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Modèles Stochastiques, Statistique & Applications

THESE

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Intitulé

Stability study of queueing systems with impatience.

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Dedication

This thesis is dedicated to

My parents, whose sincerely raised me with their caring and offered me unconditional love, a very special thank for the myriad of ways in which, throughout my life, you have actively supported me in my determination to find and realize my potential.

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Finally, I wish to express my deepest appreciation to my family for their love, understanding, and belief in me, particulary my parents, for their love and support over the years. Without the words of wisdom and teaching of my parents, I would never have persisted in the pursuit of my goals, from the smallest task to the largest achievement. I dedicate this thesis to them.

ملخص

في هذه الأطروحة نقوم بدراسة مختلف أنظمة قوائم الانتظار مع نفاذ صبر الزبائن، أولا نقوم بدراسة التقريب المحلولي لقوائم الانتظار ذات النداء المتكرر مع التغذية الراجعة و تنصل الزبائن. نتحصل على حد النشر لهذا النظام. بعد ذلك نقوم بإعطاء تحليلا لاستقرار نظام قوائم الانتظار ذات النداء المتكرر مع التغذية الراجعة و تنصل الزبائن.تم الحصول على دوال توليد الاحتمالية لاحتمالات حالة المراوحة و نستنتج الصيغ الصريحة لحالة النظام، إضافة إلى ذلك نعطي شرط لازم لاستقرار النظام. و اخبرا نقدم تحليلا لنظام ماركوفي للتغذية الراجعة عن اصطفاف في طابور مع تنصل و الاحتفاظ بالزبائن المتنصلين، وإجازات عمل متعددة ، و إعاقة جدول إجازات، حيث يعزى عدم صبر الزبائن إلى إجازات الخوادم. تم الحصول على دوال توليد الاحتمالية لاحتمالات حالة المراوحة، تم استنتاج الصيغ الصريحة لحجم النظام، عندما يكون الخادم في فترة عمل ناظمية و في حالة وجود إعاقة لجدول إجازات على التوالي، وتم اشتقاق عدة مقاييس لأداء النظام. إضافة إلى عدة أمثلة عددية للنظام.

Abstract

In this thesis we study various queueing systems with impatience. At first, we study the fluid approximation of a retrial queueing model with abandonment and feedback. The diffusion limit for the model under consideration is carried out. Then, we deal with the stability of a retrial queueing system with abandoned and feedback customers. The balance equations and generating functions of the model are derived, further the necessary stability condition is established. Finally, an analysis of a Markovian feedback queueing system with reneging and retention of reneged customers, multiple working vacations and vacation interruption, where customers' impatience is due to the servers' vacation is presented. The stationary analysis of the system is established. The probability generating functions of the stationary state probabilities is obtained, the explicit expressions of the system sizes when the server is in a normal service period and in a Bernoulli schedule vacation interruption, respectively are deduced, and various performance measures of the system are derived.

Résumé

Dans cette thèse nous étudions différents systèmes de files d'attente avec impatience, en premier lieu nous étudions l'approximation fluide de système de files d'attente avec rappel, abandon et feedback. La limite de diffusion pour le modèle considéré est effectuée. En suite nous analysons la stabilité d'un système de files d'attente avec rappel, abandon et feedback. Pour ce système nous dérivons les équations d'équilibre et les fonctions génératrices, en outre, nous établissons la condition de stabilité nécessaire. Finalement nous considérons un système de files d'attente Markovian avec feedback, multiples vacances, interruption de vacances, clients impatients et rétention de clients abandonnés, où l'impatience des clients est due aux vacances des serveurs. Nous établissons l'analyse stationnaire du système. Nous obtenons les fonctions génératrices des probabilités d'état stationnaire, nous déduisons les expressions explicites des tailles de système quand le serveur est dans une période de service normale et dans une période de vacance interrompue, respectivement. Diverses mesures de performance du système sont dérivées.

Contents

Chapter 1

Introduction and Presentation

Queueing theory is prominent and successful branch of mathematics which provided applications, results, and methods in probability theory. Since the past few decades, it has been found that in most cases, it is the probabilistic models that mark an edge over the deterministic models in terms of practical applicability. Most of the probabilistic models find their significant and great applications in wider domains of common branches such as statistics, operations research and in many complex and nontrivial areas such as applied industrial research.

Queueing theory is one of the most important and predominant areas where probabilistic models have been used for the good, this field of applied probability theory deals with an extensive and in great depth study of various service systems plagued by bottleneck. The areas of applications are profuse and include telephonic systems, the reliability of seemingly complex systems, computer, communication and telecommunication systems, manufacturing, industry, etc.

This thesis is concerned with the analysis of impatient customers in different ***"retrial and vacation"*** queueing systems. In recent decades much effort has been devoted to this type of queueing systems, because of their wide applications in many real life situations; in the performance modelling, in cellular mobile networks, in computer, communication and telecommunication networks, in local area networks, flexible manufacturing systems and divers other areas of applications.

The goal of the present chapter is to introduce a fairly broad set of results gathering important results in retrial and vacation queueing systems and queueing models with impatient customers as their applications in solving several realistic problems. In Section 1.1 we recall some fundamental vocabulary and results of retrial queueing systems. Then, in Section 1.2 the concept of queueing systems with impatient customers ***"balking and reneging"*** is presented. After that, in Section 1.3 we give succinctly the basic definitions and results from vacation queueing models. The rest of this chapter, Section 1.4 and

Section 1.5 is dedicated to the contribution and the layout of the thesis, respectively.

1.1 Retrial queueing systems

Queueing systems in which incoming customers who repeatedly attempt to get services from the server, on finding dejection, owing to the inactivity of the server to supply service at any moment of selection from the queue are known as retrial queueing systems. The customer is in "***orbit***" anytime it is in between the retrials. The customers, in retrial queueing systems, can wait in the orbit and while being in the there, they can try and attempt as many times as they desire, to benefit service from the selfsame server.

The standard queueing models do not take into consideration the case of retrials and consequently cannot be applied in solving a numerous of practically and sensibly significant problems. (Kosten (1973)) notes that "any theoretical result that does not take into consideration this repetition effect should be considered suspect". Retrial queues have been introduced to solve this insufficiency. Retrial queues have a powerful and potential areas of application like mobile cellular networks, call centers, computer networks where efficient retrial queueing systems can play a pivotal role in giving quality service in high-speed time. The general structure of a retrial queue is shown in Figure 1.1.

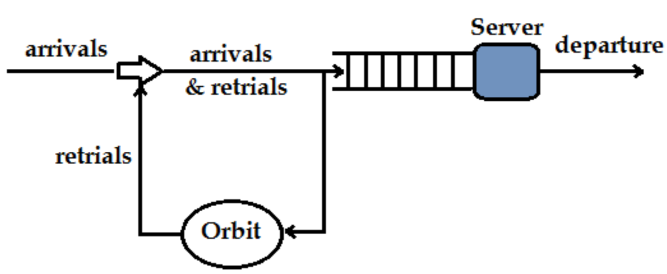

Figure 1.1: General structure of a retrial queue

It is clear from this graph that retrial queues can also be viewed as a specific sort of queueing networks.

1.1.1 Some motivating examples

In standard queueing theory it is mostly assumed that a customer who cannot get service directly after arrival either joins the waiting line (and then is served according to some queueing discipline) or leaves the system forever. Sometimes, impatient customers leave the queue, and it is also assumed that they may leave the system forever. Nevertheless,

the assumption on impatient customers which are selected to leave the system is just a first order approximation to a real situation. Generally such a customer after some random period of time he comes back to the system and essay to receive service again.
Next are a few examples explaining this general remark in more detail.

※ ***Telephone systems.***
Everyone knows that a telephone subscriber who get a busy signal retries and repeats the call until the desired connection is occurred. Consequently, the flow of calls which circulates in a telephone network is formed of two parts: the flow of elementary calls, which reflects the real wills of the telephone subscribers, and the flow of retried calls, which is the result of the lack of success of precedent attempts.
So, One of the customarily used methods to model the commonly and broadly used telephone systems is to model them as retrial queues. Owing to the repeating lawsuits of a telephonic caller, who is not permitted to get service because of network congestion, comes back and makes a call another time, this lets telephone systems a case of retrial queues. Recent progress and advancements in telecommunication sector led to an ever-increasing necessity to expand, develop and extend the retrial phenomenon. In addition the fact that under the overload conditions, most telephone systems work poorly, so, it is necessary to see again the models that we develop while modeling the telephone systems on retrial theme.

Let note that the principal purpose of any call center is to give quality service via an appliance, in our case that appliance being a telephone. In addition, when the structure, planning, management and execution of a typical call center are carefully studied, it is immediately clear how the modeling of telephone systems as retrial queues best clarifies the behavior of the customers in such a scenario.

Almost all the major players in the telecom industry use call centers as their prime way of communication and interaction with their customers. To get a perspective, a call center can be thought of as a basic queueing structure based on $M/M/c$ queue (one of the most widely used queueing model) Shekhar et al.(2016).

As performance parameters on which call centers are assessed, quality of service and speed in time of network bottleneck are the two most prominent evaluation criteria's, so here the effective call mechanisms can enhance conventional telephone systems.

As a conclusion, we realize that while modeling a telephone network or a mobile cellular network, we can not neglect the inherent existence of recurrent calls. All the factors mentioned above are quite convincing in favor of new queueing systems which properly are retrial queueing systems.

⋇ ***Retrial shopping queue.***

In a shop, a customer who finds a queue too long may want to do something else and come back later with the hope that the queue will dissolve. Alike behavior may show some impatient clients who have entered the queue but then discovered that the residual wait time is too long.

⋇ ***Random access protocols in digital communication networks.***

Consider a communication line with the slot time that is shared by diverse stations. The duration of the slot is equal to the transmission time of a single data packet. If two or more stations transmit packets at the same time, then a collision occurs, i.e all packets are destroyed and must be retransmitted. If the stations involved in the conflict would attempt to retransmit destroyed packets in the nearest slot, a collision occurs with certainty. To avoid this, each station, independently of the other stations, transmits the packet with a certain probability and delays actions until the next slot with a complementary probability, or equivalently, each station initiates a random delay before the next trial to transmit the packet.

The underlying rules governing the operations of random access protocols in computer networks provide a sound enough background and motivation at the same, for design of communication protocols with new feature and that is allowing the protocols have to have retransmission control which is nothing but thinking of protocols to have retrial capabilities in the queueing domain Shekhar et al.(2016). Thus, to understand it a little more clearly, let's take an example of a communication line that has a sharing time that is shared among various stations. The duration of the slot is equal to the total time taken during the transmission of a single data packet. A conflict occurs whenever two or more stations transmit packets together. A clash will always cause damaged packets over the entire transmission line and these damaged data packets must now be retransmitted. The immediate quick fix may be to allow the clashing stations to retransmit the damaged packets within the nearest available time, but then such a fix is bound to run again a shock, thus leading to damaged packets and to new need for retransmission. To avert the aforementioned intricate situations, we can permit each clashing station to introduce a random delay before the next transmission challenge of the packet. This sufficiently lucid and secular explanation justifies the need to incorporate the functionality of the retrial feature into computer networks.

⋇ ***Priority queues.***

To accommodate the modeling of complex real-life scenarios, a prioritized queue is requisite, then we move towards an alternative employment of retrial queue. Priority queueing systems are characterized by the widespread presence of two categories of customers: primary customers arriving by independent streams and secondly customers that were previously in the habitual queue making them the low priority entities in the new queue set up. In such a revalued retrial queueing system, primary customers are treated as a first preference and hence given higher priority and these later are queued first and also served according to a certain class service.

Due to the presence of priorities, any case of congestion between the two types of customers will be treated to take account of the interests of priority customers. Therefore, in the case of blocking, it is the low-priority customer who must leave the service area and make the task of waiting until new retrials occur to extend the waiting and service time for the low priority customers. As can be seen now, higher priority customers for priority retrial queueing systems have uncontrolled authority and priority over low-priority customers.

The most widespread and succinct example to visualize the importance of such prioritized retrial queueing system is the case of a large hospital where a higher priority queue is always preserved for emergency cases or patients with special needs, even if an habitual queue is the order of the day to put up the interests of the usual population arriving at the hospital to take advantage of treatment services.

1.1.2 Literature Review

Retrial queues have attracted a considerable attention because of their wide application in many real life situations, and because of their powerful applications in performance analysis of various systems such as call centers, computer networks and communication telecommunication systems.

It is important to note that analysis of retrial queues is harder and more complicated than that of the corresponding queueing models without retrials, and this is due to the fact that the arrival flow of customers from the orbit makes the underlying Markov chain of retrial queues nonhomogeneous, specific results are obtained just in a few particular cases (Artalejo and Gomez-Corral (2008), Falin and Templeton(1997))....

In majority literature on retrial queues, only the new arriving calls are served, and after all calls are served, the server waits either for the next arrival of a primary call or for a retrial call. Nevertheless, in real life situations there exists a case where the server has a to make outgoing phone calls. In diverse service systems like a ***call center***, an operator not only serves ingoing calls but it also makes outgoing phone calls if he/she is in free period, while this later is busy, ingoing arriving calls cannot receive a service.

At this time the calls join an orbit and retry to get a service after some random time independently of other calls.

Nowadays, call center activity is very significant because it furnishes a channel for two-way communication between companies and their customers (Aksin et al. (2007), Koole and Mandelbaum (2002)).
Characteristically, there are two types of call centers: inbound and outbound call centers. The first is used for customer support where customers call from outside for certain requests such as booking tickets or complaint regarding products. (Stolletz (2003)). The second is used for telephone marketing where a telephone dialing system haphazardly makes direct calls to potential customers for advertising or selling new products (Samuelson (1999)). Lately, modern call centers incorporate the two types of call centers "both inbound and outbound functions" to increase the productivity (Bhulai and Koole (2003), Deslauriers et al. (2007)). These centers are called blended call centers where an operator not only receives ingoing calls but also makes phone calls to customers when he/she is unoccupied. (Falin (1979)) derived the integral formulae for the partial generating functions and explicit expressions for certain expected characteristics of an $M/G/1/1$ retrial queue with two way communication in which ingoing calls and outgoing calls are are following the same service distribution. (Choi et al. (1998)) extended Falin's model to $M/G/1/K$ retrial queues where ingoing and outgoing calls are also supposed following the same service time distribution. But, from an application point of view, this hypothesis is restrictive as in general ingoing calls and outgoing calls can have different service time distributions. (Bhulai and Koole (2003)) presented a multiserver queueing model with infinite buffer for blended call centers for which optimal and nearly optimal policies are derived for the case where ingoing calls and outgoing calls are the two exponentially distributed and otherwise, respectively. (Deslauriers et al. (2007)) expanded five Markovian queueing models for blended call centers where ingoing and outgoing calls are distinguished and undistinguished. In (Deslauriers et al. (2007)) it was pointed that the models where ingoing and outgoing calls follow different distributions are more difficult than that with the same service time distribution for both types of calls. In (Artalejo and Resing (2010)), authors obtained the first partial moments for the $M/G/1/1$ retrial queue with different service time distributions of ingoing and outgoing calls by using a mean value analysis approach. (Avrachenkov et al. (2010)) used the matrix analytic approach to study a single server retrial queue with two classes of customers where retrial behaviors and service time distributions are different.

In the last few decade many efforts have addressed to the numerical investigation of complex retrial queues. In this sense, we especially mention the use of

⋇ ***Hypergeometric functions.*** These latter play a significant and pivotal role to analyze the stationary characteristics of a vast of retrial queues including the $M/M/2/2$ retrial queue (Hanschke (1987)), study of the steady state solution of an $M/M/1/1$ queue with linear repeated request presented in (Artalejo and Gomez-Corral (1997)). An $M/M/1/1$ retrial queues with Bernoulli abandonment and feedback studied in (Choi et al. (1998)), single server retrial queues with orbital search and nonpersistent customers established by (Krishnamoorthy et. al (2005)), state-dependent $M/M/c/c+r$ retrial queue with Bernoulli abandonment given in (Phung-Duc et al.(2010)), the analysis of a single server retrial queue with collision and impatience in (Kim (2010)) and more reference therein.

⋇ ***The Generating function methods.*** They were used extensively, let cite for instance the most resent works, (Amador and Moreno (2011)) gave the analysis of the successful and blocked events in the $Geo/Geo/c$ retrial queue. (Deepak et al. (2013)) introduced an $M^{(X)}/G/1$ retrial system with two types of search of customers from the orbit. (Choudhury and Deka (2013)) studied a batch arrival retrial queue with two phases of service and Bernoulli vacation schedule. (Gao and Wang (2014)) established the performance and reliability analysis of an $M/G/1$ G-retrial queue with orbital search and non-persistent customers. (Dimitriou (2015)) presented a retrial queue for modeling fault-tolerant systems with check pointing and rollback recovery. (Jain and Bhagat (2015)) introduced the embedded Markov chain approach to retrial queue with vacation, phase repair and multi optional services.

⋇ ***Matrix-analytical methods.*** (Choi et. al (1999), Diamond and Alfa (1999), Dudin and Klimenok (2000)) investigated the versatile retrial models with interarrival and interrepetition distributions of type PH, MAP, etc. And recently (Shin and Moon (2010)) presented the approximations of retrial queue with limited number of retrials. (Efrosinin and Winkler (2011)) analyzed a queueing system with a constant retrial rate, non-reliable server and threshold-based recovery.(Do et al. (2013)) for the investigation of enhanced algorithm to solve multiserver retrial queueing systems with impatient customers. (Kuo et al. (2014)) investigated the reliability-based measures for a retrial system with mixed standby components. (Rabia (2014)) improved truncation technique to analyze a $Geo/PH/1$ retrial queue with impatient customers. (Dudin et al. (2015a)) studied the Single server retrial queue with group admission of customers. (Dudin et al. (2015b)) presented a priority retrial queueing model operating in random environment with varying number and reservation of servers. (Shin (2015)) studied an algorithmic approach to Markovian multi-server retrial queues with vacations. (Laxmi and Soujanya (2015)) studied a perishable inventory system with service interruptions, retrial demands and negative customers. (Phung-Duc (2015)) investigated an asymptotic analysis for Markovian queues with two types of nonpersistent retrial customers.

⋇ ***The supplementary variable method.*** It was also employed in many research works, (Ke et al. (2011)) presented a Multi-server retrial queue with second optional service: algorithmic computation and optimization, (Rajadurai et al. (2014)) gave an analysis of a $M^{[X]}/(G_1, G_2)/1$ retrial queueing system with balking, optional re-service under modified vacation policy and service interruption. (Gao and Wang (2014)) established the performance and reliability analysis of an $M/G/1$ G-retrial queue with orbital search and non-persistent customers. Gao (2015) studied a preemptive priority retrial queue with two classes of customers and general retrial times. (Haridass and Arumuganathan (2015)) analyzed single server batch arrival retrial queueing system with modified vacations and N-policy.

⋇ ***The recursive method.*** This method is the most popular, for instance, (Avrachenkov and Yechiali (2010)) analyzed a tandem blocking queues with a common retrial queue. (Dragieva (2013)) presented a finite source retrial queue. Zhang and Wang (2013) gave the performance analysis of the retrial queues with finite number of sources and service interruptions. (Shin and Moon (2014)) studied an approximation of throughput in tandem queues with multiple servers and blocking.

1.2 Queueing models with impatient customers

Impatience generally takes three forms. The first is ***balking***; the reluctance of a customer to join a queue upon arrival, the second ***reneging***; the reluctance to remain in line after joining and waiting, and the third ***jockeying between lines*** when each of number of parallel lines has its own queue. (Gupta and Garg (2012)).

Impatience 'balking and reneging' is an interesting feature in a large variety of situations that can be met in healthcare applications, call centers, telecommunication networks, manufacturing systems where accumulated orders may be canceled, manufacturing systems of perishable goods.

1.2.1 Models incorporating customer impatience

Models including customer impatience are nearer to reality, and lead to more precise analysis. Let's cite some applications.

⋇ ***Healthcare applications.***
For various medical processes, patients are facing high risk of complication or death when treatment (for instance in the case of organ transplantation) is overdyed. In such situation, if there are many patients waiting for treatment (a queue is formed), it will be more suitable to serve the patients depending on the urgency of their requirements. When the condition of a patient deteriorates to a certain level, the treatment can become no longer required. In such a case, the patient is removed from the queue without service (an abandoned patient).

⋇ ***Perishable goods.***
There are many examples of perishable products let's cite for instance food items, chemicals, pharmaceuticals, adhesive materials used for plywood, blood, etc. (Karaesmen and Deniz (2011)) reported that, in 2004, 22% of the unsalable costs incurred by distributors of consumer packaged goods were due to expired products, and 5.8% of all components of blood processed for transfusion were outdated. Therefore, it is extremely important to understand such systems and to study the impact of the finiteness of product lifetimes on production and inventory control decisions. A literature related to the modeling of perishable inventory systems via queueing systems with impatient customers is considerable, knowing that customer abandonment and product perishing are similar phenomena. That is, a customer whose time of patience expires leaves the queue and similarly a product made to a stock whose lifetime expires is removed from the inventory.

⋇ ***Aircrafts in queue for landing, military applications and call centers.***

. Aircrafts in queue for landing is another example of impatient customers. Aircrafts are willing to wait, but only up to a point. An airplane may run out of fuel and must therefore have priority for landing.

. In military applications, abandonment is a significant feature. For instance, enemy aircraft or missiles (customers) take a finite time to transit to an area where interception is possible and they escape (abandon) if they are not intercepted (served) within that time.

- In most call center cases, customers waiting online are impatient. A customer will wait a certain amount of time for the service to begin. If the service did not begin with this time, it will give up and be lost.

1.2.2 Literature Review

The literature on queueing models with impatient customers is abundant because of the powerful importance of this feature, In this chapter we are limited to some which have been the basis of several research results including those presented in this thesis.

Queueing systems with balking, reneging or both have been studied by many researchers. (Haight (1957)) first considered an $M/M/1$ queue with balking. An $M/M/1$ queue with customers reneging was also proposed by (Haight (1959)). The combined effects of balking and reneging in an $M/M/1/N$ queue have been investigated by (Ancker and Gafarian (1963a), (1963b)). (Abou-EI-Ata and Hariri (1992)) considered the multiple servers queueing system $M/M/c/N$ with balking and reneging.

Recently queueing systems with impatience have attracted much attention in queueing literature because of explosive demands to efficiently design and manage call or contact centers, (Altman and Yechiali (2006, 2008)) studied the customer impatience in a classical vacation model and system with additional task, respectively.

(Yechiali (2007)) considered an $M/M/c$ system which as a whole suffers occasionally a disastrous breakdown, upon which all present customers (waiting and served) are cleared from the system and lost. (Chen et al. (2008)) studied $M/M/m/k$ queue with preemptive resume and impatience of the prioritized customers and derived the queue length distraction in stationary state and performance measures using the method of matrix analysis.

(Perel and Yechiali (2010)) considered a two-phase service impatient model where the customers become impatient if the server is in slow service phase. There are situations where customer's impatience is due to the absence of the server, more precisely due to the server being on vacation, and is independent of the customers in system.

(El-Paoumy and Nabwey (2011)) obtained the analytical solution of the $M/M/2/N$ queue with general balk function, reneging and two heterogeneous servers. (Kumar (2013)) presented an economic analysis of an $M/M/c/N$ queueing model with balking, reneging and retention of reneged customers. (Kumar and Sharma (2014)) gave a study of a finite capacity multi-server Markovian feedback queuing model with balking, reneging and retention of reneged customers. (Kumar and Sharma (2014)) gave an optimization of an $M/M/1/N$ feedback queue with retention of reneged customers. (Misra and Goswami (2015)) analyzed a power saving class II traffic in IEEE 802.16E with multiple sleep state and balking. Panda and (Goswami (2016)) analyzed the equilibrium balking strategies for a $GI/M/1$ queue with Bernoulli-schedule vacation (working vacation) and vacation interruption in the case where a customer can only observe the state of the server (observable queues) and when there is no information available to a customer before taking decision to join the system or balk (fully unobservable queues).

1.3 Vacation queueing models

Queueing systems with server vacation have been investigated extensively due to their wide applications in several areas including computer communication systems, manufacturing and production systems and inventory systems. In a vacation queueing system, the server may not be available for a period of time (utilize the idle time for different purposes) due to many reasons like, being checked for maintenance,working at other queues, scanning for new work (a typical aspect of many communication systems) or simply taking break. This period of time, when the server is unavailable for primary customers is referred as a vacation (Chandrasekaran et al. (2016)). For more detail on this subject wonderful surveys on server vacation models in the queueing literature may be found in (Doshi (1986), Takagi (1991), Tian and Zhang(2006)) and the references therein. A recent survey given by (Ke et al. (2010c)) and (Tian et al. (2009)) reported the more important research results on vacation and working vacation queueing systems.

A vacation in a queuing context is a period during which the server is unavailable to provide the service. Arrivals that come during the vacation can only enter service after the return of the server from its vacation. There are many situations that result in server vacation, that is, machine failures(breakdowns), systems maintenance and cyclic servers(where the server serves more than one queue in the system or more than one system).

The queueing model with server vacations (server absences) has been well studied in the past three decades and successfully applied in many areas such as manufacturing/service and computer/communication network systems and many other real life situations.

1.3.1 Different types of vacation models

Vacation queueing models can be classified according to the arrival processes, service processes, and the vacation policies. So, as it was mentioned above, excellent surveys on the earlier works of vacation models have been given by (Doshi (1986), Takagi (1991), and Tian and Zhang (2006)).

Accordingly to the previous survey chapters and books in particular that of (Doshi (1986), different types of vacation models are as follow

▶ The single vacation model; there is only one vacation after the end of each busy period. If the server returns from this vacation, it does not go for another vacation even if the system is still empty at that time. This type of vacation may come from cases such as maintenance in production systems (maintenance can be considered a vacation).

▶ The multiple vacation model, this type of vacation may come from cases such as maintenance in computer and communication systems where processors in computer and communication systems perform extensive testing and maintenance in addition to their main functions (processing telephone calls, reception and transmission of data, etc.). The required maintenance work is divided into short segments. Whenever customers are absent, the processor makes a segment of maintenance work. When the system is idle, the server takes a vacation (runs on a maintenance segment). On return from vacation, the server starts the service only if it finds K or more customers waiting in the queue, if the waiting number of customers is less than K then another vacation takes place (Maintenance segment).

▶ The limited service vacation model in which the server takes a vacation on becoming inactive or after serving m consecutive customers, or after a certain time T.

The way that the server serves a customer is connected with the vacation type. In (Doshi (1986)) some of the service models are discussed as the following:

◊ Gated service; as soon as the server comes back from the vacation it puts a gate behind the last waiting customer. It then begins to serve only customers who are within the gate, based on some rules of how many or for how long it might serve.

◊ Exhaustive service; the server is working (serves customers) until the system is emptied, after it goes on vacation.

◊ Limited service; a fixed limit of K is put on the maximum number of customers that can be served before the server leaves for vacation. The server goes on vacation either: (a) when the system is empty, or (b) when the K customers have been served.

1.3.2 Literature review

▶ ***Vacation models with variants of arrival processes.***

Considerable studies were carried out on the vacation models with Markov Arrival Process (MAP), (Gupta and Sikdar (2006)) studied an $MAP/G/1/N$ queue with single or multiple vacation policies, where the stationary distributions of number of customers at service completions, vacation terminations, pre-arrival, and arbitrary epochs were obtained, (Banik et al. (2006)) studied a finite buffer $MAP/G/1/N$ queue under single/multiple vacation policies and found the queue length distributions. Furthermore, (Wu et al. (2009)) investigated a $BMAP/G/1$ G-queues with second optional service and multiple vacations where arrivals of positive customers and negative customers follow a batch Markovian arrival process (BMAP) and Markovian arrival process (MAP), respectively. The queue length distributions and the mean of the busy period based were obtained. Very recently, (Banik and Chaudhry (2017)) investigated an efficient computational analysis of stationary probabilities for the queueing system $BMAP/G/1/N$ with or without vacation(s). Vacation models with batch arrivals were executively studied, (Arumuganathan and Ramaswami (2005)) studied a $M^{[x]}/G(a,b)/1$ queue with two service rates and multiple vacations. (Ke (2007a)) analyzed a $M^{[x]}/G/1$ queue under vacation policies (single or multiple vacation policy) with server breakdown and startup/closedown times. Later, Ke and Lin (2008) used the maximum entropy approach to examine an $M^{[x]}/G/1$ queue with N policy, server breakdowns, and single vacation policy. Recently, (Haridass and Arumuganathan (2015)) investigated a analysis of a single server batch arrival retrial queueing system with modified vacations and N-policy.

▶ ***Vacation models with variants of vacation policies.***

⋇ ***Modified vacation policy.***

(Ke and Chu (2006)) studied the operating characteristics of an $M^{[x]}/G/1$ queueing system under a modified vacation policy, where the server leaves for a vacation as soon as the system is empty. The server takes at most J vacations repeatedly until at least one customer is found waiting in the queue when the server returns from a vacation. The system size distribution at different points in time, as well as the waiting time distribution in the queue, the expected length of the busy period and idle period were derived. After that, (Ke (2007b)) extended the model in (Ke and Chu (2006)) to the case with customer balking behavior. (Ke et al. (2010a)) generalized the model to the case with N-policy. Moreover, (Ke et al. (2010b)) investigated the threshold model of (Ke et al. (2010a)) with a randomized control policy. Later, more works on the models with the modified vacation policies were given. For instance, (Ke and Chang (2009a)) considered an $M/G/1$ retrial queue with modified vacation policy, customer balking, and feedbacks. (Chang and Ke (2009)) investigated an $M^{[x]}/G/1$ retrial queue with modified vacation policy using the supplementary variable approach. (Ke and Chang (2009a)) extended Chang and Ke's model to more general cases with impatience customers and feedback behaviors. Recently, (Padmavathi et al. (2016)) investigated a finite-source inventory system with postponed demands and modified M vacation policy.

※ ***Bernoulli vacation policy.***

(Madan et al. (2003)) examined an $M/M/2$ queue with a single Bernoulli schedule vacation policy. (Choudhury and Madan (2004)) analyzed a batch arrival queueing system with two phase service and Bernoulli vacation. Further, (Choudhury and Madan (2005)) considered a system with a modified Bernoulli vacation and N-policy. Later, (Choudhury (2007)) examined a two phase batch arrival retrial queueing system with Bernoulli vacation schedule. At the same time (Choudhury et al. (2007)) considered an $M^{[x]}/G/1$ queue with two-phase service and Bernoulli vacation and multiple vacation policy. Choudhury (2008) investigated an $M/G/1$ retrial queue with two-phase service and Bernoulli vacation schedule. After that, (Kumar et al. (2009)) considered an $M/M/c$ retrial queueing system with Bernoulli vacations and obtained various system performance measures. (Ke and Chang (2009b)) studied a $M^{[x]}/(G_1, G_2)/1$ retrial queue under Bernoulli vacation schedules with general repeated attempts and starting failures. Recently, (Ye et al. (2016)) investigated an analysis of a single-sever queue with disasters and repairs under Bernoulli vacation schedule.

※ ***Working vacation policy.***

Working vacation (WV) is one kind of vacation policy under which the server provides service at a lower speed during the vacation period rather than stopping service completely. This queueing model can be used to some practical systems like network service, web service, file transfer service and mail service etc Chandrasekaran et al. (2016).

Servi and Finn (2002) first introduced the concept of working vacation in a single server system. In such a system denoted by $M/M/1/WV$, the server would work at a different rate rather than completely stop during the vacation period. After that, the research interests on working vacation models grew fast. (Liu et al. (2007)) studied stochastic decomposition structures of the queue length and waiting time in an $M/M/1/WV$ queue. (Xu et al. (2009)) extended the $M/M/1/WV$ queue to a bulk input $M^{[x]}/M/1/WV$ queue and obtained the upper and lower bounds of the mean waiting time by using the properties of the conditional Erlang distribution. After that, the finite capacity $GI/M/1$ queue with multiple working vacations was studied by (Banik et al. (2007)). The $GI/M/1$ queue with working vacation and vacation interruption was discussed by (Li et al. (2008)). Afterward, the comparison analysis between the $GI/M/1$ and the $GI/Geo/1$ queues with single working vacation was provided by (Chae et al. (2009)). For the general service time, (Li et al. (2009)) used the matrix analytic method to analyze an $M/G/1$ queue with exponentially working vacations under a specific assumption. They obtained the conditional stochastic decomposition result and the joint distribution for queue length and service status. Recently, (Lin and Ke (2009)) considered the multi-server system with single working vacation. The matrix-geometric approach was utilized to develop the computable explicit formula for the probability distributions of queue length and other performance measures. (Yang et al. (2010)) treated the F-policy $M/M/1/K$ queue with single working vacation and exponential startup times and derived the stationary distributions and related system characteristics, including an optimization numerical analysis. (Jain and Jain (2010)) investigated a single-server working-vacation model with server breakdowns of multiple types.

Baba (2012) studied the $M^X/M/1$ queue with multiple working vacations, (Arivudainambi et al. (2014)) presented the performance analysis of a single server retrial queue with working vacation, (Gao and Yao (2014)) presented the $M^X/G/1$ queue with randomized working vacations and at most J vacations, (Lee and Kim (2015)) gave a note on the sojourn time distribution of an M/G/1 queue with a single working vacation and vacation interruption, (Liu et al. (2015)) presented a cold standby repairable system with working vacations and vacation interruption following Markovian arrival process, (Laxmi and Jyothsna (2014)) obtained Performance analysis of variant working vacation queue with balking and reneging, (Laxmi and Rajesh (2015)) gave an analysis of variant working vacations on batch arrival queues. Cost-minimization analysis of a working vacation queue with N-policy and server breakdowns was given by (Yang and Wu (2015)).

Recently, there has been considerable attention paid to the retrial queueing models with working vacation, (Li et al.(2012)) studied a $Geo/Geo/1$ retrial queue with working vacations and vacation interruption, Gao et al. (2014) established the analysis of the $M/G/1$ retrial queue with general retrial times, working vacations and vacation interruption. (Upadhyaya (2015)) presented the working vacation policy for a discrete-time $Geo^X/Geo/1$ retrial queue, (Rajadurai et al. (2016)) gave the performance analysis of preemptive priority retrial queue with immediate Bernoulli feedback under working vacations and vacation interruption.

▶ ***Multi-server vacation models.***
Multi-server vacation models were studied by a number of researchers, the servers in these models can either take the same vacation together (synchronous vacation) or take individual vacations (asynchronous vacations) independently. (Zhang and Tian (2004)) first studied the multi-server model with asynchronous vacations which represents a service system with multi-task employees. More multi-server vacation models are based on synchronous vacations. (Zhang and Tian (2003a, 2003b)) first analyzed the Markovian multi-server queueing system with single/multiple synchronous vacations. Moreover, (Tian and Zhang (2003)) investigated a more general $GI/M/c$ queueing system with phase-type vacations where all servers take multiple vacations together until waiting customers exist at a vacation completion instant. (Tian and Zhang (2006)) considered a multi-server queueing system with a threshold type (d, N) vacation policy under which d idle servers keep taking multiple synchronous vacations until the number of customers reaches or exceeds a threshold N. A computational study is presented for determining the optimal values of d and N. Another multi-server vacation model with single vacation and threshold policy was treated by (Xu and Zhang (2006)). (Zhang (2005)) presented an analysis on the multi-server vacation model with three threshold policy. Yue et al. (2006) studied a finite buffer multi-server queue with balking, reneging, and single synchronous vacation policy. They obtained the stationary distributions of the queue length and some other performance measures in matrix forms. Several special cases with MAPs and numerical examples were presented, including the table of optimal values of system parameters and the corresponding system performances measures. Recently, (Ke et al. (2009)) studied the optimal (d, c) vacation policy for finite capacity $M/M/c/N$ queue with unreliable servers and repairs. (Gharbi and Ioualalen (2010)) studied the finite-source multi-server queue-

ing systems with single/multiple vacation policies and developed some the algorithms for computing the system performance measures. (Ke et al. (2013)) presented a note on a multi-server queue with vacations of multiple groups of servers.

▶ ***Vacation models with impatient customers.***

Both single server and multi-server vacation models with impatient customers were discussed by (Altman and Yechiali (2006)). (Katayama (2011)) examined an $M/G/1$ queue with multiple and single vacation, sojourn time limits and balking behavior. Via the level crossing approach, author derived the explicit solutions for the stationary virtual waiting time distribution under various assumptions on the service time distribution. (Sakuma and Inoie (2012)) considered an $M/M/c+D$ queue with multiple vacation exponentially distributed, where customers are impatient only when all servers are unavailable. Using the matrix-analytic method, the stationary distribution of the system is derived. Liu and Song (2013) established the analysis of $Geo/Geo/1$ retrial queue with non-persistent customers and working vacations. (Rajadurai et al. (2015)) Analyzed the $M/G/1$ retrial queue with balking, negative customers, working vacations and server breakdown.

1.4 Contribution of the thesis

The contribution of this thesis consists mainly in studying the impact of impatience in different queueing models, namely, retrial and vacation queueing systems. In this way we develop different and more advanced queueing systems. A number of queueing models presented by many researchers are special cases of our systems.

⋇ **First Result: A note on fluid approximation of retrial queueing system with two orbits, abandonment and feedback.**

This work deals with *multi-server retrial queueing network with two orbits, time dependent parameters, state dependent routing, abandonment and feedback. The $M_t/M_t/c_t$ queue has a (time inhomogeneous) Poisson arrival process with rate λ_{i_t}, a service rate (per server) of μ_{i_t}, $i=1,2$ and c_t servers, for all $t>0$. Two independent Poisson streams of customers flow into c servers. The incoming customer of type i, $i=1,2$ is handled by an available server, if there is any; otherwise, he waits in an infinite buffer queue. Note that the customers are served in the order of arrival. A waiting customer of type i who did not get connected to a server will lose his patience and abandon after an exponentially distributed amount of time, the abandoned one may leave the entire network (loss customer) or move into one of the orbits with some probability, from which he retries to reach the* *primary queue to get a service. A served customer may comeback to the system, to the orbit depending on its type for another service. A customer in orbit i, $i = 1, 2$ may lose his patience and abandon the entire network after an exponentially distributed amount of time (loss customer).*

. For this type of systems the sample paths for the $M_t/M_t/c_t$ queue length process is uniquely determined by this relation

$$Q_1(t) = Q_1(0) + \Pi_1\left(\int_0^t \lambda_{1_s} ds\right) + \Pi_2\left(\int_0^t \lambda_{2_s} ds\right) + \Pi_3\left(\int_0^t \alpha_{1_s} Q_2(s) ds\right) + \Pi_4 \left(\int_0^t \alpha_{2_s} Q_3(s) ds\right) - \Pi_5\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{1_s} \phi_s ds\right) - \Pi_6\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{2_s} \phi_s ds\right) - \Pi_7\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{1_s} (1 - \phi_s) ds\right) - \Pi_8\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{2_s} (1 - \phi_s) ds\right) - \Pi_9\left(\int_0^t \mu_{1_s} (Q_1(s) \wedge c_s) ds\right) - \Pi_{10}\left(\int_0^t \mu_{2_s} (Q_1(s) \wedge c_s) ds\right) \tag{1.1}$$

$$\begin{aligned} Q_2(t) = Q_2(0) &+ \Pi_1^1\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{1_s} (1 - \phi_s) ds\right) + \Pi_2^1\left(\int_0^t \omega_{1_s} ds\right) \\ &- \Pi_3^1\left(\int_0^t \alpha_{1_s} Q_2(s) ds\right) - \Pi_4^1\left(\int_0^t \rho_s \theta_{1_s} (Q_2(s) - k_{1_s})^+ ds\right). \end{aligned} \tag{1.2}$$

$$\begin{aligned} Q_3(t) = Q_3(0) &+ \Pi_1^2\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{2_s} (1 - \phi_s) ds\right) + \Pi_2^2\left(\int_0^t \omega_{2_s} ds\right) \\ &- \Pi_3^2\left(\int_0^t \alpha_{2_s} Q_3(s) ds\right) - \Pi_4^2\left(\int_0^t \rho_s \theta_{2_s} (Q_3(s) - k_{2_s})^+ ds\right), \end{aligned} \tag{1.3}$$

where $\Pi_i(\cdot)$, $\Pi_i^1(\cdot)$, and $\Pi_i^2(\cdot)$, are given independent, standard (rate 1) Poisson processes, and for all real x and y, $x \wedge y \equiv \min(x, y)$.

And the Markovian service network $\{Q(t) | t \geq 0\}$ is the $\mathbb{V}$-valued stochastic process whose sample paths are uniquely determined by $Q(0)$ and the functional equations

$$Q(t) = Q(0) + \sum_{i \in \mathbf{I}} \Pi_i\left(\int_0^t \nu_s(Q(s), i) ds\right) v_i, \text{ for all } t \geq 0$$

with

$$\{\nu_t(\cdot,i)|t \geq 0, i \in I\} \tag{1.4}$$

a collection of real-valued, non-negative Lipschitz rate functions on a separable Banach space $\mathbb{V}$.

. Further, a scaled version $Q^\eta(t) = (Q_1^\eta(t), Q_2^\eta(t), Q_3^\eta(t))$ of the process $Q(t)$ is given as

$$\begin{aligned} Q_1^\eta(t) = Q_1^\eta(0) &+ \Pi_1\left(\int_0^t \eta\lambda_{1_s}ds\right) + \Pi_2\left(\int_0^t \eta\lambda_{2_s}ds\right) + \Pi_3\left(\int_0^t \alpha_{1_s}Q_2^\eta(s)ds\right) + \Pi_4 \\ &\left(\int_0^t \alpha_{2_s}Q_3^\eta(s)ds\right) - \Pi_5\left(\int_0^t (Q_1^\eta(s)-\eta c_s)^+\delta_{1_s}\phi_s ds\right) - \Pi_6\left(\int_0^t (Q_1^\eta(s)-\eta c_s)^+\right. \\ &\left.\delta_{2_s}\phi_s ds\right) - \Pi_7\left(\int_0^t (Q_1^\eta(s)-\eta c_s)^+\delta_{1_s}(1-\phi_s)ds\right) - \Pi_8\left(\int_0^t (Q_1^\eta(s)-\eta c_s)^+\right. \\ &\left.\delta_{2_s}(1-\phi_s)ds\right) - \Pi_9\left(\int_0^t \mu_{1_s}(Q_1^\eta(s)\wedge\eta c_s)ds\right) - \Pi_{10}\left(\int_0^t \mu_{2_s}(Q_1^\eta(s)\wedge\eta c_s)ds\right) \end{aligned} \tag{1.5}$$

$$\begin{aligned} Q_2^\eta(t) = Q_2^\eta(0) &+ \Pi_1^1\left(\int_0^t (Q_1^\eta(s)-\eta c_s)^+\delta_{1_s}(1-\phi_s)ds\right) + \Pi_2^1\left(\int_0^t \eta\omega_{1_s}ds\right) \\ &-\Pi_3^1\left(\int_0^t \alpha_{1_s}Q_2^\eta(s)ds\right) - \Pi_4^1\left(\int_0^t \rho_s\theta_{1_s}((Q_2^\eta(s)-\eta k_{1_s})^+ds\right). \end{aligned} \tag{1.6}$$

$$\begin{aligned} Q_3^\eta(t) = Q_3^\eta(0) &+ \Pi_1^2\left(\int_0^t (Q_1^\eta(s)-\eta c_s)^+\delta_{2_s}(1-\phi_s)ds\right) + \Pi_2^2\left(\int_0^t \eta\omega_{2_s}ds\right) \\ &-\Pi_3^2\left(\int_0^t \alpha_{2_s}Q_3^\eta(s)ds\right) - \Pi_4^2\left(\int_0^t \rho_s\theta_{2_s}(Q_3^\eta(s)-\eta k_{2_s})^+ds\right) \end{aligned} \tag{1.7}$$

as $\eta \to \infty$.

With $\{Q^\eta|\eta > 0\}$ the rescaled process such that

$$Q^\eta(t) = Q^\eta(0) + \sum_{i\in\mathbf{I}} \Pi_i\left(\eta\int_0^t \nu_s\left(\frac{Q^\eta(s)}{\eta}, i\right)ds\right)v_i, \tag{1.8}$$

And $Q^{(1)}$ be the diffusion approximation associated with the family $\{Q^\eta(t)|t \geq 0\}$.

. The first-order asymptotic result takes the form of a functional strong law of large numbers, and yields a fluid approximation for the original process.

Theorem 1.4.1. *Let Q^{η} be the uniform acceleration as in* (1.8)*, the fluid limit for the multiserver queue with abandonment feedback and retrials is the unique solution to the differential equations*

$$\begin{aligned} \tfrac{d}{dt}Q_1^{(0)}(t) &= \lambda_{1_t} + \lambda_{2_t} + \alpha_{1_t}Q_2^{(0)}(t) + \alpha_{2_t}Q_3^{(0)}(t) - (\mu_{1_t} + \mu_{2_t})(Q_1^{(0)}(t) \wedge c_t) \\ &\quad -(\delta_{1_t} + \delta_{2_t})(Q_1^{(0)}(t) - c_t)^+ \end{aligned} \tag{1.9}$$

$$\tfrac{d}{dt}Q_2^{(0)}(t) = \omega_{1_t} - \alpha_{1_t}Q_2^{(0)}(t) + \delta_{1_t}(1-\phi_t)(Q_1^{(0)}(t) - c_t)^+ - \theta_{1_t}\rho_t(Q_2^{(0)}(t) - k_{1_t})^+. \tag{1.10}$$

$$\tfrac{d}{dt}Q_3^{(0)}(t) = \omega_{2_t} - \alpha_{2_t}Q_3^{(0)}(t) + \delta_{2_t}(1-\phi_t)(Q_1^{(0)}(t) - c_t)^+ - \theta_{2_t}\rho_t(Q_3^{(0)}(t) - k_{2_t})^+. \tag{1.11}$$

Furthermore, the diffusion limit for the multiserver queue with abandonment, feedback and retrials is the unique solution to the integral equations

$$\begin{aligned} Q_1^{(1)}(t) &= Q_1^{(1)}(0) + \Omega_1\left(\int_0^t \lambda_{1_s}ds\right) + \Omega_2\left(\int_0^t \lambda_{2_s}ds\right) + \Omega_3\left(\int_0^t \alpha_{1_s}Q_2^{(0)}(s)ds\right) + \Omega_4 \\ &\left(\int_0^t \alpha_{2_s}Q_3^{(0)}(s)ds\right) + \int_0^t \Big[\Big(\mu_{1_s}1_{\{Q_1^{(0)}(s)\le c_s\}} + (\delta_{1_s}+\delta_{2_s})1_{\{Q_1^{(0)}(s)>c_s\}}\Big)Q_1^{(1)}(s)^- \\ &-\Big((\mu_{1_s}+\mu_{2_s})1_{\{Q_1^{(0)}(s)<c_s\}} + (\delta_{1_s}+\delta_{2_s})1_{\{Q_1^{(0)}(s)\ge c_s\}}\Big)Q_1^{(1)}(s)^+ + \alpha_{1_s}Q_2^{(1)}(s) \\ &+\alpha_{2_s}Q_3^{(1)}(s)\Big]ds - \Omega_5\left(\int_0^t (Q_1^{(0)}(s)-c_s)^+\delta_{1_s}(1-\phi_s)ds\right) - \Omega_6\left(\int_0^t (Q_1^{(0)}(s)-c_s)^+\right. \\ &\left.\delta_{2_s}(1-\phi_s)ds\right) - \Omega_7\left(\int_0^t \mu_{1_s}(Q_1^{(0)}(s)\wedge c_s)ds\right) - \Omega_8\left(\int_0^t \mu_{2_s}(Q_1^{(0)}(s)\wedge c_s)ds\right) \\ &-\Omega_9\left(\int_0^t (Q_1^{(0)}(s)-c_s)^+\delta_{1_s}\phi_s ds\right) - \Omega_{10}\left(\int_0^t (Q_1^{(0)}(s)-c_s)^+\delta_{2_s}\phi_s ds\right) \end{aligned} \tag{1.12}$$

$$
\begin{aligned}
Q_2^{(1)}(t) = Q_2^{(1)}(0) &+ \Omega_1^1\left(\int_0^t (Q_1^{(0)}(s)-c_s)^+\delta_{1_s}(1-\phi_s)ds\right) + \Omega_2^1\left(\int_0^t \omega_{1_s}ds\right)\\
&+\int_0^t \left[Q_1^{(1)}(s)^+1_{\{Q_1^{(0)}(s)\geq c_s\}} - Q_1^{(1)}(s)^-1_{\{Q_1^{(0)}(s)>c_s\}}\right]\delta_{1_s}(1-\phi_s)ds\\
&-\Omega_3^1\left(\int_0^t \alpha_{1_s}Q_2^{(0)}(s)ds\right) - \Omega_4^1\left(\int_0^t \rho_s\theta_{1_s}(Q_2^{(1)}(s)-k_{1_s})^+ds\right)\\
&-\int_0^t \theta_{1_s}\rho_s\left[(Q_2^{(1)}(s))^+1_{\{(Q_2^{(0)}(s)\geq k_{1_s}\}} - (Q_2^{(1)}(s))^-1_{\{(Q_2^{(0)}(s)>k_{1_s}\}}\right]ds\\
&-\int_0^t \alpha_{1_s}Q_2^{(1)}(s)ds.
\end{aligned} \tag{1.13}
$$

$$
\begin{aligned}
Q_3^{(1)}(t) = Q_3^{(1)}(0) &+ \Omega_1^2\left(\int_0^t (Q_1^{(0)}(s)-c_s)^+\delta_{2_s}(1-\phi_s)ds\right) + \Omega_2^2\left(\int_0^t \omega_{2_s}ds\right)\\
&+\int_0^t \left[Q_1^{(1)}(s)^+1_{\{Q_1^{(0)}(s)\geq c_s\}} - Q_1^{(1)}(s)^-1_{\{Q_1^{(0)}(s)>c_s\}}\right]\delta_{2_s}(1-\phi_s)ds\\
&\quad-\Omega_3^2\left(\int_0^t \alpha_{2_s}Q_3^{(0)}(s)ds\right) - \Omega_4^2\left(\int_0^t \rho_s\theta_{2_s}(Q_3^{(1)}(s)-k_{2_s})^+ds\right)
\end{aligned} \tag{1.14}
$$

$$
\begin{aligned}
&-\int_0^t \theta_{2_s}\rho_s\left[Q_3^{(1)}(s)^+1_{\{(Q_3^{(0)}(s)\geq k_{2_s}\}} - Q_3^{(1)}(s)^-1_{\{(Q_3^{(0)}(s)>k_{2_s}\}}\right]ds\\
&-\int_0^t \alpha_{2_s}Q_3^{(1)}(s)ds.
\end{aligned} \tag{1.15}
$$

. Next, the ordinary differential equations for the mean vector and covariance matrix of $Q_i^{(1)}$ are obtained in the following Theorem.

Theorem 1.4.2. *The mean vector for the diffusion limit solves the set of differential equations*

$$
\begin{aligned}
\tfrac{d}{dt}\mathbb{E}(Q_1^{(1)}(t)) = &\left((\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)\leq c_t\}} + (\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)>c_t\}}\right)\mathbb{E}(Q_1^{(1)}(t)^-)\\
&-\left((\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)<c_t\}} + (\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)\geq c_t\}}\right)\mathbb{E}(Q_1^{(1)}(t)^+)\\
&+\alpha_{1_t}\mathbb{E}(Q_1^{(2)}(t)) + \alpha_{2_t}\mathbb{E}(Q_1^{(3)}(t)).
\end{aligned} \tag{1.16}
$$

$$\begin{aligned}\tfrac{d}{dt}\mathbb{E}(Q_2^{(1)}(t)) &= \delta_{1_t}(1-\phi_t)\left(\mathbb{E}(Q_1^{(1)}(t)^+)1_{\{Q_1^{(0)}(t)\geq c_t\}} - \mathbb{E}(Q_1^{(1)}(t)^-)1_{\{Q_1^{(0)}(t)>c_t\}}\right)\\ &\quad - \left(\rho_t\theta_{1_t}1_{\{Q_2^{(0)}(t)\geq k_{1_t}\}}\right)\times\mathbb{E}\left(Q_2^{(1)}(t)\right) - \alpha_{1_t}\mathbb{E}(Q_2^{(1)}(t)).\end{aligned} \tag{1.17}$$

$$\begin{aligned}\tfrac{d}{dt}\mathbb{E}(Q_3^{(1)}(t)) &= \delta_{2_t}(1-\phi_t)\left(\mathbb{E}(Q_1^{(1)}(t)^+)1_{\{Q_1^{(0)}(t)\geq c_t\}} - \mathbb{E}(Q_1^{(1)}(t)^-)1_{\{Q_1^{(0)}(t)>c_t\}}\right)\\ &\quad - \left(\rho_t\theta_{2_t}1_{\{Q_3^{(0)}(t)\geq k_{2_s}\}}\right)\times\mathbb{E}\left(Q_3^{(1)}(t)\right) - \alpha_{2_t}\mathbb{E}(Q_3^{(1)}(t)).\end{aligned} \tag{1.18}$$

The covariance matrix for the diffusion limit solves the differential equations

$$\begin{aligned}\tfrac{d}{dt}Var(Q_1^{(1)}(t)) &= 2\left((\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)>c_t\}} + (\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)\leq c_t\}}\right)\\ &\quad\times Cov(Q_1^{(1)}(t), Q_1^{(1)}(t)^-) + \lambda_{1_t} + \lambda_{2_t} + (\delta_{1_t}+\delta_{2_t})(Q_1^{(0)}(t)-c_t)^+\\ &\quad -2\left((\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)\geq c_t\}} + (\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)<c_t\}}\right)\\ &\quad\times Cov(Q_1^{(1)}(t), Q_1^{(1)}(t)^+) + (\mu_{1_t}+\mu_{2_t})(Q_1^{(0)}(t)\wedge c_t) + \alpha_{1_t}Q_2^{(0)}(t)\\ &\quad +\alpha_{2_t}Q_3^{(0)}(t) + 2\left[\alpha_{1_t}cov(Q_1^{(1)}(t), Q_2^{(1)}(t)) + \alpha_{2_t}cov(Q_1^{(1)}(t), Q_3^{(1)}(t))\right].\end{aligned} \tag{1.19}$$

$$\begin{aligned}\tfrac{d}{dt}Var(Q_2^{(1)}(t)) &= 2\delta_{1_t}(1-\phi_t)Cov(Q_2^{(1)}(t), Q_1^{(1)}(t)^+)1_{\{Q_1^{(0)}(t)\geq c_t\}} - 2\delta_{1_t}(1-\phi_t)\\ &\quad\times Cov(Q_2^{(1)}(t), Q_1^{(1)}(t)^-)1_{\{Q_1^{(0)}(t)>c_t\}} - 2\alpha_{1_t}Var(Q_2^{(1)}(t))\\ &\quad +\delta_{1_t}(1-\phi_t)(Q_1^{(0)}(t)-c_t)^+ + \alpha_{1_t}Q_2^{(0)}(t) + \rho_t\theta_{1_t}(Q_2^{(0)}(t)-k_{1_t})^+\\ &\quad +\omega_{1_t} - 2\rho_t\theta_{1_t}1_{\{Q_2^{(0)}(t)\geq k_{1_t}\}}Var(Q_2(t)).\end{aligned} \tag{1.20}$$

$$\begin{aligned}\tfrac{d}{dt}Var(Q_3^{(1)}(t)) &= 2\delta_{2_t}(1-\phi_t)Cov(Q_1^{(1)}(t)^+, Q_3^{(1)}(t))1_{\{Q_1^{(0)}(t)\geq c_t\}} - 2\delta_{2_t}(1-\phi_t)\\ &\quad\times Cov(Q_1^{(1)}(t)^-, Q_3^{(1)}(t))1_{\{Q_1^{(0)}(t)>c_t\}} - 2\alpha_{2_t}Var(Q_3^{(1)}(t))\\ &\quad +\delta_{2_t}(1-\phi_t)(Q_1^{(0)}(t)-c_t)^+ + \alpha_{2_t}Q_3^{(0)}(t) + \rho_t\theta_{2_t}(Q_3^{(0)}(t)-k_{2_t})^+\\ &\quad +\omega_{2_t} - 2\rho_t\theta_{2_t}1_{\{Q_3^{(0)}(t)\geq k_{2_t}\}}Var(Q_3(t)).\end{aligned} \tag{1.21}$$

$$
\begin{aligned}
\tfrac{d}{dt}Cov(Q_1^{(1)}(t),Q_2^{(1)}(t)) = {} & \Big((\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)\le c_t\}}+(\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)>c_t\}}\Big)\\
& \times Cov((Q_1^{(1)}(t))^-,Q_2^{(1)}(t))-\alpha_{1_t}Cov(Q_1^{(1)}(t),Q_2^{(1)}(t))\\
& -\Big((\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)<c_t\}}+(\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)\ge c_t\}}\Big)\\
& \times Cov((Q_1^{(1)}(t))^+,Q_2^{(1)}(t))+\delta_{1_t}(1-\phi_t)(Q_1^{(0)}(t)-c_t)^+\\
& -\Big(\theta_{1_t}\rho_t 1_{\{Q_2^{(0)}(t)\ge k_{1_t}\}}\Big)Cov(Q_1^{(1)}(t),Q_2^{(1)}(t))\\
& +\delta_{1_t}(1-\phi_t)1_{\{Q_1^{(0)}(t)\ge c_t\}}Var(Q_1^{(1)}(t))+\alpha_{1_t}Var(Q_2^{(1)}(t))\\
& +\alpha_{2_t}Cov(Q_3^{(1)}(t),Q_2^{(1)}(t))+\alpha_{1_t}Q_2^{(0)}(t).
\end{aligned}
\tag{1.22}
$$

$$
\begin{aligned}
\tfrac{d}{dt}Cov(Q_2^{(1)}(t),Q_3^{(1)}(t)) = {} & \delta_{2_t}(1-\phi_t)1_{\{Q_1^{(0)}(t)\ge c_t\}}Cov(Q_2^{(1)}(t),Q_1^{(1)}(t))+\delta_{1_t}(1-\phi_t)\\
& \times 1_{\{Q_1^{(0)}(t)\ge c_t\}}Cov(Q_3^{(1)}(t),Q_1^{(1)}(t))-\Big(\alpha_{1_t}+\alpha_{2_t}+\theta_{1_t}\rho_t\\
& \times 1_{\{Q_2^{(0)}(t)\ge k_{1_t}\}}+\theta_{2_t}\rho_t 1_{\{Q_3^{(0)}(t)\ge k_{2_t}\}}\Big)Cov(Q_2^{(1)}(t),Q_3^{(1)}(t)).
\end{aligned}
\tag{1.23}
$$

⋇ **Second Result: Stability Condition of a Retrial Queueing System with Abandoned and Feedback Customers.**

In this work we investigate the analysis of the necessary stability condition of *a Markovian retrial queueing system with two classes of jobs and constant retrial, abandonment and feedback customers. Two independent Poisson streams of jobs, S_1 and S_2, flow into a single-server service system. The service system can hold at most one job. The arrival rate of stream S_i is α_i, $i = 1, 2$, with $\alpha_1 + \alpha_2 = \alpha$. The required service time of each job is independent of its type and is exponentially distributed with mean $1/\mu$. If an arriving type-i job finds the server busy, it is routed to a dedicated retrial (orbit) queue from which jobs are re-transmitted at an exponential rate. The rates of retransmissions may be different from the rates of the original input streams. So, the blocked jobs of type i form a type-i single-server orbit queue that attempts to retransmit jobs (if any) to the main service system at a Poisson rate of γ_i, $i = 1, 2$. This creates a system with three dependent queues.*

The customer in the orbit either attempts service again after a random time or gives up receiving service and leaves the system after a random time at rate δ_i $i = 1, 2$. After the customer is served completely, it will decide either to join the retrial group again for another service with probability β or to leave the system forever with probability $\bar{\beta} = 1 - \beta$.

. The Markov process $\{(N_1(t), N_2(t), C(t)) : t \in [0, +\infty)\}$ is irreducible on the state-space $\{0, 1, ...\} \times \{0, 1, ...\} \times \{0, 1\}$ such that $C(t)$ denotes the number of jobs in the main queue which takes the values of 0 or 1, and $N_i(t)$ is the number of jobs in orbit queue i, $i = 1, 2$.

. Such a network can serve as a model for two competing job streams in a carrier sensing multiple access system "CSMA". Local Area Computer Network (LAN) can be an example of CSMA.

. In the case of our system, the set of stationary probabilities $P_{n_1 n_2}(c)$ is defined as follows:

$$P_{n_1 n_2}(c) = \lim_{t \to \infty} P(N_1(t) = n_1, N_2(t) = n_2, L(t) = c)$$

$$= P(N_1 = n_1, N_2 = n_2, C = c),$$

for $n_1, n_2 = 0, 1, ...$, and $c = 0, 1$, when these limits exist. Define the marginal probabilities

$$P_{n_1 \cdot}(c) = \sum_{n_2=0}^{\infty} P_{n_1 n_2}(c) = P(N_1 = n_1, C = c), n_1 = 0, 1, 2, ..., \quad c = 0, 1$$

and

$$P_{\cdot n_2}(c) = \sum_{n_1=0}^{\infty} P_{n_1 n_2}(c) = P(N_2 = n_2, C = c), n_2 = 0, 1, 2, ..., \quad c = 0, 1.$$

The balance equations are presented as

1. $N_2 = n_2 = 0$

1.1. $N_1 = n_1 = 0, \quad c = 0$

$$\alpha P_{00}(0) = \overline{\beta} \mu P_{00}(1) + \delta_1 P_{10}(0) + \delta_2 P_{01}(0). \tag{1.24}$$

1.2. $N_1 = n_1 \geq 1, \quad c = 0$

$$(\alpha + \gamma_1 + \delta_1) P_{n_1 0}(0) = \overline{\beta} \mu P_{n_1 0}(1) + \beta \mu P_{n_1 - 10}(1) + \delta_1 P_{n_1 + 10}(0) + \delta_2 P_{n_1 1}(0). \tag{1.25}$$

1.3. $N_1 = 0, \;\; c = 1$

$$\begin{aligned}(\alpha+\mu)P_{00}(1) = {} & \alpha P_{00}(0) + \gamma_1 P_{10}(0) + \gamma_2 P_{01}(0) + \delta_1 P_{10}(1) \\ & +\delta_2 P_{01}(1).\end{aligned} \tag{1.26}$$

1.4. $N_1 = n_1 \geq 1, \;\; c = 1$

$$\begin{aligned}(\alpha+\mu+\delta_1)P_{n_1 0}(1) = {} & \alpha P_{n_1 0}(0) + \gamma_1 P_{n_1+10}(0) + \gamma_2 P_{n_1 1}(0) \\ & +\delta_1 P_{n_1+10}(1) + \delta_2 P_{n_1 1}(1) + \alpha_1 P_{n_1-10}(1).\end{aligned} \tag{1.27}$$

2. $N_2 = n_2 \geq 1$

2.1. $N_1 = 0 \;\; c = 0$

$$\begin{aligned}(\alpha+\gamma_2+\delta_2)P_{0n_2}(0) = {} & \overline{\beta}\mu P_{0n_2}(1) + \beta\mu P_{0n_1-1}(1) + \delta_1 P_{1n_2}(1) \\ & +\delta_2 P_{0n_2+1}(0).\end{aligned} \tag{1.28}$$

2.2. $N_1 = n_1 \geq 1 \;\; c = 0$

$$\begin{aligned}(\alpha+\gamma_1+\gamma_2+\delta_1+\delta_2)P_{n_1n_2}(0) = {} & \beta\mu P_{n_1-1n_2}(1) + \beta\mu P_{n_1n_2-1}(1) \\ & +\overline{\beta}\mu P_{n_1n_2}(1) + \delta_1 P_{n_1+1n_2}(0) + \delta_2 P_{n_1n_2+1}(0).\end{aligned} \tag{1.29}$$

2.3. $N_1 = 0 \;\; c = 1$

$$\begin{aligned}(\alpha+\mu+\delta_2)P_{0n_2}(1) = {} & \alpha P_{0n_2}(0) + \gamma_1 P_{1n_2}(0) + \gamma_2 P_{0n_2+1}(0) + \delta_1 P_{1n_2}(1) \\ & +\delta_2 P_{0n_2+1}(1) + \alpha_2 P_{0n_2-1}(1).\end{aligned} \tag{1.30}$$

2.4. $N_1 = n_1 \geq 1 \;\; c = 1$

$$\begin{aligned}&(\alpha+\mu+\delta_1+\delta_2)P_{n_1n_2}(1) = \alpha P_{n_1n_2}(0) + \gamma_1 P_{n_1+1n_2}(0) + \gamma_2 P_{n_1n_2+1}(0) \\ &+\delta_1 P_{n_1+1n_2}(1) + \delta_2 P_{n_1n_2+1}(1) + \alpha_1 P_{n_1-1n_2}(1) + \alpha_2 P_{n_1n_2-1}(1).\end{aligned} \tag{1.31}$$

The probability generating function of the stationary version of the Markov process $\{(N_1(t), N_2(t), C(t)) : t \in [0, +\infty)\}$ is given by

$$F(z_1, z_2, z) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{c=0}^{1} P_{n_1n_2}(c) z_1^{n_1} z_2^{n_2} z^c. \tag{1.32}$$

The probability generating functions are defined as

$$R_{n_2}^{(c)}(z_1) = \sum_{n_1=0}^{\infty} P_{n_1 n_2}(c) z_1^{n_1}, \;\; c = 0, 1, n_2 = 0, 1, ...,$$

and

$$F^{(c)}(z_1, z_2) = \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} P_{n_1 n_2}(c) z_1^{n_1} z_2^{n_2} = \sum_{n_2=0}^{\infty} R_{n_2}^{(c)}(z_1) z_2^{n_2}, \;\; c = 0, 1. \tag{1.33}$$

$$F(z_1, z_2, z) = F^{(0)}(z_1, z_2) + zF^{(1)}(z_1, z_2), \;\; |z_1| \leq 1, \;\; |z_2| \leq 1.$$

The main result is given in the following.

Proposition 1.4.1. *The following condition*

$$\frac{\alpha(\gamma_1+\delta_1)(\gamma_2+\delta_2)}{[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1}\left(1+\frac{\alpha_i}{\gamma_i+\delta_i}\right) < 1, \tag{1.34}$$

for $i = 1, 2$ *and*

$$[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1 \neq 0 \tag{1.35}$$

is necessary for the stability of the system.

To prove the stability of our Markovian retrial queueing system with two classes of jobs and constant retrial rates, abandonment and feedback customers we showed that

$$\frac{\alpha(\gamma_1+\delta_1)(\gamma_2+\delta_2)}{[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1} \leq 1. \tag{1.36}$$

$$\frac{\alpha(\gamma_1+\delta_1)(\gamma_2+\delta_2)}{[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1}\left(1+\frac{\alpha_i}{\gamma_i+\delta_i}\right) \leq 1, \;\; i = 1, 2 \tag{1.37}$$

are necessary conditions for the existence of a steady-state.

And if

$$\frac{\alpha(\gamma_1+\delta_1)(\gamma_2+\delta_2)}{[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1}\left(1+\frac{\alpha_i}{\gamma_i+\delta_i}\right) = 1, \;\; i = 1, 2, \tag{1.38}$$

then both queues N_1 and N_2 are unbounded with probability one.

⋇ **Third Result: On feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption.**

In this work we established the analysis of *an $M/M/1$ queueing system multiple working vacations, Bernoulli schedule vacation interruptions, reneging, retention of reneged customers and feedback, where the customers become impatient due to the servers' vacation. Customers arrive according to a Poisson process with arrival rate λ. The service times during a normal service period, the service times during a working vacation period, and the vacation times are exponentially distributed with rates μ, α, and θ, respectively. The customers are supposed to be impatient during the multiple working vacations. Whenever a customer arrives at the system and finds the system is on working vacation, he activates an impatient timer T, which is exponentially distributed with rate ϑ. If the server finishes the working vacation before the impatience timer expires, the customer remains in the system till his service completion. However, if the impatience timer expires when the server is still on working vacation, the customer abandons the system, this time is reneging time of an individual customer. The reneged customer can be retained in the system with some probability σ or he may abandon the system with complementary probability $\delta(1-\sigma)$. During the working vacation period, a customer is serviced at a lower rate, and, at the instants of the service completion, the vacation is interrupted and the server resumes a regular busy period with probability $1-\beta$ (if there are customers in the queue) or remains in the vacation with probability β. The inter-arrival times, service times, vacation duration times, and the impatient times all are taken to be mutually independent. The customers are served on a first come first-served queue discipline. After completion of each service, the customer can either join the end of the queue with probability ν or he can leave the system with probability γ where $\nu+\gamma=1$. The customers both newly arrived and those that are fed back are served in FIFO discipline. We do not distinguish between the regular arrival and feedback arrival.*

- $\{(N(t), St)); t \geq 0\}$ is a continuous-time Markov process with state space $\Omega = [\{(0,0) \cup (i,j)\}, \ \ i = 1, 2, ..., \ \ j = 0, 1]$. $N(t)$ denotes the number of customers in the system at time t, and $S(t)$ is the state of the server at time t which takes two values 0 if the server is in working vacation period, and 1 if the server is in normal busy period.

- Let $\pi_{ij} = \underset{t\to\infty}{lim} \mathbb{P}\{N(t) = i, S(t) = j\}$, $(i,j) \in \Omega$, and π_{i0}, $i \geq 0$ be the probability that there are i customers in the system when the server is in working vacation period and, π_{i1}, $i \geq 1$ be the probability that there are i customers in the system when the server is in normal busy period.

. Via the Markov process theory, the set of steady-state equations is given as follow:

$$\lambda\pi_{00} = (\delta\vartheta + \gamma\alpha)\pi_{10} + \gamma\mu\pi_{11} \tag{1.39}$$

$$(\lambda + \gamma\alpha + \theta + n\delta\vartheta)\pi_{n0} = \lambda\pi_{n-10} + (\beta\gamma\alpha + (n+1)\delta\vartheta)\pi_{n+10}, \;\; n \geq 1 \tag{1.40}$$

$$(\lambda + \gamma\mu)\pi_{11} = \theta\pi_{10} + \overline{\beta}\gamma\alpha\pi_{20} + \gamma\mu\pi_{21} \tag{1.41}$$

$$(\lambda + \gamma\mu)\pi_{n1} = \theta\pi_{n0} + \lambda\pi_{n-11} + \gamma\mu\pi_{n+11} + \overline{\beta}\gamma\alpha\pi_{n+10}, \;\; n \geq 2 \tag{1.42}$$

. The probability generating functions of the number of customers in the system when the server is in a working vacation period and in a normal service period, respectively are

$$\Pi_0(z) = \frac{-\left(\gamma\mu\pi_{11} + (\theta + \overline{\beta}\gamma\alpha)\pi_{0,0} + \overline{\beta}\gamma\alpha\pi_{10}\right)\Phi_1(z) + \beta\gamma\alpha\Phi_2(z)}{\delta\vartheta e^{-(\lambda/\delta\vartheta)z} z^{\beta\gamma\alpha/\delta\vartheta}(1-z)^{(\theta+\overline{\beta}\gamma\alpha)/\delta\vartheta}}, \tag{1.43}$$

$$\Pi_1(z) = \frac{(\theta z + \overline{\beta}\gamma\alpha)\Pi_0(z) - z(\theta + \overline{\beta}\gamma\alpha)\Pi_0(1)}{(\lambda z - \gamma\mu)(1-z)} - \frac{\overline{\beta}\gamma\alpha\pi_{0,0}}{(\lambda z - \gamma\mu)}. \tag{1.44}$$

With

$$\Phi_1(z) = \int_0^z e^{-(\lambda/\delta\vartheta)x} x^{\beta\gamma\alpha/\delta\vartheta}(1-x)^{(\theta+\overline{\beta}\gamma\alpha)/\delta\vartheta - 1} dx \tag{1.45}$$

$$\Phi_2(z) = \int_0^z e^{-(\lambda/\delta\vartheta)x} x^{\beta\gamma\alpha/\delta\vartheta - 1}(1-x)^{(\theta+\overline{\beta}\gamma\alpha)/\delta\vartheta} dx, \tag{1.46}$$

where $\delta \neq 0, \;\; \vartheta \neq 0, \;\; x \neq 0, \; \delta\vartheta \neq 1$. And

$$\begin{aligned}\Pi_0(1) = &\left((\delta\vartheta + \theta + \gamma\bar{\beta}\alpha)(\gamma\mu - \lambda)\beta\Phi_2(1)\right) \times \Big((\delta\bar{\beta}\vartheta + \theta + \gamma\bar{\beta}\alpha)(\theta + \gamma\bar{\beta}\alpha)\Phi_1(1) \\ &+ (\delta\vartheta(\gamma\mu - \lambda) + \gamma(\mu - \alpha)(\theta + \gamma\bar{\beta}\alpha) - \delta\vartheta\bar{\beta}\gamma\alpha) \times \beta\Phi_2(1)\Big)^{-1}\end{aligned} \tag{1.47}$$

. Then, the stationary state probabilities are obtained as follow

$$\pi_{0,0} = \frac{(\theta + \gamma\bar{\beta}\alpha)\Phi_1(1)}{\gamma\beta\alpha\Phi_2(1)}\Pi_0(1), \tag{1.48}$$

$$\pi_{1,0} = \frac{(\theta + \gamma\bar{\beta}\alpha)\left((\lambda + \theta + \gamma\bar{\beta}\alpha)\Phi_1(1) - \gamma\beta\alpha\Phi_2(1)\right)}{(\delta\vartheta + \gamma\beta\alpha)\gamma\beta\alpha\Phi_2(1)}\Pi_0(1), \tag{1.49}$$

and

$$\pi_{1,1} = \Big(\big((\theta+\gamma\bar{\beta}\alpha)\left\{(\delta\vartheta+\gamma\alpha)\beta\gamma\alpha\Phi_2(1) - \Phi_1(1)(\lambda\gamma\bar{\beta}\alpha + (\delta\vartheta+\gamma\alpha)(\theta+\gamma\bar{\beta}\alpha))\right\}\Big) \\ \times(\gamma\beta\mu\alpha\Phi_2(1)(\delta\vartheta+\gamma\beta\alpha))^{-1}\big)\,\Pi_0(1). \tag{1.50}$$

Here, the explicit expressions for various performance measures are derived.

$\Diamond$ The expected number of customers in the system when the server is on a working vacation period $\mathbb{E}(L_0)$.

$$\mathbb{E}(L_0) = \left(\frac{\gamma\mu-\lambda}{\theta+\bar{\beta}\gamma\alpha}\right)(1-\Pi_0(1)) - \left(\frac{\bar{\beta}\gamma\alpha}{\theta+\bar{\beta}\gamma\alpha}\right)(\pi_{00}-\Pi_0(1)).$$

$\Diamond$ The expected number of customers in the system when the server is in a normal busy period $\mathbb{E}(L_1)$.

$$\mathbb{E}(L_1) = \Pi_1'(1) = \tfrac{\theta+\gamma\bar{\beta}\alpha}{\gamma\mu-\lambda}\tfrac{\Pi_0''(1)}{2} + \tfrac{1}{(\theta+\gamma\bar{\beta}\alpha)(\gamma\mu-\lambda)} \\ \times\left((\gamma\theta\mu+\lambda\gamma\bar{\beta}\alpha)(1-\Pi_0(1)) + \theta\gamma\bar{\beta}\alpha(\Pi_0(1)-\pi_{0,0})\right).$$

$\Diamond$ The expected number of customers in the system can be computed as

$$\mathbb{E}(L) = \mathbb{E}(L_0) + \mathbb{E}(L_1).$$

$\Diamond$ The sojourn times, with W the total sojourn time of a customer in the system, evaluated from the instant of arrival till departure, with the departure either due to completion of service or as a consequence of abandonment.

$$\mathbb{E}(W) = \frac{1}{\lambda}(\mathbb{E}(L_0) + \mathbb{E}(L_1)).$$

$\Diamond$ The proportion of customers served Δ.

$$\Delta = \frac{1}{\lambda}\left(\gamma\mu\Pi_1(1) + \gamma\alpha(\Pi_0(1)-\pi_{0,0})\right).$$

$\Diamond$ The rate of abandonment Θ of a customer due to impatience is given by

$$\Theta = \delta\vartheta\mathbb{E}(L_0) = \lambda - \left(\gamma\mu\Pi_1(1) + \gamma\alpha(\Pi_0(1)-\pi_{0,0})\right),$$

$\Diamond$ The probability that the system is in normal busy period Γ and the probability that the system is in working vacation Ω are, respectively, given by

$$\Gamma = \sum_{n=1}^{\infty}\pi_{n,1} = \Pi_1(1), \quad \Omega = \sum_{n=0}^{\infty}\pi_{n,0} = \Pi_0(1).$$

1.5 The layout of the thesis

Our study focuses on the analysis of queuing systems with impatience, the following chapters represent the way the research progressed. Our thesis is composed of four chapters:

The first one is an introductory chapter, it presents a work of synthesis, with which the thesis is begun; a detailed description of retrial queues, queueing systems with impatient customers and vacation queueing models is given. In this part of the thesis an important literature review is presented. These queueing models play an important role in modeling and analyzing the performance of many complex systems, such as computer networks, telecommunications systems, call centers, flexible manufacturing systems and service systems.

In the second chapter we investigate a simple intuitive approximation for retrial queueing model queueing system model with two orbits, c_t servers, $t \geq 0$, abandoned and feedback customers. Two independent Poisson streams of customers arrive to the system, an arriving one of type i; $i = 1; 2$ is handled by an available server, if there is any; otherwise, he waits in an infinite buffer queue. A waiting customer of type i may lose his patience and abandon after an exponentially distributed amount of time, this latter may leave the system or move to one of the orbits depending of its type, from which he retries to to reach the primary queue, the customer in the orbit may lose his patience and leave the system definitively after an exponentially distributed amount of time. After completion of a service, the customer may comeback to the system, to one of the orbits for another service.

This result has been the subject of an international publication in **Mathematical Sciences And Applications E-Notes Volume, 2(2),51-66, 2014.**

The third chapter is consecrated to the study of the stability of a retrial queueing system with two orbits, abandoned and feedback customers. Two independent Poisson streams of customers arrive to the system, and flow into a single-server service system. An arriving one of type i, $i = 1, 2$ is handled by the server if it is free; otherwise, it is blocked and routed to a separate type-i retrial (orbit) queue that attempts to re-dispatch its jobs at its specific Poisson rate. The customer in the orbit either attempts service again after a random time or gives up receiving service and leaves the system after a random time. After the customer is served completely, it will decide either to join the retrial group again for another service or leave the system forever with some probability.
This study was published in **Appl. Appl. Math, 10(2),667-677, 2015.**

In the fourth chapter, we establish an analysis of a Markovian feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption, where customers' impatience is due to the servers' vacation. The reneging times are assumed to be exponentially distributed. After the completion of service, each customer may reenter the system as a feedback customer for receiving another regular service with some probability or leave the system. A reneged customer can be retained in many cases by employing certain convincing mechanisms to stay in queue for completion of service. Thus, a reneged customer can be retained in the queueing system with some probability or he may leave the queue without receiving service. The stationary analysis of the system is established. The probability generating functions of the stationary state probabilities are obtained, the explicit expressions of the system sizes when the server is in a normal service period and in a Bernoulli schedule vacation interruption are deduced, respectively. Various performance measures of the system are derived.

This work was punctuated by a publication published in **Arab. J. Math, 6(1),1-11, 2017.**

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Chapter 2

A Note On Fluid Approximation Of Retrial Queueing System With Two Orbits, Abandonment And Feedback

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A note on fluid approximation of retrial queueing system with two orbits, abandonment and feedback

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Abstract
This chapter deals with a queueing system model with two orbits, abandoned and feedback customers and c_t, $t \geq 0$ servers. Two independent Poisson streams of customers arrive to the system, an arriving one of type i, $i = 1, 2$ is handled by an available server, if there is any; otherwise, he waits in an infinite buffer queue. A waiting customer of type i who did not get connected to a server will lose his patience and abandon after an exponentially distributed amount of time, the abandoned one may leave the system (loss customer) or move to the orbit depending of its type, from which he makes a new attempts to reach the primary queue, then this later when he finishes his conversation with a server, he may comeback to the system, to one of the orbits for another service.

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Keywords: Queueing system, call center, retrial queue, fluid approximation, abandonment, feedback.

2.1 Introduction

During the past few decades, there has been increasing interest in studying retrial queueing systems because they are widely used in performance analysis of many practical systems, retrial queues have been investigated extensively because of their applications in telephone switching systems, telecommunication networks and computer systems for competing to gain service from a central processing unit and so on. Moreover, retrial queues are also used as mathematical models for several computer systems: packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks, etc. Retrial queueing systems are characterized by the feature that a blocked customer (a customer who finds the server unavailable) may leave the service area temporarily and join a retrial group in order to retry his request after some random time. For excellent bibliography on retrial queues, the readers are referred to [15, 19, 16, 12, 29, 42, 8] and references therein.

Behavioral psychology concerning the use of service offered by mobile cellular networks includes repeated attempts and abandonments. Both phenomena reflect the impatience of subscribers when all channels are occupied. Following the arrival of a call, if all the available channels are occupied, a call is not be admitted into a network. Later, a subscriber initiates a repeated attempt for the admission of a call. An abandonment happens when a subscriber's call becomes rejected and the subscriber gets impatient and gives up after a certain time without getting service.

In feedback queueing model, if the service of the job is unsuccessful, it may try again and again until a successful service is completed. Takacs [40] was the first to study feedback queueing model. Studies on queue length, the total sojourn time and the waiting time for an $M/G/1$ queue with Bernoulli feedback were provided by Vanden Berg and Boxma [41]. Choudhury and Paul [9] derived the queue size distribution at random epoch and at a service completion epoch for $M/G/1$ queue with two phases of heterogeneous services and Bernoulli feedback system, Krishna Kumar *et al.*[25] considered a generalized $M/G/1$ feedback queue in which customers are either "positive" or "negative". In [17] Fayolle treated a simple telephone exchange with delayed feedback, Choi [8] considered an $M/M/c$ retrial queues with geometric loss and feedback when $c = 1, 2$.

A queueing system with two orbits and two exogenous streams of different type serves as a model for two competing job streams in a carrier sensing multiple access system, where the jobs, after a failed attempt to network access, wait in an orbit queue [34, 39]. An example of carrier sensing multiple access system is a local area computer network with bus architecture. The two types of customers can be interpreted as customers with different priority requirements.

A two-class retrial system with a single- server, no waiting room, batch arrivals and classical retrial scheme was introduced and analyzed in [26]. Then, in [14] author extended the analysis of the model in [26] to the multi-class setting with arbitrary number of classes. In [20] author has established equivalence between the multi-class batch arrival retrial queues with classical retrial policy and branching processes with immigration. In [33] a non-preemptive priority mechanism was added to the model of [14, 26]. In [28] authors have considered a multi-class retrial system where retrial classes are associated with different phases of service. Retrial queueing model $MMAP/M_2/1$ with two orbits was studied in [5], authors considered a retrial single-server queueing model with two types of customers. In case of the server occupancy at the arrival epoch, the customer moves to the orbit depending on the type of the customer, one orbit is infinite while the second one is a finite. Joint distribution of the number of customers in the orbits and some performance measures are computed. In [7] authors considered two retrial queueing system with balking and feedback, the joint generating function of the number of busy server and the queue length was found by solving Kummer differential equation, and by the method of series solution.

Call centers have become the central focus of many companies, as these centers stay in direct contact with the form's customers and form an integral part of their customer relationship management. So, at the present time, call centers are becoming an important means of communication with the customer. Therefore, the response-time performance of call centers is essential for the customer satisfaction. For call center managers, making the right staffing decisions is essential to the costs and the performances of call centers. Various models have been developed in order to decide on the right number of agents, see [18, 21], and the references therein. Thus, considering customer retrial behaviors in call centers is quite significant [18, 2, 38, 11] and reference therein.

Fluid models for call centers have been extensively studied, for instance see [43, 32]. In [31] the fluid and the diffusion approximation for time varying multiserver queue with abandonment and retrials as studied, it was shown that the fluid and the diffusion approximation can both be obtained by solving sets of non-linear differential equations. In [30] more general theoretical results for the fluid and diffusion approximation for Markovian service networks was given. In [1] authors extended the model by allowing customer balking behavior. Fluid models have also been applied in delay announcement of customers in call centers [22, 23].

And recently, in [10] authors study call centers with one redial and one orbit, using fluid limit they calculate the expected total arrival rate, which is then given as an input to the Erlang A model for the purpose of calculating service levels and abandonment rates. The performance of such a procedure is validated in the case of single intervals as well as multiple intervals with changing parameters.

In the present chapter, an analysis of $M_t/M_t/c_t$ retrial queueing model with abandonment and feedback; a system with two orbits and two exogenous streams of different types is carried out.

The layout of the chapter is given as follows. After the introduction, in section 2, we describe the mathematical model in more details and give the notations, assumptions and some results that will be used and useful throughput this chapter. In section 3, our main result is given; an asymptotic analysis of the considered model is presented.

2.2 The mathematical model

Consider retrial queueing network with time dependent parameters, state dependent routing, abandonment and feedback (figure 2.1). The $M_t/M_t/c_t$ queue has a (time in homogeneous) Poisson arrival process with rate λ_{i_t}, a service rate (per server) with mean $\frac{1}{\mu_{i_t}}$, $i = 1, 2$ and c_t servers, for all $t > 0$.

Two independent Poisson streams of customers flow into c servers. An arriving customer of type i, $i = 1, 2$ is handled by an available server in FIFO manner, if there is any; otherwise, he waits in an infinite buffer queue. The customers are handled in the order of arrival. A waiting customer of type i who did not get connected to a server will lose his patience and abandon after an exponentially distributed amount of time at rate δ_{i_t}, the abandoned one may leave the entire network (loss customer) with probability ϕ_t or move into one of the orbits with probability $1 - \phi_t$, from which he makes a new attempts to reach the primary queue at rate α_{i_t}. Each customer waiting in the retrial pool may leave his patience and thus abandon the whole system at rate θ_{i_t} if at some moment he beholds that the queue length i is greater than k_{i_t} with $0 < k_{i_t} < Q_1(t)$, so after an exponentially distributed amount of time he have to decide either he still waiting for a new attempts or give up. An abandoning customer leave the system from the orbit with some probability ρ_{i_t}. When a customer finishes his conversation with a server or if the service of the job is unsuccessful, the customer may comeback to the system to the retrial pools depending on its type for another service or try again and again for a successful service at rate ω_{i_t}. Let's note that all the arrival and service processes are constructed from mutually independent Poisson processes.

After the description of the considered model let us introduce some notations and results helpful in our study.

Let $\{\Pi_i(\cdot)\}_{i \in I}$ a sequence of mutually independent, standard (rate 1) Poisson processes, indexed by a set I which is at most countably infinite; a separable Banach space $\mathbb{V}$, with norm $|\cdot|$; a sequence of jump vectors $\{v_i \in \mathbb{V} | i \in I\}$ with

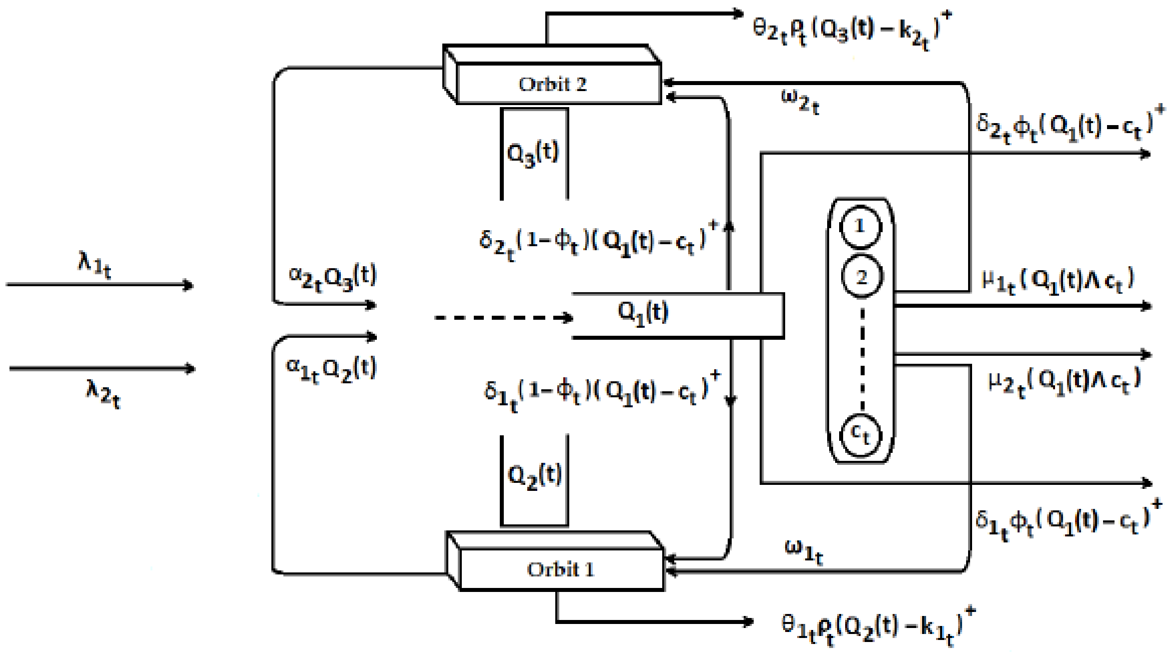

Figure 2.1: A retrial queueing model with two orbits, abandonment and feedback.

$$\sum_{i \in I} v_i < \infty \tag{2.1}$$

a random initial state vector $Q(0)$ in $\mathbb{V}$ that is assumed to be independent of the sequence of Poisson processes $\{\Pi(\cdot)\}_{i \in I}$; and a collection of real-valued, non-negative Lipschitz rate functions on $\mathbb{V}$,

$$\{\nu_t(\cdot, i) | t \geq 0, i \in I\}, \tag{2.2}$$

that together satisfy

$$\|\nu_t(\cdot, i)\| \leq \xi_t \vartheta^{(i)}, \tag{2.3}$$

with ξ_t, a locally integrable function, and $\{\vartheta^{(i)} | i \in I\}$, a sequence of real numbers; with $\| \cdot \|$ a Lipschitz norm for real-valued functions on $\mathbb{V}$. In all what follows the number of elements in I is finite, $\mathbb{V} = \mathbb{R}^N$, $1 \leq N < \infty$ and $| \cdot |$ the standard Euclidean norm on $\mathbb{R}^N$.

Let the Markovian service network $\{Q(t) | t \geq 0\}$, be the $\mathbb{V}$-valued stochastic process whose sample paths are uniquely determined by $Q(0)$ and the functional equations

$$Q(t) = Q(0) + \sum_{i \in I} \Pi_i \left(\int_0^t \nu_s(Q(s), i) ds \right) v_i, \text{ for all } t \geq 0.$$

Let $\{Q^\eta | \eta > 0\}$ be the rescaled procees such that

$$Q^\eta(t) = Q^\eta(0) + \sum_{i \in I} \Pi_i \left(\eta \int_0^t \nu_s \left(\frac{Q^\eta(s)}{\eta}, i \right) ds \right) v_i, \tag{2.4}$$

The asymptotic analysis described above was carried out in [27] for the special case of rate functions having no explicit time dependence and state dependence that is continuously differentiable. The analysis was extend to the following general class of processes [30].

$$Q^\eta(t) = Q^\eta(0) + \sum_{i \in \mathbf{I}} \Pi_i \left(\int_0^t \nu_s^\eta \left(\frac{Q^\eta(s)}{\eta}, i \right) ds \right) v_i, \tag{2.5}$$

with

$$\| \nu_t^\eta(\cdot, i) \| \leq \eta \xi_t \vartheta^{(i)}. \tag{2.6}$$

In this extension, we permit the following hypotheses:

(H1) The rate functions $\nu_t^\eta(\cdot, i)$ are functions of time as well as state.

(H2) The rate functions, indexed by the parameter η, are such that for each $i \in I$, $\nu_t^\eta(\cdot, i)$ has the following asymptotic expansion as $\eta \to \infty$;

$$\nu_t^\eta(\cdot, i) = \eta \nu_t^{(0)}(\cdot, i) + \sqrt{\eta} \nu_t^{(1)}(\cdot, i) + 0(\sqrt{\eta}). \tag{2.7}$$

(H3) The rate functions, as a function of the state space $\mathbb{V}$, have a more general type of differentiability that include functions on the real line that are everywhere left and right differentiable.

These conditions allow to apply the limit theorems to a wider class of Markov processes that arise in the study of queueing networks with large numbers of servers. Now, let's introduce the first result where the sample path representation (2.5) of $\{Q^\eta | \eta > 0\}$ is strongly presented;

Theorem 2.2.1. *[30] Assume that (2.1) and (2.6) hold. Moreover, assume that*

$$\lim_{n \to \infty} \sum_{i \in I} \left(\int_0^t \left\| \frac{\nu_t^\eta(\cdot, i)}{\eta} - \nu_t^{(0)}(\cdot, i) \right\| \right) ds = 0, \tag{2.8}$$

for all $t \geq 0$. *If* $\{Q^{\eta}(0)|\eta > 0\}$ *is any family of random initial state vectors in* $\mathbb{V}$, *then*

$$\frac{Q^{\eta}(0)}{\eta} = Q^{(0)}(0), \ \ a.s \ \ implies \ \frac{Q^{\eta}(t)}{\eta} = Q^{(0)}(t) \ \ a.s \tag{2.9}$$

where the convergence is uniform on compact sets in t, *and* $Q^{(0)}$ *is the unique deterministic process* $\{Q^{(0)}(t)|t \geq 0\}$ *that solves the integral equation*

$$Q^{(0)}(t) = Q^{(0)}(0) + \int_0^t \nu_s^{(0)}(Q^{(0)}(s))ds, \ \ t \geq 0. \tag{2.10}$$

Here $\nu_t^{(0)}$, *given by*

$$\nu_t^{(0)}(x) = \sum_{i \in I} \nu_t^{(0)}(x,i)v_i, \ \ \ x \in \mathbb{V}, \tag{2.11}$$

is a Lipschitz mapping of $\mathbb{V}$ *into itself and its Lipschitz norm* $\|\nu_t^{(0)}\|$, *is a locally integrable function of* t.

We call $Q^{(0)}$ *the fluid approximation associated with the family* $\{Q^{\eta}(t)|t \geq 0\}$. *It gives rise to first-order macroscopic fluid approximations of the form*

$$Q^{\eta}(t,\omega) = \eta Q^{(0)}(t) + o(\eta) \ \ a.s., \ \ t \geq 0. \tag{2.12}$$

We can now state the functional central limit theorem

Theorem 2.2.2. *[30] Assume that (2.1) and (2.6) hold. Moreover, assume that*

$$\sum_{i \in I} \overline{lim}_{\eta \to \infty} \int_0^t \left\| \sqrt{\eta} \left(\frac{\nu_t^{\eta}(\cdot,i)}{\eta} - \nu_t^{(0)}(\cdot,i) \right) \right\| ds < \infty, \tag{2.13}$$

and

$$\lim_{\eta \to \infty} \sum_{i \in I} \int_0^t \left\| \sqrt{\eta} \left(\frac{\nu_t^{\eta}(\cdot,i)}{\eta} - \nu_t^{(0)}(\cdot,i) \right) - \nu_t^{(1)}(\cdot,i) \right\| ds = 0. \tag{2.14}$$

It follows that $\nu_t^{(0)}$, *given by (2.11), and* $\nu_t^{(1)}$, *given by*

$$\nu_t^{(1)}(x) = \sum_{i \in I} \nu_t^{(1)}(x,i)v_i, \ \ x \in \mathbb{V}, \tag{2.15}$$

are both Lipschitz mappings of $\mathbb{V}$ *into itself, and their Lipschitz norms are locally integrable functions of* t.

Moreover, if we assume that $\nu_t^{(0)}(\cdot)$ *has a scalable Lipschitz derivative* $\wedge\nu_t^{(0)}(Q^{(0)}(t);\cdot)$ *and we have a family of random initial state vectors* $\{Q^\eta(0)|\eta > 0\}$ *in* $\mathbb{V}$*, then for all random vectors* $Q^{(0)}(0)$ *and* $Q^{(1)}(0)$ *in* $\mathbb{V}$*, it follows that*

$$\lim_{\eta\to\infty} \sqrt{\eta}\left(\frac{Q^\eta(0)}{\eta} - Q^{(0)}(0)\right) =^d Q^{(1)}(0), \tag{2.16}$$

implies

$$\lim_{\eta\to\infty} \sqrt{\eta}\left(\frac{Q^\eta(t)}{\eta} - Q^{(0)}(t)\right) =^d Q^{(1)}(t), \tag{2.17}$$

the convergence being weak-convergence in $D_{\mathbb{V}}[0,\infty)$*, the space of* $\mathbb{V}$*-valued functions that are right-continuous with left-limits, equipped with the Skorohod* J_1 *topology. Finally, the limit* $Q^{(1)} \equiv \{Q^{(1)}(t)|t \geq 0\}$ *is the unique stochastic process that solves the stochastic integral equation*

$$\begin{aligned} Q^{(1)}(t) = Q^{(1)}(0) + \int_0^t \left(\left(\wedge\nu_s^{(0)}(Q^{(0)}(s), Q^{(1)}(s))\right) + \nu_s^{(1)}(Q^{(0)}(s))\right) ds \\ + \sum_{i\in I} \Omega_i \left(\int_0^t \nu_s^{(0)}(Q^{(0)}(s), i) ds\right) v_i, \quad t \geq 0, \end{aligned} \tag{2.18}$$

where the $\{\Omega_i|i \in I\}$ *are a family of mutually independent, standard Brownian motions. We call* $Q^{(1)}$ *the diffusion approximation associated with the family* $\{Q^\eta(t)|t \geq 0\}$*. It quantifies deviations from the fluid approximations, and it gives rise to second-order mesoscopic diffusion approximations of the form*

$$Q^\eta(t) =^d \eta Q^{(0)}(t) + \sqrt{\eta} Q^{(1)}(t) + o(\sqrt{\eta}), \tag{2.19}$$

as $\eta \to \infty$ *for all* $t \geq 0$*, with the approximation being in distribution*

Now consider the case of $\mathbb{V}$ being either a finite dimensional vector space or a Banach space that can be embedded into its own dual space (like a Hilbert space), so that the notion of a transpose can be defined, denoted by a superscript "$\top$" (for $\mathbb{V} = \mathbb{R}^N$, this corresponds to the standard transpose of a matrix). One consequence of the diffusion limit is an associated set of differential equations that become useful in the computation of its mean and covariance matrix.

Theorem 2.2.3. *[30] If conditions (2.1), (2.6), (2.13), and (2.14) all hold, then the mean vector and covariance matrix for* $Q^{(1)}(t)$ *solve the following set of differential equations:*

$$\frac{d}{dt}\mathbb{E}(Q^{(1)}(t)) = \mathbb{E}(\wedge\nu_t^{(0)}(Q^{(0)}(t), Q^{(1)}(t))) + \nu_t^{(1)}(Q^{(0)}(t)). \tag{2.20}$$

$$\begin{aligned}\tfrac{d}{dt}Cov(Q^{(1)}(t),Q^{(1)}(t)) = &\Big(Cov(Q^{(1)}(t),\wedge\nu_t^{(0)}(Q^{(0)}(t),Q^{(1)}(t)))\Big)\\ &+\sum_{i\in I}\nu_t^{(0)}(Q^{(0)}(t),i)v_i^{\top}\cdot v_i.\end{aligned} \tag{2.21}$$

for almost all t, *where*

$$Cov(Q^{(1)}(t),Q^{(1)}(t)) \equiv \mathbb{E}\left((Q^{(1)}(t))^T\cdot Q^{(1)}(t)\right) - \mathbb{E}(Q^{(1)}(t))^{\top}\cdot\mathbb{E}(Q^{(1)}(t)), \tag{2.22}$$

and for all operators A *on* $\mathbb{V}$,

$$\{A\} \equiv A + A^{\top}. \tag{2.23}$$

Moreover, if $\wedge\nu_t^{(0)}(Q^{(0)}(t),\cdot)$ is a linear operator for almost all t, then $\mathbb{E}[Q^{(1)}(t)]$ is the unique solution for (2.20) and $Cov[Q^{(1)}(t),Q^{(1)}(t)]$ is the unique solution for (2.21). Finally, for all $s<t$, $Cov[Q^{(1)}(s),Q^{(1)}(t)]$ solves the same set of differential equations in t as does $\mathbb{E}[Q^{(1)}(t)]$, but with a different set of initial conditions. Now, and after having stated all these results, we are able to give our main result.

2.3 Main result

Consider our queueing model presented in figure 2.1. The $M_t/M_t/c_t$ queue has a (time inhomogeneous) Poisson arrival process with external arrival rates λ_{i_t}, a service rates (per server) of μ_{i_t}, feedback rates ω_{i_t}, abandonment rates from the primary queue δ_{i_t}, abandonment rates from retrial pool i ω_{i_t}, $i=1,2$ and c_t servers, for all $t>0$, $c_t = 1,2,3,....$ With ϕ_t, $0\le\phi_t\le 1$, the probability of no retrial at time t, ρ_t $0\le\rho_t\le 1$ the probability of leaving the network from the orbit at time t.
Let $\mathbb{V}=\mathbb{R}^3$ and $Q(t)=\{Q_1(t),Q_2(t),Q_3(t)\}$. We can construct the sample paths for the $M_t/M_t/c_t$ queue length process as the unique set of solutions to the functional equation

$$\begin{aligned}Q_1(t) = Q_1(0) &+ \Pi_1\left(\int_0^t \lambda_{1_s}ds\right) + \Pi_2\left(\int_0^t \lambda_{2_s}ds\right) + \Pi_3\left(\int_0^t \alpha_{1_s}Q_2(s)ds\right)\\ &+\Pi_4\left(\int_0^t \alpha_{2_s}Q_3(s)ds\right) - \Pi_5\left(\int_0^t (Q_1(s)-c_s)^+\delta_{1_s}\phi_s ds\right)\\ &-\Pi_6\left(\int_0^t (Q_1(s)-c_s)^+\delta_{2_s}\phi_s ds\right) - \Pi_7\left(\int_0^t (Q_1(s)-c_s)^+\delta_{1_s}(1-\phi_s)ds\right)\\ &-\Pi_8\left(\int_0^t (Q_1(s)-c_s)^+\delta_{2_s}(1-\phi_s)ds\right) - \Pi_9\left(\int_0^t \mu_{1_s}(Q_1(s)\wedge c_s)ds\right)\\ &-\Pi_{10}\left(\int_0^t \mu_{2_s}(Q_1(s)\wedge c_s)ds\right).\end{aligned} \tag{2.24}$$

$$Q_2(t) = Q_2(0) + \Pi_1^1\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{1_s}(1-\phi_s)ds\right) + \Pi_2^1\left(\int_0^t \omega_{1_s} ds\right)$$
$$-\Pi_3^1\left(\int_0^t \alpha_{1_s} Q_2(s) ds\right) - \Pi_4^1\left(\int_0^t \rho_s \theta_{1_s}(Q_2(s) - k_{1_s})^+ ds\right). \tag{2.25}$$

$$Q_3(t) = Q_3(0) + \Pi_1^2\left(\int_0^t (Q_1(s) - c_s)^+ \delta_{2_s}(1-\phi_s)ds\right) + \Pi_2^2\left(\int_0^t \omega_{2_s} ds\right)$$
$$-\Pi_3^2\left(\int_0^t \alpha_{2_s} Q_3(s) ds\right) - \Pi_4^2\left(\int_0^t \rho_s \theta_{2_s}(Q_3(s) - k_{2_s})^+ ds\right), \tag{2.26}$$

where $\Pi_i(\cdot)$, $\Pi_i^1(\cdot)$, and $\Pi_i^2(\cdot)$, are given independent, standard (rate 1) Poisson processes, and for all real x and y, $x \wedge y \equiv \min(x, y)$.

For the $M_t/M_t/c_t$ queue, we create a family of associated processes. The $M_t/M_t/c_t$ queue is indexed by η, we want to have both the arrival rate and number of servers grow large, i.e., scaled up by η, but leave the service rate unscaled. We are then interested in the asymptotic behavior of the process $Q^\eta(t) = (Q_1^\eta(t), Q_2^\eta(t), Q_3^\eta(t))$

$$\begin{aligned} Q_1^\eta(t) = {} & Q_1^\eta(0) + \Pi_1\left(\int_0^t \eta\lambda_{1_s} ds\right) + \Pi_2\left(\int_0^t \eta\lambda_{2_s} ds\right) + \Pi_3\left(\int_0^t \alpha_{1_s} Q_2^\eta(s) ds\right) \\ & +\Pi_4\left(\int_0^t \alpha_{2_s} Q_3^\eta(s) ds\right) - \Pi_5\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{1_s}\phi_s ds\right) \\ & -\Pi_6\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{2_s}\phi_s ds\right) - \Pi_7\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{1_s}(1-\phi_s) ds\right) \\ & -\Pi_8\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{2_s}(1-\phi_s) ds\right) - \Pi_9\left(\int_0^t \mu_{1_s}(Q_1^\eta(s) \wedge \eta c_s) ds\right) \\ & -\Pi_{10}\left(\int_0^t \mu_{2_s}(Q_1^\eta(s) \wedge \eta c_s) ds\right). \end{aligned} \tag{2.27}$$

$$Q_2^\eta(t) = Q_2^\eta(0) + \Pi_1^1\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{1_s}(1-\phi_s)ds\right) + \Pi_2^1\left(\int_0^t \eta\omega_{1_s} ds\right)$$
$$-\Pi_3^1\left(\int_0^t \alpha_{1_s} Q_2^\eta(s) ds\right) - \Pi_4^1\left(\int_0^t \rho_s \theta_{1_s}((Q_2^\eta(s) - \eta k_{1_s})^+ ds\right). \tag{2.28}$$

$$Q_3^\eta(t) = Q_3^\eta(0) + \Pi_1^2\left(\int_0^t (Q_1^\eta(s) - \eta c_s)^+ \delta_{2_s}(1-\phi_s)ds\right) + \Pi_2^2\left(\int_0^t \eta\omega_{2_s} ds\right)$$
$$-\Pi_3^2\left(\int_0^t \alpha_{2_s} Q_3^\eta(s)ds\right) - \Pi_4^2\left(\int_0^t \rho_s\theta_{2_s}(Q_3^\eta(s) - \eta k_{2_s})^+ ds\right) \tag{2.29}$$

as $\eta \to \infty$.

Let us note that servers and time-dependent parameters do not need to be scaled; The primary motivating models are call centers, where service involves an interaction between the customer and the server, because a customer is involved, it does not seem reasonable to scale the service rates with η. Thus, in order to accommodate the arrivals, whose rate is proportional to η, the number of servers must be scaled with η. Time dependent arrival rates should need no justification, since phenomena such as rush hours are quite common. Time dependent service rates can be used to model phenomena such as server fatigue or changes in the nature of services over the day. Finally, a time dependent number of servers arises with shift changes and in systems where the number of servers is varied to accommodate changes in the arrival rate.
The first-order asymptotic result takes the form of a functional strong law of large numbers and yields a fluid approximation for the original process.

Theorem 2.3.1. *Let Q^η be the uniform acceleration as in (2.4), the fluid limit for the multiserver queue with retrials abandonment and feedback is the unique solution to the differential equations*

$$\tfrac{d}{dt}Q_1^{(0)}(t) = \lambda_{1_t} + \lambda_{2_t} + \alpha_{1_t}Q_2^{(0)}(t) + \alpha_{2_t}Q_3^{(0)}(t) - (\mu_{1_t} + \mu_{2_t})(Q_1^{(0)}(t) \wedge c_t)$$
$$-(\delta_{1_t} + \delta_{2_t})(Q_1^{(0)}(t) - c_t)^+. \tag{2.30}$$

$$\tfrac{d}{dt}Q_2^{(0)}(t) = \omega_{1_t} - \alpha_{1_t}Q_2^{(0)}(t) + \delta_{1_t}(1-\phi_t)(Q_1^{(0)}(t) - c_t)^+ - \theta_{1_t}\rho_t(Q_2^{(0)}(t) - k_{1_t})^+. \tag{2.31}$$

$$\tfrac{d}{dt}Q_3^{(0)}(t) = \omega_{2_t} - \alpha_{2_t}Q_3^{(0)}(t) + \delta_{2_t}(1-\phi_t)(Q_1^{(0)}(t) - c_t)^+ - \theta_{2_t}\rho_t(Q_3^{(0)}(t) - k_{2_t})^+. \tag{2.32}$$

Furthermore, the diffusion limit for the multiserver queue with abandonment, feedback and *retrials is the unique solution to the integral equations*

$$
\begin{aligned}
Q_1^{(1)}(t) &= Q_1^{(1)}(0) + \Omega_1\left(\int_0^t \lambda_{1_s} ds\right) + \Omega_2\left(\int_0^t \lambda_{2_s} ds\right) + \Omega_3\left(\int_0^t \alpha_{1_s} Q_2^{(0)}(s) ds\right) \\
&+\Omega_4\left(\int_0^t \alpha_{2_s} Q_3^{(0)}(s) ds\right) - \Omega_5\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+ \delta_{1_s}(1-\phi_s) ds\right) \\
&+ \int_0^t \Big[\Big((\mu_{1_s} + \mu_{2_s}) 1_{\{Q_1^{(0)}(s) \le c_s\}} + (\delta_{1_s} + \delta_{2_s}) 1_{\{Q_1^{(0)}(s) > c_s\}}\Big) Q_1^{(1)}(s)^- \\
&- \Big((\mu_{1_s} + \mu_{2_s}) 1_{\{Q_1^{(0)}(s) < c_s\}} + (\delta_{1_s} + \delta_{2_s}) 1_{\{Q_1^{(0)}(s) \ge c_s\}}\Big) Q_1^{(1)}(s)^+ \qquad (2.33) \\
&+\alpha_{2_s} Q_3^{(1)}(s) + \alpha_{1_s} Q_2^{(1)}(s)\Big] ds - \Omega_6\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+ \delta_{2_s}(1-\phi_s) ds\right) \\
&-\Omega_7\left(\int_0^t \mu_{1_s}(Q_1^{(0)}(s) \wedge c_s) ds\right) - \Omega_8\left(\int_0^t \mu_{2_s}(Q_1^{(0)}(s) \wedge c_s) ds\right) \\
&-\Omega_9\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+ \delta_{1_s}\phi_s ds\right) - \Omega_{10}\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+ \delta_{2_s}\phi_s ds\right).
\end{aligned}
$$

$$
\begin{aligned}
Q_2^{(1)}(t) &= Q_2^{(1)}(0) + \Omega_1^1\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+ \delta_{1_s}(1-\phi_s) ds\right) + \Omega_2^1\left(\int_0^t \omega_{1_s} ds\right) \\
&+ \int_0^t \Big[Q_1^{(1)}(s)^+ 1_{\{Q_1^{(0)}(s) \ge c_s\}} - Q_1^{(1)}(s)^- 1_{\{Q_1^{(0)}(s) > c_s\}}\Big] \delta_{1_s}(1-\phi_s) ds \\
&-\Omega_3^1\left(\int_0^t \alpha_{1_s} Q_2^{(0)}(s) ds\right) - \Omega_4^1\left(\int_0^t \rho_s \theta_{1_s}(Q_2^{(1)}(s) - k_{1_s})^+ ds\right) \qquad (2.34) \\
&- \int_0^t \theta_{1_s}\rho_s \Big[(Q_2^{(1)}(s))^+ 1_{\{(Q_2^{(0)}(s) \ge k_{1_s}\}} - (Q_2^{(1)}(s))^- 1_{\{(Q_2^{(0)}(s) > k_{1_s}\}}\Big] ds \\
&- \int_0^t \alpha_{1_s} Q_2^{(1)}(s) ds.
\end{aligned}
$$

$$
\begin{aligned}
Q_3^{(1)}(t) &= Q_3^{(1)}(0) + \Omega_1^2\left(\int_0^t (Q_1^{(0)}(s) - c_s)^+ \delta_{2_s}(1-\phi_s) ds\right) + \Omega_2^2\left(\int_0^t \omega_{2_s} ds\right) \\
&+ \int_0^t \Big[Q_1^{(1)}(s)^+ 1_{\{Q_1^{(0)}(s) \ge c_s\}} - Q_1^{(1)}(s)^- 1_{\{Q_1^{(0)}(s) > c_s\}}\Big] \delta_{2_s}(1-\phi_s) ds
\end{aligned} \qquad (2.35)
$$

$$-\Omega_3^2\left(\int_0^t \alpha_{2_s} Q_3^{(0)}(s)ds\right) - \Omega_4^2\left(\int_0^t \rho_s\theta_{2_s}(Q_3^{(1)}(s)-k_{2_s})^+ds\right)$$

$$-\int_0^t \theta_{2_s}\rho_s\left[Q_3^{(1)}(s)^+1_{\{(Q_3^{(0)}(s)\geq k_{2_s}\}} - Q_3^{(1)}(s)^-1_{\{(Q_3^{(0)}(s)>k_{2_s}\}}\right]ds \tag{2.36}$$

$$-\int_0^t \alpha_{2_s}Q_3^{(1)}(s)ds.$$

Getting these equations is based essentially on the theorem 2.2.1 and 2.2.2. The following result provides ordinary differential equations for the mean vector, variance and covariance matrices of $Q_i^{(1)}$.

Theorem 2.3.2. *The mean vector for the diffusion limit solves the set of differential equations*

$$\begin{aligned}\tfrac{d}{dt}\mathbb{E}(Q_1^{(1)}(t)) &= \left((\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)\leq c_t\}} + (\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)>c_t\}}\right)\mathbb{E}(Q_1^{(1)}(t)^-)\\ &-\left((\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)<c_t\}} + (\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)\geq c_t\}}\right)\mathbb{E}(Q_1^{(1)}(t)^+)\\ &+\alpha_{1_t}\mathbb{E}(Q_1^{(2)}(t)) + \alpha_{2_t}\mathbb{E}(Q_1^{(3)}(t)).\end{aligned} \tag{2.37}$$

$$\begin{aligned}\tfrac{d}{dt}\mathbb{E}(Q_2^{(1)}(t)) &= \delta_{1_t}(1-\phi_t)\left(\mathbb{E}(Q_1^{(1)}(t)^+)1_{\{Q_1^{(0)}(t)\geq c_t\}} - \mathbb{E}(Q_1^{(1)}(t)^-)1_{\{Q_1^{(0)}(t)>c_t\}}\right)\\ &-\left(\rho_t\theta_{1_t}1_{\{Q_2^{(0)}(t)\geq k_{1_t}\}}\right)\times\mathbb{E}\left(Q_2^{(1)}(t)\right) - \alpha_{1_t}\mathbb{E}(Q_2^{(1)}(t)).\end{aligned} \tag{2.38}$$

$$\begin{aligned}\tfrac{d}{dt}\mathbb{E}(Q_3^{(1)}(t)) &= \delta_{2_t}(1-\phi_t)\left(\mathbb{E}(Q_1^{(1)}(t)^+)1_{\{Q_1^{(0)}(t)\geq c_t\}} - \mathbb{E}(Q_1^{(1)}(t)^-)1_{\{Q_1^{(0)}(t)>c_t\}}\right)\\ &-\left(\rho_t\theta_{2_t}1_{\{Q_3^{(0)}(t)\geq k_{2_s}\}}\right)\times\mathbb{E}\left(Q_3^{(1)}(t)\right) - \alpha_{2_t}\mathbb{E}(Q_3^{(1)}(t)).\end{aligned} \tag{2.39}$$

The covariance matrix for the diffusion limit solves the differential equations

$$
\begin{aligned}
\tfrac{d}{dt}Var(Q_1^{(1)}(t)) = 2&\left((\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)>c_t\}}+(\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)\le c_t\}}\right)\\
&\times Cov(Q_1^{(1)}(t),Q_1^{(1)}(t)^-)+\lambda_{1_t}+\lambda_{2_t}+(\delta_{1_t}+\delta_{2_t})(Q_1^{(0)}(t)-c_t)^+\\
&-2\left((\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)\ge c_t\}}+(\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)<c_t\}}\right)\\
&\times Cov(Q_1^{(1)}(t),Q_1^{(1)}(t)^+)+(\mu_{1_t}+\mu_{2_t})(Q_1^{(0)}(t)\wedge c_t)+\alpha_{1_t}Q_2^{(0)}(t)\\
&+\alpha_{2_t}Q_3^{(0)}(t)+2\left[\alpha_{1_t}cov(Q_1^{(1)}(t),Q_2^{(1)}(t))+\alpha_{2_t}cov(Q_1^{(1)}(t),Q_3^{(1)}(t))\right].
\end{aligned}
\tag{2.40}
$$

$$
\begin{aligned}
\tfrac{d}{dt}Var(Q_2^{(1)}(t)) &= 2\delta_{1_t}(1-\phi_t)Cov(Q_2^{(1)}(t),Q_1^{(1)}(t)^+)1_{\{Q_1^{(0)}(t)\ge c_t\}}-2\delta_{1_t}(1-\phi_t)\\
&\times Cov(Q_2^{(1)}(t),Q_1^{(1)}(t)^-)1_{\{Q_1^{(0)}(t)>c_t\}}-2\alpha_{1_t}Var(Q_2^{(1)}(t))\\
&+\delta_{1_t}(1-\phi_t)(Q_1^{(0)}(t)-c_t)^+ +\alpha_{1_t}Q_2^{(0)}(t)+\rho_t\theta_{1_t}(Q_2^{(0)}(t)-k_{1_t})^+\\
&+\omega_{1_t}-2\rho_t\theta_{1_t}1_{\{Q_2^{(0)}(t)\ge k_{1_t}\}}Var(Q_2(t)).
\end{aligned}
\tag{2.41}
$$

$$
\begin{aligned}
\tfrac{d}{dt}Var(Q_3^{(1)}(t)) &= 2\delta_{2_t}(1-\phi_t)Cov(Q_1^{(1)}(t)^+,Q_3^{(1)}(t))1_{\{Q_1^{(0)}(t)\ge c_t\}}-2\delta_{2_t}(1-\phi_t)\\
&\times Cov(Q_1^{(1)}(t)^-,Q_3^{(1)}(t))1_{\{Q_1^{(0)}(t)>c_t\}}-2\alpha_{2_t}Var(Q_3^{(1)}(t))\\
&+\delta_{2_t}(1-\phi_t)(Q_1^{(0)}(t)-c_t)^+ +\alpha_{2_t}Q_3^{(0)}(t)+\rho_t\theta_{2_t}(Q_3^{(0)}(t)-k_{2_t})^+\\
&+\omega_{2_t}-2\rho_t\theta_{2_t}1_{\{Q_3^{(0)}(t)\ge k_{2_t}\}}Var(Q_3(t)).
\end{aligned}
\tag{2.42}
$$

$$\frac{d}{dt}Cov(Q_1^{(1)}(t),Q_2^{(1)}(t)) = \Big((\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)\le c_t\}} + (\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)>c_t\}}\Big)$$
$$\times Cov((Q_1^{(1)}(t))^-,Q_2^{(1)}(t)) - \alpha_{1_t}Cov(Q_1^{(1)}(t),Q_2^{(1)}(t))$$
$$-\Big((\mu_{1_t}+\mu_{2_t})1_{\{Q_1^{(0)}(t)<c_t\}} + (\delta_{1_t}+\delta_{2_t})1_{\{Q_1^{(0)}(t)\ge c_t\}}\Big)$$
$$\times Cov((Q_1^{(1)}(t))^+,Q_2^{(1)}(t)) + \delta_{1_t}(1-\phi_t)(Q_1^{(0)}(t)-c_t)^+ \qquad (2.43)$$
$$-\Big(\theta_{1_t}\rho_t 1_{\{Q_2^{(0)}(t)\ge k_{1_t}\}}\Big)Cov(Q_1^{(1)}(t),Q_2^{(1)}(t))$$
$$+\delta_{1_t}(1-\phi_t)1_{\{Q_1^{(0)}(t)\ge c_t\}}Var(Q_1^{(1)}(t)) + \alpha_{1_t}Var(Q_2^{(1)}(t))$$
$$+\alpha_{2_t}Cov(Q_3^{(1)}(t),Q_2^{(1)}(t)) + \alpha_{1_t}Q_2^{(0)}(t).$$

$\frac{d}{dt}Cov(Q_1^{(1)}(t),Q_3^{(1)}(t))$ *will be given easily, in the same manner.*

$$\frac{d}{dt}Cov(Q_2^{(1)}(t),Q_3^{(1)}(t)) = \delta_{2_t}(1-\phi_t)1_{\{Q_1^{(0)}(t)\ge c_t\}}Cov(Q_2^{(1)}(t),Q_1^{(1)}(t)) + \delta_{1_t}(1-\phi_t)$$
$$\times 1_{\{Q_1^{(0)}(t)\ge c_t\}}Cov(Q_3^{(1)}(t),Q_1^{(1)}(t)) - \Big(\alpha_{1_t}+\alpha_{2_t}+\theta_{1_t}\rho_t$$
$$\times 1_{\{Q_2^{(0)}(t)\ge k_{1_t}\}} + \theta_{2_t}\rho_t 1_{\{Q_3^{(0)}(t)\ge k_{2_t}\}}\Big)Cov(Q_2^{(1)}(t),Q_3^{(1)}(t)). \qquad (2.44)$$

The proof of this theorem is based on Theorems 2.2.2 and 2.2.3; Given the integral equations (2.33)-(2.36) that $Q_i^{(1)}(t)$ solves, we immediately have for i=1,2,3

$$\mathbb{E}(Q_i^{(1)}(t)) = \mathbb{E}(Q_i^{(1)}(0)) + \int_0^t \mathbb{E}(\wedge\nu_s^{(0)}(Q_i^{(0)}(s),Q_i^{(1)}(s))ds + \int_0^t \nu_s^{(1)}(Q^{(0)}(s)). \qquad (2.45)$$

Differentiating this equation we get (2.37), (2.38) and (2.39).

Then The solution to the integral equations (2.33)-(2.36) also solves the stochastic differential equation

$$d(Q_i^{(1)})(t) = (\wedge\nu_t^{(0)}(Q_i^{(0)}(t),Q_i^{(1)}(t))) + \nu_s^{(1)}(Q^{(0)}(t))dt + \sum_{i\in I}\sqrt{\nu_t^{(0)}(Q_i^{(0)}(t),i)}vi\ \ d\Omega_i^*(t). \qquad (2.46)$$

Using Ito's formula [24] (page 149) we get

$$
\begin{aligned}
d((Q_i^{(1)}(t))^\top, Q_i^{(1)}(t)) = (\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t))^\top \cdot Q_i^{(1)}(t)dt \\
+\sum_{i\in I}\sqrt{\nu_t^{(0)}(Q_i^{(0)}(t), i)}v_i^\top \cdot Q_i^{(1)}(t)d\Omega_i^*(t) + (Q_i^{(1)}(t))^\top \\
\cdot(\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t))^\top dt \\
+\sum_{i\in I}\sqrt{\nu_t^{(0)}(Q_i^{(0)}(t), i)}(Q_i^{(1)}(t))^\top \cdot v_i d\Omega_i^*(t) \\
+\sum_{i\in I}\nu_t^{(0)}(Q_i^{(0)}(t), i)v_i^\top \cdot v_i dt.
\end{aligned}
$$

Taking the expectations, we get

$$
\begin{aligned}
\tfrac{d}{dt}\mathbb{E}(Q_i^{(1)}(t))^\top, Q_i^{(1)}(t)) = \mathbb{E}\left((\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t))^\top \cdot Q_i^{(1)}(t)\right) \\
+\mathbb{E}\left(Q_i^{(1)}(t)\cdot(\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t))^\top\right) \\
+\sum_{i\in I}\nu_t^{(0)}(Q_i^{(0)}(t), i)v_i^\top \cdot v_i.
\end{aligned} \tag{2.47}
$$

for almost all t. Using the derivative of (2.45), we obtain

$$
\begin{aligned}
\mathbb{E}(Q_i^{(1)}(t))^\top)\mathbb{E}(Q_i^{(1)}(t)) = \mathbb{E}(\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t))^\top \cdot \mathbb{E}(Q_i^{(1)}(t) \\
+\mathbb{E}(Q_i^{(0)}(t))^\top \cdot \mathbb{E}(\wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) + \nu_t^{(1)}(Q_i^{(0)}(t)).
\end{aligned} \tag{2.48}
$$

Subtracting (2.49) from (2.47) gives us (2.40)-(2.44).

Now, observe that (2.45) can be written as

$$
\mathbb{E}(Q_i^{(1)}(t)) = \mathbb{E}(Q_i^{(1)}(0)) + \int_0^t \mathbb{E}(Q_i^{(1)}(s))A_s ds + \int_0^t \nu_s^{(1)}(Q^{(0)}(s)). \tag{2.49}
$$

With A_t is the matrix that represents its action on $\mathbb{V}$;

$$
\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t))) = Q_i^{(1)}(t))A_t,
$$

and $|A_t| \le \|\nu_t^{(0)}\|$ So, let

$$
cov(Q_i^{(1)}(t), \wedge\nu_t^{(0)}(Q_i^{(0)}(t), Q_i^{(1)}(t)))) = cov(Q_i^{(1)}(t), Q_i^{(1)}(t))))A_t,
$$

for almost all t, and so the integral equation for the covariance matrix is

$$cov(Q_i^{(1)}(t), Q_i^{(1)}(t)) = cov(Q_i^{(1)}(0), Q_i^{(1)}(0)) + \int_0^t cov(Q_i^{(1)}(s), Q_i^{(1)}(s))A_s ds$$
$$+ \int_0^t \sum_{i \in I} \nu_t^{(0)}(Q_i^{(0)}(t), i) v_i^\top \cdot v_i ds.$$

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Chapter 3

Stability Condition of a Retrial Queueing System with Abandoned and Feedback Customers

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Stability Condition of a Retrial Queueing System with Abandoned and Feedback Customers

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Abstract

This paper deals with the stability of a retrial queueing system with two orbits, abandoned and feedback customers. Two independent Poisson streams of customers arrive to the system, and flow into a single-server service system. An arriving one of type i, $i = 1, 2$ is handled by the server if it is free; otherwise, it is blocked and routed to a separate type-i retrial (orbit) queue that attempts to re-dispatch its jobs at its specific Poisson rate. The customer in the orbit either attempts service again after a random time or gives up receiving service and leaves the system after a random time. After the customer is served completely, it will decide either to join the retrial group again for another service or leave the system forever with some probability.

Keywords: Queueing system, call center, retrial queue, abandonment, feedback

2010 MSC No: 60K25; 68M20; 90B22

3.1 Introduction

In classical Queueing theory, it is assumed that any customer who cannot get service immediately upon arrival, either joins a waiting line or leaves the system forever. But there are real situations where the blocked customers leave the service area temporarily but returns to repeat their demand after some random time. This Queueing behavior is referred as retrial queues (Parveen and Begum, 2014).

Retrial queues are characterized by the feature that a customer who finds the server busy or down or on vacation, he/she may decide to join a group of blocked customers (called orbit) for repeating their demand, or request after some random amount of time, or leave the system immediately.

Retrial queues have wide applications, in case of many real life systems, these later can be applied in the performance modelling, for instance, in modelling magnetic disk memory system, cellular mobile networks, computer networks, and local area networks with non-persistent CSMA/CD protocols, with star topology, with random access protocols, and with multiple access protocols.

The study of retrial queue in queueing theory has been focused by many authors because of its wide applicability in web access, telephone switching systems, telecommunication networks and computer networks, and many daily life situations. Extensive survey articles in retrial queues are due to (Yang and Templeton, 1987) and (Falin, 1990). For an excellent scenario of retrial queues, monograph on this topic is given by (Falin and Templeton, 1997).

Feedback in queueing literature represents customer dissatisfaction because of inappropriate quality of service. In case of feedback, after getting partial or incomplete service, customer retries for service. In computer communication, the transmission of protocol data unit is sometimes repeated due to occurrence of an error. This usually happens because of non-satisfactory quality of service. Rework in industrial operations is also an example of a queue with feedback (Sharma and Kumar, 2014). (Takacs, 1963) was the first to study feedback queueing model, author studied queue with feedback to determine the stationary process for the queue size, and the the first two moments of the distribution function of the total time spent in the system by a customer. Studies on queue length, the total sojourn time and the waiting time for an $M/G/1$ queue with Bernoulli feedback were provided by (Vanden Berg and Boxma, 1991). (Choi et al., 1998) studied an $M/M/c$ retrial queueing model with geometric loss and feedback. (Santhakumaran and Thangaraj, 2000) considered a single server feedback queue with impatient and feedback customers. (Choudhury and Paul 2005) derived the queue size distribution at random epoch, and at a service completion epoch for $M/G/1$ queue with two phases of heterogeneous services, and Bernoulli feedback system. (Krishna Kumar et al., 2006) considered a generalized $M/G/1$ feedback queue in which customers are either "positive" or "negative".

(Thangaraj and Vanitha 2009) obtained transient solution of $M/M/1$ feedback queue with catastrophes using continued fractions, the steady-state solution, moments under steady state and busy period analysis are calculated. (Ayyapan et al., 2010) studied $M/M/1$ retrial queueing system with loss and feedback, under non preemptive priority service, by matrix geometric method. (Kumar and Sharma, 2012) analyzed a single server queueing system with retention of reneged customers. (Arivudainambi and Godhandaraman, 2012) considered a batch arrival queueing system with two phases of service, feedback and K optional vacations under a classical retrial policy. (Bouchentouf et al., 2014) analyzed a queueing model with two heterogeneous servers balking, reneging and feedback.

A queueing system with two orbits, and two exogenous streams of different types serves as a model for two competing job streams in a carrier sensing multiple access system, where the jobs, after a failed attempt to network access, wait in an orbit queue (Nain, 1985; Szpankowski, 1994). The retrial queueing systems with a constant retrial rate and a single type of jobs were considered in (Fayolle, 1986; Choi et al., 1993a; Choi et al., 1993b; Artalejo et al., 2001; Avrachenkov & Yechiali, 2010). A two-class retrial system with a single server, no waiting room, batch arrivals and classical retrial scheme was introduced and analyzed in (Kulkarni, 1986). Then, in (Falin, 1988) author extended the analysis of the model in (Kulkarni, 1986) to the multi-class setting with arbitrary number of classes. In (Grishechkin, 1992) author has established equivalence between the multi-class batch arrival retrial queues with classical retrial policy and branching processes with immigration. In (Moutzoukis & Langaris, 1996) a non-preemptive priority mechanism was added to the model of (Falin, 1988 & Kulkarni, 1986). In (Langaris & Dimitriou, 2010) authors considered a multi-class retrial system where retrial classes are associated with different phases of service. Retrial queueing model $MMAP/M_2/1$ with two orbits was studied in (Avrachenkov et al., 2010), authors considered a retrial single-server queueing model with two types of customers. In case of the server occupancy at the arrival epoch, the customer moves to the orbit depending on the type of the customer, one orbit is infinite while the second one is a finite. Joint distribution of the number of customers in the orbits, and some performance measures are computed. In (Bouchentouf & Belarbi, 2013) authors considered two retrial queueing system with balking and feedback, the joint generating function of the number of busy server, and the queue length was found by solving Kummer differential equation, and by the method of series solution. In (Avrachenkov et al., 2014) authors analyzed a retrial model with two input streams and two orbit queues. (Bouchentouf et al., 2014) gave a note on fluid approximation of retrial queueing system with two orbits, abandonment and feedback. And in (Bouchentouf & Sakhi, 2015) authors presented a note on an $M/M/s$ queueing system with two reconnect and two redial orbits.

So, motivated by the need to analyze retrial queueing networks, and by the need to develop analytical tools that support performance analysis of many large telecommunication systems (call centers), where abandonments, retrial and feedback arise naturally and are prevalent, we study in the present paper, a retrial queueing model with abandonment and feedback customers; system with two orbits, constant retrials, abandoned and feedback customers is carried out.

The layout of the paper is given as follows. After the introduction (section 1), the retrial queueing model is described (section 2). In section 3, we give the main result; we formulate our retrial system as a three-dimensional Markovian queueing network, then we derive balance equations and generating functions, and finally we give its necessary stability condition.

3.2 The model

The Markovian retrial queueing system with two classes of jobs and constant retrial, abandonment and feedback customers is considered (Figure 3.1).

Two independent Poisson streams of jobs, S_1 and S_2, flow into a single-server service system. The service system can hold at most one job. The arrival rate of stream S_i is α_i, $i = 1, 2$, with $\alpha_1 + \alpha_2 = \alpha$. The required service time of each job is independent of its type and is exponentially distributed with mean $1/\mu$. If an arriving type-i job finds the (main) server busy, it is routed to a dedicated retrial (orbit) queue from which jobs are re-transmitted at an exponential rate. The rates of retransmissions may be different from the rates of the original input streams. So, the blocked jobs of type i form a type-i single-server orbit queue that attempts to retransmit jobs (if any) to the main service system at a Poisson rate of γ_i, $i = 1, 2$. This creates a system with three dependent queues. The customer in the orbit either attempts service again after a random time or gives up receiving service and leaves the system after a random time at rate δ_i $i = 1, 2$. After the customer is served completely, it will decide either to join the retrial group again for another service with probability β or to leave the system forever with probability $\bar{\beta} = 1-\beta$.

Let $C(t)$ denotes the number of jobs in the main queue. $C(t)$ takes the values of 0 or 1.

Let $N_i(t)$ be the number of jobs in orbit queue i, $i = 1, 2$.
The Markov process $\{(N_1(t), N_2(t), C(t)) : t \in [0, +\infty)\}$ is irreducible on the state-space $\{0, 1, ...\} \times \{0, 1, ...\} \times \{0, 1\}$.

Such a network can serve as a model for two competing job streams in a carrier sensing multiple access system "CSMA". Local Area Computer Network (LAN) can be an example of CSMA.

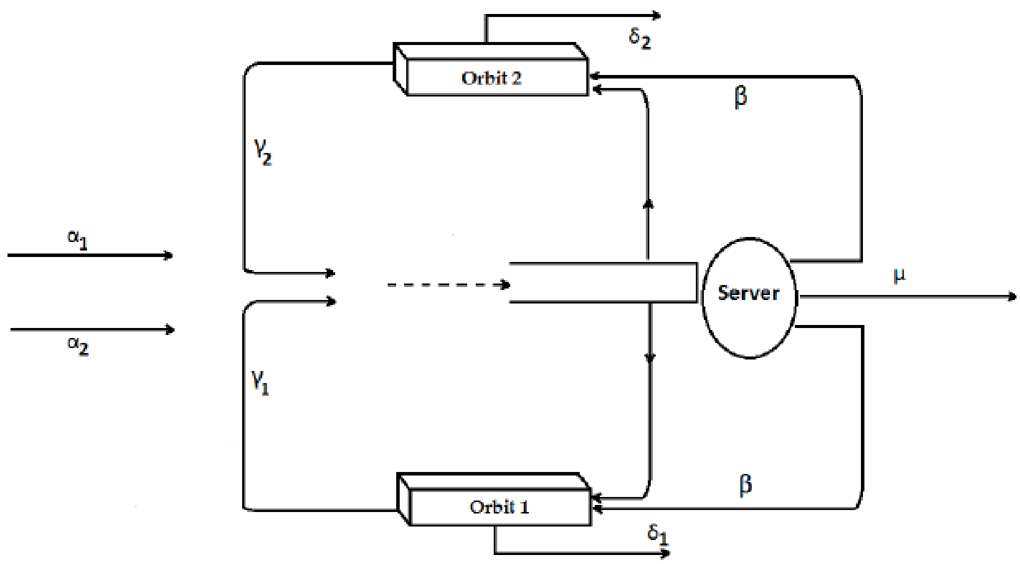

Figure 3.1: A queueing model with two orbits, abandonment and feedback

3.3 Main Result: Necessary stability condition

The main objective of this work is to give the necessary stability condition of a retrial queueing system with two orbits, constant retrials, abandoned and feedback customers, the main result is given in the following proposition.

Proposition 3.3.1. *The following condition*

$$\frac{\alpha(\gamma_1 + \delta_1)(\gamma_2 + \delta_2)}{[\alpha + (\beta + 1)\mu](\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\delta_1\gamma_2 - \alpha_2\delta_2\gamma_1} \left(1 + \frac{\alpha_i}{\gamma_i + \delta_i}\right) < 1, \quad (3.1)$$

for $i = 1, 2$ and

$$[\alpha + (\beta + 1)\mu](\gamma_1 + \delta_1)(\gamma_2 + \delta_2) - \alpha\gamma_1\gamma_2 - \alpha_1\delta_1\gamma_2 - \alpha_2\delta_2\gamma_1 \neq 0 \quad (3.2)$$

is necessary for the stability of the system.

Proof.

To prove that our Markovian retrial queueing system with two classes of jobs and constant retrial rates, abandonment and feedback customers is stable.

At first, we have to show that

$$\frac{\alpha(\gamma_1+\delta_1)(\gamma_2+\delta_2)}{[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1}\leq 1 \tag{3.3}$$

and for $i=1,2$

$$\frac{\alpha(\gamma_1+\delta_1)(\gamma_2+\delta_2)}{[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1}\left(1+\frac{\alpha_i}{\gamma_i+\delta_i}\right)\leq 1 \tag{3.4}$$

are necessary conditions for the existence of a steady-state.

♦ Let us present the balance equations, and generating functions. So, consider the system in steady-state, where we define by (N_1, N_2, C) the stationary version of the Markov chain $\{(N_1(t), N_2(t), C(t)) : t \in [0,\infty)\}$. Define the set of stationary probabilities $P_{n_1n_2}(c)$ as follows:

$$P_{n_1n_2}(c) = \lim_{t\to\infty} P(N_1(t)=n_1, N_2(t)=n_2, L(t)=c)$$

$$= P(N_1=n_1, N_2=n_2, C=c),$$

for $n_1, n_2 = 0, 1, ...,$ and $c = 0, 1$, when these limits exist. Define the marginal probabilities

$$P_{n_1\cdot}(c) = \sum_{n_2=0}^{\infty} P_{n_1n_2}(c) = P(N_1=n_1, C=c), n_1 = 0,1,2,..., \quad c=0,1$$

and

$$P_{\cdot n_2}(c) = \sum_{n_1=0}^{\infty} P_{n_1n_2}(c) = P(N_2=n_2, C=c), n_2 = 0,1,2,..., \quad c=0,1.$$

Now, let us write the balance equations

1. $N_2 = n_2 = 0$

1.1. $N_1 = n_1 = 0, \quad c = 0$

$$\alpha P_{00}(0) = \overline{\beta}\mu P_{00}(1) + \delta_1 P_{10}(0) + \delta_2 P_{01}(0). \tag{3.5}$$

1.2. $N_1 = n_1 \geq 1, \;\; c = 0$

$$(\alpha + \gamma_1 + \delta_1) P_{n_1 0}(0) = \overline{\beta}\mu P_{n_1 0}(1) + \beta\mu P_{n_1-10}(1) + \delta_1 P_{n_1+10}(0) + \delta_2 P_{n_1 1}(0). \tag{3.6}$$

1.3. $N_1 = 0, \;\; c = 1$

$$\begin{aligned}(\alpha + \mu) P_{00}(1) &= \alpha P_{00}(0) + \gamma_1 P_{10}(0) + \gamma_2 P_{01}(0) + \delta_1 P_{10}(1) \\ &+ \delta_2 P_{01}(1).\end{aligned} \tag{3.7}$$

1.4. $N_1 = n_1 \geq 1, \;\; c = 1$

$$\begin{aligned}(\alpha + \mu + \delta_1) P_{n_1 0}(1) &= \alpha P_{n_1 0}(0) + \gamma_1 P_{n_1+10}(0) + \gamma_2 P_{n_1 1}(0) \\ &+ \delta_1 P_{n_1+10}(1) + \delta_2 P_{n_1 1}(1) + \alpha_1 P_{n_1-10}(1).\end{aligned} \tag{3.8}$$

2. $N_2 = n_2 \geq 1$

2.1. $N_1 = 0 \;\; c = 0$

$$\begin{aligned}(\alpha + \gamma_2 + \delta_2) P_{0n_2}(0) &= \overline{\beta}\mu P_{0n_2}(1) + \beta\mu P_{0n_1-1}(1) + \delta_1 P_{1n_2}(1) \\ &+ \delta_2 P_{0n_2+1}(0).\end{aligned} \tag{3.9}$$

2.2. $N_1 = n_1 \geq 1 \;\; c = 0$

$$\begin{aligned}(\alpha + \gamma_1 + \gamma_2 + \delta_1 + \delta_2) P_{n_1 n_2}(0) &= \beta\mu P_{n_1-1n_2}(1) + \beta\mu P_{n_1 n_2-1}(1) \\ &+ \overline{\beta}\mu P_{n_1 n_2}(1) + \delta_1 P_{n_1+1n_2}(0) + \delta_2 P_{n_1 n_2+1}(0).\end{aligned} \tag{3.10}$$

2.3. $N_1 = 0 \;\; c = 1$

$$\begin{aligned}(\alpha + \mu + \delta_2) P_{0n_2}(1) &= \alpha P_{0n_2}(0) + \gamma_1 P_{1n_2}(0) + \gamma_2 P_{0n_2+1}(0) + \delta_1 P_{1n_2}(1) \\ &+ \delta_2 P_{0n_2+1}(1) + \alpha_2 P_{0n_2-1}(1).\end{aligned} \tag{3.11}$$

2.4. $N_1 = n_1 \geq 1 \;\; c = 1$

$$\begin{aligned}&(\alpha + \mu + \delta_1 + \delta_2) P_{n_1 n_2}(1) = \alpha P_{n_1 n_2}(0) + \gamma_1 P_{n_1+1n_2}(0) + \gamma_2 P_{n_1 n_2+1}(0) \\ &+ \delta_1 P_{n_1+1n_2}(1) + \delta_2 P_{n_1 n_2+1}(1) + \alpha_1 P_{n_1-1n_2}(1) + \alpha_2 P_{n_1 n_2-1}(1).\end{aligned} \tag{3.12}$$

The probability generating function of the stationary version of the Markov process $\{(N_1(t), N_2(t), C(t)) : t \in [0, +\infty)\}$ is given by

$$F(z_1, z_2, z) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{c=0}^{1} P_{n_1 n_2}(c) z_1^{n_1} z_2^{n_2} z^c. \tag{3.13}$$

Let us also define the following (partial) probability generating function

$$R_{n_2}^{(c)}(z_1) = \sum_{n_1=0}^{\infty} P_{n_1 n_2}(c) z_1^{n_1}, \quad c = 0, 1, n_2 = 0, 1, ...,$$

and

$$F^{(c)}(z_1, z_2) = \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} P_{n_1 n_2}(c) z_1^{n_1} z_2^{n_2} = \sum_{n_2=0}^{\infty} R_{n_2}^{(c)}(z_1) z_2^{n_2}, \quad c = 0, 1. \tag{3.14}$$

$$F(z_1, z_2, z) = F^{(0)}(z_1, z_2) + zF^{(1)}(z_1, z_2), \quad |z_1| \le 1, \quad |z_2| \le 1.$$

For $n_2 = 0$ and $c = 0$ we multiply (3.5) and (3.6) by $z_1^{n_1}$, we get

$$\begin{aligned} &((\alpha + \gamma_1 + \delta_1)z_1 - \delta_1)\, R_0^{(0)}(z_1) - ((\gamma_1 + \delta_1)z_1 - \delta_1)\, P_{00}(0) \\ &= \left(\overline{\beta}\mu z_1 + \beta\mu z_1^2\right) R_0^{(1)}(z_1) + \delta_2 z_1 R_1^{(0)}(z_1). \end{aligned} \tag{3.15}$$

For $n_2 = 0$ and $c = 1$ we multiply (3.7) and (3.8) by $z_1^{n_1}$, we get

$$\begin{aligned} &((\alpha + \mu + \delta_1)z_1 - \delta_1 - \alpha_1 z_1^2)\, R_0^{(1)}(z_1) - (\delta_1 z_1 - \delta_1) P_{00}(1) \\ &= (\alpha z_1 + \gamma_1) R_0^{(0)}(z_1) - \gamma_1 P_{00}(0) + \delta_2 z_1 R_1^{(1)} + \gamma_2 z_1 R_1^{(0)}. \end{aligned} \tag{3.16}$$

For $n_2 \ge 1$ and $c = 0$ we multiply (3.9) and (3.10) by $z_1^{n_1}$, we get

$$\begin{aligned} &((\alpha + \gamma_1 + \gamma_2 + \delta_1 + \delta_2)z_1 - \delta_1)\, R_{n_2}^{(0)}(z_1) - ((\gamma_1 + \delta_1)z_1 - \delta_1)\, P_{0n_2}(0) \\ &= \left(\overline{\beta}\mu z_1 + \beta\mu z_1^2\right) R_{n_2}^{(1)}(z_1) + \beta\mu z_1^2 R_{n_2-1}^{(1)}(z_1) + \delta_2 z_1 R_{n_2+1}^{(1)}(z_1). \end{aligned} \tag{3.17}$$

For $n_2 \ge 1$ and $c = 1$ we multiply (3.11) and (3.15) by $z_1^{n_1}$, we get

$$\begin{aligned} &((\alpha + \mu + \delta_1 + \delta_2)z_1 - \delta_1 - \alpha_1 z_1^2)\, R_{n_2}^{(1)}(z_1) - (\delta_1 z_1 - \delta_1) P_{0n_2}(1) \\ &= (\alpha z_1 + \gamma_1) R_{n_2}^{(0)} z_1 + \gamma_2 z_1 R_{n_2+1}^{(0)}(z_1) + \alpha_2 z_1 R_{n_2-1}^{(1)}(z_1) \\ &+ \delta_2 z_1 R_{n_2+1}^{(1)}(z_1) - \gamma_1 P_{0n_2}(0). \end{aligned} \tag{3.18}$$

Using equations (3.15) and (3.17) then multiplying by $z_2^{n_2}$, we get

$$\begin{aligned}
&(z_1(\alpha+\gamma_1+\gamma_2+\delta_1+\delta_2)z_2-\delta_1 z_2-\delta_2 z_1)\, F^{(0)}(z_1,z_2)\\
&-(z_1(\gamma_2+\delta_2)z_2-\delta_2 z_1)\, F^{(0)}(z_1,0)-(z_1(\gamma_1+\delta_1)z_2-\delta_1)\, z_2 F^{(0)}(0,z_2)\\
&=\left(\overline{\beta}\mu z_1 z_2+\beta\mu z_1^2 z_2+\beta\mu z_1^2 z_2^2\right) F^{(1)}(z_1,z_2).
\end{aligned}\tag{3.19}$$

We do the same with equations (3.16) and (3.18)

$$\begin{aligned}
&(z_1(\alpha+\mu+\delta_1+\delta_2)z_2-\alpha_1 z_1^2 z_2-\delta_1 z_2-\delta_2 z_1-\alpha_2 z_1 z_2^2)\, F^{(1)}(z_1,z_2)\\
&-(z_1 z_2\delta_2-\delta_2 z_1)F^{(1)}(z_1,0)+\gamma_2 z_1 F^{(0)}(z_1,0)\\
&-z_2(\delta_1 z_1-\delta_1)F^{(1)}(0,z_2)+\gamma_1 F^{(0)}(0,z_2)\\
&=((\alpha z_1+\gamma_1)z_2+\gamma_2 z_1)\, F^{(0)}(z_1,z_2).
\end{aligned}\tag{3.20}$$

Let

$$\alpha_1 P_{n_1\cdot}(1)=\gamma_1 P_{n_1+1\cdot}(0)+\delta_1 P_{n_0+1\cdot}(1).\tag{3.21}$$

Summing over n_1, we get

$$1-F^{(0)}(0,1)=\left(1+\frac{\alpha_1}{\gamma_1+\delta_1}\right)F^{(1)}(1,1).\tag{3.22}$$

Then by symmetry

$$1-F^{(0)}(1,0)=\left(1+\frac{\alpha_2}{\gamma_2+\delta_2}\right)F^{(1)}(1,1).\tag{3.23}$$

Then

$$F^{(1)}(1,1)=\frac{\alpha(\gamma_1+\delta_1)(\gamma_2+\delta_2)}{[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1},\tag{3.24}$$

with $[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1\neq 0$.

Secondly, we have to prove that for i=1,2, if

$$\frac{\alpha(\gamma_1+\delta_1)(\gamma_2+\delta_2)}{[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1}\left(1+\frac{\alpha_i}{\gamma_i+\delta_i}\right)=1,\tag{3.25}$$

then both queues N_1 and N_2 are unbounded with probability one.

✠ Assume, for instance, that

$$\frac{\alpha(\gamma_1+\delta_1)(\gamma_2+\delta_2)}{[\alpha+(\beta+1)\mu](\gamma_1+\delta_1)(\gamma_2+\delta_2)-\alpha\gamma_1\gamma_2-\alpha_1\delta_1\gamma_2-\alpha_2\delta_2\gamma_1}\left(1+\frac{\alpha_i}{\gamma_i+\delta_i}\right)=1,$$

so that $F^{(0)}(1,0)=0$ from (3.23). Since $F^{(0)}(1,0)=\sum_{n_1=0} P_{n_1,0}(0)$ (see (3.14)), the condition $F^{(0)}(1,0)=0$ implies that

$$P_{n_1,0}(0)=0 \text{ for } n_1=0,1,..., \tag{3.26}$$

so that from (3.5) to (3.6)

$$P_{n_1,0}(1)=0 \text{ for } n_1=0,1,.... \tag{3.27}$$

We now use an induction argument to prove that

$$P_{n_1,n_2}(0)=0 \text{ for } n_1,n_2=0,1,.... \tag{3.28}$$

We have already shown in (3.26) that (3.28) is true for $n_2=0$. Assume that (3.28) is true for $n_2=0,1,...,k$ and let us show that it is still true for $n_2=k+1$.

From (3.10) and the induction hypothesis we get that $P_{n_1,k}(0)=P_{n_1,k}(1)=0$ for $n_1=1,2,....$ The latter equality implies, using (3.15), that $P_{n_1,k+1}(0)=0$. This shows that (3.28) holds for $n_1=0,1,...,$ and $n_2=k+1$, and completes the induction argument, proving that (3.28) is true. We have therefore proved that $P_{n_1,n_2}(0)=0$ for all $n_1,n_2=0,1,....$

Let us prove that $P_{n_1,n_2}(1)=0$ for all $n_1,n_2=0,1,....$ The latter is true for $n_1,n_2=1,2,...,$ (3.10). It is also true for $n_2=0, n_1=0,1,...,$ from (3.27).

It remains to investigate the case where $n_1=0$ and $n_2=0,1,....$

By (3.9) and (3.28) we get that $P_{0,n_2}(1)=0$ for $n_2=1,2,...,$ whereas we have already noticed that $P_{0,0}(1)=0$. In summary, $P_{n_1,n_2}(0)=P_{n_1,n_2}(1)=0$ for all $n_1,n_2=0,1,...,$ so that $P(N_1=n_1,N_2=n_2)=P_{n_1,n_2}(0)+P_{n_1,n_2}(1)$ for all $n_1,n_2=0,1,...,$ which completes the proof.

4. Conclusion

In this article, a Markovian retrial queueing system with two classes of jobs and constant retrial, abandonment and feedback customers is studied. A necessary condition for the stability of this system is derived. For further work, it will be interesting to analyze the sufficient condition for the stability of the system, to this end we have to obtain the generating functions for this system via the solution of a Riemannian Hilbert boundary value technique.

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Chapter 4

On feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption

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On feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption

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Abstract

This paper presents an analysis of a Markovian feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption, where customers' impatience is due to the servers' vacation. The reneging times are assumed to be exponentially distributed. After the completion of service, each customer may reenter the system as a feedback customer for receiving another regular service with some probability or leave the system. A reneged customer can be retained in many cases by employing certain convincing mechanisms to stay in queue for completion of service. Thus, a reneged customer can be retained in the queueing system with some probability or he may leave the queue without receiving service.

We establish the stationary analysis of the system. The probability generating functions of the stationary state probabilities is obtained, we deduce the explicit expressions of the system sizes when the server is in a normal service period and in a Bernoulli schedule vacation interruption, respectively. Various performance measures of the system are derived. Finally, we present some numerical examples to demonstrate how the various parameters of the model influence the behavior of the system.

Keywords: Queueing systems, Markovian model, reneging, feedback, multiple working vacations, Bernoulli schedule vacation interruption

subclass:60K25; 68M20; 90B22

4.1 Introduction

Queueing is a prevalent phenomenon in our daily lives. At this time, queueing theory is very important in studying scheduling and system performance, it is also an all powerful tool to solve various problems in many complex systems, such as computer systems, telecommunication systems, call centers, flexible manufacturing systems and service systems. During the past few decades, there has been increasing interest in studying queueing systems.

Queueing systems with customers' impatience and server vacations have been widely studied because of their wide applications in real-life congestion problems such as communication systems, telecommunication systems, traffic systems, and manufacturing/production systems. The customer's impatient behavior should be needed in the study of queueing system to model real conditions precisely. Occasional operation of a service may be economically invoking when entire time service would result in substantial server idle time or would prevent the utilization of the server in different productive capacities. On the other hand, the server remaining not working for periods of time might gain the probability of customer losses due to balking and reneging Goswami [10].

Vacation queues have been greatly analyzed, Ke et al. [13] provided a succinct summary of the most recent research works on vacation queueing systems in the last decade, Tian and Zhang [25] discussed many variations of vacation policy, and a variety of typical vacation model applications that include call centers with multi-task employees, customized manufacturing, telecommunication systems, maintenance activities, etc are also studied. Yue et al. [27] presented an analysis for an $M/M/1/N$ queueing system with balking, reneging and server vacations. By using the Markov process method, authors developed the equations of the steady state probabilities, then, they derived the matrix form solution of the steady-state probabilities, and gave some performance measures of the system, after that they formulated a cost model to determine the optimal service rate.

There are several situations where the server stays active during the vacation period. The server can provide service at a lower speed during the working vacation period instead of stopping service completely. If the queue is empty at the end of a vacation, the server takes another vacation; otherwise a service period begins with normal service rate Goswami [10].

Queueing systems with vacation interruption have been investigated by many authors, Baba [6] studied an $M/PH/1$ queue with phase type working vacation and vacation interruption where the vacation time follows a phase type distribution, Chen et al. [7] Considered a $GI/M/1$ queue with phase-type working vacations and vacation interruption where the vacation time follows a phase-type distribution. Li and Tian [18] studied the $M/M/1$ queue with working vacations and vacation interruptions, Zhang and Hou [29] analyzed an $M/G/1$ queue with a working vacations and vacation interruption. Using the method of a supplementary variable and the matrix-analytic method, authors obtained the queue length distribution and service status at an arbitrary epoch under steady state conditions. Zhang and Shi [30] presented an $M/M/1$ queue with Bernoulli schedule vacation and vacation interruption. Altman and Yechiali [2] considered the impatience of customers only when the servers are on vacation and unavailable for service. Selvaraju and Goswami [21] analyzed impatient customers in a single server Markovian queue with single and multiple working vacations.

Many practical queueing systems especially those with balking and reneging have been widely applied to many real-life problems, such as the situations involving impatient telephone switchboard customers, the hospital emergency rooms handling critical patients, and the inventory systems with storage of perishable goods Robert [19]. Haight [11] considered an $M/M/1$ queue with balking. An $M/M/1$ queue with customers reneging was also proposed by Haight [12]. The combined effects of balking and reneging in an $M/M/1/N$ queue have been investigated by Ancker and Gafarian [3, 4]. Abou-EI-Ata and Hariri [1] considered the multiple servers queueing system $M/M/c/N$ with balking and reneging. Wang and Chang [26] extended this work to study an $M/M/c/N$ queue with balking, reneging and server breakdowns. Laxmi et al. [17] studied $M/M/1/N$ working vacations queue with balking and reneging. Yue et al. [28] analyzed an $M/M/1$ queueing system with working vacations and impatient customers, authors derived the probability generating functions of the number of customers in the system when the server is in a service period and a working vacation, respectively, then they obtained the closed-form expressions for various performance measures.

Feedback in queueing literature represents customer dissatisfaction because of inappropriate quality of service. In case of feedback, after getting partial or incomplete service, customer retries for service. In computer communication, the transmission of protocol data unit is sometimes repeated due to occurrence of an error. This usually happens because of non-satisfactory quality of service. Rework in industrial operations is also an example of a queue with feedback Kumar and Sharma [16]. Takacs [23] studied queue with feedback to determine the stationary process for the queue size and the first two moments of the distribution function of the total time spent in the system by a customer. In [8] D'Avignon and Disney studied single server queues with state dependent feedback. Santhakumaran and Thangaraj [20] considered a single server feedback queue with impatient and feedback customers, they studied $M/M/1$ queueing model for queue length at arrival epochs and obtained result for stationary distribution, mean and variance of queue length. Thangaraj and Vanitha [24] obtained transient solution of $M/M/1$ feedback queue with catastrophes using continued fractions, the steady-state solution, moments under steady state and busy period analysis were calculated. Ayyapan et. al [5] studied $M/M/1$ retrial queueing system with loss and feedback under non preemptive priority service by matrix geometric method. Kumar and Sharma [14] studied a single server queueing system with retention of reneged customers. Kumar and Sharma [15] studied a single server queueing system with retention of reneged customers and balking. Sharma and Kumar [22] considered a single server, finite capacity Markovian feedback queue with reneging, balking and retention of reneged customers in which the inter-arrival and service times follow exponential distribution. Mahdy El-Paoumy and Hossam Nabwey [9] studied the $M/M/2/N$ queue with general balk function, reneging and two heterogeneous servers. In [10], Goswami analyzed customers' impatience in Markovian queueing system with multiple working vacations and Bernoulli schedule vacation interruption, where customers' impatience is due to the servers' vacation.

In this paper, we consider a single-server Markovian feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption, where customers' impatience is due to the servers' vacation. During the working vacation period, if there are customers in the queue, the vacation can be interrupted at a service completion instant and the server begins a regular busy period with probability $1-\beta$ or continues the vacation with probability β. The reneging times are assumed to be exponentially distributed. After the completion of service (which can be partial or incomplete), each customer may rejoin the system as a feedback customer for receiving another regular service with probability ν or he can leave the system with probability γ where $\nu+\gamma=1$. A reneged customer can be retained in many cases by employing certain convincing mechanisms to stay in queue for completion of service. Thus, a reneged customer can be retained in the queuing system with some probability σ or he may leave the queue without receiving service with probability δ $(=1-\sigma)$.

We obtain the probability generating functions of the stationary state probabilities and deduce the explicit expressions of the system sizes when the server is in a normal service period and in a Bernoulli schedule vacation interruption, respectively. Various performance measures such as the mean system size, the proportion of customers served, the rate of abandonment due to impatience, and the mean sojourn time of a customer served are derived. Finally, we present some numerical examples to demonstrate how the various parameters of the model influence the behavior of the system.

The rest of the paper is arranged as follows: In section 4.2, we describe the model, then we give in section 3 the main result; the probability generating functions of the stationary state probabilities are obtained, then the explicit expressions of the system sizes when the server is in a normal service period and in a Bernoulli schedule vacation interruption are given, respectively. After that in section 4.3, various performance measures such as the expected number of customers in the system when the server is on a working vacation period and in a normal busy period, the expected number of customers in the system, the proportion of customers served, the rate of abandonment due to impatience, and the mean sojourn time of a customer served are derived. Finally, we finish the paper by a small conclusion.

4.2 Description of the model

We consider the multiple working vacations $M/M/1$ queueing system with Bernoulli schedule vacation interruptions, reneging, retention of reneged customers and feedback, where the customers become impatient due to the servers' vacation. Customers arrive according to a Poisson process with arrival rate λ. The service times during a normal service period, the service times during a working vacation period, and the vacation times are exponentially distributed with rates μ, α, and θ, respectively. The customers are assumed to be impatient during the multiple working vacations. Whenever a customer arrives at the system and finds the system is on working vacation, the customer activates an impatient timer T, which is exponentially distributed with rate ϑ. If the server finishes the working vacation before the impatience timer expires, the customer remains in the system till his service completion. However, if the impatience timer expires when the server is still on working vacation, the customer abandons the system, this time is reneging time of an individual customer. The reneged customer can be retained in the system with some probability σ or he may abandon the system with complementary probability $\delta(1-\sigma)$. During the working vacation period, a customer is serviced at a lower rate, and, at the instants of the service completion, the vacation is interrupted and the server resumes a regular busy period with probability $1-\beta$ (if there are customers in the queue) or remains in the vacation with probability β. The inter-arrival times, service times, vacation duration times, and the impatient times all are taken to be mutually independent. The customers are served on a first come first-served queue discipline. After completion of each service, the customer can either join the end of the queue with probability ν or he can leave the system with probability γ where $\nu + \gamma = 1$. The customers both newly arrived and those that are fed back are served in order in which they join the tail of original queue. We do not distinguish between the regular arrival and feedback arrival.

Let $N(t)$ denote the number of customers in the system at time t, and let $S(t)$ denote the state of the server at time t with

$$S(t) = \begin{cases} 0, & \text{if the server is in working vacation period,} \\ 1, & \text{if the server is in normal busy period.} \end{cases}$$

Then, the pair $\{(N(t), St)); t \geq 0\}$ is a continuous-time Markov process with state space $\Omega = [\{(0,0) \cup (i,j)\},\ \ i = 1, 2, ...,\ \ j = 0, 1]$. Let $\pi_{ij} = \underset{t\to\infty}{lim} \mathbb{P}\{N(t) = i, S(t) = j\},\ \ (i,j) \in \Omega$.

Let π_{i0}, $i \geq 0$ be the probability that there are i customers in the system when the server is in working vacation period and let π_{i1}, $i \geq 1$ be the probability that there are i customers in the system when the server is in normal busy period.

4.3 Main Result

In this part of paper, we study a stationary analysis for our model. At first, we develop the probability generating functions of the number of customers in the system when the server is in a working vacation period and in a normal service period, respectively. Then, we derive the explicit expressions for various performance measures.

Via the Markov process theory, we get the following set of steady-state equations:

$$\lambda\pi_{00} = (\delta\vartheta + \gamma\alpha)\pi_{10} + \gamma\mu\pi_{11} \tag{4.1}$$

$$(\lambda + \gamma\alpha + \theta + n\delta\vartheta)\pi_{n0} = \lambda\pi_{n-10} + (\beta\gamma\alpha + (n+1)\delta\vartheta)\pi_{n+10},\ \ n \geq 1 \tag{4.2}$$

$$(\lambda + \gamma\mu)\pi_{11} = \theta\pi_{10} + \overline{\beta}\gamma\alpha\pi_{20} + \gamma\mu\pi_{21} \tag{4.3}$$

$$(\lambda + \gamma\mu)\pi_{n1} = \theta\pi_{n0} + \lambda\pi_{n-11} + \gamma\mu\pi_{n+11} + \overline{\beta}\gamma\alpha\pi_{n+10},\ \ n \geq 2 \tag{4.4}$$

where $\overline{\beta} = 1 - \beta$. Let us define the probability generating functions as

$$\Pi_0(z) = \sum_{n=0} \pi_{n0} z^n,\ \ \Pi_1(z) = \sum_{n=1} \pi_{n1} z^n, \tag{4.5}$$

where $\Pi_0(1) + \Pi_1(1) = 1$ and $\Pi_0'(z) = \sum_{n=1} n z^{n-1} \pi_{n0}$.

Multiplying the appropriate power of z^n in (4.1) and (4.2), in (4.3) and (4.4), respectively, then summing over all possible values of n yield

$$\begin{aligned} &\delta\vartheta z(1-z)\Pi_0'(z) + \left(\lambda z^2 - (\lambda + \gamma\alpha + \theta)z + \beta\gamma\alpha\right)\Pi_0(z) + \\ &\left(\gamma\mu\pi_{11} + (\theta + \overline{\beta}\gamma\alpha)\pi_{0,0} + \overline{\beta}\gamma\alpha\pi_{10}\right)z + \beta\gamma\alpha\pi_{0,0}(1-z) = 0, \end{aligned} \tag{4.6}$$

and

$$(\lambda z-\gamma\mu)(1-z)\Pi_1(z) = (\theta z+\overline{\beta}\gamma\alpha)\Pi_0(z)-\big(\gamma\mu\pi_{11} + (\theta + \overline{\beta}\gamma\alpha)\pi_{0,0}\big)+\overline{\beta}\gamma\alpha\pi_{10}z-\overline{\beta}\gamma\alpha(1-z)\pi_{0,0}. \tag{4.7}$$

Then, solving Equation (4.6), we get

$$\Pi_0(z) = \frac{-\left(\gamma\mu\pi_{11} + (\theta + \overline{\beta}\gamma\alpha)\pi_{0,0} + \overline{\beta}\gamma\alpha\pi_{10}\right)\Phi_1(z) + \beta\gamma\alpha\Phi_2(z)}{\delta\vartheta e^{-(\lambda/\delta\vartheta)z} z^{\beta\gamma\alpha/\delta\vartheta}(1-z)^{(\theta+\overline{\beta}\gamma\alpha)/\delta\vartheta}}, \tag{4.8}$$

where

$$\Phi_1(z) = \int_0^z e^{-(\lambda/\delta\vartheta)x} x^{\beta\gamma\alpha/\delta\vartheta}(1-x)^{(\theta+\overline{\beta}\gamma\alpha)/\delta\vartheta-1}dx \tag{4.9}$$

$$\Phi_2(z) = \int_0^z e^{-(\lambda/\delta\vartheta)x} x^{\beta\gamma\alpha/\delta\vartheta-1}(1-x)^{(\theta+\overline{\beta}\gamma\alpha)/\delta\vartheta}dx, \tag{4.10}$$

where $\delta \neq 0,\ \ \vartheta \neq 0,\ \ \ x \neq 0$ and $\delta\vartheta \neq 1$.

Now, we should find the probabilities $\pi_{0,0}$, $\pi_{1,0}$, and $\pi_{1,1}$, then some important performance measures are obtained.

Assume that $\mathbb{E}(L_0)$ and $\mathbb{E}(L_1)$ are the expected number of customers in the system when the server is on a working vacation period and in a normal busy period, respectively.

So, adding (4.3) and (4.4) over all possible values of n, we obtain

$$(\theta + \overline{\beta}\gamma\alpha)\Pi_0(1) = \big(\gamma\mu\pi_{11} + (\theta + \overline{\beta}\gamma\alpha)\pi_{0,0}\big) + \overline{\beta}\gamma\alpha\pi_{10}. \tag{4.11}$$

And using (4.7), we have

$$\Pi_1(z) = \frac{(\theta z + \overline{\beta}\gamma\alpha)\Pi_0(z) - z(\theta + \overline{\beta}\gamma\alpha)\Pi_0(1)}{(\lambda z - \gamma\mu)(1-z)} - \frac{\overline{\beta}\gamma\alpha\pi_{0,0}}{(\lambda z - \gamma\mu)}. \tag{4.12}$$

Using L'Hopital's rule, we get

$$\Pi_1(1) = \left(\frac{\theta + \bar{\beta}\gamma\alpha}{\gamma\mu - \lambda}\right)\Pi_0'(1) + \left(\frac{\bar{\beta}\gamma\alpha}{\gamma\mu - \lambda}\right)(\pi_{00} - \Pi_0(1)), \tag{4.13}$$

where $\Pi_0'(1) = \mathbb{E}(L_0)$; the expected number of customers in the system when the server is on a working vacation period.

Since $\Pi_1(1) = 1 - \Pi_0(1)$, and by using (4.13), we obtain

$$\mathbb{E}(L_0) = \left(\frac{\gamma\mu - \lambda}{\theta + \bar{\beta}\gamma\alpha}\right)(1 - \Pi_0(1)) - \left(\frac{\bar{\beta}\gamma\alpha}{\theta + \bar{\beta}\gamma\alpha}\right)(\pi_{00} - \Pi_0(1)). \tag{4.14}$$

Now, We have to deduce the proportion of time the server is on a working vacation period $(\Pi_0(1))$, so that $\mathbb{E}(L_0)$ can be derived. Adding (4.2)-(4.4) and rearranging the terms, we get

$$\begin{aligned}\lambda\pi_{n0} + \lambda\pi_{n1} - ((\gamma\alpha + (n+1)\delta\vartheta)\pi_{n+1,0} + \gamma\mu\pi_{n+1,1}) \\ = \lambda\pi_{n-1,0} + \lambda\pi_{n-1,1} \\ + ((\gamma\alpha + n\delta\vartheta)\pi_{n,0} + \gamma\mu\pi_{n,1}), \quad n \geq 1\end{aligned} \tag{4.15}$$

Using recursively (4.15), and applying (4.1)

$$\lambda\pi_{n,0} + \lambda\pi_{n,1} = (\gamma\alpha + (n+1)\delta\vartheta)\pi_{n+1,0} + \gamma\mu\pi_{n+1,1}, \quad n \geq 0 \tag{4.16}$$

Adding over all possible values of n in (4.16), we obtain

$$\lambda\Pi_0(1) + \lambda\Pi_1(1) = \gamma\mu\Pi_1(1) + \gamma\alpha(\Pi_0(1) - \pi_{00}) + \delta\vartheta\sum_{n=0}^{\infty}(n+1)\pi_{n+1,0}, \quad n \geq 0. \tag{4.17}$$

Note that $\mathbb{E}(L_0) = \sum_{n=0}^{\infty}(n+1)\pi_{n+1,0}$ and $\Pi_1(1) = 1 - \Pi_0(1)$.

By substituting the value of $\mathbb{E}(L_0)$ from (4.14) in (4.17), we get

$$\begin{aligned}(\delta\vartheta + \theta + \bar{\beta}\gamma\alpha)(\gamma\mu - \lambda) = \big(\delta\vartheta(\gamma\mu - \lambda) + (\gamma\mu - \gamma\alpha)(\theta + \bar{\beta}\gamma\alpha) - \delta\vartheta\bar{\beta}\gamma\alpha\big)\,\Pi_0(1) \\ + \big(\delta\vartheta\bar{\beta}\gamma\alpha + \gamma\alpha(\theta + \bar{\beta}\gamma\alpha)\big)\,\pi_{0,0}.\end{aligned} \tag{4.18}$$

When $z \to \infty$ in (4.8) and using (4.6), (4.7) and (4.12), we get

$$\Pi_0(1) = \frac{e^{\lambda/\delta\vartheta}}{\delta\vartheta}\big(-(\theta + \bar{\beta}\gamma\alpha)\Pi_0(1)\Phi_1(1) + \bar{\beta}\gamma\alpha\pi_{00}\Phi_2(1)\big)\lim_{z\to+\infty}(1-z)^{-(\theta+\bar{\beta}\gamma\alpha)/\delta\vartheta} \tag{4.19}$$

As, $0 \leq \Pi_0(1) = \sum_{n=0}^{\infty}\pi_{n,0} \leq 1$ and $\lim_{z\to 1}(1-z)^{-(\theta+\gamma\bar{\beta}\alpha)/\delta\vartheta} \to \infty$, so we should have

$$-(\theta + \bar{\beta}\gamma\alpha)\Pi_0(1)\Phi_1(1) + \beta\gamma\alpha\pi_{00}\Phi_2(1) = 0. \tag{4.20}$$

Then, using (4.19) and (4.20), we get

$$\pi_{0,0} = \frac{(\theta + \gamma\bar{\beta}\alpha)\Phi_1(1)}{\gamma\beta\alpha\Phi_2(1)}\Pi_0(1), \tag{4.21}$$

and

$$\Pi_0(1) = \left((\delta\vartheta + \theta + \gamma\bar{\beta}\alpha)(\gamma\mu - \lambda)\beta\Phi_2(1)\right) \times \left((\delta\bar{\beta}\vartheta + \theta + \gamma\bar{\beta}\alpha)(\theta + \gamma\bar{\beta}\alpha)\Phi_1(1) + (\delta\vartheta(\gamma\mu - \lambda) + \gamma(\mu - \alpha)(\theta + \gamma\bar{\beta}\alpha) - \delta\vartheta\bar{\beta}\gamma\alpha) \times \beta\Phi_2(1)\right)^{-1} \tag{4.22}$$

Thus, $\mathbb{E}[L_0]$ is found from (4.14). Using (4.1) and (4.11) the unknowns $\pi_{1,0}$ and $\pi_{1,1}$ are obtained as follows:

$$\pi_{1,0} = \frac{(\theta + \gamma\bar{\beta}\alpha)\left((\lambda + \theta + \gamma\bar{\beta}\alpha)\Phi_1(1) - \gamma\beta\alpha\Phi_2(1)\right)}{(\delta\vartheta + \gamma\beta\alpha)\gamma\beta\alpha\Phi_2(1)}\Pi_0(1), \tag{4.23}$$

$$\pi_{1,1} = \left(\left((\theta + \gamma\bar{\beta}\alpha)\left\{(\delta\vartheta + \gamma\alpha)\beta\gamma\alpha\Phi_2(1) - \Phi_1(1)(\lambda\gamma\bar{\beta}\alpha + (\delta\vartheta + \gamma\alpha)(\theta + \gamma\bar{\beta}\alpha))\right\}\right) \times (\gamma\beta\mu\alpha\Phi_2(1)(\delta\vartheta + \gamma\beta\alpha))^{-1}\right)\Pi_0(1). \tag{4.24}$$

Now, the stationary probabilities $\pi_{n,0}$ and $\pi_{n,1}$ can be derived by using (4.2)-(4.4) in terms of $\pi_{0,0}$, $\pi_{1,0}$, and $\pi_{1,1}$.

The expected number of customers in the system when the server is in a normal busy period $\mathbb{E}(L_1)$ can be obtained from (4.12).
By using L'Hopital's rule we get

$$\mathbb{E}(L_1) = \Pi_1'(1) = \tfrac{\theta+\gamma\bar{\beta}\alpha}{\gamma\mu-\lambda}\tfrac{\Pi_0''(1)}{2} + \tfrac{1}{(\theta+\gamma\bar{\beta}\alpha)(\gamma\mu-\lambda)} \times \left((\gamma\theta\mu + \lambda\gamma\bar{\beta}\alpha)(1 - \Pi_0(1)) + \theta\gamma\bar{\beta}\alpha(\Pi_0(1) - \pi_{0,0})\right). \tag{4.25}$$

Differentiating (4.6) twice at $z = 1$, we obtain

$$f''(1)\Pi_0(1) + 2(f'(1) - \delta\vartheta)\Pi_0'(1) + (f(1) - 2\delta\vartheta)\Pi_0''(1) = 0, \tag{4.26}$$

where $f(1) = -(\theta + \gamma\bar{\beta}\alpha)$, $f'(1) = \lambda - (\gamma\alpha + \theta)$ and $f''(1) = 2\lambda$.

Then, from (4.26), we get

$$\frac{1}{2}\Pi_0''(1) = \left(\frac{\lambda}{\theta + \gamma\bar{\beta}\alpha + 2\delta\vartheta}\right)\Pi_0(1) - \left(\frac{\delta\vartheta + \gamma\alpha + \theta - \lambda}{\theta + \gamma\bar{\beta}\alpha + 2\delta\vartheta}\right) \times \mathbb{E}(L_0) \tag{4.27}$$

Using (4.27) and (4.14) in (4.25), we get $\mathbb{E}(L_1)$. The expected number of customers in the system can be computed as $\mathbb{E}(L) = \mathbb{E}(L_0) + \mathbb{E}(L_1)$. Now, we define the sojourn times, let W be the total sojourn time of a customer in the system, evaluated from the instant of arrival till departure, with the departure either due to completion of service or as a consequence of abandonment. We have by Little's rule

$$\mathbb{E}(W) = \frac{1}{\lambda}(\mathbb{E}(L_0) + \mathbb{E}(L_1)). \tag{4.28}$$

Let Δ be the proportion of customers served, Θ be the rate of abandonment due to impatience. The expected number of customers served per unit of time is $\gamma\mu\Pi_1(1) - \gamma\alpha(\Pi_0(1) - \pi_{0,0})$ signifying that the proportion of customers served is

$$\Delta = \frac{1}{\lambda}\left(\gamma\mu\Pi_1(1) + \gamma\alpha(\Pi_0(1) - \pi_{0,0})\right). \tag{4.29}$$

The rate of abandonment Θ of a customer due to impatience is given by

$$\Theta = \delta\vartheta\mathbb{E}(L_0) = \lambda - \left(\gamma\mu\Pi_1(1) + \gamma\alpha(\Pi_0(1) - \pi_{0,0})\right), \tag{4.30}$$

which follows from (4.17). The probability that the system is in normal busy period Γ and the probability that the system is in working vacation Ω are, respectively, given by

$$\Gamma = \sum_{n=1}^{\infty} \pi_{n,1} = \Pi_1(1), \quad \Omega = \sum_{n=0}^{\infty} \pi_{n,0} = \Pi_0(1). \tag{4.31}$$

4.4 Numerical Results

In this part of this paper, we present some numerical examples to demonstrate how the various parameters of the model influence the behavior of the system, and to show the impact of different parameters and its relationship with the expected number of customers when the system is on working vacation $\mathbb{E}(L_0)$, the expected number of customers when the system is on busy period $\mathbb{E}(L_1)$, the expected number of customers in the system $\mathbb{E}(L)$, the expected waiting time in the system $\mathbb{E}(W)$, the proportion of customers served Δ, the rate of abandonment Θ, the probability that the system is in normal busy period Γ and the probability that the system is on working vacation Ω.

✓ Firstly, Let us present the evolution of the system by varying β, ϑ, and α.

• The parameters for table 1 are taken as $\lambda = 2$, $\mu = 6$, $\theta = 0.8$, $\alpha = 3$, $\gamma = 0.5$, and $\delta = 0.65$.

Table 1. Impact of ϑ and β on some performance measures.

β	ϑ	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	$\mathbb{E}(L)$	$\mathbb{E}(W)$	Δ	Θ	Γ	Ω
	0.4	0,38322	0,67317	1,05638	0,52819113	0,95018	0,09964	0,52912	0,47088
0.15	1.2	0,36227	0,42851	0,79078	0,395392473	0,85871	0,28257	0,45684	0,54316
	2	0,33808	0,29520	0,63328	0,316639316	0,78025	0,43950	0,40317	0,59683
	0.4	0,44204	0,75376	1,19581	0,597903865	0,94253	0,11493	0,51473	0,48527
0.5	1.2	0,40317	0,43885	0,84203	0,421012957	0,84276	0,31447	0,43569	0,56431
	2	0,36790	0,28641	0,65430	0,327150257	0,76087	0,47826	0,38126	0,61874
	0.4	0,55594	0,91678	1,47272	0,73636023	0,92773	0,14455	0,48943	0,51057
0.9	1.2	0,46844	0,45451	0,92294	0,461472108	0,81731	0,36538	0,40222	0,59778
	2	0,41155	0,27219	0,68374	0,341872387	0,73249	0,53502	0,34925	0,65075
	0.4	0,60040	0,98064	1,58104	0,790519493	0,92195	0,15610	0,48032	0,51968
1	1.2	0,48940	0,45930	0,94870	0,47434832	0,80913	0,38173	0,39152	0,60848
	2	0,42461	0,26764	0,69225	0,346124654	0,72400	0,55200	0,33969	0,66031

Table 1 shows at first that for fixed ϑ, $\mathbb{E}(L_0)$, $\mathbb{E}(L_1)$, $\mathbb{E}(L)$, $\mathbb{E}(W)$, Ω and Θ increase as β increases, otherwise Δ and Γ decease as β increases. Now, for fixed β, $\mathbb{E}(L_0)$, $\mathbb{E}(L_1)$, $\mathbb{E}(L)$, $\mathbb{E}(W)$, Γ and Δ decrease as ϑ increases. But the rate of abandonment Θ of a customer due to impatience and the probability that the system is in working vacation Ω increase as ϑ increases.

- The parameters for table 2 are taken as $\lambda = 3$, $\mu = 7$, $\theta = 0.4$, $\vartheta = 0.6$, $\gamma = 0.6$, and $\delta = 0.7$.

Table 2. Impact of β and α on some performance measures.

β	α	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	$\mathbb{E}(L)$	$\mathbb{E}(W)$	Δ	Θ	Γ	Ω
	2	0,73780	2,14719	2,88498	0,961660758	0,89671	0,30987	0,55645	0,44355
0.3	4	0,50582	1,85779	2,36360	0,787867677	0,92919	0,21244	0,51678	0,48322
	5	0,44269	1,74599	2,18868	0,72955893	0,93802	0,18593	0,49605	0,50395
	2	0,90276	2,11320	3,01595	1,00531821	0,87361	0,37916	0,52898	0,47102
0.6	4	0,63494	1,77538	2,41032	0,803440036	0,91111	0,26667	0,47878	0,52122
	5	0,55563	1,64233	2,19796	0,732652234	0,92221	0,23336	0,45344	0,54656
	2	1,07762	2,07083	3,14845	1,049484825	0,84913	0,45260	0,50055	0,49945
0.8	4	0,78531	1,67606	2,46137	0,820456865	0,89006	0,32983	0,43623	0,56377
	5	0,68812	1,51894	2,20706	0,735688075	0,90366	0,28901	0,40535	0,59465
	2	1,36460	1,99159	3,35620	1,118731794	0,80896	0,57313	0,45487	0,54513
1	4	1,07313	1,47924	2,55237	0,85079094	0,84976	0,45072	0,35771	0,64229
	5	0,94565	1,27516	2,22081	0,740269278	0,86761	0,39717	0,31522	0,68478

Table 2 shows for fixed α, $\mathbb{E}(L_0)$, $\mathbb{E}(L)$ $\mathbb{E}(W)$ Ω and Θ increase as β increases, otherwise

$\mathbb{E}(L_1)$, Δ and Γ decrease.

However, for fixed β, $\mathbb{E}(L_0)$, $\mathbb{E}(L_1)$, $\mathbb{E}(L)$, $\mathbb{E}(W)$, Γ and Θ decrease as α increases, otherwise, Δ and Ω increase as α increases. All these results (tables 1 and 2) agree absolutely with our intuition.

✓ Now, let us present the impact of service rate during vacation α on the expected number of customers in the system for different values of vacation rates θ, while $\lambda = 2$ $\mu = 5$, $\vartheta = 1$, $\beta = 1$, $\gamma = 0.6$ and $\delta = 0.7$. The numerical results are given in Figure 4.1.

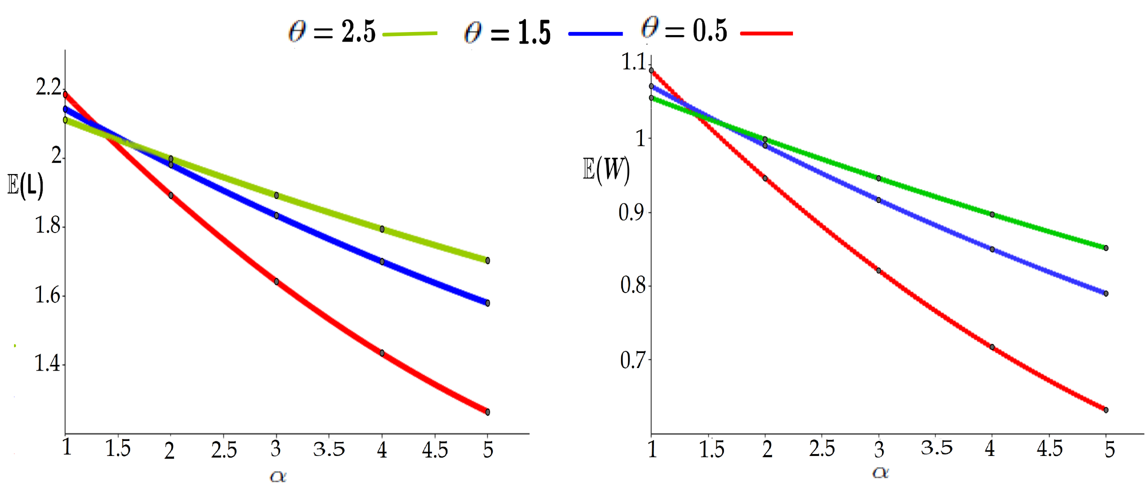

Figure 4.1: Effect of α on $\mathbb{E}(L)$, Effect of α on $\mathbb{E}(W)$

From Figure 4.1, we observe that the expected number of customers in the system decreases with the increase of α, furthermore when $\alpha > 1.5$, $\mathbb{E}(L)$ increases as θ increases, however, when $\alpha < 1.5$, $\mathbb{E}(L)$ decreases when the vacation rate increases.

✓ Next, we present the effect of arrival rate λ on the rate of abandonment of a customer due to impatience, Θ, and the effect of arrival rate λ on $\mathbb{E}(L)$ and on $\mathbb{E}(W)$ for various parameters ϑ and β. We take $\mu = 6$, $\theta = 0.5$, $\alpha = 4$, $\gamma = 0.5$, and $\delta = 0.5$.

Figure 4.2 shows that Θ first increases then diminishes with increasing of the arrival rate when $\beta = 0.5$ and $\beta = 1$, which agree absolutely with our expectation; the rate of abandonment of a customers increases when ϑ increases because more numbers of customers renege and leave the system.

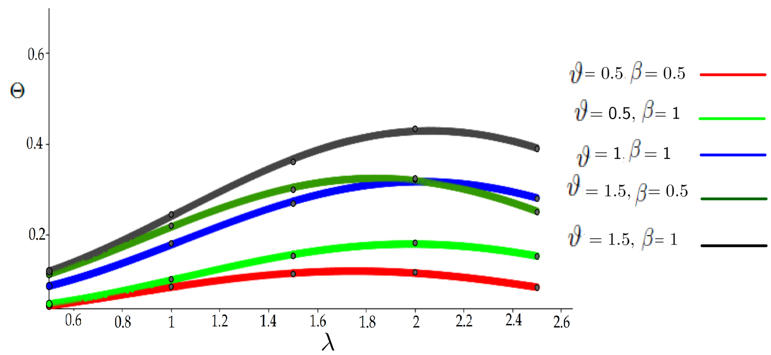

Figure 4.2: Arrival rate λ versus θ

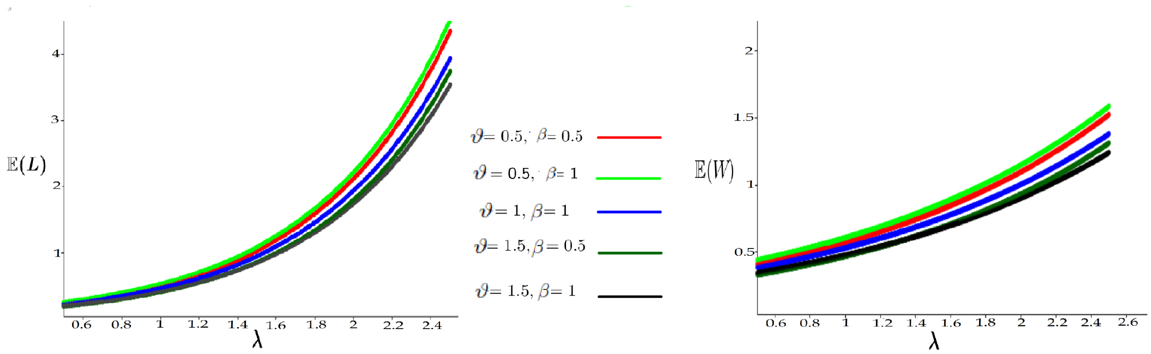

Figure 4.3: Arrival rate λ versus $\mathbb{E}(L)$, Arrival rate λ versus $\mathbb{E}(W)$

Figure 4.3 shows the increases of $\mathbb{E}(L)$ and $\mathbb{E}(W)$ with the increases of λ. This result is absolutely reasonable.

✓ Next, we present the dependence of the proportion of customers served and abandonment rate Δ and Θ with θ and ϑ. Let $\lambda = 3$, $\mu = 5$, $\alpha = 0.65$, $\beta = 0.5$, $\delta = 0.4$, and $\gamma = 0.65$.

Figure 4.4 shows that for fixed θ, Δ decreases and Θ increases when ϑ increases. Moreover, for fixed ϑ, Δ increases and Θ decreases when θ increases.

✓ Now, we present the dependence of the proportion of customers served and abandonment rate Δ and Θ to θ and δ. Let $\lambda = 3$, $\mu = 5$, $\alpha = 3$, $\beta = 0.5$, $\vartheta = 0.4$, and $\gamma = 0.65$.

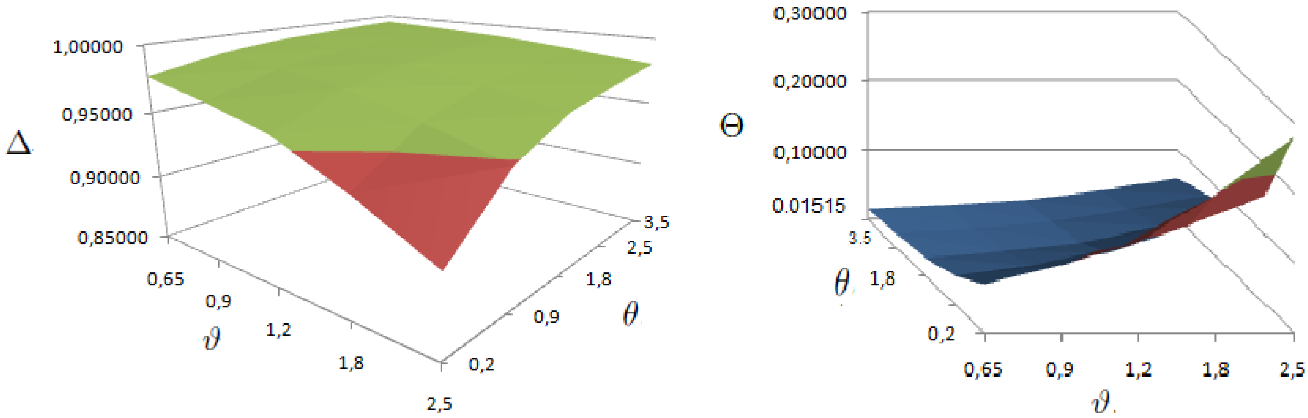

Figure 4.4: Effect of θ and ϑ on Δ Effect of θ and ϑ on Θ

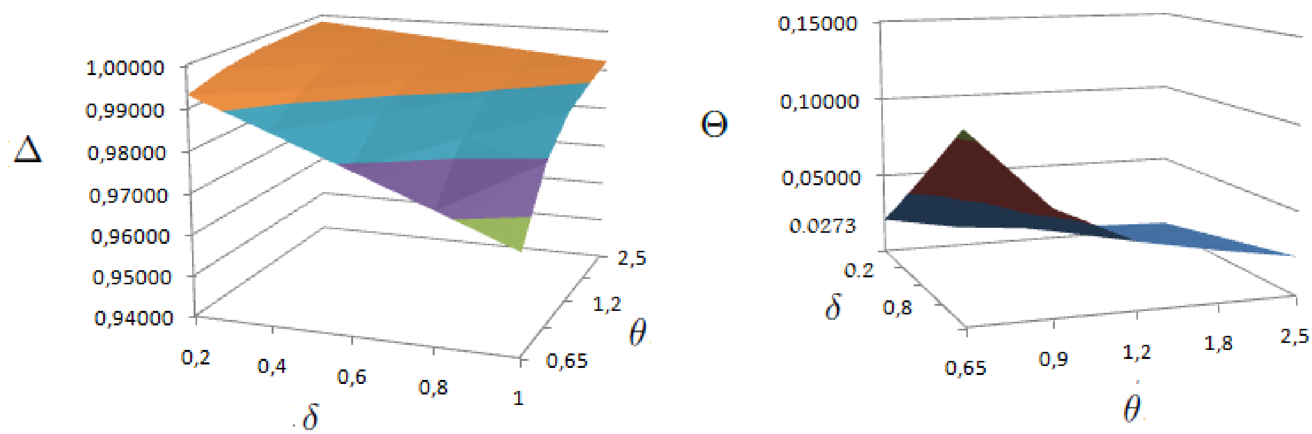

Figure 4.5: Effect of θ and δ on Δ, Effect of θ and δ on Θ

Figure 4.5 shows that for fixed θ, Δ decreases and Θ increases when δ increases. Moreover, for fixed δ, Δ increases and Θ decreases when θ increases.

✓ Finally, let's present dependence of the proportion of customers served Δ and abandonment rate Θ on θ and γ. We take $\lambda = 3$, $\mu = 7$, $\alpha = 3$, $\beta = 0.5$, $\vartheta = 0.4$, and $\delta = 0.65$.

Figure 4.6 shows that for fixed γ, Θ decreases and Δ increases when θ increases. Moreover, for fixed θ, Θ increases and Δ decreases when γ increases.

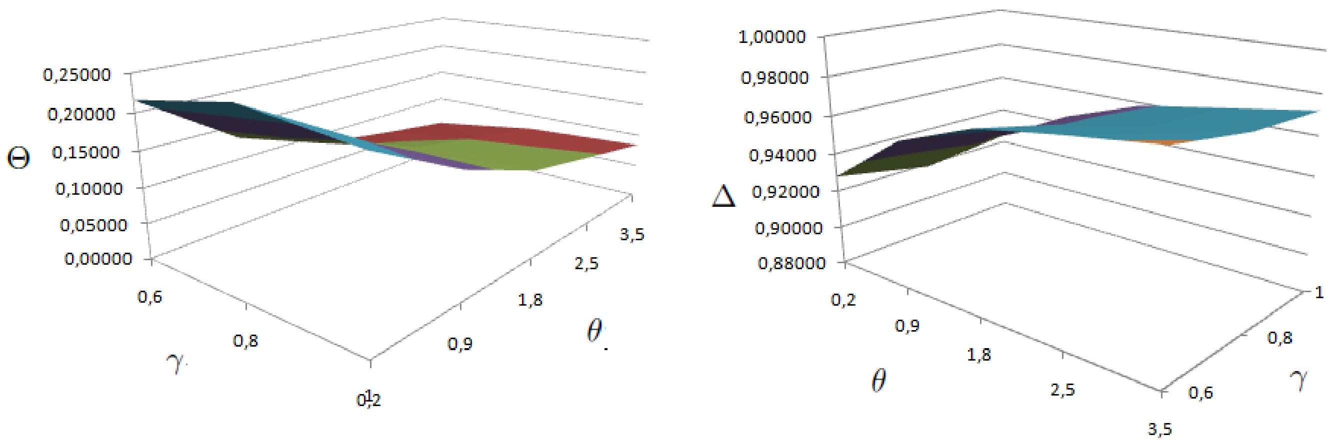

Figure 4.6: Effect of θ and γ on Θ, Effect of θ and γ on Δ

4.5 Conclusion

In this paper, an analysis of a feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption is carried out, the explicit expressions for various performance measures are derived. Some numerical examples are presented to demonstrate how the various parameters of the model influence the behavior of the system. For further work, this model can be studied under the provision of time dependent arrival and service rate. The cost-profit analysis of the model can also be carried to study its economic analysis.

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General Conclusion and Future Work

1. General Conclusion

In this thesis we studied different queueing systems with impatient customers.

▶ *Chapter One is an introduction to various queueing systems; retrial queues, impatience queues and queueing system with vacation were presented on the basis of the literature.*

▶ *In the last three chapters, we investigated*

⋇ *The study of fluid approximation of retrial queueing system with two orbits, abandonment and feedback is considered, An $M_t/M_t/c_t$ retrial queueing model with abandonment and feedback is considered, where two independent Poisson streams of customers arrive to the system, an arriving one of type i, $i = 1, 2$ is handled by an available server, if there is any; otherwise, he waits in an infinite buffer queue. A waiting customer of type i who did not get connected to a server may abandon the system amount of time, the abandoned one may leave the system (loss customer) or move to the orbit depending of its type, from which he makes a new attempts to reach the primary queue, then this later when he finishes his conversation with a server, he may comeback to the system, to one of the orbits for another service. The diffusion limit for the model under consideration is carried out.*

⋇ *Stability analysis of the of a queueing system with feedback, two orbits, abandonment and feedback, a retrial queueing system with two orbits, abandoned and feedback customers is considered, at which two independent Poisson streams of customers arrive to the system, and flow into a single-server service system. An arriving one of type i, $i = 1, 2$ is handled by the server if it is free; otherwise, it is blocked and routed to a separate type-i retrial (orbit) queue that attempts to re-dispatch its jobs at its specific Poisson rate. The customer in the orbit either attempts service again after a random time or gives up receiving service and leaves the system after a random time. After the customer is served completely, it will decide either to join the retrial group again for another service or leave the system forever with some probability. The balance equations and generating functions* *of the model are derived, further, the necessary stability condition is established.*

⋇ *The analysis of a Markovian queueing system with feedback, multiple vacations, vacation interruption, impatient customers and retention of abandoned customers, where the impatience of the customers is due to the vacations of the servers. The stationary analysis of the system is established, the generating functions of the steady state probabilities is given, the explicit expressions of the system sizes when the server is in a normal period and in an interrupted vacation period are deduced, respectively. Various system performance measures are derived.*

The obtained results have many practical queueing systems especially those with balking and reneging have been widely applied to many real-life problems, such as the situations involving impatient telephone switchboard customers, the hospital emergency rooms handling critical patients, and the inventory systems with storage of perishable goods.

2. Future Works

The following queueing systems are suggested to be developed according to the results found in this research:

- *Batch arrival retrial queueing system, random breakdowns, general repair times and general stand-by server works during every main server interruption, single/multiple vacations and impatient customers.*

- *Batch arrival retrial queueing system, random breakdowns, general repair times, single/multiple vacations, working vacation and impatient customers.*

- *Batch arrival retrial queueing systems with random breakdowns, general delay times and two types of general repairs, multiple vacation, working vacation, interruption of working vacation and impatient customers.*

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ملخص . في هذه الاطروحة نقوم بدراسة مختلف أنظمة قوائم الانتظار مع نفاذ صبر الزبائن، أولا نقوم بدراسة التقريب المحلولي لقوائم الانتظار ذات النداء المتكرر مع التغذية الراجعة و تنصل الزبائن. نتحصل على حد النشر لهذا النظام. بعد ذلك نقوم بإعطاء تحليلا لاستقرار نظام قوائم الانتظار ذات النداء المتكرر مع التغذية الراجعة و تنصل الزبائن. تم الحصول على دوال توليد الاحتمالية لاحتمالات حالة المراوحة و نستنتج الصيغ الصريحة لحالة النظام، اضافة إلى ذلك نعطي شرط لازم لاستقرار النظام . و اخيرا نقدم تحليلا لنظام ماركوفي للتغذية الراجعة عن اصطفاف في طابور مع تنصل و الاحتفاظ بالزبائن المتنصلين، واجازات عمل متعددة، و اعاقة جدول اجازات، حيث يعزى عدم صبر الزبائن إلى اجازات الخوادم. تم الحصول على دوال توليد الاحتمالية لاحتمالات حالة المراوحة، تم استنتاج الصيغ الصريحة لحجم النظام، عندما يكون الخادم في فترة عمل ناظمية و في حالة وجود اعاقة لجدول اجازات على التوالي، وتم اشتقاق عدة مقاييس لأداء النظام. اضافة إلى عدة أمثلة عددية للنظام.

Abstract. In this thesis we study various queueing systems with impatience. At first, we study the fluid approximation of a retrial queueing model with abandonment and feedback. The diffusion limit for the model under consideration is carried out. Then, we deal with the stability of a retrial queueing system with abandoned and feedback customers. The balance equations and generating functions of the model are derived, further the necessary stability condition is established. Finally, an analysis of a Markovian feedback queueing system with reneging and retention of reneged customers, multiple working vacations and vacation interruption, where customers' impatience is due to the servers' vacation is presented. The stationary analysis of the system is established. The probability generating functions of the stationary state probabilities is obtained. The explicit expressions of the system sizes when the server is in a normal service period and in a Bernoulli schedule vacation interruption respectively are deduced and various performance measures of the system are derived.

Résumé. Dans cette thèse nous étudions différents systèmes de files d'attente avec impatience, en premier lieu nous étudions l'approximation fluide de système de files d'attente avec rappel, abandon et feedback. La limite de diffusion pour le modèle considéré est effectuée. En suite nous analysons la stabilité d'un système de files d'attente avec rappel, abandon et feedback. Pour ce système nous dérivons les équations d'équilibre et les fonctions génératrices, en outre, nous établissons la condition de stabilité nécessaire. Finalement nous considérons un système de files d'attente Markovian avec feedback, multiples vacances, interruption de vacances, clients impatients et rétention de clients abandonnés, où l'impatience des clients est due aux vacances des serveurs. Nous établissons l'analyse stationnaire du système. Nous obtenons les fonctions génératrices des probabilités d'état stationnaire, nous déduisons les expressions explicites des tailles de système quand le serveur est dans une période de service normale et dans une période de vacance interrompue, respectivement. Diverses mesures de performance du système sont dérivées.