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# STOCHASTIC DIFFERENTIAL EQUATIONS AND STOCHASTIC FLOWS OF DIFFEOMORPHISMS

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par

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# Contents

# Introduction

The study of stochastic equation flows is an essential tool in geometry stochastic differential. It was Neveu [37], the first who has demonstrated a theorem of continuity of the solution of a stochastic differential equation as a function of the initial value in 1973 for the classical type equations governed by a Brownian motion.

The fundamental result in the study of the differentiability of solutions of a stochastic differential equation according to the initial conditions says that, if we consider the solution of a very good differential equation stochastic, corresponding to the initial value, there is a differentiable version. This result is due to Malliavin for the classical type equations on varieties, this is one of the important steps in his probabilistic demonstration of hypo-ellipticity results. In 1979, Paul André Meyer [38] demonstrated the same result in $\mathbb{R}^n$ for an equation governed by a semi-martingale discontinuous.

In the deterministic case, the flow is a group with a parameter of diffeomorphisms. For general stochastic differential equations, we can only hope for injectivity if semi-martingale is continuous. In the case of the Wiener process, Malliavin has actually demonstrated it by means of the natural time-reversal argument, and Bismut has demonstrated surjectivity in the case of $\mathbb{R}^n$. More recently (1980), the general case of injectivity has been treated without reversal of time, firstly the so-called weak injectivity by Emery [39]. Following the injectivity called strong by Kunita [40]. In 1982, Are Uppman [40] would have by its method of using exponential semi-martingales, an improved demonstration of the results of strong injectivity. Finally, recent developments in the stochastic flow are developed by Bismut.

The first goal of this work is to present the basics of stochastic calculus versus semi-

martingales. The second one demonstrates that the stochastic flow generated by a stochastic differential equation governed by semi-martingales is a diffeomorphism.
This memory is organized as follows:
An introduction where we place our work and its plan.
The first chapter is devoted to the theory of stochastic calculus for continuous semi-martingales.
In the second chapter, we will discuss the properties of stochastic flows.

# Chapter 1

# Stochastic calculus for continuous semi-martingales

## 1.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a family of sub $\sigma$-fields $\{\mathcal{F}_t, t \in [0, a]\}$ with following properties, where $a$ is a finite positive constant:
(i) Each $\mathcal{F}_t$ contains all null sets of $\mathcal{F}$.
(ii) $\{\mathcal{F}_t\}$ is increasing, i.e.$\mathcal{F}_t \supset \mathcal{F}_s$ if t$\geq$ s.
(iii) $\{\mathcal{F}_t\}$ is right continuous, i.e. $\bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon} = \mathcal{F}_t$ for any $t < a$.

The probability space $(\Omega, \mathbb{F}, \mathcal{F}_t, \mathbb{P})$ will be fixed throughout this chapter.

- Let $X_t$ , $t \in [0, a]$ be a stochastic process with values in $\mathbb{R} = (-\infty, \infty)$.

- We will assume, unless otherwise mentioned, that it is $(\mathcal{F}_t)$ - adapted , i. e.$X_t$ is $\mathcal{F}_t$-measurable for any t $\in$[0,a]. The process $X_t$ is called continuous if $X_t(\omega)$ is a continuous function of $t$ for almost all $\omega$.

- Let $L_c$ be the linear space consisting of all continuous stochastic processes. We introduce the metric $\rho$ by

$$\rho(X - Y) = \rho(X, Y) = \mathbb{E}\left[\frac{\sup_t \mid X_t - Y_t \mid^2}{1 + \sup_t \mid X_t - Y_t \mid^2}\right]^{\frac{1}{2}}$$

- It is equivalent to the topology of the uniform convergence in probability:
  A sequence $\{X^n\}$ of $L_c$ is a Cauchy sequence if and only if for any $\epsilon > 0$,

  $$\mathbb{P}(\sup_t \mid X_t^n - X_t^m \mid > \epsilon) \underset{n,m\to\infty}{\rightarrow} 0$$

  Obviously $L_c$ is a complete metric space.

- We introduce the norm $\|.\|$ by $\|X\| = \mathbb{E}\left[\sup_t |X_t|^2\right]^{\frac{1}{2}}$ and denote by $L_c^2$ the set of all elements in $L_c$ with finite norms. We may say that the topology of $L_c^2$ is the uniform convergence in $L^2$. Since $\rho(X) \leq \|X\|$, the topology by $\|.\|$ is stronger than that by $\rho$. It is easy to see that $L_c^2$ is a dense subset of $L_c$.

**Definition 1.1.1.** *Let $X_t$ , $t \in [0, a]$ be a continuous $(\mathcal{F}_t)$-adapted process.*
*(i) It is called a martingale if $\mathbb{E} \mid X_t \mid < \infty$ for any $t$ and satisfies $\mathbb{E}[X_t/\mathcal{F}_s] = X_s$ for any $t > s$.*
*(ii) It is called a local martingale if there is an increasing sequence of stopping times $\{T_n\}$ such that $T_n \uparrow \infty$ and each stopped process $X_t^{T_n} = X_{t\wedge T_n}$ is a martingale.*
*(iii) It is called an increasing process if $X_t(\omega)$ is an increasing function of $t$ a.s.*
*(iv) It is called a process of bounded variation if it is written as the difference of two increasing processes.*
*(v) It is called a semi-martingale if it is written as the sum of a local martingale and a process of bounded variation.*

We will quote two famous results of Doob's concerning martingales without giving proofs.

**Theorem 1.1.1.** *[40] Let $X_t$, $t \in [0, a]$ be a martingale.*
*(i) Optional sampling theorem:*
*Let $S$ and $T$ be stopping times with values in $[0, a]$. Then $X_s$ is integrable and satisfies $\mathbb{E}[X_S/\mathcal{F}_T] = X_{S\wedge T}$.*
*(ii) Inequality:*
*Suppose $\mathbb{E}[|X_a|^p] < \infty$ with $p > 1$. Then $\mathbb{E}\left[\sup_S |X_S|^p\right] < q^p \mathbb{E}\left[|X_a|^p\right]$ where $q$ is the conjugate of $p$.*

**Remark 1.1.1.** *Let $S$ be a stopping time. If $X_t$ is a martingale, the stopped process $X^S$ is also a martingale. In fact, by Doob's optional sampling theorem, we have for $t \geq S$*

$$\mathbb{E}[X_t^S|\mathcal{F}_S] = X_{t\wedge S\wedge s} = X_{S\wedge s} = X_s^S$$

*Similarly if $X$ is a local martingale, the stopped process $X^S$ is s a local martingale.*

A martingale is a local martingale, obviously, the following theorem gives us a criterion that a local martingale is a martingale.

**Theorem 1.1.2.** *[40] Let $X_t$ be a continuous local martingale.*
*(i) If $\mathbb{E}[\sup_t |X_t|] < \infty$ then $X$ is a martingale.*
*(ii) Let $p > 1$, then $X$ is an $L^P$-martingale if and only if $\mathbb{E}[\sup_t \mid X_t \mid^P] < \infty$.*

**Remark 1.1.2.** *Let $X$ be a local martingale. Then there is an increasing sequence of stopping times $S_k \uparrow \infty$ such that each stopped process $X^{S_k}$ is a bounded martingale. In fact, define $S_k$ by*

$$S_k = \inf\{t > 0; |X_t| \geq k\}$$

*Then $S_k \uparrow \infty$ and it holds $\sup_t |X_t^{S_K}| \leq k$, so that each $X^{S_K}$ is a martingale.*

Let $M_c$ be the set of all square integrable martingales $X_t$ with $X_0 = 0$. Because of Doob's inequality, the norm $\|X\|$ is finite for any $X$ of $M_c$, Hence $M_c$ is a subset of $L_c^2$.

We denote by $M_c^{loc}$ the set of all continuous local martingales $X_t$ such that $X_0 = 0$. It is a subset of $L_c$.

**Theorem 1.1.3.** *[40] $M_c$ is a closed subspace of $L_c^2$. $M_c^{loc}$ is a closed subspace of $L_c$ Furthermore, $M_c$ is dense in $M_c^{loc}$.*

## 1.2 Quadratic variations of continuous semi-martingales

This section is devoted to the study of the quadratic variation of a continuous stochastic process $X_t, t \in [0,a]$. Let $\triangle$ be a partition of the interval $[0,a]$ : $\Delta = \{0 = t_0 < ... < t_n = a\}$ and let $|\Delta| = \max(t_{i+1} - t_i)$. Associated with the partition $\Delta$, we define a continuous process $< X >_t^{\Delta}$ as

$$< X >_t^{\Delta} = \sum_{i=0}^{k-1} (X_{t_{i+1}} - X_{t_i})^2 + (X_t - X_{t_k})^2.$$

where $k$ is the number such that $t_k \leq t < t_{k+1}$ . We call it the quadratic variation of $X_t$ associate with the partition $\Delta$.

Now let $\{\Delta_m\}$ be a sequence of partitions such that $|\Delta_m| \to 0$. If the limit of $< X >_t^{\Delta_m}$ exists in probability and it is independent of the choice of sequences $\{\Delta_m\}$ a.s., it is called the quadratic variation of $X_t$ and is denoted by $< X >_t$.

The quadratic variation is not well defined to any continuous stochastic process. We will see in the sequel that a natural class of processes where quadratic variations are well defined is that of continuous semi-martingales.

We begin the discussion with a process of bounded variation.

**Lemma 1.2.1.** *[40] Let $X$ be a continuous process of bounded variation. Then the quadratic variation exists and equals 0 a.s.*

**Theorem 1.2.1.** *[40] Let $M$ be a bounded continuous martingale. Let $\{\Delta_n\}$ be a sequence of partitions such that $|\Delta_n| \to 0$. Then $< M >_t^{\Delta_n}$, $t \in [0,a]$ converges uniformly to a continuous increasing process $< M >_t$ in $L^2$ -sense, i.e.,*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_t | < M >_t^{\Delta_n} - < M >_t |^2\right] = 0.$$

**Lemma 1.2.2.** *[40]For any $t > s$, it holds*

$$\mathbb{E}\left[< M >_t^{\Delta} / \mathcal{F}_s\right] - < M >_s^{\Delta} = \mathbb{E}\left[(M_t - M_s)^2 / \mathcal{F}_s\right] = \mathbb{E}[M_t^2 / \mathcal{F}_s] - M_s^2.$$

*In particular, $M_t^2 - < M >_t^{\Delta}$ is a continuous martingale.*

**Lemma 1.2.3.** *[40] It holds* $\lim_{n,m\longrightarrow\infty} \mathbb{E}\left[| < M >_a^{\Delta_n} - < M >_a^{\Delta_m} |^2\right] = 0.$

**Theorem 1.2.2.** *[40] Let $M_t$ be a continuous local martingale. Then there is a continuous increasing process $< M >_t$ such that $< M >_t^{\Delta}$ converges uniformly to $< M >_t$ in probability.*

**Remark 1.2.1.** *Let $M_t$ be a continuous local martingale and let $T$ be a stopping time. Then it holds $< M^T >_t = < M >_t^T$ for all $t$ a.s. In fact, it is easy to see that $< M^T >_t^{\Delta} = (< M >^{\Delta})_t^T$ holds for any partition $\Delta$. Letting $|\Delta|$ tend to 0, we get the desired relation.*

**Corollary 1.2.1.** *[40] $M_t^2 - < M >_t$ is a local martingale if $M_t$ is a continuous local martingale.*

**Corollary 1.2.2.** *[40] An element $M$ of $M_c^{loc}$ of belongs to $M_c$ if and only if $< M >_a$ is integrable. In this case, $M_t^2 - < M >_t$ is a martingale.*

**Theorem 1.2.3.** *[40] Let $M_t$ be a continuous local martingale. A continuous increasing process $A_t$ satisfying $A_0 = 0$ coincides with the quadratic variation of $M_t$ if and only if $M_t^2$ - $A_t$ is a local martingale.*

**Remark 1.2.2.** *Corollary 1.2.2 indicates that the sub-martingale $M_t^2$ is decomposed into the sum of martingale $N_t = M_t^2 - < M >_t$ and increasing process $< M >_t$. The decomposition is known as the Doob-Meyer decomposition of the sub-martingale. Note that we did not use the decomposition theorem for the proof of Theorem 1.2.1. If one knows the theorem and apply it, then one can prove the theorem more easily.*

We will finally consider the quadratic variation of a continuous semi-martingale. Let $X_t$ be a continuous semi-martingale and let $X_t = M_t + A_t$ be the decomposition to the local martingale $M_t$ and a process of bounded variation $A_t$. The quadratic variation $< X >_t^\Delta$ associated with the partition $\Delta$ satisfies

$$\left| < X >_t^\Delta - < M >_t^\Delta - < A >_t^\Delta \right| \leq 2 \left\{ < M >_t^\Delta < A >_t^\Delta \right\}^{\frac{1}{2}}.$$

$< M >_t^\Delta$ converges uniformly to $< M >_t$ in probability and $< A >_t$ converges uniformly to 0 a. s. Therefore $< X >_t^\Delta$ converges uniformly to $< M >_t$ in probability. We then have the following theorem.

**Theorem 1.2.4.** *[40] Let $X_t$ be a continuous semi-martingale. Then $< X >_t^\Delta$ converges uniformly to $< M >_t$ in probability as $|\Delta| \to 0$, where $M_t$ is the local martingale part of $X_t$.*

## 1.3 Continuity of quadratic variations in $M_c$ and $M_c^{loc}$

Quadratic variations are continuous in the space $M_c$ and $M_c^{loc}$ in their topologies.

**Theorem 1.3.1.** *[40]*

1. *Let $M^n$ be a sequence in $M_c$. It Converges to $M$ of $M_c$ if and only if $\{< M^n - M >_a\}$ converges to 0 in $L^1$ -norm.*

2. *Let $\{M^n\}$ be a sequence in $M_c^{loc}$. It converges to $M$ of $M_c^{loc}$ if and only if $\{< M^n - M >_a\}$ converges to 0 in probability.*

**Theorem 1.3.2.** *[40]*

1. *Let $\{M^n\}$ be a sequence in $M_c$ converging to $M$ of $M_c$. Then it holds*

$$\sup_\Delta \mathbb{E} \left[ \sup_t < M^n - M >_t^\Delta \right] \underset{n,m \to \infty}{\to} 0$$

2. *Let $\{M^n\}$ be a sequence in $M_c^{loc}$ converging to $M$ of $M_c^{loc}$. Then it holds for any $\epsilon > 0$*

$$\sup_{\Delta} \mathbb{P}\left[\sup_{t} < M^n - M >_t^{\Delta} > \epsilon\right] \underset{n,m\to\infty}{\to} 0$$

## 1.4 Joint quadratic variations

Let $M$ and $N$ be elements of $M_c^{loc}$. The joint quadratic variation of $M$, $N$ associated with the partition $\Delta = \{0 = t_0 < .... < t_n = a\}$ is defined by

$$< M, N >_t^{\Delta} = \sum_{i=0}^{k-1} (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) + (M_t - M_{t_k})(N_t - N_{t_k}),$$

where $k$ is the number such that $t_k \le t < t_{k+1}$.

**Theorem 1.4.1.** *[40] $< M, N >^{\Delta}$ converges uniformly to a continuous process of bounded variation $< M, N >$ in probability as $|\Delta| \to 0$.*

**Remark 1.4.1.** *$< M, M >=< M >$.*

The following is immediate from Theorem 1.2.3.

**Corollary 1.4.1.** *[40] Given $M$, $N$ of $M_c^{loc}$, a continuous process of bounded variation $A$ coincides with the joint quadratic variation $< M, N >$ if and only if $MN - A$ is a local martingale.*

**Theorem 1.4.2.** *[40] Joint quadratic variations have the following properties.*

*(i) Bilinear:* $< aM^1 + bM^2, N >= a < M^1, N > +b < M^2, N >$ *holds for any* $M^1$ *,*$M^2$*,* $N$ *of* $M_c^{loc}$ *and real numbers* $a$*,* $b$*.*

*(ii) Symmetric:* $< M, N >=< N, M >$ *for any* $M$*,* $N$ *of* $M_c^{loc}$*.*

*(iii) Positive definite:* $< M >_t - < M >_s \ge 0$ *holds for any* $t \ge s$ *and the equality holds a.s. if and only if* $M_r = M_s$ *holds for all* $r \in [s, t]$ *a.s.*

*(iv) Schwarz's inequality:*

$$|< M,N >_t - < M,N >_s| \leq (< M >_t - < M >_s)^{\frac{1}{2}} (< N >_t - < N >_s)^{\frac{1}{2}}.$$

*(v) Extended Schwarz's inequality: Let $f_u$, $g_u$, $u \in [0,a]$ be processes measurable with respect to the smallest $\sigma$-field on $[0,a]\times\Omega$ for which all continuous stochastic processes are measurable. Suppose*

$$\int_0^t |f_u|^2 d < M >_u < \infty, \int_0^t |g_u|^2 d < N >_u.$$

*Then*

$$\left| \int_0^t f_u g_u d < M,N >_u \right| \leq \left( \int_0^t |f_u|^2 d < M >_u \right)^{\frac{1}{2}} \left( \int_0^t |g_u|^2 d < N >_u \right)^{\frac{1}{2}}.$$

**Theorem 1.4.3.** *[40]*

1. *Let $\{M^n\}$ be a sequence of $M_c$ converging to $M$. Then it holds for any $N$ of $M_c$*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_t |< M^n - M, N >_t|\right] = 0$$

$$\lim_{n\to\infty} \sup_{\Delta} \mathbb{E}\left[\sup_t | < M^n - M, N >_t^{\Delta} |\right] = 0$$

2. *Let $\{M^n\}$ be a sequence of $M_c^{loc}$ converging to $M$. Then it holds for any $\epsilon > 0$ and $N$ of $M_c^{loc}$*

$$\lim_{n\to\infty} \mathbb{P}[\sup_t |< M^n - M, N >_t| > \epsilon] = 0$$

$$\lim_{n\to\infty} \sup_{\Delta} \mathbb{P}[\sup_t |< M^n - M, N >_t^{\Delta}| > \epsilon] = 0$$

Finally we will mention the joint quadratic variations of continuous semi-martingales. Let $X$ and $Y$ be continuous semi-martingales. The joint quadratic variation associated with the partition $\Delta$ is defined as before and is written as $< X,Y >^{\Delta}$. The following theorem is immediate.

**Theorem 1.4.4.** *[40] $< X,Y >^{\Delta}$ converges uniformly in probability to a continuous process of bounded variation $< X,Y >_t$. If $M$ and $N$ are local martingale parts of $X$ and $Y$, respectively, then $< X,Y >$ coincides with $< M,N >$.*

## 1.5 Stochastic integrals

Let $M_t$ be a continuous local martingale and let $f_t$ be a continuous($\mathcal{F}_t$)-adapted process. We will define the stochastic integral of $f_t$ by the differential $dM_t$. Here, the differential does not mean a signed measure, since the sample function of a continuous local martingale is not of bounded variation, except a trivial martingale $M_t \equiv$ constant a. s. Nevertheless, the integral is well defined if the integrand $f_t$ is ($\mathcal{F}_t$)-adapted: our discussion will be based on the properties of martingales, specially those of quadratic variations. Let $\Delta = 0 = t_0 < .... < t_n = a$ be a partition of the interval $[0, a]$. For any $t \in [0, a]$, choose $t_k$ of $\Delta$ such that $t_k \leq t < t_{k+1}$ and define

$$L_t^{\Delta} = \sum_{i=0}^{k-1} f_{t_i}(M_{t_{i+1}} - M_{t_i}) + f_{t_k}(M_t - M_{t_k}) \tag{1.1}$$

It is easy to see that $L_t^{\Delta}$ is a continuous local martingale. The quadratic variation is computed directly as

$$< L^{\Delta} >_t = \sum_{i=0}^{k-1} f_{t_i}^2(< M >_{t_{i+1}} - < M >_{t_i} + f_{t_k}^2(< M >_t - < M >_{t_k})$$

$$< L^{\Delta} >_t = \int_0^t \mid f_s^{\Delta} \mid^2 d < M >_s \tag{1.2}$$

where $f_s^{\Delta}$ is a step process defined from $f_s$ by $f_s^{\Delta} = f_{t_k}$ if $t_k \leq s < t_{k+1}$. Let $\Delta'$ be another partition of $[0, a]$. We define $L_t^{\Delta'}$ similarly using the same $f_s$ and $M_s$ Then it holds

$$< L^{\Delta} - L^{\Delta'} >_t = \int_0^t |f_s^{\Delta} - f_s^{\Delta'}|^2 d < M >_s .$$

Now let $\{\Delta_n\}$ be a sequence of partitions of $[0, a]$ such that $|\Delta_n| \to 0$. Then $< L^{\Delta_n} - L^{\Delta_m} >_a$ converges to 0 in probability as $n, m \to \infty$. Hence $\{L^{\Delta_n}\}$ is a Cauchy sequence in $M_c^{loc}$ by Theorem 1.3.1. We denote the limit as $\mathrm{L}_t$.

**Definition 1.5.1.** *The above $L_t$ is called the Itô integral of $f_t$ by $dM_t$ and is denoted by* $\int_0^t f_s dM_s$.

The Itô integral can be defined to more general class of stochastic processes called predictable ones. Here the predictable $\sigma$-field is, by definition, the least $\sigma$-field on the product space $[0, a] \times \Omega$ for which all continuous $(\mathcal{F}_t)$-adapted processes $f_t(\omega)$ are measurable. A predictable process is, by definition, a process measurable to the predictable $\sigma$-field. A continuous $(\mathcal{F}_t)$-adapted process is predictable, obviously.
Now let $M_t$ be a continuous local martingale and let $< M >_t$ be the quadratic variation. We denote by $L^2(< M >)$ the set of all predictable processes $f_t$ such that $\int_0^a |f_s|^2 d < M >_s < \infty$ a.s. Then the set of continuous $(\mathcal{F}_t)$-adapted processes is dense in $L^2(< M >)$, i.e., for any $f$ of $L^2(< M >)$, there is a sequence of continuous $(\mathcal{F}_t)$-adapted processes $f_t^n$ such that $\int_0^a |f_s^n - f_s|^2 d < M >_s$ converges to 0 a. s. Then the sequence of stochastic integrals $\int_0^t f_s^n dM_s$, $n = l, 2, ...$ forms a Cauchy sequence in $M_c^{loc}$. Denote the limit as $\int_0^t f_s dM_s$ and call it the Itô integral of $f_t$ by $dM_t$.

**Theorem 1.5.1.** *[40]*

1. *Let $M \in M_c^{loc}$ and $f \in L^2(< M >)$. Then Itô integral satisfies the following relation:*
$$< \int f dM, N >_t = \int_0^t f_s \, d < M, N >_s, \quad \forall N \in M_c^{loc} \tag{1.3}$$

2. *Conversely suppose that $L$ of $M_c^{loc}$ satisfies:*
$$< L, N >_t = \int_0^t f_s d < M, N >_s \quad \forall N \in M_c^{loc} \tag{1.4}$$

   *Then $L$ is the Itô integral of $f_t$ by $dM_t$ , i.e., Itô integral is characterized as the unique element $L$ in $M_c^{loc}$ satisfying 1.4.*

**Corollary 1.5.1.** *[40] It holds*
$$< \int f dM >_t = \int_0^t f_s^2 d < M >_s .$$

We will list a few properties of Itô integrals.

**Theorem 1.5.2.** *[40] Let $M$ be an element of $M_c^{loc}$.*

1. *If $f$, $g$ are in $L^2(<M>)$ and $a$, $b$ are constants, then $af+bg$ is in $L^2(<M>)$ and satisfies*

$$\int_0^t (af_s + bg_s)dM_s = a\int_0^t f_s dM_s + b\int_0^t g_s dM_s$$

2. *Let $f \in L^2(<M>)$ and $L_t = \int_0^t f_s dM_s$. Let $g_s$ be a predictable process such that $\int_0^t f_s^2 g_s^2 d<M>_s < \infty$ Then $g$ is in $L^2(<L>)$ and*

$$\int_0^t g_s dL_s = \int_0^t g_s f_s dM_s \tag{1.5}$$

3. *Let $T$ be a stopping time. Then it holds*

$$\int_0^{t\wedge T} f_s dM_s = \int_0^t f_s dM_s^T = \int_0^{t\wedge T} f_s dM_s^T$$

**Definition 1.5.2.** *Let $X$ be a continuous semi-martingale decomposed to the sum of a continuous local martingale $M$ and a continuous process of bounded variation $A$. Let $f$ be a predictable process such that $f \in L^2(<M>)$ and $\int_0^a |f_s|d|A_s| < \infty$. Then the Itô integral of $f$ by $dX_t$ is defined as*

$$\int_0^t f_s dX_s = \int_0^t f_s dM_s + \int_0^t f_s dA_s$$

We will define another stochastic integral by the differential $\circ\, dX_t$:

$$\int_0^t f_s \circ dX_s = \lim_{|\Delta|\to 0}\left\{\sum_{i=0}^{k-1}\frac{1}{2}(f_{t_{i+1}} + f_{t_i})(X_{t_{i+1}} - X_{t_i}) + \frac{1}{2}(f_t + f_{t_k})(X_t - X_{t_k})\right\}$$

**Definition 1.5.3.** *If the above limit exists, it is called the Stratonovich integral of $f$ by $dX_s$.*

**Theorem 1.5.3.** *[40] If $f$ is a continuous semi-martingale, the Stratonovich integral is well defined and satisfies*

$$\int_0^t f_s \circ dX_s = \int_0^t f_s dX_s + \frac{1}{2} < f, X >_t .$$

## 1.6 Stochastic integrals of vector valued processes

Let $B$ a separable reflexive Banach space and let $f_s$ be a $B$-valued process. Let $M$ be a real valued continuous local martingale. In this and the next section, we will discuss the stochastic integral of the form $\int_0^t f_s dM_s$, which is to be a $B$-valued local martingale.

We begin with introducing conditional expectations for Banach space valued random variables. Let $B$ be a separable reflexive Banach space with norm $\|.\|$, and let $\mathcal{B}$ be the topological Borel field of $B$. We denote by $B'$ the dual space of $B$. Let $f(\omega)$ be a mapping from $\Omega$ into $B$. It is called a $B$-valued random variable if it is a measurable mapping from $(\Omega, \mathbb{F})$ into $(B, \mathcal{B})$. This is equivalent to saying that $(f, \phi)$ is a real valued random variable for any $\phi$ of $B'$, where $(.,.)$ is the canonical bilinear form on $B \times B'$.

**Definition 1.6.1.** *Let $\mathcal{G}$ be a sub $\sigma$-field of $\mathbb{F}$. A measurable mapping $g : (\Omega, \mathcal{G}) \to (B, \mathcal{B})$ is called a conditional expectation of $f$ with respect to $\mathcal{G}$ if*

$$(g, \phi) = \mathbb{E}[(f, \phi)/\mathcal{G}] \tag{1.6}$$

*is satisfied a.s. for any $\phi$ of $B'$.*

**Lemma 1.6.1.** *[40] Suppose that $\|f\|$ is integrable. Then a conditional expectation exists uniquely.*

**Definition 1.6.2.** *The conditional expectation of $f$ with respect to $\mathcal{G}$ is denoted by $\mathbb{E}[f/\mathcal{G}]$ .*

The following lemma corresponds to Doob's convergence theorem of a real $L^P$ martingale.

**Lemma 1.6.2.** *[40] Let $f$ be a $B$-valued random variable such that $\mathbb{E}[\|f\|^p] < \infty$ for some $p \geq 1$. Let $\mathcal{G}_n$, $n = l, 2, ...$ be an increasing sequence of $\sigma$-fields such that $\vee_n \mathcal{G}_n = \mathbb{F}$. Set $f_n = \mathbb{E}[f/\mathcal{G}_n]$. Then it holds $\mathbb{E}[\|f - f_n\|^p] \to 0$ as $n \to \infty$ .*

**Definition 1.6.3.** *Let $M_t$ be a real continuous local martingale and let $f_s$ be a $B$-valued predictable process such that*

$$\int_0^a \| f_s \|^2 \, d < M >_s < \infty \tag{1.7}$$

*A $B$-valued local martingale $L_t$ is called the stochastic integral of $f_t$ by $dM_t$ if it satisfies*

$$(L_t, \phi) = \int_0^t (f_s, \phi) dM_s \tag{1.8}$$

*for any $\phi$ of $B'$. If it exists, we denote it by $\int_0^t f_s dM_s$.*

It is obvious that stochastic integral is at most one. The existence is easily seen if $f_t$ is a step process, i. e., $f_t = f_{t_i}$. holds for all t $\in (t_i, t_{i+1}]$, where $0 = t_0 < t_1 < ... < t_n = a$. In fact,

$$L_t \equiv \sum_{i=0}^{k-1} f_{t_i}(M_{t_{i+1}} - M_{t_i}) + f_{t_k}(M_t - M_{t_k}), t_k \leq t < t_{k+1} \tag{1.9}$$

is a $B$-valued local martingale satisfying 1.8. However, it is not an easy problem to show in general the existence of stochastic integrals of any $B$-valued predictable processes satisfying 1.7.

**Lemma 1.6.3.** *[40] Suppose that the step process $f_t$ satisfies $\mathbb{E}\left[\int_0^a \|f_s\|^2 d < M >_s\right] < \infty$ Then $L_t$ defined by 1.9 is a martingale and satisfies*

$$\mathbb{E}\left[\| L_t \|^2\right] = \mathbb{E}\left[\int_0^t \| f_s \|^2 \, d < M >_s\right], \; \forall t \in [0, a] \tag{1.10}$$

**Lemma 1.6.4.** *[40] Let $f_t$ be a predictable $B$-valued process such that*

$$\mathbb{E}\left[\int_0^a \| f_s \|^2 d < M >_s\right] < \infty \tag{1.11}$$

*Then there is a sequence of $B$-valued step processes $f_t^n$ such that*

$$\mathbb{E}\left[\int_0^a \| f_s - f_s^n \|^2 d < M >_s\right] \underset{n\to\infty}{\to} 0 \tag{1.12}$$

**Theorem 1.6.1.** *[40] For any predictable Hilbert space valued process $f_t$ satisfying 1.7, the stochastic integral $\int_0^t f_s dM_s$ is well defined. Furthermore, it is a strongly continuous Hilbert space valued local martingale.*

## 1.7 Regularity of integrals with respect to parameters

Let $f_s(\lambda)$ be a real valued predictable process with parameter $\lambda \in \Lambda$ and let $M_t$ be a continuous local martingale. If $\int_0^a |f_s(\lambda)|^2 d < M >_s < \infty$ a.s. for any $\lambda$, Itô's stochastic integral $\int_0^t f_s(\lambda) dM_s$ is well defined except for a null set for each $\lambda$. However, the exceptional set may depend on the parameter $\lambda$. Therefore, in order to discuss the regularity of the integral with respect to the parameter, we have to choose a good modification of the integrals so that the exceptional set does not depend on $\lambda$. For this purpose, we shall consider that $f_s(\lambda)$ is a Sobolev space valued process and we shall define the integral as a Sobolev space valued local martingale.
Let us introduce some notations concerning Sobolev space. The parameter space $\Lambda$ is assumed to be a bounded domain in $\mathbb{R}^d$. Let $\lambda = (\lambda_1 \cdots \lambda_d) \in \Lambda$ and $k = (k_1 \cdots k_d)$ be a multi-index of non-negative integers. We denote by $D^k$ the differential operator $\left(\frac{\partial}{\partial \lambda_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial \lambda_d}\right)^{k_d}$. Let $p$ be a real number greater than 1 and $m$ be a nonnegative

integer. A Sobolev space of type $p$, $m$, denoted by $W_p^m$, is the set of all $L_p$ functions $\phi$ on $\Lambda$ such that derivatives $D^k\phi$, $|k| = (k_1 + \cdots + k_d) \le m$ in the distributional sense are all $L_p$ functions. For $\phi \in W_p^m$, we define the norm $\|.\|_{p,m}$ by

$$\|\phi|_{p,m} = \left| \sum_{|k \le m|} \int_\Lambda |D^k\phi(\lambda)|^p d\lambda \right|^{\frac{1}{p}}$$

Then $W_p^m$ is a separable reflexive Banach space.
Now let $C_b^m$ be the set of all $m$-times continuously differentiable functions whose derivatives up to $m$ are all bounded. For $\phi \in C_b^m$, we define the norm

$$\|\phi\|_{\infty,m} = \sum_{|k \le m|} \sup_\lambda |D^k\phi(\lambda)|$$

Then $C_b^m$ is a separable Banach space. A fundamental result concerning Sobolev space is the following.

**Theorem 1.7.1.** *[40] Let $\ell$ be a nonnegative integer less than $m - \dfrac{d}{p}$. Then it holds $W_p^m \subset C_b^\ell$ and there is a positive constant $k_{p,m}^\ell$ such that*

$$\|\phi\|_{\infty,\ell} \le k_{p,m}^\ell \|\phi\|_{p,m}, \qquad \forall \phi \in W_p^m$$

In the following, we will fix $p$ ,$m$ and omit it from the notation of the norm.
We shall now define the stochastic integral of $W_p^m$ -valued process. If $f_t$ is a predictable $W_p^m$-valued step process, the stochastic integral was defined by 1.9 in the previous section. In order to define the integral for more general class of $f_t$ , we need a lemma analogous to lemma 1.6.3.

**Lemma 1.7.1.** *[40] Let $p \ge 2$. There exists a positive constant $C$ such that*

$$\mathbb{E}[\| L_t \|^p] \le C\mathbb{E}[(\int_0^t \| f_s \|^p \, d < M >_s) < M >_t^{\frac{p}{2}-1}], \ \forall t \in [0, a] \tag{1.13}$$

*holds for any step process $f_t$.*

For the proof, we require Burkholder's inequality.

**Theorem 1.7.2.** *[40] Let $p \geq 2$. Then there is a positive constant $C^{(p)}$ such that*

$$\mathbb{E}\left[|M_t|^p\right] \leq C^{(p)}\mathbb{E}\left[< M >_t^{\frac{p}{2}}\right], \ \forall t \in [0, a] \tag{1.14}$$

*holds for any $M \in M_c$ such that $\mathbb{E}\left[|M_a|^p\right] < \infty$.*

**Lemma 1.7.2.** *[40] Let $p \geq 2$. Let $f_t$ be a predictable process such that*

$$\mathbb{E}\left[(\int_0^a \parallel f_s \parallel^p d < M >_s) < M >_a^{\frac{p}{2}-1}\right] < \infty.$$

*Then there is a sequence of step processes $f_t^n$ such that*

$$\mathbb{E}\left[(\int_0^a \parallel f_s - f_s^n \parallel^p d < M >_s) < M >_a^{\frac{p}{2}-1}\right] \to 0$$

**Theorem 1.7.3.** *[40] Let $p \geq 2$. Let $f_t$ be a predictable $W_p^m$-valued process satisfying*

$$\int_0^a \parallel f_s \parallel^p d < M >_s < \infty \, a.s. \tag{1.15}$$

*Then the stochastic integral $\int_0^t f_s dM_s$ is well defined. It is a strongly continuous $W_p^m$-valued local martingale.*

We shall apply the above theorem to the regularity problem of the real valued stochastic integral $\int_0^t f_s(\lambda)dM_s$ with parameter $\lambda$ .

**Theorem 1.7.4.** *[40] Suppose $p \geq 2$ and $mp > d$. Let $f_s(\lambda)$, $\lambda \in \Lambda$ be a predictable $C_b^m$ -valued process satisfying*

$$\int_0^a \|f_s\|_{\infty,m}^p d < M >_s < \infty a.s. \tag{1.16}$$

*Then the real valued stochastic integral $\int_0^t f_s(\lambda)dM_s$ with parameter $\lambda$ has a modification $L_t(\lambda)$ which satisfies the following properties.*

1. *$L_t(\lambda)$ is continuous in $(t, \lambda)$ and $\ell$-times continuously differentiable in $\lambda$ where $\ell < m - \frac{d}{p}$.*

2. *If $|k| < m - \frac{d}{p}$, then $D^k L_t(\lambda)$ is continuous in $(t, \lambda)$ and satisfies:*

$$D^k L_t(\lambda) = \int_0^t D^k f_s(\lambda) dM_s \quad \forall t \, a.s. \tag{1.17}$$

*for any $\lambda$.*

**Theorem 1.7.5.** *[40] Suppose that $mp > d$ and $p \geq 2$. Let $\{f_s^n\}$ be a sequence of predictable $W_p^m$-valued processes such that $\int_0^a \| f_s - f_s^n \|^p \, d < M >_s$ converges to 0 in probability. Let $L_t^n = \int_0^t f_s^n dM_s$. Then $\sup_t \| L_t^n - L_t \|$ converges to 0 in probability as $n \to \infty$.*

**Corollary 1.7.1.** *[40] Suppose that $f_s$ is a predictable strongly continuous $W_p^m$-valued process. Then there is a sequence of partitions $\Delta_n$ of $[0, a]$ with $|\Delta_n| \to 0$ such that*

$$\sup_t \| \int_0^t f_s dM_s - \int_0^t f_s^{\Delta_n} dM_s \| \to 0 \ a.s.$$

*If $\ell < m - \frac{d}{p}$, then $\int_0^t f_s^{\Delta_n}(\lambda) dM_s$ converges to $\int_0^t f_s(\lambda) dM_s$ by the norm $\|.\|_{\infty,\ell}$ a.s.*

**Corollary 1.7.2.** *[40] Let $Y$ be a $\Lambda$-valued $\mathcal{F}_0$-measurable random variable. If $f_s(\lambda)$ is continuous in $(s, \lambda)$ and continuously differentiable in $\lambda$, then it holds*

$$\int_0^t f_s(\lambda) dM_s|_{\lambda = Y} = \int_0^t f_s(Y) dM_s$$

**Theorem 1.7.6.** *[40] Let $f_t(\lambda)$ be a continuous random field satisfying the following properties.*

1. *It is $m + 1$-times continuously differentiable in $\lambda$ a.s.*

2. *For each $\lambda$, $f_t(\lambda)$ is a continuous semi-martingale represented as*

$$f_t(\lambda) = f_0(\lambda) + \sum_{j=1}^{n} \int g_s^j(\lambda) dN_s^j,$$

*where $N_t^1, \cdots, N_t^n$ are continuous semi-martingales, $g_s^j(\lambda)$ are continuous random fields satisfying*

- *$g_s^j(\lambda)$ is $m+1$-times continuously differentiable a.s.*
- *For each $\lambda$, it is $\mathbb{F}$-adapted.*

*Then the Stratonovich integral $\int_0^t f_s(\lambda) \circ dM_s$ has a modification which is continuous in $(t, \lambda)$ and $m$-times continuously differentiable in $\lambda$. Furthermore, it holds for any $k$ such that $|k| \leq m$,*

$$D^k \int_0^t f_s(\lambda) \circ dM_s = \int_0^t D^k f_s(\lambda) \circ dM_s$$

## 1.8 Itô's formula

One of the fundamental tool for studying stochastic differential equations is so called Itô's formula, which describes the differential rule for change of variables or composition of functions. We present here a differential rule for the composition of two stochastic processes, which is a generalization of the well known Itô's formula.

**Theorem 1.8.1.** *[40] Let $F_t(x)$, $t \in [O, a]$, $x \in \mathbb{R}^d$ be a random field continuous in $(t, x)$ a.s., satisfying*

- *$F_t(x)$ is twice continuously differentiable in $x$.*
- *For each $x$, $F_t(x)$ is a continuous semi-martingale and it satisfies*

$$F_t(x) = F_0(x) + \sum_{j=1}^{n} \int_0^t f_s^j(x) dY_s^j, \; \forall x \in \mathbb{R}^d \tag{1.18}$$

  *a.s., where $Y_s^1, \cdots, Y_s^m$ are continuous semi-martingales, $f_s^j(x)$, $s \in [0, a]$, $x \in \mathbb{R}^d$ are random fields which are continuous in $(s, x)$ and satisfy*

1. *$f_s^j(x)$ are twice continuously differentiable in $x$.*
2. *For each $x$, $f_s^j(x))$ are adapted processes.*

Let now $X_t = (X_t^1, \cdots, X_t^d)$ be continuous semi-martingales. Then we have

$$\begin{aligned} F_t(X_t) &= F_0(X_0) + \sum_{j=1}^m \int_0^t f_s^j(X_s)dY_s^j + \sum_{i=1}^d \int_0^t \frac{\partial F_s}{\partial x_i}(X_s)dX_s^i \\ &+ \sum_{i=1}^d \sum_{j=1}^m \int_0^t \frac{\partial f_s^j}{\partial x_i}(X_s)d < Y^j, X^i >_s \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_j}(X_s)d < X^i, X^j >_s \end{aligned}$$

Observe that the above formula is not like the classical formula for the differential of composite functions, where the last two terms do not appear. We will see later that if we replace Itô integrals by Stratonovich integrals, then we have a rule similar to the classical rule. See theorem 1.8.2.

If we take $F_t(x)$ as a $C^2$ function $F(x)$ in the theorem, we obtain a well known Itô's formula.

**Corollary 1.8.1.** *[40] Let $F : \mathbb{R}^d \to \mathbb{R}^1$ be a $C^2$ function and let $X_t = (X_t^1, \cdots, X_t^d)$ be continuous semi-martingales. Then we have*

$$F_t(X_t)=F_0(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F_s}{\partial x_i}(X_s)dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_j}(X_s)d < X^i, X^j >_s$$

In applications it is sometimes useful to rewrite the above formula using Stratonovich integral. The new formula is close to the classical formula for the differential rule of composite function. We need, however, additional assumption for processes.

**Theorem 1.8.2.** *[40] Let $F_t(x)$, $t \in [0, a]$, $x \in \mathbb{R}^d$ be a random field continuous in $(t, x)$ a. s., satisfying*

- *For each $t$, $F_t(.)$ is a $C^3$-map from $\mathbb{R}^d$ into $\mathbb{R}^1$ a.s. $\omega$.*

- *For each $x$, $F_t(x)$ is a continuous semi-martingale and it satisfies*

$$F_t(x) = F_0(x) + \sum_{j=1}^{m} \int_0^t f_s^j(x) \circ dY_s^j, \quad \forall x \in \mathbb{R}^d \tag{1.19}$$

  *where $Y_s^1, \ldots\ldots, Y_s^m$ are continuous semi-martingales, $f_t^j(x)$ are random fields satisfying conditions of Theorem 1.8.1.*

*Let now $X_t = (X_t^1 \ldots, X_t^d)$ be continuous semi-martingales. Then we have*

$$F_t(X_t) = F_0(X_0) + \sum_{j=1}^{m} \int_0^t f_s^j(X) \circ dY_s^j + \sum_{j=1}^{d} \int_0^t \frac{\partial F_s}{\partial x_i}(X_s) \circ dX_s^i \tag{1.20}$$

**Corollary 1.8.2.** *[40] Let $F : \mathbb{R}^d \to \mathbb{R}^1$ be a $C^3$-class function and let $X_t = (X_t^1, \cdots, X_t^d)$ be continuous semi-martingales. Then we have*

$$F(X_t) = F(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x_i}(X_s) o\, dX_s^i$$

## 1.9 Brownian motion and stochastic intagrals

Let $B_t = (B_t^1, \cdots, B_t^m)$ be an $m$-dimensional standard Brownian motion defined on $(\Omega, \mathbb{F}, \mathcal{F}_t, \mathbb{P})$, we will call it an $(\mathcal{F}_t)$-Brownian motion if it is $(\mathcal{F}_t)$-adapted and the future of Brownian motion $B_u - B_t$ ; $u \geq t$ and the past $\sigma$-field $(\mathcal{F}_t)$ are independent for any $t$. The following theorem characterizes $(\mathcal{F}_t)$-Brownian motion by martingales and their joint quadratic variations.

**Theorem 1.9.1.** *[40] Let $B_t = (B_t^1, \cdots, B_t^m)$ be an $m$- dimensional $(\mathcal{F}_t)$-adapted continuous stochastic process. It is an $(\mathcal{F}_t)$-Brownian motion if and only if each $B_t^1, \cdots, B_t^m$ are square integrable martingales such that $< B^i, B^j >_t = \delta_{ij} t$.*

**Theorem 1.9.2.** *[40] Let $M_t = (M_t^1, \cdots, M_t^m)$ be a continuous local martingale. Suppose that there is a strictly increasing process $A_t$ with $\lim_{t \uparrow a} A_t = \infty$ a.s. such that $< M^i, M^j >_t = \delta_{ij}\, A_t$. Let $\tau_S$ be the inverse function of $A_t$. Then the time-changed process $\hat{M}_s = (M_{\tau_S}^1, \ldots\ldots, M_{\tau_S}^m)$ is a standard Brownian motion.*

## 1.10 Kolmogorov's theorem

We shall introduce a criterion for the Holder continuity of random fields, which is a generalization of the well known Kolmogorov's criterion for the continuity of stochastic processes. It will provide us another method of deriving the regularity of stochastic integrals with respect to the parameter.

**Theorem 1.10.1.** *[40] Let $X_\lambda(\omega)$ be a real valued random field with parameter $\lambda = (\lambda_1, ..., \lambda_d) \in \Lambda = [0, 1]$. Suppose that there are constants $\gamma > 0$, $\alpha_i > d$, $i = 1, ...., d$ and $C > 0$ such that*

$$\mathbb{E}\left[|X_\lambda - X_\mu|^\gamma\right] \leq C \sum_{i=1}^{d} |\lambda_i - \mu_i|^{\alpha_i}, \quad \forall \lambda, \mu \in \Lambda$$

*Then $X_\lambda$ has a continuous modification $\tilde{X}_\lambda$.*

# Chapter 2

# Stochastic differential equations and stochastic flows of homeomorphisms

## 2.1 Stochastic differential equation with Lipschitz continuous coefficients

A primitive and intuitive way of expressing a stochastic differential equation could be

$$\frac{d\xi_t}{dt} = X_0(t, \xi_t) + \sum_{k=1}^{m} X_k(t, \xi_t)\dot{B}_t^k$$

where $\dot{B}_t^k$, $k = 1, ..., m$ are independent white noises. It is intended to describe the motion of a particle driven by random forces or the motion perturbed by random noises. However, the equation fails to have a rigorous meaning, since $X_k(t, \xi_t)\dot{B}_t^k$ are not well defined. For the rigorous argument, we will introduce Itô's stochastic differential equation.

Let $B_t = (B_t^1, ..., B_t^m)$, $t \in [0, a]$ be an $m$-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a pair $s$, $t$ of $[0, a]$ such that $s < t$, we denote by $\mathcal{F}_{s,t}$ the least complete $\sigma$-field for which all $B_u - B_v$ ; $s \leq v \leq u \leq t$ are measurable. Then the family of $\sigma$-fields $\{\mathcal{F}_{s,t}\}$ is increasing in $t$, decreasing in $s$; $\mathcal{F}_{s,t} \subset \mathcal{F}_{s',t'}$ if $s' < s$ and $t' < t$. Then $B_t - B_s$, $t \geq s$ is an $\mathcal{F}_{s,t}$-martingale for any $s$.

Given continuous mappings $X_k(t,x)$, $k = 0, \cdots, m$; $[0,a]\times\mathbb{R}^d \to \mathbb{R}^d$, we shall consider an Itô's stochastic differential equation (SDE):

$$d\xi_t = \sum_{k=1}^{m} X_k(t,\xi_t)\, dB_t^k + X_0(t,\xi_t)\, dt \tag{2.1}$$

**Definition 2.1.1.** *Given a time $s \in [0,a]$ and a state $x \in \mathbb{R}^d$, a continuous stochastic process $\xi_t$, $t \in [s,a]$ with values in $\mathbb{R}^d$ is called a solution of 2.1 with the initial condition $\xi_s = x$, if it is $(\mathcal{F}_{s,t})$-adapted for each $t \geq s$ and satisfies*

$$\xi_t = x + \sum_{k=1}^{m} \int_s^t X_k(r,\xi_r) dB_r^k + \int_s^t X_0(r,\xi_r) dr \tag{2.2}$$

*For the convenience of notations, we will often write $dt$ as $dB_t^0$ and write SDE 2.2 as*

$$\xi_t = x + \sum_{k=0}^{m} \int_s^t X_k(r,\xi_r) dB_r^k \tag{2.3}$$

In this section we will show following Itô that equation 2.3 has a unique solution for any initial condition if coefficients $X_0, \cdots, X_m$ are globally Lipschitz continuous, i.e., there is a positive constant $L$ such that

$$|X_k(t,x) - X_k(t,y)| \leq L\,|x-y| \tag{2.4}$$

holds for all $t \in [0,a]$ and $x, y \in \mathbb{R}^d$.

**Theorem 2.1.1.** *Suppose that coefficients $X_0, \cdots, X_m$ of equation 2.3 are globally Lipschitz continuous. Then the equation has a unique solution for any given initial condition. Further it is in $L^P$ for any $p \geq 1$.*

*Proof.* We shall construct a solution starting from $x$ at time $s$, by the method of successive approximation. Define a sequence of $(\mathcal{F}_{s,t})$-adapted continuous stochastic processes by induction:

$$\begin{aligned} \xi_t^0 &= x \\ \xi_t^n &= x + \sum_{k=0}^{m} \int_s^t X_k(r,\xi_r^{n-1}) dB_r^k, \qquad n \geq 1 \end{aligned}$$

Then it holds

$$\xi_t^{n+1} - \xi_t^n = \sum_{k=0}^m \int_s^t \left\{ X_k(r, \xi_r^{n-1}) - X_k(r, \xi_r^n) \right\} dB_r^k.$$

Therefore we have for p≥2,

$$\mathbb{E}[\sup_{s\le u\le t} \mid \xi_u^{n+1} - \xi_u^n \mid^p] \le (m+1)^p \sum_{k=0}^m \mathbb{E}[\sup_{s\le u\le t} \mid \int_u^t \{X_k(r,\xi_r^{n-1}) - X_k(r,\xi_r^n)\} dB_r^k \mid^p].$$

By Doob's inequality and Burkholder's inequality, each term corresponding to $k \ge 1$ is dominated by

$$\begin{aligned} q^p \mathbb{E}\left[\mid \int_s^t \{\cdots\} dB_r^k \mid^p\right] &\le q^p C^{(p)} \mid t-s \mid^{\frac{p}{2}-1} \mathbb{E}\left[\int_s^t \mid \{\cdots\} \mid^p dr\right] \\ &\le q^p C^{(p)} \mid t-s \mid^{\frac{p}{2}-1} L^p \,\mathbb{E}\left[\int_s^t \mid \xi_r^n - \xi_r^{n-1} \mid^p dr\right]. \end{aligned}$$

The term corresponding to $k = 0$ is dominated by

$$\mid t-s \mid^{\frac{p}{q}} L^p \,\mathbb{E}\left[\int_s^t \mid \xi_r^n - \xi_r^{n-1} \mid^p dr\right]$$

Therefore we get

$$\mathbb{E}\left[\sup_{s\le u\le t} \mid \xi_u^{n+1} - \xi_u^n \mid^p\right] \le c_1 \mathbb{E}\left[\int_s^t \mid \xi_r^n - \xi_r^{n-1} \mid^p dr\right] \tag{2.5}$$

Denote the left hand side by $\rho_t^{(n)}$. Then the above implies $\rho_t^{(n)} \le c_1 \int_s^t \rho_r^{(n-1)} dr$. By iteration, we get $\rho_t^{(n)} \le \frac{c_1^n}{n!} a^n \rho_t^{(0)}$. Then

$$\sum_{n=0}^\infty \mathbb{E}[\sup_{s\le u\le t} \mid \xi_u^{n+1} - \xi_u^n \mid^p]^{\frac{1}{p}} \le \sum_{n=0}^\infty \{\frac{C_1^n}{n!} a^n \rho_t^{(0)}\}^{\frac{1}{p}} < +\infty,$$

since $\rho_t^{(0)} < \infty$. Therefore, $\{\xi_t^n\}$ converges uniformly in $[s,t]$ a.s. and in $L^P$-norm. Denote the limit as $\xi_t$. It is a continuous $(\mathcal{F}_{s,t})$-adapted process. Furthermore, $\int_s^t X_k(r,\xi_r^n)dB_r^k$ converges to $\int_s^t X_k(r,\xi_r)dB_r^k$ in $L^P$-norm, since the quadratic variation of $\int_s^t \{X_k(r,\xi_r^n) - X_k(r,\xi_r)dB_r^k$ converges to 0 in $L^P$-norm. The convergence is valid for $k=0$, obviously. Consequently $\xi_t$ is a solution of equation 2.3.

We will next prove the uniqueness of the solution. Let $\xi_t$ and $\tilde{\xi}_t$ be solutions of equation 2.3. Define $T_n = \inf\{t > 0; \mid \xi_t \mid \geq n \text{ or } \mid \tilde{\xi}_t \mid \geq n\}$ and $(= \infty$ if $\{...\} = \emptyset)$. Then it holds

$$\xi_t^{T_n} - \tilde{\xi}_t^{T_n} = \sum_{k=0}^{n} \int_s^{t\wedge T_n} \{X_k(r,\xi_r^{T_n}) - X_k(r,\tilde{\xi}_r^{T_n})\}dB_r^k.$$

Then by a similar calculation as the above, we obtain

$$\mathbb{E}[\sup_{s\leq u\leq t} \mid \xi_u^{T_n} - \tilde{\xi}_u^{T_n} \mid^p] \leq c_1\mathbb{E}[\int_s^{t\wedge T_n} \mid \xi_r^{T_n} - \tilde{\xi}_r^{T_n} \mid^p dr].$$

Set $\rho_t = \mathbb{E}[\sup_{s\leq u\leq t} \mid \xi_u^{T_n} - \tilde{\xi}_u^{T_n} \mid^p]$, where $n$ is fixed. Then we get $\rho_t \leq \int_s^t \rho_r dr$. By Gronwall's lemma, we get $\rho_t = 0$. This proves $\xi_t^{T_n} = \tilde{\xi}_t^{T_n}$. Since $T_n \uparrow \infty$, we have $\xi_t = \tilde{\xi}_t$. The proof is complete. □

**Definition 2.1.2.** *The unique solution is denoted by $\xi_{s,t}(x)$. The solution $\xi_{s,t}(x)$ has many properties analogous to those of ordinary differential equation. Instead of 2.3, consider a control system of ordinary differential equation on $\mathbb{R}^d$;*

$$\frac{d\phi_t}{dt} = X_0(t,\phi_t) + \sum_{k=0}^{m} X_k(t,\phi_t)u_t^k \tag{2.6}$$

*where $u_t = (u_t^1, \cdots, u_t^m)$ is a piecewise smooth function. We denote the solution starting from $(s,x)$ as $\phi_{s,t}(x)$. It is a well known fact that if coefficients $X_0,\cdots,X_m$ are globally Lipschitz continuous, $\phi_{s,t}(x)$ defines a flow of homeomorphisms:*

- *$\phi_{s,t}(x)$ is Lipschitz continuous in $(s,t,x)$,*
- *For $r < s < t$, $\phi_{r,t}(x) = \phi_{s,t} \circ \phi_{r,t}(x)$,*
- *For each $s < t$, $\phi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$ is a homeomorphism.*

In the subsequent sections we will prove the similar property for the solution $\xi_{s,t}(x)$ of equation 2.3. In Section 2, we will prove the Hölder continuity of $\xi_{s,t}(x)$ in $(s,t,x)$. In Section 3, more smoothness of the solution with respect to $x$ will be shown under additional smoothness assumptions for coefficients $X_0, ...., X_m$ . The homeomorphic property of the map $\xi_{s,t}(x) : \mathbb{R}^d \to \mathbb{R}^d$ will be shown at Section 4.
We will introduce some notations for a class of smooth functions.

**Definition 2.1.3.** *Let $k$ be a nonnegative integer and let $\alpha$ be a number such that $0 < \alpha \leq 1$. A real function $f$ on $\mathbb{R}^d$ is called a $C^{k,\alpha}$ function if it is $k$-th continuously differentiable and the $k$-th derivatives are locally Hölder continuous of order $\alpha$. If the $k$-th derivatives are globally Hölder continuous we will call it a $C_g^{k,\alpha}$ function . In particular if $k = 0$, $C^{0,\alpha}$ (or $C_g^{0,\alpha}$ function is a locally (or globally) Hölder continuous function.*

## 2.2 Continuity of the solution with respect to the initial data

Let $\xi_{s,t}(x)$ be the solution of Itô's stochastic differential equation with globally Lipschitz continuous coefficients starting from $(s,x)$;

$$\xi_{s,t}(x) = x + \sum_{k=0}^{m} \int_s^t X_k(r, \xi_{s,r}(x))\, dB_r^k \tag{2.7}$$

The purpose of this section is to prove that there is a continuous modification of the solution $\xi_{s,t}(x)$ and Itô integrals $\int_s^t X_k(r,\xi_{s,r}(x))dB_r^k$ with respect to three variables $(s,t,x)$ so that the equation 2.7 is satisfied for all $(s,t,x)$ a.s. Our argument is based on the following $L^P$ -estimate of the solution.

**Theorem 2.2.1.** *For any $p$ greater than 2, there is a positive constant $C_1^{(p)}$ such that*

$$\mathbb{E} \mid \xi_{s,t}(x) - \xi_{s',t'}(x') \mid^p \leq C_1^{(p)} \{ \mid x - x' \mid^p + (1 \mid x \mid^p + \mid x' \mid^p) + (\mid t - t' \mid^{\frac{p}{2}} + \mid s - s' \mid^{\frac{p}{2}}) \} \tag{2.8}$$

*holds for all $(s, t, x)$ and $(s', t', x')$ such that $s < t$ and $s' < t'$.*

**Remark 2.2.1.** *If coefficients $X_0, \cdots, X_m$ of equation 2.7 are bounded functions, we have an estimate*

$$\mathbb{E} \mid \xi_{s,t}(x) - \xi_{s',t'}(x') \mid^p \leq C_2^{(p)} \{ \mid x - x' \mid^p + \mid t - t' \mid^{\frac{p}{2}} + \mid s - s' \mid^{\frac{p}{2}} \}.$$

The following will be immediate from the above, applying Kolmogorov's theorem.

**Theorem 2.2.2.** *[40] There are modifications of the solution and the stochastic integrals in 2.7 with following properties. $\xi_{s,t}(x)$ and $\int_s^t X_k(r, \xi_{s,r}(x)) dB_r^k$, $k = O, ..., m$ are continuous in $(s, t, x)$ and the equality 2.7 holds for any $(s, t, x)$ a.s.*

Furthermore, the solution $\xi_{s,t}(x)$ is $(\beta, \beta, \alpha)$-Hölder continuous in $(s, t, x)$, where $\beta$ is an arbitrary number less than $\frac{1}{2}$ and $\alpha$ is an arbitrary less than 1.
The rest of this section is devoted to the proof of theorem 2.2.1. We will consider the case $s < s' < t < t'$ only. Other cases will be treated quite similarly. Since

$$\begin{aligned} \xi_{s',t'}(x') &= x' + \sum_{k=0}^{m} \int_{s'}^{t} X_k(r, \xi_{s',r}(x')) dB_r^k + \sum_{k=0}^{m} \int_{t}^{t'} X_k(r, \xi_{s',r}(x')) dB_r^k, \\ \xi_{s,t}(x) &= \xi_{s,s'}(x) + \sum_{k=0}^{m} \int_{s'}^{t} X_k(r, \xi_{s,r}(x)) dB_r^k, \end{aligned}$$

we have

$$\begin{aligned} \mid \xi_{s,t}(x) - \xi_{s',t'}(x') \mid^p &\leq (2m+3)^p \{ \sum_{k=0}^{m} \int_{t}^{t'} X_k(r, \xi_{s',r}(x')) dB_r^k \mid^p + \mid \xi_{s',s}(x) - x' \mid^p \\ &+ \sum_{k=0}^{m} \int_{s'}^{t} \{ X_k(r, \xi_{s,r}(x)) - X_k(r, \xi_{s',r}(x')) \} dB_r^k \mid^p \}. \end{aligned}$$

Consequently it is sufficient to prove the following three estimates:

$$\mathbb{E}\left[ \mid \int_t^{t'} X_k(r, \xi_{s',r}(x'))dB_r^k \mid^p \right] \le C_3 \mid t' - t \mid^{\frac{p}{2}} (1+ \mid x' \mid^p), \tag{2.9}$$

$$\mathbb{E} \mid \xi_{s,s'}(x) - x' \mid^p \le C_4\{\mid x - x' \mid^p + \mid s - s' \mid^{\frac{p}{2}} (1+ \mid x \mid^p)\}, \tag{2.10}$$

$$\mathbb{E}\left[ \int_{s'}^t \{X_k(r, \xi_{s,r}(x)) - X_k(r, \xi_{s',r}(x'))\}dB_r^k \mid^p\} \right] \le C_5\{\mid x - x' \mid^p + \mid s - s' \mid^{\frac{p}{2}} (1+ \mid x \mid^p)\}. \tag{2.11}$$

For the proofs of 2.9 and 2.10, we claim a lemma.

**Lemma 2.2.1.** *[40] Let $p$ be any real number and $\epsilon > 0$. Then there is a positive constant $C_6^{(p,\epsilon)}$ such that*

$$\mathbb{E}\left[(\epsilon + \mid \xi_{s,t}(x) \mid^2)^p\right] \le C_6^{p,\epsilon} (\epsilon + \mid x \mid^2)^p$$

*holds for all $s, t \in [0, a]$ and $x \in \mathbb{R}^d$ .*

*Proof.* Set $f(x) = (\epsilon + |x|^2)$ and apply Itô's formula to $F(x) = f(x)^p$ and $M_t = \xi_t = \xi_{s,t}(x)$, where $(s, x)$ is fixed. Set $x = (x_1, \cdots, x^d)$ and observe

$$\begin{aligned} \frac{\partial F}{\partial x_i}(x) &= 2\, p\, f(x)^{p-1} x_i, \\ \frac{\partial^2 F}{\partial x_i \partial x_j}(x) &= 2\, p\, f(x)^{p-2}\{f(x)\delta_{ij} + 2(p-1)x_i x_j\}. \end{aligned}$$

By setting $X_k(r, x) = (X_k(r, x)^1, \cdots, X_k(r, x)^d)$ and $\xi_t = (\xi_t^1, \cdots, \xi_t^d)$ ,

$$F(\xi_t) - F(x) = 2\;\; p \sum_{i,k \ge 1} \int_s^t f(\xi_r)^{p-1} \xi_r^i X_k^i(r, \xi_r) dB_r^k + 2p \sum_{i \ge 1} \int_s^t f(\xi_r)^{p-1} \xi_r^i X_0^i(r, \xi_r) dr$$

$$+ p \sum_{i,j \ge 1} \int_s^t f(\xi_r)^{p-2}\{f(\xi_r)\delta_{ij} + 2(p-1)\xi_r^i \xi_r^j\}(\sum_{k \ge 1} X_k^i(r, \xi_r) X_k^j(r, \xi_r))dr \tag{2.12}$$

Here we have used the relation

$$\begin{aligned} d < \xi^i, \xi^j >_t &= \sum_{k,\ell\geq 1} X_k^i(t,\xi_t) X_\ell^j(t,\xi_t)\, d < B^k, B^\ell >_t \\ &= \sum_{k\geq 1} X_k^i(t,\xi_t)\, X_k^j(t,\xi_t) dt \end{aligned}$$

The first member of the right hand side of 2.12 is of mean 0. Observe the inequalities $|X_k^i(r,x)| \leq Cf(x)^{\frac{1}{2}}$, $|x^i| \leq f(x)^{\frac{1}{2}}$. Then we see that the second and the third members are dominated by a constant times $\int_0^t F(\xi_r)dr$. Therefore, taking expectations in 2.12, we have

$$\mathbb{E}[F(\xi_t)] - F(x) \leq C_7^{(p,\epsilon)} \int_s^t \mathbb{E}[F(\xi_r)]dr,$$

where $C_7^{(P,\epsilon)}$ is a positive constant. By Gronwall's lemma, we get $\mathbb{E}[F(\xi_t)] \leq F(x) \exp C_7^{(P,\epsilon)}(t-s)$. The proof is complete. □

**Proof of the estimate 2.9:**
Let $k \geq 1$, By Burkholder's inequality, we have

$$\mathbb{E}\left[ |\int_t^{t'} X_k(r,\xi_{s',r}(x'))dB_r^k|^p \right] \leq C_0^{(p)} |t'-t|^{\frac{p}{2}-1} \int_t^{t'} \mathbb{E}\left[ |X_k(r,\xi_{s',r}(x'))|^p \right] dr.$$

Since it holds $|X_k(r,x)| \leq C(1+|x|)$ with some positive constant $C$, lemma 2.2.1 implies inequality 2.9 immediately. The case $k=0$ can be proved similarly.

**Proof of the estimate 2.10:**

$$\xi_{s,s'}(x) - x' = x - x' + \sum_{k=0}^m \int_s^{s'} X_k(r,\xi_{s,r}(x))\, dB_r^k,$$

we have, using 2.9,

$$\begin{aligned}\mathbb{E}[\mid \xi_{s,s'}(x) - x' \mid^p] &\leq (m+2)^p\{\mid x - x' \mid^p + \sum_{k=0}^{m} \mathbb{E}[\mid \int_s^{s'} X_k(r, \xi_{s,r}(x)) dB_r^k \mid^p]\} \\ &\leq (m+2)^p\{\mid x - x' \mid^p + (m+1)C_3 \mid s' - s \mid^{\frac{p}{2}} (1 + \mid x \mid^p)\}.\end{aligned}$$

This proves 2.10.

For the proof of estimate 2.11, we require a lemma.

**Lemma 2.2.2.** *[40] For any real number $p$, there is a positive constant $C_8^{(p)}$ not depending on $\epsilon > 0$ such that*

$$\mathbb{E}[(\epsilon + \mid \xi_{s,t}(x) - \xi_{s,t}(y) \mid^2)^p] \leq C_8^{(p)}(\epsilon + \mid x - y \mid^2)^p \tag{2.13}$$

*holds for all $s < t$ and $x, y$.*

*Proof.* Apply Itô's formula to $F(x) = f(x)^p$, $f(x) = \epsilon + |x|^2$ and $M_t = \eta_t = \xi_{s,t}(x) - \xi_{s,t}(y)$, where $s, x, y$ are fixed. Since

$$\eta_t = x - y + \sum_{k=0}^{m} \int_s^t \{X_k(r, \xi_{s,r}(x)) - X_k(r, \xi_{s,r}(y))\} dB_r^k$$

we have

$$\begin{aligned}F(\eta_t) - F(\eta_s) = 2p \sum_{i,k} \int_s^t f(\eta_r)^{p-1} \eta_r^i \{X_k^i(r, \xi_{s,r}(x)) - X_k^i(r, \xi_{s,r}(y))\} dB_r^k \\ + p \sum_{i,j} \int_s^t f(\eta_r)^{p-2}(f(\eta_r)\delta_{ij} + 2(p-1)\eta_r^i \eta_r^j) \\ \times \{\sum_{k \geq 1} (X_k^i(r, \xi_{s,r}(x)) - X_k^i(r, \xi_{s,r}(y))(X_k^j(r, \xi_{s,r}(x)) - X_k^j(r, \xi_{s,r}(y))\} dr\end{aligned} \tag{2.14}$$

The expectation of the first of the right hand side is 0 except for the term corresponding to $k = 0$. Observe $|\eta_r^i| \leq (f\eta_r)^{\frac{1}{2}}$ and

$$|X_k^i(r, \xi_{s,r}(x)) - X_k^j(r, \xi_{s,r}(y)| \leq L|\eta_r| \leq Lf(\eta_r)^{\frac{1}{2}}$$

by the Lipschitz condition. Then the expectation of the term $\int \cdots dB_r^0$ plus that of the last member in 2.14 is dominated by $C_9 \int_s^t \mathbb{E}[F(\eta_r)]dr$. Then we get

$$\mathbb{E}[F(\eta_t)] - F(x-y) \leq C_9 \int_s^t \mathbb{E}[F(\eta_r)]dr.$$

The assertion follows from Gronwall's lemma. □

**Proof of the estimate 2.11:**

By Burkholder's inequality, we have

$$\begin{aligned}
&\mathbb{E}[\mid \int_{s'}^t \{X_k(r, \xi_{s,r}(x)) - X_k(r, \xi_{s',r}(y))\} dB_r^k \mid^p] \\
&\leq C_0^{(p)} \mid t - s' \mid^{\frac{p}{2}-1} \int_{s'}^t \mathbb{E}[\mid X_k(r, \xi_{s,r}(x)) - X_k(r, \xi_{s',r}(y)) \mid^p] dr \\
&\leq C_0^{(p)} L^p \mid t - s' \mid^{\frac{p}{2}-1} \int_{s'}^t \mathbb{E}[\mid \xi_{s,r}(x) - \xi_{s',r}(x') \mid^p] dr \qquad (2.15)
\end{aligned}$$

Note that $\xi_{s,r}(x) = \xi_{s',r}(x) \circ \xi_{s,s'}(x)$ and that $\xi_{s',r}(y)$ and $\xi_{s,s'}(x)$ are independent. Apply lemma 2.2.2 and estimate 2.10. Then we have

$$\begin{aligned}
\mathbb{E}[\mid \xi_{s,r}(x) - \xi_{s',r}(x') \mid^p] &= \int \mathbb{E}[\mid \xi_{s',r}(y) - \xi_{s',r}(x') \mid^p] \mathbb{P}(\xi_{s,s'}(x) \in dy) \\
&\leq C_8^{(p)} \int \mid y - x' \mid^p \mathbb{P}(\xi_{s,s'}(x) \in dy) \\
&\leq C_8^{(p)} \mathbb{E}[\mid \xi_{s,s'}(x) - x' \mid^p] \\
&\leq C_8^{(p)} C_4 \{\mid x - x' \mid^p + \mid s' - s \mid^{\frac{p}{2}} (1 + \mid x \mid^p)\}.
\end{aligned}$$

Substitute the above inequality to 2.15, we get the estimate 2.11.

**Proof of theorem 2.2.2:**

If 2.8 is satisfied, then by Kolmogorov's theorem, $\xi_{s,t}(x)$ has a modification which is locally $(\beta, \beta, \alpha)$-Hölder continuous with respect to $(s, t, x)$, where $\beta < p^{-1}(\frac{p}{2} - d)$ and $\alpha < 2p^{-1}(\frac{p}{2} - d)$. Since $p$ is arbitrary, $\beta$ can take any value less than half and $\alpha$ can take any value less than 1.

We will next prove the continuity of the integral $\int_s^t X_k(r, \xi_{s,r}(x))dB_r^k$. Since the case $k = 0$ is obvious, we will consider the case $k \geq 1$. Suppose $s < s' < t < t'$ as before. Then

$$\int_s^t X_k(r, \xi_{s,r}(x))dB_r^k - \int_{s'}^{t'} X_k(r, \xi_{s',r}(x'))dB_r^k = \int_s^{s'} X_k(r, \xi_{s,r}(x))dB_r^k$$

$$+ \int_{s'}^{t} \{X_k(r, \xi_{s,r}(x)) - X_k(r, \xi_{s',r}(x'))\}dB_r^k - \int_t^{t'} X_k(r, \xi_{s',r}(x'))dB_r^k$$

$L_p$ -estimates of the first and the third terms of the right hand side have been given in 2.9. $L_p$-estimate of the second term is given by 2.11. Therefore, $L^P$-norm of the left hand side is again dominated by a quantity like the right hand side of 2.8. Therefore the stochastic integrals $\int_s^t X_k(r, \xi_{s,r}(x))dB_r^k$, $k = 1, \cdots, m$ have the same kind of continuity as that of $\xi_{s,t}(x)$. Other properties of the theorem will be obvious from the above.

## 2.3 Smoothness of the solution with respect to the initial data

We have seen in the previous section that the solution $\xi_{s,t}(x)$ of a SDE is locally Hölder continuous of order $\alpha < 1$, provided that coefficients of the SDE are Lipschitz continuous. In this section we will see more smoothness of the solution under additional smoothness assumption for coefficients.

**Theorem 2.3.1.** *Suppose that coefficients $X_0, \cdots, X_m$ of an Itô SDE are $C_g^{1,\alpha}$ functions for some $\alpha > 0$ and their first derivatives are bounded. Then the solution $\xi_{s,t}(x)$*

*is a $C^{1,\beta}$ function of $x$ for any $\beta$ less than $\alpha$ for each $s < t$ a.s. Furthermore, the derivative $\partial_\ell \xi_{s,t}(x) = \left(\frac{\xi_{s,t}(x)}{\partial x_\ell}\right)$ satisfies the following SDE:*

$$\partial_\ell \xi_{s,t}(x) = e_\ell + \sum_{k=0}^{m} \int_s^t X'_k(r, \xi_{s,r}(x)) \partial_\ell \xi_{s,r}(x) dB_r^k \tag{2.16}$$

*for all $(s,t,x)$ a.s., where $X'_k(r,x)$ is a matrix valued function $\left(\frac{\partial X_k^i(r,x)}{\partial x_j}\right)_{i,j=1,\cdots,d}$ and $e_\ell$ is the unit vector $(0,\cdots,0,1,0,\cdots,0)$. For $y \in \mathbb{R} - \{0\}$, define*

$$\eta_{s,t}(x,y) = \frac{1}{y}\{\xi_{s,t}(x + ye_\ell) - \xi_{s,t}(x)\} \tag{2.17}$$

*Then the existence of the partial derivative $\frac{\xi_{s,t}(x)}{\partial x_\ell}$, for any $s,t,x$, a.s. can be assured if $\eta_{s,t}(x,y)$ has a continuous extension at $y = 0$ for any $s,t,x$ a.s. This follows from the following lemma and Kolmogorov's theorem.*

**Lemma 2.3.1.** *[40] For any $p \geq 2$, there is a positive constant $C_{10}^{(p)}$ such that*

$$\mathbb{E} \mid \eta_{s,t}(x,y) - \eta_{s',t'}(x',y') \mid^p$$

$$\leq C_{10}^{(p)}\{\mid x - x' \mid^{\alpha p} + \mid y - y' \mid^{\alpha p} + (1 + \mid x \mid + \mid x' \mid)^{\alpha p}(\mid s - s' \mid^{\frac{\alpha p}{2}} + \mid t - t' \mid^{\frac{\alpha p}{2}})\} \tag{2.18}$$

*Proof.* We first show the boundedness of $\mathbb{E}|\eta_{s,t}(x)|^P$. By the mean value theorem, it holds

$$\eta_{s,t}(x,y) = e_\ell + \sum_{k=0}^{m} \int_s^t \left\{\int_0^1 X'_k(r, \xi_{s,r}(x) + v(\xi_{s,r}(x + ye_\ell) - \xi_{s,r}(x)))dv\right\} \times \eta_{s,r}(x,y) dB_r^k \tag{2.19}$$

Therefore we have

$$\mathbb{E} \mid \eta_{s,t}(x,y) \mid^p \leq (m+2)^p \left\{1 + \sum_{k=0}^{m} \mathbb{E}\left[\mid \int_s^t (\int_0^1 X'_k(...)dv) \eta_{s,r}(x,y) dB_r^k \mid^p\right]\right\} \tag{2.20}$$

Using Burkholder's inequality, we have for k≥1,

$$\begin{aligned}
&\mathbb{E}[| \int_s^t (\int_0^1 X_k'(...)dv)\eta_{s,r}(x,y)dB_r^k \ |^p] \\
\leq \ & C_{11}^{(p)} \mid t-s \mid^{\frac{p}{2}-1} \mathbb{E}[\int_s^t \mid \int_0^1 X_k'(...)dv\eta_{s,r}(x,y) \mid^p dr] \\
\leq \ & C_{11}^{(p)} \mid t-s \mid^{\frac{p}{2}-1} \| X_k' \| \int_s^t \mathbb{E} \mid \eta_{s,r}(x,y) \mid^p dr.
\end{aligned}$$

Here $\| X_k' \| = \sup_{(r,x)} |X_k'(r,x)|$ and $|A|$ denotes the norm of the matrix $A = (a_{ij})$ defined by $|A| = \sqrt{\sum_{i,j} a_{ij}^2}$. Similar estimate is valid for $k = 0$. Then from 2.20, we obtain

$$\mathbb{E} \mid \eta_{s,t}(x,y) \mid^p \leq C_{12}^{(p)} + C_{13}^{(p)} \int_s^t \mathbb{E} \mid \eta_{s,r}(x,y) \mid^p dr,$$

where constants $C_{12}^{(p)}$ and $C_{13}^{(p)}$ do not depend on $s,t,x,y$. Therefore, by Gronwall's inequality, we see that $\mathbb{E} \mid \eta_{s,t}(x,y) \mid^p$ is bounded.

We next show 2.18 in case $t = t'$. We assume $s < s' < t$. Other cases will be treated similarly. Note that $\eta_{s,t}(x,y) - \eta_{s,t}(x',y')$ is a sum of the following terms:

$$\int_s^{s'} \left( \int_0^1 X_k'(r, \xi_{s,r}(x) + v(\xi_{s,r}(x+ye_\ell) - \xi_{s,r}(x)))dv \right) \eta_{s,r}(x,y)dB_r^k \tag{2.21}$$

$$\begin{aligned}
&\int_{s'}^t \Bigg[ \left( \int_0^1 X_k'(r, \xi_{s,r}(x) + v(\xi_{s,r}(x+ye_\ell) - \xi_{s,r}(x)))dv \right) \eta_{s,r}(x,y) \\
&- \left( \int_0^1 X_k'(r, \xi_{s',r}(x') + v(\xi_{s',r}(x'+y'e_\ell) - \xi_{s',r}(x')))dv \right) \eta_{s',r}(x',y') \Bigg] dB_r^k
\end{aligned} \tag{2.22}$$

Using Burkholder's inequality, the expectation of the $p$-th power of **??** is estimated in case $k \geq 1$ as

$$
\begin{aligned}
& \mathbb{E}[| \int_s^{s'} (\int_0^1 X_k'(...)dv)\eta_{s,r}(x,y)dB_r^k |^p] \\
\leq \ & C_{14}^{(p)} \mid s'-s \mid^{\frac{p}{2}-1} \mathbb{E}[\int_s^{s'} \mid (\int_0^1 X_k'(...)dv)\eta_{s,r}(x,y) \mid^p dr] \\
\leq \ & C_{14}^{(p)} \parallel X_k' \parallel \mid s'-s \mid^{\frac{p}{2}-1} \int_s^{s'} \mathbb{E} \mid \eta_{s,r}(x,y) \mid^p dr,
\end{aligned}
$$

which is dominated by $C_{15}^{(p)} \mid s-s' \mid^{p/2}$ by the argument of the previous paragraph. We will calculate the expectation of the $p$-th power of 2.22. Note that the integrant $[\cdots]$ in 2.22 estimated as

$$
\begin{aligned}
& |\text{integrant}[...]| \\
\leq \ & \int_0^1 \mid X_k'(r, \xi_{s,r}(x) + vy\eta_{s,r}(x,y)) \mid dv \times \mid \eta_{s,r}(x,y) - \eta_{s',r}(x',y') \mid \\
+ \ & \int_0^1 \mid X_k'(r, \xi_{s,r}(x) + vy\eta_{s,r}(x,y)) - X_k'(r, \xi_{s',r}(x') + vy'\eta_{s',r}(x',y')) \mid dv \times \mid \eta_{s',r}(x',y') \mid \\
\leq \ & \parallel X_k' \parallel \mid \eta_{s,r}(x,y) - \eta_{s',r}(x',y') \mid \\
+ \ & L \int_0^1 \{(1-v)^\alpha \mid \xi_{s,r}(x) - \xi_{s',r}(x') \mid^\alpha + v^\alpha \mid \xi_{s,r}(x+ye_\ell) - \xi_{s',r}(x'+y'e_\ell) \mid^\alpha\} dv \times \mid \eta_{s',r}(x',y') \mid \\
\leq \ & \parallel X_k' \parallel \mid \eta_{s,r}(x,y) - \eta_{s',r}(x',y') \mid + L \mid \xi_{s,r}(x) - \xi_{s',r}(x') \mid^\alpha \times \mid \eta_{s',r}(x',y') \mid \\
+ \ & L \mid \xi_{s,r}(x+ye_\ell) - \xi_{s',r}(x'+y'e_\ell) \mid^\alpha \times \mid \eta_{s',r}(x',y') \mid .
\end{aligned}
$$

Here $L$ is a Hölder constant; $|X_k'(r,x) - X_k'(r,x')| \leq L|x-x'|^\alpha$. Therefore, by

Burkholder's inequality,

$$\begin{aligned}
& C^{(p)^{-1}}\mathbb{E}[| \int_{s'}^{t}[...]dB_r^k |^p] \\
\leq & \mid t-s' \mid^{\frac{p}{2}-1} \int_{s'}^{t}\mathbb{E}[| \, [...] \, |^p]dr \\
\leq & \mid t-s' \mid^{\frac{p}{2}-1} 3^p\{\| X_k' \|^p \int_{s'}^{t}\mathbb{E}[| \, \eta_{s,r}(x,y)-\eta_{s',r}(x',y') \, |^p]dr \\
+ & L^p(\int_{s'}^{t}\mathbb{E}[| \, \xi_{s,r}(x)-\xi_{s',r}(x') \, |^{2\alpha p}]^{\frac{1}{2}}\mathbb{E}[| \, \eta_{s',r}(x',y') \, |^{2p}]^{\frac{1}{2}}dr \\
+ & \int_{s'}^{t}\mathbb{E}[| \, \xi_{s,r}(x+ye_\ell)-\xi_{s',r}(x'+y'e_\ell) \, |^{2\alpha p}]^{\frac{1}{2}}\mathbb{E}[| \, \eta_{s',r}(x',y') \, |^{2p}]^{\frac{1}{2}}dr\}.
\end{aligned}$$

Apply theorem 2.2.1 to $\mathbb{E}|\xi_{s,r}(x)-\xi_{s',r}(x')|^{\alpha p}$. Then the above is dominated by

$$\begin{aligned}
& C_{15}\{(1+\mid x\mid+\mid x'\mid)^{\alpha p}\mid s-s'\mid^{\frac{\alpha p}{2}}+\mid x-x'\mid^{\alpha p}+\mid y-y'\mid^{\alpha p}\} \\
& +C_{16}\int_{s'}^{t}\mathbb{E}[| \, \eta_{s,r}(x,y)-\eta_{s',r}(x',y') \, |^p]dr.
\end{aligned}$$

Summing up these calculations for 2.21 and 2.22, we arrive at

$$\begin{aligned}
& \mathbb{E}[| \, \eta_{s,t}(x,y)-\eta_{s',t}(x',y') \, |^p] \\
\leq & C_{17}\{\mid s-s'\mid^{\frac{p}{2}}+(1+\mid x\mid+\mid x'\mid)^{\alpha p}\mid s-s'\mid^{\frac{\alpha p}{2}}+\mid x-x'\mid^{\alpha p}+\mid y-y'\mid^{\alpha p}\} \\
+ & C_{18}\int_{s'}^{t}\mathbb{E}[| \, \eta_{s,r}(x,y)-\eta_{s',r}(x',y') \, |^p]dr.
\end{aligned}$$

By Gronwall's inequality, we have

$$\begin{aligned}
& \mathbb{E}[| \, \eta_{s,t}(x,y)-\eta_{s',t}(x',y') \, |^p] \\
\leq & C_{17}\{(1+\mid x\mid+\mid x'\mid)^{\alpha p}\mid s-s'\mid^{\frac{\alpha p}{2}}+\mid x-x'\mid^{\alpha p}+\mid y-y'\mid^{\alpha p}\}\exp C_{18}(t-t').
\end{aligned}$$

This proves 2.18 in case $t = t'$.It remains to prove 2.18 in case $t \neq t'$. Assuming $t < t'$, we have

$$\begin{aligned} & \eta_{s,t}(x,y) - \eta_{s',t}(x',y') \\ = \; & \eta_{s,t}(x,y) - \eta_{s',t}(x',y') - \sum_{k=0}^{m} \int_t^{t'} (\int_0^1 X_k'(...)dv)\eta_{s',r}(x',y')dB_r^k. \end{aligned}$$

It holds

$$\begin{aligned} & C^{(p)^{-1}}\mathbb{E}[| \int_t^{t'} (\int_0^1 X_k'(...)dv)\eta_{s',r}(x',y')dB_r^k \,|^p] \\ \leq \; & | \, t'-t \, |^{\frac{p}{2}-1} \, \mathbb{E}[\int_t^{t'} | \, (\int_0^1 X_k'(...)dv)\eta_{s',r}(x',y') \, |^p \, dr] \\ \leq \; & | \, t'-t \, |^{\frac{p}{2}-1} \| \, X_k' \, \|^p \int_t^{t'} \mathbb{E} \, | \, \eta_{s',r}(x',y') \, |^p \, dr \\ \leq \; & C_{19} \, | \, t'-t \, |^{\frac{p}{2}} \end{aligned}$$

Therefore we get the desired estimation 2.18. The proof is complete.

**Proof of theorem 2.3.1:**

By Kolmogorov's theorem, $\eta_{s,t}(x,y)$ has a continuous extension at $y = 0$ for all $s < t$ and $x \in \mathbb{R}^d$ a.s. This means that $\xi_{s,t}(x)$ is continuously differentiable in the domain $\{(s.t, x)/s < t, x \in \mathbb{R}^d\}$ and the derivative $\partial_\ell \xi_{s,t}(x)$ is $\beta$-Hölder continuous for any $\beta < \alpha$. Let $y$ tend to 0 in 2.19. Then we obtain 2.16. The proof is complete. □

**Theorem 2.3.2.** *Let $k$ be a positive integer and $\alpha$ be $0 < \alpha \leq 1$. Suppose that coefficients $X_0, \cdots, X_m$ are $C_g^{k,\alpha}$ functions of $x$ for some $\alpha$ and their derivatives up to $k$-th order are bounded. Then the solution $\xi_{s,t}(x)$ is a $C^{k,\beta}$ function of $x$ for any $\beta$ less than $\alpha$.*

*Proof.* We will consider the case $k = 2$. Let $y \in \mathbb{R} - \{0\}$ and set

$$\zeta_{s,t}(x,y) = \frac{1}{y}\{\partial_i \xi_{s,t}(x + ye_\ell) - \partial_i \xi_{s,t}(x)\}.$$

Then similarly as the proof of lemma 2.3.1, we obtain an estimate

$$\begin{aligned}&\mathbb{E}[\mid \zeta_{s,t}(x,y)-\zeta_{s',t'}(x',y')\mid^p]\\ \leq\;& C_{20}\{\mid x-x'\mid^{\alpha p}+\mid y-y'\mid^{\alpha p}(1+\mid x\mid+\mid x'\mid)^{\alpha p}(\mid s-s'\mid^{\frac{\alpha p}{2}}+\mid t-t'\mid^{\frac{\alpha p}{2}})\}\end{aligned}$$

for all $s<t$, $s'<t'$, $x,x'\in\mathbb{R}^d$, $y,y'\in\mathbb{R}-\{0\}$. This implies the existence of the partial derivative $\partial_\ell\xi_{s,t}(x)$ for all $s<t$ and $x$ a.s. and the partial derivative is $\beta$-Hölder continuous for any $\beta<\alpha$. □

## 2.4 Stochastic flow of homeomorphisms

In section 2, we saw that if coefficients of an Itô SDE are globally Lipschitz continuous, then there is a modification of the solution $\xi_{s,t}(x)$ which is continuous in three variables $(s,t,x)$ a.s. Then for any $s<t$, $\xi_{s,t}(.,\omega)$ defines a continuous map $\mathbb{R}^d\to\mathbb{R}^d$ for almost all $\omega$. We will prove in this section that the map is actually a homeomorphism of $\mathbb{R}^d$ onto itself a.s.
We will first consider the "one to one" property of the map $\xi_{s,t}(.,\omega)$ lemma 2.2.2 implies the inequality

$$\mathbb{E}[\mid \xi_{s,t}(x)-\xi_{s,t}(y)\mid^{2p}]\leq C_8^{(p)}\mid x-y\mid^{2p} \tag{2.23}$$

for negative $p$. This shows that if $x\neq y$, then $\xi_{s,t}(x)-\xi_{s,t}(y)$ a.s. for any $s<t$. But this does not imply immediately that the map $\xi_{s,t}(.,\omega)$ is one to one a.s. To prove the latter assertion, we require a lemma.

**Lemma 2.4.1.** *[40] Set*

$$\eta_{s,t}(x,y)=\frac{1}{\mid \xi_{s,t}(x)-\xi_{s,t}(y)\mid}.$$

*Then for any $p>2$, there is a constant $C_{21}^{(p)}$ such that for any $\delta>0$*

$$\begin{aligned}&\mathbb{E}[\mid \eta_{s,t}(x,y)-\eta_{s',t'}(x',y')\mid^p]\\ \leq\;& C_{21}^{(p)}\delta^{-2p}\{\mid x-x'\mid+\mid y-y'\mid+(1+\mid x\mid^p+\mid x'\mid^p+\mid y\mid^p+\mid y'\mid^p)(\mid t-t'\mid^{\frac{p}{2}}+\mid s-s'\mid^{\frac{p}{2}})\}\end{aligned}$$

*holds for all* $s < t$ *and* $x, y, x', y'$ *such that* $| x - y | \geq \delta$ *and* $| x' - y' | \geq \delta$.

*Proof.* A simple computation yields

$$\begin{aligned} & | \eta_{s,t}(x,y) - \eta_{s',t'}(x',y') |^p \\ \leq \; & 2^p \eta_{s,t}(x,y)^p - \eta_{s',t'}(x',y')^p \{ | \xi_{s,t}(x) - \xi_{s',t'}(x') |^p + | \xi_{s,t}(y) - \xi_{s',t'}(y') |^p \}. \end{aligned}$$

Take expectations for both sides and use Hölder's inequality. Then,

$$\begin{aligned} & \mathbb{E}[| \eta_{s,t}(x,y) - \eta_{s',t'}(x',y') |^p] \\ \leq \; & 2^p \mathbb{E}[| \eta_{s,t}(x,y) |^{4p}]^{\frac{1}{4}} \mathbb{E}[| \eta_{s',t'}(x',y') |^{4p}]^{\frac{1}{4}} \\ \times \; & \{ \mathbb{E}[| \xi_{s,t}(x) - \xi_{s',t'}(x') |^{2p}]^{\frac{1}{2}} + \mathbb{E}[| \xi_{s,t}(y) - \xi_{s',t'}(y') |^{2p}]^{\frac{1}{2}} \}. \end{aligned}$$

It holds by 2.23

$$\mathbb{E}[| \eta_{s,t}(x,y) |^{4p}]^{\frac{1}{4}} \leq C_{22} | x - y |^{-p} \leq C_{22} \delta^{-p},$$

where $| x - y | \geq \delta$. Also by theorem 2.2.1,

$$\mathbb{E}[| \xi_{s,t}(x) - \xi_{s',t'}(x') |^{2p}]^{\frac{1}{2}} \leq C_{23} \{ | x - x' |^p + (1 + | x |^p + | x' |^p)(| t - t' |^{\frac{p}{2}} + | s - s' |^{\frac{p}{2}}) \}.$$

Therefore we get the lemma. □

We can prove the "one to one" property of the map $\xi_{s,t}$. Take $p$ as large as $\frac{p}{2} > 2(d+1)$ in lemma 2.4.1. Kolmogorov's theorem states that $\eta_{s,t}(x,y)$ is continuous in $(s,t,x,y)$ in the domain $\{(s,t,x,y) \mid s < t, | x - y | \geq \delta\}$. Since $\delta$ is arbitrary, it is also continuous in the domain $\{(s,t,x,y) \mid s < t, x \neq y\}$. This proves that the map $\xi_{s,t}$; $\mathbb{R}^d \to \mathbb{R}^d$ is one to one for any $0 < s < t < a$ a. s.
We will next consider the onto property of the map $\xi_{s,t}$ . We claim a lemma.

**Lemma 2.4.2.** *[40] Let* $\bar{\mathbb{R}}^d = \bar{\mathbb{R}}^d \cup \{\infty\}$ *be the one point campactification of* $\mathbb{R}^d$. *Set* $\hat{x} = | x |^{-2} x$ *and define*

$$\eta_{s,t}(\hat{x}) = \frac{1}{1 + \xi_{s,t}(x)} \quad if \hat{x} \in \mathbb{R}^d, \; \hat{x} = 0.$$

*Then for any positive $p$, there is a constant $C_{24}^{(p)}$ such that*

$$\mathbb{E}[\mid \eta_{s,t}(\hat{x}) - \eta_{s',t'}(\hat{x'}) \mid^p] \leq C_{24}^{(p)}\{\mid \hat{x} - \hat{x'} \mid^p + \mid t - t' \mid^{\frac{p}{2}} + \mid s - s' \mid^{\frac{p}{2}}\}.$$

*Proof.* Since

$$\mid \eta_{s,t}(\hat{x}) - \eta_{s',t'}(\hat{x'}) \mid^p \leq \eta_{s,t}(\hat{x})^p - \eta_{s',t'}(\hat{x'})^p \mid \xi_{s,t}(x) - \xi_{s',t'}(x') \mid^p,$$

we have by Hölder's inequality

$$\mathbb{E}[\mid \eta_{s,t}(\hat{x}) - \eta_{s',t'}(\hat{x'}) \mid^p] \leq \mathbb{E}[\mid \eta_{s,t}(\hat{x}) \mid^{4p}]^{\frac{1}{4}} \mathbb{E}[\mid \eta_{s',t'}(\hat{x'}) \mid^{4p}]^{\frac{1}{4}}$$

$$\times \mathbb{E}[\xi_{s,t}(x) - \xi_{s',t'}(x') \mid^{2p}]^{\frac{1}{2}}$$

Apply lemma 2.2.1 and theorem 2.2.1. Then the right hand side is dominated by

$$C_{25}(1+ \mid x \mid)^{-p}(1+ \mid x' \mid)^{-p}\{\mid x - x' \mid^p + (1+ \mid x \mid + \mid x' \mid)^p(\mid t - t' \mid^{\frac{p}{2}} + \mid s - s' \mid^{\frac{p}{2}})\}$$

$$\leq C_{25}\{\mid \hat{x} - \hat{x'} \mid^p \mid t - t' \mid^{\frac{p}{2}} + \mid s - s' \mid^{\frac{p}{2}}\}$$

if $x$ and $x'$ are finite. Here we have used the inequality $(1+ \mid x \mid)^{-1} \times (1+ \mid x' \mid)^{-1} \mid x - x' \mid \leq \mid \hat{x} - \hat{x}' \mid$. In case $x = \infty$, we have

$$\mathbb{E}[\mid \eta_{s',t'}(\hat{x'}) \mid^p] \leq C_{26}(1+ \mid x' \mid)^{-p} \leq C_{26} \mid x' \mid^{-p}.$$

Therefore the inequality of the lemma follows.
The "onto" property of the map $\xi_{s,t}$ follows from lemma 2.4.2. Take $p$ greater than $2(d+3)$. Then by Kolmogorov's theorem, $\eta_{s,t}(\hat{x})$ is continuous at $\hat{x} = 0$. Therefore, $\xi_{s,t}(., \omega)$ can be extended to a continuous map from $\bar{\mathbb{R}}^d$ into itself for any $s < t$ a.s. The extension $\tilde{\xi}_{s,t}(x, \omega)$ is continuous in $(s, t, x)$ a.s. We will fix such $\omega$. The map $\tilde{\xi}_{s,t}(., \omega)$; $\mathbb{R}^d \to \mathbb{R}^d$ is then homotopic to the identity map $\tilde{\xi}_{s,s}(., \omega)$, so that it is an onto map by a well known theorem of homotopic theory. The restriction of $\tilde{\xi}_{s,t}(., \omega)$ to $\mathbb{R}^d$ is again an "onto" map since $\tilde{\xi}_{s,t}(\infty, \omega) = \infty$.

The map $\xi_{s,t}(.,\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is one to one and onto. Hence the inverse map $\xi_{s,t}^{-1}(.,\omega)$ is also one to one and onto. It is continuous. Indeed, the inverse map $\xi_{s,t}^{-1}(.,\omega)$ ; $\bar{\mathbb{R}}^d \to \bar{\mathbb{R}}^d$ is continuous since $\xi_{s,t}(.,\omega)$ is a one to one, continuous map from the compact space $\bar{\mathbb{R}}^d$ into itself. □

We will summarize the result.

**Theorem 2.4.1.** *[40] Suppose that coefficients of an Itô SDE are globally Lipschitz continuous. Then there is a modification of the solution, denoted by $\xi_{s,t}(x,\omega)$ which satisfies the following properties.*

- *For each $s < t$ and $x$, $\xi_{s,t}(x,.)$ is $(\mathcal{F}_{s,t})$ -measurable.*
- *For almost all $\omega$, $\xi_{s,t}(x,\omega)$ is continuous in $(s,t,x)$ and satisfies $\lim_{t \downarrow s} \xi_{s,t}(x,\omega) = x$.*
- *For almost all $\omega$, $\xi_{s,t+u}(x,\omega) = \xi_{t,t+u}(\xi_{s,t}(x,\omega),\omega)$ is satisfied for all $s < t$ and $u > 0$.*
- *For almost all $\omega$, the map $\xi_{s,t}(.,\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is an onto homeomorphism for all $s < t$.*

**Theorem 2.4.2.** *Let $k$ be a positive integer. Suppose that coefficients of an Itô equation are $C_g^{k,\alpha}$ functions for some $\alpha > 0$ and their derivatives up to $k$-th order are bounded. Then the map $\xi_{s,t}(.,\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a $C^k$ - diffeomorphism for all $s < t$ a.s.*

*Proof.* Smoothness of the map $\xi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$ was shown in theorem 2.3.2. It is enough to show that the Jacobian matrix $\partial\xi_{s,t}(x) = (\frac{\partial \xi_{s,t}(x)}{\partial x})$ is nonsingular for any $x$ a.s. If it were shown then the implicit function theorem states that the inverse map is again of $C^k$-class. Now by theorem 2.3.1, the Jacobian matrix satisfies following linear SDE:

$$\partial\xi_{s,t} = I + \sum_{k=0}^{m} \int_s^t X_k'(r, \xi_{s,r}(x)) \partial\xi_{s,r} dB_r^k.$$

Consider an adjoint equation of the above:

$$\begin{aligned} K_{s,t}(x) = I \quad & - \quad \sum_{k=0}^{m} \int_s^t K_{s,r}(x) X_k'(r, \xi_{s,r}) dB_r^k \\ & - \quad \sum_{k=1}^{m} \int_s^t K_{s,r}(x) X_k'(r, \xi_{s,r})^2 dr. \end{aligned}$$

Obviously it has unique matrix solution $K_{s,t}(x)$. We can prove similarly as before

$$\mathbb{E}[\mid K_{s,t}(x) - K_{s',t'}(x') \mid^p] \leq C_{27}^{(p)} \{\mid x - x' \mid^{\alpha p} + (1 + \mid x \mid + \mid x' \mid)^{\alpha p} (\mid s' - s \mid^{\frac{\alpha p}{2}} + \mid t' - t \mid^{\frac{\alpha p}{2}})\}.$$

Hence $K_{s,t}(x)$ is continuous in $(s, t, x)$ a.s. By Itô's formula, it holds

$$\begin{aligned} K_{s,t}(x)\partial\xi_{s,t}(x) \quad & = \quad I + \int_s^t (dK_{s,r}(x))\partial\xi_{s,r}(x) + \int_s^t K_{s,r}(x) d\partial\xi_{s,r}(x) + < K(x), \xi(x) >_t \\ & = \quad I \end{aligned}$$

Therefore $\partial\xi_{s,t}(x)$ has the inverse matrix $K_{s,t}(x)$ for any $(s, t, x)$, proving $\partial\xi_{s,t}(x, \omega)$ is nonsingular for any $(s, t, x)$ a.s. □

# Conclusion

The subject we are discussing in this work is a topical issue and one that is very important. Many authors have made publications in this research path. What is now fashionable is the study of the properties of the stochastic flow in the case where the coefficients are non-Lipschitzian, and also in the case where the initial condition itself is a stochastic process.

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