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On the study of some class of self-similar stochastic processes

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وَقُلِ ٱعْمَلُوا۟ فَسَيَرَى ٱللَّهُ عَمَلَكُمْ وَرَسُولُهُۥ وَٱلْمُؤْمِنُونَ ۖ وَسَتُرَدُّونَ إِلَىٰ
عَٰلِمِ ٱلْغَيْبِ وَٱلشَّهَٰدَةِ فَيُنَبِّئُكُم بِمَا كُنتُمْ تَعْمَلُونَ ١٠٥

And say, "Do (as you will), for Allah will see your deeds, and (so, will) His Messenger and the believers. And you will be returned to the Knower of the unseen and the witnessed, and He will inform you of what you used to do."

"***Pure mathematics is, in its way, the poetry of logical ideas.***"[Albert Einstein]

"***Mathematics is the language of nature***." [Sunny Singh]

Dedication

*All praise to **Allah**, today we fold the day's tiredness and the errand summing up between the cover of this humble work.*

I dedicate my work to:

*The utmost knowledge lighthouse, to our greatest and most honored prophet **Mohammed-peace and grace from Allah be upon him**.*

*The one that in feebleness upon feebleness did carried me, and taught me the love of science, to the which who taught me life, to the one that helped me by her prayers, to the Light of my eyes, and the sun of my spirit, my dear precious mother, **Khayera**. God save her.*

*The one how save me what he deprived of it, and he taught me the basics of patience, and was in dark, the light for me and he guide me along the way of my study, to my father, **Kaddour**. God save him.*

My grandmother, God rest her soul.

*The one who was the affectionate chest and was with me in distress and prosperity, my sister **Amoura**.*

*Whose love flows in my veins, and my heart always remembers them, to my brothers **Mohammad**, **Mahmoud**.*

*Whom I have truly tasted the taste of life my sisters **Aicha**, **karima**, **Souhila**.*

My uncles and aunts.

To Kaddouri and Mammeri family.

*To met them in distress and prosperity in joy tropical sores my companions: **Assmaa**, **Souad**, **Naima**, **Asma**.*

All those if my pen forget them, my heart will not forgotten them.

All those who are looking glory and pride in Islam and nothing else .

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Contents

Abstract

This work provides an important step in the construction, definition and the study of a class of H-sssi stochastic processes (self-similar with stationary increments), which have marginal probability density function that evolves in time according to a differential equation of fractional type. This construction is based on the theory of finite measures on functional spaces.

First, we brought the reader through the fundamental notions of stochastic processes and stochastic integration as well. In particular, within the study of $H-$sssi processes. Then, we focused on fractional Brownian motion (fBm), and introduced the theory of fractional integrals and derivatives, which indeed turns out to be very appropriate for studying and modeling systems with long-memory properties. We introduced and studied the generalized grey Brownian motion (ggBm), which is actually a parametric class of $H-$sssi processes. The ggBm has been defined canonically in the so called grey noise space. However, we have been able to provide a characterization notwithstanding the underline probability space. We also pointed out that the generalized grey Brownian motion is a direct generalization of a Gaussian process and in particular it generalizes Brownain motion and fractional Brownain motion as well.

Key words: Self-similar processes, Brownian motion, Fractional Brownain motion, Fractional derivatives and integrals, Mittag-Leffler function, Grey noise, Grey Brownian motion.

Résumé

Ce travail constitue une étape importante dans la construction, la définition et l'étude d'une classe des processus stochastiques H-sssi (auto-similaires à accroissements stationnaires), qui ont une densité de probabilité marginale qui évolue dans le temps selon une équation différentielle fractionnaire. Cette construction est basée sur la théorie des mesures finies sur les espaces fonctionnels.

D'abord, nous avons donné au lecteur les notions fondamentales des processus stochastiques, l'intégration stochastiques. En particulier, dans l'étude de H- sssi processus. Ensuite, nous nous sommes concentrés sur le mouvement Brownien fractionnaire (fBm), et nous avons introduit ensuite la théorie des intégrales et des dérivées fractionnaires, ce qui s'avère très approprié pour étudier et modéliser des systèmes avec la propriété de la longue mémoire. On a introduit et étudié le mouvement brownien gris généralisé (ggBm), qui est en fait une classe des processus H-sssi. Le ggBm a été défini canoniquement dans l'espace de bruit gris. Cependant, nous avons été en mesure de donner la caractérisation de ce processus sur un espace de probabilité. Nous avons également souligné que le mouvement Brownien gris généralisé est une généralisation directe d'un processus Gaussien et en particulier il généralise le mouvement Brownien et le mouvement Brownien fractionnaire.

Mots clés: Processus auto-similaires, mouvement Brownien, mouvement Brownien fractionnaire, dérivées et intégrales fractionnaires, fonction Mittag-Leffler, bruit gris, mouvement Brownien gris.

الملخص

يعتبر هذا العمل خطوة هامة في بناء، تعريف ودراسة فئة من العمليات العشوائية (ذات تشابه ذاتي مع الزيادات الثابتة H-SSSI)، والتي لها دالة الكثافة الاحتمالية الهامشية تتغير بمرور الوقت وفقا لمعادلة تفاضلية ذات رتب كسرية من نوع كابيتو. يعتمد هذا البناء على نظرية قياسات المحدودة في الفضاءات الوظيفية.
أولا، أعطينا القارئ المفاهيم الأساسية للعمليات العشوائية، التكامل العشوائي على وجه الخصوص، ثم ركزنا على الحركة البراونية ذات الرتب الكسرية، ولقد قدمنا بعدها ملخصا وجيزا لنظرية التكاملات الجزئية ومشتقاته، والتي تبين انها مناسبة جدا لنمذجة ولدراسة الانظمة ذات خاصية الذاكرة الطويلة. لنخلص في الأخير إلى تقديم ودراسة الحركة البراونية الرمادية المعممة (ggBm)، وهو في الواقع فئة من العمليات العشوائية ذات التشابه الذاتي مع زيادات ثابتة. والتي لها تعريف قانوني في ما يسمى فضاء الضوضاء الرمادية. ومع ذلك، تمكنا من تقديم تعريف هذه الاخيرة على فضاء احتمالي. كما أشرنا إلى أن الحركة البراونية الرمادية المعممة هي تعميم مباشرلعمليات غوس وعلى وجه الخصوص انه تعميم الحركة البراونية العادية و الحركة البراونية ذات الرتب الكسرية.

الكلمات المفتاحية: العمليات المتشابهة ذاتيا، الحركة البراونية العادية، الحركة البراونية ذات الرتب الكسرية، المشتقات والتكاملات ذات الرتب الكسرية، دالة ميتاج- لفلر، ضوضاء رمادية، الحركة البراونية الرمادية.

List Of Notations And Symbols & Acronyms

List Of Notations And Symbols

- H: Hurst parameter.
- E_α : M-L function.
- $E_{\alpha,\beta}$: Generalized M-L function.
- $\Gamma(\alpha)$: Gamma function.
- $\mathcal{I}_{a+}^{\alpha}\phi$: Left side R-L fractional integral.
- $\mathcal{I}_{b-}^{\alpha}\phi$: Right side R-L fractional integral.
- $\mathcal{D}_{a+}^{\alpha}\phi$: Left side R-L fractional derivative.
- $\mathcal{D}_{b-}^{\alpha}\phi$: Right side R-L fractional derivative.
- $*\mathcal{D}_{t}^{\alpha}\phi$: Caputo fractional derivative.

Acronyms

- M-L: Means "Mettag -Leffler".
- R-L: Means "Riemann-Liouville".
- sBm: Standard Brownian motion.
- fBm: Fractional Brownian motion.
- fGn: Fractional Gaussian noise.
- gBm: Grey Brownian motion.
- ggBm: Generalized grey Brownian motion.
- sssi: Self-similar with stationary increments.
- LRD: Long Range Dependence.

Introduction

In everyday life as well as in the sciences, conceptual models are used to try to understand the world around us. One time-honored theme is the idea that nothing changes, that history repeats itself. More recently, as people learned about the rules governing galaxies and microscopic worlds, the idea emerged that size doesn't make much difference. Mathematical models are used along with the conceptual ones to understand and solve practical problems. Mathematically, these two conceptual models above are closely related to the concepts of stationarity and self-similarity, respectively.

In the last few decades, new developments in self-similarity have been obtained, including the appearance of new classes of (Gaussian or non-Gaussian) self-similar processes and new techniques to study their behavior, related to the stochastic calculus. Self-similar processes are stochastic processes that are invariant in distribution under suitable scaling of time and space. The first rigorous probabilistic study of self-similar processes is due to J. Lamperti [12] at the beginning of the sixties. These processes can be used to model many space-time scaling random phenomena and have been applied with some success in diverse areas such as physics, biology , hydrology (cf. Montanari (2003), [13]), geophysics, medicine, genetics (growth and genealogy of population), financial economics (option pricing , Willinger and al. (1999) , [34]) and more recently in modeling Internet traffic patterns (Leland and al. (1994), [34]), various areas of image processing, climatology, environmental science and other fields. Additional applications are given in Goldberger and West (1987), Stewart and al. (1993), Buldyrev and al. (1993), Ossandik and al. (1994), Percival and Guttorp (1994) and Peng and al. (1992,)[28, 20, 21].

Self-similar processes appear in various parts of probability theory, it is well known that Brownian motion is selfsimilar, some well known examples are: stable Lévy process, Feller branching diffusion, Bessel processes, Brownian sheet. Fractional Brownian motion is also a Gaussian selfsimilar process with stationary increments.

In the same line of thought, long-range dependence is related to the concept of self-similarity for a stochastic process in that the increments of a self-similar process with stationary increments exhibit long-range dependence under some conditions. The probability theory of self-similarity and long-range dependence is discussed in [23]. In recent years fractional Brownian motion and processes related to fractional dynamics have become more and more an object of study. The reason for this interest from the mathematical and applied science point of view is two fold: on one hand the processes in general lack both the Markov and the (semi-)martingale property, which make them mathematically challenging and not accessible by classic methods from stochastic analysis.

On the other hand, due to these properties, it is possible to model processes with long-range self interaction and memory effects with the help of fractional differential equations. By investigating the time-fractional equation, i.e. where the time derivative is a Caputo derivative of fractional order, Schneider introduced the notion of grey Brownian motion in [32]. This stochastic processes, is more general than the classical standard and fractional Brownian motions. The link between grey Brownian motion and fractional differential equations is also studied by Mura and Mainardi [20, 21] in the framework of fractional diffusion equations, where they could show that the marginal probability density function of grey Brownian motion solves the time-fractional heat equation, and they introduced the generalized grey Brownian motion (denoted by ggBm for short, in [20]) and from now denoted simply by gBm, which, by construction, is made up of self-similar processes with stationary increments. Grey Brownian motion is constructed on a probability space with a non-Gaussian measure, called Mittag-Lefler measure (or grey noise (reference) measure), whose characteristic function is given by the Mittag-Lefler function, see e.g. [2, 20, 21, 16].

The Present master thesis is devoted to the study of this class of $H-s$ssi stochastic processes. And it's divided into four chapters, it's organized as follows:

In **Chapter 1** we gather some preliminary results. In particular, we introduce the notion and the basic theory of stochastic processes. We give the notion of filtration and martingales. Then, we define stationary processes and self-similarity. After introduced the mathematical definition of Brownian motion and its properties, we focus on Markov processes and we study H-sssi processes. Then, we introduce the Itô integral and the so called Itô calculus.

Chapter 2 focuses on fractional Brownian motion. Fractional Brownian motion is the most well known self-similar process with stationary increments. It includes standard Brownian motion as a particular case. The applications of this process are now widely recognized. In this chapter, we first survey the definition of fBm and some of the basic properties of this process. Then, we study integral representation of fractional Brownian motion.

In **Chapter 3** we introduced the theory of fractional integrals and derivatives, which turns out to be very appropriate for studying and modeling systems which exhibit long-memory properties. We first introduce the notion of fractional integral/derivatives. Then, we study many of its properties. Finally, we see how fractional Brownian motion can be represented as a fractional integral of a Gaussian white noise, and this actually gives reason of its name.

The core of this master thesis is the **Chapter 4** in this chapter, we introduce an other related processes that have recently emerged in scientific research, is grey Brownian motion. we introduce the parametric class of generalized grey Brownian motion. This class includes non-Markovian stochastic models either for slow and fast-anomalous diffusion. After having presented and motivated the mathematical construction (Nuclear spaces, characteristic functionals, Minlos' theorem, canonical noises), in particular the development of the grey noise space, we show that this class is made up of H-sssi processes and contain either Gaussian and non-Gaussian processes, like fractional Brownian motion and grey Brownian motion. We study different characterizations of the ggBm notwithstanding the underline probability space.

Finally, a conclusion is given. In witch we summarize the main results of this master thesis, an appendix is also given.

Chapter 1

Background on stochastic calculus

In this chapter, the definitions and main properties of stochastic processes are discussed and some concepts are clarified, especially self-similarity and long-range-dependence. Clearly distinguishing these concepts will help the understanding of the models to be discussed in the next chapters. This chapter provides theoretical basis for the rest of this work. There already exists a vast literature that treats different aspects of stochastic process, we refer the reader to the most of them [5, 17, 24, 22, 6, 8, 10] for more detail.

1.1 Basics in stochastic processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable X is a rule for assigning to every outcome ω of an experiment Ω a number $X(\omega)$. A stochastic process X_t is a rule for assigning to every $\omega \in \Omega$ a function $X_t(\omega)$. Thus, a stochastic process could be seen as a family of time functions depending on the parameter ω (a collection of paths or trajectories) or, equivalently, a family of random variables depending on a time parameter t, or a function of t and ω as well.

1.1.1 Stochastic processes and filtration

Definition 1.1.1. *(**Stochastic process**) We define real valued* $(one - dimensional)$ ***stochastic process*** *a family of random variables* $\{X_t\}_{t \in I}$ *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$*:*

$$X_t : \Omega \longrightarrow \mathbb{R}, \quad t \in \mathrm{I} \subseteq \mathbb{R}_+.$$

A stochastic process could be a discrete time or a continuous time process, according to the set I *is countable or continuous.*

Definition 1.1.2. ***(Finite dimensional distributions)*** *For any natural number $k \in \mathbb{N}$ and a "time" sequence $\{t_i\}_{i=1,\dots,k} \in \mathrm{I}$, the finite-dimensional distributions of the real valued stochastic process $X_t = \{X_t\}_{t\in I}$ are the measures $\mu_{t_1,\dots,t_k}$, defined on $\mathbb{R}^k$, such that*

$$\mu_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = \mathbb{P}(\{X_{t_1} \in A_1, \cdots, X_{t_k} \in A_k\}), \tag{1.1}$$

where $\{A_1, \dots, A_k\}$ are Borel sets on $\mathbb{R}, (\mathcal{B})$.

Theorem 1.1.1. ***(Kolmogorov extension theorem,*** *[17]) For all $\{t_i\}_{i=1,\dots,k} \in \mathrm{I}$, $k \in \mathbb{N}$ let $\nu_{t_1,\dots,t_k}$ be probability measures on $\mathbb{R}^k$,such that :*

1. *for all permutations π on $\{1, 2, \dots, k\}$,*

$$\nu_{t_{\pi(1)},\dots,t_{\pi(k)}}(A_1 \times \dots \times A_k) = \nu_{t_1,\dots,t_k}(A_{\pi^{-1}(1)} \times \cdots \times A_{\pi^{-1}(k)})$$

2. *for any $m \in \mathbb{N}$,*

$$\nu_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = \nu_{t_1,\dots,t_k,t_{k+1},\dots,t_{k+m}}(A_1 \times \cdots \times A_k \times \mathbb{R} \times \cdots \times \mathbb{R}),$$

where of course the set on the right side as a total of $k + m$ factors. Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a real valued stochastic process X defined on it, such that:

$$\nu_{t_1,\dots,t_k}(A_1 \times \dots \times A_k) = \mathbb{P}(\{X_{t_1} \in A_1, \dots, X_{t_k} \in A_k\}),$$

for any $t_i \in \mathrm{I}$, $k \in \mathbb{N}$ and $A_i \in \mathcal{B}$.

Definition 1.1.3. ***(Filtration)*** *An increasing family $\mathcal{F}_t = \{\mathcal{F}_t\}_{t\in I}$ of complete sub σ-fields of $\mathcal{F}$ is said a **filtration** on $(\Omega, \mathcal{F}, \mathbb{P})$.*

Consider a stochastic process $X = \{X_t\}_{t\in I}$ and let:

$$\mathcal{F}_t^X = \sigma(\{X_s; 0 \le s \le t\}) = \sigma(\{\mathcal{N} \cup \{X_s^{-1}(H); 0 \le s \le t, H \in \mathcal{B}\}\}),$$

where $\mathcal{B}$ is the Borel σ-algebra and $\mathcal{N}$ indicates the class of null-sets. Clearly if $0 \le s \le t$ one has

$$\mathcal{F}_s^X \subseteq \mathcal{F}_t^X \subseteq \mathcal{F}.$$

*Therefore ,$\mathcal{F}^X = \{\mathcal{F}_t^X\}_{t\in I}$ defines a filtration, termed **natural filtration** of $\{X_t\}_{t\in I}$.*

Definition 1.1.4. *(**Adapted process**) A stochastic process $\{X_t\}_{t\in I}$ is said adapted to the filtration $\{\mathcal{F}_t\}_{t\in I}$ if for each $t \geq 0$:*

$$\mathcal{F}_t^X \subseteq \mathcal{F}_t.$$

In other words, for each t, the r.v. $X(t)$ is $\mathcal{F}_t$-measurable.

Definition 1.1.5. *(**Martingale**) A stochastic process $M = \{M_t\}_{t\geq 0}$ is a **martingale** with respect to the filtration $\mathcal{F}_t$ and the measure $\mathbb{P}$ if, for any $t \geq 0$, one has:*

1. *$M_t \in L^1(\Omega, \mathbb{P})$;*
2. *$\mathbb{E}(M_t|\mathcal{F}_s) = M_s$, for any $0 \leq s \leq t$.*

This means that M_t is $\mathcal{F}_t$-adapted. Moreover, the expected value of M_t does not depend on time. Indeed,

$$\mathbb{E}(M_t) = \mathbb{E}(\mathbb{E}(M_t|\mathcal{F}_0)) = \mathbb{E}(M_0).$$

Definition 1.1.6. *(**Gaussian process**) A real stochastic process $\{X_t\}_{t\in I}$ is Gaussian if and only if, for every finite sequence $\{t_1, t_2, ..., t_k\} \in I$,*

$$X_{t_1,...,t_k} = (X_{t_1}, \cdots, X_{t_k}),$$

has a multivariate normal distribution.

Definition 1.1.7. *(**Version**) Let $X = \{X_t\}_{t\in \mathrm{I}}$ be a stochastic process on the space $(\Omega, \mathcal{F}, \mathbb{P})$. A process $\widetilde{X}$, defined in the same probability space, is a version of the process X if, for any $t \in \mathrm{I}, \mathbb{P}(X_t = \widetilde{X}_t) = 1$.*

1.1.2 Stationary processes

In many stochastic processes that appear in applications their statistics remain invariant under time translations. Such stochastic processes are called stationary.

Definition 1.1.8. *(**Stationary process**) A stochastic process $\{X_t\}_{t\geq 0}$ is said a stationary process if any collection $\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$ has the same distribution of $\{X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_n+\tau}\}$ for each $\tau \geq 0$. That is,*

$$\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\} \stackrel{d}{=} \{X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_n+\tau}\}.$$

Let X be a stationary process, then the following elementary properties hold:

- Varying t, all the random variables X_t have the same low;i.e. $X_{t_1} \stackrel{d}{=} X_{t_2}$ for any t_1, $t_2 \geq 0$.
- All the moments, if they exist, are constant in time.
- The distribution of X_{t_1} and X_{t_2} depends only on the difference $\tau = t_1 - t_2$ (time lag).

Therefore, the autocovariance function $\gamma(t_1, t_2) = \gamma(t_1 - t_2)$ depends only on $\tau = t_1 - t_2$.We write

$$\gamma(\tau) = \mathbb{E}(X_t - \mu)(X_{t-\tau} - \mu) = Cov(X_t, X_{t-\tau}), \tag{1.2}$$

where $\mu = \mathbb{E}(X(t))$ and $\gamma(\tau)$ indicates the autocovariance coefficient at the lag τ.

Definition 1.1.9. ***(Stationary increment process)*** *A stochastic process $\{X_t\}_{t\geq 0}$ is said a stationary increment process, shortly **si**, if for any $h \geq 0$:*

$$\{X_{t+h} - X_h\}_{t\geq 0} \stackrel{d}{=} \{X_t - X_0\}_{t\geq 0}. \tag{1.3}$$

1.1.3 Self-similar processes

Self-similar(shortly **ss**) processes, introduced by Lamperti [12, 6], are the ones that are invariant under suitable translations of time and scale. In the last few years there has been an explosive growth in the study of self-similar processes.

Definition 1.1.10. ***(Self-similar processes)*** *A real valued stochastic process $X = \{X_t\}_{t\geq 0}$ is said self-similar with index $H \geq 0$, shortly **H-ss**, if for any $a \geq 0$:*

$$\{X_{at}\}_{t\geq 0} \stackrel{d}{=} \{a^H X_t\}_{t\geq 0}.$$

We observe that the transformation scales differently "space" and "time", for this reason one often prefers using the word **self-affine** process. The index H is said **Hurst's***exponent* or **scaling** exponent of the process.

Remark 1.1.1. *Observe that, if X is an **H-ss** process, then all the finite-dimensional distributions of X in $[0, \infty[$ are completely determined by the distribution in any finite real interval.*

Corollary 1.1.1. *For $H > 0$, a $H - ss$ process starts at 0 a.s.*

Proof: We have $\forall\ a$ that $X_0 = X_{a0} \overset{d}{\sim} a^H X_0$. Then letting $a \to 0$, we obtain the result. ■

Proposition 1.1.1. *Let $X = \{X_t\}_{t\geq 0}$ be a non-degenerate[1] stationary process, then X can not be an **H-ss** process.*

Proof: Indeed, for any $a \geq 0$:

$$X_t \overset{d}{=} X_{at} \overset{d}{=} a^H X_t,$$

by stationarity and self-similarity of the process X. Let $a \longrightarrow \infty$. Then the family of random variables on the right diverge with positive probability, whereas the family of random variable on the left is finite with probability one, leading to a contradiction. ■

Nevertheless, there is an important connection between self-similar and stationary processes.

Proposition 1.1.2. *Let $\{X_t\}_{t\geq 0}$ be an **H-ss** process; then the process*

$$Y(t) = e^{-tH} X(e^t), \quad t \in \mathbb{R} \tag{1.4}$$

is stationary. We have the converse, in the sense that if $(Y_t)_{t\in\mathbb{R}}$ is stationary, then

$$X_t = t^H Y(ln(t)), \quad t \geq 0 \tag{1.5}$$

is $H-ss$.

Proof: Let $\theta_1, ...\theta_d$ be real numbers. If $\{X(t), 0 < t < \infty\}$ is H-ss, then for any $t_1, ..., t_d \in \mathbb{R}^1$ and $h > 0$,

$$\begin{aligned}
\sum_{j=1}^{d} \theta_j Y(t_j + h) &= \sum_{j=1}^{d} \theta_j e^{-t_j H} e^{-hH} X(e^h e^{t_j}), \\
&\overset{d}{=} \sum_{j=1}^{d} \theta_j e^{-t_j H} X(e^{t_j}), \\
&= \sum_{j=1}^{d} \theta_j Y(t_j).
\end{aligned}$$

[1]A process $\{X_t\}_{t\geq 0}$ is said to be degenerate if for any $t \geq 0$, $X_t = 0$ almost surely.

proving that $\{Y(t), -\infty < t < \infty\}$ is stationary.

Conversely, if $\{Y(t), -\infty < t < \infty\}$ is stationary , then for $t_1, ..., t_d > 0$ and $a > 0$

$$\begin{aligned} \sum_{j=1}^{d} \theta_j X(at_j) &= \sum_{j=1}^{d} \theta_j a^H t_j^H Y(ln(a) + ln(t_j)), \\ &\stackrel{d}{=} \sum_{j=1}^{d} \theta_j a^H t_j^H Y(ln(t_j)), \\ &= \sum_{j=1}^{d} \theta_j a^H X(t_j). \end{aligned}$$

proving that $\{X(t), 0 < t < \infty\}$ is H-ss. ■

The transformation defined by (eq 1.4) is called the Lamperti transformation.

1.1.4 Brownian motion

The motivating example of a stochastic process is Brownian motion, the physical phenomenon of Brownian motion was discovered by Robert Brown, a 19th century scientist who observed through a microscope the random swarming motion of pollen grains in water, now understood to be due to molecular bombardment. The theory of Brownian motion was developed by Bachelier in his 1900 PhD Thesis, and independently by Einstein in his 1905 paper. For further history of Brownian motion and related processes we cite Meyer[17], Klebaner [10] and J. Pitman [24].

1.1.4.1 Definition of Brownian Motion

Definition 1.1.11. *(**Brownian motion**) A stochastic process $\{B(t) : t \geq 0\}$ is said to be a Brownian motion* [2] *with variance parameter $\sigma^2 > 0$ if:*

(i) $B(0) = 0$.

*(ii) (**Independent increments.**) For each $0 \leq t_1 < t_2 < \ldots < t_m$,*

$$B(t_1), B(t_2) - B(t_1), \ldots, B(t_m) - B(t_{m-1}),$$

[2] A Brownian motion is also called a Wiener process since, it is the canonical process defined on the Wiener space.

are independent r.v.'s.

*(iii) (**Stationary increments.**) For each* $0 \leq s < t, B(t) - B(s)$ *has a normal distribution with mean zero and variance* $\sigma^2(t-s)$.

*(iv) (**Continuity of paths.**)* $\{B(t)\}_{t\geq 0}$ *are continuous functions of* t.

If $\sigma^2 = 1$, we said that $\{B(t) : t \geq 0\}$ is a standard Brownian motion.

Remark 1.1.2.

- *Notice that the natural filtration of the Brownian motion is* $\mathcal{F}_t^B = \sigma(B_s, s \leq t)$.
- *We can define the Brownian motion without the last condition of continuous paths, because with a stochastic process satisfying the second and the third conditions, by applying the Kolmogorov's continuity theorem, there exists a modification of* $(B_t)_{t\in\mathbb{R}_+}$ *which has continuous paths a.s.*

1.1.4.2 Properties of Brownian motion

1- Martingale property for Brownian motion

A martingale is a very special type of stochastic process. Examples of martingales constructed from Brownian motion are given in the next result.

Theorem 1.1.2. *Let* $B(t)$ *be a Brownian motion. Then*

1. $B(t)$ *is a martingale.*

2. $B^2(t) - t$ *is a martingale.*

3. For any $u, e^{uB(t)-\frac{u^2}{2}t}$ *is a martingale.*

Proof: We refer the reader to ([8, 10]).

2- Markov property

Markov processes form a fundamental class of stochastic processes, with many applications in real life problems outside mathematics. The reason why Markov processes are so important comes from the so-called Markov property, which enables many explicit calculations that would be intractable for more general random processes [8].

Definition 1.1.12. X *is a Markov process if for any* t *and* $s > 0$ *the conditional distribution of* $X(t+s)$ *given* $\mathcal{F}_t$ *is the same as the conditional distribution of* $X(t+s)$ *given*

$X(t)$, that is,

$$\mathbb{P}\left(X(t+s) < y|\mathcal{F}_t\right) = \mathbb{P}\left(X(t+s) < y|X(t)\right). \qquad a.s$$

Theorem 1.1.3. *([10]) The Brownian motion $B(t)$ possesses Markov property.*

3- Self-similarity

Theorem 1.1.4. *B is a H-ss process with $H = 1/2$.*

Proof: It is enough to show that for every $a > 0, \{a^{1/2}B(at)\}$ is also Brownian motion. Conditions (i), (ii)and (iv) follow from the same conditions for $\{B(t)\}$. As to (iii), Gaussianity and mean-zero property also follow from the properties of $\{B(t)\}$.
As to the variance , $\mathbb{E}\left[\left(a^{1/2}B(t)^2\right)\right] = t$. and for all $-\infty < t_1, t_2 < \infty$, the autocovariance function $\mathbb{E}\left[(B(at_1)B(at_2))\right] = \min(at_1, at_2) = a\min(t_1, t_2) = \mathbb{E}\left[\left(a^{1/2}B(t_1)a^{1/2}B(t_2)\right)\right]$. Thus $\{a^{1/2}B(t)\}$ is a Brownian motion. ■

4- Non-differentiability

An occurrence of Brownian motion observed from time 0 to time t, is a random function of t on the interval $[0, t]$. It is called a '*realization*', a '*path*' or '*trajectory*'.

Theorem 1.1.5. *For any t almost all trajectories of Brownian motion are not differentiable at t.*

Proof: We refer the reader to ([10]).

5- Hölder continuity

Proposition 1.1.3. *A Brownian motion has its paths a.s. locally γ-Hölder continuous for $\gamma \in [0, 1/2)$.*

Proof: We refer the reader to ([10]).

6- Quadratic variation

Definition 1.1.13. *The quadratic variation of Brownian motion $B(t)$ is defined as*

$$[B, B](t) = [B, B]([0, t]) = \lim_{n\to\infty} \sum_{i=1}^{n} \left| B_{t_i^n} - B_{t_{i-1}^n} \right|^2,$$

where for each n, $\{t_i^n, 0 \le i \le n\}$ *is a partition of* $[0,t]$*, and the limit is taken over all partitions with* $\delta_n = \max_i(t_{i+1}^n - t_i^n) \to 0$ *as* $n \to \infty$*, and in the sense of convergence in probability.*

Theorem 1.1.6. *([10]) Quadratic variation of a Brownian motion over* $[0,t]$ *is* t.

Example 1.1.1. 1. *For any* $T > 0, \{T^{-1/2}B(Tt)\}$ *is Brownian motion.*

2. *The process*

$$\frac{t}{\sqrt{\pi}} + \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{sin(jt)}{j}\xi_j,$$

where $\xi_j; j = 0, 1, \ldots,$ *are independent standard normal random variables, is Brownian motion on* $[0;\pi]$.

3. $\{-B(t); t \ge 0\}$ *is also a Brownian motion.*

4. $\{tB(1/t); t > 0\}$ *is also a Brownian motion.*

5. *If* $B(t)$ *is a Brownian motion on* $[0,1]$*, then* $(t+1)B(1/t+1) - B(1)$ *is a Brownian motion on* $[0,\infty)$.

1 *The first statement is the self-similarity property of the Brownian motion.*

2 *The second is the random series representation of Brownian motion.*

3 *The third is the symmetry of Brownian motion.*

4 *The fourth allows to transfer results on the behavior of the paths of Brownian motion for large* t *to that of small* t.

5 *The last (and the second) show the existence of Brownian motion.*

1.1.5 H-sssi processes

The Proposition 1.1.2 shows that there is a set of different self-similar processes. From the point of view of practical implementation those with stationary increments are of special interest, as they lead to stationary sequences with a special behavior. An H-ss process having stationary increments is specifically marked as H-sssi.

Definition 1.1.14. *A stochastic process* $X = \{X_t\}_{t\in I}$*,* $\mathcal{F}$*-adapted, which is* ***H-ss*** *with stationary increments, is said* ***H-sssi*** *process with exponent* H.

In the following we always suppose that $\mathbb{E}(X_t^2) < \infty, t \in I$. let $X = \{X_t\}_{t\in I}$,$\mathcal{F}$-adapted ,be an **H-sssi** process with finite variance [3], the following properties hold:

1. $X_0 = 0$ almost surely.

2. If $H \neq 1$, then for any $t \geq 0$, $\mathbb{E}(X_t) = 0$.

3. One has:
$$X(-t) \stackrel{d}{=} -X(t);$$
it follows from the first property and the stationarity of the increments:
$$X(-t) \stackrel{a.s.}{=} X(-t) - X(0) \stackrel{d}{=} X(0) - X(t) \stackrel{a.s.}{=} -X(t).$$
The above property allows us to extend the definition of an H-sssi process to the whole real line: $\{X_t\}_{t\in\mathbb{R}}$.

4. Let $\sigma^2 = \mathbb{E}(X_1^2)$. Then,
$$\mathbb{E}(X_t^2) = |t|^{2H}\sigma^2. \tag{1.6}$$
Indeed, from the third property and the self-similarity:
$$\mathbb{E}X(t)^2 = \mathbb{E}X^2(|t|sign(t)) = |t|^{2H}\mathbb{E}X^2(sign(t)) = |t|^{2H}\mathbb{E}(X_1^2) = |t|^{2H}\sigma^2.$$

5. The autocovariance function of an **H-sssi** process [4] X, with $\mathbb{E}(X_1^2) = \sigma^2$, turns out to be:
$$\gamma_{s,t}^H = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \tag{1.7}$$
It follows from the fourth property and the stationarity of the increments
$$\mathbb{E}(X_sX_t) = \frac{1}{2}(\mathbb{E}X_s^2 + \mathbb{E}X_t^2 - \mathbb{E}(X_t - X_s)^2).$$

6. If $X = \{X_t\}_{t\in I}$ is an H-sssi process, then one must have $H \leq 1$.
The constraint of the scaling exponent follows directly from the stationarity of the increments:
$$2^H\mathbb{E}|X_1| = \mathbb{E}|X_2| = \mathbb{E}|X_2 - X_1 + X_1| \leq \mathbb{E}|X_2 - X_1| + \mathbb{E}|X_1| = 2\mathbb{E}|X_1|,$$
therefore, $2^H \leq 2 \Longleftrightarrow H \leq 1$.

[3]We always consider finite variance H-sssi process because it have many interesting properties
[4]Sometimes, we refer to the **H-sssi** process $\{X_t\}_{t\in I}$ with the word standard if $\sigma^2 = 1$

Remark 1.1.3. *The case $H = 1$ corresponds a.s. to $X_t = tX_1$. Indeed, on the $L^2(\Omega, \mathbb{P})$ norm:*

$$\mathbb{E}(X_t - tX_1)^2 = \mathbb{E}(X_t^2 + t^2 X_1^2 - 2tX_tX_1) = \sigma^2(2t^2 - 2t^2) = 0.$$

1.1.6 Long-range dependence

The notion of long memory (or long range dependence) has intrigued many at least since B. Mandelbrot brought it to the attention of the scientific community in the 1960's in a series of papers (Mandelbrot (1965); Mandelbrot and Van Ness (1968) and Mandelbrot and Wallis (1968, 1969)[13]) that, among other things, explained the so-called Hurst phenomenon, today this notion has become especially important as potentially crucial applications arise in new areas. In this subsection we attempt to describe the important ways in which one can think about long memory and connections between long memory and other notions of interest, most importantly scaling and self-similarity.

1.1.6.1 Long-range dependence definition

We introduce the concept of slowly varying function:

Definition 1.1.15. ***(Slowly varying function)*** *A real valued function $L : \mathbb{R} \longrightarrow \mathbb{R}$ is called slowly varying function in zero (infinity) if it is bounded on any finite interval $I \subseteq \mathbb{R}$ and if, for each $a > 0$, one has*

$$\frac{L(ax)}{L(x)} \longrightarrow 1,$$

as x tends to zero (infinity).

Such a functions vary slower than any power function. Logarithms and constants are typical examples of slowly varying functions.

Definition 1.1.16. ***(Long-range dependence)*** *Let $\gamma(k)$ be the autocovariance function of a stationary process X. The following three definitions of long-range dependence are quite common in literature*

1. *The process X has long-range dependence if there exists $0 < a < 1$ such that:*

$$\sum_{k=0}^{n} \gamma(k) \sim n^{\alpha} L_1(n), \quad n \longrightarrow \infty, \tag{1.8}$$

where L_1 is a slowly varying function at infinity.

2. *The process X has long-range dependence if there exists $0 < \beta < 1$ such that:*

$$\gamma(k) \sim k^{-\beta} L_2(k), \quad k \longrightarrow \infty, \tag{1.9}$$

where L_2 is a slowly varying function at infinity.

3. *The process X has long-range dependence if there exists $0 < \gamma < 1$ such that:*

$$f(\nu) \sim |\nu|^{-\gamma} L_3(|\nu|), \quad \nu \longrightarrow 0, \tag{1.10}$$

where L_3 is a slowly varying function at zero.

The parameters α, β, γ measure the LRD intensity, in the sense that greater the long-range dependence greater the values of α and γ and smaller the values of β.

1.1.6.2 Selfsimilarity and Long-Range Dependency

This tow notions are closely related, but they are different and should not be confused. The main difference between self-similar processes and processes with LRD is that self-similar processes are non-stationary, while LRD processes are stationary. However, these two kinds of processes are related by a parameter (the Hurst parameter), and one process can be derived from the other.

Proposition 1.1.4. *([3]) Let $\{X_t\}_{t\geq 0}$ be a H-self-similar process with stationary increments such that $\mathbb{E}(X_1^2) < \infty$. Define, for any integer $n \geq 1$*

$$\gamma(n) = \mathbb{E}(X_1(X_{n+1} - X_n)).$$

Then, if $H \neq 1/2$, as $n \to \infty$

$$\gamma(n) \sim H(2H-1)n^{2H-2}\mathbb{E}(X_1^2).$$

Definition 1.1.17. *We say that a process X exhibits long-range dependence (or it is a long-memory process) if*

$$\sum_{n\geq 0} \gamma_n = \infty$$

where $r(n) = \mathbb{E}((X_1 - X_0)(X_{n+1} - X_n))$. Otherwise, if

$$\sum_{n\geq 0} \gamma_n < \infty$$

we say that X is a short-memory process.

From this definition and Proposition 1.1.4 we conclude that if $(X_t)_{t\geq 0}$ is $H-$ self-similar process with stationary increments and with $\mathbb{E}(X_1^2) < \infty$ then X is with long-range dependence for $H > 1/2$ and with short-memory if $H \leq 1/2$.

1.2 Introduction to stochastic integration

Let us consider the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of the Bm $B(t)$, $t \geq 0$. We introduce the following class of functions.

Definition 1.2.1. *Let $\mathcal{V}(S,T)$ be the class of real **measurable** functions $f(t,\omega)$, defined on $[0,\infty) \times \Omega$, such that:*

1. *$f(t,\omega)$ is $\mathcal{F}_t$-adapted ;*
2. $\mathbb{E}\left(\int_S^T f(t,\cdot)^2 dt\right) < \infty.$

1.2.1 Itô integral

1.2.1.1 Itô integral definition

Let $f \in \mathcal{V}(S,T)$. We want to define the Itô integral of f in the interval $[S,T)$. Namely:

$$\mathcal{I}(f)(\omega) = \int_S^T f(t,\omega) dB_t(\omega), \tag{1.11}$$

where B_t is a standard $(\mathbb{E}(B(1)^2) = 1)$ one dimensional Brownian motion. We begin defining the integral for a special class of functions:

Definition 1.2.2. *(**Simple functions**) A function $\phi \in \mathcal{V}(S,T)$ is called **simple function** (or elementary), if it can be expressed as a superposition of characteristic functions.*

$$\phi(t,\omega) = \sum_{k\geq 0} e_k(\omega) \mathbf{1}_{[t_k,t_{k+1})}(t), \tag{1.12}$$

Definition 1.2.3. *Let $\phi \in \mathcal{V}(S,T)$ be a simple function of the form of (1.12), then we define the stochastic integral:*

$$\int_S^T \phi(t,\omega) dB_t = \sum_{k\geq 0} e_k(\omega)(B_{t_{k+1}} - B_{t_k})(\omega); \tag{1.13}$$

Lemme 1.2.1. *(**Ito isometry**, [20]) Let $\phi \in \mathcal{V}(S,T)$ be a simple function, then:*

$$\mathbb{E}\left(\left(\int_S^T \phi(t,\cdot) dB_t\right)^2\right) = \mathbb{E}\left(\int_S^T \phi(t,\cdot)^2 dt\right). \tag{1.14}$$

Remark 1.2.1. *Observe that (1.14) is indeed an isometry. In fact, it can been written as equality of norms in L^2 spaces:*

$$\left\| \int_S^T \phi(t,\cdot) dB_t \right\|_{L^2(\Omega,\mathbb{P})} = \|\phi\|_{L^2([S,T]\times\Omega)} .$$

We have the following important proposition.

Proposition 1.2.1. *Let $f \in \mathcal{V}$, then there exists a sequence of simple functions $\phi_n \in \mathcal{V}, n \in \mathbb{N}$, which converges to f in the L^2-norm. Namely,*

$$\lim_{n\longrightarrow\infty} \int_S^T \mathbb{E}\left((f(t,\cdot) - \phi_n(t,\cdot))^2\right) dt = \lim_{n\longrightarrow\infty} \|f - \phi_n\|^2_{L^2([S,T]\times\Omega)} = 0. \tag{1.15}$$

Proof: See [20]

Definition 1.2.4. *(**Itô integral**) Let $f \in \mathcal{V}(S,T)$ the Itô integral from S to T of f is defined as the $L^2(\Omega,\mathbb{P})$ limit:*

$$\mathcal{I}(f) = \int_S^T f(t,\omega) dB_t(\omega) = \lim_{n\longrightarrow\infty} \int_S^T \phi_n(t,\omega) dB_t(\omega), \tag{1.16}$$

where $\phi_n \in \mathcal{V}$, $n \in \mathbb{N}$, is a sequence of simple functions which converges to $f \in L^2([S,T]\times\Omega)$.

Remark 1.2.2. *Observe, in view of (eq ,1.15), that the definition above does not depend on the actual choice of $\{\phi_n, n \in \mathbb{N}\}$.*

By definition, we have that Itô isometry holds for Itô integrals:

Corollary 1.2.1. *(**Itô isometry for Ito integrals**, [20]) Let $f \in \mathcal{V}(S,T)$, then:*

$$\mathbb{E}\left(\left(\int_S^T f(t,\cdot) dB_t\right)^2\right) = \mathbb{E}\left(\int_S^T f(t,\cdot)^2 dt\right). \tag{1.17}$$

Moreover,

Corollary 1.2.2. *[20] If $f_n(t,\omega) \in \mathcal{V}(S,T)$ converges to $f(t,\omega) \in \mathcal{V}(S,T)$ as $n \longrightarrow \infty$ in the $L^2([S,T]\times\Omega)$-norm, then:*

$$\int_S^T f_n(t,\cdot) dB_t \longrightarrow \int_S^T f(t,\cdot) dB_t, \tag{1.18}$$

in the $L^2(\Omega,\mathbb{P})$-norm.

1.2.1.2 Properties of the Itô integral

Proposition 1.2.2. *[20] Let $f, g \in \mathcal{V}(0,T)$ and let $0 \leq S < U < T$. Then:*

1. $\displaystyle\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t.$
2. *For some constant $a \in \mathbb{R}$,* $\displaystyle\int_S^T (af+g) dB_t = a\int_S^T f dB_t + \int_S^T g dB_t.$
3. $\mathbb{E}\left[\displaystyle\int_S^T f dB_t\right] = 0.$
4. $\displaystyle\int_S^T f dB_t$ *is $\mathcal{F}_T$-measurable.*
5. *The process $M_t(\omega) = \displaystyle\int_0^T f(t,\omega) dB_s(\omega)$ where $f \in \mathcal{V}(0,T)$ for any $t > 0$, is a martingale with respect to $\mathcal{F}_t$.*

1.2.2 Extensiens of Itô integral

The construction of the Itô Integral can be extended to a class of function $f(t,\omega)$ which satisfies a weak integration condition. This generalization is indeed necessary because it is not difficult to find functions which do not belong to $\mathcal{V}$. For instance, take a function of Bm which increase rapidly $f(t,\omega) = exp(B_t(\omega)^2)$. Therefore, we introduce the following class of functions:

Definition 1.2.5. *Let $\mathcal{W}(S,T)$ be the class of real measurable functions $f(t,\omega)$, defined on $[0,\infty) \times \Omega$, such that*

1. *$f(t,\omega)$ is $\mathcal{F}_t$-adapted;*
2. $\mathbb{P}\left(\displaystyle\int_S^T f(t,\cdot)^2 dt < \infty\right) = 1.$

Remark 1.2.3. *Clearly, $\mathcal{V} \subset \mathcal{W}$.*

In the construction of stochastic integrals for the class of functions belonging to Ω we can no longer use the L^2 notion of convergence, but rather we have to use convergence in probability. In fact, for any $f \in \mathcal{W}$, one can show that there exists a sequence of simple functions $\phi_n \in \mathcal{W}$ such that

$$\int_S^T |\phi_n(t,\cdot) - f(t,\cdot)|^2 dt \longrightarrow 0 \tag{1.19}$$

in probability. For such a sequence one has that the sequence $\left\{\int_S^T \phi_n(t,\omega)dB_t(\omega), n \in \mathbb{N}\right\}$ converges in probability to some random variable. Moreover, the limit does not depends on the approximating sequence ϕ_n. Thus, we may define:

Definition 1.2.6. ***(Itô integral II)*** *Let $f \in \mathcal{W}(S,T)$. The Itô integral from S to T of f is defined as the limit in probability:*

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n \longrightarrow \infty} \int_S^T \phi_n(t,\omega)dB_t(\omega), \tag{1.20}$$

where $\phi_n \in \mathcal{W}, n \in \mathbb{N}$, is a sequence of simple functions which converges to f in probability.

Remark 1.2.4. *Note that this integral is not in general a martingale. However, it is a local martingale.*

Chapter 2

Fractional Brownian motion

The aim of this chapter is to provide a comprehensive overview of fractional Brownian motion. However, for the reader's convenience, in this chapter we review the main properties that make fractional Brownian motion interesting for many applications in different fields. The main references for this chapter are [1, 33].

2.1 Fractional Brownian motion definition

The fractional Brownian motion was first introduced within a Hilbert space framework by Kolmogorov in 1940 [6], where it was called ***Wiener Helix***. The name fractional Brownian motion is due to Mandelbrot and Van Ness, who in 1968 provided in [13] a stochastic integral representation of this process in terms of a standard Brownian motion.

Definition 2.1.1. *(**Fractional Brownian motion**) A fractional Brownian motion (fBm in short) with Hurst parameter $H \in (0,1)$ is a centered continuous Gaussian process $B^H = (B_t^H)_{t>0}$ with covariance function*

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} + |t-s|^{2H}). \tag{2.1}$$

Let us consider the increment process of B^H, (also called fractional gaussian noise)

$$X_k := B_k^{(H)} - B_{k-1}^{(H)}; \qquad X_{k+n} := B_{k+n}^{(H)} - B_{k+n-1}^{(H)}.$$

Proposition 2.1.1. *[33] The covariance of the fractional gaussian noise is given by:*

$$\rho^H(n) = \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right); \tag{2.2}$$

Proposition 2.1.2. *Let B^H be a fractional Brownian motion, with Hurst index $H \in (0, 1]$.*

- *If $H = \frac{1}{2}$, then fBm is nothing but a classical Brownian motion.*
- *If $H = 1$, then $B_1^H(t) = tB_1^H(1)$, almost surely for all $t > 0$.*

Proof:

1. We immediately see that the covariance of $B^{\frac{1}{2}}$ reduces to $(s, t) \longmapsto s \wedge t$, so that $B^{\frac{1}{2}}$ is a classical Brownian motion.
2. When $H = 1$, we have, for all $t \geq 0$,

$$\mathbb{E}[(B_t^H - tB_1^H)^2] = \mathbb{E}[(B_t^H)^2] - 2t\mathbb{E}[B_t^H B_1^H] + t^2\mathbb{E}[(B_1^H)^2] = t^2 - t(t^2 + 1 - (1-t)^2) + t^2 = 0,$$

that is, $B_1^H(t) = tB_1^H$. almost surely, this case won't be considered because it corresponds to the trivial case of a line with random slope (Remark1.1.3):

2.2 Fractional Brownian motion characterization.

The following proposition characterizes the fBm and gives a useful criterium which allows to recognize whether a given process is a fractional Brownian motion.

Proposition 2.2.1. ***(fBm characterization)*** *Let $X = \{X_t\}_{t \geq 0}$ be a stochastic process, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,such that:*

- $\mathbb{P}(X_0 = 0) = 1$.
- *X is a zero-mean Gaussian process such that, for any $t > 0, \mathbb{E}(X_t^2) = \sigma^2 t^\alpha$ for some $\sigma > 0$ and $0 < \alpha < 2$.*
- *X is a **si**-process.*

Then, $\{X_t\}_{t>0}$ is a (one-sided) fractional Brownian motion of order $H = \alpha/2$.

Proof: Since X is a zero-mean Gaussian process, its finite-dimensional distributions are completely characterized by its autocovariance function. Given that, for any $t > 0$:

$$\mathbb{E}(X_t^2) = \sigma^2 |t|^\alpha$$

and X has stationary increments, it follows that the autocovariance function is given by (eq. 2.1), which is the autocovariance of a fBm with $H = \alpha/2$. ∎

Corollary 2.2.1. *[33] Let $X = \{X_t\}_{t\geq 0}$ be a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $0 < H < 1$ and $\sigma^2 = E(X_1^2)$. The following statements are equivalent:*

1. *X is an **H-sssi** Gaussian process.*
2. *X is a (one-sided) fractional Brownian motion with scaling exponent H.*
3. *X is Gaussian with zero mean and covariance function given by (eq. 2.1).*

2.3 Fractional Brownian motion proprieties

The fractional Brownian motion has the following properties:

2.3.1 Selfsimilarity

There is an other classic definition of the fBm using selfsimilar properties, which we give as a theorem.

Theorem 2.3.1. *For $H \in (0,1)$, the fBm $(B_t^H)_{t\in\mathbb{R}_+}$ is a gaussian $H-$sssi process.*

Proof: First, let us prove the selfsimilarity property. We have that

$$\begin{aligned}
\mathbb{E}\left(B_{at}^{(H)} B_{as}^{(H)}\right) &= \frac{1}{2}\left((at)^{2H} + (as)^{2H} - (a|t-s|)^{2H}\right), \\
&= a^{2H}\mathbb{E}\left(B_t^{(H)} B_s^{(H)}\right), \\
&= \mathbb{E}\left((a^H B_t^{(H)})(a^H B_s^{(H)})\right).
\end{aligned}$$

Thus, since all processes are centered and gaussian, it implies that

$$\left(B_{at}^{(H)}\right) \stackrel{d}{=} \left(a^H B_t^{(H)}\right).$$

Second, we show that it has stationary increments. Note that if, for $h > 0$, we have

$$\mathbb{E}\left((B_{t+h}^{(H)} - B_h^{(H)})(B_{s+h}^{(H)} - B_h^{(H)})\right) = \mathbb{E}\left(B_t^{(H)} B_s^{(H)}\right), \tag{2.3}$$

we conclude that $(B^{(H)}_{t+h} - B^{(H)}_{h}) \stackrel{d}{=} B^{(H)}_{t}$. We have,

$$\begin{aligned}
\mathbb{E}\left((B^{(H)}_{t+h} - B^{(H)}_{h})(B^{(H)}_{s+h} - B^{(H)}_{h})\right) &= \mathbb{E}\left((B^{(H)}_{t+h}B^{(H)}_{s+h})\right) - \mathbb{E}\left((B^{(H)}_{t+h}B^{(H)}_{h})\right) - \mathbb{E}\left((B^{(H)}_{s+h}B^{(H)}_{h})\right), \\
&+ \mathbb{E}\left((B^{(H)}_{h})^2\right), \\
&= \frac{1}{2}\Big(((t+h)^{2H} + (s+h)^{2H} - |t-s|^{2H}), \\
&- ((t+h)^{2H} + h^{2H} - t^{2H}), \\
&- ((s+h)^{2H} + h^{2H} - s^{2H}) + 2h^{2H}\Big), \\
&= \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \\
&= \mathbb{E}\left(B^{(H)}_{t}B^{(H)}_{s}\right).
\end{aligned}$$

Therefore the fBm is a H-sssi process. ■

2.3.2 Markovian property

Remark 2.3.1. *Fractional Brownian motion is non-Markovian provided that $H \neq 1/2$.*

Proof: See ([9]) ■

2.3.3 Hölder continuity

Theorem 2.3.2. *(**Kolmogorov continuity theorem**, [1]) A stochastic process $\{X_t\}_{t\in I}$ has a version with continuous trajectories if there exist: $p \geq 1$ and $\eta > 1$ and a constant c, such that, for any $t_1, t_2 \in I$:*

$$\mathbb{E}|X_{t_2} - X_{t_1}|^p \leq c|t_2 - t_1|^{\eta}. \tag{2.4}$$

Theorem 2.3.3. *Let $H \in (0,1)$. The fbm $B^{(H)}$ admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*

Proof: We recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Hölder continuous of order α, $0 < \alpha \leq 1$ and write $f \in \mathcal{C}^{\alpha}(\mathbb{R})$, if there exists $M > 0$ such that

$$|f(t) - f(s)| \leq M|t-s|^{\alpha},$$

for every $s, t \in \mathbb{R}$. For any $\alpha > 0$ we have

$$\mathbb{E}\big[|B^H(t) - B^H(s)|^{\alpha}\big] = \mathbb{E}\big[|B^H(1)|^{\alpha}\big]|t-s|^{\alpha H};$$

hence, by the Kolmogorov criterion we get that the sample paths of B^H are almost everywhere Hölder continuous of order strictly less than H. Moreover, by ([1]) we have

$$\limsup_{t\longrightarrow 0_+} \frac{|B^{(H)}(t)|}{t^H\sqrt{log(log(t^{-1}))}} = c_H$$

with probability one, where c_H is a suitable constant. Hence B^H can not have sample paths with Hölder continuity's order greater than H. ∎

2.3.4 Differentiability

By ([15]) we also obtain that the process B^H is not mean square differentiable and it does not have differentiable sample paths.

Proposition 2.3.1. *Let $H \in (0,1)$. The fBm sample path $B^H(.)$ is not differentiable. In fact, for every $t_0 \in [0,\infty)$*

$$\limsup_{t\to t_0} \left|\frac{B^H(t)-B^H(t_0)}{t-t_0}\right| = \infty$$

with probability one.

Proof: Here we recall the proof of ([15]). Note that we assume $B^H(0) = 0$. The result is proved by exploiting the self-similarity of B^H. Consider the random variable

$$\mathcal{R}_{t,t_0} := \frac{B^H(t)-B^H(t_0)}{t-t_0}$$

that represents the incremental ratio of B^H. Since B^H is self-similar (see[1]), we have that the law of $\mathcal{R}_{t,t_0}$ is the same of $(t-t_0)^{H-1}B^H(1)$. If one considers the event

$$A(t,w) := \left\{\sup_{0\le s\le t}\left|\frac{B^H(s)}{s}\right| > d\right\},$$

then for any sequence $(t_n)_{n\in\mathbb{N}}$ decreasing to 0, we have

$$A(t_n,w) \supseteq A(t_{n+1},w),$$

and

$$A\left(t_n,w\right) \supseteq \left(|\frac{B^H(t_n)}{t_n}| > d\right) = \left(|B^H(1)| > t_n^{1-H}d\right).$$

The proof follows since the probability of the last term tends to 1 as $n \longrightarrow \infty$. ∎

2.3.5 The FBm is not a semimartingale

Proposition 2.3.2. *The fBm is not a semimartingale except when $H = 1/2$.*

Proof: The fact that the fBm is not a semimartingale for $H \neq \frac{1}{2}$ has been proved by several authors. In order to verify that $B^H$is not a semimartingale for $H \neq \frac{1}{2}$, it is sufficient to compute the p-variation of B^H.

Definition 2.3.1. *Let $(X(t))_{t\in[0;T]}$ be a stochastic process and consider a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$. Put*

$$S_p(x,\pi) := \sum_{i=1}^{n} |X(t_i) - X(t_{i-1})|^p$$

The p-variation of X over the interval $[0,T]$ is defined as

$$V_p(X,[0,T]) := sup_\pi S_p(X,\pi),$$

where π is a finite partition of $[0,T]$.

Definition 2.3.2. *The index of p-variation of a process is defined as*

$$I(X,[0,T]) := inf\{p > 0; V_p(X,[0,T]) < \infty\}$$

We claim that

$$I(B^H,[0,T]) = \frac{1}{H}$$

In fact, consider for $p > 0$,

$$Y_{n,p} = n^{pH-1}|B^H(i) - B^H(i-1)|^p$$

Since B^H has the self-similarity property, the sequence $Y_{n,p}, n \in \mathbb{N}$ has the same distribution as

$$\widetilde{Y}_{n,p} = n^{-1}|B^H(\tfrac{i}{n}) - B^H(\tfrac{i-1}{n})|^p$$

and By the Ergodic theorem (see, [1]) the sequence $\widetilde{Y}_{n,p}$ converges almost surely and in L^1 to $\mathbb{E}[|B^H(1)|^p]$ as n tends to infinity. It follows that

$$V_{n,p} = \sum_{i=1}^{n} |B^H(\tfrac{i}{n}) - B^H(\tfrac{i-1}{n})|^p$$

converges in probability respectively to 0 if $pH > 1$ and to infinity if $pH < 1$ as n tends to infinity. Thus we can conclude that $I(B^H,[0,T]) = \frac{1}{2}$, Since for every semimartingale X, the index $I(X,[0,T])$ must belong to $[0,1]\bigcup\{2\}$, the fBm B^H cannot be a semimartingale unless $H = \frac{1}{2}$. ∎

2.3.6 Long-Range Dependence

Note also that the fBm is one of the simplest process which exhibits long-range dependency.

Lemme 2.3.1. *For fBm $B(H)$ of Hurst index $H \in (1/2, 1)$, the three definitions of long-range dependence of Definition 1.1.16 are equivalent. They hold with the following choice of parameters and slowly varying functions:*

1. $\alpha = 2H - 1, L_1(x) = 2H.$

2. $\beta = 2 - 2H, L_2(x) = H(2H - 1).$

3. $\gamma = 2H - 1, L_3(x) = \pi^{-1} H\Gamma(2H) \sin \pi H.$

Proof: For the proof, we refer [33].

Depending on the qualitative behavior of the fBm trajectories, it is common the following fBm partitioning, which can be actually used to characterize any $H-$sssi process:

1. If $0 < H < 1/2$, the fBm is termed anti-persistent.

2. If $H = 1/2$, the fBm is termed purely random, or chaotic.

3. If $1/2 < H < 1$, the fBm is termed persistent.

This division is due to the behavior of the autocovariance function of the increment process. In the first case the covariance of two consecutive increments is negative. Therefore, the fBm increments tend to have opposite sign. On the other hand, in the third case the covariance is always positive and one has a less "zigzaging" behavior of the paths The case $H = 1/2$ corresponds to the Bm, which has independent identically distributed increments, i.e. a purely random increment process.

2.4 Representation of fBm

In this section, we show that fractional Brownian motion can be represented as Wiener integral in two different ways:

2.4.1 Moving averge representation of fBm

The proposition below provides a first representation of fractional Brownian motion in terms of stochastic integrals of Brownian motion.

Proposition 2.4.1. *(**Moving average representation,**[14]) Let $\{B_H(t), t \geq 0\}$ be a standard one-sided fractional Brownian motion with $0 < H < 1$. Then, for any $t \geq 0$*

$$B_H(t) = \int_{\mathbb{R}} f_t(x)dB(x) = \frac{1}{C_1(H)} \int_{\mathbb{R}} \left((t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}}\right) dB(x), \tag{2.5}$$

where:

$$C_1(H) = \left(\int_0^\infty ((1+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}})^2 dx + \frac{1}{2H}\right)^{1/2} \tag{2.6}$$

$$= \frac{\Gamma(H+1/2)}{(\Gamma(2H+1)sin\pi H)^{1/2}}. \tag{2.7}$$

such that the function $f_t(x)$ is called representation kernel.
*This representation is called **moving average representation**.*

Remark 2.4.1. *We observe that for $H = 1/2$ one obtains,*

$$B(t) = \int_0^t dB_x, t \geq 0,$$

that is a standard Bm.

Proof: Let $X(t)$ denoted the integral in (eq 2.5) and let $f_t(x)$ denoted the integrand. In order to verify that $X(t)$ is well defined ,

1. We show firstly that $\int_{-\infty}^{\infty} f_t^2(x)dx < \infty$. This relation is obvious when $H = 1/2$, because $\int_{-\infty}^{\infty} f_t^2(x)dx = \int_t^{|t|} dx < \infty$.
2. Suppose now $0 < H < 1, H \neq 1/2$. Then

   - as $x \longrightarrow -\infty$, $f_t(x) \sim (H-1/2)(-x)^{H-3/2}$ whose square is integrable around $-\infty$,
   - $f_t(x) \sim (t-x)_+^{H-1/2}$ as $x \longrightarrow t$ whose square is integrable around $x = t$,
   - and similarly for $x = 0$ and $x = \infty$.

   Hence $\int_{-\infty}^{\infty} f_t^2(x)dx < \infty$ and (eq 2.5) is well defined.

3. We now verify that $\{X(t), t \in \mathbb{R}\}$ has the following autocovariance function (eq 2.1) with $Var(X(1)) = 1$. Notice that $X(0) = 0$ a.s, and, for every $t > 0$, $\mathbb{E}X^2(t)$ equals

$$\frac{1}{C_1(H)^2}\int_{-\infty}^{\infty}\left((t-s)_+^{H-1/2} - (tx)_+^{H-1/2}\right)^2 dx$$

$$= \frac{1}{C_1(H)^2}t^{2H}\int_{-\infty}^{\infty}\left((1-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right)^2 du,$$

$$= \frac{1}{C_1(H)^2}t^{2H}\left[\int_{-\infty}^{0}((1-u)^{H-1/2} - (-u)^{H-1/2})^2 du + \int_0^t (1-u)^{2H-1}\right],$$

$$= \frac{1}{C_1(H)^2}t^{2H}\left[\int_0^{\infty}((1+x)_+^{H-1/2} - x_+^{H-1/2})^2 dx + \frac{1}{2H}\right] = t^{2H},$$

and similarly $\mathbb{E}X^2(t) = |t|^{2H}$ for $t < 0$. Further, for any $t, s \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}(X(t) - X(s))^2 &= \frac{1}{C_1(H)^2}\int_{-\infty}^{\infty}\left((t-x)_+^{H-1/2} - (s-x)_+^{H-1/2}\right)^2 dx;\\ &= \frac{1}{C_1(H)^2}\int_{-\infty}^{\infty}\left((t-s-x)_+^{H-1/2} - (-x)_+^{H-1/2}\right)^2 dx;\\ &= |t-s|^{2H},\end{aligned}$$

by the previous calculation and hence (eq 2.1) follows. ■

Remark 2.4.2. *Heuristically, we observe that the process $\{X(t), t \geq 0\}$ defined by (eq 2.5), is indeed self-similar with scaling exponent H. In fact,*

$$X(at) = \frac{1}{C_1(H)}\int_{\mathbb{R}}\left((at-x)_+^{H-1/2} - (-x)_+^{H-1/2}\right) dB(x)$$

(with the change of variables $x = ax'$)

$$= \frac{1}{C_1(H)}\int_{\mathbb{R}}\left((at-ax')_+^{H-1/2} - (-ax')_+^{H-1/2}\right) dB(x)$$

$$\stackrel{d}{=} \frac{a^{H-1/2}a^{1/2}}{C_1(H)}\int_{\mathbb{R}}\left((t-x)_+^{H-1/2} - (-x)_+^{H-1/2}\right) dB(x) = a^H X_t.$$

2.4.2 Spectral representation of fBm.

Our second representation of fBm is the Spectral representation of fBm type (also called harmonizable representation). In fact, following ([30])

Let f_t be the kernel of the moving average representation . Since $f_t \in L^2(\mathbb{R})$, one can evaluate its Fourier transform :

$$\widetilde{f}_t\,(v) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixv} f_t(x)dx; \quad f_t(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixv}\, \widetilde{f}_t\,(v)dv. \tag{2.8}$$

Because $\widetilde{f}_t \in L^2(\mathbb{R})$, it is possible to define an integral representation of fBm based on $\widetilde{f}_t\,(v)$:

$$\widetilde{X}_t = \int_{\mathbb{R}} \widetilde{f}_t\,(v)\widetilde{M}(dv), \tag{2.9}$$

where $\widetilde{M}$ is a suitable complex Gaussian random measure, which satisfies :

$$\widetilde{M}(dv) = \overline{\widetilde{M}(-dv)}, \tag{2.10}$$

$$\mathbb{E}|\widetilde{M}(dv)|^2 = dv \tag{2.11}$$

Remark 2.4.3. *The previous conditions, together with the hermitian property of the Fourier transform, i.e.* $\widetilde{f}_t(v) = \widetilde{f}_t(-v)$, *ensure that the process* $\widetilde{X}_t$ *is a real valued process.*

Remark 2.4.4. *We observe that, if* $\widetilde{M}$ *is a complex Gaussian measure as above, then for any* $h > 0$,

$$\widetilde{M}_*(dn) = e^{ivh}\widetilde{M}(dn),$$

is another Gaussian measure which still verifies (eq. 2.10) and (eq.2.11).

In order to define the measure $\widetilde{M}$, we introduce two independent Brownian motions $B_t^{(1)}$ et $B_t^{(2)}$, and we define

$$M^{(i)}(A) = \frac{1}{\sqrt{2}} \int_A dB^{(i)}(n), A \subset \mathbb{R}_+;$$

while, for any $A \subset \mathbb{R}_-$:

$$M^{(1)}(A) = M^{(1)}(-A), M^{(2)}(A) = -M^{(2)}(-A).$$

Then, we define:

$$\widetilde{M} = M^{(1)} + iM^{(2)}. \tag{2.12}$$

Proposition 2.4.2. *(**Spectral representation of fBm**). Let $\widetilde{M}$ be defined as above and let $\widetilde{f}_t$ be the Fourier transform of a fBm representation kernel f_t, for instance (2.5). Then, the process*

$$\widetilde{X}_t = \int_{\mathbb{R}} \widetilde{f}_t\,(n)\;\widetilde{M}\;(dv), \tag{2.13}$$

is a fractional Brownian motion.

*This proposition introduces the so called **spectral representation of fBm**.*

Proof : $\tilde{X}_t$ is a real valued (Remark 2.4.3) zero mean Gaussian process. Furthermore, let $X_t = \int f_t(x)dB(x), t \geq 0$, be a fBm of order H, then by using Parseval theorem [20]:

$$\mathbb{E}(X_{t_1}X_{t_2}) = \int_{\mathbb{R}} f_{t_1}(x) f_{t_2}(x)dx = \int_{\mathbb{R}} \tilde{f}_{t_1}(x)\tilde{f}_{t_2}(v)dv = \mathbb{E}(\tilde{X}_{t_1}\tilde{X}_{t_2})$$

Therefore, $\tilde{X}_t$ is a zero mean Gaussian process with the covariance of a fBm of order H. Then, by (Corollary 2.2.1), $\tilde{X}_t$ is a fBm. ■

Cosider the representation 2.5 of fractional Brownian motion

Proposition 2.4.3. *let $0 < H < 1$. Then the standard fractional Brownian motion has the integral representation:*

$$\frac{1}{C_2(H)} \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} |x|^{-(H-1/2)} \widetilde{M} d(x), \quad t \in \mathbb{R}, \tag{2.14}$$

where

$$C(H) = \left(\frac{\pi}{H\Gamma(2H)\sin H_\pi} \right)^{1/2}. \tag{2.15}$$

Proof: See [30]

Chapter 3

Introduction to Fractional calculus

The fractional calculus is the theory of integrals and derivatives of arbitrary real or even complex order (called integrals and derivatives), which is unifies and a generalizes the integer -order integration and differentiation. In this chapter we will introduce the basic notions of fractional calculus and its properties, providing useful examples and applications. Interested readers are referred to [19, 29].

3.1 Fractional integrals

We first define the fractional integral operator according to Riemann-Liouville, which is the most widely used definition in fractional calculus.

Let $a < b$ be two real numbers and f a function defined on $I = [a, b]$. Then, we have the following result

Proposition 3.1.1. *[20] By induction it is easy to show that, for any integer $n \geq 1$, a multiple integral of ϕ can be expressed as:*

$$\int_a^{t_n} \cdots \left\{ \int_a^{t_2} \left\{ \int_a^{t_1} \phi(s)ds \right\} dt_1 \right\} \cdots dt_{n-1} = \frac{1}{(n-1)!} \int_a^{t_n} (t_n - s)^{n-1} \phi(s) ds. \tag{3.1}$$

3.1.1 Fractional integrals definitions

By replacing the integer n by a positive real number α, we obtain the following definition:

Definition 3.1.1. ***(Riemann-Liouville fractional integral)*** *Let $\phi \in L^1([a,b])$ and $\alpha > 0$. Then, for any $t \in (a,b)$, the integrals*

$$\mathcal{I}^{\alpha}_{a+}\phi(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\phi(s)ds = \frac{1}{\Gamma(\alpha)}\int_a^b (t-s)_+^{\alpha-1}\phi(s)ds, \tag{3.2}$$

$$\mathcal{I}^{\alpha}_{b-}\phi(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}\phi(s)ds = \frac{1}{\Gamma(\alpha)}\int_a^b (s-t)_+^{\alpha-1}\phi(s)ds, \tag{3.3}$$

are called left-side and right-side Riemann-Liouville fractional integrals of order $\alpha > 0$.

Remark 3.1.1. *The fractional integrals $\mathcal{I}^{\alpha}_{a+}\phi(t)$ and $\mathcal{I}^{\alpha}_{b-}\phi(t)$ are well defined for any $\phi(t) \in L^p([a,b]), p \geq 1$ In fact, suppose $\phi(t) \in L^p([a,b])$ then, for any $t \in (a,b)$,*

$$\Gamma(\alpha)\int_a^b |\mathcal{I}^{\alpha}_{a+}\phi(t)|dt \leq \Gamma(\alpha)\int_a^b \mathcal{I}^{\alpha}_{a+}|\phi(t)|dt = \int_a^b\int_a^b |\phi(s)|(t-s)^{\alpha-1}dsdt$$

(by changing the integration order)

$$= \alpha^{-1}\int_a^b |\phi(s)|(b-s)^{\alpha}ds \leq \alpha^{-1}(b-a)^{\alpha}\int_a^b |\phi(s)ds < \infty,$$

and the same for the right-side integral.

Example 3.1.1. *Evaluate the α-th fractional integral of $\phi(t) = (t-a)^{-\alpha}, 0 < \alpha < 1$. One has,*

$$\frac{1}{\Gamma(\alpha)}\int_s^t (t-s)^{\alpha-1}(s-a)^{-\alpha}ds$$

(after the change of variables $s = a + (t-a)z$)

$$= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-z)^{\alpha-1}z^{-\alpha}dz = \frac{1}{\Gamma(\alpha)}\mathbb{B}(\alpha, 1-\alpha) = \Gamma(1-\alpha),$$

where:

$$\boldsymbol{B}(x,y) = \int_0^1 s^{x-1}1(1-s)^{y-1}ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y > 0,$$

is the Beta function. Thus

$$\mathcal{I}^{\alpha}_{a+}(t-a)^{\alpha} = \Gamma(1-\alpha). \tag{3.4}$$

Example 3.1.2. *In the same way, one can easily show that, for any $\gamma > -1$ and $\alpha > 0$:*

$$\mathcal{I}^{\alpha}_{a+}(t-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(t-a)^{\gamma+\alpha}, \quad t \in (a,b).$$

Remark 3.1.2. *By convention is set: $\mathcal{I}^0\phi(t) = \phi(t)$.*

3.1.2 Properties of fractional integrals

Let $\alpha > 0$, then the fractional integrals, both the left-side and right-side, Ia have the following properties:

1. **Reflection property** : let Q be the reflection operator: $Q\phi(t) = \phi(a - b - t)$, $t \in [a, b]$. Then,
$$Q\mathcal{I}^{\alpha}_{a+} = \mathcal{I}^{\alpha}_{b-}Q. \tag{3.5}$$

2. **Semigroup property** : for any $\phi \in L^1([a, b])$
$$\mathcal{I}^{\alpha}\mathcal{I}^{\beta}\phi(t) = \mathcal{I}^{\alpha+\beta}\phi(t), \quad \alpha, \beta > 0. \tag{3.6}$$

3. $\mathcal{I}^{\alpha}\phi = 0$ implies that $\phi = 0$ almost everywhere.

4. **Fractional integration by parts formula** : let $\phi \in L^p([a, b])$ and $\psi \in L^q([a, b])$ either with $\alpha \geq 1$, $p = q = 1$, or with $0 < \alpha < 1$, $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha, p, q > 1$. Then,
$$\int_a^b \phi(s)\mathcal{I}^{\alpha}_{a+}\psi(s)ds = \int_a^b \mathcal{I}^{\alpha}_{b-}\phi(s)\psi(s)ds. \tag{3.7}$$

Remark 3.1.3. *Observe that the second property implies that if $f = \mathcal{I}^{\alpha}_{a+}\phi_1$ and $f = \mathcal{I}^{\alpha}_{a+}\phi_2$, then $\phi_1 = \phi_2$ a.e., that is, the inverse of the fractional integral is a.e. unique. This property is used in the definition of fractional derivatives.*

3.2 Fractional derivatives

3.2.1 Fractional derivatives definitions

Definition 3.2.1. ***(Rimeann-Liouville fractional derivatives)*** *Let $0 < \alpha < 1$. Then, for any $t \in (a, b)$, the integrals*
$$\mathcal{D}^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t - s)^{-\alpha}f(s)ds, \tag{3.8}$$
$$\mathcal{D}^{\alpha}_{b-}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b (t - s)^{-\alpha}f(s)ds, \tag{3.9}$$
are called left-side and right-side Riemann-Liouville fractional derivatives of order α.

Fractional derivatives of order $0 < \alpha < 1$ are also well defined if, for example, f is differentiable.

Example 3.2.1. *Let* $f(t) = (t-a)^{\alpha-1}, t \in (a,b)$ *and* $0 < \alpha < 1$. *Then,*

$$\mathcal{D}^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^t (t-s)^{-\alpha}(s-a)^{\alpha-1} ds = \Gamma(\alpha) \frac{d}{du} 1 = 0 \tag{3.10}$$

Example 3.2.2. *Let* $C \in \mathbb{R}$, *then:*

$$\mathcal{D}^{\alpha}_{a+} C = \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^t (t-s)^{-\alpha} C ds = \frac{C}{\Gamma(1-\alpha)} (t-a)^{-\alpha}. \tag{3.11}$$

The Riemann-Liouville fractional derivative of a constant in not zero.

3.2.2 Properties of fractional derivatives

Let $0 < \alpha < 1$. The fractional derivative $\mathcal{D}^{\alpha}_{a+}$ has the following properties :

1. For any $\phi \in L^1([a,b])$, we have that:

$$\mathcal{D}^{\alpha}_{a+} \mathcal{I}^{\alpha}_{a+} \phi = \phi. \tag{3.12}$$

2. For any $f = \mathcal{I}^{\alpha}_{a+} \phi$, we have that:

$$\mathcal{I}^{\alpha}_{a+} \mathcal{D}^{\alpha}_{a+} f = f. \tag{3.13}$$

3. The latter can be generalized. In fact, if the function $\mathcal{I}^{1-\alpha}_{a+} f$ is absolutely continuous on $[a,b]$, then:

$$\mathcal{I}^{\alpha}_{a+} \mathcal{D}^{\alpha}_{a+} f(t) = f(t) - \frac{\mathcal{I}^{1-\alpha}_{a+} f(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}, \quad t \in (a,b), \tag{3.14}$$

where $\mathcal{I}^{1-\alpha}_{a+} f(a) = \lim_{s \longrightarrow a+} (\mathcal{I}^{1-\alpha}_{a+} f(s))$, which is in general non-zero.

3.2.3 Two forms of fractional derivatives

It is possible to define fractional derivative operators for $\alpha \geq 1$ as well. The idea is to use usual integer order derivative operators. Consider an integer m such that $m-1 < \alpha \leq m$. The first step is to integrate f by order $m-\alpha$ and then differentiate by m to obtain a resultant differentiation of order α . That is,

$$\mathcal{D}^{\alpha}_{a+} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_a^t (t-s)^{m-\alpha-1} f(s) ds, & m-1 < \alpha < m \\ \\ \dfrac{d^m}{dt^m} f(t). & \alpha = m \end{cases}$$

There is another way to define fractional derivatives, which is one just has to invert the integration and derivation operations. Then, one can define:

Definition 3.2.2. *(**Caputo derivative**) Let $\alpha > 0$, then for any $t \in (a, b)$*

$$*\mathcal{D}^{\alpha}_{a+} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^t (t-s)^{m-\alpha-1} \frac{d^m}{ds^m} f(s) ds, & m-1 < \alpha < m, \\ \\ \dfrac{d^m}{dt^m} f(t). & \alpha = m \end{cases}$$

is called (left-side) Caputo derivative of order $\alpha > 0$.

Remark 3.2.1. *Let $0 < \alpha < 1$ and $a = 0$. In this case we shall write $\mathcal{I}^{\alpha}_{0+} = \mathcal{I}^{\alpha}_t$. Consider a well defined function f, for instance take $f \in \mathcal{C}^1(\mathbb{R}+)$. Then, one has:*

$$\mathcal{I}^{\alpha}_t * \mathcal{D}^{\alpha}_{a+} f(t) = \mathcal{I}^{\alpha}_t \mathcal{I}^{1-\alpha}_t \frac{d}{dt} f(t) = \mathcal{I}_t \frac{d}{dt} f(t) = f(t) - f(0_+) \tag{3.15}$$

Therefore,(by using eq.3.11) we have the following relationships between the R-L and Caputo derivatives of order $0 < \alpha < 1$:

$$* \mathcal{D}^{\alpha}_{a+} f(t) = \mathcal{D}^{\alpha}_{a+}(f(t) - f(0_+)) = \mathcal{D}^{\alpha}_{a+} f(t) - \frac{f(0_+)}{\Gamma(1-\alpha)} t^{-\alpha}. \tag{3.16}$$

A generalization of (eq. 3.15)

$$* \mathcal{D}^{\alpha}_t f(t) = \mathcal{D}^{\alpha}_t \left(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+) \frac{t^k}{k!} \right) = \mathcal{D}^{\alpha}_t f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0_+) t^{k-\alpha}}{\Gamma(k-\alpha+1)}, \tag{3.17}$$

where $\alpha > 0$ and $m - 1 < \alpha < m$.

Remark 3.2.2. *The Caputo fractional derivative represents a sort of regularization (in the origin) of the RL fractional derivative. Moreover, in order for the Caputo derivative to exist, all the limiting values $f^{(k)}(0_+)$ are required to be finite for any $k \leq m-1$. Then, because the derivative of order m is required to exist and is subjected to some regularity conditions, the Caputo fractional derivative is in this sense more restrictive than the Riemann-Liouville derivative. The Caputo fractional derivative turns out to be very useful in treating initial-value problems for physical and engineering applications. In fact, in this case the initial conditions can be expressed in terms of integer-order derivatives.*

3.3 Fractionls integrals and derivatives on the real line

Fractional integral and derivatives can be defined also in the real line:

Definition 3.3.1. *(**Fractional integrals on the real line**) Let $\alpha > 0$. The integrals,*

$$\mathcal{I}^{\alpha}_{+} \phi(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} \phi(s) ds = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} (t-s)^{\alpha-1} \phi(s) ds, \quad t \in \mathbb{R} \tag{3.18}$$

$$\mathcal{I}^{\alpha}_{-}\phi(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(s-t)^{\alpha-1}\phi(s)ds = \frac{1}{\Gamma(\alpha)}\int_{\mathbb{R}}(s-t)^{\alpha-1}\phi(s)ds, \quad t\in\mathbb{R} \tag{3.19}$$

are called fractional integrals of order $\alpha > 0$ on the real line.

Definition 3.3.2. ***(Fractional derivatives on the real line)*** *Let $0 < \alpha < 1$. For any $t \in \mathbb{R}$, the integrals,*

$$\mathcal{D}^{\alpha}_{+}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{-\infty}^{t}(t-s)^{-\alpha}f(s)ds, \tag{3.20}$$

$$\mathcal{D}^{\alpha}_{-}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t}^{\infty}(s-t)^{-\alpha}f(s)ds, \tag{3.21}$$

are called fractional derivatives of order $0 < \alpha < 1$ on the real line.

Example 3.3.1. *Let $a < b$, it is easy to show that, for any $a > 0$ and $t \in \mathbb{R}$,*

$$\mathcal{I}^{\alpha}_{\pm}\mathbf{1}_{[a,b)}(t) = \frac{1}{\Gamma(1+\alpha)}[(b-t)^{-\alpha}_{\mp} - (a-t)^{-\alpha}_{\mp}]. \tag{3.22}$$

where we remember that $x_{-} = -\min(x,0) = \max(-x,0) = (-x)_{+}$. Moreover, for any $0 < \alpha < 1$, we have that:

$$\mathcal{D}^{\alpha}_{\pm}\mathbf{1}_{[a,b)}(t) = \frac{1}{\Gamma(1-\alpha)}[(b-t)^{-\alpha}_{\mp} - (a-t)^{-\alpha}_{\mp}]. \tag{3.23}$$

Remark 3.3.1. *The definitions of fractional derivatives on the real line can be extended to the case $\alpha \geq 1$, see ([29]) for details.*

3.4 Applications

The fractional calculus fined applications in different field of sciences. This theory is deeply used to develop mathematical models in which differential equations of fractional order appear and fundamental solutions are available in terms of the M-L function and its generalizations. We will study the basic M-L function, its generalization and some of their properties.

3.4.1 The Mettag-Leffler function

The Mittag-Leffler function was introduced by Magnus Gustaf (Gösta) Mittag-Leffler in 1903 ([16]). Its importance is not recognized before the last 20 years, it is discovered due to the various types of applications in engineering, physics, biological sciences and in many different areas. Gorenflo and Mainardi call Mittag-Leffler function as the queen function in fractional calculus.

Definition 3.4.1. *The basic Mittag-Leffler function is denoted by $E_\alpha(z)$ and it is defined as:*

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \tag{3.24}$$

Later The two-parameter generalization Mittag-Leffler function is given by :

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0, \ \alpha, \beta \in \mathbb{R}, \ z \in \mathbb{C}. \tag{3.25}$$

it is called the two-parameter function of Mittag-Leffler type.

Remark 3.4.1. *The Mittag-Leffler function provides a simple generalization of the exponential function, which is indeed obtained when $\alpha = 1$.*

Some of its interesting examples are ([20])
$E_{1,1}(z) = e^z, E_{2,1}(z^2) = \cosh(z), E_{2,2}(z^2) = \frac{\sinh(z)}{z}, E_{\alpha,1}(z) = E_\alpha(z)$.

Proposition 3.4.1. *[20] Let $E_{\alpha,\beta}(z)$ be the generalized Mittag-Leffler function (eq. 3.25). Then, one has the following useful relations:*

- *Let $z \in \mathbb{C}$ then,*

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z). \tag{3.26}$$

- *Moreover,*

$$E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z), \tag{3.27}$$

- *For any $x, q > 0$ and $0 < a < 1$,*

$$qx^{\alpha-1} E_{\alpha,\alpha}(-qx^\alpha) = -\frac{d}{dx} E_\alpha(-qx^\alpha). \tag{3.28}$$

- *Finally, for any $x > 0$ and $p \in \mathbb{N}$*

$$\frac{d^p}{dx^p} \left[x^{\beta-1} E_{\alpha,\beta}(x^\alpha)\right] = x^{\beta-p-1} E_{\alpha,\beta-p}(x^\alpha). \tag{3.29}$$

3.4.2 Fractional representation of fractional Brawnian motion

The name fractional Brownian motion suggests that it can be in some way regarded as a fractional integral of Brownian motion. Here, we want to show that it is best viewed as a fractional integral of a Gaussian white noise.

Let $B^H(t), t \in \mathbb{R}$, be a (two-sided) fractional Brownian motion of index $0 < H < 1$. In the context of fractional integration, it is convenient to use another parameterization of fBm: we set $k = H - 1/2$, and we indicate with $B^k(t)$ a fBm of order k. Clearly, the range $0 < H < 1$ corresponds to the range $-1/2 < k < 1/2$.

We start with the moving average representation of standard fractional Brownian motion (eq 2.5), namely:

$$B^k(t) \stackrel{d}{=} \frac{1}{C_1(k)} \int_{\mathbb{R}} \left((t-s)_+^k - (-s)_+^k\right) dB^0(s), \quad t \in \mathbb{R}, \tag{3.30}$$

where

$$C_1(k)^2 = \int_0^\infty ((1+s)^k - s^k)^2 ds + \frac{1}{2k+1}. \tag{3.31}$$

Proposition 3.4.2. *[20] Let $-1/2 < k < 1/2$ and suppose that B^k is a standard fBm. Then,*

$$B^k(t) \stackrel{d}{=} \frac{\Gamma(k+1)}{C_1(k)} \int_{\mathbb{R}} \mathcal{I}_-^k \mathbf{1}_{[0,t)}(s) dB^0(s), \quad t \in \mathbb{R}. \tag{3.32}$$

Heuristically, the representation (3.32) says that the fractional Gaussian noise is the k-fractional integral of the white noise $\dot{B}^0$. That is, formally speaking:

$$\dot{B}^k(t) = \mathcal{I}_+^k \dot{B}^0(t). \tag{3.33}$$

In fact, we have naively that

$$B^k(t) = \int_{\mathbb{R}} \mathbf{1}_{[0,t)}(s)\dot{B}^k(s)ds = \int_{\mathbb{R}} \mathcal{I}_-^k \mathbf{1}_{[0,t)}(u)\dot{B}^0(u)du = \int_{\mathbb{R}} \mathbf{1}_{[0,t)}(u)\mathcal{I}_+^k \dot{B}^0(u)du,$$

where we have used the integration by parts formula (eq. 3.7).

Remark 3.4.2. *It is also possible to relate fractional Brownian motion to fractional integrals defined on an interval. (see [27]).*

Chapter 4

Grey Brawnian motion

Grey Brownian motion was introduced by W. Schneider in [32, 31], as a stochastic model for slow-anomalous diffusion described by the time-fractional diffusion equation. Later F. Minardi, A. Mura and G. Pagnini [20, 21], extended this class, so called "generalized" grey Brownian motion which includes stochastic models for slow and fast-anomalous diffusion,i.e., the time evolution of the marginal density function is described by differential equations of fractional type.

In this chapter, we introduce an extended class of stochastic processes which is called 'generalized' grey Brownian motion (ggBm). This class includes non-Markovian stochastic models either for slow and fast-anomalous diffusion. For a good introduction to the theory of grey noise, we refer the reader to the book of [32].

4.1 Preliminaries

We begin by introducing some basic concepts and facts. Let X be a vector space over a $\mathbb{K}-$field and let $\{\|\cdot\|_p, p \in I\}$ be a countable family of Hilbert norms defined on it. The space X along with the Hilbert norms $\{\|\cdot\|_p, p \in I\}$ is called a **topological vector space** if it carries as natural topology the initial topology[1] of the norms and the vector space operations. We indicate with X_p the **completion** of X with respect to the norm $\|\cdot\|_p$. Let $\langle\cdot,\cdot\rangle$, denotes the natural bilinear pairing between X and its dual space X'. We equip X' with the so-called **weak topology**, which is the coarsest topology such that the functional $\langle\cdot, x\rangle$ is continuous for any $x \in X$.

[1] The coarsest topology on X which makes those functions continuous.

Definition 4.1.1. ***(Schwartz space)*** *The space $\mathcal{S}(\mathbb{R}^n)$ is the space of all the functions $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, such that for any multi-indices $j = (j_1, j_2, ..., j_n)$ and $k = (k_1, k_2, ..., k_n)$:*

$$\sup_{x \in \mathbb{R}^n} |x^j D^k f(x)| < \infty. \tag{4.1}$$

Definition 4.1.2. ***(Nuclear space)*** *A topological vector space X, with the topology defined by a family of Hilbert norms, is said a nuclear space if for any Hilbert norm $\|\cdot\|_p$ there exists a larger norm $\|\cdot\|_q$ such that the inclusion map $X_q \hookrightarrow X_p$ is an Hilbert−Schmidt operator.*

Remark 4.1.1. *Nuclear spaces have many of the good properties of the finite-dimensional Euclidean spaces $\mathbb{R}^d$. For example, a subset of a nuclear space is compact if and only if is bounded and closed. Moreover, spaces whose elements are '**smooth**' in some sense tend to be nuclear spaces.*

In the following example, we see how nuclear spaces could be constructed naturally starting from an Hilbert space and an operator (see [11]).

Example 4.1.1. *Let H be an Hilbert space and A an operator defined on it. Suppose that there exists an orthonormal bases $\{h_n, n = 1, 2, ...\}$ satisfying the following properties:*

1. *They are eigenvectors of A; i.e. for any $n > 0 : Ah_n = \lambda_n h_n, \lambda_n \in \mathbb{R}$.*

2. *$\{\lambda_n\}_{n>0}$ is a non−decreasing sequence such that:$1 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.*

3. *There exists a positive integer a such that $\sum_{n=1}^{\infty} \lambda_n^{-a} < \infty$.*

For any non−negative rational number $p \in \mathbb{Q}_+$, we define a sequence of norms $\{\|\cdot\|_p, p \in \mathbb{Q}_+\}$ such that: $\|\xi\|_p = \|A^p \xi\|, \quad \xi \in H$. That is,

$$\|\xi\|_p = \left(\sum_{n=1}^{\infty} \lambda_n^{2p} (\xi, h_n)^2 \right)^{1/2}, \tag{4.2}$$

where $(\cdot, \cdot)$ indicates the H-inner product.

Remark 4.1.2. *For any $p \in \mathbb{Q}_+$, the norm $\|\cdot\|_p$ is an Hilbert norm. Indeed, it comes from the scalar product:*

$$(\xi, \eta) = \sum_{n=1}^{\infty} \lambda_n^{2p} (\xi, h_n)(\eta, h_n). \tag{4.3}$$

For any $p \in \mathbb{Q}_+$ we define: $X_p = \{\xi \in H, \|\xi\|_p < \infty\}$. In view of the above remark, X_p is an Hilbert space. Moreover, it is easy to see that for any $p \geq q \geq 0$:

$$X_p \subset X_q. \tag{4.4}$$

We have the following proposition

Proposition 4.1.1. *For any $p \in \mathbb{Q}_+$, the inclusion map $X_{p+a/2} \hookrightarrow X_p$ is an Hilbert–Schmidt operator.*

Proof: We set $h_n^p = \frac{1}{\lambda_n^p} h_n$. The collection $h_n^p, n = 1, 2, ...$ is an orthonormal bases of X_p . In fact, for any positive integers n and m:

$$(h_n^p, h_m^p)_p = \sum_{k=1}^{\infty} \lambda_k^p (h_n^p, h_k)(h_m^p, h_k) = \sum_{k=1}^{\infty} \frac{\lambda_k^{2p}}{\lambda_n^p \lambda_m^p} \delta_{nk}\delta_{mk} = \delta_{nm}.$$

For each $\xi \in X_{p+a/2}$, we indicate with $i(\xi) = \xi \in X_p$ the inclusion map. Therefore, for any n > 0:

$$i(h_n^{p+a/2}) = h_n^{p+a/2} = \frac{1}{\lambda_n^{p+a/2}} \lambda_n^p h_n^p = \lambda^{-a/2} h_n^p,$$

and thus by hypothesis

$$\sum_{n=1}^{\infty} \|i(h_n^{p+a/2})\|_p^2 = \sum_{n=1}^{\infty} \lambda_n^{-a} < \infty.$$

∎

Consider the vector space $X = \bigcap_{p \in \mathbb{Q}_+} X_p$. In view of the above proposition, X along with the family of Hilbert norms $\{\| \cdot \|_p, p \in \mathbb{Q}_+\}$ is a nuclear space.

Definition 4.1.3. *A continuous map $\Phi : X \longrightarrow \mathbb{C}$ is called a* ***characteristic functional*** *on X if it is:*

1. *Normalized:* $\Phi(0) = 1$,
2. *Positive defined:* $\sum_{i,j=1}^m \overline{c_i} \Phi(\xi_i - \xi_j) c_j \geq 0, \ m \in \mathbb{Z}, \ \{c_i\}_{i=1,\dots,m} \in \mathbb{C}, \ \{\xi_i\}_{i=1,\dots,m} \in X$

Example 4.1.2. *For instance, consider the so called generating functional of a probability measure μ defined on $\mathbb{R}^n$, that is the Fourier transform:*

$$\chi_\mu(\xi) = \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} d\mu(x), \quad \xi \in \mathbb{R}^n \tag{4.5}$$

where (ξ, x) indicates the Euclidean scalar product. Then, χ_μ is a characteristic functional. In fact, it is continuous and normalized. Moreover, if we define

$$f(x) = \sum_{i=1}^m c_i e^{i\langle \xi, x \rangle},$$

then we have:

$$\sum_{i,j=1}^m \bar{c}_i \chi(\xi_i - \xi_j) c_j = \int_{\mathbb{R}^n} |f(x)|^2 d\mu(x) = \|f\|_\mu^2 \geq 0,$$

where $\| \cdot \|_\mu$ is the $L^2(\mathbb{R}^n, \mu)$ norm.

The finite dimensional Bochner theorem states that the converse is also true

Theorem 4.1.1. *(**Bochner's theorem**[20]) For any characteristic functional Φ on $\mathbb{R}^n$ there exists a **unique** probability measure μ on $\mathbb{R}^n$ such that Φ is its generating functional. Namely,*

$$\Phi(\xi) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} d\mu(x), \quad \xi \in \mathbb{R}^n.$$

In the characterization of typical configurations of measures on infinite-dimensional spaces, the so-called Minlos theorem plays a very important role. This theorem is an infinite-dimensional generalization of the Bochner theorem:

Theorem 4.1.2. *(**Minlos theorem** [20]) Let X be a nuclear space. For any characteristic functional Φ defined on X there exists a unique probability measure μ defined on the measurable space $(X', \mathcal{B})$, where $\mathcal{B}$ is regarded as the Borel σ-algebra generated by the weak topology on X', such that:*

$$\int_{X'} e^{i\langle w,\xi\rangle} d\mu(w) = \Phi(\xi), \quad \xi \in X. \tag{4.6}$$

Characteristic functional on Hilbert spaces can be defined starting from completely monotonic functions. In fact, we have the following proposition:

Proposition 4.1.2. *[20] Let F be a completely monotonic function defined on the positive real line.Therefore, there exists a unique characteristic functional , defined on a real separable Hilbert space H, such that:*

$$\Phi(\xi) = F(\|\xi\|^2), \xi \in H.$$

Remark 4.1.3. *This is obvious because completely monotonic functions are associated to non-negative measure defined on the positive real line (see [7]). The converse is also true (see [31],[32]).*

4.2 White noise

Consider the Schwartz space $\mathcal{S}(\mathbb{R})$. Equip $\mathcal{S}(\mathbb{R})$ with the usual scalar product

$$(\xi, \eta) = \int_{\mathbb{R}} \xi(t)\eta(t)dt, \quad \xi, \eta \in \mathcal{S}(\mathbb{R}). \tag{4.7}$$

We indicate the completion of $\mathcal{S}(\mathbb{R})$ with respect to Equation (4.7) with $\mathcal{S}_0(\mathbb{R}) = L^2(\mathbb{R})$. We consider the orthonormal system $\{h_n\}_{n\geq 0}$ of the Hermite functions

$$h_n(x) = \frac{1}{\sqrt{(2^n n! \sqrt{\pi})}} H_n(x) e^{-x^2/2}, \tag{4.8}$$

where $H_n(x) = (-1)^n e^{x^2} (d/dx)^n e^{-x^2}$ are the Hermite polynomials of degree n. Let A be the 'harmonic oscillator' operator:

$$A = -\frac{d^2}{dx^2} + x^2 + 1; \tag{4.9}$$

A is densely defined on $S_0(\mathbb{R})$ and the Hermite functions are eigenfunctions of A :

$$Ah_n = \lambda_n h_n = (2n+2)h_n, \quad n = 0, 1, \ldots.$$

We observe that $1 \leq \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n$ and $\sum_n \lambda_n^{-2} < \infty$. We are in the condition of Example 4.1.1. Therefore, for any non-negative integer p, we can define:

$$\|\xi\|_p = \|A^p \xi\| = \left(\sum_{n=0}^{\infty} (2n+2)^{2p} (\xi, h_n)^2 \right)^{1/2},$$

where $\|\cdot\|$ indicates the $L^2(\mathbb{R})$ norm. The Schwartz space $\mathcal{S}(\mathbb{R})$ could be then reconstructed as the projective limit of the Hilbert spaces $\mathcal{S}_p(\mathbb{R}) = \{\xi \in L^2(\mathbb{R}); \quad \|\xi\|_p < \infty\}$. That is,

$$\mathcal{S}(\mathbb{R}) = \bigcap_{p\geq 0} \mathcal{S}_p(\mathbb{R}). \tag{4.10}$$

Therefore, the topological Schwartz space, with the topology defined by the $\|\cdot\|_p$ norms, is a nuclear space. Since $\mathcal{S}(\mathbb{R})$ is a nuclear space, we can apply the Minlos theorem in order to define probability measures on its dual space $\mathcal{S}'(\mathbb{R})$.

Consider the positive function $F(t) = e^{-t}, t \geq 0$. It is obvious that F is a completely monotonic function. Therefore, the functional $\Phi(\xi) = F(\|\xi\|^2), \xi \in L^2(\mathbb{R})$, defines a characteristic functional on $\mathcal{S}(\mathbb{R})$. By Minlos theorem, there exists a unique probability measure μ, defined on $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$, such that:

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle w,\xi\rangle} d\mu(w) = e^{-\|\xi\|^2}, \quad \xi \in \mathcal{S}(\mathbb{R}). \tag{4.11}$$

The probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ is called **white noise space** and the measure μ is called **white noise measure**, or **standard Gaussian measure**, on $\mathcal{S}'(\mathbb{R})$.

Consider the generalized stochastic process X, defined on the white noise space, such that for each test function $\varphi \in \mathcal{S}(\mathbb{R})$:

$$X(\varphi)(\cdot) = \langle \cdot, \varphi \rangle. \tag{4.12}$$

Clearly, for any $\varphi \in \mathcal{S}(\mathbb{R}), X(\varphi)$ is a Gaussian random variable with zero mean and variance $\mathbb{E}(X(\varphi)^2) = 2\|\varphi\|^2$. Moreover, for any $\varphi, \phi \in \mathcal{S}(\mathbb{R})$:

$$\mathbb{E}(X(\varphi)X(\phi)) = 2(\varphi, \phi), \tag{4.13}$$

where $\mathbb{E}(w)$ indicates the expectation value of the random variable w. We refer to the generalized process X as the canonical noise of $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$.

Remark 4.2.1. *In view of the above properties the process X is a white noise [11], and this also motivates the name '**white noise space**' for the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$*

We have the following:

Proposition 4.2.1. *For any $h \in L^2(\mathbb{R})$, $X(h)$ is defined almost everywhere on $\mathcal{S}'(\mathbb{R})$. Moreover, it is Gaussian with zero mean and variance $2\|h\|^2$.*

Proof: We indicate with $L^2 = L^2(\mathcal{S}'(\mathbb{R}), \mu)$. Clearly, for any $\xi \in \mathcal{S}(\mathbb{R})$, we have that $X(\xi) \in L^2$ and

$$\|X(\xi)\|^2_{L^2} = \mathbb{E}(X(\xi)^2) = 2\|\xi\|^2_{L^2}. \tag{4.14}$$

For each $h \in L^2(\mathbb{R})$, there exists a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of $\mathcal{S}(\mathbb{R})$-elements which converges to h in the $L^2(\mathbb{R})$-norm. Therefore, from Equation 4.14, the sequence $\{X(\xi_n)\}_{n\in\mathbb{N}}$ is Cauchy in L^2 and converges to a limit function $X(h)$, defined on $\mathcal{S}'(\mathbb{R})$. ■

The latter proposition states that for every sequence $\{f_t\}_{t\in\mathbb{R}}$ of $L^2(\mathbb{R})-$functions, depending continuously on a real parameter $t \in \mathbb{R}$, there exists a Gaussian stochastic process

$$\{Y(t)\}_{t\in\mathbb{R}} = \{X(f_t)\}_{t\in\mathbb{R}}, \tag{4.15}$$

defined on the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$, which has zero mean, variance $\mathbb{E}(Y_t)^2 = 2\|f_t\|^2$ and covariance $\mathbb{E}(Y(t_1)Y(t_2)) = 2(f_{t_1}, f_{t_2})$.

Remark 4.2.2. *Observe that if $W(x)$, $x \in \mathbb{R}$, is a Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the functional*

$$X(\varphi) = \int \varphi(x)dW(x), \quad \varphi \in L^2(\mathbb{R}) \tag{4.16}$$

is a white noise on the space $(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, if we indicate with $\mathbf{1}_{[0,t)}(x), t \geq 0$, the indicator function of the interval $[0, t)$, the process

$$X(\mathbf{1}_{[0,t)}) = \int_0^t dW(x) = W(t), \quad t \geq 0, \tag{4.17}$$

is a one-sided Brownian motion.

Example 4.2.1. ***(Brownian motion)*** *Let X be a white noise defined canonically on the white noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$. Looking at Equation 4.17, it is natural to state that the stochastic process*

$$\{B(t)\}_{t\geq 0} = \{X(\mathbf{1}_{[0,t)})\}_{t\geq 0} \tag{4.18}$$

is a standard Brownian motion. Indeed, the process $\{X(\mathbf{1}_{[0,t)})\}_{t\geq 0}$ is Gaussian with covariance:

$$\mathbb{E}[X(\mathbf{1}_{[0,t)})X(\mathbf{1}_{[0,s)})] = 2(\mathbf{1}_{[0,t)}\mathbf{1}_{[0,s)}) = 2\min(t,s), \quad t,s \geq 0.$$

Example 4.2.2. ***(Fractional Brownian motion)*** *The stochastic process:*

$$\{B_{\alpha/2}(t)\}_{t\geq 0} = \{X(f_{\alpha,t})\}_{t\geq 0}, \qquad 0 < \alpha < 2, \tag{4.19}$$

where

$$f_{\alpha,t}(x) = \frac{1}{C_1(\alpha)}((t-x)_+^{\alpha-1/2} - (-x)_+^{\alpha-1/2}), \qquad x_+ = \max(x,0), \tag{4.20}$$

and

$$C_1(\alpha) = \frac{\Gamma(\alpha+1/2)}{(\Gamma(\alpha+1)\sin(\pi\alpha/2))^{1/2}}, \tag{4.21}$$

is a 'standard' fractional Brownian motion of order $H = \alpha/2$ (see[33]).

4.3 Grey noises

We have seen that the white noise is a generalized stochastic process X defined canonically on the white noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$, with space of test functions $L^2(\mathbb{R})$. We have remarked that the white noise could also be defined starting from stochastic integrals with respect to the Brownian motion. In this case the space of test function turns out to be the space of integrands of the stochastic integral. Then, the Brownian motion $B(t)$ could be obtained from the white noise by setting $B(t) = X(\mathbf{1}_{[0,t)})$ [8]. We generalize the previous construction in order to define a general class of **H-sssi** processes that includes Brownian motion, fractional Brownian motion and more general processes.

Consider a one-sided fractional Brownian motion $\{B_{\alpha/2}(t)\}_{t\geq 0}$ with self-similarity parameter $H = \alpha/2$ and $0 < \alpha < 2$, defined on a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The fractional Brownian motion has a spectral representation :

$$B_{\alpha/2} = \sqrt{C(\alpha)} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \frac{e^{itx}-1}{ix} |x|^{\frac{1-\alpha}{2}} d\widetilde{B}(x), \quad t \geq 0, \tag{4.22}$$

where $d\widetilde{B}(x)$ is a complex Gaussian measure such that $d\widetilde{B}(x) = dB_1(x) + idB_2(x)$ with $dB_1(x) = dB_1(-x)$, $dB_2(x) = -dB_2(-x)$ and where B_1 and B_2 are independent Brownian motion. Moreover,

$$C(\alpha) = \Gamma(\alpha+1)\sin\frac{\pi\alpha}{2}. \tag{4.23}$$

We observe that

$$\frac{1}{\sqrt{2\pi}}\frac{e^{itx}-1}{ix} = \widetilde{\mathbf{1}}_{[0,t)}(x), \tag{4.24}$$

where we have indicated with $\widetilde{f}(x)$ the Fourier transform of the function f evaluated on $x \in \mathbb{R}$:

$$\widetilde{f}(x) = \mathcal{F}(f)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ixy} f(y)dy. \tag{4.25}$$

In view of Equation 4.24 we have

$$B_{\alpha/2} = \sqrt{C(\alpha)}\int_{\mathbb{R}} \widetilde{1}_{[0,t)}(x)|x|^{\frac{1-\alpha}{2}} d\widetilde{B}(x). \tag{4.26}$$

Therefore, if one defines a generalized stochastic process X such that for a suitable choice of a test function φ

$$X_\alpha(\varphi) = \sqrt{C(\alpha)}\int_{\mathbb{R}} \widetilde{\varphi}(x)|x|^{\frac{1-\alpha}{2}} d\widetilde{B}(x), \tag{4.27}$$

one can write

$$B_{\alpha/2} = X_\alpha(\mathbf{1}_{[0,t)}), \quad t \geq 0. \tag{4.28}$$

Remark 4.3.1. *The space of test function can be the space*

$$\widetilde{\Lambda}_\alpha = \{f \in L^2(\mathbb{R}); \quad \|f\|_\alpha^2 = C(\alpha)\int_{\mathbb{R}} |\widetilde{f}(x)|^2 |x|^{1-\alpha} dx < \infty\}, \tag{4.29}$$

which coincides with a space of deterministic integrands for fractional Brownian motion (see [25],[27]).

Consider now the Schwartz space $\mathcal{S}(\mathbb{R})$ equipped with the scalar product:

$$(\xi,\eta)_\alpha = C(\alpha)\int_{\mathbb{R}} \overline{\widetilde{\xi}(x)}\widetilde{\eta}(x)|x|^{1-\alpha}dx, \quad \xi,\eta \in \mathcal{S}(\mathbb{R}), \quad 0<\alpha<2, \tag{4.30}$$

where $C(\alpha)$ is given by Equation (4.23). This scalar product generate the α-norm in Equation (4.29). We indicate with $\mathcal{S}_0^{(\alpha)}(\mathbb{R})$ the completion of $\mathcal{S}(\mathbb{R})$ with respect to Equation (4.30).

Remark 4.3.2. *If we set $\alpha = 1$ in Equation (4.30), we have $C(1) = 1$ and*

$$(\xi,\eta)_1 = \int_{\mathbb{R}} \overline{\widetilde{\xi}(x)}\widetilde{\eta}(x)dx = \int_{\mathbb{R}} \xi(y)\eta(y)dy, \tag{4.31}$$

so that, we recover the $L^2(\mathbb{R})$-inner product. Moreover, $\mathcal{S}_0^{(1)}(\mathbb{R}) = \mathcal{S}_0(\mathbb{R}) = L^2(\mathbb{R})$.

Starting from the Hilbert space $(\mathcal{S}_0^{(\alpha)}(\mathbb{R}), \|\cdot\|_\alpha)$, it is possible to reproduce the construction of Example (4.1.1). Then, the space $\mathcal{S}(\mathbb{R})$ turns out to be a nuclear space with respect to the topology generated by the α-norm $\|\cdot\|_\alpha$ and an operator $A^{(\alpha)}$.

We need to find an orthonormal bases for the space $\mathcal{S}_0^{(\alpha)}(\mathbb{R})$ For this purpose, we introduce the following definition:

Definition 4.3.1. *(**Generalized Laguerre polynomials**) The generalized Laguerre polynomials are defined, for any non-negative integer n, by:*

$$L_n^\gamma(x) = \frac{x^{-\gamma}e^x}{\Gamma(n+1)}\frac{d^n}{dx^n}(e^{-x}x^{n+\gamma}), \quad \gamma > -1, \quad x \geq 0, \tag{4.32}$$

They are orthogonal with respect to the weighting function $x^\gamma e^{-x}$:

$$\int_0^\infty x^\gamma e^{-x} L_n^\gamma(x) L_m^\gamma(x) = \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)}\delta_{nm}. \tag{4.33}$$

Proposition 4.3.1. *[20] The generalized Laguerre polynomials satisfies the Laguerre equation:*

$$\left(x\frac{d^2}{dx^2} + (\gamma+1-x)\frac{d}{dx}\right) L_n^\gamma(x) = -nL_n^\gamma(x). \tag{4.34}$$

Define now a set of $\mathcal{S}_0^{(\alpha)}(\mathbb{R})$ functions $\{h_n^\alpha\}_{n\in\mathbb{Z}_+}$ in terms of their Fourier transform by:

$$\begin{cases} \widetilde{h}_{2n}^\alpha(x) = a_{n,\alpha} e^{-x^2/2} L_n^{-\alpha/2}(x^2), & n \in \mathbb{Z}_+; \\ \\ \widetilde{h}_{2n+1}^\alpha(x) = b_{n,\alpha} e^{-x^2/2} x L_n^{1-\alpha/2}(x^2), & n \in \mathbb{Z}_+. \end{cases}$$

One has:

Proposition 4.3.2. *The set of functions $\{h_n^\alpha\}_{n\in\mathbb{Z}_+}$ is an orthonormal basis for $\mathcal{S}_0^{(\alpha)}(\mathbb{R})$ with the choice:*

$$a_{\alpha,n} = \left(\frac{\Gamma(n+1)}{C(\alpha)\Gamma(n+1-\alpha/2)}\right)^{1/2}, \tag{4.35}$$

$$b_{\alpha,n} = \left(\frac{\Gamma(n+1)}{C(\alpha)\Gamma(n+2-\alpha/2)}\right)^{1/2}. \tag{4.36}$$

Proof: See [20]

Then, using Equation (4.34), one can show that the orthonormal bases $\{h_n^\alpha\}_{n\in\mathbb{Z}_+}$ is a set of eigenfunction of an operator $A^{(\alpha)}$, defined on $\mathcal{S}_0^{(\alpha)}(\mathbb{R})$, with eigenvalues $\lambda_n^{(\alpha)} = 2n+2-\alpha+1$.

Remark 4.3.3. *We recall the well-known relationships between Laguerre and Hermite polynomials:*

$$\begin{cases} H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-1/2}(x^2) \\ \\ H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{1/2}(x^2). \end{cases}$$

In view of the above relations, when $\alpha = 1$ the orthonormal bases $\{h_n^\alpha\}_{n\in\mathbb{Z}_+}$ reduces to the Hermite bases of $L^2(\mathbb{R})$ (Equation 4.8), which is preserved under Fourier transformation.

4.3.1 Generalised grey noise space

By Proposition (4.1.2) starting from a completely monotonic function F, we can define characteristic functionals on $\mathcal{S}(\mathbb{R})$ by setting $\Phi(\xi) = F(\|\xi\|_\alpha^2)$. Then, we can use **Minlos theorem** in order to define probability measures on $\mathcal{S}'(\mathbb{R})$. We consider the **Mittag-Leffler function** of $\beta > 0$:

$$E_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + 1)}, \quad x \in \mathbb{R}. \tag{4.37}$$

It is known that the function $F_\beta(t) = E_\beta(-t), t \geq 0$, is a completely monotonic function if $0 \leq \beta \leq 1$ [18]. For example, if $\beta = 1$ we recover $F_1(t) = e^{-t}$. Therefore, the functional $\Phi_{\alpha,\beta}(\xi) = F_\beta(\|\xi\|_\alpha^2), \xi \in \mathcal{S}_0^{(\alpha)}(\mathbb{R})$ defines a characteristic functional on $\mathcal{S}(\mathbb{R})$. By Minlos theorem, there exists a unique probability measure $\mu_{\alpha,\beta}$, defined on $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$, such that:

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle w,\xi\rangle} d\mu_{\alpha,\beta}(w) = F_\beta(\|\xi\|_\alpha^2), \quad \xi \in \mathcal{S}(\mathbb{R}). \tag{4.38}$$

When $\alpha = \beta$ and $0 < \beta \leq 1$, the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_{\beta,\beta})$ is called **grey noise space** and the measure $\mu_{\beta,\beta}$ is called **grey noise measure**$(see$[31], [32]).
In this master theses, we focus on the more general case $0 < \alpha < 2$ and we call the space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_{\alpha,\beta})$ **generalized** grey noise space and $\mu_{\alpha,\beta}$ **generalized** grey noise measure.

Definition 4.3.2. ***The generalized stochastic process*** *$X_{\alpha,\beta}$, defined canonically on the 'generalized' grey noise space$(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_{\alpha,\beta})$, is called generalized grey noise. Therefore, for each test function $\varphi \in \mathcal{S}(\mathbb{R})$:*

$$X_{\alpha,\beta}(\varphi)(\cdot) = \langle \cdot, \varphi \rangle. \tag{4.39}$$

Remark 4.3.4. *By the definition of 'generalized' grey noise measure Equation (4.38), for any $\varphi \in \mathcal{S}(\mathbb{R})$, we have:*

$$\mathbb{E}(e^{iyX_{\alpha,\beta}(\varphi)}) = E_\beta(-y^2\|\varphi\|_\alpha^2), \quad y \in \mathbb{R}. \tag{4.40}$$

Using Equations (4.40) and (4.37), it easy to show that the generalized grey noise has moments of any order:

$$\begin{cases} \mathbb{E}(X_{\alpha,\beta}(\xi)^{2n+1}) = 0, \\ \\ \mathbb{E}(X_{\alpha,\beta}(\xi)^{2n}) = \frac{2n!}{\Gamma(\beta n+1)}\|\xi\|_\alpha^{2n}, \end{cases}$$

for any integer $n \geq 0$ and $\xi \in \mathcal{S}(\mathbb{R})$. It is possible to extend the space of test functions to the whole. In fact, for any $\xi \in \mathcal{S}(\mathbb{R})$ we have $X_{\alpha,\beta}(\xi) \in L^2 = L^2(\mathcal{S}'(\mathbb{R}), \mu_{\alpha,\beta})$. Thus, for any $h \in \mathcal{S}_0^{(\alpha)}(\mathbb{R})$, the function $X_{\alpha,\beta}(h)$ is defined as a limit of a sequence $X_{\alpha,\beta}(\xi_n)$, where $\{\xi_n\}$ belong to $\mathcal{S}(\mathbb{R})$. Therefore, we have the following:

Proposition 4.3.3. *[20] For any $h \in \mathcal{S}_0^{(\alpha)}(\mathbb{R}), X_{\alpha,\beta}(h)$ is defined almost everywhere on $\mathcal{S}'(\mathbb{R})$ and belongs to L^2.*

Summarizing, the "generalized" grey noise is defined canonically on the grey noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_{\alpha,\beta})$ with the following properties:

1. For any $\mathcal{S}_0^{(\alpha)}(\mathbb{R}), X_{\alpha,\beta}(h)$ is well defined and belong to L^2.

2. $\mathbb{E}(e^{iyX_{\alpha,\beta}(h)}) = E_\beta(-y^2\|h\|_\alpha^2)$ for any $y \in \mathbb{R}$.

3. $\mathbb{E}(X_{\alpha,\beta}(h)) = 0$ and $\mathbb{E}(X_{\alpha,\beta}(h)^2) = (2/\Gamma(\beta+1))\|h\|_\alpha^2$.

4. For any h and g that belong to $\mathcal{S}_0^{(\alpha)}(\mathbb{R})$, one has:

$$\mathbb{E}(X_{\alpha,\beta}(h)X_{\alpha,\beta}(g)) = \frac{1}{\Gamma(\beta+1)}[(h,g)_\alpha + \overline{(h,g)}_\alpha]. \tag{4.41}$$

4.4 Generalesed grey brownian motion

If we put $\beta = 1$, the measure $\mu_{\alpha,1} := \mu_\alpha$ is a Gaussian measure and $X_{\alpha,1} := X_\alpha$ is a Gaussian noise. In fact, for any $h \in \mathcal{S}_0^{(\alpha)}(\mathbb{R})$, the random variable $X_\alpha(h)$ is Gaussian with zero mean and variance $\mathbb{E}(X_\alpha(h)^2) = 2\|h\|_\alpha^2$ (see Equation 4.40). Moreover, for any sequence $\{f_t\}_{t\in\mathbb{R}}$ of $\mathcal{S}_0^{(\alpha)}(\mathbb{R})$-functions,depending continuously on a real parameter $t \in \mathbb{R}$ the stochastic process $Y(t) = X_\alpha(f_t)$ is Gaussian with autocovariance given by Equation (4.41)

$$\mathbb{E}(Y(t)Y(s)) = \mathbb{E}(X_\alpha(f_t)X_\alpha(f_s)) = (f_t, f_s)_\alpha + \overline{(f_t, f_s)}_\alpha. \tag{4.42}$$

When $\alpha = 1$, X_α reduces to a "standard" white noise (see Remark 4.3.2 and Remark 4.2).

Example 4.4.1. *(**Fractional Brownian motion**) For any $t \geq 0$ the function $\mathbf{1}_{[0,t)}$ belongs to $\mathcal{S}_0^{(\alpha)}(\mathbb{R})$. In fact, it is easy to show that $\|\mathbf{1}_{[0,t)}\|_\alpha^2 < \infty$ when $0 < \alpha < 2$ and*

$$\|\mathbf{1}_{[0,t)}\|_\alpha^2 = \frac{C(\alpha)}{2\pi}\int_{\mathbb{R}} \frac{2}{|x|^{1+\alpha}}(1-\cos tx)dx = t^\alpha. \tag{4.43}$$

Therefore, we can define the process

$$B_{\alpha/2}(t) = X_\alpha(\mathbf{1}_{[0,t)}), \quad t \geq 0. \tag{4.44}$$

The process $B_{\alpha/2}(t)$ is a 'standard' fractional Brownian motion with parameter $H = \alpha/2$. Indeed, it is Gaussian with variance $\mathbb{E}(B_{\alpha/2}(t)^2) = 2\|\mathbf{1}_{[0,t)}\|_\alpha^2 = 2t^\alpha$ and autocovariance:

$$\begin{aligned}
\mathbb{E}(B_{\alpha/2}(t)B_{\alpha/2}(s)) &= (\mathbf{1}_{[0,t)}, \mathbf{1}_{[0,s)})_\alpha + \overline{(\mathbf{1}_{[0,t)}, \mathbf{1}_{[0,s)})}_\alpha \\
&= \frac{C(\alpha)}{2\pi}\int_{\mathbb{R}} \frac{2}{|x|^{1+\alpha}}(1-\cos tx + 1 - \cos sx - 1 + \cos(t-s)x)dx \\
&= t^\alpha + s^\alpha - |t-s|^\alpha = \gamma_\alpha(t,s), \quad t,s \geq 0.
\end{aligned}$$

In view of the above example, X_α could be regarded as a fractional Gaussian noise defined on the space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_\alpha)$.

Example 4.4.2. *(**Deconvolution of Brownian motion**) The stochastic process*

$$\{B(t)\}_{t\geq 0} = \{X_\alpha(g_{\alpha,t})\}_{t\geq 0}, \tag{4.45}$$

where, for each $t \geq 0$, the function $g_{\alpha,t}$ is defined by

$$\tilde{g}_{\alpha,t}(x) = \frac{1}{\sqrt{C(\alpha)}}\tilde{\mathbf{1}}_{[0,t)}(x)(ix)^{\alpha-1/2}, \tag{4.46}$$

is a 'standard' Brownian motion. Indeed, it is Gaussian, with zero mean, variance

$$\mathbb{E}(B(t)^2) = 2\int_{\mathbb{R}} |X|^{1-\alpha}|\tilde{\mathit{1}}_{[0,t)}(x)|^2|X|^{\alpha-1}dx = 2\int_{\mathbb{R}} |\tilde{\mathit{1}}_{[0,t)}(x)|^2 dx = 2t, \tag{4.47}$$

and autocovariance

$$\mathbb{E}(B(t)B(s)) = \int_{\mathbb{R}} \left(\overline{\tilde{\mathit{1}}_{[0,t)}}(x)\tilde{\mathit{1}}_{[0,s)}(x) + \overline{\tilde{\mathit{1}}_{[0,s)}}(x)\tilde{\mathit{1}}_{[0,s)}(x)\right) dx = 2\min(t,s). \tag{4.48}$$

Remark 4.4.1. *The representation of Brownian motion in terms of the fractional Gaussian noise Equation (4.45) corresponds to a particular case of the so-called deconvolution formula, which expresses the Brownian motion as a stochastic integral with respect to a fractional Brownian motion of order $H = \alpha/2$ (see[26]). More generally, we can represent a fractional Brownian motion $B_{\gamma/2}(t)$ of order $H = \gamma/2, 0 < \gamma < 2$ in terms of a fractional Gaussian noise of order α, which corresponds to a representation of $B_{\gamma/2}$ in terms of a stochastic integral of a fractional Brownian motion $B_{\alpha/2}$ of order $H = \alpha/2,\ 0 < \alpha < 2$ (see Example 4.4.3).below*

Example 4.4.3. ***(Deconvolution of fractional Brownian motion)*** *The stochastic process,*

$$\{B_{\gamma/2}(t)\}_{t\geq 0} = \{X_\alpha(g_{\alpha,\gamma,t})\}_{t\geq 0}, \tag{4.49}$$

where

$$\tilde{g}_{\alpha,\gamma,t}(x) = \sqrt{\frac{C(\gamma)}{C(\alpha)}}\,\tilde{\mathit{1}}_{[0,t)}(x)(ix)^{(\alpha-\gamma/2)}, \quad 0 < \gamma < 2, \tag{4.50}$$

is a 'standard' fractional Brownian motion of order $H = \gamma/2$.

4.4.1 Generalized grey Brownian motion definition

We now consider the general case $0 < \alpha < 2,\ 0 < \beta \leq 1$.

Definition 4.4.1. *The stochastic process*

$$\{B_{\alpha,\beta}(t)\}_{t\geq 0} = \{X_{\alpha,\beta}(\mathit{1}_{[0,t)})\}_{t\geq 0}. \tag{4.51}$$

*is called **'generalized' (standard) grey Brownian motion**.*

The generalized grey Brownian motion $B_{\alpha,\beta}$ has many properties that come directly from the grey noise properties and Equation (4.43):

1. $B_{\alpha,\beta}(0) = 0$ a.s. Moreover, for each $t \geq 0$, $\mathbb{E}(B_{\alpha,\beta}(t)) = 0$ and
$$\mathbb{E}(B_{\alpha,\beta}(t)^2) = \frac{2}{\Gamma(\beta+1)} t^{\alpha}. \tag{4.52}$$

2. The autocovariance function is:
$$\mathbb{E}(B_{\alpha,\beta}(t)B_{\alpha,\beta}(s)) = \gamma_{\alpha,\beta}(t,s) = \frac{1}{\Gamma(\beta+1)}(t^{\alpha} + s^{\alpha} - |t-s|^{\alpha}). \tag{4.53}$$

3. For any $t, s \geq 0$, the characteristic function of the increments is:
$$\mathbb{E}(e^{iy(B_{\alpha,\beta}(t) - B_{\alpha,\beta}(s))}) = E_{\beta}(-y^2|t-s|^{\alpha}), \quad y \in \mathbb{R}. \tag{4.54}$$

The third property follows from the linearity of the grey noise definition. In fact, suppose $0 \leq s < t$, we have $y(B_{\alpha,\beta}(t) - B_{\alpha,\beta}(s)) = yX_{\alpha,\beta}(\mathbf{1}_{[0,t)} - \mathbf{1}_{[0,s)}) = X_{\alpha,\beta}(y\mathbf{1}_{[s,t)})$, and $\|y\mathbf{1}_{[s,t)}\|_{\alpha}^2 = y^2(t-s)^{\alpha}$. All these properties are enclosed in the following:

Proposition 4.4.1. *For any $0 < \alpha < 2$ and $0 < \beta \leq 1$, the process $B_{\alpha,\beta}(t), t \geq 0$, is a self-similar with stationary increments process (**H-sssi**), with $H = \alpha/2$.*

Proof: This result is actually a consequence of the linearity of the noise definition. Given a sequence of real numbers $\{\theta_j\}_{j=1,\ldots,n}$ we have to show that for any $0 < t_1 < t_2 < \cdots < t_n$ and $a > 0$:

$$\mathbb{E}\left(\exp\left(i\sum_j \theta_j B_{\alpha,\beta}(at_j)\right)\right) = \mathbb{E}\left(\exp\left(i\sum \theta_j a^{\alpha/2} B_{\alpha,\beta}(t_j)\right)\right).$$

The linearity of the grey noise definition allows to write the above equality as

$$\mathbb{E}\left[\exp\left(iX_{\alpha,\beta}\left(\sum_j \theta_j \mathbf{1}_{[0,at_j)}\right)\right)\right] = \mathbb{E}\left[\exp\left(iX_{\alpha,\beta}\left(a^{\alpha/2}\sum_j \theta_j \mathbf{1}_{[0,t_j)}\right)\right)\right].$$

Using Equation 4.40 we have

$$F_{\beta}\left(\|\sum_j \theta_j \mathbf{1}_{[0,at_j)}\|_{\alpha}^2\right) = F_{\beta}\left(\|a^{\alpha/2}\sum_j \theta_j \mathbf{1}_{[0,t_j)}\|_{\alpha}^2\right)$$

which, because of the complete monotonicity, reduces to

$$\|\sum_j \theta_j \mathbf{1}_{[0,at_j)}\|_{\alpha}^2 = a^{\alpha}\|\sum_j \theta_j \mathbf{1}_{[0,t_j)}\|_{\alpha}^2.$$

In view of the definition Equations 4.30 and 4.24, the above equality is checked after a simple change of variable in the integration. In the same way we can prove the stationarity of the increments.We have to show that for any $h \in \mathbb{R}$:

$$\mathbb{E}\left[\exp\left(i\sum_j \theta_j(B_{\alpha,\beta}(t_j+h)-B_{\alpha,\beta}(h))\right)\right] = \mathbb{E}\left[\exp\left(i\sum_j \theta_j(B_{\alpha,\beta}(t_j))\right)\right].$$

We use the linearity property to write

$$\mathbb{E}\left[\exp\left(iX_{\alpha,\beta}\left(\sum_j \theta_j \mathbf{1}_{[h,t_j+h)}\right)\right)\right] = \mathbb{E}\left[\exp\left(iX_{\alpha,\beta}\left(\sum_j \theta_j \mathbf{1}_{[0,t_j)}\right)\right)\right]$$

By using the definition and the complete monotonicity, we have

$$\|\sum_j \theta_j \mathbf{1}_{[h,t_j+h)}\|_\alpha^2 = \|\sum_j \theta_j \mathbf{1}_{[0,t_j)}\|_\alpha^2$$

which is true because

$$\tilde{\mathbf{1}}_{[h,t_j+h)}(x) = \frac{1}{\sqrt{2\pi}}\frac{e^{ixh}}{ix}(e^{ixt_j}-1).$$

$\blacksquare$

Remark 4.4.2. *In view of Proposition 4.4.1,* $\{B_{\alpha,\beta}(t)\}$ *forms a class of* ***H-sssi*** *stochastic processes indexed by two parameters* $0<\alpha<2$ *and* $0<\beta\leq 1$*. This class includes fractional Brownian motion* ($\beta=1$)*, grey Brownian motion* ($\alpha=\beta$) *and Brownian motion* ($\alpha=\beta=1$).

In Figure 4.1, we present a diagram that allows us to identify the elements of this class of processes. The long range dependence domain corresponds to the region $1<\alpha<2$. The horizontal line represents the processes with purely random increments, that is, processes that possess uncorrelated increments. The fractional Brownian motion is identified by the vertical line ($\beta=1$). The lower diagonal line represents the grey Brownian motion.

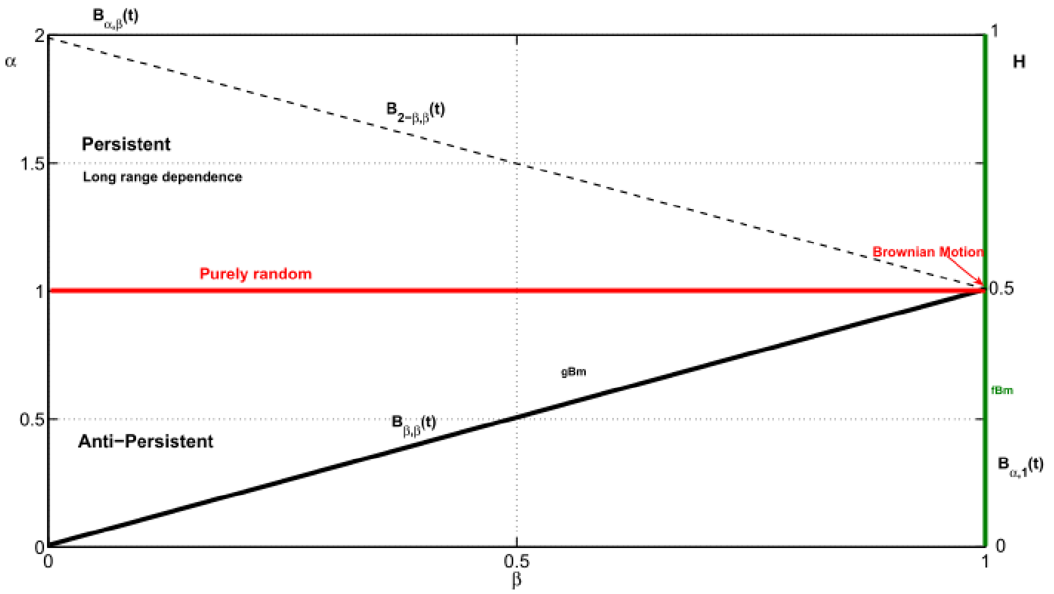

Figure 4.1: Parametric class of generalized grey Brownian motion. The upper diagonal line indicates the 'conjugated' process of grey Brownian motion.

4.4.2 The p-variation of generalized grey Brownian motion

This subsection is devoted to the study of the p-variation of ggBm. The approach taken is inspired from the one used for the fBm.

Proposition 4.4.2. *We have the following limit in probability*

$$\lim_{n\longrightarrow+\infty} n^{p\frac{\alpha}{2}-1}\sum_{j=1}^{n}\left|B_{\alpha,\beta}\left(\frac{j}{n}\right)-B_{\alpha,\beta}\left(\frac{j-1}{n}\right)\right|^{p}=\mathbb{E}(|B_{\alpha,\beta}(1)|^{p}).$$

Proof: See [4]

Proposition 4.4.3. *[4]We have the following limit in probability*

$$V_{p,n}:=\sum_{j=1}^{n}\left|B_{\alpha,\beta}(\frac{j}{n})-B_{\alpha,\beta}(\frac{j-1}{n})\right|^{p}\xrightarrow[n\longrightarrow+\infty]{}\begin{cases}0 & a.s.\quad if\quad p\alpha/2>1\\ \infty & a.s.\quad if\quad p\alpha/2<1\\ \mathbb{E}(|B_{\alpha,\beta}(1)|^{p}) & a.s.\quad if\quad p=2/\alpha.\end{cases}$$

Remark 4.4.3. *The ggBm is not a semimartingale. In addition, $B_{\alpha,\beta}$ cannot be of finite variation on $[0,1]$ and by scaling and stationarity of the increment on any interval.*

Proof: Indeed there is a subsequence such that $V_{p,n}$ converge almost surely to ∞ for $p=1$ and $\alpha\in(0,2)$. If $\alpha\in(1,2)$ we can choose $p\in(2/\alpha,2)$ such that $V_{p,n}$ converge to

0 for some subsequence. This implies that the quadratic variation of $B_{\alpha,\beta}$ is zero. if $\alpha \in (0,1)$ we can choose $p > 2$ such that $2p/\alpha < 1$ and the p-variation of $B_{\alpha,\beta}$ must be infinite. So, in any case $B_{\alpha,\beta}$ can not be a semimartingale. ■

4.5 The ggBm master equation

The following proposition characterizes the marginal density function of the process $\{B_{\alpha,\beta}(t), t \geq 0\}$:

Proposition 4.5.1. *The marginal probability density function $f_{\alpha,\beta}(x,t)$ of the process $\{B_{\alpha,\beta}(t), t \geq 0\}$: is the fundamental solution of the 'stretched' time-fractional diffusion equation:*

$$u(x,t) = u_0(x) + \frac{1}{\Gamma(\beta)} \int_0^t \frac{\alpha}{\beta} s^{\alpha/\beta - 1} (t^{\alpha/\beta} - s^{\alpha/\beta})^{\beta-1} \frac{\partial^2}{\partial x^2} u(x,s) ds, \quad t \geq 0. \tag{4.55}$$

Proof: Equation 4.54 with $(s=0)$ states that $\widetilde{f}_{\alpha,\beta}(y,t) = E_\beta(-y^2 t^\alpha)$. Using Equation (4.37), we can show that the Mittag-Leffler function satisfies

$$\begin{aligned} E_\beta(-y^2(t^{\alpha/\beta})^\beta) &= 1 - \frac{y^2}{\Gamma(\beta)} \int_0^{t^{\alpha/\beta}} (t^{\alpha/\beta} - s')^{\beta-1} E_\beta(-y^2 s'^\beta) ds' \\ &= 1 - \frac{y^2}{\Gamma(\beta)} \int_0^t \frac{\alpha}{\beta} s^{\alpha/\beta-1} (t^{\alpha/\beta} - s^{\alpha/\beta})^{\beta-1} E_\beta(-y^2 s^\alpha) ds, \end{aligned}$$

where we have used the change of variables $s' = s^{\alpha/\beta}$. Thus, $f_{\alpha,\beta}(x,t)$ solves Equation 4.55 with initial condition $u_0(x) = f_{\alpha,\beta}(x,0) = \delta(x)$. ■

We refer to Equation 4.55 as the master equation of the marginal density function of the 'generalized' grey Brownian motion, the fundamental solution of (eq. 4.55) is just:

$$u(x,t) = M_\beta(x, t^{\alpha/\beta}) = \frac{t^{-\alpha/2}}{2} M_\beta/2(|x| t^{-\alpha/2}). \tag{4.56}$$

The M-Wright function M_β(see Appendix) emerges as a natural generalization of the Gaussian distribution.

4.6 Characterization of th ggBm

We have seen that the generalized grey Brownian motion (ggBm), is made up off self-similar with stationary increments processes (Prop. 4.4.1) and depends on two real parameters $\alpha \in (0,2)$ and $\beta \in (0,1]$.

The ggBm is defined through the explicit construction of the underline probability space. However, we are now going to show that it is possible to define it in an unspecified probability space. For this purpose, we write down explicitly all the finite dimensional probability density functions. Moreover, we shall provide different ggBm characterizations.

Proposition 4.6.1. *Let $B_{\alpha,\beta}$ be a ggBm, then for any collection $\{B_{\alpha,\beta}(t_1), ..., B_{\alpha,\beta}(t_n)\}$, the joint probability density function is given by:*

$$f_{\alpha,\beta}(x_1, x_2, ..., x_n; \gamma_{\alpha,\beta}) = \frac{(2\pi)^{-\frac{n-1}{2}}}{\sqrt{2\Gamma(1+b)^n \det \gamma_{\alpha,\beta}}} \int_0^\infty \frac{1}{\tau^{n/2}} M_{1/2}\left(\frac{\zeta}{\tau^{1/2}}\right) M_\beta(\tau) d\tau. \quad (4.57)$$

with:
$\zeta = \left(2\Gamma(1+\beta)^{-1} \sum_{i,j=1}^{n} x_i \gamma^{-1}_{\alpha,\beta}(t_i, t_j) x_j\right)^{1/2}, \gamma_{\alpha,\beta}(t_i, t_j) = \frac{1}{\Gamma(1+\beta)}(t_i^\alpha + t_j^\alpha - |t_i - t_j|^\alpha), i, j = 1, ..., n.$

Proof: See ([20]).

Using the Kolmogorov extension theorem (see Theorem 1.1.1), the above proposition allows us to define the ggBm in an unspecified probability space. In fact, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the following proposition characterizes the ggBm:

Proposition 4.6.2. *[20] Let $X(t), t \geq 0$, be a stochastic process, defined in a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that*

1. *$X(t)$ has covariance matrix indicated by $\gamma_{\alpha,\beta}$ and finite-dimensional distributions defined by (eq. 4.57).*
2. *$\mathbb{E}X^2(t) = \frac{2}{\Gamma(1+\beta)} t^\alpha$ for $0 < \beta \leq 1$ and $0 < \alpha < 2$.*
3. *$X(t)$ has stationary increments,*

then $X(t)$, $t \geq 0$, is a generalized grey Brownian motion.

In fact condition 2) together with condition 3) imply that $\gamma_{\alpha,\beta}$ must be the ggBm autocovariance matrix (eq. 4.53).

Corollary 4.6.1. *[20] Let $X(t)$, $t \geq 0$, be a stochastic process defined in a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $H = \alpha/2$ with $0 < \alpha < 2$ and suppose that $\mathbb{E}X(1)^2 = 2/\Gamma(1+\beta)$.*

The following statements are equivalent:

*i) X is H-**sssi** with finite-dimensional distribution defined by (eq. 4.57);*

ii) X is a generalized grey Brownian motion with scaling exponent $\alpha/2$ and "fractional order" parameter β;

iii) X has zero mean, covariance function $\gamma_{\alpha,\beta}(t,s)$, $t,s \geq 0$, defined by (eq. 4.53) and finite dimensional distribution defined by (eq. 4.57).

4.7 Representation of ggBm

Up to now, we have seen that the ggBm $B_{\alpha,\beta}(t), t \geq 0$, is an H-sssi process, which generalizes Gaussian processes (it is indeed Gaussian when $\beta = 1$) and is defined only by its autocovariance structure. These properties make us think that $B_{\alpha,\beta}(t)$ may be equivalent to a process $\Lambda_\beta X_\alpha(t), t \geq 0$, where $X_\alpha(t)$ is a Gaussian process and Λ_β is a suitable chosen independent random variable. Indeed, the following proposition holds:

Proposition 4.7.1. *It was shown in [20] that the gBm $B_{\alpha,\beta}$ admits the following representation*

$$\{B_{\alpha,\beta}(t), t \geq 0\} \stackrel{d}{=} \{\sqrt{L_\beta} X_\alpha(t), t \geq 0\}, \tag{4.58}$$

where $X_\alpha(t)$ is a standard fBm , L_β is an independent nonnegative random variable with probability density function $M_\beta(\tau), \tau \geq 0$.

The representation (4.58) is particularly interesting. In fact, a number of question, in particularly those related to the distribution properties of $B_{\alpha,\beta}(t)$, can be reduced to question concerning the fBm $X_\alpha(t)$, which are easier since $X_\alpha(t)$ is a Gaussian process.

Remark 4.7.1. . *It follows from the representation (4.58) that the Hölder continuity of the trajectories of ggBm reduces to the Hölder continuity of the fBm.*

4.8 ggBm trajectories

In order to obtain examples of the $B_{\alpha,\beta}(t) = \sqrt{L_\beta} X_\alpha(t)$ trajectories, we just have to simulate the fractional Brownian motion $X_\alpha(t)$. For this purpose (See [20]). Some typical path simulations of $B_{\beta,\beta}(t)$ (shortly $B_\beta(t)$ and $B_{2-\beta,\beta}(t)$), with $\beta = 1/2$ are shown in Figures 4.2, 4.3. The first process provides an example of stochastic model for slow-diffusion (short-memory), the second provides a stochastic model for fast-diffusion (long-memory), see [20].

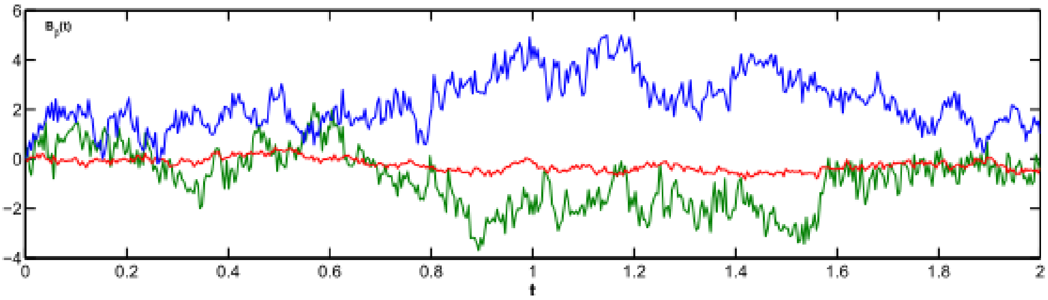

Figure 4.2: $B_\beta(t)$ trajectories in the case $\beta = 0.5$ for $0 \le t \le 2$

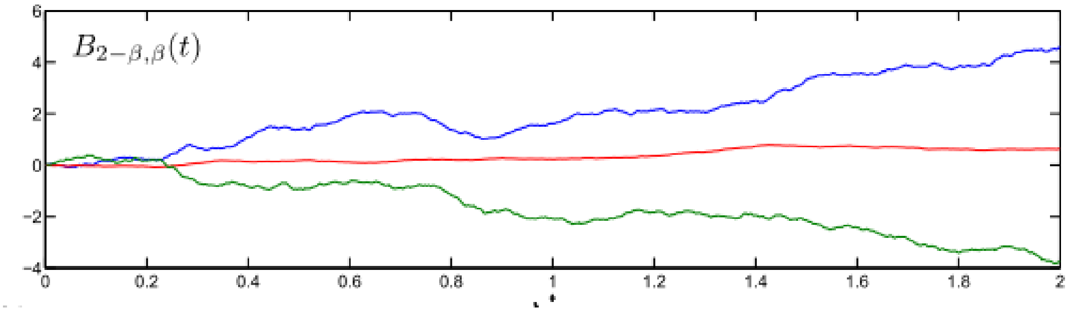

Figure 4.3: $B_{2-\beta,\beta}(t)$ trajectories in the case $\beta = 0.5$ for $0 \le t \le 2$.

Conclusion

Our interest through this work has been focused on structural properties of the generalized grey Brownian motion $\{B_{\alpha,\beta}(t), 0 < \alpha < 2, 0 < \beta \leq 1\}$ (ggBm) which is class of self-similar stochastic processes with stationary increments.

we brought the reader through the fundamental notions of stochastic processes, stochastic integration. In particular, within the study of H-sssi processes, we focused on Brownian motion, fractional Brownian motion (fBm), then we introduced the theory of fractional integrals and derivatives, which turns out to be very appropriate for studying and modeling systems which exhibit long-memory properties.

We showed that this process (ggBm) is made up of self-similar with stationary increments H-sssi of order $H = \frac{\alpha}{2}$. Which for $\alpha = \beta$ actually reduces to the usual time-factional diffusion equation of order β. We have shown that: when, $0 < \alpha < 1$, the diffusion is slow. The increments of the ggBm turn out to be negatively correlated and this implies that the trajectories are very "zigzaging" (antipersistent). When $\alpha = 1$, the diffusion is normal. The increments of the process are uncorrelated, but not independent unless $\beta = 1$. And for $1 < \alpha < 2$, the diffusion is fast. The increments of this process are positively correlated, so that the trajectories are more regular (persistent). In this case the increments exhibits long-range dependence see Figures 4.2, 4.3.

The ggBm is of course Non-Markovian. We also pointed out that the generalized grey Brownian motion is a direct generalization of a Gaussian process. Question related to gBm may be reduced to questions concerning the fBm which is easier since it is Gaussian. From this point and as future work we tray to focuss on the problem of stochastic differential equation driven by gBm, and before that the problem of stochastic integration with respect to this process.

Appendix

The Hurst parameter

Is one of the most important parameters that can characterize a self-similar or LRD signal. It takes a value between 0 and 1, and mostly we deal with processes with a Hurst parameter larger than 0.5, which are known to possess long memory (called the selfsimilarity parameter).

Hilbert-Schmidt Operators

An Hilbert-Schmidt operator is a bounded operator A, defined on an Hilbert space H, such that there exists an orthonormal basis $\{e_i\}_{i\in\mathcal{I}}$ of H with the property $\sum\limits_{i\in\mathcal{I}} ||Ae_i||^2 < \infty$.

Gamma function

Euler made the first step in the right direction in 1729 with the Gamma function , is the generalization of the factorial function $n!$:

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt; \tag{4.59}$$

which is defined for all $s \in \mathbb{C}\ \{0, 1, 2, 3, ...\}$. It is easy to see that $\Gamma(1) = 1$, and integration by parts reveals the identity:

$$\Gamma(s+1) = s\Gamma(s), \forall s. \tag{4.60}$$

From these two facts we deduce that the Gamma function extends the factorial function:

$$\Gamma(n) = (n-1)!, \forall n \in \mathbb{N}. \tag{4.61}$$

Some of the most important examples are :

$$\Gamma(1) = \Gamma(2) = 1, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(n+1/2) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!, n \in \mathbb{N}.$$

Anomalous diffusion

Anomalous diffusion is characterized by the asymptotic time power-law behaviour of the variance for large times: $\sigma^2(t) \sim t^\gamma$. Namely, the diffusion is slow if the exponent γ is lesser than one, normal if it is equal to one and fast if it is greater than one.

Completely monotone function

A function $F(t)$ is completely monotone if it is non-negative and possesses derivatives of any order such that:

$$(-1)^k \frac{d^k}{dt^k} F(t) \geq 0, t > 0, k \in \mathbb{Z}_+$$

The Wright function $W_{\lambda,\mu}(z)$

The Wright function, that we denote by $W_{\lambda,\mu}(z)$, is so named in honour of E. Maitland Wright, the eminent British mathematician, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the theory of partitions. The function is defined by the series representation, convergent in the whole complex plane,

$$W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in C,$$

so $W_{\lambda,\mu}(z)$ is an entire function. Originally, Wright assumed $\lambda > 0$, and, only in 1940, he considered $-1 < \lambda < 0$.

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