy $(1),(3)$ and $(4)$. However, for the smoothness of $c$, to meet our purpose we need more discussion. First we take a sequence en $\epsilon_n\downarrow 0$. Set $c_{\epsilon_n}=c1\!\!1_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}}(z)$, and

$$c^m(t,x,z)=1\!\!1_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}}(z)\cdot\tilde{c}^m(t,x,z),$$

where

$$\tilde{c}^m(t,x,z)=\int_{R^1\times\mathbb{R}^d\times Z}c_{\epsilon_n}(t-m^{-1}\bar{t},x-m^{-1}\bar{x},z-m^{-1}\bar{z})J(\bar{t},\bar{x},\bar{z})d\bar{t}d\bar{x}d\bar{z},$$

where we define $c(t,x,z)=0$, as $t<0$. That is, $\tilde{c}^m(t,x,z)$ is the smoothness function of $c(t,x,z)$ on $A_n=[0,\infty)\times\mathbb{R}^d\times\{\epsilon_n<|z|<\epsilon_n^{-1}\}$. Then

$$\int_Z|c^m|^2\pi(dz)\leq\int_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}}\int_{\mathbb{R}^d}\int_{R^1}\int_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}}$$

$$\frac{|c(t-m^{-1}\bar{t},x-m^{-1}\bar{x},z-m^{-1}\bar{z})|^2}{|z-m_n^{-1}\bar{z}|^{d+1}}dz\frac{|z-m^{-1}\bar{z}|^{d+1}}{|z|^{d+1}}J(\bar{t},\bar{x},\bar{z})d\bar{t}d\bar{x}d\bar{z}$$

$$\leq k_0\int_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}}\int_{\mathbb{R}^d}\int_{R^1}2J(\bar{t},\bar{x},\bar{z})d\bar{t}d\bar{x}d\bar{z}\leq 2k_0,$$

where we have used the fact that for $\epsilon_n\leq|\bar{z}|\leq\epsilon_n^{-1}$, and $\epsilon_n\leq|z|\leq\epsilon_n^{-1}$

$$\frac{|z-m^{-1}\bar{z}|^{d+1}}{|z|^{d+1}}\leq(1+\left|\frac{\bar{z}/m}{z}\right|)^{d+1}\leq 2,$$

if we take $m>\frac{1}{\epsilon_0}\epsilon_n^{-2}$, and $\epsilon_0>0$ is a constant such that $(1+\epsilon_0)_{d+1}\leq 2$. Thus we have proved that $\int_Z|c^m|^2\pi(dz)\leq 2k_0$, as $m>\frac{1}{\epsilon_0}\epsilon_n^{-2}$. Now for each $\epsilon_n$ by assumption

$$k_0\geq\int_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}}|c|^2\frac{dz}{|z|^{d+1}}\geq\int_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}}|c|^2\epsilon_n^{d+1}dz.$$

So for each $\epsilon_n, S_N$ and $T < \infty \int_{[0,T]\times S_N\times\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}} |c|^2 dtdxdz < \infty$. Thus by the property of the smoothness functions as $m \to \infty$,

$$\int_{[0,T]\times S_N\times\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}} |c-c^m|^2 dtdxdz$$

$$= \int_{[0,T]\times S_N\times\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}} |c-\tilde{c}^m|^2 dtdxdz \to 0.$$

Hence as $m \to \infty$,

$$\int_{[0,T]\times S_N\times\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}} |c-c^m|^2 \frac{dtdxdz}{|z|^{d+1}}$$

$$\leq \int_{[0,T]\times S_N\times\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}} |c-c^m|^2 \frac{dtdxdz}{\epsilon_n^{d+1}} \to 0,$$

for each fixed $n, N$ and $T < \infty$. This deduces that as $m \to \infty$,

$$\int_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}} |c-c^m|^2 \frac{dz}{|z|^{d+1}} \to 0, \text{ a.e.}(t,x) \in [0,T]\times S_N.$$

(Otherwise, a contradiction is easily derived). Now applying Lebesgue's dominated convergence theorem one finds that for any $q \geq 1$ as $m \to \infty$,

$$\| \int_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}} |c-c^m|^2 \frac{dz}{|z|^{d+1}} \|_{L^q([0,T],\times S_N)} \to 0, \text{for each} n, N \text{ and } T.$$

From this for each $n$ one easily choose a $m_n$ such that $m_n > \frac{1}{\epsilon_0}\epsilon_n^{-2}$ and

$$\| \int_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}} |c-c^{m_n}|^2 \frac{dz}{|z|^{d+1}} \|_{L^q([0,T],\times S_N)} < \frac{1}{2^n}.$$

For simplicity write $c^n$ for $c^{m_n}$. Since $c^n$ is smooth in $t$ and $x$, so

$$\int_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}} |c^n(t,x_1,z)-c^n(t,x_2,z)|^2 \frac{dz}{|z|^{d+1}}$$

$$\leq \int_{\{\epsilon_n\leq|z|\leq\epsilon_n^{-1}\}} k_n[|x_1-x_2|^2+|t_1-t_2|^2] \frac{dz}{|z|^{d+1}}$$

$$\leq k_n'[|x_1 - x_2|^2 + |t_1 - t_2|^2].$$

Finally, as $n \to \infty$,

$$\| \int_Z |c^n - c|^2 \pi(dz)\|_{q,[0,T]\times S_N} \leq \| \int_{\{0<|z|<\epsilon_n^{-1}\}\cup\{\epsilon_n^{-1}<|z|\}} |c|^2 \pi(dz)\|_{q,[0,T]\times S^N}$$

$$+\| \int_{\{\epsilon_n \leq |z| \leq \epsilon_n^{-1}\}} |c^n - c|^2 \pi(dz)\|_{q,[0,T]\times S_N} \to 0.$$

The Proof is complete. $\square$

Now let us prove Theorem 3.5.1

*Proof.* For $b^n, \sigma^n$ and $c^n$, which are constructed in the previous lemma, for each $n = 1, 2, \cdots$ there exists a unique strong solution $(x_t^n)$ of the following SDE, $t \geq 0$:

$$x_t^n = x_0 + \int_0^t b^n(s, x_s^n)ds + \int_0^t \sigma^n(s, x_s^n)dw_s$$

$$+ \int_0^t \int_Z c^n(s, x_{s-}^n, z)\widetilde{N}_k(ds, dz). \tag{3.17}$$

Now applying Lemma 3.2.1 "the result of SDE from the Skorokhod weak convergence technic" holds. So we only needs to show 3.10 Remark 3.2.1 holds. As in the proof of Theorem 3.3.1 we may assume that $|\tilde{x}_t^n| \leq k_0, \forall t \in [0,T], \forall n = 0, 1, 2, \cdots$. Notice that for any given $\epsilon > 0$

$$\begin{aligned}
\widetilde{\mathbb{P}}(\left|\int_0^t (b^n(s, \tilde{x}_s^n) - b(s, \tilde{x}_s^0))ds\right| > \epsilon) \quad &\leq \quad \frac{\epsilon}{3}\mathbb{E}^{\mathbb{P}}[\int_0^t |(b^n - b^{n_0})(s, \tilde{x}_s^n)|1\!\!1_{|\tilde{x}_s^n|\leq k_0} ds \\
&+ \quad \widetilde{\mathbb{P}}(\left|\int_0^t (b^{n_0}(s, \tilde{x}_s^n) - b^{n_0}(s, \tilde{x}_s^0))ds\right| > \frac{\epsilon}{3}) \\
&+ \quad \frac{\epsilon}{3}\mathbb{E}^{\mathbb{P}}\int_0^t |(b^{n_0} - b)(s, \tilde{x}_s^0)|1\!\!1_{|\tilde{x}_s^0|\leq k_0} ds] \\
&= \quad I_1^{n,n_0} + I_2^{n_0,n} + I_3^{n_0}.
\end{aligned}$$

Obviously, by (4) in Lemma 3.5.1 and by the Krylov type estimate there exists a $\widetilde{N}$ such that as $n \geq \widetilde{N}, n_0 \geq N$,

$$I_1^{n,n_0} + I_3^{n_0} \le 2 \cdot \frac{\epsilon}{3} \|b^n - b^{n_0}\|_{d+1,[0,T]\times S_{k_0}} < 2\bar{\epsilon}/4.$$

Now for each $n_0 \ge \widetilde{N}$, by 3.9 as $n \to \infty, \forall t \in [0,T]$.

$$I_2^{n_0,n} \to 0$$

Thus the first limit in 3.10 is proved. Now notice that for each $n^0 = 1, 2, \cdots$

$$\begin{aligned}
&\widetilde{\mathbb{P}}(\Big|\int_0^t \int_Z c^n(s,\tilde{x}^n_{s-},z)\tilde{q}^n(ds,dz) - \int_0^t \int_Z c(s,\tilde{x}^0_{s-},z)\tilde{q}^0(ds,dz)\Big| > \epsilon) \\
\le\ & \left(\frac{3}{\epsilon}\right)^2 \mathbb{E}\int_0^t \int_Z \Big|(c^n - c^{n^0})(s,\tilde{x}^n_s,z)\Big|^2 1\!\!1_{|\tilde{x}^n_s|\le k_0}\pi(dz)ds \\
+\ & \widetilde{\mathbb{P}}(|\int_0^t \int_Z c^{n^0}(s,\tilde{x}^n_{s-},z)\tilde{q}^n(ds,dz) \\
-\ & \int_0^t \int_Z c^{n^0}(s,\tilde{x}^0_{s-},z)\tilde{q}^0(ds,dz)| 1\!\!1_{\sup_{t\in[0,T]}|\tilde{x}^n_t| + \sup_{t\in[0,T]}|\tilde{x}^0_t| \le 2k_0} > \frac{\epsilon}{3}) \\
+\ & \left(\frac{3}{\epsilon}\right)^2 \mathbb{E}\int_0^t \int_Z \Big|(c^{n^0} - c)(s,\tilde{x}^0_s,z)\Big|^2 1\!\!1_{|\tilde{x}^0_s|\le k_0}\pi(dz)ds \\
=\ & I_2^{n,n^0} + I_3^{n^0,n} + I_4^{n^0}.
\end{aligned}$$

For an arbitrary given $\bar{\epsilon} > 0$, as above (by using the Krylov estimate) one can show that there exist a large enough $\widetilde{N}$ such that for any fixed $n^0 \ge \widetilde{N}$, as $n \ge \widetilde{N}$

$$I_2^{n,n^0} + I_4^{n^0} < \frac{3}{4}\bar{\epsilon}.$$

On the other hand, one also finds that as $n \to \infty, \forall n^0$

$$I_3^{n^0,n} \to 0. \tag{3.18}$$

$$
\begin{aligned}
I_3^{n^0,n} &\leq \widetilde{\mathbb{P}}\left(\left|\int_0^t\int_Z (c^{n^0}(s,\tilde{x}_{s-}^n,z)-c^{n^0}(s,\tilde{x}_{s-}^0,z))1_{|\tilde{x}_s^n|\leq k_0,|\tilde{x}_s^0|\leq k_0}\tilde{q}^n(ds,dz)\right|>\frac{\epsilon}{6}\right)\\
&+ \widetilde{\mathbb{P}}\left(\left|\int_0^t\int_Z c^{n^0}(s,\tilde{x}_{s-}^n,z)\tilde{q}^n(ds,dz)-\int_0^t\int_Z c^{n^0}(s,\tilde{x}_{s-}^0,z)\tilde{q}^0(ds,dz)\right|>\frac{\epsilon}{6}\right)\\
&\leq \left(\frac{6}{\epsilon}\right)^2\mathbb{E}\int_0^t\int_Z\left|c^{n^0}(s,\tilde{x}_s^n,z)-c^{n^0}(s,\tilde{x}_s^0,z)\right|^2 1_{|\tilde{x}_s^n|\leq k_0,|\tilde{x}_s^0|\leq k_0}\pi(dz)ds\\
&+ \left(\frac{12}{\epsilon}\right)^2\mathbb{E}\int_0^t\int_{\{|z|\leq\delta\}\cup\{|z|\leq\delta^{-1}\}}\left|c^{n^0}(s,\tilde{x}_s^0,z)\right|^2\pi(dz)ds\\
&+ \widetilde{\mathbb{P}}\left(\left|\int_0^t\int_{\delta<|z|<\delta^{-1}}1_{|\tilde{x}_s^0|\leq k_0}c^{n^0}(s,\tilde{x}_{s-}^0,z)\tilde{p}^n(ds,dz)\right.\right.\\
&- \left.\left.\int_0^t\int_{\delta<|z|<\delta^{-1}}1_{|\tilde{x}_s^0|\leq k_0}c^{n^0}(s,\tilde{x}_{s-}^0,z)\tilde{p}^0(ds,dz)\right|>\frac{\epsilon}{12}\right)\\
&= \sum_{i=1}^3 I_{3i}^{n^0,n}.
\end{aligned}
$$

Notice that $\forall s\geq 0, \tilde{x}_s^n\to\tilde{x}_s^0$, in probability, as $n\to\infty$, so applying Lebesgue's dominated convergence theorem,

$$I_{31}^{n^0,n}\leq k_{n^0}\mathbb{E}\int_0^t|\tilde{x}_s^n-\tilde{x}_s^0|^2 1_{|\tilde{x}_s^n|\leq k_0,|\tilde{x}_s^0|\leq k_0}ds\to 0,\text{ as } n\to\infty.$$

Now by $\mathbb{E}\int_0^t\int_z|c^{n^0}(s,\tilde{x}_s^0,z)|^2\pi(dZ)ds\leq k_0t<\infty$ for any $\bar{\epsilon}>0$ one can choose a small enough $\delta>0$ such that

$$I_{32}^{n^0,n}<\bar{\epsilon}/3.$$

Let us show that for any $\delta>0$ and $n^0$, as $n\to\infty$,

$$I_{33}^{n^0,n}\to 0. \tag{3.19}$$

To show this we make a division: $0=s_0<s_1<\cdots<s_{m+1}=t$. Then

$$\begin{aligned}
I_{33}^{n^0,n,N} &\leq \widetilde{\mathbb{P}}(|\int_0^t \int_{\delta<|z|<\delta^{-1}} 1_{|\tilde{x}_s^0|\leq k_0} c^{n^0}(s,\tilde{x}_{s-}^0,z)\tilde{p}^n(ds,dz) \\
&- \sum_{i=0}^k \int_{s_i}^{s_{i+1}} \int_{\delta<|z|<\delta^{-1}} 1_{|\tilde{x}_s^0|\leq k_0} c^{n^0}(s_i,\tilde{x}_{s_i-}^0,z)\tilde{p}^n(ds,dz)| > \frac{\epsilon}{12}) \\
&+ \widetilde{\mathbb{P}}(|\sum_{i=0}^k \int_{s_i}^{s_{i+1}} \int_{\delta<|z|<\delta^{-1}} 1_{|\tilde{x}_s^0|\leq k_0} c^{n^0}(s_i,\tilde{x}_{s_i-}^0,z)\tilde{p}^n(ds,dz) \\
&- \sum_{i=0}^k \int_{s_i}^{s_{i+1}} \int_{\delta<|z|<\delta^{-1}} 1_{|\tilde{x}_s^0|\leq k_0} c^{n^0}(s_i,\tilde{x}_{s_i-}^0,z)\tilde{p}^0(ds,dz)| > \frac{\epsilon}{12} \\
&+ \widetilde{\mathbb{P}}(|\int_0^t \int_{\delta<|z|<\delta^{-1}} 1_{|\tilde{x}_s^0|\leq k_0} c^{n^0}(s_i,\tilde{x}_{s_i-}^0,z)\tilde{p}^0(ds,dz) \\
&- \sum_{i=0}^k \int_{s_i}^{s_{i+1}} \int_{\delta<|z|<\delta^{-1}} 1_{|\tilde{x}_s^0|\leq k_0} c^{n^0}(s_i,\tilde{x}_{s_i-}^0,z)\tilde{p}^0(ds,dz)| > \frac{\epsilon}{12} \\
&= I_{331}^{n^0,n} + I_{332}^{n^0,n} + I_{333}^{n^0,n}.
\end{aligned}$$

Because $c^{n^0}$ is a smooth function satisfying the condition that as $\epsilon_{n^0} \leq \delta$

$$\int_{\delta<|z|<\delta^{-1}} |c^{n^0}(s,x,z) - c^{n^0}(s',x',z)|^2 \pi(dz)$$

$$\leq k_{n_0}\tilde{k}_0[|x-x'|^2 + |s-s'|], \text{ and } |c^{n_0}(s,x,z)| \leq \tilde{k}_{n_0,\delta,k_0},$$

$$\text{as } (s,x,z) \in [0,T] \times \{|x| \leq k_0\} \times \{\delta < |z| < \delta^{-1}\},$$

where $k_{n_0,\delta} > 0$ is a constant only depending on $n_0$ and $\delta$; $\tilde{k}_{n_0,\delta,k_0} > 0$ is a constant only depending on $n_0, \delta$ and $k_0$; and $\tilde{x}_s^0$ is right continuous such that $s \downarrow s_i \Rightarrow x_s \to s_i$. So by using Lebesgue's dominated convergence theorem, one finds that as $\lambda = \max_{i=0,\cdots,m}(s_{i+1} - s_i) \to 0$

$$I_{331}^{n^0,n}, I_{33}^{n^0,n} \leq \left(\frac{12}{\epsilon}\right)^2 \mathbb{E}\int_0^t \int_{\delta<|z|<\delta^{-1}} |1_{|\tilde{x}_s^0|\leq k_0} c^{n^0}(s,\tilde{x}_{s-}^0,z)$$

$$-\sum_{i=0}^k 1_{|\tilde{x}_s^0|\leq k_0} c^{n^0}(s_i,\tilde{x}_{s_i}^0,z) 1_{(s_{i+1},s_i]}(s)|^2 \pi(dz)ds \to 0$$

For any given division,

$$\lim_{n\to\infty} I_{332}^{n^0,n} = 0.$$

Therefore, 3.19 holds. Thus 3.18 is proved, and the third limit in 3.10 is also established. Finally, the second limit in 3.10 can be similarly proved. In fact, for arbitrary $\epsilon > 0$

$$\mathbb{P}(|\int_0^t \sigma^n(s,\tilde{x}_s^n)d\widetilde{w}_s^n - \int_0^t \sigma(s,\tilde{x}_s^0)d\widetilde{w}_s^0| > \epsilon)$$

$$\leq \left(\frac{3}{\epsilon}\right)^2 \mathbb{E}\int_0^t |(\sigma^n - \sigma^{n^0})(s,\tilde{x}_s^n)|^2 1\!\!1_{|\tilde{x}_s^n|\leq k_0} ds$$

$$+\widetilde{\mathbb{P}}(|\int_0^t \sigma^{n^0}(s,\tilde{x}_s^n)d\widetilde{w}_s^n - \int_0^t \sigma^{n^0}(s,\tilde{x}_s^0)d\widetilde{w}_s^0|$$

$$\times 1\!\!1 \sup_{t\in[0,T]} |\tilde{x}_t^n| \leq 2k_0 + \sup_{t\in[0,T]} |\tilde{x}_t^0| \leq 2k_0 > \frac{\epsilon}{3})$$

$$+\left(\frac{3}{\epsilon}\right)^2 \mathbb{E}\int_0^t \int_Z |(\sigma^{n^0} - \sigma)(s,\tilde{x}_s^0)|^2 I_{|\tilde{x}_s^0|\leq k_0} ds$$

$$= I_2^{n,n^0} + I_3^{n,n^0} + I_4^{n^0}.$$

Now the proof can be completed. So the second limit in 3.10 is established. Thus we have proved that $\{\tilde{x}_t^0\}_{t\geq 0}$ satisfies the following SDE on probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{\mathbb{P}})$ for any $T < \infty$ as $t \in [0,T]$

$$\tilde{x}_t^0 = x_0 + \int_0^t b(s,\tilde{x}_s^0)ds + \int_0^t \sigma(s,\tilde{x}_s^0)d\widetilde{w}_s^0 + \int_0^t \int_Z c(s,\tilde{x}_{s-}^0,z)\tilde{q}^0(ds,dz),$$

where $w_t^0$ and $\tilde{q}^0(dt,dz)$ are a BM and a Poisson martingale with the compensator $\pi(dz)dt$, respectively. □

By using Theorem 3.5.1 and the Girsanov type theorem we can obtain the existence of a weak solution to a BSDE with jumps under much weaker conditions.

**Theorem 3.5.2.** *Assume that $b, \sigma$ and $c$ are Borel measurable functions such that*

1. *$Z = \mathbb{R}^d - \{0\}$, and $\pi(dz) = dz/|z|^{d+1}$;*

2. $|\sigma(t,x)| + \int_Z |c(t,x,z)|^2 \pi(dz) \le k_0,$
   *where* $k_0 > 0$ *is a constant;*

3. *there exists a* $\delta_0 > 0$ *such that for all* $\mu \in \mathbb{R}^d$,
$$\langle \sigma(t,x)\mu, \mu\rangle \ge |\mu|^2 \delta_0.$$

4. $\langle x, b(t,x)\rangle \le c_1(t)(1 + |x|^2 \Pi_{k=1}^m g_k(x)),.$
   *Furthermore,* $b(t,x)$ *is locally bounded for* $x$*, that is, for each* $r > 0$*, as* $|x| \le r, |b(t,x)| < k_r$*,*
   *where* $k_r > 0$ *is a constant only depending on* $r$*.*
   *Then there exists a weak solution for 3.15.*

*Proof.* In fact, by Theorem 3.5.1 there exists a weak solution for the following SDE with jumps: $\forall t \ge 0$,

$$x_t = x_0 + \int_0^t \sigma(s, x_s) dw_s + \int_0^t \int_Z c(s, x_{s-}, z) \widetilde{N}_k(ds, dz).$$

Notice that by Skorokhod theorem we know that the above SDE holds in a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, where $\Omega = [0,1], \mathfrak{F} = \mathfrak{B}([0,1]), \mathbb{P} =$ Lebesgue measure on $[0,1]$. Since such $(\Omega, \mathfrak{F})$ is a standard measurable space. So the conclusion follows on $\forall t \ge 0$.

□

In the above theorem we assume that $\sigma$ is bounded. Now we relax the coefficient $\sigma$ to be less than linear growth, (so, it can be unbounded). In this case we have to assume that $\sigma$ and $c$ are jointly continuous.

**Theorem 3.5.3.** *Assume that conditions* $(1), (3)$ *and* $(4)$ *in the previous theorem hold, and assume that*

1. $\int_Z |c(t,x,z)|^2 \pi(dz) \le c_1(t),$

2. $\sigma(t,x)$ *is jointly continuous in* $(t,x)$*; and*

$$\lim_{h,h'\to 0}\int_Z |c(t+h', x+h, z) - c(t,x,z)|^2\pi(dz) = 0;$$

*3. there exists a* $\delta_0 > 0$ *such that* $\sigma|(t,x)| \geq \delta_0$*, and*

$$|\sigma(t,x)|^2 \leq c_1(t)(1+|x|^2)$$

*Then for any given constant* $x_0 \in \mathbb{R}^d$ *3.15 has a weak solution on* $t \geq 0$*.*

Now let us prove Theorem 3.5.3.

*Proof.* The proof can be completed by applying Theorem 3.3.1 and the Girsanov type theorem. Since it is completely similar to the proof of the previous theorem. We do not repeate it.

□

Finally, applying the above results and applying the Yamada-Watanabe type theorem we immediately obtain the following theorems on the existence of a pathwise unique strong solution to SDE 3.15.

**Theorem 3.5.4.** *[70] Under the assumption of Theorem 3.5.3 if, in addition, the (PWU1) condition in Theorem 3.4.1 holds, then 3.15 has a pathwise unique strong solution.*

# Conclusion

In many cases we need to minimize some target functional subject to a controlled dynamical system; for example, to minimize the energy expended by the controlled system during a period of time, like, minimizing $\mathbb{E}\int_0^T |x_t^u|^2 dt$, where $u(\cdot)$ is a control, $x_t^u$ is the solution of the system corresponding to the applied control $u(\cdot)$. We will find that the minimal value of the target functional will be obtained when we can apply some extreme solution of the dynamic system. For this example the idea is that at each time when the trajectory of the state process leaves the point 0, we should immediately use a feedback control to fully pull back the trajectory directed towards 0, because if the state $x_t^u$ is closer to 0, then the energy $|x_t^u|^2$ expended is also closer to zero and so it is smaller, even though it cannot be 0. Such an extreme feedback control is called a Bang-Bang control. Obviously, such a feedback control is not Lipschitz continuous, and so it also makes the coefficients of the system non-Lipschitzian, for example, when the system is linear with respect to the control $u(\cdot)$ : the system coefficient is $A(t)x_t + B(t)u_t$. However, we need the state of the system, that is, the solution, to exist for such a control, so the system can be controlled. Therefore, discussing solutions for stochastic differential equations (SDEs) with jumps and with non-Lipschitzian coefficients, is necessary and useful from the practical point of view. The interesting thing is also that in the ordinary differential equation (ODE) case, if its coefficients are only continuous then a solution, even when it exists, is not necessary unique. However, in the SDE case we can have a unique solution even when the coefficients are not continuous. This means that a stochastic perturbation can some- times improve the nice properties of the solution. The stochastic integral term is very important in the financial market. Actually, its coefficient corresponds to a part of a portfolio of investment of the stocks by an investor in the financial market. In the optimal consumption problem the SDEs with non-Lipschitzian coefficients also need to be considered.

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