République Algérienne Démocratique et Populaire

Ministère de l'enseignement supérieure et de la recherche scientifique

$N^{\circ}$ Attribué par la bibliothéque

Année: 2017/2018

# Asymptotic study for single-index conditional models

Thèse présentée en vue de l'obtention du grade de

Docteur

Universitaire de Saida

Discipline : MATHEMATIQUES

Spécialité: Modèles Stochastiques, Statistique et Applications

par

**D**iaa eddine **H**amdaoui[1]

Sous la direction de

*Pr* **T.G**uendouzi

| | | | |
|---|---|---|---|
| **Mr.A. K**andouci | Professeur | Univ. Saida | Président |
| **Mr.T. G**uendouzi | Professeur | Univ. Saida | Directeur de Thése |
| **Mlle.A.A. B**ouchentouf | Professeur | Univ. SBA | Co-Directeur de Thése |
| **Mlle.S. R**ahmani | Maître de conférences | Univ. Saida | Examinatrice |
| **Mr.A. R**abhi | Professeur | Univ. SBA | Examinateur |

[1]e-mail: diaa-eddine@hotmail.fr

# *Dedication*

This thesis is dedicated to the memory of my Professeur **Benamar Chouaf**

# *Acknowledgments*

First and fore most, praises and thanks to **Allah** ,the Almighty, for His showers of blessings throughout my research work to complete the research successfully.

I would like to express my deep and sincere gratitude to my research supervisors, professors **Amina Angelika Bouchentouf** and **Toufik Guendouzi**, for giving me the opportunity to do research and providing invaluable guidance throughout this research.Their dynamism, vision, sincerity and motivation have deeply inspired me.They have taught me the methodology to carry out the research and to present the research works as clearly as possible.It was a great privilege and honor to work and study under their guidance.I am extremely grateful for what they have offered me.I would also like to thank them for their friendship, empathy, and great sense of humor.I am extending my heartfelt thanks to their families for their acceptance and patience during the discussion I had with them on research work and thesis preparation. I appreciate all their contributions of time, ideas, and funding to make my Ph.D.Experience productive and stimulating.The joy and enthusiasm they have for my research was contagious and motivational for me, even during tough times in the Ph.D.Pursuit.I am also thankful for the excellent example they have provided as a successful mathematicians and professors.

Besides my research supervisors, I would like to thank the rest of my thesis committee: Prof.**Abdeljebbar Kandouci**, Prof.**Abbes Rabhi** and Dr.**Saadia Rahmani** for their encouragement, insightful comments, and hard questions.

I am extremely grateful to my parents for their love, prayers, caring and sacrifices for educating and preparing me for my future. I am very much thankful to my sisters, brother for their love, understanding, prayers and continuing support to complete this research work.

# Contents

# ملخص

في هذه الأطروحة، ندرس بعض الوسائط الوظيفية عندما يتم توليد البيانات من نموذج الانحدار ذي مؤشر بسيط.

أولا نقترح دراسة مشكلة النمذجة غير الوسيطية عندما تكون المتغيرات الإحصائية منحنيات وبشكل أدق، نحن مهتمون بالتنبؤ بالمشاكل من متغير تفسري ذي قيم في حيز من البعد اللانهائي (الفضاء الوضيفي) ونحن نسعى إلى تطوير بدائل لطريقة الانحدار.

في الواقع، نحن نفترض أن لدينا متغير عشوائي حقيقي (استجابة) وغالبا يرمز له y و متغير وظيفي (تفسيري) غالبا ما يرمزله x والنموذج الغير وسيطي المستخدم لدراسة العلاقة بين x و y هو التوزيع الشرطي الذي يفترض أن وظيفة التوزيع (على التوالي الكثافة) التي تشير إلى F (على التوالي f) تنتمي إلى حيز وظيفي مناسب.

وفي خطوة ثانية، نعتبر سلسلة من الملاحظات ذات توزيع غير مرتبط ونقوم ببناء تقدير بواسطة طريقة النواة للوظيفة من دالة التوزيع المشروط وكذلك الكثافة المشروطة، نتعامل مع الخصائص المتناظرة لهذه المقدرين في الحالة المستقلة، نحصل على الحالة الطبيعية المتقاربة للمقدرات التي تم بناؤها، حيث نناقش تأثير هذه النتيجة في التوقع من خلال تقدير الوضع الشرطي

# *Summary*


In this thesis, we propose to study single-index conditional functional models. The problem of non-parametric modeling when statistical variables are curves is studied.

More precisely, we are interested in forecasting problems using an explicative variable with values in an infinite dimensional space (functional space), and we seek to develop alternatives to the regression method. Indeed, we assume that we have a real random variable (response), often denoted Y and a (explanatory) functional variable, often denoted X. The nonparametric model used to study the link between X and Y concerns the conditional distribution whose distribution function (respectively the density), denoted F (respectively f), when the data are generated from a simple index regression model.

We suppose that the explanatory variable has values in a semi-metric space (infinite dimension) and we consider the estimation of the conditional models by the kernel method. We treat the asymptotic properties of these estimators in the independent case under standard conditions.

The asymptotic results obtained are well suited to the topological structure of the functional space of our observations and the functional character of the models considered.

# *Résumé*


Dans cette thèse, nous proposons d'étudier des modèles fonctionnels conditionnels à direction révélatrice. Le problème de la modélisation non paramétrique lorsque les variables statistiques sont des courbes est étudié.

Plus précisément, nous nous intéressons à des problèmes de prévisions à partir d'une variable explicative à valeurs dans un espace de dimension infinie (espace fonctionnel), et nous cherchons à développer des alternatives à la méthode de régression. En effet, nous supposons que nous disposons d'une variable aléatoire réelle (réponse), souvent notée Y et d'une variable fonctionnelle (explicative), souvent notée X. Le modèle non paramétrique utilisé pour étudier le lien entre X et Y concernent la distribution conditionnelle dont la fonction de répartition (respectivement la densité), notée F (respectivement f), lorsque les données sont générées à partir d'un modèle de régression à indice simple.

Nous supposons que la variable explicative est à valeurs dans un espace semi-métrique (dimension infinie) et nous considérons l'estimation des modèles conditionnelles par la méthode de noyau. Nous traitons les propriétés asymptotiques de ces estimateurs dans le cas indépendant sous des conditions standard.

Les résultats asymptotiques obtenus exploitent bien la structure topologique de l'espace fonctionnel de nos observations et le caractère fonctionnel des modèles considérés.

# *The List Of Works*

**Publications**

1. Diaa eddine Hamdaoui, Amina Angelika Bouchentouf, Abbes Rabhi and Toufik Guendouzi.Asymptotic normality of conditional distribution estimation in the single index model, Acta Univ. Sapientiae, Mathematatica, 9, 1, (2017), 162-175

2. Diaa Eddine Hamdaoui, Aboubacar Traore, Amina Angelika Bouchentouf, Abbes Rabhi.Nonparametric conditional density estimation in the single index model for independent functional data. Submitted in International Journal of Statistics and Economics

# Chapter 1

# Introduction to Functional Nonparametric Statistics

Over the last two decades, an immense innovation on measuring instruments has emerged and realized that enabling several objects to be monitored continuously, such as stock market indices, pollution, climatology, satellite images,... this technological development required the modernization of statistical methods as tools for analysis and control.Thus, a new branch of statistics, called functional statistics, has been developed to treat observations as functional random elements.The first contributions on the subject were devoted to the study of parametric models (see the monographs of Ramsay and Silverman (1997, 2002, 2005) for the i.i.d case or Bosq (2000) for the dependent case).However, statistical analysis via linear models is based on a preliminary knowledge of the nature of covariability between observations, which is very difficult to verify in functional statistics, contrary to the classical statistic where graphic tools are available such as the scatterplot which gives an overview on the relation between the observations.This justifies the importance of modeling functional data by nonparametric methods.

Nonparametric processing of functional data is much more recent than parametric analysis.Indeed, the first results were obtained by Gasser et al (1998).Authors were interested in the nonparametric estimation of the mode of distribution of a functional variable verifying a fractal condition.Considering the same fractal condition Ferraty and Vieu (2000) studied the almost complete convergence of a kernel estimator of the regression function, when the observations are independent and identically distributed.Dabo-Niang (2002) obtained, the almost sure convergence and the asymptotic normality of a histogram type estimator of the density of a random variable in a space of infinite dimension.Using the concentration property of the probability measure of the functional explanatory variable, Dabo-Niang and Rhomari (2004) studied the convergence in $L^p$ norm of the kernel estimator of the nonparametric regression.The almost complete convergence for the strongly mixing case was studied by Ferraty et al (2004).Masry (2005) showed asymptotic normality in the case of $\alpha$-mixing functional observations.The first results on the conditional models were obtained by Ferraty et al (2006).They have specified the almost complete rate of convergence of the kernel estimators for the conditional distribution function, the conditional density and its derivatives, the conditional mode and conditional quantiles.We refer to Ferraty and Vieu (2006) for a wide range of applications of these models in functional statistics.Dabo-Niang and Laksaci (2007) added results on the convergence in $L^p$ norm of the kernel estimator of the conditional mode in the i.i.d case.The determination of the dominant terms of the quadratic error of the kernel estimator of the conditional density was obtained by Laksaci (2007).Ferraty et al (2008) discussed the estimation of conditional hazard function and established the almost complete convergence of a kernel estimator of this nonparametric model.The asymptotic normality of the kernel estimators of the conditional mode and the conditional quantiles has been studied by Ezzahrioui and Ould-saïd (2008a, 2008b, 2008c) by treating two cases (i.i.d case and mixing case).Considering $\alpha$-mixing observations, Quantela-del Rio (2008) established the almost complete convergence and asymptotic normality of the estimator proposed by Ferraty et al (2008) on the conditional random function.Author illustrated these asymptotic results by applying them to seismic data.An alternative estimator of conditional quantiles has been proposed by Lemdani et al (2009).They treated conditional quantiles as a robust model belonging to the M-estimator class.The asymptotic results of the paper were almost complete convergence and asymptotic normality in the i.i.d case.

The contribution of Ferraty et al (2010) on uniform convergence is very important.Dabo-Niang in collaboration with Laksaci (2010) have generalized their results of the convergence in the $L^p$ norm of the kernel estimator of the conditional mode in the i.i.d case to the strongly mixing case.Considering the same structure of dependence, Lemdani et al (2011) studied the almost complete convergence and asymptotic normality of the $L^1$ estimator of the conditional quantiles.While the convergence in $L^p$ norm for the double-kernel estimator of conditional quantiles has recently been obtained by Dabo-Niang and Laksaci (2012).The question of the choice of the smoothing parameter in the estimation of the conditional variable with functional explanatory variable was considered by Laksaci et al (2012).Other authors have been interested in estimating the regression function using other approaches, such as the k nearest neighbors method of Burba et al (2008), robust techniques, Azzidine et al (2008), Crambes et al (2008), and Attouch et al (2009, 2010).For estimation by the method of local polynomials, we can see Baìllo et

Grané (2009), Barrientos-Marin et al (2010), Berlinet et al (2011) and Demongeot et al (2012).The literature on the case of a functional response variable is very restricted in functional statistics.We will cite, in this context, the article by Dabo-Niang and Rhomari (2009) for the convergence in the $L^p$ norm of the kernel estimator of the regression operator as a banach element.The almost complete convergence of this estimator is obtained by Ferraty et al (2011).Van Keilegom in collaboration with Ferraty and Vieu (2012) established the asymptotic normality of the estimator of the regression function, when the two variables (response, explanatory) are functional in nature.All these results have been obtained in the case where the observations are independent identically distributed.The dependent case was recently considered by Ferraty et al (2012).In their paper, the authors demonstrated the almost complete convergence of the kernel estimator of the regression operator for $\beta$-mixing observations.
In this thesis we are interested in functional data and giving a general bibliografic context on different functional models.

## 1.1 Functional Variable

There is actually an increasing number of situations coming from different fields of applied sciences (environmetrics, chemometrics, biometrics, medicine, econometrics, ...) in which the collected data are curves.Indeed, the progress of the computing tools, both in terms of memory and computational capacities, allows us to deal with large sets of data.In particular, for a single phenomenon, we can observe a very large set of variables.For instance, look at the following usual situation where some random variable can be observed at several different times in the range $(t_{\min}, t_{\max})$.An observation can be expressed by the random family $\{X(t_j)\}_{j=1,\dots,J}$.In modern statistics, the grid becomes finer and finer meaning that consecutive instants are closer and closer.One way to take this into account is to consider the data as an observation of the continuous family $X = \{X(t); t \in (t_{\min}, t_{\max})\}$.

**Definition 1.1.1.** *(Ferraty and Vieu 2006) A random variable $X$ is called functional variable (f.v.) if it takes values in an infinite dimensional space (or functional space).An observation $x$ of $X$ is called a functional data.*

Note that, when $X$ ($resp.x$) denotes a random curve (resp.Its observation), we implicitly make the following identification $X = \{X(t); t \in \mathrm{T}\}$(resp $x = \{x(t); t \in \mathrm{T}\}$).In this situation, the functional feature comes directly from the observations.The situation when the variable is a curve is associated with an unidimensional set $\mathrm{T} \subset \mathbb{R}$.Here, it is important to remark that the notion of functional variable covers a larger area than curves analysis.(Ferraty and Vieu 2006)

## 1.2 Nonparametric Statistics for Functional Data

Traditional statistical methods fail as soon as we deal with functional data.Indeed, if for instance we consider a sample of finely discretized curves, two crucial statistical problems appear.The first comes from the ratio between the size of the sample and the number of variables (each real variable corresponding to one discretized point).The second, is due to the existence of strong correlations between the variables and becomes an ill-conditioned problem in the context of multivariate linear model.So, there is a real necessity to develop statistical methods/models in order to take into account the functional structure of this kind of data.Most of existing statistical methods dealing with functional data use linear modelling for the object to be estimated.Key references on methodological aspects are those by Ramsay and Silverman (1997), (2005), while applied issues are discussed by Ramsay and Silverman (2002) and implementations are provided by Clarkson, Fraley and Ramsay (2005).Note also that, for some more specific problem, some theoretical support can be found in Bosq (2000).On the other hand, nonparametric statistics have been developped intensively.Indeed, since the beginning of the sixties, a lot of attention has been paid to free-modelling (both in a free-distribution and in a free-parameter meaning) statistical models and/or methods.The functional feature of these methods comes from the nature of the object to be estimated (such as for instance a density function, a regression function, ...) which is not assumed to be parametrizable by a finite number of real quantities.In this setting, one is usually speaking of **Nonparametric Statistics** for which there is an abundant literature.

**Definition 1.2.1.** *(Ferraty and Vieu 2006) Let $\mathcal{Z}$ be a random variable valued in some infinite dimensional space* $\mathrm{F}$ *and let $\phi$ be a mapping defined on* $\mathrm{F}$ *and depending on the distribution of $\mathcal{Z}$. A model for the estimation of $\phi$ consists in introducing some constraint of the form*

$$\phi \in \mathcal{C}$$

*The model is called a functional parametric model for the estimation of $\phi$ if $\mathcal{C}$ is indexed by a finite number of elements of* $\mathrm{F}$. *Otherwise, the model is called a functional nonparametric model.*

The appellation **Functional Nonparametric Statistics** covers all statistical backgrounds involving a nonparametric functional model.In the terminology **Functional Nonparametric Statistics**, the adjective **nonparametric** refers to the form of the set of constraints whereas the word **functional** is linked with the nature of the data.In other words, nonparametric aspects come from the infinite dimensional feature of the object to be estimated and functional designation is due to the infinite dimensional feature of the data.(Ferraty and Vieu 2006)

## 1.3 Functional data

One never observes an integral function over its entire trajectory.This would require a measuring instrument with an in- ternal recording speed.Even the fastest quotes on the fully computerized financial markets are spanned by a few milliseconds.When the functional data arrive they are for these reasons always in vector form.Thus we shall not observe, for example $X(t)$ $\forall$ t but we shall have $[X(t_1), X(t_2), ..., X(t_p)]$ where the $t_j$ constitute a discretization grid.According to the phenomenon studied p can vary between several units and several million.This type of data is not new and has been studied for a long time using multivariate techniques (seeing X as a random vector in $\mathbb{R}^p$ to continue the previous example).But there are two problems.

- If the frequency of discretization of the curves is high (i.e. if p is large) we can find ourselves in situations where the size of X is of the order or even greater than the size of the sample itself.This situation can pose prohibitive problems both from the theoretical point of view and from the numerical aspects.This problem is common with that of many problems of statistics in large dimensions.

- By treating X as a vector, we completely lose its true nature, that of process in continuous time or more generally of function.The derivation operation, for example, does not make sense in this context.It is logical then to ask the question of alternative methods in which, by failing to grasp X (t) $\forall$ t, one could be satisfied with an approximation $\tilde{X}$ which would be a real function.

Next, we present different types of functional data sets given in Delsol (2008).

## 1.4 Some functional datasets

We present in this section two particular functional data sets coming from quite varied domains (climatology and chemometrics).To illustrate the benefits and challenges associated with their study through a functional approach.

**Study of the El Niño phenomenon.** – We are now interested in a set of data from the study of a fairly important climatological phenomenon commonly called El Niño. It is a great marine current that occurs exceptionally (on average once or twice per decade) along the Peruvian coast at the end of winter.This current is causing climate-related disruptions on the planet.The dataset available to us consists of monthly sea surface temperature records made since 1950 in an area off northern Peru (At the coordinates 0 - $10^0$ South, 80 - $90^0$ West) where the El Niño marine current can appear.These data and descriptions are available on the US Climate Prediction Center : http ://www.cpc.ncep.noaa.gov/data/indices/.The evolution of temperatures over time is really a continuous phenomenon.The number of measurements we have is fairly well represented (see Figure 1.1) and allows us to consider the functional nature of the data.

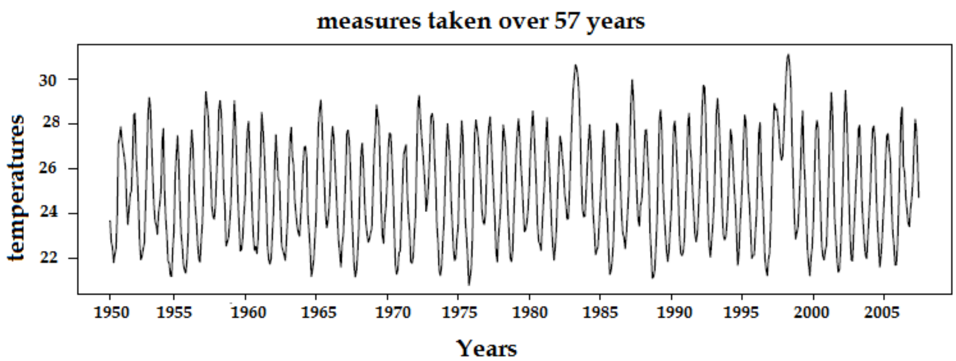


Figure 1.1: Monthly measurements of surface temperature around the El Niño marine current since 1950 (Delsol 2008).

On the basis of these data one may be interested in trying to predict the evolution of the phenomenon from the data collected in previous years. Several approaches have been introduced to try to address this problem.The first work attempts to predict the temperature of the following month from the previous monthly temperatures (see Katz, 2002, for more references).However, this modeling does not allow to take into account the functional nature of the phenomenon studied nor to benefit from it.More recently, following the approach introduced by Bosq (1991) we chose to consider the process no longer through its discretized version but rather as a continuous process that is broken down into successive curves of 12 months (between June of one year and May of the following year, (see Figure 1.2)).

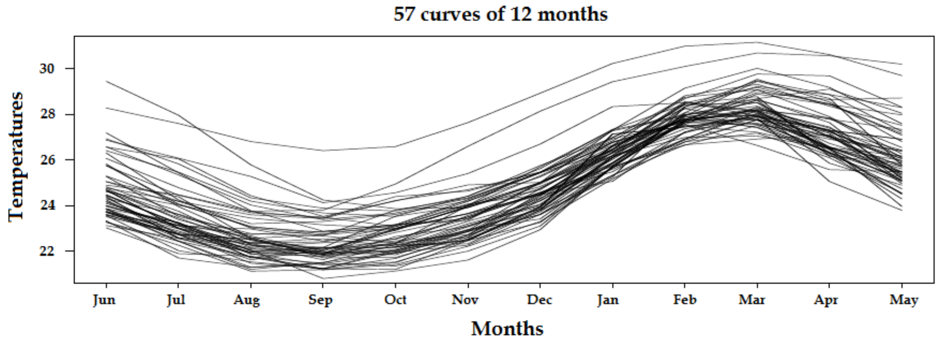


Figure 1.2: 57 curves representing the surface temperature of the El Niño marine current in 12 - month by slices since June 1950(Delsol 2008).

We can then try to predict the curve of the following twelve months (or certain particular values of it) from one or more preceding curves.This functional approach has the advantage of being adapted even when the number of discretization points increases and makes it possible to take into account, for each period of twelve months, the temperature curve as a whole.The work of Besse et al (2000), Valderama et al (2002), Antoniadis and Sapatinas (2003) or Ferraty et al (2005) illustrate different ways of responding to this problem and modeling the dependence of successive curves

**Grease and Spectrometric Measurements.** – We consider a set of data from quantitative chemistry (branch of chemistry using mathematical tools, also called chemometry) the data were collected to address a quality control problem in the agri-food industry.When packing minced meat, it is mandatory to put on the packaging the fat content.A chemical analysis can accurately give the fat content in a piece of meat.However this method takes time, partially deteriorates the piece studied and costs quite expensive, this is why we are interested in other more profitable methods thus, it has been envisaged to predict the fat content from spectrometric curves whose obtaining is less costly (time and money) and does not require partial deterioration of the meat being studied.The spectrometric data are obtained by measuring for each piece of meat the absorbance of lights of different wavelengths.These are intrinsically functional data, as emphasized Leurgans et al (1993) : the spectra observed are to all intents and purposes functional observations.They can therefore be summarized by curves (called spectrometric curves) representing the absorbance as a function of the wavelength.We are interested in a set of data from the study by chemical analysis and by spectrometry of 215 pieces of meat.Thus, we dispose 215 spectrometric measurements (see figure 1.3) and corresponding fat levels (obtained by chemical analysis).These data and their precise description are available on the StatLib website (http ://lib.stat.cmu.edu/datasets/tecator).

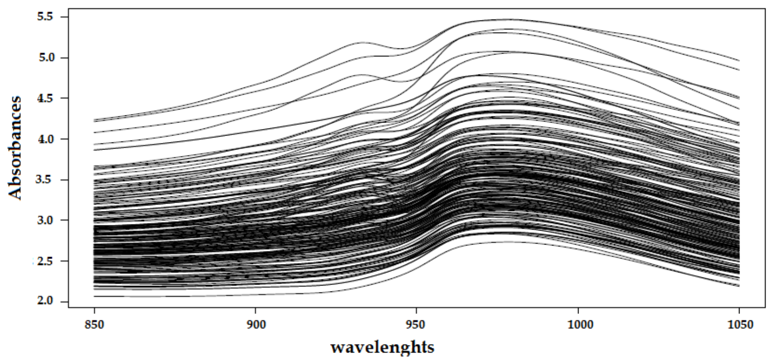


Figure 1.3: Spectrometric curves obtained from the 215 pieces of meat studied.

It can be seen that the spectrometric curves are very regular and of similar shapes outside a vertical translation.One wonders if this shift contains important information to predict the fat content.If so, it should be taken into account in choosing the semi-metric.Otherwise, since the curves are very smooth and of similar shapes, it may be useful to use the derived curves rather than the curves themselves.We are interested in predicting the fat content of a new piece of meat from the spectrometric curve associated with it.The earliest work on this problem is due to Borggaard and Thodberg (1992) and uses neural network methods.Since other articles including Ferraty and Vieu (2002), Ferré and Yao (2005).Ferraty and Vieu (2006), Ferraty, Mas and Vieu (2007) proposed and applied other methods to answer this problem.On the other hand, the results given in the first part of this paper complement the literature dedicated to the prediction of a real variable from a variable functional and can therefore be used to predict the fat level from of the spectrometric curve.

## 1.5 Other fields of application.

The two examples of datasets that we have presented are from quite different areas and reflect various issues.In order to illustrate the wide variety of fields where one can be confronted with data of a functional nature, we propose in this section a brief overview of these fields of application.Given the rise of functional statistics and the very large number of cases where it is used, we present only an incomplete panorama of its domains application. We have chosen to limit our references to publications bearing on the study of data corresponding to curves using functional statistics, giving preference to the most recent or precursor works in the existing literature.However, as with all non-parametric functional statistical results, the results of this thesis apply directly to samples of functional data of a different nature such as images, surfaces,...

- In the field of medicine, we can observe the use of functional statistics through studies of different phenomena such as the evolution of certain cancers (see for intance Ramsay and Silverman, 2005, Cao and Ramsay, 2007), Cardiac activity (see for instance Clot, 2002, Ratcliffe et al., 2002a, 2002b and Harezlak et al., 2007), Knee movements during effort Under constraints (see Abramovich and Angelini, 2006, and Antoniadis and Sapatinas, 2007), Or certain deformations of the cornea (see Locantore et al., 1999).

- In recent years, the genetics sector is booming. Thanks to advances in measuring equipment and methods the biologists manage to make several measurements of gene expression during time.The aim of these measures is to allow a better understanding of the function of the genes and the interactions between some of their effects (for example the phenomena of regularization of a substance by another).We are also interested in identifying responsible gene groups of the evolution of a complex biological phenomenon observed over time.Recently, several methods of functional statistics have been applied to the expression profiles of the genes over time through the work of Araki et al (2004), Leng et Müller (2006), Song et al (2007).

- In the field of animal biology, functional statistics were used to study the evolution of certain phenomena over time.We can, for example, mention the different studies on the evolution of laying of flies by Müller et Stadtmüller (2005), Cardot (2006), Chiou and Müller (2007).The study of phenomena related to the environment and their evolution is very often related to the study of functional data which may correspond to the evolution of a phenomenon over time or as a function of another parameter (Altitude, temperature, ...).Outside of the study of the El Niño marine current mentioned earlier, we find work on the prediction of ozone peaks (see for instance Aneiros-Perez et al, 2004, Cardot et al. 2004, 2006), the study of the evolution of ozone levels at different altitudes (Meiring, 2005), of pollution caused by certain greenhouse gases (Febrero et al., 2007), of water quality (Henderson, 2006, Nerini and Ghattas, 2007) or tests concerning radioactivity readings at different altitudes over time (Cardot et al., 2007).

- Measurements, and in particular images collected by satellites, are also data that can be studied using functional statistical methods.For example, the work of Vidakovic (2001) in the field of meteorology or those of Dabo-Niang et al (2004b, 2007) in the field of geophysics.They are interested in classifying curves collected by the satellite in different places of the Amazon to identify the nature of the soil.One can also evoke Cardot et al (2003), Cardot and Sarda (2006) which study the evolution of vegetation from satellite data

- In the field of econometrics we are confronted with numerous phenomena which can be modeled by functional variables.We can cite for example studies on the dynamics of the monthly index of perishable food production (Ramsay, 2002), The prediction of electrical consumption (Ferraty et al., 2002), the volatility of financial markets (Müller et al., 2007), the performance of a company (Kawasaki et Ando, 2004), the price evolution of an item at auction (Reddy and Das, 2006, Wang et al., 2007), electronic commerce (Jank et Shmueli, 2006) or the intensity of financial transactions (Laukaitis and Rackauskas, 2002 and Laukatis, 2006).One can refer to Kneip and Utikal (2001), Benko (2006) for additional references.

- Finally, we can study functional random variables even if we have independent real or multivariate initial data.This is the case when one wants to compare or study functions that can be estimated from the data.Among the typical examples of this type of situation, it is possible to evoke the comparison of different density functions (see Kneip and Utikal, 2001, Ramsay and Silverman, 2002, Delicado, 2007, regression functions (Härdle and Marron, 1990, Heckman and Zamar, 2000).

## 1.6 Other examples of functional data

The statistic for functional data or functional data analysis studies observations which are not real or vector variables but random curves.
Examples:

- The temperature curve recorded at a given point on the globe is a completely random continuous process.If the temperature is observed during N days it may be interesting to cut out the starting curve on N curves which plot the temperature for each of the observation days.Each of these daily curves can then be seen as an element of a sample of size N constituted of functional data

- Currently, experiments are carried out on the INRA campus to study the growth of maize plants from different varieties and subjected to explicit conditions, different errors.For each maize plant the measuring instruments collect a function which is indeed random (it depends on the varietal of maize, experimental conditions and other fluctuations ...)

-In the two preceding examples the random curves depend on time but the situation may be different.The spectrometric analysis of the materials (which aims to deduce physicochemical properties by examining a light spectrum from the material).Also produces random curves indexed by a wavelength (and more by time).
Next, here we presente breifly the semi metric spaces and the small ball probabilities; this post have been already given in Delsol (2008).

## 1.7 Semi-metric spaces

To study data it is often necessary to have a notion of distance between them.It is well known that in finite dimension all the metrics are equivalent.This is no longer the case in infinite dimension, which is why the choice of the metric (and therefore of the associated topology) is even more crucial for the study of functional random variables than it is in multivariate statistics.Many authors define or study functional variables as random variables with values in $\mathbb{L}^2([0;1])$ (See for example Crambes, Kneip and Sarda 2007), more generally in Hilbert space (see for example Preda, 2007), Banach (see for example Cuevas and Fraiman, 2004) or metric (see for example Dabo-Niang and Rhomari , 2003).On the other hand, Boscq (2000) considers samples of dependent functional random variables with value in Hilbert space (or Banach) obtained by cutting the same continuous-time process.In addition to the available metrics, it is often interesting to consider semi-metrics allowing a wider range of possible topologies that can be chosen depending on the nature of the data and the problem under consideration.This is why we have chosen in this thesis to consider and study functional variables defined as random variables with values in a semi-metric space of infinite dimension.
Apart from allowing the modeling of more general phenomena, another interest of using a semi-metric rather than a metric is that it can constitute an alternative to the problems

posed by large data dimensions.Indeed, we can take a semi-metric defined from a projection of our functional data in a space of smaller size than by performing in functional principal component analysis of our data (Besse and Ramsay, 1986, Yao and Lee, 2006), or by projecting them on a finite basis (wavelets, splines, ...).This reduces the size of the data and increases the speed of convergence of the methods used while preserving the functional nature of the data.We can choose the basis on which we project based on the knowledge we have of the nature of the functional variable.For example, we can choose the fourier basis if we assume that the functional variable observed is periodic.We can refer to Ramsay and Silverman (1997, 2005) for more complete discussion of the different approximation methods by projection of functional data.Further discussion of the value of using different types of semi-metric is made in Ferraty and Vieu (2006) (especially in Section 3.4).

It can be remembered that the choice of the semi-metric makes it possible both to take account of more varied situations and to be able to circumvent the scourge of the dimension.This choice, however, should not be made lightly but taking into account the nature of the data and the problem under study.

### 1.7.1 Semimetrics and Small Ball Probabilities

The curse of dimensionality is a well-known phenomenon in nonparametric regression on multivariate variable (see Stone, 1982).In multivariate nonparametric regression, convergence rates (for the dispersion part) are expressed in terms of $h_n^d$.In the functional case we adopt more general concentration notions called small ball probabilities and express our asymptotic results in function of these quantitities.Small ball probabilities are defined by :

$$\phi_x(h) = \mathrm{P}\big(X \in \mathrm{B}(x, h)\big)$$

The way they decrease to zero have a great influence on the convergence rate of the kernel estimator.One can find in many probability papers asymptotic equivalents for these small ball probabilities when d is a norm (see for instance Lifshits et al., 2006, Shmileva,2006, Gao et Li, 2007) or a specific seminorm (Aurzada et Simon, 2007).In the case of unsmooth processes (Brownian motion, Ornstein-Ullenbeck process,...), these small ball probabilities have an exponential form (with regard to $h_n$) and hence the convergence rate is a power of log (n) (see Ferraty et al., 2006, Section 5, Ferraty et Vieu, 2006a, Paragraph 13.3.2.).The choice of the semimetric d has a direct influence on the topology

and consequently on small ball probabilities.The diversity of semimetrics allows, in various situations, to find a topology that gives a relevant notion of proximity between curves (see Ferraty and Vieu, 2006a, Chapter 3).
One may wonder how to choose the semimetric in practice.A first method has been proposed by Ferraty et al.(2002b).One firstly has to choose a family of semimetrcs from the information one gets on the data.Then, one determines, for instance by cross-validation, the semimetric (among this family) that is the most adapted to the data.Theoretical justification of the usefulness of a particular semimetric is still an open problem.

### 1.7.2 The choice of the semi-metric, the banwidth and the kernel in practice.

The nonparametric functional approach presented in this paper depends on three parameters : the semimetric d, the bandwidth $h_n$ and the kernel function K.In this paragraph we discuss the way to choose them in practice.See the monograph by Ferraty and Vieu (2006) for a deeper discussion on this topic.

The metric choice is important in the multivariate case where all metrics are equivalent.In the functional case that is no longer true hence the metric choice is more crucial.Moreover in addition to the usual functional metrics it is worth considering semimetrics because it may be an alternative to the curse of dimensionality, it enables to take into account more general situations and it may be more relevant when observed curves are very smooth.See the monograph by Ferraty and Vieu (2006) for a deeper discussion on some arguments to choose the semimetric from the curves nature.For instance, when the observed curves are smooth, it may be interesting to use semimetrics based on derivatives (see for instance the spectrometric dataset studied in Ferraty and Vieu, 2006, p.106). In other cases, when the curves are not smooth, it may be useful to consider projection semimetrics (based for instance on functional principal components, first coefficients in Fourrier's decomposition, ...) to avoid the curse of dimensionality.Despite these considerations, it is allways possible to consider a family of semimetrics and choose the one that is the best adapted to the considered dataset by cross-validation.

The most popular way to choose the smoothing parameter from the dataset is to take the one given by cross-validation criterion : $h_{CV}$.The choice of the optimal smoothing parameter in functional regression by cross-validation criterion has been addressed in a theoretical way in the recent paper by Rachdi and Vieu (2007).To avoid the estimation of the bias term we have assumed the smoothing parameter to be small enough to make the bias term negligible with regard to the variance one.That is why we obtain in simulations that the best smoothing parameter seems to be smaller than $h_{CV}$.Moreover, there is no automatic method to choose the optimal bandwidth in practice that is why we use $h_{CV}$ .

As in the real case there exists various kinds of kernel functions.The main difference is that in the functional case we consider kernel functions with compact support $[0;1]$ such that $K(1) > 0$.Most standard kernel functions are the restriction of classical indicator, triangular, quadratic or gaussian kernels to the set $[0;1]$.The choice of the kernel function is linked we the smoothness of the operator we want to estimate.(Delsol 2008)

## 1.8 Conditional models in non-parametric statistics

The study of nonparametric models related to the conditional distribution has been widely considered in nonparametric statistics.Historically, the first results on these models were obtained by Roussas (1969).He treated the estimation of the conditional distribution function by the kernel method using Markov observations.He established the convergence in probability of the constructed estimator.An alternative estimator for the same model was developed by Stone (1977).The latter studied the empirical estimator of the conditional distribution function and applied the results obtained to the estimation of conditional quantiles as the generalized inverse of the conditional distribution function.Stute (1986) added results on the almost complete convergence of the kernel estimator of the distribution function of a vector random variable conditionally to a vector explanatory variable.The estimation of the conditional mode was treated for the first time by Collomb et al (1987).These authors showed the uniform convergence of the kernel estimator of this conditional model when the observations are $\phi - mixing$.In 1989, Samanta studied the asymptotic normality of the kernel estimator of conditional quantiles when the observations are independent and identically distributed.The latter, in collaboration with Thavaneswaran in 1990, obtained the same asymptotic property for a kernel estimator of the conditional mode by considering the i.i.d case. Roussas (1991) established the almost sure convergence of a kernel estimator.Of the conditional quantiles when the observations come from a Markov process.The contribution of Youndjé (1993) on the estimation of the conditional density is decisive.He addressed the question of the choice of the smoothing parameter by considering the two independent and dependent cases.Quintela and Vieu (1997) have treated the conditional mode as being the point that cancels the first order derivative of the conditional density and constructed an estimator for this model using the kernel estimator of the derivative of the conditional density.Ould-saïd (1997) studied the kernel estimator of the conditional mode from ergodic observations.We refer to Berlin et al.(1998a), Louani and Ould-Saïd (1999) for the convergence in law of the kernel estimator of the conditional mode in the $\alpha$-mixing case.The Berlinet et al.(1998b) gives a general theorem of the asymptotic normality of the conditional quantile estimators, independently of the correlation of the observations.Zhou and Liang (2000) used the $L^1$ approach to construct a conditional median estimator using $\alpha$-mixing observations.They showed the asymptotic normality of this estimator.The convergence in $L^p$ norm of the kernel estimator of the conditional density of a stationary Markov process was obtained by Laksaci and Yousfate (2002).Ioannides and Matzner (2002) constructed an estimator for the conditional mode, when, the observations are tainted by errors.In this article the authors focus on the almost sure convergence of the proposed estimator.While its asymptotic normality has been demonstrated by the same authors in Ioannides and Matzner (2004).Gannoun et al (2003) have approached the estimation of conditional quantiles by the $L^1$ method, they have established almost complete convergence and asymptotic normality.Considering the same model and the same estimation method, Lin and Li (2007) studied asymptotic normality from the associated variables.Other authors have been interested in estimating conditional models from censored or truncated observations (see, for example, Lemdani et al (2009) Liang and Uña-Àlvarez (2010, 2011), Khardani et al (2010, 2011 and 2012), Ould Saïd and Tatachak (2011) or Ould Saïd and Djabrane (2011)).

### 1.8.1 On the conditional distribution function

The estimation of the conditional distribution function in a functional framework was introduced by Ferraty et al (2006).They constructed a double kernel estimator for the conditional distribution function and specified the almost complete convergence rate of this estimator when the observations are independent and identically distributed.The case of $\alpha$-mixing observations was studied by Ferraty et al (2005).An example of application on prediction via the conditional median, as well as the determination of prediction intervals have been considered in this paper.Several authors have treated the estimation of the conditional distribution function as a preliminary study of the estimation of conditional quantiles.For example, Ezzahrioui and Ould-Saïd (2005,2006) which studied the asymptotic normality of this estimator in both cases (i.i.d. and $\alpha$-mixing).An alternative estimation method for conditional quantiles has been proposed by Laksaci et al (2009).The asymptotic results of this paper are almost complete convergence and asymptotic normality in the i.i.d case.We refer to Cardot et al (2004) for a linear approach to conditional quantiles in functional statistics.

### 1.8.2 On the conditional density

The estimation of the conditional density function and its derivatives, in functional statistics, was introduced by Ferraty et al (2006).These authors obtained almost complete convergence in the i.i.d case.Since this article, an abundant literature has developed on the estimation of the conditional density and its derivatives, in particular in order to use it to estimate the conditional mode.Indeed, considering $\alpha$-mixing observations, Ferraty et al (2005) established the almost complete convergence of a kernel estimator of the conditional mode defined by the random variable maximizing the conditional density.Alternatively, Ezzahrioui and Ould-Saïd (2005, 2006) estimated the conditional mode by the point which cancels the derivative of the kernel estimator of the conditional density.The latter focused on the asymptotic normality of the proposed estimator in both cases (i.i.d. and $\alpha$ mixing). The precision of the dominant terms of the quadratic error of the kernel estimator of the conditional density was obtained by Laksaci (2007).We refer to Laksaci et al (2010) for the choice of the smoothing parameter in the estimation of the conditional variable with a functional explanatory variable.

## 1.9 Some kernal types

For the sake of simplicity, we will consider only three kinds of kernels.

**Definition 1.9.1.** *(Ferraty and Vieu 2006)*

*i) A function $K$ from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ is called a kernel of type I if there exist two real constants $0 < \mathrm{C}_1 < \mathrm{C}_2 < \infty$ such that:*

$$\mathrm{C}_1 1_{[0,1]} \leq K \leq \mathrm{C}_2 1_{[0,1]}.$$

*ii)A function $K$ from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ is called a kernel of type $II$ if its support is $[0,1]$ and if its derivative $K'$exists on $[0,1]$ and satisfies for two real constants $-\infty < \mathrm{C}_2 < \mathrm{C}_1 < 0$:*

$$\mathrm{C}_2 \le K' \le \mathrm{C}_1.$$

The first kernel family contains the usual discontinuous kernels such as the asymmetrical box one while the second family contains the standard asymmetrical continuous ones (as the triangle, quadratic,...).

**Definition 1.9.2.** *(Ferraty and Vieu 2006) A function $K$ from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ with compact support $[-1,1]$ and such that $\forall u \in (0,1), K(u) > 0$ is called a kernel of type 0.*

## 1.10 Different Approaches to the Prediction Problem

Let us start by recalling some notation.Let $(X_i, Y_i)_{i=1,\dots,n}$, be $n$ independent pairs, identically distributed as $(X, Y)$ and valued in $E \times \mathbb{R}$, where $(E, d)$ is a semi-metric space (i.e.X is a f.r.v. and $d$ a semi-metric).Let $x$ (resp.$y$) be a fixed element of $E$ (resp.$\mathbb{R}$), let $\mathcal{N} \subset \mathrm{E}$ be a neighboorhood of $x$ and $S$ be a fixed compact subset of $\mathbb{R}$.Given $x$, let us denote by $\hat{y}$ a predicted value for the scalar response.We propose to predict the scalar response $Y$ from the functional predictor X by using various methods all based on the conditional distribution of $Y$ given X.This leads naturally to focus on some conditional features such as conditional expectation, median and mode.The regression (nonlinear) operator $r$ of $Y$ on X is defined by:

$$r(x) = \mathbb{E}(Y|X = x). \tag{1.1}$$

and the conditional cumulative distribution function $(c.d.f.)$ of $Y$ given X is defined by:

$$\forall y \in \mathbb{R}, F_Y^X(x, y) = \mathbb{P}(Y \le y|X = x). \tag{1.2}$$

In addition, if the probability distribution of $Y$ given X is absolutely continuous with respect to the Lebesgue measure, we note $f_Y^X(x, y)$ the value of the corresponding density function at $(x, y)$.Note that under a differentiability assumption on $F_Y^X(X, .)$, this functional conditional density can be written as

$$\forall y \in \mathbb{R}, f_Y^X(x,y) = \frac{\partial}{\partial y} F_Y^X(x,y). \tag{1.3}$$

For these two last definitions, we are implicitly assuming that there exists a regular version of this conditional probability.In the remainder of this thesis, this assumption will be done implicitly as long as we will need to introduce this conditional cdf $F_Y^X(x,y)$ or the conditional density $f_Y^X(x,y)$. It is clear that each of these nonlinear operators gives information about the link between X and $Y$ and thus can be useful for predicting $y$ given $x$.Indeed, each of them will lead to some specific prediction method.

- The first way to construct such a prediction is obtained directly from the regression operator by putting:

  $$\widehat{y} = \widehat{r}(x). \tag{1.4}$$

  $\widehat{r}$ being an estimator of $r$.

- The second one consists of considering the median $m(x)$ of the conditional c.d.f.$F_Y^X$:

  $$m(x) = inf\{y \in \mathbb{R}, F_Y^X(x,y) \geq \frac{1}{2}\}. \tag{1.5}$$

  and to use as predictor:

  $$\widehat{y} = \widehat{m}(x). \tag{1.6}$$

  where $\widehat{m}(x)$ is an estimator of this functional conditional median m$(x)$.Note that such a conditional median estimate will obviously depend on some previous estimation of the nonlinear operator $F_Y^X$.

- the third predictor is based directly on the mode $\theta(x)$ of the conditional density of $Y$ given X:

$$\theta(x) = \arg\sup_{y \in S} f_Y^X(x, y). \tag{1.7}$$

This definition assumes implicitly that $\theta(x)$ exists on $S$.The predictor is defined by:

$$\widehat{y} = \widehat{\theta}(x), \tag{1.8}$$

where $\widehat{\theta}(x)$ is an estimator of this functional conditional mode $\theta(x)$.Once again note that this conditional mode estimate will directly depend on some previous estimation of the nonlinear operator $f_Y^X$.(Ferraty and Vieu 2006)

## 1.11 Kernel Estimators

Once the nonparametric modelling has been introduced, we have to find ways to estimate the various mathematical objects exhibited in the previous models, namely the [nonlinear] operators $r$, $F_Y^X$ and $f_Y^X$.So kernel estimators are good candidates.They combine both of the following advantages: simple expression and ease of implementation.

### 1.11.1 Estimating the conditional c.d.f

We focus now on the estimator $\widehat{F}_Y^X$ of the conditional c.d.f. $F_Y^X$, but let us first explain how we can extend the idea previously used for the construction of the kernel regression estimator.Clearly, $F_Y^X(x, y) = \mathbb{P}(Y \leq y | X = x)$ can be expressed in terms of conditional expectation:

$$F_Y^X(x, y) = \mathbb{E}(1_{(-\infty, y]}(Y) | X = x),$$

and by analogy with the functional regression context, a naive kernel conditional c.d.f. estimator could be defined as follows:

$$\tilde{F}_Y^X(x, y) = \frac{\sum_{i=1}^n K(h_K^{-1} d(x, X_i)) \mathbb{1}_{(-\infty, y]}(Y_i)}{\sum_{i=1}^n K(h_K^{-1} d(x, X_i))} \tag{1.9}$$

By following the ideas previously developed by Roussas (1969) and Samanta (1989) in the finite dimensional case, it is easy to construct a smooth version of this naive estimator.To do so, it suffices to change the basic indicator function into a smooth c.d.f.Let $K_0$ be an usual symmetrical kernel (see examples in Section 1.9), let $H$ be defined as:

$$\forall u \in \mathbb{R} \qquad H(u) = \int_{-\infty}^{u} K_0(v)dv, \tag{1.10}$$

and define the kernel conditional c.d.f. estimator as follows:

$$\widehat{F}_Y^X(x,y) = \frac{\sum_{i=1}^{n} K(h_K^{-1} d(x,X_i)) H(h_H^{-1}(y-Y_i))}{\sum_{i=1}^{n} K(h_K^{-1} d(x,X_i))} \tag{1.11}$$

where $h_H$ is a strictly positive real number (depending on n)(Ferraty and Vieu 2006).

### 1.11.2 Estimating the conditional density

It is known that, under some differentiability assumption, the conditional density function can be obtained by derivating the conditional c.d.f. (see(1.3)).Since we have now at hand some estimator $\widehat{F}_Y^X$ of $F_Y^X$ it is natural to propose the following estimate:

$$\widehat{f}_Y^X(x,y) = \frac{\partial}{\partial y}\widehat{F}_Y^X \tag{1.12}$$

Assuming the differentiability of $H$, we have

$$\frac{\partial}{\partial y}\widehat{F}_Y^X = \frac{\sum_{i=1}^{n} K(h_K^{-1} d(x,X_i)) \frac{\partial}{\partial y} H(h_H^{-1}(y-Y_i))}{\sum_{i=1}^{n} K(h_K^{-1} d(x,X_i))} \tag{1.13}$$

and this is motivating the following expression for the kernel functional conditional density estimate:

$$\widehat{f}_Y^X(x,y) = \frac{\sum_{i=1}^{n} K(h_K^{-1} d(x,X_i)) \frac{1}{h_H} H'(h_H^{-1}(y-Y_i))}{\sum_{i=1}^{n} K(h_K^{-1} d(x,X_i))} \tag{1.14}$$

Next we can easily get the following kernel functional conditional mode estimator of $\theta(\chi)$:

$$\widehat{\theta}(x) = \arg\sup_{y \in S} \widehat{f}_Y^X(x, y) \tag{1.15}$$

(Ferraty and Vieu 2006)

## 1.12 Topological considerations

### 1.12.1 Kolmogorov's entropy

The purpose of this section is to emphasize the topological components of our study.Indeed, as indicated in Ferraty and Vieu (2006), all the asymptotic results in nonparametric statistics for functional variables are closely related to the concentration properties of the probability measure of the functional variable X.Here, we have moreover to take into account the uniformity aspect.To this end, let $\mathcal{S}_{\mathcal{F}}$ be a fixed subset of $\mathcal{H}$; we consider the following assumption:

$$\forall x \in \mathcal{S}_{\mathcal{F}}, 0 < \mathrm{C}\phi_x(h) \leq \mathbb{P}(X \in \mathrm{B}(x, h)) \leq \mathrm{C}^{'}\phi_x(h) < \infty$$

We can say that the first contribution of the topological structure of the functional space can be viewed through the function $\phi_x$ controlling the concentration of the measure of probability of the functional variable on asmall ball.Moreover, for the uniform consistency, where the main tool is to cover a subset $\mathcal{S}_{\mathcal{F}}$ with a finite number of balls, one introduces an other topological concept defined as follows:

**Definition 1.12.1.** *Let $\mathcal{S}_{\mathcal{F}}$ be a subset of a semi-metric space $\mathcal{H}$, and let $\varepsilon > 0$ be given.A finite set of points $x_1, x_2, ..., x_N$ in $\mathcal{F}$ is called an $\varepsilon$-net for $\mathcal{S}_{\mathcal{F}}$ if $\mathcal{S}_{\mathcal{F}} \subset \bigcup_{k=1}^{N} \mathrm{B}(x_k, \varepsilon)$*
*The quantity $\psi_{\mathcal{S}_{\mathcal{F}}} = log(N_\varepsilon(\mathcal{S}_{\mathcal{F}}))$, where $N_\varepsilon(\mathcal{S}_{\mathcal{F}})$ is the minimal number of open balls in $\mathcal{F}$ of radius $\varepsilon$ wich is necessary to cover $\mathcal{S}_{\mathcal{F}}$, is called the Kolmogorov's $\varepsilon$-entropy of the set $\mathcal{S}_{\mathcal{F}}$*

This concept was introduced by Kolmogorov in the mid-1950's (see, Kolmogorov and Tikhomirov, 1959) and it represents a measure of the complexity of a set, in sense that, high entropy means that much information is needed to describe an element with an accuracy $\varepsilon$.Therefore, the choice of the topological structure (with other words, the choice of the semi-metric) will play a crucial role when one is looking at uniform (over some subset $\mathcal{S}_{\mathcal{F}}$) of $\mathcal{F}$ asymptotic results.More precisely, we will see thereafter that a good semi-metric can increase the concentration of the probability measure of the functional variable X as well as minimize the $\varepsilon$-entropy of the subset $\mathcal{S}_{\mathcal{F}}$. In an earlier contribution (see, Ferraty et al., 2006) we highlighted the phenomenon of concentration of the probability measure of the functional variable by computing the small ball probabilities in various standard situations.We will devote Section 1.12.2 to discuss the behaviour of the Kolmogorov's $\varepsilon$-entropy in these standard situations.Finally, we invite the readers interested in these two concepts (entropy and small ball probabilities) or/and the use of the Kolmogorov's $\varepsilon$-entropy in dimensionality reduction problems to refer to respectively, Kuelbs and Li (1993) or/and Theodoros and Yannis (1997).

### 1.12.2 Some examples

We will start example (1.12.1) by recalling how this notion behaves in unfunctional case (that is when $\mathcal{F} = \mathbb{R}^P$).More interestingly (from statistical point of view) is example (1.12.2) since it allows to construct, in any case, a semi-metric with reasonably "small" entropy.

**Example 1.12.1.** *(Compact subset in finite dimensional space) :*
*A standard theorem of topology guaranties that for each compact subset $\mathcal{S}_{\mathcal{F}}$ of $\mathbb{R}^P$ and for each $\varepsilon > 0$ there is a finite $\varepsilon$-net and we have for any $\varepsilon > 0$,*

$$\psi_{\mathcal{S}_{\mathcal{F}}}(\varepsilon) \leq \mathrm{C}_p\, log(1/\varepsilon).$$

*More precisely, Chate and Courbage (1997) have shown that, for any $\varepsilon > 0$ the regular polyhedron in $\mathbb{R}^P$ with length $r$ can be covered by $([2r\sqrt{p}/\varepsilon] + 1)$ balls, where [m] is the largest integer which is less than or equal to m.Thus, the Kolmogorov's $\varepsilon$-entropy of a polyhedron $P_r$ in $\mathbb{R}^P$ with length $r$ is*

$$\forall \varepsilon > 0, \psi_{P_r}(\varepsilon) \sim p\, log([2r\sqrt{p}/\varepsilon] + 1).$$

**Example 1.12.2.** *(Compact subset in a Hilbert space with a projection semimetric):*
*The projection-based semi-metrics are constructed in the following way.Assume that $\mathcal{H}$ is a separable Hilbert space, with inner product $< .,. >$ and with orthonormal basis $\{e_1, ..., e_j, ...\}$, and let $k$ be a fixed integer, $k > 0$.As shown in Lemma (13.6) of Ferraty*

*and Vieu (2006), a semi-metric $d_k$ on $\mathcal{H}$ can be defined as follows*

$$d_k(x, x') = \sqrt{\sum_{i=1}^{k} < x - x', e_j >^2}. \tag{1.16}$$

*Let $\chi$ be the operator defined from $\mathcal{H}$ into $\mathbb{R}^k$ by*

$$\chi(x) = (< x, e_1 >, ..., < x, e_k >)$$

*and let $d_{eucl}$ be the euclidian distance on $\mathbb{R}^k$, and let us denote by $\mathbf{B}_{eucl}(.,\ .)$ an open ball of $\mathbb{R}^k$ for the associated topology.Similarly, let us note by $\mathbf{B}_k(.,.)$ an open ball of $\mathcal{H}$ for the semi-metric $d_k$.Because is a continuous map from $(\mathcal{H}, d_k)$ into $(\mathbb{R}^k, d_{eucl})$, we have that for any compact subset $\mathcal{S}$ of $(\mathcal{H}, d_k)$, $\chi(\mathcal{S})$ is a compact subset of $\mathbb{R}^k$.Therefore, for each $\varepsilon > 0$ we can cover $\chi(\mathcal{S})$ with balls of centers $z_i \in \mathbb{R}^k$:*

$$\chi(\mathcal{S}) \subset \bigcup_{i=1}^{d} \mathbf{B}_{eucl}(z_i, r), \quad with \quad dr^k = \mathbf{C} \quad for\, some \quad \mathbf{C} > 0 \tag{1.17}$$

*For $i = 1, ..., d$, let $x_i$ be an element of $\mathcal{H}$ such that $\chi(x_i) = z_i$.The solution of the equation $\chi(x) = z_i$ is not unique in general, but just take $x_i$ to be one of these solutions.Because of (1.16), we have that*

$$\chi^{-1}(\mathbf{B}_{eucl}(z_i, r)) = \mathbf{B}_k(x_i, r) \tag{1.18}$$

*Finally, (1.17) and (1.18) are enough to show that the Kolmogorov's $\varepsilon$-entropy of $\mathcal{S}$ is*

$$\psi\mathcal{S}(\varepsilon) \approx \mathbf{C}k\, log\left(\frac{1}{\varepsilon}\right).$$

## 1.13 On the problematic of single index models

For several years, an increasing interest has been worn to models which incorporating by both parts parametric and nonparametric.Such models types are called semi-parametric model.This is mainly due to the problems associated with the poor specification of certain models.Tackle a problem of misspecification semiparametric way consists in not specifying the functional form of some model components.This approach completes those non-parametric models, which can not be useful in small samples, or with a large number of variables.For example, in the classical regression case, the important parameter whose one assumed its existence is the regression function of Y knowing the covariate X, denoted $r(x) = \mathbb{E}(Y \mid X = x).X, Y \in \mathbb{R}^d \times \mathbb{R}$ for this model, the non-parametric method considers only regularity assumptions on the function r.Obviously, this method has some drawbacks.One can cite the problem of curse of dimensionality.This problem appears when the number of regressors d increases, the rate of convergence of the nonparametric estimator r which is supposed k times differentiable is $O(n^{-k/2k+d})$ deteriorate.The second drawback is the lack of means to quantify the effect of each explanatory variable.To alleviate in these drawbacks, an alternative approach is naturally provided by the semi-parametric model which supposes the introduction of a parameter on the regressors, by writing that the regression function is of the form

$$\mathbb{E}_\theta(Y \mid X) = \mathbb{E}(Y \mid < X, \theta >= x),$$

The models defined are known in the literature as the single-index models.These models allow to obtain a compromise between parametric models, generally too restrictive and nonparametric model where the rate of convergence of the estimators deteriorate quickly in the presence of a large number of explanatory variables.In this area, differents types of models have been studied in the literature : amongst the most famous, there may be mentioned additive models, partially linear models or single index models.The idea of these models, in the case of estimating the conditional density or regression consists in bringing to the covariates a dimension smaller than dimension of the variable space, thus allowing overcome the problem of curse of dimensionality.For example, in the partially linear model,we decompose the quantity to be estimated, into a linear part and a functional part.This latter quantity does not pose estimation problem since it's expressed as a function of explanatory variables of finite dimension, thus avoiding the problems associated with curse of dimensionality.In order to treat the problem of curse of dimensionality in the case of chronological series, several semi-parametric approaches have been proposed.Without preting to exhaustivity, we quote for example: Xia and An (2002) for the index model.A general presentation of this type of model is given in Ichimura et al.(1993) where the convergence and asymptotic normality are obtained.In the case of M-estimators, Delecroix et al (1999) proves the consistency and asymptotic normality of the estimate the index and they study its effectiveness.The statistical literature on these methods is rich, quote Huber (1985) and Hall (1989) present an estimation method which consists projecting the density and the regression function on a space of dimension one, to bring a non-parametric estimation for dimensional covariates.These amounts exactly estimate these functions in a single index model.Attaoui et al (2011) have established the pointwise and the uniform almost complete convergence (with the rate) of the kernel estimate of this model.The interest of their study is to show how the estimate of the conditional density can be used to obtain an estimate of the simple functional index if the latter is unknown. More precisely, this parameter can be estimated by pseudo-maximum likelihood method which is based on the preliminary estimate of the conditional density.Recently Mahiddine et al (2014) have established the pointwise almost complete convergence and the uniform almost complete convergence (with the rate) of some characteristics of the conditional distribution and the successive derivatives of the conditional density when the observations are linked with a single-index structure and they are applied to the estimations of the conditional mode and conditional quantiles.

The single-index approach is widely applied in econometrics as a reasonable compromise between nonparametric and parametric models.Such kind of modelization is intensively studied in the multivariate case.Without pretsenting to exhaustivity, we quote for example Härdle et al (1993), Hristache et al (2001).Based on the regression function, Delecroix et al (2003) studied the estimation of the single-index and established some asymptotic properties.The literature is strictly limited in the case where the explanatory variable is functional (that is a curve).The first asymptotic properties in the fixed functional single-model were obtained by Ferraty et al (2003).They established the almost complete convergence, in the i.i.d.case, of the link regression function of this model.Their results were extended to dependent case by Aït-Saidi et al (2005). Aït-Saidi et al (2008) studied the case where the functional single-index is unknown.They proposed an estimator of this parameter, based on the cross-validation procedure.

## 1.14 Structure of the thesis

In this thesis we are interested in the estimation of some functional parameters in the conditional models.We deal with the estimation of conditional density and conditional distribution.The explanatory variable for the two functional parameters is of infinite dimension.

The first chapter is an introductory chapter where our objective is to recall some notions of functional estimation (functional data) and a bibliographic context on the density function and the conditional distribution function as well as notations and asymptotic results obtained

After briefly describing the thesis, we will present in the second chapter the conditional distribution function in the case where the variables are identically distributed.In this case, we construct a kernel estimator for this functional parameter.We establish the asymptotic normality of this estimator.Our study takes into account the concentration of the probability measure of the explanatory variable via the structure of the functional index in small balls.

We study the estimation of conditional density function based on the single-index model for independent functional data.Under general conditions, the asymptotic normality of the conditional density estimator is established, a numerical study is presented in order to illuminate our theoretical result.

Finally, we conclude the thesis with a conclusion and some extensions of our work related to the content of this research area.

# Bibliography


Aït-Saidi, A. Ferraty, F. Kassa, R. (2005). Single functional index model for a time series. R. Roumaine Math. Pures et Appl. 50, 321-330.

Aït-Saidi, A. Ferraty, F. Kassa, R. Vieu, P. (2008). Cross-validated estimation in the single functional index model. Statistics. 42, 475-494.

Akritas, M. Politis, D.(2003) (ed.) Recent advances and trends in nonparametric statistics. Elsevier, Amsterdam .

Antoniadis, A. Sapatinas, T. (2003). Wavelet methods for continuous-time prediction using Hilbert-valued autoregressive processes. J.Multivariate Anal. 87 (1) 133-158.

Attouch, M. Laksaci, A. Ould-Saïd, E. (2009). Asymptotic distribution of robust estimator for functional nonparametric models. Comm. Statist. Theory and Methods. 38, 1317-1335.

Attouch, M. Laksaci, A. Ould-Saïd, E. (2010). Asymptotic normality of a robust estimator of the regression function for functional time series data. J. of the Korean Statist. Soc. 39,489-500.

Attaoui, S. Laksaci, A. Ould-Saïd, E. (2011). A note on the conditional density estimate in the single functional index model. Statist. Probab. Lett. 81 (1) 45-53.

Aurzada, F. Simon, T. (2007). small ball probabilities for stable convolutions.ESAIM Probab. Stat. 11, 327-343 (electronic).

Azzedine, N. Laksaci, A. Ould SaAïd, E. (2008). On the robust nonparametric regression estimation for functional regressor. Statist. Probab. Lett.78, 3216-3221

Baìllo, A. and Grané, A. (2009). Local linear regression for functional predictor and scalar response, Journal of Multivariate Analysis. 100, 102-111

Barrientos-Marin, J. Ferraty, F. Vieu, P. (2010). Locally Modelled Regression and Functional Data. Journal of Nonparametric Statistics. 22, 617-632

Benko. M, Härdle, W. Kneip, A.(2006). Common Functional Principal Components SFB 649 Discussion Papers SFB649 DP 2006-010, Humboldt University, Berlin, Germany.

Berlinet, A. Gannoun, A. Matzner-Lober, E. (1998a). Normalité asymptotique d'estimateurs convergents du mode conditionnel. La Rev. Canad. de Statist. 26, 365-380.

Berlinet, A. Gannoun, A. Matzner-Lober, E. (1998b). Propriétés asymptotiques d'estimateurs convergents des quantiles conditionnels. C. R. Acad. Sci., Paris, Sér. I, Math. 326,611-614.

Besse, P. Ramsay, J.O. (1986). Principal components analysis of sampled functions. Psychometrika. 51 (2) 285-311.

Borggaard, C. Thodberg, H.H.(1992). Optimal minimal neural interpretation of spectra Analytical chemistry. 64 (5) 545 - 551.

Bosq, D. Lecoutre, J.P.(1987). Théorie de l'estimation fo.nctionnelle (in french). Economica.

Bosq, D. (1991). Modelization, nonparametric estimation and prediction for continuous time processes. In Nonparametric functional estimation and related topics (Spetses, 1990), 509-529, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.335, Kluwer Acad. Publ., Dordrecht.

Bosq, D. (2000). Linear processes in function spaces. Theory and Applications, Lecture Notes in Statistics, 149, Springer-Verlag, New York.

Cao, J. Ramsay, J.O. (2007). Parameter cascades ans profiling in functional data analysis Comp. Stat. 22 (3) 335-351.

Cardot, H. Crambes, C. Sarda, P. (2004). Spline estimation of conditionalquantiles for functional covariates, C. R. Math. Acad. Sci. Paris. 339,141-144.

Chate, H., Courbage, M. (1997). Lattice systems. Physica D. 103, 1 -612.

Clarkson, D.B. Fraley, C. Gu, C.C. Ramsay, J.S.(2005). S+ functional data analysis user's guide. Comp. Statist. Series, Springer, New York .

Clot, D. (2002). Using functional PCA for cardiac motion exploration Proceedings of the. IEEE International Conference on Data Mining. 91-98.

Collomb, G. Härdle, W. and Hassani, S. (1987). A note on prediction via conditional mode estimation. J. Statist. Plann. and Inf. 15, 227-236.

Crambes, C. Delsol, L. and Laksaci, A. (2008). Robust nonparametric estimation for functional data. J. Nonparametric Statist. 20, 573-598.

Dabo-Niang, S. (2002). Estimation de la densité dans un espace de dimension infinienie : Application aux diffusions. C. R. Math. Acad. Sci. Paris . 334, 213-216.

Dabo-Niang, S. Rhomari, N. (2003). Estimation non paramétrique de la régression avec variable explicative dans un espace métrique. C. R., Math., Acad. Sci. Paris. 336, 75-80.

Dabo-Niang, S. (2004). Kernel density estimator in an infinite dimensional space with a rate of convergence in the case of difusion process. Applied Math. Lett. 17, 381-386.

Dabo-Niang, S. Yao, A-F. (2007). Kernel regression estimation for continuous spatial processes. Math. Methods. Statist. 16, 298-317.

Dabo-Niang, S. Laksaci, A. (2007). Propriétées asymptotiques d'un estimateur à noyau du mode conditionnel pour variable explicative fonctionnelle. Ann. I.S.U.P. 51, 27-42.

Dabo-Niang, S. Rhomari, N. (2009). Kernel regression estimation in a Banach space. J. Statist. Plann. Inference. 139, 1421-1434.

Dabo-Niang, S. and Laksaci, A. (2010). Note on conditional mode estimation for functional dependent data, Statistica. 70, 83-94.

Dabo-Niang,S. Kaid,Z. Laksaci, A. (2012). On spatial conditional mode estimation for a functional regressor Statist. Probab. Lett. 82, 1413-1421.

Dabo-Niang,S. Kaid,Z. Laksaci, A. (2012). Spatial conditional quantile regression : Weak consistency of a kernel estimate. 57 (4) 311-339 .

Dabo-Niang,S. Kaid,Z. Laksaci, A. (2012). Asymptotic properties of the kernel estimate of the spatial conditional mode when the regressor is functional

Delecroix, M, Härdle, W. Hristache, M. (1999). M-estimateurs semiparamétriques dans les modéles à direction révélatrice unique. Bull. Belg. Math. Soc. Simon Stevin. 6 (2) 161-185.

Delecroix, M. Härdle, W. Hristache, M. (2003). Efficient estimation in conditional single-index regression. J. Multivariate Anal. 86, 213-226.

Delsol, L. (2007). CLT and Lq errors in nonparametric functional regression. C. R. Math. Acad. Sci. Paris. 345, 411-414.

Demongeot, J., Laksaci, A., Madani, F. and Rachdi, M. (2011). Functional data : Local linear estimation of the conditional density and its application. Statistics. 85-90.

Ezzahrioui, M., Ould-Saïd, E. (2005). Asymptotic normality of nonparametricestimators of the conditional mode for functional data. Technical report, (249), LMPA, Univ. Littoral Cote d'Opale.

Ezzahrioui, M. Ould-Saïd, E. (2006). On the asymptotic properties of a nonparametric estimator of the conditional mode for functional dependent data. Preprint, LMPA (277), Univ. du Littoral Cote d'Opale.

Ezzahrioui, M. and Ould-Saïd, E. (2008a). Asymptotic normality of a nonparametric estimator of the conditional mode function for functional data. J. Nonparametr. Stat. 20, 3-18.

Ezzahrioui, M. and Ould-Saïd, E. (2008b). Asymptotic normality of the kernel estimator of conditional quantiles in a normed space. Far East J. Theor. Stat. 25, 15-38.

Ezzahrioui, M. and Ould-Saïd, E. (2008c). Asymptotic results of a nonparametric conditional quantile estimator for functional time series. Comm. Statist. Theory Methods. 37, 2735-2759.

Faden, A.(1985). The existence of regular conditional probabilities: necessary and sufficient conditions. Ann. Probab. 13, 288-298 .

Ferraty, F. Vieu, P. (2000). Dimension fractale et estimation de la régression dans des espaces vectoriels semi-normés. C. R. Acad. Sci., Paris. 330, 139-142.

Ferraty, F. Vieu, P.(2002). The functional nonparametric model and application to spectrometric data. Comput. Statist. 17 (4) 545-564.

Ferraty, F. Goia, A. Vieu, P. (2002a). Régression non-paramétrique pour des variables aléatoires fonctionnelles mélangeantes. (French) [Nonparametric regression for mixing functional random variables] C. R. Math. Acad. Sci.Paris. 334 (3) 217-220.

Ferraty, F. Goia, A. Vieu, P. (2002b). Functional nonparametric model for time series : a fractal approach for dimension reduction. Test. 11 (2) 317-344

Ferraty, F. Vieu, P.(2003). Functional nonparametric statistics: a double infinite dimensional framework. In: M. Akritas and D. Politis (eds.) Recenadvances and trends in nonparametric statistics. Elsevier, Amsterdam. 61-78.

Ferraty, F. Peuch, A. and Vieu, P. (2003). Modèle à indice fonctionnel simple,C. R. Acad. Sci., Paris. 336, 1025-1028.

Ferraty, F. Vieu, P.(2003). Curves discrimination: a nonparametric functional approach. Computational Statistics and Data Analysis. 44, 161-173.

Ferraty, F. Vieu, P. (2004). Nonparametric models for functional data, with application in regression times series prediction and curves discrimination. J. Nonparametric Statist. 16, 111-127.

Ferraty, F. Laksaci, A. Vieu, P. (2005). Functional time series prediction via conditional mode estimation. C. R. Math. Acad. Sci. Paris. 340, 389-392.

Ferraty, F. Rabhi, A. Vieu, P. (2005). Conditional quantiles for dependent functional data with application to the climatic El Niño phenomenon. Sankhy*ā*. 67 (2) 378-398.

Ferraty, F. Vieu, P. (2006). Nonparametric functional data analysis. Theory and Practice. Springer-Verlag.

Ferraty, F. Laksaci, A. Vieu, P.(2006). Estimation some characteristics of the conditional distribution in nonparametric functional models. Stat Inference Stoch. Process. 9, 47-76.

Ferraty, F.Mas, A. Vieu, P. (2007). Advances on nonparametric regression for fonctionnal data. ANZ Journal of Statistics. 49, 267-286.

Ferraty, F. Rabhi, A. et Vieu, P. (2008). Estimation non-paramétrique de la fonction de hasard avec variable explicative fonctionnelle. Rev.Roumaine Math. Pures Appl. 53, 1-18.

Ferraty, F. Laksaci, A. Tadj, A. and Vieu, P. (2010). Rate of uniform consistency for nonparametric estimates with functional variables. J. Statist. Plann. Inference. 140, 335-352.

Ferraty, F. Laksaci, A. Tadj, A. and Vieu, P. (2011). Kernel regression with functional response. Electron. J. Stat. 5 , 159-171.

Ferraty, F. and Romain, Y. (2011). The Oxford handbook of functional data analysis. Oxford University Press.

Ferraty, F. Laksaci, A. Tadj, A. and Vieu, P. (2012). Estimation de la fonction de régression pour variable explicative et réponse fonctionnelles dépendantes C. R. Acad. Sci. Maths. Paris. 350, 13-14

Ferraty, F. Van Keilegom, I. Vieu, P. (2012). Regression when both response and predictor are functions J. Multivariate Anal. 109, 10-28.

Ferré, L. Yao, A.-F. (2005). Smoothed functional inverse regression. Statist.Sinica. 15 (3) 665-683.

Gannoun, A. Saracco, J. and Yu, K. (2003). Nonparametric prediction by conditional median and quantiles,J. Statist. Plann. Inference. 117, 207-223.

Gao, F. et Li, W.V. (2007). Small ball probabilities for the Slepian Gaussian fields. Trans. Amer. Math. Soc. 359 (3) 1339-1350 (electronic).

Gasser, T. Hall, P. and Presnell, B. (1998). Nonparametric estimation of the mode of a distribution of random curves. J. R. Stat. Soc.Ser. B, Stat. Methodol. 60, 681-691.

Hall, P. (1989). On projection pursuit regression. Ann. Statist. 17 (2) 573-588.

Hall. P, Vial. C.(2006a). Assessing extrema of empirical principal component. functions. Ann. Statist. 34, 1518-1544.

Härdle, W.(1990). Applied nonparametric regression. Cambridge Univ. Press, UK .

Härdle, W. Hall, P. Ichumira, H. (1993). Optimal smoothing in singleindex models, Ann. Statist. 21, 157-178.

Härdle, W. Müller, M.(2000). Multivariate and semiparametric regression. In:M. Schimek (Ed) Smoothing and regression; Approaches, Computation, and Application. Wiley Series in Probability and Statistics, Wiley, New York. 357-392.

Harezlak, J. Coull, B.A. Laird, N.M. Magari, S.R. Christiani, D.C.(2007). Penalized solutions to functional regression problems Comp. Stat. and Data. Anal.51 (10) 4911-4925.

Helland, I.(1990). Pls regression and statistical models, Scand. J. Statist. 17, 97-114.

Huber, P. J. (1985). Projection pursuit. Ann. Statist. 13 (2) 435-475.

Hristache, M. Juditsky, A. Spokoiny, V. (2001). Direct estimation of the index coefficient in the single-index model. Ann. Statist. 29, 595-623.

Ichimura, H. (1993). Semiparametric least squares (SLS) and weighted SLS estimation of single-index models. Journal of Econometrics. 58, 71-120.

Ioannides, D. A. Matzner-Lober, E.(2002). Nonparametric estimation of the conditional mode with errors-in-variables : strong consistency for mixing processes. J. Nonparametr.Stat. 14, 341-352

Ioannides, D. A. Matzner-Lober, E. (2004). A note on asymptotic normality of convergent estimates of the conditional mode with errors-in-variables. J. Nonparametr. Stat. 16, 515-524

Khardani, S. Lemdani, M.Ould Saïd, E. (2010). Some asymptotic properties for a mooth kernel estimator of the conditional mode under random censorship. J. Korean Statist. Soc. 39, 455-469.

Khardani, S. Lemdani, M. and Ould Saïd, E. (2011). Uniform rate of strong consistency for a smooth kernel estimator of the conditional mode for censored time series. J. Statist. Plann. Inference 141, 3426-3436

Khardani, S. Lemdani, M. Ould Saïd, E. (2012). On the strong uniform consistency of the mode estimator for censored time series. Metrika 75, 229-241.

Kolmogorov, A. N. Tikhomirov, V. M. (1959). $\varepsilon-$entropy and .$\varepsilon-$capacity. Uspekhi Mat. Nauk. 14, 3-86. 2, 277-364 (1961).

Kuelbs, J. Li, W. (1993). Metric entropy and the small ball problem for Gaussian measures. J. Funct. Anal. 116, 133-157.

Laksaci, A. Yousfate, A. (2002). Estimation fonctionnelle de la densité de l'opérateur de transition d'un processus de Markov à temps discret C. R., Math., Acad. Sci. Paris. 334, 1035-1038.

Laksaci, A. (2007). Erreur quadratique de l'estimateur à noyau de la densité conditionnelle à variable explicative fonctionnelle. C. R. Math. Acad. Sci. Paris. 345, 171-175.

Laksaci, A. Lemdani, M. Ould-Saïd, E. (2009). A generalized $L_1$-approach for a kernel estimator of conditional quantile with functional regressors: consistency and asymptotic normality. Statist. Probab. Lett. 79, 1065-1073.

Laksaci, A. Mechab, B. (2010). Estimation non parametrique de la fonction de hasard avec variable explicative fonctionnelle cas des donnees spatiales.Rev : Roumaine, Math Pures Appl. 55, 35-51.

Leao, D. Fragoso, M. Ruffino, P.(2004). Regular conditional probability, integration of probability and Radon spaces. Proyecciones. 23, 15-29.

Lemdani, M. Ould-Saïd, E. Poulin, N. (2009). Asymptotic properties of a conditional quantile estimator with randomly truncated data. J. Multivariate Anal. 100, 546-559.

Leurgans, S. E. Moyeed, R. A. Silverman, B. W. (1993). Canonical correlation analysis when the data are curves. J. Roy. Statist. Soc. Ser. B. 55 (3)725-740

Liang, H. de U$\tilde{n}$a- Àlvarez, J. (2010). Asymptotic normality for estimator of conditional mode under left-truncated and dependent observations. Metrika. 72, 1-19.

Liang, H. de U$\tilde{n}$a- Àlvarez, J. (2011). Asymptotic properties of conditional quantile estimator for censored dependent observations. Ann. Inst. Statist.Math. 63, 267-289

Lifshits, M.A. Linde, W. Shi, Z. (2006). Small deviations of Riemann-Liouville processes in $L_q$-spaces with respect to fractal measures. Proc. London Math. Soc. (3) 92 (1) 224-250.

Lin,Z. Li, D.(2007). Asymptotic normality for $L^1$-norm kernel estimator of conditional median under association dependence,J. Multivariate Anal. 98, 1214-1230.

Louani, D. and Ould-Saïd, E. (1999). Asymptotic normality of kernel estimators of the conditional mode under strong mixing hypothesis. J. Nonparametric Stat. 11, 413-442.

Ma, Z.M. (1985). Some results on regular conditional probabilities. Acta Math.Sinica (N.S.). 1, 302-307.

A. Mahiddine, A. A. Bouchentouf and A. Rabhi.(2014). Nonparametric estimation of some characteristics of the conditional distribution in single functional index model, Malaya Journal of Matematik (MJM). 2 (4) 392-410.

Manté, C. Yao, A.F. Degiovanni, C.(2007). Principal component analysis of measures, with special emphasis on grain-size curves Comp. Stat. Data Anal. 51 (10) 4969-4984.

Masry, E. (2005). Nonparametric regression estimation for dependent functional data :Asymptotic normality. Stoch. Proc. and their Appl. 115, 155-177.

Ould-Saïd, E. (1997). A note on ergodic processes prediction via estimation of the conditional mode function. Scand. J. Stat. 24, 231-239.

Ould-Saïd, E. Cai, Z. (2005). Strong uniform consistency of nonparametric estimation of the censored conditional mode function. Nonparametric Statistics. 17, 797-806.

Ould Saïd, E. Djabrane, Y. (2011). Asymptotic normality of a kernel conditional quantile estimator under strong mixing hypothesis and left-truncation. Comm. Statist. Theory Methods. 40, 2605-2627

Ould Saïd, E. Tatachak, A. (2011). A nonparametric conditional mode estimate under RLT model and strong mixing condition. it Int. J. Stat. Econ. 6 , 76-92.

Quintela del Rio, A. Vieu, P. (1997). A nonparametric conditionnal mode estimate. Nonparametric Statistics. 8, 253-266.

Quintela-del-Rio, A. (2008). Hazard function given a functional variable:Non-parametric estimation under strong mixing conditions.J. Nonparametr. Stat. 20, 413-430.

Rachdi, M. Vieu, P. (2007). Nonparametric regression for functional data : automatic smoothing parameter selection. J. Statist. Plann. Inference. 137 (9) 2784-2801.

Ramsay, J.O. Bock, R. Gasser, T. (1995). Comparison of height acceleration curves in the Fels, Zurich, and Berkeley growth data, Annals of Human Biology. 22, 413-426

Ramsay, J. Silverman, B.W.(1997). Functional Data Analysis. Springer-Verlag, New York.

Ramsay, J. O. Silverman, B. W. (1997). Functional data analysis. Springer, New York.

Ramsay. J.O,.(2000). Différential equation models for statistical functions. Canad. J. Statist. 28 (2) 225-240.

Ramsay, J. O. and Silverman, B. W. (2002). Applied functional data analysis ; Methods and case studies. Springer-Verlag, New York.

Ramsay, J.O. Silverman, B.W. (2005). Functional Data Analysis, Second Edition. Springer, New York.

Roussas, G. (1969). Nonparametric estimation of the transition distribution function of a Markov process. Ann. Math. Statist. 40, 1386-1400.

Roussas, G. (1991). Estimation of transition distribution function and its quantiles in Markov processes : strong consistency and asymptotic normality. it Nonparametric functional estimation and related topics , (Spetses, 1990). 443-462.

Samanta, M.(1989). Non-parametric estimation of conditional quantiles. Statist. Proba. Letters. 7, 407-412.

Samanta, M. Thavaneswaran, A. (1990). Nonparametric estimation of the conditional mode. Comm. Statist. Theory Methods, 19 (12) 4515-4524

Schimek, M.(2000). (ed.): Smoothing and regression; Approaches, Computation, and Application. Wiley Series in Probability and Statistics, Wiley, New York.

Shmileva, E. (2006). Small ball probabilities for jump Lévy processes from the Wiener domain of attraction. Statist. Probab. Lett. 76 (17) 1873-1881.

Stone, C. J., (1977). Consistent nonparametric regression. Discussion. Ann. Stat. 5, 595-645.

Stone, C.J. (1982). Optimal global rates of convergence for nonparametric regression. Ann. Statist. 10 (4) 1040-1053.

Stute, W. (1986). On almost sure convergence of conditional empirical distribution functions. Ann. Probab. 14, 891-901.

Theodoros, N., Yannis, G. Y. (1997). Rates of convergence of estimate, Kolmogorov entropy and the dimensioanlity reduction principle in regression. The Annals of Statistics, 25 (6) 2493-2511.

Valderrama, M.J. Ocana, F.A. Aguilera, A.M. (2002). Forecasting PCARIMA models for functional data. COMPSTAT (Berlin). 25-36.

Van der Vaart, A. W. van Zanten, J. H. (2007). Bayesian inference with rescaled Gaussian process priors. Electronic Journal of Statistics. 1, 433-448.

Xia, X. An H. Z. (2002). An projection pursuit autoregression in time series.J. of Time Series Analysis. 20 (6) 693-714.

Yao, F. Lee, T.C.M. (2006). Penalised spline models for functional principal component analysis J.R. Stat. Soc. B. 68 (1) 3-25.

Youndjé, E. (1993). Estimation non paramétrique de la densité conditionnelle par la méthode du noyau. Thése de Doctarat, Université de Rouen.

Zhou,Y. Liang, H. (2000). Asymptotic normality for $L^1$-norm kernel estimator of conditional median under $\alpha$-mixing dependence, J. Multivariate Anal.73, 136-154.

# Chapter 2

# Asymptotic normality of conditional distribution estimation in the single index model

This chapter is published in Acta Univ. Sapientiae, Mathematica, 9, 1 (2017) 162-175

# Asymptotic normality of conditional distribution estimation in the single index model

**Diaa Eddine HAMDAOUI[1], Amina Angelika BOUCHENTOUF[2]**
**Abbes RABHI[3], Toufik GUENDOUZI[4]**
[1,4] Department of Mathematics, Dr. Moulay Taher University, Saida 20000, Algeria
E-mail:[1]diaa-eddine@hotmail.fr, [4]tf.guendouzi@gmail.com
[2,3] Department of Mathematics, Djillali Liabes university of Sidi Bel Abbes,
B. P. 89, Sidi Bel Abbes 22000, Algeria
E-mail:[2]bouchentouf_amina@yahoo.fr, [3]rabhi_abbes@yahoo.fr

**Abstract**
This paper deals with the estimation of conditional distribution function based on the single-index model. The asymptotic normality of the conditional distribution estimator is established. Moreover, as an application, the asymptotic $(1-\gamma)$ confidence interval of the conditional distribution function is given for $0 < \gamma < 1$.

**subclass [2000]:**Mathematics Subject Classification.Primary 62G07; Secondary 62G5; Thirdly 62G20 .
**Keywords:**Asymptotic normality, Single functional index model, Conditional distribution

## 2.1 Introduction

The single functional index models have received a considerable attention because of their wide applications in many areas such as economics, medicine, financial econometric and so on. The study of these models has been developed rapidly, see Ait-Saidi *et al.* (2005, 2008a, 2008b). Recently, Attaoui *et al.* (2011) investigated the kernel estimator of the conditional density of a scalar response variable $Y$, given a Hilbertian random variable $X$ when the observations are from a single functional index model. The pointwise and the uniform almost complete convergence of the estimator with rates in this model were obtained for independent observations. Furthermore, Ling *et al.* (2012) obtained the asymptotic normality of the conditional density estimator and the conditional mode estimator for the $\alpha$-mixing dependence functional time series data. Ling *et al.* (2014)

investigated the pointwise almost complete consistency and the uniform almost complete convergence of the kernel estimation with rate for the conditional density in the setting of the $\alpha$-mixing functional data, which extend the i.i.d case in Attaoui et al. (2011) to the dependence setting, the convergence rate of the kernel estimation for the conditional mode was also obtained.

The main contribution of this paper is to establish the asymptotic normality for the estimator of conditional distribution function in the i.i.d. case when the single functional index $\theta$ is fixed. As an application, the asymptotic $(1-\gamma)$ confidence interval for the conditional density function $F(\theta, y, x)$ is presented. The outline of the present paper is as follows. In section 2, we introduce the model as well as basic assumptions that are necessary in deriving the main result of this paper. In section 3, we state the main result of the paper; the asymptotic normality of the estimator for the conditional distribution function. As an application, the asymptotic $(1-\gamma)$ confidence interval of the conditional distribution function is given for $0 < \gamma < 1$. Finally, the technical proofs are related to section 4.

## 2.2 Model and some basic assumptions

Let $\{(X_i, Y_i), 1 \leq i \leq n\}$ be n random variables, identically distributed as the random pair $(X, Y)$ with values in $\mathcal{H} \times \mathbb{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\|.\|$ generated by an inner product $< .,. >$. Under such topological structure and for a fixed functional $\theta$, we suppose that the conditional probability distribution of $Y$ given $< X, \theta > = < x, \theta >$ exists and is given by

$$\forall y \in \mathbb{R}, F(\theta, y, x) = \mathbb{P}(Y \leq y | < X, \theta > = < x, \theta > . \tag{2.1}$$

The nonparametric kernel estimator $\widehat{F}(\theta, y, x)$ of $F(\theta, y, x)$ is defined as follows,

$$\widehat{F}(\theta, y, x) = \frac{\sum_{i=1}^{n} K(h_K^{-1}(< x - X_i, \theta >))H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^{n} K(h_K^{-1}(< x - X_i, \theta >))}, \tag{2.2}$$

where $K$ is a kernel, $H$ is a cumulative distribution function ($cdf$) and $h_K = h_{K,n}$(resp,$h_H = h_{H,n}$) is a sequence of positive real numbers which goes to zero as $n$ tends to infinity, and with the convention $0/0 = 0$.

Let, for any $x \in \mathcal{H}$, $i = 1, \ldots, n$ and $y \in \mathbb{R}$

$$K_i(\theta, x) := K(h_K^{-1} | < x - X_i, \theta > |), \text{ and } H_i(y) := H(h_H^{-1}(y - Y_i)).$$

We denote by $B_\theta(x, h) = \{X \in \mathcal{H}/0 < | < x - X, \theta > | < h\}$ the ball centered at $x$ with radius $h$, let $\mathcal{N}_x$ be a fixed neighborhood of $x$ in $\mathcal{H}$, $S_R$ will be a fixed compact subset of $\mathbb{R}$.

Now, we introduce the following basic assumptions that are necessary in deriving the main result of this paper.

(H1) $\mathbb{P}(X \in B_\theta(x, h_K)) =: \phi_{\theta,x}(h) > 0, \quad \phi_{\theta,x}(h) \to 0$ as $h \to 0$.

(H2) The conditional cumulative distribution $F(\theta, y, x)$ satisfies the Hölder condition, that is:

$\forall (y_1, y_2) \in S_R \times S_R$ , $\forall (x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x$.

$$|F(\theta, y_1, x_1) - F(\theta, y_2, x_2)| \leq C_{\theta,x}(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}), \; b_1 > 0, \; b_2 > 0.$$

(H3) For $j = 0, 1$, $H^{(j)}$ satisfies the lipschitz conditions and

$$m := \inf_{t \in [0,1]} K(t)H'(t) > 0,$$

with

$$\int H'(t)dt = 1, \quad \int H^2(t)dt < \infty \text{ and } \int |t|^{b_2} H^{(1)}(t)dt < \infty$$

(H4) The kernel $K$ is nonnegative, with compact support $[0, 1]$ of class $\mathcal{C}^1$ on $[0, 1)$ such that $K(1) > 0$ and its derivative $K'$ exists on $[0, 1)$ and $K'(t) < 0$.

(H5) For all $u \in [0, 1]$, $\displaystyle lim_{h \to 0} \frac{\phi_{\theta,x}(uh)}{\phi_{\theta,x}(h)} = lim_{h \to 0} \xi_h^{\theta,x}(u) = \xi_0^{\theta,x}(u)$.

(H6) The bandwidth $h_H$ satisfies,

(i) $\displaystyle \frac{\log n}{n\, \phi_{\theta,x}(h_K)} \to 0$, as $n \to \infty$.

(ii) $n h_H^2 \phi_{\theta,x}^2(h_K) \longrightarrow \infty$, and $\frac{n h_H^3 \phi_{\theta,x}(h_K)}{\log^2 n} \longrightarrow \infty$ as $n \to \infty$.

(iii) $nh_H^2\phi_{\theta,x}^3(h_K) \longrightarrow 0$, as $n \to \infty$.

(H7) (i) $\frac{\phi_{\theta,x}(h)}{n} + \phi_x(h) = \mathcal{O}(\frac{1}{n})$.

(ii) $\sqrt{n\phi_{\theta,x}(h)} \to 0$ as $n \to \infty$.

**Comments on the assumptions.** Assumption (H1) is the same as one given in Ferraty *et al.* (2005). Assumption (H2) is a regularity conditions which characterize the functional space of our model and is needed to evaluate the bias term of our asymptotic results. Assumptions (H3) and (H5) and (H6) are technical conditions for the proofs. Assumptions (H4) is classical in functional estimation for finite or infinite dimension spaces.

**remark 2.2.1.** *Assumption (H5) is known as (for small h) the "concentration assumption acting on the distribution of X" in infinite dimensional spaces.*

*The function $\xi_h^x(\cdot)$ intervening in assumption (H5) is increasing for all fixed $h$. Its pointwise limit $\xi_0^x(\cdot)$ plays a determinant role. It is possible to specify this function (with $\xi_0(u) := \xi_0^x(u)$ in the above examples by:*

1. $\xi_0(u) = u^\gamma$,
2. $\xi_0(u) = \delta_1(u)$, *where* $\delta_1(\cdot)$ *is Dirac function,*
3. $\xi_0(u) = \mathbf{1}_{]0,1]}(u)$.

## 2.3 Main results: Asymptotic normality of the estimator $\widehat{F}(\theta, y, x)$

In this part of paper, we give the asymptotic normality of the conditional cumulative distribution function in the single functional index model. The main result is given in the following theorem.

**Theorem 2.3.1.** *Under Assumptions (H1)-(H7) we have*

$$\sqrt{\frac{n\phi_{\theta,x}(h_K)}{\sigma^2(\theta, y, x)}}(\widehat{F}(\theta, y, x) - F(\theta, y, x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

*Where*

$$\sigma^2(\theta, y, x) = \frac{C_2(\theta, x)F(\theta, y, x)(1 - F(\theta, y, x))}{C_1^2(\theta, x)},$$

*with* $C_j(\theta, x) = K^j(1) - \int_0^1 sK'(s)\beta_{\theta,x}(s)ds$ *for* $j = 1, 2$,

" $\xrightarrow{\mathcal{D}}$ " *means the convergence in distribution.*

*Proof.* Consider, for $i = 1, \dots, n$,

$$K_i(\theta, x) = K(h_K^{-1}(< x - X_i, \theta >)), \ H_i(y) = H\left(h_H^{-1}(y - Y_i)\right),$$

$$\begin{aligned}
\widehat{F}_N(\theta, y, x) &= \frac{1}{n\,\mathbb{E}(K_1(\theta, x))} \sum_{i=1}^{n} K_i(\theta, x) H_i(y), \\
\widehat{F}_D(\theta, x) &= \frac{1}{n\,\mathbb{E}(K_1(\theta, x))} \sum_{i=1}^{n} K_i(\theta, x), \\
\Delta_i(x, \theta) &= \frac{K(h_K^{-1}(< x - X_i, \theta >))}{\mathbb{E}K_1(\theta, x)}.
\end{aligned}$$

In order to establish the asymptotic normality of $\widehat{F}(\theta, y, x)$ we have to consider the following decomposition

$$\begin{aligned}
\widehat{F}(\theta, y, x) - F(\theta, y, x) &= \frac{\widehat{F}_N(\theta, y, x)}{\widehat{F}_D(\theta, x)} - \frac{C_1(\theta, x) F(\theta, y, x)}{C_1(\theta, x)} \\
&= \frac{1}{\widehat{F}_D(\theta, x)} \left(\widehat{F}_N(\theta, y, x) - \mathbb{E}\widehat{F}_N(\theta, y, x)\right) \\
&\quad - \frac{1}{\widehat{F}_D(\theta, x)} \left(C_1(\theta, x) F(\theta, y, x) - \mathbb{E}\widehat{F}_N(\theta, y, x)\right) \\
&\quad + \frac{F(\theta, y, x)}{\widehat{F}_D(\theta, x)} \left(C_1(\theta, x) - \mathbb{E}\left[\widehat{F}_D(\theta, x)\right]\right) \\
&\quad - \frac{F(\theta, y, x)}{\widehat{F}_D(\theta, x)} \left(\widehat{F}_D(\theta, x) - \mathbb{E}\widehat{F}_D(\theta, x)\right) \\
&= \frac{1}{\widehat{F}_D(\theta, x)} A_n(\theta, y, x) + B_n(\theta, y, x) \qquad (2.3)
\end{aligned}$$

where

$$
\begin{aligned}
A_n(\theta, y, x) &= \frac{1}{n\mathbb{E}K_1(x,\theta)} \sum_{i=1}^{n} \Big\{ \left(H_i(y) - F(\theta, y, x)\right) K_i(\theta, x) \\
&\qquad - \mathbb{E}\left[\left(H_i(y) - F(\theta, y, x)\right) K_i(\theta, x)\right] \Big\} \\
&= \frac{1}{n\mathbb{E}K_1(x,\theta)} \sum_{i=1}^{n} N_i(\theta, y, x),
\end{aligned}
$$

and

$$
N_i(\theta, y, x) = \left(H_i(y) - F(\theta, y, x)\right) K_i(\theta, x) - \mathbb{E}\left[\left(H_i(y) - F(\theta, y, x)\right) K_i(\theta, x)\right].
$$

It follows that,

$$
\begin{aligned}
n\phi_{\theta,x}(h_K) Var\left(A_n(\theta, y, x)\right) &= \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2 K_1(x,\theta)} Var(N_1) + \frac{\phi_{\theta,x}(h_K)}{n\mathbb{E}^2 K_1(x,\theta)} \sum_{|i-j|>0}^{n}\sum Cov(N_i, N_j) \\
&= V_n(\theta, y, x) \qquad (2.4)
\end{aligned}
$$

Then, the rest of the proof is based on the following Lemmas

**lemma 2.3.1.** *Under hypotheses (H1)-(H3), (H5) and (H7), as $n \to \infty$ we have*

$$
n\phi_{\theta,x}(h_K) Var\left(A_n(\theta, y, x)\right) \longrightarrow V(\theta, y, x),
$$

*where*

$$
V(\theta, y, x) = \frac{C_2(\theta, x)}{(C_1(\theta, x))^2} F(\theta, y, x) \left(1 - F(\theta, y, x)\right).
$$

**lemma 2.3.2.** *Under hypotheses (H1)-(H3) and (H5)-(H7), as $n \to \infty$ we have*

$$
\left(\frac{n\phi_{\theta,x}(h_K)}{V(\theta, y, x)}\right)^{1/2} A_n(\theta, y, x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1),
$$

*where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.*

**lemma 2.3.3.** *Under assumptions (H1)-(H3) and (H5)-(H7); as $n \to \infty$ we have*

$$
\sqrt{n\phi_{\theta,x}(h_K)} B_n(\theta, y, x) \longrightarrow 0 \quad \textit{in Probabilty.}
$$

□

Now, because the unknown functions $C_j(\theta,x)$ and $F(\theta,y,x)$ intervening in the expression of the variance, we need to estimate the quantities $C_1(\theta,x)$, $C_2(\theta,x)$ and $F(\theta,y,x)$, respectively.

By assumptions (H1)-(H4) we know that $a_j(\theta,x)$ can be estimated by $\widehat{C_j}(\theta,x)$ which is defined as

$$\widehat{C}_j(\theta,x)=\frac{1}{n\widehat{\phi}_{\theta,x}(h_K)}\sum_{i=1}^{n}K_i^j(\theta,x)\ ,\ j=1,2$$

where

$$\widehat{\phi}_{\theta,x}(h_K)=\frac{1}{n}\sum_{i=1}^{n}\mathrm{I}_{\{|<x-X_i,\theta>|<h_k\}}.$$

By applying the kernel estimator of $F(\theta,y,x)$ given above, the quantity $\sigma^2(\theta,x)$ can be estimated finally by:

$$\widehat{\sigma}^2(\theta,x)=\frac{\widehat{C}_2(\theta,x)\widehat{F}(\theta,y,x)}{\widehat{C}_1^2(\theta,x)}\int H^2(t)dt.$$

Next, we can derive the following corollary:

**corollary 2.3.1.** *Under assumptions of Theorem 1, we have*

$$\sqrt{\frac{n\widehat{\phi}_{\theta,x}(h_K)}{\widehat{\sigma}^2(\theta,y,x)}}(\widehat{F}(\theta,y,x)-F(\theta,y,x))\xrightarrow{\mathcal{D}}\mathcal{N}(0,1).$$

Thus, following this Corollary we can approximate $(1-\gamma)$ confidence interval of $F(\theta,y,x)$ by

$\widehat{F}(\theta,y,x)\pm t_{\gamma/2}\times\dfrac{\widehat{\sigma}(\theta,x)}{\sqrt{n\widehat{\phi}_{\theta,x}(h_K)}}$, where $t_{\gamma/2}$ is the upper $\gamma/2$ quantile of standard Normal $\mathcal{N}(0,1)$.

## 2.4 Proofs of technical lemmas

*Proof of Lemma 2.3.1.* Let

$$\begin{aligned} V_n(\theta,y,x) &= \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2 K_1(\theta,x)}\mathbb{E}\left[K_1^2(\theta,x)\left(H_1(y)-F(\theta,y,x)\right)^2\right] \\ &= \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2 K_1(\theta,x)}\mathbb{E}\left[K_1^2(\theta,x)\mathbb{E}\left((H_1(y)-F(\theta,y,x))^2 \,|< \theta, X_1 >\right)\right] \quad (2.5)\end{aligned}$$

Using the definition of conditional variance, we have

$$\mathbb{E}\left[\left(H(h_H^{-1}(y-Y_1))-F(\theta,y,x)\right)^2 | < \theta, X_1 >\right] = J_{1n}+J_{2n},$$

where

$$J_{1n} = Var\left(H(h_H^{-1}(y-Y_1))| < \theta, X_1 >\right),$$

and

$$J_{2n} = \left[\mathbb{E}\left(H(h_H^{-1}(y-Y_1))| < \theta, X_1 >\right) - F(\theta,y,x)\right]^2$$

$\rightsquigarrow$**Concerning** $J_{1n}$. Let

$$\begin{aligned} J_{1n} &= \mathbb{E}\left[H^2\left(\frac{y-Y_1}{h_H}\right) | < \theta, x >\right] - \left(\mathbb{E}\left[H\left(\frac{y-Y_1}{h_H}\right) | < \theta, X_1 >\right]\right)^2 \\ &= \mathcal{J}_1+\mathcal{J}_2\end{aligned}$$

• By the property of double conditional expectation, we get that

$$\begin{aligned} \mathcal{J}_1 &= \mathbb{E}\left[H^2\left(\frac{y-Y_1}{h_H}\right) | < \theta, X_1 >\right] \\ &= \int_{\mathbb{R}} H^2\left(\frac{y-v}{h_H}\right) dF(\theta,v,X_1) \\ &= \int_{\mathbb{R}} H^2(t) dF(\theta,y-h_H t,X_1). \quad (2.6)\end{aligned}$$

On the other hand, by integrating by part and under assumption (H3), we have

$$\begin{aligned} \mathcal{J}_1 &= \int_{\mathbb{R}} 2H(t)H'(t)F(\theta,y-h_H t,X_1)dt \\ &= \int_{\mathbb{R}} 2H(t)H'(t)\left(F(\theta,y-h_H t,X_1)-F(\theta,y,x)\right)dt \\ &\quad + \int_{\mathbb{R}} 2H(t)H'(t)F(\theta,y,x)dt.\end{aligned}$$

Clearly, we have

$$\int_{\mathbb{R}} 2H(t)H'(t)F(\theta,y,x)du = \Big[H^2(t)F(\theta,y,x)\Big]_{-\infty}^{+\infty} = F(\theta,y,x), \tag{2.7}$$

thus

$$\int_{\mathbb{R}} H^2(t)dF(\theta,y-h_Ht,X_1) = F(\theta,y,x) + \mathcal{O}(h_K^{b_1}+h_H^{b_2}). \tag{2.8}$$

$\rightsquigarrow$ **Concerning** $\mathcal{J}_2$. Let

$$\begin{aligned}
I &= \mathbb{E}\left(H_i(y)| < X_1,\theta >\right) \\
\mathbb{E}\left(H\left(\frac{y-Y_1}{h_H}\right)| < X_1,\theta >\right) \\
&= \int_{\mathbb{R}} H\left(\frac{y-u}{h_H}\right) f(\theta,y,X_1)du, \\
&= \int_{\mathbb{R}} H\left(\frac{y-u}{h_H}\right) dF(\theta,y,X_1), \\
&= \int_{\mathbb{R}} H'\left(\frac{y-u}{h_H}\right) F(\theta,u,X_1)du, \\
&= \int_{\mathbb{R}} H'(t)\left(F(\theta,y-h_Ht,X_1)-F(\theta,y,x)\right)dt \\
&\quad +F(\theta,y,x)\int_{\mathbb{R}} H'(t)dt.
\end{aligned}$$

Because $H'$ is a probability density and by hypotheses (H2) and (H3), we can write:

$$\begin{aligned}
I &\leq C_{x,\theta}\int_{\mathbb{R}} H'(t)\left(h_K^{b_1}+|t|^{b_2}h_H^{b_2}\right)dt + F(\theta,y,x) \\
&= \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right) + F(\theta,y,x).
\end{aligned}$$

Finally, by hypothesis (H3) we get

$$\mathcal{J}_2 \longrightarrow F^2(\theta,y,x), \quad \text{as } n\to\infty. \tag{2.9}$$

The last equality is due to the fact that $H'$ is a probability density, thus we have by hypothesis (H3)

$$\int_{\mathbb{R}} H'(t)\left(F(\theta,y-h_Ht,X_1)-F(\theta,y,x)\right)dt \leq \int_{\mathbb{R}} H'(t)\left(|t|^{b_2}h_H^{b_2}+h_K^{b_1}\right)dt \underset{n\to\infty}{\longrightarrow} 0.$$

$\rightsquigarrow$ **Concerning** $J_{2n}$.

We have by integration by parts and changing variables

$$\begin{aligned} J_{2n} &= \mathbb{E}\left(H_1(y)| < \theta, X_1 >\right) \\ &= \mathbb{E}\left(H\left(\frac{y-Y_1}{h_H}\right) | < \theta, X_1 >\right) \\ &= \int H\left(\frac{y-v}{h_H}\right) f(\theta, v, X_1)dv \\ &= \int H\left(\frac{y-v}{h_H}\right) dF(\theta, v, X_1) \\ &= \int H'(t)F(\theta, y-h_H t, X_1)dt \\ &= F(\theta, y, x)\int H'(t)dt + \int H'(t)\left(F(\theta, y-h_H t, x) - F(\theta, y, x)\right)dt, \end{aligned}$$

the last equality is due to the fact that $H'$ is a probability density.

Thus, we have:

$$J_{2n} = F(\theta, y, x) + \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) \tag{2.10}$$

Finally, we obtain that $J_{2n} \underset{n\to\infty}{\longrightarrow} 0$.

Meanwhile, by (H1), (H2), (H4) and (H5), it follows that:

$$\frac{\phi_{\theta,x}(h_K)\mathbb{E}K_1^2(\theta, x)}{\mathbb{E}^2 K_1(\theta, x)} \underset{n\to\infty}{\longrightarrow} \frac{C_2(\theta, x)}{(C_1(\theta, x))^2},$$

Then, by combining equations (2.5)-(2.10), it leads to

$$V_n(\theta, y, x) \underset{n\to\infty}{\longrightarrow} \frac{C_2(\theta, x)}{(C_1(\theta, x))^2} F(\theta, y, x)\left(1 - F(\theta, y, x)\right). \tag{2.11}$$

□

*Proof of Lemma 2.3.2.* We will establish the asymptotic normality of $A_n(\theta, y, x)$ suitably normalized.

We have

$$
\begin{aligned}
\sqrt{n\phi_{\theta,x}(h_K)}A_n(\theta,y,x) &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{n\mathbb{E}K_1(\theta,x)}\sum_{i=1}^{n} N_i(\theta,y,x) \\
&= \frac{\sqrt{\phi_{\theta,x}(h_K)}}{\sqrt{n}\mathbb{E}K_1(\theta,x)}\sum_{i=1}^{n} N_i(\theta,y,x) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Xi_i(\theta,y,x) = \frac{1}{\sqrt{n}}S_n
\end{aligned}
\tag{2.12}
$$

Now, we can write,

$$
\Xi_i = \frac{\sqrt{\phi_{\theta,x}(h_K)}}{\mathbb{E}K_1(\theta,x)}N_i,
$$

Thus

$$
Var(\Xi_i) = \frac{\phi_{\theta,x}(h_K)}{\mathbb{E}^2K_1(\theta,x)}Var(N_i) = V_n(\theta,y,x).
$$

Note that by (2.11), we have $Var(\Xi_i) \longrightarrow V(\theta,y,x)$ as $n$ goes to infinity.
Obviously, we have

$$
\sqrt{\frac{n\phi_{\theta,x}(h_K)}{V(\theta,y,x)}}\left(A_n(\theta,y,x)\right) = \left(nV(\theta,y,x)\right)^{-1/2} S_n.
$$

Thus, the asymptotic normality of $\left(nV(\theta,y,x)\right)^{-1/2} S_n$, is deduced from the following results

$$
\left|\mathbb{E}\left\{\exp\left(izn^{-1/2}S_n\right)\right\} - \prod_{j=0}^{n}\mathbb{E}\left\{\exp\left(izn^{-1/2}\Xi_j\right)\right\}\right| \longrightarrow 0, \tag{2.13}
$$

$$
\frac{1}{n}\sum_{j=0}^{n}\mathbb{E}\left(\Xi_j^2\right) \longrightarrow V(\theta,y,x), \tag{2.14}
$$

$$
\frac{1}{n}\sum_{j=0}^{n}\mathbb{E}\left(\Xi_j^2\mathbf{1}_{\{|\Xi_j|>\varepsilon\sqrt{nV(\theta,y,x)}\}}\right) \longrightarrow 0, \text{ for every } \varepsilon > 0. \tag{2.15}
$$

While equations (2.13) and (2.14) show that the $\Upsilon_j$ are asymptotically independent, verifying that the sum of their variances tends to $V(\theta,y,x)$. Expression (2.15) is the

Lindeberg-Feller's condition for a sum of independent terms. Asymptotic normality of $S_n$ is a consequence of equations (2.13)-(2.15).

- **Proof of (2.13)** We make use of Volkonskii and Rozanov's lemma (see the appendix in Masry (2005) and the fact that the process $(X_i)$ is i.i.d.

  Note that using that $V_j = \exp\left(izn^{-1/2}S_n\right)$, we have

$$\left|\mathbb{E}\left\{\exp\left(izn^{-1/2}S_n\right)\right\} - \prod_{j=0}^{n}\mathbb{E}\left\{\exp\left(izn^{-1/2}\Xi_j\right)\right\}\right| \longrightarrow 0$$

  as $n$ goes to infinity.

- **Proof of (2.14)** Note that $Var(S_n) \longrightarrow V(\theta, y, x)$ by equation (2.11) and (2.12) (by the definition of the $\Xi_i$). Then because

$$\mathbb{E}\left(S_n\right)^2 = Var\left(S_n\right) = \sum_{j=0}^{n} Var\left(\Xi_j\right),$$

  and, using the same arguments as those previously used in the proof of first term of equation (2.5), we obtain

$$\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left(\Xi_j^2\right) = Var\left(\Xi_1\right),$$

  as $Var\left(\Xi_1\right) \longrightarrow V(\theta, y, x)$.

- **Proof of (2.15)** Recall that

$$\Xi_j \ = \ \sum_{i=0}^{n}\Upsilon_i.$$

  Finally, to establish (2.15) it suffices to show that the set

$$\{|\Xi_j| > \varepsilon\sqrt{nV(\theta, y, x)}\}$$

  is negligible for $n$ large enough.

By using assumptions (H4) and (H5), we have

$$\left|\Upsilon_i\right| \leq C\left(\phi_{\theta,x}(h_K)\right)^{-1/2},$$

therefore

$$\left|\Xi_j\right| \leq Cn\left(\phi_{\theta,x}(h_K)\right)^{-1/2},$$

which goes to zero as $n$ goes to infinity.

Since

$$|H_i(y) - F(\theta,y,x)| \leq 1.$$

Then for $n$ large enough, the set $\left\{|\Xi_j| > \varepsilon\left(nV(\theta,y,x)\right)^{-1/2}\right\}$ becomes empty, this completes the proof and therefore that of the asymptotic normality of $\left(nV(\theta,y,x)\right)^{-1/2} S_n$ and the Lemma 2.3.2.

□

*Proof of Lemma 2.3.3.* We have

$$\begin{aligned}\sqrt{n\phi_{\theta,x}(h_K)}B_n(\theta,y,x) &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{\widehat{F}_D(\theta,x)}\Big\{\mathbb{E}\widehat{F}_N(\theta,y,x) - C_1(\theta,x)F(\theta,y,x)\\ &\quad +F(\theta,y,x)\left(C_1(\theta,x) - \mathbb{E}\widehat{F}_D(\theta,x)\right)\Big\}.\end{aligned}$$

Firstly, observe that as $n \to \infty$

$$\frac{1}{\phi_{\theta,x}(h_K)}\mathbb{E}\left[K^l\left(\frac{< x - X_i, \theta >}{h_K}\right)\right] \longrightarrow C_l(\theta,x), \quad \text{for } l = 1,2 \tag{2.16}$$

$$\mathbb{E}\left[\widehat{F}_D(\theta,x)\right] \longrightarrow C_1(\theta,x), \tag{2.17}$$

and

$$\mathbb{E}\left[\widehat{F}_N(\theta,y,x)\right] \longrightarrow C_1(\theta,x)F(\theta,y,x), \tag{2.18}$$

can be proved in the same way as in Ezzahrioui and Ould Said (2008) corresponding to their Lemmas 5.1 and 5.2. Then the proofs of (2.16)-(2.18) are omitted.

Secondly, making use of (2.16), (2.17) and (2.18), we have as $n \to \infty$

$$\left\{\mathbb{E}\widehat{F}_N(\theta,y,x) - C_1(\theta,x)F(\theta,y,x) + F(\theta,y,x)\Big(C_1(\theta,x) - \mathbb{E}\widehat{F}_D(\theta,x)\Big)\right\} \longrightarrow 0.$$

On other hand

$$\frac{\sqrt{n\phi_{\theta,x}(h_K)}}{\widehat{F}_D(\theta,x)} = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\widehat{F}(\theta,y,x)}{\widehat{F}_D(\theta,x)\widehat{F}(\theta,y,x)} = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\widehat{F}(\theta,y,x)}{\widehat{F}_N(\theta,y,x)}. \tag{2.19}$$

Because $K(\cdot)H'(\cdot)$ is continuous with support on $[0,1]$, then by hypotheses (H3) and (H4) $\exists\ m = \inf_{t\in[0,1]} K(t)H'(t)$ such that

$$\widehat{F}_N(\theta,y,x) \geq \frac{m}{h_H\phi_{\theta,x}(h_K)}$$

which gives

$$\frac{n\phi_{\theta,x}(h_K)}{\widehat{F}_N(\theta,y,x)} \leq \frac{\sqrt{nh_H^2\phi_{\theta,x}(h_K)^3}}{m}$$

Finally, using (H6), the proof of Lemma 2.3.3 is completed.

$\square$

# Bibliography


A. Ait Saidi., F. Ferraty., R. Kassa., P. Vieu, Cross-validated estimations in the single functional index model, *Statistics*, 42, (2008a), 475-494.

A. Ait Saidi, F. Ferraty, R. Kassa, P. Vieu, Choix optimal du paramètre fonctionnel dans le modèle à indice fonctionnel simple, *C. R. Acad. Sci. Paris, Ser. I,* 346, (2008b), 217-220.

A. Ait Saidi., F. Ferraty., R. Kassa, Single functional index model for time series, *Romanian J. Pure & Applied Mathematics,* 50, (2005), 321-330.

S, Attaoui., A, Laksaci., E. Ould-Said, A note on the conditional density estimate in the single functional index model, *Statist. Probab. Lett,* **81**(1), (2011), 45-53.

M. Ezzahrioui., E. Ould-Saïd, Asymptotic results of a nonparametric conditional quantile estimator for functional time series, *Comm. Statist. Theory Methods,* **37**(16-17), (2008), 2735-2759.

F. Ferraty., A. Rabhi. and P. Vieu, Conditional quantiles for functional dependent data with application to the climatic El Ninõ phenomenon, *Sankhyã: The Indian Journal of Statistics, Special Issue on Quantile Regression and Related Methods*, **67**(2), (2005), 378-399.

N. Ling. and Q. Xu, Asymptotic normality of conditional density estimation in the single index model for functional time series data, *Statistics & Probability Letters,* **82**(12), (2012), 2235-2243.

Nengxiang Ling , Zhihuan Li & Wenzhi Yang (2014) Conditional Density Estimation in the Single Functional Index Model for a-Mixing Functional Data, Communications in Statistics - Theory and Methods, 43:3, 441-454, DOI: 10.1080/03610926.2012.664236

E. Masry, Nonparametric regression estimation for dependent functional data: Asymptotic normality. *Stoch. Proc. and their Appl*, **115**, (2005), 155-177.

# Chapter 3

# Nonparametric conditional density estimation in the single index model for independent functional data

This chapter is submitted in international journal of Statistics and Economics

# Nonparametric conditional density estimation in the single index model for independent functional data

**Diaa Eddine HAMDAOUI**[1], **Aboubacar Traore** [2]
**Amina Angelika BOUCHENTOUF** [3], **Abbes RABHI**[4]

[1] Department of Mathematics, Dr. Moulay Taher University, Saida 20000, Algeria
[2] Department of Mathematics, Faculty of sciences and Techniques of Bamako, Mali
E-mail:[1]diaa-eddine@hotmail.fr, [2]ababu02@yahoo.fr
[3,4] Department of Mathematics, Djillali Liabes university of Sidi Bel Abbes,
B. P. 89, Sidi Bel Abbes 22000, Algeria
E-mail:[3]bouchentouf_amina@yahoo.fr, [4]rabhi_abbes@yahoo.fr

**Abstract**
The main objective of this paper is to study the estimation of conditional density function based on the single-index model for independent functional data. Under general conditions, the asymptotic normality of the conditional density estimator is established. The asymptotic $(1-\xi)$ confidence intervals of conditional density function are given, for $0 < \xi < 1$. As an application the conditional mode in functional single-index model is presented. Furthermore, a simulation study is carried out.

**subclass [2000]:**Mathematics Subject Classification.Primary 62G07; Secondary 62G5; Thirdly 62G20 .
**Keywords:**Asymptotic normality, single functional index model, conditional density function

## 3.1 Introduction

Conditional density function estimation is one of the crucial problems in non-parametric statistics, one can refer to De Gooijer and Zerom (2003). As far as the problem is concerned, it is usual that the explanatory variable $X$ takes value in $\mathbb{R}^d$ and the response variable $Y$ takes value in $\mathbb{R}$. Moreover, the sample of $(X, Y)$ is always supposed to be independent identically distributed (i.i.d) or to be some dependence (see for instance Masry (1989), Cai (1991), Quintela-Del-Rio and Vieu (1997), De Gooijer and Zerom (2003), Gannoun and colleagues (2003)). The conditional density plays an important role in nonparametric prediction, because the several prediction tools in nonparametric statistic, such as the conditional mode, the conditional median are based on the preliminary estimate of this functional parameter. Nonparametric estimation of the conditional density has been widely studied, when the data is real. The first related result in non-parametric functional statistic was obtained by Ferraty et al. (2006). Authors established the almost complete consistency in the independent and identically distributed (i.i.d.) random variables of the kernel estimator of the conditional probability density. These result have been extend to dependent data by Ezzahrioui and Ould Saïd (2010).

The single-index models are becoming incrementally important and popular, and have been attracting considerable attention last few years because of their importance in several areas of science such as econometrics, biostatistics, medicine, financial. The single-index approach is extensively and largely used in econometrics. Such kind of modelization is extensively studied in the multivariate case, we quote for example Härdle et al. (1993), Hristache et al. (2001). Based on the regression function, Delecroix et al. (2003) studied the estimation of the single-index and established some asymptotic properties. The literature is strictly limited in the case where the explanatory variable is functional (that is a curve). The first asymptotic properties in the fixed functional single-model were obtained by Ferraty et al. (2003), authors established the almost complete convergence, in the i.i.d. case, of the link regression function of this model. Their results were extended to dependent case by Aït Saidi et al. (2005). Aït Saidi et al. (2008) studied the case where the functional single-index is unknown. They proposed an estimator of this parameter, based on the cross-validation procedure. Ling and Xu (2012) investigated the estimation of conditional density function based on the single-index model for functional time series data. The asymptotic normality of the conditional density estimator and the conditional mode estimator for the $\alpha$- mixing dependence functional time series data are obtained, respectively. Attaoui (2014) investigated a nonparametric estimation of the conditional density of a scalar response variable given a random variable taking values in separable Hilbert space. Author established under general conditions the uniform almost complete convergence rates and the asymptotic normality of the conditional density kernel estimator, when the variables satisfy the strong mixing dependency, based on the single-index structure.

The goal of this paper is to present the asymptotic normality for the estimators of conditional density function in the single functional index model when the data are independent. The paper is organized as follows. We present our model and some basic assumptions in section 2. In section 3 we state the main results. Section 4 is devoted to the proofs of some lemmas and the main result. In section 5 an application: the conditional mode in functional single-index model is presented. Section 6 is consecrated to a simulation study.

## 3.2 Model description and some basic assumptions

All along the paper, we will denote by $\mathcal{C}$, $\mathcal{C}'$ or/and $C_{\theta,x}$ some generic constant in $\mathbb{R}_+^*$, and in the following, any real function with an integer in brackets as exponent denotes its derivative with the corresponding order. Let $X$ be a functional random variable *frv* . Let $(X_i, Y_i)$ be a sample of independent pairs, each one have the same distribution as $(X, Y)$, our aim is to build nonparametric estimates of several functions related with the conditional density of $Y$ given $< X, \theta >=< x, \theta >$.
Let

$$\forall y \in \mathbb{R},\ f(y|x) =: f(y| < x, \theta >) \tag{3.1}$$

be the conditional density of $Y$ given $< X, \theta >=< x, \theta >$, for $x \in \mathcal{H}$, which also shows the relationship between $X$ and $Y$ but is often unknown.

In the following, we denote by $f(\theta, ., x)$ the conditional density of Y given $< x, \theta >$ and we define the kernel estimator $\widehat{f}(\theta, ., x)$ of $f(\theta, ., x)$ by:

$$\widehat{f}(\theta, ., x) = \frac{h_H^{-1} \sum_{i=1}^{n} K(h_K^{-1}(< x - X_i, \theta >)) H'(h_H^{-1}(y - Y_i))}{\sum_{i=1}^{n} K(h_K^{-1}(< x - X_i, \theta >))} \tag{3.2}$$

with the convention$0/0 = 0$, where K and H are kernels function and $h_K := h_{n,K}$ (resp. $h_H := h_{n,H}$) is a sequence of bandwidths that decrease to zero as n goes to infinity.

Let, for any $x \in \mathcal{H}$ and $i = 1, ..., n$ and y $\in \mathbb{R}$:

$$K_i(\theta, x) := K(h_K^{-1}| < x - X_i, \theta > |),\ \ H_i(y) := H(h_H^{-1}(y - Y_i)).$$

We denote by $B_\theta(x, h) = \{X \in \mathcal{H}/0 < | < x - X, \theta > | < h\}$ the ball centered at $x$ with radius $h$.

Let $\mathcal{N}_x$ be a fixed neighborhood of $x$ in $\mathcal{H}$, $S_\mathbb{R}$ will be a fixed compact subset of $\mathbb{R}$.

Now ,we will make use of the following basic assumptions that are necessary in deriving the main result of this paper.

(H1) $\mathbb{P}(X \in B_\theta(x, h_K)) =: \phi_{\theta,x}(h_k) > 0$ ; $\ \phi_{\theta,x}(h_k) \to 0$ *as* $h_K \to 0$.

(H2) The conditional density $f(\theta, y, x)$ satisfies the Hölder condition, that is:

$$\forall (y_1, y_2) \in S_\mathbb{R} \times S_\mathbb{R}\ ,\ \forall (x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x$$

$$|f(\theta, y_1, x_1) - f(\theta, y_2, x_2)| \leq C_{\theta,x}(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}), \ b_1 > 0, \ b_2 > 0.$$

(H3) The kernel $H$ is a positive bounded function with:

$\int H(t)dt = 1$ ; $\int |t|^{b_2} H(t)dt < \infty$ and $\int H^2(t)dt < \infty$.

$\forall y_1, y_2 \in \mathbb{R}, |H(y_1) - H(y_2)| \leq C|y_1 - y_2|$.

(H4) The kernel $K$ is a positive bounded function with support $[0,1]$ or such that

$$\begin{cases} (H4a)\ 0 < \mathcal{C}\mathrm{I}_{[0,1]}(t) < K(t) < \mathcal{C}'\mathrm{I}_{[0,1]}(t) < \infty. \\ (H4b)\ \forall t_1, t_2 \in \mathbb{R}, |K(t_1) - K(t_2)| \leq \mathcal{C}|t_1 - t_2|. \end{cases}$$

(H5) $\forall s \in [0,1], \lim_{n \longrightarrow +\infty} \frac{\phi_{\theta,x}(sh_K)}{\phi_{\theta,x}(h_K)} = \beta_{\theta,x}(s)$.

(H6) $\lim_{n \longrightarrow +\infty} h_K = 0 \quad \lim_{n \longrightarrow +\infty} h_H = 0 \quad \lim_{n \longrightarrow +\infty} \frac{logn}{nh_H\phi_{\theta,x}(h_K)} = 0$.

(H7) The kernel $K$ is a differentiable function satisfying (H4) and its derivative $K'$ exists and is such that there exist two constants $\mathcal{C}$ and $\mathcal{C}'$ with$-\infty < \mathcal{C} < K'(t) < \mathcal{C}' < 0$,for t$\in [0,1]$.

(H8) $\sqrt{nh_H\phi_{\theta,x}(h_K)}h_H^{b_2} \to 0, \ as \ n \to \infty$.

## 3.3 Main result

**Theorem 3.3.1.** *Under Assumptions we have (H1)-(H8) for all $x \in \mathcal{H}$*

$$\sqrt{\frac{nh_H\phi_{\theta,x}(h_K)}{\sigma^2(\theta, y, x)}}\Big(\hat{f}(\theta, y, x) - f(\theta, y, x)\Big) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \quad as\ n \to \infty \tag{3.3}$$

*Where :*

$$\sigma^2(\theta, y, x) = \frac{C_2 f(\theta, y, x)}{C_1^2} \int H^2(t)dt$$

*with*

$$C_j = K^j(1) - \int_0^1 sK'(s)\beta_{\theta,x}(s)ds$$

*for* $j = 1, 2$; " $\xrightarrow{\mathcal{D}}$ "*means the convergence in density.*

Because the unknown functions $C_j := C_j(\theta, x)$ and $f(\theta, y, x)$ intervening in the expression of the variance. So we need to estimate the quantities $C_1(\theta, x)$ , $C_2(\theta, x)$ and $f(\theta, y, x)$ ,respectively .

By the assumptions (H1)-(H4) we know that $C_j(\theta, x)$can be estimated by $\widehat{C}_j(\theta, x)$ which is defined as :

$$\widehat{C}_j(\theta, x) = \frac{1}{n\widehat{\phi}_{\theta,x}(h)} \sum_{i=1}^{n} K_i^j(\theta, x) \tag{3.4}$$

where :

$$\widehat{\phi}_{\theta,x}(h) = \frac{1}{n} \sum_{i=1}^{n} \mathrm{I}_{\{|<x-X_i,\theta>|<h_k\}}$$

.

By applying the kernel estimator of $f(\theta, y, x)$ given above, the quantity $\sigma^2(\theta, y, x)$ can be estimated finally by:

$$\widehat{\sigma}^2(\theta, y, x) = \frac{\widehat{C}_2(\theta, x)\widehat{f}(\theta, y, x)}{\widehat{C}_1^2(\theta, x)} \int H^2(t)dt \tag{3.5}$$

so we can derive the following corollary:

**corollary 3.3.1.** *Under the assumptions of Theorem 3.3.1 ,we have*

$$\sqrt{\frac{nh_H\widehat{\phi}_{\theta,x}(h_K)}{\widehat{\sigma}^2(\theta, y, x)}} \left(\widehat{f}(\theta, y, x) - f(\theta, y, x)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as } n \to \infty \tag{3.6}$$

Thus, following this Corollary we can approximate $(1 - \xi)$ confidence interval of $f(\theta, y, x)$ by

$$\widehat{f}(\theta, y, x) \pm t_{\xi/2} \times \frac{\widehat{\sigma}(\theta, x)}{\sqrt{nh_H\widehat{\phi}_{\theta,x}(h_K)}} \tag{3.7}$$

where $t_{\xi/2}$ is the upper $\xi/2$ quantile of standard Normal $\mathcal{N}(0, 1)$.

## 3.4 Proofs the main result

In this section, we demonstrate the main result given above. To this end, let us introduce the following decomposition:

$$\begin{aligned}\widehat{f}(\theta,y,x)-f(\theta,y,x) &= \frac{1}{\widehat{f}_D(\theta,x)}\{(\widehat{f}_N(\theta,y,x)-\mathbb{E}\widehat{f}_N(\theta,y,x))-(f(\theta,y,x)-\mathbb{E}\widehat{f}_N(\theta,y,x))\}\\ &+ \frac{f(\theta,y,x)}{\widehat{f}_D(\theta,x)}\{1-\widehat{f}_D(\theta,x)\}\end{aligned}$$

where

$$\widehat{f}_N(\theta,y,x)=\frac{1}{nh_H\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x)H_i(y)\ \left(resp.\widehat{f}_D(\theta,x)=\frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x)\right).$$

The proof is based on the following Lemmas.

**lemma 3.4.1.** *Under conditions of Theorem 3.3.1, we have*

$$\sqrt{nh_H\phi_{\theta,x}(h_K)}\left(\widehat{f}_N(\theta,y,x)-\mathbf{E}\big(\widehat{f}_N(\theta,y,x)\big)\right)\xrightarrow{\mathcal{D}}\mathcal{N}(0,\sigma^2(\theta,y,x))\quad asn\to\infty \tag{3.8}$$

*where $\sigma^2(\theta,y,x)$ is defined as Theorem 3.3.1.*

**Proof of Lemma 3.4.1**
We have:

$$\begin{aligned}\hat{f}_N(\theta,y,x)-\mathbb{E}\big(\hat{f}_N(\theta,y,x)\big) &= \frac{1}{nh_H\mathbb{E}[K_1(\theta,x)]}\sum_{i=1}^{n}\left(K_i(\theta,x)H_i(y)-\mathbb{E}\Big[K_i(\theta,x)H_i(y)\Big]\right)\\ &= \frac{1}{nh_H\mathbb{E}[K_1(\theta,x)]}\sum_{i=1}^{n}\Delta_i \qquad (3.9)\end{aligned}$$

with

$$\Delta_i=K_i(\theta,x)H_i(y)-\mathbb{E}\Big[K_i(\theta,x)H_i(y)\Big]$$

For the variance of $\Delta_1$: since $\mathbb{E}[\Delta_i]=0,\forall i=1,...,n$, by the definition of conditional expectation we can write:

$$\begin{aligned} Var(\Delta_1) &= \mathbb{E}[\Delta_1^2] \\ &= \mathbb{E}\bigg( K_1^2(\theta,x)\mathbb{E}\big[H_1^2(y)| < y, X_1 > \big]\bigg) \end{aligned}$$

Now, by a change of variable in the following integral and by applying (H2) and (H3), one gets

$$\begin{aligned} \Big|\mathbb{E}\big[H_1^2(y)| < \theta, X_1 > \Big| &= \Big| \int_{\mathbb{R}} H^2\big(h_H^{-1}(y-z)\big) f(\theta,z,x)dz \Big| \\ &\leq h_H \int_{\mathbb{R}} H^2(t) \Big| f(\theta, y - h_H t, x) - f(\theta,y,x)\Big| dt + h_H f(\theta,y,x) \int_{\mathbb{R}} H^2(t)dt \\ &\leq h_H^{1+b2} \int_{\mathbb{R}} |t|^{b2} H^2(t)dt + h_H f(\theta,y,x) \int_{\mathbb{R}} H^2(t)dt \\ &= h_H \bigg( o(1) + f(\theta,y,x) \Big( \int_{\mathbb{R}} H^2(t)dt \Big) \bigg) \end{aligned} \tag{3.10}$$

as $n \to \infty$, we have for,$j = 1, 2$, $\mathbb{E}\big[K_1^j(\theta,x)\big] \to C_j \phi_{\theta,x}(h_K)$(see Ferraty et al. 2007). So,

$$Var(\Delta_1) = o\big(h_H \phi_{\theta,x}(h_K)\big)$$

Define:

$$Z_i(\theta,y,x) = \frac{\sqrt{\phi_{\theta,x}(h_K)}}{\sqrt{nh_H}\mathbb{E}(K_1(\theta,x))} \Big( \Psi_i(\theta,y,x) - \mathbb{E}(\Psi_i(\theta,y,x)) \Big)$$

and

$$S_n = \sum_1^n Z_i(\theta,y,x).$$

Thus,

$$S_n = \sqrt{nh_H \phi_{\theta,x}(h_K)} \Big( \hat{f}_N(\theta,y,x) - \mathbb{E}\big(\hat{f}_N(\theta,y,x)\big) \Big)$$

So,our claimed result is now

$$S_n \to \mathcal{N}(0, \sigma^2(\theta,y,x)) \tag{3.11}$$

Therefore,we have

$$\begin{aligned} Var(S_n) &= nh_H\phi_{\theta,x}(h_K)Var\Big(\hat{f}_N(\theta,y,x)-\mathbb{E}\big(\hat{f}_N(\theta,y,x)\Big) \\ &= nh_H\phi_{\theta,x}(h_K)Var\Big(\hat{f}_N(\theta,y,x)\Big) \end{aligned}$$

Now, we need to evaluate the variance of $f(\theta,y,x)$. For this we have for all $1\le i\le n, \Psi_i(\theta,y,x)=H_i(y)K_i(\theta,x)$, so we have

$$\begin{aligned} Var\Big(\hat{f}_N(\theta,y,x)\Big) &= \frac{1}{\Big(nh_H\mathbb{E}[K_1(\theta,x)]\Big)^2}\sum_{i=1}^{n}\sum_{j=1}^{n}Cov\Big(\Psi_i(\theta,y,x),\Psi_j(\theta,y,x)\Big) \\ &= \frac{1}{\Big(nh_H\mathbb{E}[K_1(\theta,x)]\Big)^2}Var\Big(\Psi_1(\theta,y,x)\Big) \\ &= J'_{1.n} \end{aligned} \tag{3.12}$$

Where

$$\begin{aligned} Var\Big(\Psi_1(\theta,y,x)\Big) &\le \mathbb{E}\Big[H_1^2(y)K_1^2(\theta,x)\Big] \\ &\le \mathbb{E}\Big[K_1^2(\theta,x)\mathbb{E}\big[H_1^2(y)|<\theta,X_1>\big]\Big] \end{aligned} \tag{3.13}$$

By means of (3.10) and the fact that, as $n\to\infty$ ,$\mathbb{E}\Big[K_1^2(\theta,x)\Big]\to C_2\phi_{\theta,x}(h_K)$,one gets

$$Var\Big(\Psi_1(\theta,y,x)\Big)=C_2\phi_{\theta,x}(h_K)h_H\Big(o(1)+f(\theta,y,x)\Big(\int_{\mathbb{R}}H^2(t)dt\Big)\Big)$$

$$\begin{aligned} J'_{1.n} &= \frac{C_2\phi_{\theta,x}(h_K)}{n\big(C_1h_H\phi_{\theta,x}(h_K)\big)^2}h_H\Big(o(1)+f(\theta,y,x)\Big(\int_{\mathbb{R}}H^2(t)dt\Big)\Big) \\ &= o\Big(\frac{1}{nh_H\phi_{\theta,x}(h_K)}\Big)+\frac{C_2f(\theta,y,x)}{C_1^2nh_H\phi_{\theta,x}(h_K)}\Big(\int_{\mathbb{R}}H^2(t)dt\Big) \\ &\to \frac{C_2f(\theta,y,x)}{C_1^2nh_H\phi_{\theta,x}(h_K)}\Big(\int_{\mathbb{R}}H^2(t)dt\Big)\,as\,n\to\infty \end{aligned}$$

Finally we get

$$Var(S_n) \to \frac{C_2 f(\theta, y, x)}{C_1^2}\Big(\int_{\mathbb{R}} H^2(t)dt\Big) := \sigma^2(\theta, y, x)$$

**lemma 3.4.2.** *If the assumptions (H2), (H3) and (H8) are satisfied, then, we have*

$$\underset{n \longrightarrow +\infty}{lim} \sqrt{nh_H \phi_{\theta,x}(h_K)}(\mathbf{E}\widehat{f}_N(\theta, y, x) - (f(\theta, y, x)) = 0 \tag{3.14}$$

**Proof of Lemma 3.4.2** One has

$$\begin{aligned}
\mathbf{E}\big[\widehat{f}_N(\theta, y, x)\big] - (f(\theta, y, x)) &= \mathbf{E}\Bigg[\frac{1}{nh_H K_1(\theta, x)}\sum_{i=1}^{n}\Big(K_i(\theta, x)H_i(y)\Big)\Bigg] - \big(f(\theta, y, x)\big) \\
&= \frac{1}{h_H \mathbb{E}\big(K_1(\theta, x)\big)}\mathbb{E}\Big[(K_1(\theta, x)H_1(y)\Big] - f(\theta, y, x) \\
&= \frac{1}{h_H \mathbb{E}\big(K_1(\theta, x)\big)}\mathbb{E}\Big[\mathbb{E}\Big(K_1(\theta, x)H_1(y)| < \theta, X_1 > \Big) - f(\theta, y, x)\Big] \\
&= \frac{1}{h_H \mathbb{E}\big(K_1(\theta, x)\big)}\mathbb{E}\Big[K_1(\theta, x)\Big(\mathbb{E}\Big(H_1(y)| < \theta, X_1 > \Big) - f(\theta, y, x)\Big)\Big]
\end{aligned}$$

Let $H_i(y) = H(h_H^{-1}(y - Y_i)$ we have

$$\mathbb{E}\Big(H_1(y)| < \theta, X_1 > \Big) = \int_{\mathbb{R}} H\big(h_H^{-1}(y - z)\big)f(\theta, z, X_1)dz$$

Condition (H3)allows to write

$$\Big|\mathbb{E}\big(H_1(y)| < \theta, X_1 > \big) - f(\theta, y, x)\Big| \leq \int_{\mathbb{R}} H(t)\Big|f(\theta, y - h_H t, X_1) - f(\theta, y, x)\Big|dt$$

(H2)allows to write

$$\left|\mathbb{E}\big(H_1(y)| < \theta, X_1 > \big) - f(\theta, y, x)\right| \leq C_{\theta,x} \int_{\mathbb{R}} H(t)\left(h_k^{b_1} + |t|^{b_2} h_H^{b_2}\right) dt$$

Finaly we get

$$\lim_{n \longrightarrow +\infty} \sqrt{n h_H \phi_{\theta,x}(h_K)}(\mathbb{E}\widehat{f}_N(\theta, y, x) - (f(\theta, y, x)) = 0$$

**lemma 3.4.3.** *Under the assumptions (H1)-(H7), then*

$$\sqrt{n h_H \phi_{\theta,x}(h_K)}(1 - \widehat{f}_D(\theta, x)) \xrightarrow{\mathcal{P}} 0, as\, n \to \infty \tag{3.15}$$

**Proof of Lemma 3.4.3**

by the definition of $\widehat{f}_D(\theta, x)$,we have $\sqrt{n h_H \phi_{\theta,x}(h_K)}(\widehat{f}_D(\theta, x) - 1) =: A_n - \mathbb{E}A_n$ where

$$A_n = \frac{\sqrt{n h_H \phi_{\theta,x}(h_K)} \sum_{i=1}^{n} K_i}{n\mathbb{E}K_1}$$

In order to prove (3.15) , similar to Attouch et al. (2010), we only need to prove $VarA_n \to 0$ as n$\to \infty$ . in fact ,since

$$\begin{aligned} VarA_n &= \frac{h_H \phi_{\theta,x}(h_K)}{\mathbb{E}^2 K_1}(VarK_1) \\ &\leq \frac{h_H \phi_{\theta,x}(h_K)}{\mathbb{E}^2 K_1}\mathbb{E}K_1^2 \\ &=: B_1 \end{aligned} \tag{3.16}$$

then, using the boundedness of function K allows us to get that:

$$B_1 \leq C h_H \phi_{\theta,x}(h_K) \to 0 \text{ as n} \to \infty$$

## 3.5 Application: The conditional mode in functional single-index model

The main objective of this section is to establish the asymptotic normality a of the kernel estimator of the conditional mode of $Y$ given $< X, \theta > = < x, \theta >$ denoted by $M_\theta(x)$

We estimate the conditional mode $\widehat{M_\theta}(x)$ with a random variable $M_\theta$ such that

$$\widehat{M_\theta}(x) = \arg\sup_{y \in \mathcal{S}_{\mathbb{R}}} \widehat{f}(\theta, y, x). \tag{3.17}$$

Let's note that in all the remaining of our paper we will consider any value $\widehat{M_\theta}$ satisfying (3.17).

In order to present the estimation of the conditional mode in the functional single-index model, we introduce the following additional smoothness condition.

$(U1)$ $f(\theta, ., x)$ is twice continuously differentiable around the point $M_\theta(x)$
with $f^{(1)}(\theta, M_\theta(x), x) = 0$
and $f^{(2)}(\theta, ., x)$ is uniformly continuous on $S_{\mathbb{R}}$ such that $f^{(2)}(\theta, M_\theta(x), x) \neq 0$
where $f^{(q)}(\theta, ., x)$ $(q = 1, 2)$ is the qth order derivative of the conditional density $f(\theta, y, x)$.

$(U2)$ $\forall \varepsilon > 0$ , $\exists \eta > 0$ ,$\forall y \in S_{\mathbb{R}}$

$$|M_\theta(x) - y| \geq \varepsilon \Rightarrow |f(\theta, M_\theta(x), x) - f(\theta, y, x)| \geq \eta$$

$(U3)$ The conditional density function $f(\theta, y, x)$ satisfies: $\exists \beta_0 > 0$, $\forall (y_1, y_2) \in S_{\mathbb{R}} \times S_{\mathbb{R}}$,

$$|f^q(\theta, y_1, x) - f^q(\theta, y_2, x)| \leq C\big(|y_1 - y_2|^{\beta_0}\big) \;\forall q = 1, 2.$$

$(U4)$ $G'$ and $G''$ are bounded respectively with

$$\int \left(G'(t)\right)^2 dt < \infty \,, \int |t|^{\beta_0} G(t) dt < \infty$$

$(U5)$ $nh^3\phi_{\theta,x}(h) \to \infty$ ,as n $\to \infty$

**Theorem 3.5.1.** *Suppose that hypotheses (H1)-(H7) and (U1)-(U5) are satisfied, we have*

$$\sqrt{\frac{nh^3\phi_{\theta,x}(h)}{\sigma_1^2(\theta, x)}}(\widehat{M_\theta}(x) - M_\theta(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), as\, n \to \infty$$

*Where*

$$\sigma_1^2(\theta, x) = \frac{C_2(\theta, x) f(\theta, M_\theta(x), x)}{C_1^2(\theta, x)\big[f^2(\theta, M_\theta(x), x)\big]^2} \int (G'(t))^2 dt$$

In order to show the asymptotic $(1-\gamma)$ confidence interval of $M_\theta(x)$, we need to consider the estimator of $\sigma_1^2(\theta,x)$ as follows:

$$\widehat{\sigma}_1^2(\theta,x) = \frac{\widehat{C}_2(\theta,x)\widehat{f}(\theta,M_\theta(x),x)}{\widehat{C}_1^2(\theta,x)\left[\widehat{f}^2(\theta,M_\theta(x),x)\right]^2}\int (G^{'}(t))^2dt$$

Thus, the following corollary is obtained.

**corollary 3.5.1.** *Under conditions of Theorem 3.5.1, we have*

$$\sqrt{\frac{nh^3\widehat{\phi}_{\theta,x}(h)}{\widehat{\sigma}_1^2(\theta,x)}}(\widehat{M}_\theta(x)-M_\theta(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), as\, n\to\infty$$

**remark 3.5.1.** *By Corollary 3.5.1, the asymptotic* $(1-\gamma)$ *confidence interval of*$(M_\theta(x),x)$ *is given by*

$$(M_\theta(x),x)\pm\mu_{\gamma/2}\times\frac{\widehat{\sigma}_1(\theta,x)}{\sqrt{nh^3\widehat{\phi}_{\theta,x}(h)}}$$

## 3.6 Simulation study

The best way to know the behavior of the estimator of conditional density is to compute its mean square error. So, in this part of paper we compare between the conditional density estimation in the SIM (single index model) which is our model and the conditional density estimation in the NPM (non-parametric model) defined in (3.18) .

$$\widehat{f}^x(y) = \frac{\sum_{i=1}^n K\left(h_K^{-1}d\left(x,\,X_i\right)\right)H'\left(h_H^{-1}\left(y-Y_i\right)\right)}{h_H\sum_{i=1}^n K\left(h_K^{-1}d\left(x,\,X_i\right)\right)}. \tag{3.18}$$

So, we have to compare their respective conditional density estimators by computing and comparing their respective mean square errors for some values of the scalar response $Y$. Let $MSE1$ be the mean square error in SIM case and $MSE2$ the mean square error in NPM case.

Next, let us consider the following regression model where the covariate is a curve and the response is a scalar:

$$T_i = R\left(X_i\right)+\epsilon_i,\; i=1,......,n \tag{3.19}$$

where $\epsilon_i$ is the error supposed to be generated by an autoregressive model defined by

$$\epsilon_i = \frac{1}{\sqrt{2}} \left( \epsilon_{i-1} + \eta_i \right), \; i = 1, ......., n$$

with $(\eta_i)_i$ a sequence of i.i.d. random variables normally distributed with a variance equal to 0.1.

Suppose that the functional covariate $X$ is a diffusion process defined on [0, 1] and generated by the following equation:

$$X(t) = A(2 - \sin(\pi t W)) + (1 - A)\sin(\pi t W), \; t \in [0, 1]$$

where $W \to \mathcal{N}(0, 1)$and $A$ is a Bernoulli random variable with parameter 1/2.

Figure 3.6 shows a sample of 215 curves representing the realization of the functional random variable $X$. Here a nonlinear regression function is considered such that

$$R(X) = \frac{1}{4} \int_0^1 \left( X'(t) \right)^2 dt.$$

On the other hand, $n$ i.i.d. random variables $(C_i)_i$ are simulated through the exponential distribution $\mathcal{E}(1.5)$.

Given $X = x$, we can easily see that $T \to \mathcal{N}(R(x), 0.2)$, and therefore, the conditional median, the conditional mode and the conditional mean functions will coincide and will be equal to $R(x)$, for any fixed $x$.

The computation of our estimator is based on the observed data $(X_i, Y_i, \delta_i)_{i=1,....,n}$ where $Y_i = min(T_i, C_i)$; $\delta_i = \mathbb{I}_{\{T_i \leq C_i\}}$ and the single index $\theta$ which is unknown and has to be estimated.

In practice this parameter can be selected by cross-validation approach (see Aït Saidi et al. (2008)). In this passage, it may be that one can select the real-valued function $\theta(t)$ among the eigenfunctions of the covariance operator

$$\mathbb{E}\left[ (X' - \mathbb{E}X') < X', . >_{\mathcal{H}} \right],$$

where $X(t)$ is a diffusion processes defined on a real interval $[a, b]$ and $X'(t)$ its first derivative (see Attaoui & Ling (2016)).

Next, The sample $\mathcal{L}$ will be chosen by applying the principal component analysis (PCA) method. The computation of the eigenvectors of the covariance operator estimated by its empirical covariance operator:

$$\frac{1}{|\mathcal{L}|}\sum_{i\in\mathcal{L}}(X_i' - \mathbb{E}X')\,{}^t(X_i' - \mathbb{E}X')$$

will be the best approximation of our functional parameter $\theta$.

Now, $\theta^\star$ denotes the first eigenfunction corresponding to the first higher eigenvalue, which will replace $\theta$ during the simulation step.
In practice, some tuning parameters have to be fixed, the kernel $K\,(.)$ is chosen to be the quadratic function defined as

$$K\,(u) = \frac{3}{2}\left(1 - u^2\right)\mathbb{I}_{[0,1]};$$

and the kernel $H\,(.)$ is given as

$$H\,(z) = \frac{3}{4}\left(1 - z^2\right)\mathbb{I}_{[-1,1]}\,(z)\,.$$

Taking into account the smoothness of the curves $X_i\,(t)$ (Figure 3.6), we choose the distance in $\mathcal{H}$ as

$$d\,(\chi_1, \chi_2) = \left(\int_0^1 \left(\chi_1'\,(t) - \chi_2'\,(t)\right)^2 dt\right)^{1/2}$$

as semi-metric.

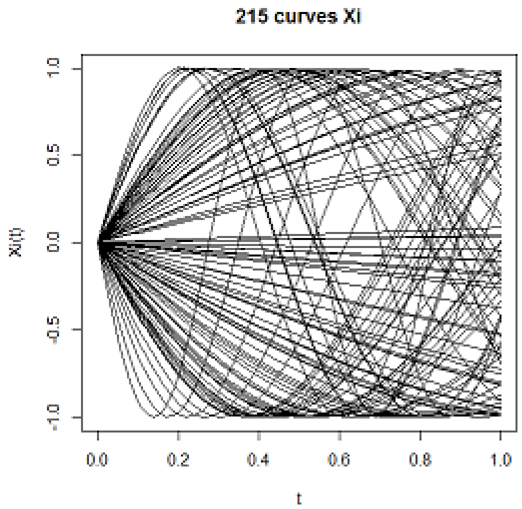


Figure 3.1: A sample of curves $\{X_i\,(t)\,,\, t \in [0, 1]\}_{i=1,....,215}$

In the following graphs, the covariance operator for $\mathcal{L} = \{1, ......., 215\}$ gives the discretization of 3 (the first eigenfunction $\theta$ is presented by a continuous curve), 20 and 215 eigenfunctions $\theta_i(t)$ respectively (Fig. 3.2, 3.3, 3.4).

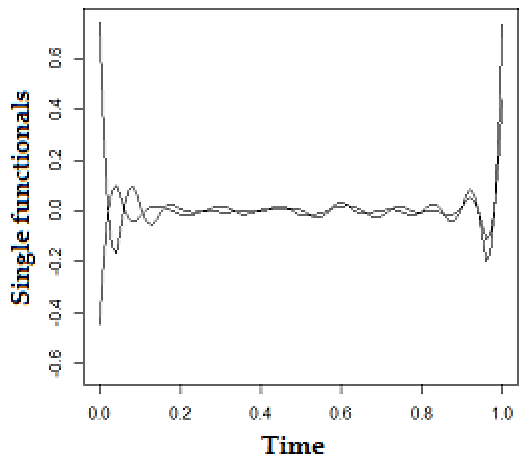


Figure 3.2: The curves $\theta_{i=1,2,3}\,(t_j)\,,\; t_j \in [0,\; 1]$

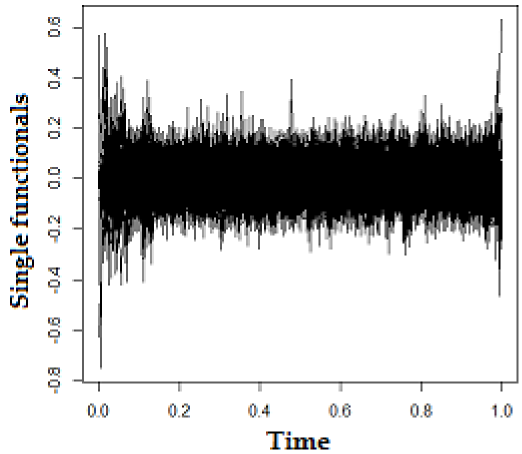


Figure 3.3: The curves $\theta_{i=1,\ldots.,215}\,(t_j)\,,\ t_j \in [0,\,1]$

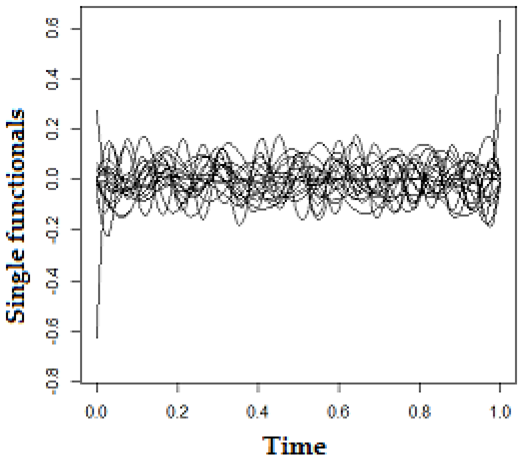


Figure 3.4: The curves $\theta_{i=1,\ldots.,20}\,(t_j)\,,\ t_j \in [0,\,1]$

Next, for simplifying the implementation of our methodology, we take the bandwidths $h_H \sim h_K = h$, where $h$ will be chosen by the cross-validation method on the k-nearest neighbors (see Ferraty and Vieu, 2006, p. 102) and we denote by $\theta^\star$ the first eigenfunction corresponding to the first higher eigenvalue of the empirical covariance operator: $\frac{1}{|\mathcal{L}|}\sum_{i\in\mathcal{L}}(X'_i - \mathbb{E}X')\,^t(X'_i - \mathbb{E}X')$

Now, we compute the mean square errors of the two estimators of conditional density for some values of the simulated responses $Y_i$. The results are presented as follow:

Table 3.1: Comparison between SIM and NPM

| $Y_i$ | $y = -14.68$ | $y = -5.47$ | $y = -3.34$ | $y = 0.57$ | $y = 1.75$ |
|---|---|---|---|---|---|
| $MSE1$ | $1.13\,10^{-7}$ | $7.19\,10^{-7}$ | $1.20\,10^{-6}$ | $1.48\,10^{-5}$ | $1.60\,10^{-5}$ |
| $MSE2$ | $1.08\,10^{-7}$ | $7.19\,10^{-7}$ | $1.18\,10^{-6}$ | $1.45\,10^{-5}$ | $1.57\,10^{-5}$ |

Finally, according to the results given in Table 3.1, it is clearly seen that there is not a big difference between $MSE1$ and $MSE2$. Thus, our conditional density estimator in the single index model is the same in the non-parametric model.

# Bibliography


Aït-Saidi, A., Ferraty, F., Kassa, R., (2005), Single functional index model for a time series. Revue Roumaine de Mathématique Pures et Appliquées, 50, 321-330.

Aït-Saidi, A., Ferraty, F., Kassa, R., Vieu, P., (2008), Cross-validated estimation in the single functional index model, Statistics 42, 475-494.

Attaoui, S., Laksaci A., Ould-Saïd, E., (2011), A note on the conditional density estimate in the single functional index model. Statist. Probab. Lett., 81(1), 45-53.

Attaoui, S., (2014), Strong uniform consistency rates and asymptotic normality of conditional density estimator in the single functional index modeling for time series data, 98, 257-286

Attaoui, S. & Ling, N., Metrika (2016) 79: 485. https://doi.org/10.1007/s00184-015-0564-6

Attouch, M., Laksaci, A., Ould-Saïd, E., (2010), Asymptotic normality of a robust estimator of the regression function for functional time series data, Journal of the Korean Statistical Society, 39, 489-500.

Bouchentouf, A. A., Djebbouri, T., Rabhi, A., Sabri, K., (2014), Strong uniform consistency rates of some characteristics of the conditional distribution estimator in the functional single-index model. Appl. Math. (Warsaw)., 41(4),301-322.

Cai, Z., (1991), Strong consistency and rates for recursive nonparametric conditional probability density estimates under $(\alpha, \beta)$-mixing conditions. Stochastic Process. Appl., 38(2), 323-333.

Delecroix, M, Härdle, W., Hristache, M., (2003), Efficient estimation in conditional single-index regression, J. Multivariate Anal., 86, 213-226.

De Gooijer, J. G. and Zerom, D., (2003), On Conditional Density Estimation. Statistica Neerlandica, 57, 159-176.

Ezzahrioui, M. and Ould-Saïd, E., (2008)., Asymptotic results of a nonparametric conditional quantile estimator for functional time series, Comm. Statist. Theory Methods, 37(16-17), 2735-2759.

Ezzahrioui, M., Ould-Saïd, E., (2008), Asymptotic Results of a Nonparametric Conditional Quantile Estimator for Functional Time Series, Communications in Statistics - Theory and Methods, 37(17), 2735-2759.

Ezzahrioui, M., Ould Saïd, E., (2010), Some asymptotic results of a nonparametric conditional mode estimator for functional time series data, Statist. Neerlandica 64, 171-201.

Ferraty, F., Vieu, P., (2003), Curves discrimination: a nonparametric functional approach, Computat. Statist. Data Anal., 44, 161-173.

Ferraty, F., Laksaci, A. and Vieu, P., (2006), Estimating some characteristics of the conditional distribution in nonparametric functional models, Statist. Inference Stoch. Process, 9, pp. 47-76.

Ferraty, F. and Vieu, P., (2006), Nonparametric functional data analysis, Springer Series in Statistics, New York.

Ferraty, F., Mas, A., Vieu, P., (2007), Advances in nonparametric regression for functional variables, Aust. New Zeal. J. Stat., 49, 1-20

Ferraty, F., Rabhi, A. and Vieu, P.,(2008), estimation non-paramétrique de la fonction de hasard avec variable explicative fonctionelle ,53,1-18

Gannoun, A., Saracco, J., Yu, K., (2003), Nonparametric prediction by conditional median and quantiles, J. Statist. Plann. Inference, 117(2), 207-223.

Härdle, W., Hall, P., Ichumira, H., (1993), Optimal smoothing in single-index models, Ann. Statist. 21, 157-178.

Hristache, M., Juditsky, A., Spokoiny, V. (2001). Direct estimation of the index coefficient in a single-index model. Ann. Statist., 29(3), 595-623.

Ling, N., Xu, Q., (2012), Asymptotic normality of conditional density estimation in the single index model for functional time series data, Statistics & Probability Letters, 82, 2235-2243

Masry, E., (1989), Nonparametric estimation of conditional probability densities and expectations of stationary processes: strong consistency and rates, Stochastic Process. Appl, 32(1):109-127.

A. Mahiddine , A.A. Bouchentouf, A. Rabhi., (2014), Nonparametric estimation of some characteristics of the conditional distribution in single functional index model, 2(4), 392-410

Quintela del Rio, A. and Vieu, P., (1997), A nonparametric conditionnal mode estimate, Nonparametric Statistics, 8, 253-266 .

A. Quintela del Rio and P. Vieu., (1997), A nonparametric conditional mode estimate, Nonparametr. J. Statist., 8, 253-266.

# Chapter 4

# General Conclusion and prospects

We were interested specifically in this thesis to single-index conditional models that treat the case of functional variables in which "response" variable is true while the explanatory variable is functional. The objective was the estimation of the distribution function as well as the density function by the kernel method. The case in question deals with complete data. The richness of this functional statistical research area offers many perspectives both theoretically and practically.

The work developed in this thesis offers many perspectives, let us cite

- The asymptotic normality of our estimators can allow us to test and build confidence intervals.
- We can also consider an asymptotic study for our esitmators in the ergodic case
- Other themes can be addressed in the long term, such as conditioning by p functional variables or a linear combination of these p functional variables. Another estimator can also be envisaged using another method than the kernel estimate (Fourier, wavelets,...)
- Another possible perspective is to assume that not only the explanatory variable is functional but also the variable of interest.

# Bibliography


Aït-Saidi, A. Ferraty, F. Kassa, R. (2005). Single functional index model for a time series. R. Roumaine Math. Pures et Appl. 50, 321-330.

Aït-Saidi, A. Ferraty, F. Kassa, R. Vieu, P. (2008). Cross-validated estimation in the single functional index model. Statistics. 42, 475-494.

A. Ait Saidi. F. Ferraty. R. Kassa. P. Vieu.(2008a). Cross-validated estimations in the single functional index model, Statistics, 42, 475-494.

A. Ait Saidi, F. Ferraty, R. Kassa, P. Vieu.(2008b). Choix optimal du paramètre fonctionnel dans le modèle à indice fonctionnel simple,C. R. Acad. Sci. Paris, Ser. I, 346, 217-220.

Akritas, M. Politis, D.(2003) (ed.) Recent advances and trends in nonparametric statistics. Elsevier, Amsterdam .

Antoniadis, A. Sapatinas, T. (2003). Wavelet methods for continuous-time prediction using Hilbert-valued autoregressive processes. J. Multivariate Anal. 87 (1) 133-158.

Attaoui, S. Laksaci, A. Ould-Saïd, E. (2011). A note on the conditional density estimate in the single functional index model. Statist. Probab. Lett. 81(1) 45-53.

Attaoui, S. (2014). Strong uniform consistency rates and asymptotic normality of conditional density estimator in the single functional index modeling for time series data. 98, 257-286

Attaoui, S. Ling, N. Metrika .(2016).Asymptotic results of a nonparametric conditional cumulative distribution estimator in the single functional index modeling for time series data with applications. 79 (5) 485-511

Attouch, M. Laksaci, A. Ould-Saïd, E. (2009). Asymptotic distribution of robust estimator for functional nonparametric models. Comm. Statist. Theory and Methods. 38, 1317-1335.

Attouch, M. Laksaci, A. Ould-Saïd, E. (2010). Asymptotic normality of a robust estimator of the regression function for functional time series data. J. of the Korean Statist. Soc. 39, 489-500.

Aurzada, F. Simon, T. (2007). Small ball probabilities for stable convolutions.ESAIM Probab. Stat. 11, 327-343 (electronic).

Azzedine, N. Laksaci, A. Ould Saïd E. (2008). On the robust nonparametric regression estimation for functional regressor. Statist. Probab. Lett.78, 3216-3221

Baìllo, A. Grané, A. (2009). Local linear regression for functional predictor and scalar response, Journal of Multivariate Analysis. 100, 102-111

Barrientos-Marin, J. Ferraty, F. Vieu, P. (2010). Locally Modelled Regression and Functional Data. Journal of Nonparametric Statistics. 22, 617-632

Benko, M. Härdle. W, Kneip. A.(2006). Common Functional Principal ComponentsSFB 649 Discussion Papers SFB 649 DP 2006-010, Humboldt University, Berlin, Germany.

Berlinet, A. Gannoun, A. Matzner-Lober, E. (1998a). Normalité asymptotique d'estimateurs convergents du mode conditionnel. La Rev. Canad. de Statist. 26, 365-380.

Berlinet, A. Gannoun, A. Matzner-Lober, E. (1998b). Propriétés asymptotiques d'estimateurs convergents des quantiles conditionnels. C. R. Acad. Sci., Paris, Sér. I, Math. 326, 611-614.

Besse, P. Ramsay, J.O. (1986). Principal components analysis of sampled functions. Psychometrika. 51 (2) 285-311.

Besse. P, Cardot. H, Stephenson. D.(2000). Autoregressive forecasting of some functional climatic variations, Scand. J. Statist. 27, 673-687.

Borggaard, C. Thodberg, H.H.(1992). Optimal minimal neural interpretation of spectra Analytical chemistry. 64 (5) 545 - 551.

Bosq, D. Lecoutre, J.P.(1987). Théorie de l'estimation fo.nctionnelle (in french). Economica.

Bosq, D. (1991). Modelization, nonparametric estimation and prediction for continuous time processes. In Nonparametric functional estimation and related topics (Spetses, 1990), 509-529, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.335, Kluwer Acad. Publ., Dordrecht.

Bosq, D. (2000). Linear processes in function spaces. Theory and Applications, Lecture Notes in Statistics, 149, Springer-Verlag New York.

Bouchentouf, A. A., Djebbouri, T., Rabhi, A., Sabri, K., (2014), Strong uniform consistency rates of some characteristics of the conditional distribution estimator in the functional single-index model. Appl. Math. (Warsaw). 41 (4) 301-322.

Cai, Z.,(1991). Strong consistency and rates for recursive nonparametric conditional probability density estimates under $(\alpha, \beta)$-mixing conditions. Stochastic Process. Appl. 38 (2) 323-333.

Cao, J. Ramsay, J.O. (2007). Parameter cascades ans profiling in functional data analysis Comp. Stat. 22 (3) 335-351.

Cardot, H. Crambes, C. Sarda, P. (2004). Spline estimation of conditionalquantiles for functional covariates, C. R. Math. Acad. Sci. Paris. 339, 141-144.

Chate, H. Courbage, M. (1997). Lattice systems. Physica D. 103, 1 -612.

Clarkson, D.B. Fraley, C. Gu, C.C. Ramsay, J.S.(2005). S+ functional data analysis user's guide. Comp. Statist. Series, Springer, New York .

Clot, D. (2002). Using functional PCA for cardiac motion exploration Proceedings of the. IEEE International Conference on Data Mining. 91-98.

Collomb, G. Härdle, W. Hassani, S. (1987). A note on prediction via conditional mode estimation. J. Statist. Plann. and Inf. 15, 227-236.

Crambes, C., Kneip, A. Sarda, P. (2007). Smoothing splines estimators for functional linear regression 37 (1) 35-72.

Crambes, C. Delsol, L. Laksaci, A. (2008). Robust nonparametric estimation for functional data. J. Nonparametric Statist. 20, 573-598.

Cuevas, A. Fraiman, R. (2004). On the bootstrap methodology for functional data. (English summary) COMPSTAT 2004-Proceedings in Computational Statistics 127-135 Physica, Heidelberg.

Dabo-Niang, S. (2002). Estimation de la densité dans un espace de dimension infinienie : Application aux diffusions. C. R. Math. Acad. Sci. Paris. 334, 213-216.

Dabo-Niang, S. Rhomari, N. (2003). Estimation non paramétrique de la régression avec variable explicative dans un espace métrique. C. R., Math., Acad. Sci. Paris. 336, 75-80.

Dabo-Niang, S. (2004). Kernel density estimator in an infinite dimensional space with a rate of convergence in the case of difusion process. Applied Math. Lett. 17, 381-386.

Dabo-Niang, S. Yao, A-F. (2007). Kernel regression estimation for continuous spatial processes. Math. Methods. Statist. 16, 298-317.

Dabo-Niang, S. Laksaci, A. (2007). Propriétés asymptotiques d'un estimateur à noyau du mode conditionnel pour variable explicative fonctionnelle. Ann. I.S.U.P. 51, 27-42.

Dabo-Niang, S., Rhomari, N. (2009). Kernel regression estimation in a Banach space. J. Statist. Plann. Inference. 139, 1421-1434.

Dabo-Niang, S. Laksaci, A. (2010). Note on conditional mode estimation for functional dependent data, Statistica. 70, 83-94.

Dabo-Niang, S. Kaid, Z. Laksaci, A. (2012). On spatial conditional mode estimation for a functional regressor Statist. Probab. Lett. 82, 1413-1421.

Dabo-Niang, S. Kaid, Z. Laksaci, A. (2012). Spatial conditional quantile regression : Weak consistency of a kernel estimate. 57 (4) 311-339.

Dabo-Niang, S. Kaid, Z. Laksaci, A. (2012). Asymptotic properties of the kernel estimate of the spatial conditional mode when the regressor is functional. 99 (2) 131-160

Delecroix, M. Härdle, W. Hristache, M. (1999). M-estimateurs semiparamétriques dans les modèles à direction révélatrice unique. Bull. Belg. Math. Soc. Simon Stevin. 6 (2) 161-185.

Delecroix, M. Härdle, W. Hristache, M. (2003). Efficient estimation in conditional single-index regression. J. Multivariate Anal. 86, 213-226.

Delsol, L. (2007). CLT and $L_q$ errors in nonparametric functional regression. C. R. Math. Acad. Sci. Paris. 345, 411-414.

De Gooijer, J. G. Zerom, D., (2003). On Conditional Density Estimation. Statistica Neerlandica. 57, 159-176.

Demongeot, J., Laksaci, A., Madani, F. and Rachdi, M. (2011). Functional data : Local linear estimation of the conditional density and its application. Statistics. 85-90 .

Hamdaoui, D. Bouchentouf, A.A . Rabhi, A. Guendouzi,T.(2017).Asymptotic normality of conditional distribution estimation in the single index model , Acta Univ. Sapientiae, Mathematica. 9 (1) 162-175

Ezzahrioui, M. Ould-Saïd, E. (2005). Asymptotic normality of nonparametric estimators of the conditional mode for functional data. Technical report, No.249, LMPA, Univ. Littoral Côte d'Opale.

Ezzahrioui, M., Ould-Saïd, E. (2006). On the asymptotic properties of a nonparametric estimator of the conditional mode for functional dependent data. Preprint, LMPA No 277, Univ. du Littoral Côte d'Opale.

Ezzahrioui, M. Ould-Saïd, E. (2008a). Asymptotic normality of a nonparametric estimator of the conditional mode function for functional data. J. Nonparametr. Stat. 20, 3-18.

Ezzahrioui, M. Ould-Saïd, E. (2008b). Asymptotic normality of the kernel estimator of conditional quantiles in a normed space. Far East J. Theor. Stat. 25, 15-38.

Ezzahrioui, M. Ould-Saïd, E. (2008c). Asymptotic results of a nonparametric conditional quantile estimator for functional time series. Comm. Statist. Theory Methods. 37, 2735-2759.

Ezzahrioui, M. Ould Saïd, E. (2010). Some asymptotic results of a nonparametric conditional mode estimator for functional time series data Statist. Neerlandica. 64, 171-201.

Faden, A.(1985). The existence of regular conditional probabilities: necessary and sufficient conditions. Ann. Probab. 13, 288-298 .

Ferraty, F. Vieu, P. (2000). Dimension fractale et estimation de la régression dans des espaces vectoriels semi-normés. C. R. Acad. Sci. Paris. 330, 139-142.

Ferraty. F, Vieu. P.(2002). The functional nonparametric model and application to spectrometric data. Comput. Statist. 17 (4) 545-564.

Ferraty, F., Goia, A. Vieu, P. (2002a). Régression non-paramétrique pour des variables aléatoires fonctionnelles mélangeantes. (French) [Nonparametricregression for mixing functional random variables] C. R. Math. Acad. Sci.Paris. 334 (3) 217-220.

Ferraty F., Goia A. Vieu P. (2002b). Functional nonparametric model for time series : a fractal approach for dimension reduction. Test. 11 (2) 317-344

Ferraty, F. Vieu, P.(2003). Functional nonparametric statistics: a double infinite dimensional framework. In: M. Akritas and D. Politis (eds.) Recenadvances and trends in nonparametric statistics. Elsevier, Amsterdam. 61-78.

Ferraty, F. Peuch, A. Vieu, P. (2003). Modéle à indice fonctionnel simple,C. R. Acad. Sci., Paris. 336, 1025-1028.

Ferraty, F., Vieu, P.(2003). Curves discrimination: a nonparametric functional approach. Computational Statistics and Data Analysis. 44, 161-173.

Ferraty, F., Vieu, P. (2004). Nonparametric models for functional data, with application in regression times series prediction and curves discrimination. J. Nonparametric Statist. 16, 111-127.

Ferraty, F. Laksaci, A. Vieu, P. (2005). Functional time series prediction via conditional mode estimation. C. R. Math. Acad. Sci. Paris. 340, 389-392.

Ferraty, F. Rabhi, A. Vieu, P.(2005). Conditional quantiles for functional dependent data with application to the climatic El Ninõ phenomenon, *Sankhyã: The Indian Journalof Statistic,Special Issue on quantile regression and related methods*. 67 (2) 378-399.

Ferraty, F. Vieu, P. (2006). Nonparametric functional data analysis. Theory and Practice. Springer-Verlag.

Ferraty, F. Laksaci, A. Vieu, P.(2006). Estimation some characteristics of the conditional distribution in nonparametric functional models. Stat Inference Stoch. Process. 9, 47-76.

Ferraty, F. Mas, A. Vieu, P. (2007). Advances on nonparametric regression for fonctionnal data. ANZ Journal of Statistics. 49, 267-286.

Ferraty, F., Rabhi, A. Vieu, P. (2008). Estimation non-paramétrique de la fonction de hasard avec variable explicative fonctionnelle. Rev.Roumaine Math. Pures Appl. 53, 1-18.

Ferraty, F. Laksaci, A. Tadj, A. Vieu, P. (2010). Rate of uniform consistency for non-parametric estimates with functional variables. J. Statist. Plann. Inference. 140, 335-352.

Ferraty, F. Laksaci, A. Tadj, A. Vieu, P. (2011). Kernel regression with functional response. Electron. J. Stat. 5, 159-171.

Ferraty, F. Romain, Y. (2011). The Oxford handbook of functional data analysis. Oxford University Press.

Ferraty, F. Laksaci, A. Tadj, A. Vieu, P. (2012). Estimation de la fonction de régression pour variable explicative et réponse fonctionnelles dépendantes C. R. Acad. Sci. Maths. Paris. 350, 13-14

Ferraty, F. Van Keilegom, I. Vieu, P. (2012). Ression when both response and predictor are functions J. Multivariate Anal. 109, 10-28.

Ferré, L. ; Yao, A.-F. (2005). Smoothed functional inverse regression. Statist.Sinica. 15 (3) 665-683.

Gannoun, A., Saracco, J. Yu,K. (2003). Nonparametric prediction by conditional median and quantiles,J. Statist. Plann. Inference. 117, 207-223.

Gao, F. Li, W.V. (2007). Small ball probabilities for the Slepian Gaussian fields. Trans. Amer. Math. Soc. 359 (3) 1339-1350.

Gasser, T. Hall, P. Presnell, B. (1998). Nonparametric estimation of the mode of a distribution of random curves. J. R. Stat. Soc.Ser. B, Stat. Methodol. 60, 681-691.

Hall, P. (1989). On projection pursuit regression. Ann. Statist. 17 (2) 573-588.

Hall, P. Vial. C.(2006a). Assessing extrema of empirical principal component. functions. Ann. Statist. 34, 1518-1544.

Härdle, W.(1990). Applied nonparametric regression. Cambridge Univ. Press, UK .

Härdle, W. Hall, P. Ichumira, H. (1993). Optimal smoothing in singleindex models, Ann. Statist. 21, 157-178.

Härdle, W. Müller, M.(2000). Multivariate and semiparametric regression. In: M. Schimek (Ed) Smoothing and regression; Approaches, Computation, and Application. Wiley Series in Probability and Statistics, Wiley, New York. 357-392.

Harezlak, J. Coull, B.A. Laird, N.M. Magari, S.R. Christiani, D.C.(2007). Penalized solutions to functional regression problems Comp. Stat. and Data. Anal. 51 (10) 4911-4925.

Helland, I.(1990). Pls regression and statistical models, Scand. J. Statist. 17, 97-114.

Huber, P. J. (1985). Projection pursuit. Ann. Statist. 13 (2) 435-475.

Hristache, M., Juditsky, A., Spokoiny, V. (2001). Direct estimation of the index coefficient in the single-index model. Ann. Statist. 29, 595-623.

Ichimura, H. (1993). Semiparametric least squares (SLS) and weighted SLS estimation of single-index models. Journal of Econometrics. 58, 71-120.

Ioannides, D. A. Matzner-Lober, E.(2002). Nonparametric estimation of the conditional mode with errors-in-variables : strong consistency for mixing processes. J. Nonparametr.Stat. 14 , 341-352

Ioannides, D. A. Matzner-Lober, E. (2004). A note on asymptotic normality of convergent estimates of the conditional mode with errors-in-variables. J. Nonparametr. Stat. 16, 515-524

Khardani, S. Lemdani, M. Ould Saïd, E. (2010). Some asymptotic properties for a mooth kernel estimator of the conditional mode under random censorship. J. Korean Statist. Soc. 39 , 455-469.

Khardani, S. Lemdani, M. Ould Saïd, E. (2011). Uniform rate of strong consistency for a smooth kernel estimator of the conditional mode for censored time series. J. Statist. Plann. Inference. 141 , 3426-3436

Khardani, S. Lemdani, M. Ould Saïd, E. (2012). On the strong uniform consistency of the mode estimator for censored time series. Metrika. 75, 229-241.

Kolmogorov, A. N. Tikhomirov, V. M. (1959). $\varepsilon-$entropy and .$\varepsilon-$capacity. Uspekhi Mat. Nauk. 14, 3-86., 2, 277-364 (1961).

Kuelbs, J. Li, W. (1993). Metric entropy and the small ball problem for Gaussian measures. J. Funct. Anal. 116, 133-157.

Laksaci, A. Yousfate, A. (2002). Estimation fonctionnelle de la densité de l'opérateur de transition d'un processus de Markov à temps discret C. R., Math., Acad. Sci. Paris. 334, 1035-1038.

Laksaci, A. (2007). Erreur quadratique de l'estimateur à noyau de la densité conditionnelle à variable explicative fonctionnelle. C. R. Math. Acad. Sci. Paris. 345, 171-175.

Laksaci, A. Lemdani, M. Ould-Saïd, E. (2009). A generalized $L^1$-approach for a kernel estimator of conditional quantile with functional regressors: consistency and asymptotic normality. Statist. Probab. Lett. 79, 1065-1073.

Laksaci, A. Mechab, B. (2010). Estimation non parametrique de la fonction de hasard avec variable explicative fonctionnelle cas des donnees spatiales.Rev : Roumaine, Math Pures Appl. 55, 35-51.

Leao, D. Fragoso, M. Ruffino, P.(2004). Regular conditional probability, integration of probability and Radon spaces. Proyecciones. 23, 15-29.

Lemdani, M. Ould-Saïd, E. ; Poulin, N. (2009). Asymptotic properties of a conditional quantile estimator with randomly truncated data. J. Multivariate Anal. 100, 546-559.

Leurgans, S. E. Moyeed, R. A. Silverman, B. W. (1993). Canonical correlation analysis when the data are curves. J. Roy. Statist. Soc. Ser. B. 55 (3)725-740

Liang, H. de Uña-Álvarez, J. (2010). Asymptotic normality for estimator of conditional mode under left-truncated and dependent observations. Metrika. 72, 1-19.

Liang, H. de Uña-Álvarez, J. (2011). Asymptotic properties of conditional quantile estimator for censored dependent observations. Ann. Inst. Statist.Math. 63 , 267-289

Lifshits, M.A. Linde, W. Shi, Z. (2006). Small deviations of Riemann-Liouville processes in Lq-spaces with respect to fractal measures. Proc. London Math. Soc. (3) 92 (1) 224-250.

Lin, Z. Li, D.(2007). Asymptotic normality for $L_1$-norm kernel estimator of conditional median under association dependence,J. Multivariate Anal. 98 , 1214-1230.

Ling, N. Xu, Q.(2012). Asymptotic normality of conditional density estimation in the single index model for functional time series data, Statistics & Probability Letters, 82 (12) 2235-2243.

Louani, D. Ould-Saïd, E. (1999). Asymptotic normality of kernel estimators of the conditional mode under strong mixing hypothesis. J. Nonparametric Stat. 11, 413-442.

Ma, Z.M. (1985). Some results on regular conditional probabilities. Acta Math.Sinica (N.S.). 1, 302-307.

Mahiddine, A. Bouchentouf, A.A. Rabhi,A. (2014). Nonparametric estimation of some characteristics of the conditional distribution in single functional index model, Malaya Journal of Matematik (MJM), 2 (4) 392-410.

Manté. C, Yao. A.F. Degiovanni. C. (2007). Principal component analysis of measures, with special emphasis on grain-size curves Comp. Stat. Data Anal. 51 (10) 4969-4984.

Masry, E. (1989). Nonparametric estimation of conditional probability densities and expectations of stationary processes: strong consistency and rates, Stochastic Process. Appl. 32(1):109-127.

Masry, E. (2005). Nonparametric regression estimation for dependent functional data :Asymptotic normality. Stoch. Proc. and their Appl. 115, 155-177.

Nengxiang, L , Zhihuan, Li. Wenzhi, Y. (2014). Conditional Density Estimation in the Single Functional Index Model for a-Mixing Functional Data, Communications in Statistics - Theory and Methods 43 (3) 441-454

Ould-Saïd, E. (1997). A note on ergodic processes prediction via estimation of the conditional mode function. Scand. J. Stat. 24, 231-239.

Ould-Saïd, E. Cai, Z. (2005). Strong uniform consistency of nonparametric estimation of the censored conditional mode function. Nonparametric Statistics. 17, 797-806.

Ould Saïd, E. Djabrane Y. (2011). Asymptotic normality of a kernel conditional quantile estimator under strong mixing hypothesis and left-truncation. Comm. Statist. Theory Methods. 40, 2605-2627

Ould Saïd, E. Tatachak, A. (2011). A nonparametric conditional mode estimate under RLT model and strong mixing condition. it Int. J. Stat. Econ. 6 , 76-92.

Preda, C. Saporta, G. (2007a). PCR and PLS for Clusterwise Regression on Functional Data Selected Contributions in Data Analysis and Classification 589-598, Springer, Berlin, Heidelberg

Preda, C. Saporta, G. Lévéder, C. (2007b). PLS classification of functional data . Computational Statistics 22 (2) 223-235.

Quintela del Rio, A. Vieu, Ph. (1997). A nonparametric conditionnal mode estimate. Nonparametric Statistics. 8, 253-266.

Quintela-del-Rio, A. (2008). Hazard function given a functional variable : Nonparametric estimation under strong mixing conditions.J. Nonparametr. Stat. 20, 413-430.

Rachdi, M. Vieu, P. (2007). Nonparametric regression for functional data :automatic smoothing parameter selection. J. Statist. Plann. Inference 137 (9) 2784-2801.

Ramsay, J.O. Bock, R. Gasser, T. (1995). Comparison of height acceleration curves in the Fels, Zurich, and Berkeley growth data, Annals of Human Biology. 22, 413-426

Ramsay, J. Silverman, B.W.(1997). Functional Data Analysis. Springer-Verlag, New York.

Ramsay, J. O. Silverman, B. W. (1997). Functional data analysis. Springer, New York.

Ramsay. J.O.(2000). Différential equation models for statistical functions. Canad. J. Statist. 28 (2) 225-240.

Ramsay, J. O. Silverman, B. W. (2002). Applied functional data analysis ; Methods and case studies. Springer-Verlag, New York.

Ramsay, J.O. Silverman, B.W. (2005). Functional Data Analysis, Second Edition. Springer, New York.

Roussas, G. (1969). Nonparametric estimation of the transition distribution function of a Markov process. Ann. Math. Statist. 40, 1386-1400.

Roussas, G. (1991). Estimation of transition distribution function and its quantiles in Markov processes : strong consistency and asymptotic normality. it Nonparametric functional estimation and related topics , (Spetses, 1990), 443-462.

Samanta, M.(1989). Non-parametric estimation of conditional quantiles. Statist. Proba. Letters. 7, 407-412.

Samanta, M. ; Thavaneswaran, A. (1990). Nonparametric estimation of the conditional mode. Comm. Statist. Theory Methods. 19 (12) 4515-4524

Schimek, M.(2000). (ed.): Smoothing and regression; Approaches, Computation, and Application. Wiley Series in Probability and Statistics, Wiley, New York.

Shmileva, E. (2006). Small ball probabilities for jump Lévy processes from the Wiener domain of attraction. Statist. Probab. Lett. 76 (17) 1873-1881.

Stone, C. J. (1977). Consistent nonparametric regression. Discussion. Ann. Stat. 5, 595-645.

Stone, C.J. (1982). Optimal global rates of convergence for nonparametric regression.Ann. Statist. 10 (4) 1040-1053.

Stute, W. (1986). On almost sure convergence of conditional empirical distribution functions. Ann. Probab. 14, 891-901.

Theodoros, N. Yannis, G. Y. (1997). Rates of convergence of estimate, Kolmogorov entropy and the dimensioanlity reduction principle in regression. The Annals of Statistics, 25 (6) 2493-2511.

Valderrama, M.J. Ocaña, F.A. Aguilera, A.M. (2002). Forecasting PCARIMA models for functional data. COMPSTAT (Berlin) 25-36.

van der Vaart, A. W. van Zanten, J. H. (2007). Bayesian inference with rescaled Gaussian process priors. Electronic Journal of Statistics., 1, 433-448.

Xia, X. An, H. Z. (2002). An projection pursuit autoregression in time series.J. of Time Series Analysis. 20 (6) 693-714.

Youndjé, E. (1993). Estimation non paramétrique de la densité conditionnelle par la méthode du noyau. Thése de Doctarat, Université de Rouen.

Yao, F. Lee, T.C.M. (2006). Penalised spline models for functional principal component analysis J.R. Stat. Soc. B 68 (1) 3-25.

Zhou,Y. Liang, H. (2000). Asymptotic normality for $L^1$-norm kernel estimator of conditional median under $\alpha$-mixing dependence, J. Multivariate Anal. 73, 136-154.