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# Intégration stochastique par rapport aux mouvements Browniens fractionnaire et sous-fractinnaire et applications aux équations différentielles stochastiques

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par

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# Introduction

Stochastic calculus allows for a consistent theory of integration of a stochastic process (integrand) with respect to an other (integrator), in order to solve stochastic differential equations. It is used to modelize systems that behave randomly. The best known and widely used process that carry out this calculus is Brownian motion. It is used in financial mathematics and economics, for example in modelizing the evolution in time of stock prices and bond interest rates.

The theory of stochastic integration and stochastic differential equations was developed by N. Wiener in 1923 [27], K. Itô 1942, 1944 [23, 24] and P. Lévy in 1948 [26]. The best known theory of stochastic calculus is that of Itô the father of stochastic integration theory. In this theory the integrator has to be semimartigale. However in recent years the well-studied theory of semimartingale turns out to be insufficient to describe many phenomena. On the one hand telecommunication connections, asset prices and other objects have long memory, this effect can not be modelized processes such as the Brownian motion, which has independent increments and no memory. On the other hand there are some concepts that can be described by self-similar fields with stationary dependent increments likewise turbulence in hydrodynamics and also the long-range property.

A suitable generalization of the standard Brownian motion that exhibits these previous properties, is the so-called fractional Brownian motion (fBm). This process was introduced by Kolmogrov in 1941 [25] and its relevance was recognized later by Mandelbrot and Van Ness in 1968 [28] who provided a stochastic integral representation of fBm in terms of the standard Brownian motion. Fractional Brownian motion is the only centered Gaussian process which is self-similar, and has stationary increments with dependence with Hurst parameter $H \in (0, 1)$. The parameter $H$ is named after the hydrologist Hurst who made a statistical study in 1951 of yearly water run-offs of the Nile river. He discovered that behavior of normalized values of the amplitude was approximately $cn^H$, $H = 0.7$. Because of this study Mandelbrot introduced the name Hurst index. Fractional Brownian motion appears in the modeling of many situation for example:

- The widths of consecutive annual rings of a tree.
- The temperature at specific area as a function of time.

- The characters of solar activity as a function of time.
- The price of electricity liberated electricity market.

Note that Fractional Brownian motion is neither a semimartingale nor a Markov process except when $H = \frac{1}{2}$, where it is a Brownian motion. So, we cannot apply the classical stochastic calculus developed by Itô. Different approaches have bein proposed in order to build an integral with respect to it. The most important contributions which are:

- *Pathwise calculus*: the stochastic integral is defined pathwise with Rieman-Stieltjes methods i.e. path by path integration. Since the fractional Brownian motion has Hölder paths for $H > \frac{1}{2}$, we can often use in this case Young integral [2, 39]. Which is a generalization of Rieman-Stieltjes integral, where the integrator must have finite $p$-variation. In the other case we have to use symmetric integral introduced by Gradinaru, Nourdin, Russo and Vallois in 2005 [29].
  An other approach was done by Lyons in 1998 [32], who built an absolutely pathwise method based on Lévy stochastic area using what is called a *rough paths theory*. The case where the integrator is fractional Brownian motion has been studied by Coutin and Qian in 2002 [30] when the Hurst index satisfies $H > \frac{1}{4}$.
- *Malliavin calculus*; also known as stochastic calculus of variation [20]. This is the base of the modern approach to the Skorohod integral with respect to fractional Brownian motion, since fbm is a Gaussian process. This calculus has been investigated for fractional Brownian motion by Decreusfond and Üstünel in 1998 [33], Carmona and Coutin in 2002 [31].
- *Wick product*: A new type of integral with zero mean defined using Wick product (a particular way of defining an adjusted product of a set of random variables) was introduced by Hu and Pasik-Dunkan in 2002 [34] for $H > \frac{1}{2}$.

The study of a class of stochastic differential equations driven by fractional Brownian motions with arbitrary Hurst parameter $H \in (0, 1)$ was generally treated when the coefficients are constant, However, in most of the existing literature the diffusion coefficient $\sigma$ has to be very carefully specified, so that the subtle restrictions on the stochastic integrals are satisfied. For example, it is usually assumed that $\sigma$ is a deterministic function, or, even more,a deterministic linear function. In fact, to our best knowledge, there has not been any study on the case when the coefficients $b$ and $\sigma$ are allowed to be both random, anticipating, and at the same time the Hurst parameter is allowed to be arbitrary. In the first case the SDE is of the so-called additive noise type, and the SDE involves only the Wiener integrals, the path regularity does not affect the solvability directly, and the SDE can be treated as an ODE with random input. We refer to, e.g., [37, 18, 12, 39] for such cases. The other case, when the coefficients are not constant, is much more complicated, since the path regularity of the fBM varies with the Hurst parameter $H$, and the requirement for the path regularity of the solution varies accordingly. The SDE is defined in Skorohod sense, it has been studied by Yu Juan Jien, Jin Ma in 2009 [21, 38].

Many other researchers have proposed to use more general self-similar processes and random fields as stochastic models. As an extension of Brownian motion, recently Bojdecki in 2004 [3] introduced and studied a rather special class of processes which preserves many properties of fractional Brownian motion except for the stationarity of increments. This process is called *sub fractional Brownian motion*. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments. Therefore, it seems interesting to study sub-fractional Brownian motion, for this we refer to [10, 11].
In 2009, C. Tudor characterized the domain of the Wiener integral with respect to a sub-fractional Brownian motion $S^H$ with index $H \in (0,1)$ [14], Shen and Chen [19] defined a stochastic integral with respect to sub-fractional Brownian motion with index $H \in (0,1/2)$ that extends the divergence integral from Malliavin calculus, and established versions of the formulas of Itô and Tanaka that hold for all $H \in (0,1/2)$, see also [9, 8, 7]. Stochastic differential equations driven by sub-fractional Brownian motion have been considered only by Mendy in 2010 (we could not obtain this paper). Zhi Li, Guoli Zhou and Jiaowan Luo in 2015 have investigate the existence and uniqueness of mild solutions to the stochastic delay differential equation [35] and study its longtime behavior as well.
The purpose of this work is to study stochastic calculus with respect to those processes. We consider different types of integration: Young integral, Skorohod and Wiener integral and other approaches that help us to solve stochastic differential equation driven by fractional Brownian motion and sub-fractional Brownian motion, and we give the Itô's formula according to them.

This dissertation is organized as follows. In Chapter 1 we revisit some background and preliminaries about the fractional and sub-fractional Brownian motions providing definitions and properties with some simulations of the fBm paths. In Chapter 2 we present theories of stochastic integration w.r.t these processes, and we give the corresponding Itô formula. The study of stochastic differential equations appears in Chapter 3. We tried to collect the most useful results about the existence and uniqueness of solution (weak or strong) in the sense of integrals in the previous chapter, with simulation of the solution behavior of SDE driven fBm in the sense of Young and Russo-Vallois integral.

# Chapter 1

# Preliminary Background

## 1.1 Basic definitions

### 1.1.1 Gaussian processes

**Definition 1.1.1.** A real-valued stochastic process $(X_t)_{t\geq 0}$ is a *Gaussian process* if every finite linear combinition of $(X_t)_{t\geq 0}$ is a Gaussian $r.v, i.e.$

$$\forall n, \forall t, 1 \leq i \leq n, \forall a, \sum_{i=1}^{n} a_i X_{t_i} \;\; is \;\; a \;\; Gaussian \;\; r.v.$$

**Definition 1.1.2.** Let $X = (X_t)_{t\geq 0}$ et $Y = (Y_t)_{t\geq 0}$ be two stochastic processes defined on the same probability space. If $\mathbb{P}(X_t = Y_t) = 1$ for all $t \geq 0$, we say that $X$ and $Y$ are *modification* of each other.

**Definition 1.1.3.** Let $X$ and $X'$ be defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then $X$ and $X'$ are *indistinguishable* if and only if

$$\mathbb{P}(\{w \in \Omega : X_t(w) = X'_t(w) \, \forall t \geq 0\}) = 1.$$

There is a chain of implications:

$$indistinguishable \Rightarrow modification.$$

**Definition 1.1.4.** let $X = (X_t)_{t\in\mathbb{T}}$ and $Y = (Y_t)_{t\in\mathbb{T}}$ be two stochastic processes, possibly defined on two different probability space. We say that $X$ and $Y$ have the *same law*, and we write $X \overset{law}{=} Y$, to indicate that $(X_{t_1}, ...., X_{t_d})$ and $(Y_{t_1}, ...., Y_{t_d})$ have the same law for all $d \geq 0$ and all $t_1, ..., t_d \in \mathbb{T}$.

**Proposition 1.1.1.** Two Gaussian processes have the same law if and only if they have the same mean and covariance functions.

**Definition 1.1.5.** A symmetric function $\Gamma : \mathbb{T}^2 \to \mathbb{R}$ is of *positive type* if

$$\sum_{k,l=1}^{d} a_k a_l \Gamma(t_k, t_l) \geq 0$$

for all $d \geq 1$, $t_1, ..., t_d \in \mathbb{T}$ and $a_1, ..., a_d \in \mathbb{R}$.

**Theorem 1.1.** ***(Kolmogrov)***
*Consider a symmetric function $\Gamma : \mathbb{T}^2 \to \mathbb{R}$. Then, there exists a centered Gaussian process $X = (X_t)_{t\in\mathbb{T}}$ having $\Gamma$ for covariance function if and only if $\Gamma$ is of positif type.*

## 1.1.2 Continuity

**Definition 1.1.6.** A stochastic process $(X_t)_{t\geq 0}$ is said to be *continuous* if $\mathbb{P}(\{w \in \Omega : t \to X_t(w) \text{ is continuous}\}) = 1$, *i.e.* its sample paths are continuous *a.s.*

**Definition 1.1.7.** A stochastic process $(X_t)_{t\geq 0}$ is said to be *stochastically continuous* at $t$ if $X_{t+h} \overset{\mathbb{P}}{\to} X_t$ as $h \to 0$.

**Definition 1.1.8.** A stochastic process is said to be *càdlàg* (resp. *càglàd*) if every sample paths are right-continuous with left-hand limits (resp. left-continuous with right-hand limits.

**Lemma 1.1.1.** ***(Kolmogrov-Čentsov)***
*Fix a compact interval $\mathbb{T} = [0, T] \subset \mathbb{R}_+$, and let $X = (X_t)_{t\in\mathbb{T}}$ be a centered Gaussian process. Suppose that there exists $C, \eta > 0$ such that, for all $s, t \in \mathbb{T}$,*

$$\mathbb{E}[(X_t - X_s)^2] \leq C \mid t - s \mid^\eta . \tag{1.1}$$

*Then, for all $\alpha \in (0, \eta/2)$, there exists a modification $Y$ of $X$ with $\alpha$-*Hölder *continuous paths. In particular, $X$ admits a continuous modification.*

**Proof.** Fix $t > s$. Since $X$ is Gaussian and centered, we have that

$$X_t - X_s \overset{law}{=} \sqrt{\mathbb{E}[(X_t - X_s)^2]}\, G$$

where $G \sim \mathcal{N}(0, 1)$. We deduce from 1.1 that, for all $p \geq 1$,

$$\mathbb{E}[\mid X_t - X_s \mid^p] \leq C^{p/2}\mathbb{E}[\mid G \mid^p] \mid t - s \mid^{\eta p/2} .$$

Therefor, the general version of the classical Kolmogrov-Čentsov lemma applies and gives the desired result.

### 1.1.3 Filtration and measurability

**Definition 1.1.9.** A *filtration* on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family $(\mathcal{F}_t)_{t\in\mathrm{T}}$ of sub $\sigma$-field of $\mathcal{F}$.
A measurable space endowed with a filtration $(\mathcal{F}_t)_{t\in\mathrm{T}}$ is said to be a filtered space.

**Definition 1.1.10.** The filtration is said to be *right continuous* if $\mathcal{F}_{t_+} = \mathcal{F}_t$, $\forall t \geq 0$, where $\forall t > 0$ we set, $\mathcal{F}_{t_+} = \underset{s>t}{\cap} \mathcal{F}_s$

**Definition 1.1.11.** A filtration is said to be *complete* if the $\mathbb{P}$-negligible set of $\mathcal{F}_\infty$ are in $\mathcal{F}_0$ and if the probability space is complete.

**Definition 1.1.12.** A filtration satisfies the *usual condition* if it is right continuous and complete.

**Remark 1.1.1.** *The interests to work with filtration which are satisfying the usual condition are that every kind of limit of adapted processes is still adapted. Moreover, every modification of a progressively measurable processes stay progressively measurable.*

**Definition 1.1.13.** ***(Measurable Process)***
A stochastic process $(X_t)_{t\geq 0}$ is *measurable* if the application $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is measurable w.r.t $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ i.e. if

$$\forall A \in \mathcal{B}(\mathbb{R}),\ \{(t, w) : X_t(w) \in A\} \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$$

The process $(X_t)_{t\geq 0}$ is said to be $(\mathcal{F}_t)_{t\geq 0}$ *adapted*, if $X_t$ is $\mathcal{F}_t$ measurable for each $t \geq 0$.
The process $(X_t)_{t\geq 0}$ is obviously adapted with respect to the natural filtration.

**Proposition 1.1.2.** *A continuous stochastic process is measurable.*

***Proof.*** Let $(X_t)_{t\geq 0}$ a continuous stochastic process. First, we show that for $A \in \mathcal{B}(\mathbb{R}_+)$, we have

$$\{(t, w) \in [0, 1] \times \Omega,\ \ X_t(w) \in A\}\ \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}. \tag{1.2}$$

For $n \in \mathbb{N}$, let

$$X_t^n = X_{\frac{[2^n t]}{2^n}},\ \ t \in [0, 1],$$

since the paths of $X^n$ are piecewise constant, we have that

$$\{(t, w) \in [0, 1] \times \Omega,\ \ X_t^n(w) \in A\}\ \ \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}.$$

Beside, $\forall t \in [0, 1],\ \ w \in \Omega$, we have

$$\lim_{n\to\infty} X_t^n(w) = X_t(w).$$

Then we have 1.2. By the same argument we can prove that $\forall k \in \mathbb{N}$,

$$\{(t,w) \in [k,k+1] \times \Omega,\ \ X_t(w) \in A\}\ \ \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}.$$

Since

$$\bigcup_{k \in \mathbb{N}} \{(t,w) \in [k,k+1] \times \Omega,\ \ X_t(w) \in A\} = \{(t,w) \in \mathbb{R} \times \Omega,\ \ X_t(w) \in A\},$$

we have the result.

**Definition 1.1.14.** ***(Progressively Measurable Process)***
A process is *progressively measurable* if for each $t$ its restriction to the time interval $[0,t]$, is measurable with respect to $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$, where $\mathcal{B}_{[0,t]}$ is the Borel $\sigma$-algebra of subsets of $[0,t]$.

**Remark 1.1.2.** *Note that every progressively measurable process is adapted (and measurable). Besides, as well as in the Proposition 1.1.2, a continuous process adapted to $(\mathcal{F}_t)$ is progressively measurable. More precisely, any càdlàg or càglàd process are progressively measurable.*

**Definition 1.1.15.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ a filtered space. A process $(X_t)_{t \in \mathrm{T}}$ is said to be *predictable* (resp. *optional*) if it is an càglàd (resp. càdlàg ) $\mathcal{F}_t$-adapted process. We note the $\sigma$-field generated by càglàd (resp. càdlàg ) $\mathcal{F}_t$-adapted process by $\mathcal{P}$ (resp. $\mathcal{O}$).

In fact, there is this inclusion chain

$$\underbrace{\mathcal{P}}_{predictable\ processes} \subset \underbrace{\mathcal{O}}_{optional\ processes} \subset \underbrace{Prog}_{progressively\ measurable} \subset \underbrace{\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_\infty}_{measurable}$$

### 1.1.4 Martingales and Semimartingales

**Definition 1.1.16.** Let $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ be an integrable process then $X$ is a:

**i) Martingale** if and only if $\mathbb{E}(X_t|\mathcal{F}_s) = X_s \quad a.s.$ for $0 \leq s \leq t < \infty$

**ii) Supermartingale** if and only if $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s \quad a.s.$ for $0 \leq s \leq t < \infty$

**iii) Submartingale** if and only if $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s \quad a.s.$ for $0 \leq s \leq t < \infty$

**Definition 1.1.17.** $M = \{M_t, \mathcal{F}_t, t \geq 0\}$ is a *local-martingale* if and only if there exists a sequence of stopping times $T_n$ tending to infinity such that $M^{T_n}$are martingales for all $n$. The space of local martingales is denotes $\mathcal{M}_{loc}$, and the subspace of continuous local martingales is denotes $\mathcal{M}^c_{loc}$.

**Definition 1.1.18.** A process $X$ is a *semimartingale* if it is an adapted càdlàg process which has a decomposition

$$X = X_0 + M + A,$$

where $M$ is a local martingale, null at zero and $A$ is a process null at zero, with paths of finite variation.

Note that the decomposition is not necessarily unique as there exist martingales which have finite variation.

## 1.2 Brownian Motion

**Definition 1.2.1.** A stochastic process $(B_t)_{t\in\mathbb{R}_+}$ is called a *standard Brownian motion* if it satisfies the following conditions:

1. $\mathbb{P}(w \in \Omega : B_t(w) = 0) = 1$
2. $\forall n, \forall t_i, 0 \leq t_0 \leq t_1 \leq ... \leq t_n$, the r.v. $(B_{t_n} - B_{t_{n-1}}, ..., B_{t_1} - B_{t_0}, B_{t_0})$ are independent.
3. For any $s \leq t$, $B_t - B_s$ is a centered real valued r.v. normally distributed with variance $t - s$, i.e.
$$B_t - B_s \sim \mathcal{N}(0, t - s)$$
4. $\mathbb{P}(w \in \Omega : t \to B_t(w)$is continuous$)= 1$

**Remark 1.2.1.** *1. we can rewrite the second condition by : for $s \leq t$, the r.v. $B_t - B_s$ is independent from the "past" $\sigma-$field $\sigma(B_r, r \leq s)$.*

*2. The natural filtration of the Brownian motion is $\mathcal{F}_t^B = \sigma(B_s, s \leq t)$.*

*3. We can define the Brownian motion without the last condition of continuous paths, because with a stochastic process satisfying the second and the third conditions, by applying the Kolmogorov's continuity theorem, there exists a modification of $(W_t)_{t\in\mathbb{R}_+}$ which has continuous paths a.s.*

**Proposition 1.2.1.** *The Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ is a Gaussian process with mean* 0 *and covariance function $Cov(W_t, W_s) = s \wedge t$.*

***Proof.*** We have that $W_t = W_t - W_0$. Thus $W_t \sim \mathcal{N}(0, t)$ by definition. Moreover, without loss of generality, we assume $s < t$. Hence, we have

$$E(W_sW_t) = E(W_s(W_t - W_s) + W_s^2) = 0 + s = s.$$

□

Note that since the Brownian motion is a continuous Gaussian process, the proposition 1.2.1 characterizes uniquely the Brownian motion.

We will give here some properties of the standard Brownian motion.

**Properties 1.2.0.1.** *[1] Let $(W_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion*

1. *Self-similarity. For any $a > 0$, $\{a^{-1/2}W_{at}\}$ is Brownian motion.*
2. *Symmetry. $\{-W_t, t \geq 0\}$ is also a Brownian motion.*
3. *$\{tW_{\frac{1}{t}}, t > 0\}$ is also a Brownian motion.*
4. *If $W_t$ is a Brownian motion on $[0,1]$, then $(t+1)W_{\frac{1}{t+1}} - W_1$ is a Brownian motion on $[0,\infty)$.*

## 1.2.1 Quadratic variation and Brownian motion

**Proposition 1.2.2.** *Let $(W_t)_{t\in\mathbb{R}+}$ be a Brownian motion. For $t \geq 0$, for any sequence of subdivisions $\Delta_n[0,t]$, such that $\lim_{n\to\infty} |\Delta_n[0,t]| = 0$ we have*

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} \left( W_{\frac{it}{2^n}} - W_{\frac{(i-1)t}{2^n}} \right)^2 = t, \qquad a.s.$$

*Proof.* The proof can be found in ([5], p.38).

## 1.2.2 Brownian paths

**Proposition 1.2.3.** *A Brownian motion has its paths a.s., locally $\gamma$-Hölder continuous for $\gamma \in [0, 1/2)$.*

***Proof.*** Let $T > 0$, $n \in \mathbb{N}$ and $0 \leq s \leq t$. Then we have,

$$\mathbb{E}((W_t - W_s)^{2n}) = \frac{(2n)!}{2^n n!}(t-s)^n.$$

Hence, by using the Kolmogrov-Centsov lemma 1.1.1, there exists a continuous modification $(\tilde{W}_t)_{0\leq t\leq T}$ of $(W_t)_{0\leq t\leq T}$, whose the paths are locally $\gamma$-Hölder continuous for $\forall\gamma \in [0, \frac{n-1}{2n})$. Moreover, we have

$$\mathbb{P}(\forall t \in [0,T], W_t = \tilde{W}_t) = 1,$$

because the two processes are continuous, It implies that also almost all the paths of $(W_t)_{0\leq t\leq T}$ are locally $\gamma$-Hölder continuous.

**Proposition 1.2.4.** *[1] The Brownian motion's sample paths are a.s., nowhere differentiable.*

There is an intuitive way to understand this property of Brownian paths. Indeed, consider the increment for $h > 0$, $W_{t+h} - W_t \sim \mathcal{N}(0, h)$. Then we have that $\frac{W_{t+h}}{\sqrt{h}} \sim \mathcal{N}(0, 1)$. But the derivative is defined to be the limit, as h tends to 0,of the quantity $\frac{W_{t+h}-W_t}{h} \sim \mathcal{N}(0, \frac{1}{h})$. it is clear ,now, that when we let $h$ tends to 0, we obtain an "infinite" variance, so that there would not be a limit.

### 1.2.3 Brownian motion and martingales

The standard Brownian motion and several functions of it, are martingales.

**Proposition 1.2.5.** *[1] Let $(W_t)_{t\in\mathbb{R}_+}$ be a Brownian motion. Then the following processes are $(\mathcal{F}_t^W)$-martingales:*

1. $(W_t)_{t\in\mathbb{R}_+}$ ,
2. $(W_t^2 - t)_{t\in\mathbb{R}_+}$,
3. *For any* $u \in \mathbb{R}$, $(e^{uW(t)-\frac{u^2}{2}t})_{t\in\mathbb{R}_+}$.

## 1.3 Fractional Brownian Motion

### 1.3.1 Existence of the fractional Brownian Motion

The next proposition shows us the existence of the fractional Brownian motion.

**Proposition 1.3.1.** *Let $H > 0$ be a real parameter. Then, there exists a continuous centered Gaussian process $B^H = (B_t^H)_{t\geq 0}$ with covariance function given by*

$$\Gamma_H(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - \mid t - s \mid^{2H}), \quad s,t \geq 0 \tag{1.3}$$

*if and only if $H \leq 1$. In this case, the sample paths of $B^H$ are, for any $\alpha \in (0, H)$ $\alpha$-Hölder continuous on each compact set.*

***Proof.*** According to Kolmogrov's theorem 1.1, to get our first claim, we must show that $\Gamma_H$ is of positive type if and only if $H \leq 1$.
Assume first that $H > 1$. When $t_1 = 1$, $t_2 = 2$, $a_1 = -2$ and $a_2 = 1$, we have

$$a_1^2\Gamma_H(t_1,t_1) + 2a_1a_2\Gamma_H(t_1,t_2) + a_2^2\Gamma_H(t_2,t_2) = 4 - 2^{2H} < 0$$

As a consequence, $\Gamma_H$ is not of positive type when $H > 1$.
The function $\Gamma_1$ is of positive type, indeed $\Gamma_1(s,t) = st$ so that, for all $d \geq 1$,$t_1, ..., t_d \geq 0$ and $a_1, ..., a_d \in \mathbb{R}$,

$$\sum_{k,l=1}^{d} \Gamma_1(t_k,t_l)a_ka_l = (\sum_{k=1}^{d} t_ka_k)^2 \geq 0.$$

Consider now the case $H \in (0,1)$. For any $x \in \mathbb{R}$, the change of variable $v = u \mid x \mid$ (whenever $x \neq 0$) leads to the representation

$$\mid x \mid^{2H} = \frac{1}{C_H} \int_0^\infty \frac{1 - e^{-u^2 x^2}}{u^{1+2H}} du,$$

Where $C_H = \int_0^\infty (1 - e^{-u^2}) u^{-1-2H} du < \infty$. Therefor, for any $s, t \geq 0$, we have

$$\begin{aligned} s^{2H} + t^{2H} - \mid t - s \mid^{2H} &= \frac{1}{C_H} \int_0^\infty \frac{(1 - e^{-u^2 t^2})(1 - e^{-u^2 s^2})}{u^{1+2H}} du \\ &\quad + \frac{1}{C_H} \int_0^\infty \frac{e^{-u^2 t^2}(e^{2u^2 ts} - 1) e^{-u^2 s^2}}{u^{1+2H}} du \\ &= \frac{1}{C_H} \int_0^\infty \frac{(1 - e^{-u^2 t^2})(1 - e^{-u^2 s^2})}{u^{1+2H}} du \\ &\quad + \frac{1}{C_H} \sum_{n=1}^\infty \frac{2^n}{n!} \int_0^\infty \frac{t^n e^{-u^2 t^2} s^n e^{-u^2 s^2}}{u^{1-2n+2H}} du \end{aligned}$$

so that, for all $d \geq 1$, $t_1, ..., t_d \geq 0$ and $a_1, ..., a_d \in \mathbb{R}$,

$$\begin{aligned} \sum_{k,l=1}^d \frac{1}{2}(t_k^{2H} + t_l^{2H} - \mid t_k - t_l \mid^{2H}) a_k a_l &= \frac{1}{2C_H} \int_0^\infty \frac{(\sum_{k=1}^d (1 - e^{-u^2 t_k^2}) a_k)^2}{u^{1+2H}} du \\ &\quad + \frac{1}{2C_H} \sum_{n=1}^\infty \frac{2^n}{n!} \int_0^\infty \frac{(\sum_{k=1}^d t_k^n e^{-u^2 t_k^2} a_k)^2}{u^{1-2n+2H}} du \end{aligned}$$

that is $\Gamma_H$ is of positif type when $H \in (0,1)$.

To conclude the second part of the proposition we suppose that $H \in (0,1)$ and consider a centered Gaussian process $B^H$ with covariance function given by 1.3. Then we have

$$\mathbb{E}[(B_t^H - B_t^H)^2] = \mid t - s \mid^{2H}, \quad s, t \geq 0,$$

so that Kolmogrov-Čentsov lemma 1.1.1 applies and shows that the sample paths of $B^H$ are $\alpha$-Hölder continuous.

### 1.3.2 Definition and properties

**Definition 1.3.1.** A *fractional Brownian motion* (fBm for short)of Hurst parameter $H$ is a centered continuous Gaussian process $B^H = (B_t^H)_{t \geq 0}$ with covariance function

$$\mathbb{E}(B_t^H - B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - \mid t - s \mid^{2H}).$$

According to proposition 1.3.1, the fBm exists and has Hölder continuous paths.

**Remark 1.3.1.** *Trivially when $H = \frac{1}{2}$ the fBm is the standard Brownian motion.*

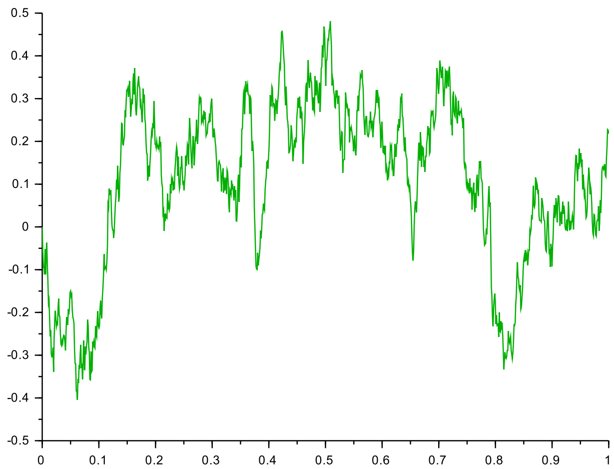


Figure 1.1: sample path of fBm when $H = \frac{1}{2}$

#### 1.3.2.1 Basic properties

**Proposition 1.3.2.** *Let $B^H$ be a fractional Brownian Motion of Hurst parameter $H \in (0,1)$. Then:*

1. *[Selfsimilarity] For all $a > 0$, $(B^H_{at}) \stackrel{d}{=} (a^H B^H_t)$.*
2. *[Stationarity of increments] For all $h > 0$, $(B^H_{t+h} - B^H_h) \stackrel{d}{=} B^H_t$.*
3. *[ Hölder continuity] For each $0 < \varepsilon < H$ and each $T > 0$ there exists a random variable $K_{\varepsilon,T}$ such that*

$$\mid B^H(t) - B^H(s) \mid \leq K_{\varepsilon,T} \mid t - s \mid^{H-\varepsilon}$$

4. *[Differentiability] The sample paths of fBm are nowhere differentiable.*

***Proof*** First, let us prove the selfsimilarity property. We have that

$$\begin{aligned}
\mathbb{E}(B^H_{at}B^H_{as}) &= \frac{1}{2}((at)^{2H} + (as)^{2H} - (a \mid t - s \mid)^{2H}) \\
&= a^{2H}\mathbb{E}(B^H_t B^H_s) \\
&= \mathbb{E}((a^H B^H_t)(a^H B^H_s))
\end{aligned}$$

Thus, since all processes are centered and Gaussian, it implies that

$$(B^H_{at}) \stackrel{d}{=} (a^H B^H_t).$$

Seconde, we show that it has stationary increments. Note that for all $h > 0$, we have

$$\begin{aligned}\mathbb{E}((B^H_{t+h} - B^H_h)(B^H_{s+h} - B^H_h)) &= \mathbb{E}(B^H_{t+h}B^H_{s+h}) - \mathbb{E}(B^H_{t+h}B^H_h) - \mathbb{E}(B^H_{s+h}B^H_h) + \mathbb{E}((B^H_h)^2) \\ &= \frac{1}{2}\left[((t+h)^{2H} + (s+h)^{2H} - \mid t-s \mid^{2H}) \right. \\ &\quad \left. - \left((t+h)^{2H} + h^{2H} - t^{2H}\right) - \left((s+h)^{2H} + h^{2H} - s^{2H}\right) + 2h^{2H}\right] \\ &= \frac{1}{2}(t^{2H} + s^{2H} - \mid t-s \mid^{2H}) = \mathbb{E}(B^H_t B^H_s).\end{aligned}$$

Therefore the fBm is of stationary increments.

For the Hölder continuity it follows from Kolmogrov-Čentsov lemma 1.1.1 and the fact that for any $\alpha > 0$, we have

$$\mathbb{E}(\mid B^H_t - B^H_s \mid^\alpha) = \mathbb{E}(\mid B^H_1 \mid^\alpha) \mid t-s \mid^{2H}$$

Finally, lets prove the differentiability, indeed for every $t_0 \in [0, \infty]$,

$$\mathbb{P}\left(\limsup_{t\to t_0} \mid \frac{B^H_t - B^H_{t_0}}{t - t_0} \mid = \infty\right) = 1.$$

let us denote by $\mathfrak{B}_{t,t_0} = \frac{B^H_t - B^H_{t_0}}{t-t_0}$, using the selfsimilarity property, we have

$$\mathfrak{B}_{t,t_0} \overset{d}{=} (t - t_0)^{H-1} B^H_1$$

We define $\mathfrak{u}(t,\omega) = \{\sup_{0\leq s\leq t} \mid \frac{B^H_s}{s} \mid > d\}$. Then, for any any sequence $(t_n)_{n\in\mathbb{N}}$ decreasing to 0,

we have $\mathfrak{u}(t_n,\omega) \supseteq \mathfrak{u}(t_{n+1},\omega)$, Thus,

$$\mathbb{P}(\lim_{n\to\infty} \mathfrak{u}(t_n)) = \lim_{n\to\infty} \mathbb{P}(\mathfrak{u}(t_n))$$

and

$$\mathbb{P}(\mathfrak{u}(t_n)) \geq \mathbb{P}(\mid \frac{B^{(H)}_{t_n})}{t_n} \mid > d) = \mathbb{P}(\mid B^{(H)}_1 \mid > t_n^{1-H} d) \overset{n\to\infty}{\to} 1.$$

### 1.3.3 Lack of Semimartingale Property

In this subsection, we study the asymptotic behavior of the $p$-variations of the fractional Brownian motion, we will show that the fBm is never a semimartingale, except for $H = \frac{1}{2}$ when it is the classical Brownian motion.

**Definition 1.3.2.** Let $(X_t)_{t\in[0,T]}$ be a stochastic process and consider a partition $\pi = 0 = t_0 < t_1 < ... < t_n = T$. Put

$$\mathcal{S}_p(X,\pi) = \sum_{i=1}^{n} |X(t_k) - X(t_{k-1})|^p .$$

The *p-variation* of $X$ over the interval $[0,T]$ is defined as

$$V_p(X,[0,T]) = \mathbb{P} - \lim_{\pi} \mathcal{S}_p(X,\pi),$$

where $p$ is a finite partition of $[0,T]$. The index of $p$-variation of a process is defined as

$$I(X,[0,T]) := \inf\{p>0; V_p(X,[0,T])<\infty\}.$$

We claim that

$$I(B^{(H)},[0,T]) = \frac{1}{H}.$$

In fact, consider for $p>0$,

$$Y_{n,p} = n^{pH-1}\sum_{i=1}^{n}\left|B^{(H)}_{\frac{i}{n}} - B^{(H)}_{\frac{i-1}{n}}\right|^p.$$

Since $B^{(H)}$ has the self-similarity property, the sequence $(Y_{n,p})_{n\in\mathbb{N}}$ has the same distribution as

$$\tilde{Y}_{n,p} = n^{-1}\sum_{i=1}^{n}\left|B^{(H)}_{i} - B^{(H)}_{i-1}\right|^p.$$

By the Ergodic theorem the sequence $\tilde{Y}_{n,p}$ converges almost surely and in $L^1$ to $\mathbb{E}[|B^{(H)}(1)|^p]$ as $n$ tends to infinity; hence, it converges also in probability to $\mathbb{E}[| B^{(H)}(1) |^p]$. It follows that

$$V_{n,p} = \sum_{i=1}^{n}\left|B^{(H)}_{\frac{i}{n}} - B^{(H)}_{\frac{i-1}{n}}\right|^p \xrightarrow[\mathbb{P}]{n\to\infty} \begin{cases} 0 & \text{if pH} >1 \\ \infty & \text{if pH}<1 \end{cases}$$

Thus we can conclude that $I(B^{(H)},[0,T]) = 1/H$. Since for every semimartingale $X$, the index $I(X,[0,T])$ must belong to $[0,1]\cup 2$, the fBm $B^{(H)}$ cannot be a semimartingale unless $H=1/2$.

## 1.3.4 Lack of Markov Property

**Theorem 1.2.** *Let $B^H$ be a fractional Brownian motion of Hurst index $H\in(0,1)-\{\frac{1}{2}\}$. Then $B^H$ is not a Markov process.*

Since the fBm is a Gaussian centered process, to prove this result we need the next lemma.

**Lemma 1.3.1.** If $X$ is a Gaussian centered Markovian process, then for all $s<t<u$

$$\mathbb{E}(X_tX_s)\mathbb{E}(X_tX_u) = \mathbb{E}(X_tX_t)\mathbb{E}(X_uX_s)$$

**Proof.** Note that $R_{st} = cov(X_s,X_t)$. Since $X$ is a Markov process then $\forall\ s<t<u$

$$\mathbb{E}(X_u/X_t,X_s) = \mathbb{E}(X_u/X_t) = \mathbb{E}(X_u) + \frac{cov(X_t,X_u)}{var(X_t)}(X_t - \mathbb{E}(X_t))$$

Therefore,

$$\begin{cases} \mathbb{E}(X_u/X_t) = \dfrac{R_{ut}}{R_{tt}} X_t, \\ \mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u) + \theta_{uv}\theta_v^{-1}(v - \mathbb{E}(v)) \end{cases}$$

where $v = \begin{pmatrix} X_t \\ X_s \end{pmatrix}$ and $\theta_{uv} = \mathbb{E}[X_u v^t]$, $\theta_v = \mathbb{E}(v^t v)$

We have that,

$$\theta_{uv} = (R_{ut} R_{us}) \ \ and \ \ \theta_v = \begin{pmatrix} R_{tt} & R_{ts} \\ R_{st} & R_{ss} \end{pmatrix}$$

$$\theta_v^{-1} v = \frac{1}{R_{tt}R_{ss} - R_{ts}^2} \begin{pmatrix} R_{ss}X_t - R_{ts}X_s \\ R_{tt}X_s - R_{st}X_t \end{pmatrix}$$

We observe that,

$$\begin{aligned} \mathbb{E}(X_u/X_t, X_s) &= \theta_{uv}\theta_v^{-1} v \\ &= \frac{1}{R_{tt}R_{ss} - R_{ts}^2}(R_{ut}R_{ss}X_t - R_{ut}R_{ts}X_s - R_{us}R_{st}X_t + R_{us}R_{tt}X_s). \end{aligned}$$

Hence, $\mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u/X_t)$ we have

$$\frac{R_{ut}}{R_{tt}} X_t = \frac{1}{R_{tt}R_{ss} - R_{ts}^2}(R_{ut}R_{ss}X_t - R_{ut}R_{ts}X_s - R_{us}R_{st}X_t + R_{us}R_{tt}X_s)$$

Moreover,

$$X_t(R_{tt}R_{ut}R_{ss} - R_{tt}R_{ut}R_{ss} - R_{ut}R_{st}^2 + R_{tt}R_{us}R_{st}) + X_s(R_{tt}R_{ut}R_{st} - R_{tt}^2 R_{us}) = 0$$

$$R_{st}X_t(R_{tt}R_{us} - R_{ut}R_{st}) - R_{tt}X_s(R_{tt}R_{us} - R_{ut}R_{st}) = 0$$

Or,

$$(R_{tt}R_{us} - R_{ut}R_{st})(R_{st}X_t - R_{tt}X_s) = 0,$$

then,

$$R_{tt}R_{us} - R_{ut}R_{st} = 0$$

which is the result.

***Proof of theorem* 1.2** We proceed by contradiction. Assume that $B^H$ is a Markov process. Since it is a Gaussian process as well, by the previous lemma we have, for $s = 1 < t = 2 < u = 3$

$$\mathbb{E}(B_1^H B_2^H)\mathbb{E}(B_2^H B_3^H) = \mathbb{E}(B_2^H B_2^H)\mathbb{E}(B_1^H B_3^H)$$

So,

$$\frac{1}{4}(1 + 2^{2H} - 1)(2^{2H} + 3^{2H} - 1) = 2^{2H}\frac{1}{2}(1 + 3^{2H} - 2^{2H})$$

$$2^{2H}(2^{2H} + 3^{2H} - 1) = 2^{2H}[2(1 + 3^{2H} - 2^{2H})]$$

by differentiating

$$3 + 3^{2H} + 3(2^{2H}) = 0$$

$$1 + 3^{2H-1} + 2^{2H} = 0$$

we deduce that, $1 + 3^{2H-1} + 2^{2H} = 0$ only if $H = \frac{1}{2}$ which leads to a contradiction.

### 1.3.5 Long and Short-Range Dependence

Process with long-range dependence have many application, such as in telecommunication specially in Internet traffic problems. Basically, the notion of long-range dependence is that the variance of the sum of stationary sequence grows non-linearly with respect to $n$.

**Definition 1.3.3.** A stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits long-range dependence if $\rho(n) = cov(X_k, X_{k+n})$ satisfies

$$\lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1$$

for $\alpha \in (0,1)$ and some constant $c$.

**Remark 1.3.2.** *If a stationary sequence $(X_n)_{n\in\mathbb{N}}$ is long-range dependent, then the dependence between $X_k$ and $X_{k+1}$ decays slowly as $n$ tends to infinity and $\sum_{n=1}^{\infty} \rho(n) = \infty$.*

**Proposition 1.3.3.** The fBm is one of the simplest processes which exhibit long-range dependency.

***Proof.*** let us consider its increments

$$X_k = B_k^H - B_{k-1}^H, \qquad X_{k+1} = B_{k+n}^H - B_{k+n-1}^H.$$

Since the fBm is centered then

$$\begin{aligned}
\rho(n) &= \mathbb{E}(X_k, X_{k+n}) \;=\; \mathbb{E}\left[(B_k^H - B_{k-1}^H)(B_{k+n}^H - B_{k+n-1}^H)\right] \\
&= \mathbb{E}\left[(B_{n+1}^H - B_n^H)B_1^H\right] \;=\; \mathbb{E}(B_{n+1}^H B_1^H) - \mathbb{E}(B_n^H B_1^H) \\
&= \frac{1}{2}\left[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right] \\
&= \frac{1}{2}n^{2H}\left[(1+\frac{1}{n})^{2H} - 2 + (1-\frac{1}{n})^{2H}\right] \\
&= \frac{n^{2H}}{2}\left[1 + \frac{2H}{n} + \frac{H(2H-1)}{n^2} - 2 + 1 - \frac{2H}{n} + \frac{H(2H-1)}{n^2} + o(\frac{1}{n^2})\right] \\
&= H(2H-1)n^{2H-2} + o(n^{2H-2})
\end{aligned}$$

it follows that for $H > \frac{1}{2}$, we have

$$\rho(n) > 0 \;\; and \;\; \sum_n \rho(n) = \infty.$$

and for $H < \frac{1}{2}$, we have

$$\rho(n) < 0 \quad and \quad \sum_n \rho(n) < \infty.$$

Therefore, we say that the fBm has long-range dependence property if and only if $H > \frac{1}{2}$ and for the other case has short-range dependence.

## 1.3.6 Representation of the FBm

Fractional Brownian motion can be expressed as a Wiener integral with respect to the Wiener process in several ways. Let us recall two of them.

### 1.3.6.1 Lévy-Hida Representation

Let $B^H$ be a fractional Brownian motion with parameter $H \in (0,1)$. The fBm admits a representation as a Wiener integral of the form

$$B^H = \int_0^t K_H(t,s)dWs,$$

where $W = (W_t)_{t\in T}$ is a Wiener process, and $K_H(t,s)$ is the kernel

$$K_H(t,s) = d_H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right),$$

$d_H$ being a constant and

$$F_1(z) = d_H\left(\frac{1}{2} - H\right)\int_0^{z-1} \theta^{H-\frac{3}{2}}\left(1 - (\theta+1)^{H-\frac{1}{2}}\right) d\theta.$$

If $H > \frac{1}{2}$ ,the kernel $K_H$ has the simpler expression

$$K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

where $t > s$ and $c_H = \left(\frac{H(H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$. The fact that the process $B^H$ is a fBm follows is from the equality

$$\int_0^{t\wedge s} K_H(t,u)K_H(s,u)du = R_H(t,s).$$

The kernel $K_H$ satisfies the condition

$$\frac{\partial K_H}{\partial t}(t,s) = d_H\left(H - \frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}.$$

### 1.3.6.2 Moving Average Representation

FBm can be represented as an integral with respect to a standard Brownian motion on the whole real line. Let $(B_s)_{s\in\mathbb{R}}$ be a standard Brownian motion. Then

$$B_t^H = \frac{1}{C(H)}\int_{\mathbb{R}}\left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right] dB_s, \tag{1.4}$$

with $C(H) > 0$ an explicit normalizing constant, is a fractional Brownian motion.

#### 1.3.6.3 Harmonizable Representation

There is another representation which uses the complex-valued Brownian motion (but the fBm is real-valued). In fact, for a fBm $(B_t^H)_{t\in\mathbb{R}}$, we obtain

$$B_t^H = \frac{1}{C_2(H)} \int_{\mathbb{R}} \frac{e^{itx}-1}{ix} |x|^{-(H-\frac{1}{2})} d\tilde{B}_x, \quad t \in \mathbb{R},$$

where $(\tilde{B}_t)_{t\in\mathbb{R}}$ is a complex Brownian measure and

$$C_2(H) = \left( \frac{\pi}{H\Gamma(2H) sin(H\pi)} \right)^{1/2}.$$

Let us note that the complex Brownian measure on $\mathbb{R}$ can be splitted as $\tilde{B} = B_1 + iB_2$ and is such that $B_1(A) = B_1(-A)$, $B_2(A) = -B_2(-A)$ and $\mathbb{E}(B_1(A))^2 = \frac{|A|}{2}$, $\forall A \in \mathcal{B}(\mathbb{R})$.
We also call this representation, the spectral representation.

## 1.4 Sub Fractional Brownian motion

As an extension of Brownian motion, recently, Bojdecki et al.[3] introduced and studied a rather special class of self-similar Gaussian processes, which preserve many properties of the fractional Brownian motion. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. This process is called the sub-fractional Brownian motion.

### 1.4.1 Definition and properties

**Definition 1.4.1.** *Sub-fractional Brownian motion (sub-fBm) is defined as a centered Gaussian process* $(S_t^H)_{t\geq 0}$*with covariance*

$$C_H(t,s) = s^{2H} + t^{2H} - \frac{1}{2}\left((s+t)^{2H} + |t-s|^{2H}\right), \quad s,t \geq 0$$

*with* $H \in (0,1)$.

Sub-fractional Brownian motion has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths, and it is neither a Markov processes nor a semimartingale).
Moreover, sub-fBm has non-stationary increments and the increments over non-overlapping intervals are more weakly correlated and their covariance decays polynomially at a higher rate in comparison with fBm (for this reason, it is called sub-fBm). The above mentioned properties make sub-fBm a possible candidate for models which involve long-dependence, self-similarity and nonstationarity.

**Remark 1.4.1.** *Trivially, for* $H = \frac{1}{2}$ *the sub-fBm reduces to the standard Brownian motion.*

**Lemma 1.4.1.** *The sfBm* $(S_t^H)_{t\in\mathbb{R}_+}$ *satisfies the following properties:*

**i)** $S^H$ *is a centered Gaussian process.*

**ii)** *For all* $(s,t)\in\mathbb{R}_+^2, s\leq t,$

$$\mathbb{E}(S_t^H - S_s^H)^2 = -2^{2H-1}(t^{2H}+s^{2H}) + (t+s)^{2H} + (t-s)^{2H}.$$

**iii)** *The increments of the sfBm are not stationary: For all* $s\leq t$

$$\mathbb{E}(\mid S_t^H \mid^2) = (2-2^{2H})t^{2H+1}$$

**iv)** *for all* $t\in\mathbb{R}_+$ $\quad S_t^H = \frac{B_t^H + B_{-t}^H}{\sqrt{2}}$

For the proofs of his properties see [6].

### 1.4.2 Representation of Sub-fBm

The sfbm has the moving average representation (see Bojdecki et al.,2004) [3].

$$S_t^k = \frac{1}{c_1(k)}\int_{\mathbb{R}} \left[(t-s)_+^k + (t+s)_-^k - 2(-s)_+^k\right] dW_s,$$

where $(W_t)_{t\in\mathbb{R}}$ is a Brownian motion,

$$c_1(k) = \left[2\left(\int_0^\infty ((1+s)^k - s^k)^2 ds\right) + \frac{1}{2k+1}\right]^{\frac{1}{2}}$$

For more Properties of the Sub fractional Brownian motion see [10].

### 1.4.3 Comparison between the FBm and the Sub-fBm

Before providing the comparison of the properties of sub-fBm considered to those of fBm lets announce these definitions

**Definition 1.4.2.** A real continuous process $D$ is is called a $(\mathcal{F}_t)$ *Dirichlet process* if it admits a decomposition $D = M + A$ where $M$ is an $(\mathcal{F}_t)$-local martingale and $A$ is a zero quadratic variation process. For convenience, we suppose $A_0 = 0$.

**Definition 1.4.3.** A square integrable process $(X_t)_{t\geq 0}$ is *quasi-Dirichlet* if for every $T > 0$

$$\sum_{j=0}^{k-1} \mathbb{E}\left(\left|\mathbb{E}[X_{t_{j+1}} - X_{t_j}/\mathcal{F}_{t_j}^X]\right|^2\right) \underset{\|\delta\|\to 0}{\to} 0$$

where $\delta : 0 =\leq t_0 < t_1 < ... < t_k = T$ is a partition of $[0,T]$ and $(\mathcal{F}_t^X)_{t\geq 0}$ is the canonical filtration of $X$.

Here are the main properties of the fBm and Sub-fBm :

1. fBm and sub-fBm become similar for large t in the sense that for each $\tau > 0$,

$$\lim_{t\to\infty} \frac{C_H(t, t+\tau)}{R_H(t, t+\tau)} = 2 - 2^{2H-1};$$

2. The fBm has long-memory if $H > \frac{1}{2}$ and short-memory if $H < \frac{1}{2}$:
The sub-fBm has short-memory.

3. The mixed processes (sums of independent Bm and fBm and of independent Bm and subfBm) are semi-martingales equivalent in law with the Bm if $H > \frac{3}{4}$.

4. The fBm is Dirichlet if $H > \frac{1}{2}$ and it is not Dirichlet if $H < \frac{1}{2}$
The sub-fBm is Dirichlet if $H > \frac{1}{2}$ and it is quasi-Dirichlet if $H < \frac{1}{2}$
The question whether fBm is quasi-Dirichlet is open.

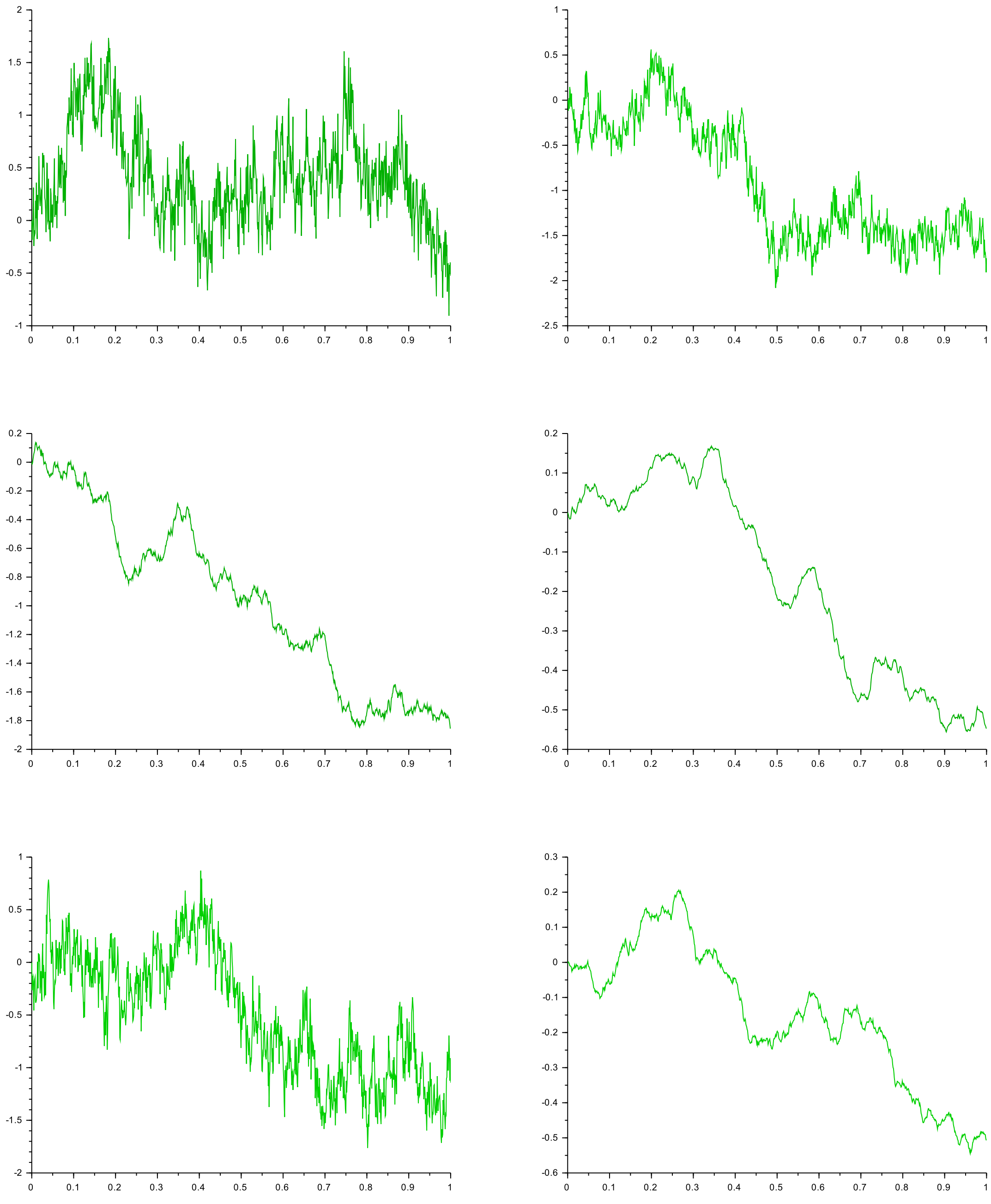

Figure 1.2: sample paths of fBm with $H = 0.3, H = 0.2, H = 0.6, H = 0.8, H = 1/4, H = 3/4$ (resp).

# Chapter 2

# Stochastic Calculus on fBm and sfBm

## 2.1 Stochastic Integration with respect to Fractional Brownian motion

### 2.1.1 Wiener Integration for fBm

In this subsection we develop the stochastic calculus for deterministic integrands with respect to fBm. We shall deal with a generalization of the Riemann-Stieltjes Integral which we will develop later : we have a fractional Brownian motion as integrator.

In Section 1.3.3, we have seen that fBm is not a semimartingale. But the classical stochastic integration namely the Itô calculus, is up to semimartingales as integrators. Therefore, we cannot apply directly this theory. Moreover, the Lebesgue-Stieltjes integration cannot be used since the paths of the fBm have unbounded variation, see Section 1.3.3. Hence we need to construct another integral. This work has been performed by several authors with different ideas. The different approaches are, among others:

- Malliavin calculus, also known as Stochastic calculus of variation, which exploits the Gaussianity of the fBm in general Wiener spaces (see [15]);
- Wick Calculus approach (see [16]);
- Pathwise Calculus (see [17]);
- Rough path analysis (see [18]).

However, We shall study the basic one : The Wiener integration w.r.t fBm. The aims of this section are twofold: to define the Wiener integral and the space of integrands. Lets us firstly define Riemann-Stieltjes Integral.

#### 2.1.1.1 Riemann-Stieltjes Integral

Riemann-Stieltjes integral is an important notion to understand the stochastic integration. But first, let us recall the basic Riemann integral.

**Definition 2.1.1.** Let $f : \mathbb{R} \to \mathbb{R}$ continuous. We define the *Riemann integral* over $[a, b] \subset \mathbb{R}$ by

$$\int_a^b f(t)dt = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1}),$$

if the limit exists, where $\Delta_n = \{t_0, t_1, ..., t_n\}$ is a partition of $[a, b]$ such that $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$, $\|\Delta_n\| = \max_{1\le i\le n}(t_i - t_{i-1})$ and $\tau_i$ is an evaluation point in the interval $[t_{i-1}, t_i]$.

**Definition 2.1.2.** The *p-variation* of a function $f : [a, b] \to \mathbb{R}$ is defined as

$$\sum_{i=1}^{n}(f(t_k^n) - f(t_{k-1}^n))^p,$$

if the limit exists, where $\Delta_n = \{t_0, t_1, ..., t_n\}$ is a partition of $[a, b]$ and the mesh goes to 0 as $n \to \infty$.

**Definition 2.1.3.** A function of *bounded variation* is a function $g : [a, b] \to \mathbb{R}$ such that $\forall t > 0$,

$$\sup_{\pi\in\mathcal{P}} \sum_{i=1}^{nP} |g(t_i) - g(t_{i-1})| < \infty,$$

where the supremum is taken over the set $\mathcal{P} = \{\pi = \{t_0, ..., t_{nP}\}, \pi \text{ is a partition of } [a, b]\}$.
We denote by $BV$ the set of functions of bounded variation.

**Definition 2.1.4.** Let $f : [a, b] \to \mathbb{R}$ continuous and $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. We define the *Riemann-Stieltjes* integral as follows:

$$\int_a^b f(t)dg(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(\tau_i)(g(t_i) - g(t_{i-1})),$$

if the limit exists, where $\Delta_n = \{t_0, t_1, ..., t_n\}$ is a partition of $[a, b]$ and the mesh goes to 0 as $n \to \infty$.

**Remark 2.1.1.** *Note that if $g(t) = t$ then the Riemann-Stieltjes integral is the Riemann integral.*

**Proposition 2.1.1.** *[5] If $f$ is continuous and $g \in \mathcal{C}^1$, then*

$$\int_a^b f(t)dg(t) = \int_a^b f(t)g'(t)dt$$

*and if $f, g \in BV$ then*

$$\int_a^b f(t)dg(t) = f(b)g(b) - f(a)g(a) - \int_a^b g(t)df(t). \tag{2.1}$$

#### 2.1.1.2 Wiener Integral

The Wiener integral is an integral where we have deterministic integrands and a Gaussian process as an integrator. It generalize the theory of Riemann-Stieltjes integral. Let us define the integral:

$$I(f) = \int_a^b f(t)dB_t^H \tag{2.2}$$

In fact, we could think of applying the integration by parts formula of the Riemann-Stieltjes integral (2.1), and obtain

$$\int_a^b f(t)dB_t^H = f(b)B_b^H - f(a)B_a^H - \int_a^b B_t^H df(t), \tag{2.3}$$

where the integrals are Riemann-Stieltjes integrals. But the problem is, as we saw, that $B_t^H \notin BV$. Hence equation (2.3) is not well defined as a Riemann-Stieltjes integral in this case. Therefore, we need a new approach to define the integral (2.2): the so-called Wiener Integral.

### 2.1.2 Construction of the Wiener Integral w.r.t FBm

The basic idea is to extend the isometry map from the set of step functions $\mathcal{E}$ into the space $L^2(\Omega)$ generated by the integrator, to an isometry defined on a larger space of integrands, usually noted $\tilde{\mathcal{H}}$ and such that $\bar{\mathcal{E}} = \tilde{\mathcal{H}}$. Let us recall that the Wiener integral (w.r.t. a Gaussian process) of a function $f \in \mathcal{H}$ is a random variable. More explicitly, it is a centered Gaussian random variable. With variance $\int_T f(t)^2 dt$ in the case of standard Brownian motion. Therefore the Wiener integral generates a Gaussian space. Let us denote this subspace of $L^2(\Omega, \mathcal{F}^{(Z)}, (\mathcal{F}_t^{(Z)})_{t\in T}, \mathbb{P}^Z)$ by $\overline{S_P(Z)}$ (Note that if $f \in \mathcal{E}$, $\int_T f(t)dZ_t$ generates $S_P(Z)$.) In our case, we take the Gaussian process $Z = B^{(H)}$, as a fBm, so we obtain $\overline{S_P(Z)} = \overline{S_{PT}(B^{(H)})} \subset L^2(\Omega, \mathcal{F}^{(H)}, (\mathcal{F}_t^{(H)})_{t\in T}, \mathbb{P}^H)$.

#### 2.1.2.1 Integrands as step functions

Let us denote by $\mathcal{E}$ the set of step functions. For $f \in \mathcal{E}$, i.e. $f = \sum_{i=1}^n f_{i-1}\mathbb{1}_{(t_{i-1},t_i]}$, where $t_0 = a$ and $t_n = b$, we define the Wiener Integral as follows:

**Definition 2.1.5.** For a fBm $(B_t^{(H)})_{t\in T}$ we define the *Wiener integral* w.r.t. fBm for $f \in \mathcal{E}$ by

$$I^H(f) \overset{not}{=} \int_T f(u)dB_u^{(H)} = \sum_{k=0}^{n-1} f_k(B_{u_{k+1}}^{(H)} - B_{u_k}^{(H)}),$$

where

$$f(u) = \sum_{k=0}^{n-1} f_k \mathbb{1}_{(u_k,u_{k+1}]}(u), \quad u \in T.$$

Let us observe that the step Wiener integral induces a Gaussian space denoted by $S_p(B^{(H)}) \subset L^2(\Omega, \mathcal{F}^{(H)}, (\mathcal{F}_t^{(H)})_{t\in T}, \mathbb{P}^H)$. Now, we would like to do as in the classical case and consider the square integrable function over $T$, because of denseness of $\mathcal{E}$ in $L^p(T)$ and to have a finite variance ($p = 2$). But it is not sufficient to take the integrands in $L^2(T)$, as in the Brownian case, due the non independency of the increments.

#### 2.1.2.2 General integrands

We then extend the isometry $I^H$ to a space of integrands which is at least an inner product space, denoted by $\tilde{\mathcal{H}}$, where, $\bar{\mathcal{E}} = \tilde{\mathcal{H}}$.

**Definition 2.1.6.** The Wiener integral with respect to the fractional Brownian motion is the isometric map $I^H$ defined as:

$$I^H : \tilde{\mathcal{H}} \to \overline{S_{pT}(B^{(H)})}$$

$$f \to I^H(f) = X$$

We can then define $S_{PT}(B^{(H)}) := \{X : I^H(f_n) \xrightarrow{L^2} X, (f_n)_{n\in\mathbb{N}} \subset \mathcal{E}\}$. Therefore, we associate with $X$ an equivalence class of sequences of step functions, $(f_n)_{n\in\mathbb{N}}$, such that $I^H(f_n) \xrightarrow{L^2} X$. Furthermore, we can write $X = \int_T f_X(t)dB_t^{(H)}$ , where $f_X$ is element of the equivalence class.

Recall our main question: which classes of integrands in the definition of the Wiener integral w.r.t. fBm are isometric to $S_{PT}(B^{(H)})$ or to some of its subspaces? The following theorem, is the basis of this investigation for the space of integrands $\tilde{\mathcal{H}}$

**Theorem 2.1.** *[5, Theorem 14]* Let $\tilde{\mathcal{H}}$ be some class of integrands and let $\mathcal{E} \subset \tilde{\mathcal{H}}$ be the class of step functions and $I^H(f)$ be an integral of $f \in \mathcal{E}$ w.r.t. fBm $(B_t^{(H)})_{t\in T}$, $H \in (0, 1)$. Under the assumptions

**i)** $\tilde{\mathcal{H}}$ is an inner product space with an inner product $< f, g >_{\tilde{\mathcal{H}}}$, $f, g \in \tilde{\mathcal{H}}$,

**ii)** for $f, g \in \mathcal{E}$,

$$< f, g >_{\tilde{\mathcal{H}}} = \mathbb{E}(I^H(f)I^H(g)),$$

**iii)** the set $\mathcal{E}$ is dense in $\tilde{\mathcal{H}}$,

we have the following:

1. there is an isometry between the space $\tilde{\mathcal{H}}$ and a linear subspace of $S_{PT}(B^{(H)})$ which is an extension of the map $f \to I^H(f)$ for $f \in \mathcal{E}$;

2. $\tilde{\mathcal{H}}$ is isometric to $S_{PT}(B^{(H)})$ if and only if $\tilde{\mathcal{H}}$ is complete.

If $\tilde{\mathcal{H}}$ is complete, we have $\tilde{\mathcal{H}} = \mathcal{H}$ The isometry constructed via this theorem is exactly the definition of the Wiener integral w.r.t. the fBm.

$$I^H(f) = \int_T f(s)dB_s^{(H)}, \quad f \in \tilde{\mathcal{H}}$$

**Proof.**

1. Let $f \in \tilde{\mathcal{H}}$. By (iii), there exists $f_n \in \mathcal{E}$ such that $f_n \to f$ in $\tilde{\mathcal{H}}$ In particular it is a Cauchy sequence in $\tilde{\mathcal{H}}$. By (ii) $I^H(f_n)$ is a Cauchy sequence in $L^2(\Omega)$, hence it converges to some random variable $\varsigma \in L^2(\Omega)$, since $L^2(\Omega)$ is complete. We set $I^H(f) = \varsigma$, which means
$$I^H(f) = \lim_{n\to\infty} I^H(f_n)$$
in the $L^2(\Omega)$-sense. Since $(I^H(f_n))_{n\in\mathbb{N}} \subset \overline{S_{pT}(B^H)}$ and $\overline{S_{pT}(B^H)}$ is a closed subspace of $L^2(\Omega)$, we obtain that $I^H(f) \in \overline{S_{pT}(B^H)}$. Therefore, we can define the map $I^H : \tilde{\mathcal{H}} \to \overline{S_{PT}(B^H)}$. We can verify that this construction does not depend on the choice of the sequence $(f_n)_{n\in\mathbb{N}}$. So it is well-defined. Moreover, for any $f,\ g \in \tilde{\mathcal{H}}$ it holds that
$$< f, g >_{\tilde{\mathcal{H}}} = \lim_{n\to\infty} < f_n, g_n >_{\tilde{\mathcal{H}}} = \lim_{n\to\infty} \mathbb{E}(I^H(f_n)I^H(g_n)) = \mathbb{E}(I^H(f)I^H(g))$$
Since $I^H$ is linear, we get an isometry between $\tilde{\mathcal{H}}$ and some subspace of $\overline{S_{PT}(B^H)}$.

2. If $\tilde{\mathcal{H}}$ is isometric to $\overline{S_{PT}(B^H)}$ itself, then $\hat{\mathcal{H}}$ is complete because the space $\overline{S_{PT}(B^H)}$ is complete since it is a closed subset of the complete space $L^2(\Omega)$. Conversely, if $\tilde{\mathcal{H}}$ is complete, then for any $\theta \in \overline{S_{PT}(B^H)}$, we have $\theta = \lim_{n\to\infty} \theta_n$, $\theta_n = I^H(f_n) \in \overline{S_p(B^H)}$, $f_n \in \mathcal{E}$. So $I^H(f_n) \xrightarrow{L^2} \theta$. Therefore, from (ii) it follows that $f_n$ is a Cauchy sequence in $\tilde{\mathcal{H}}$, and from completeness, $f_n \to f$ in $\tilde{\mathcal{H}}$, $\theta = I^H(f)$.

**Remark 2.1.2.** Let us emphasize that a priori the Wiener integral is different for each case we shall consider. Effectively, $I^H$ might depend on the inner product space $\tilde{\mathcal{H}}$ we chose. If $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ are two different classes of integrands, then their corresponding integral are in general not equal even in $\tilde{\mathcal{H}}_1 \cap \tilde{\mathcal{H}}_2$. In fact, to obtain the equality we must have that their corresponding inner product are equal for functions in their intersection.

There are other approaches that give a construction of the integrands space of the integrands using the integral representations ( we have defined previously in section 1.3.6), using different point of views which are in fact essentially the same. We took the basic one that we need in the following chapter. For more details we refer to [5].

### 2.1.3 Young Integral

Since, for $H \in (0,1)$, $(B_t)_{t\geq 0}$ does not have absolutely continuous paths, we can not directly use the theory of Riemann-Stieltjes integrals to give a sense to integrals like $\int_0^t f(s)dB_s$ for every continuous functions $f$. However, as it was understood by L.C. Young, if $f$ is regular enough in the Hölder sense, then $\int_0^t f(s)dB_s$ can still be constructed as a limit of Riemann sums. In the sequel, we shall denote by $C^\alpha(I)$ the space of $\alpha$- Hölder continuous functions that are defined on an interval $I$. The basic result of L.C. Young is the following:

**Theorem 2.2.** *[39]* Let$f \in C^\beta([0,T])$ and $g \in C^\gamma([0,T])$. If $\beta+\gamma>1$, then for every subdivision $t_i^n$ of $[0,T]$, whose mesh tends to 0, the Riemann sums

$$\sum_{i=0}^{n-1} f(t_i^n)(g(t_{i+1}^n)-g(t_i^n))$$

converge, when $n\to\infty$ to a limit which is independent of the subdivision $t_i^n$ . This limit is denoted $\int_0^T fdg$ and called *the Young's integral* of $f$ with respect to $g$.

#### 2.1.3.1 Fractional calculus

Another way to handle Young's integrals is to use the so-called fractional calculus. Let $f \in L^1(a,b)$ and $\alpha>0$. The left-sided and right-sided fractional integrals of $f$ of order $\alpha$ are defined respectively by:

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y)dy$$

and

$$I_{b-}^\alpha f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)}\int_x^b (y-x)^{\alpha-1} f(y)dy,$$

where $(-1)^\alpha = e^{i\pi\alpha}$ and $\Gamma(\alpha)=\int_0^\infty u^{\alpha-1}e^{-u}du$ is the Gamma function. Let us denote by $I_{a+}^\alpha(L^p)$ (respectively $I_{b-}^\alpha(L^p)$) the image of $L^p(a,b)$ by the operator $I_{a+}^\alpha$ (respectively $I_{b-}^\alpha$). If $f\in I_{a+}^\alpha(L^p)$ (respectively $f\in I_{b-}^\alpha(L^p)$) and $0<\alpha<1$, we define for $x\in(a,b)$ the left and right Weyl derivatives by:

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^\alpha} + \alpha\int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}}dy\right)\mathbb{1}_{(a,b)}(x)$$

and respectively,

$$D_{b-}^\alpha f(x) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(b-x)^\alpha} + \alpha\int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}}dy\right)\mathbb{1}_{(a,b)}(x)$$

We have the following property:

$$D_{a+}^\alpha D_{a+}^\beta = D_{a+}^{\alpha+\beta}, \qquad D_{b-}^\alpha D_{b-}^\beta = D_{b-}^{\alpha+\beta}$$

and for $f \in I^{\alpha}_{a+}(L^p), \quad g \in I^{\alpha}_{b-}(L^p)$

$$\int_a^b D^{\alpha}_{a+} f(t) g(t) dt = (-1)^{-\alpha} \int_a^b f(t) D^{\alpha}_{b-} g(t) dt$$

The key point that allows to use fractional calculus to study Young's integrals is the following Proposition which is due to M. Zähle [?].

**Proposition 2.1.2.** *[17]Let $f \in C^{\lambda}([a,b])$ and $g \in C^{\beta}([a;b])$ with $\lambda + \beta > 1$: Let $1 - \beta < \alpha < \lambda$. Then the Young's integral exists and it can be expressed as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f(t) D^{1-\alpha}_{b-} g_{b-}(t) dt;$$

*where $g_{b-}(t) = g(t) - g(b)$.*

### 2.1.4 Russo-Vallois Integral

**Definition 2.1.7.** Let $X, Y$ be two real continuous processes defined on $[0,T]$. The symmetric integral (in the sense of *Russo-Vallois*) is defined by

$$\int_0^T Y_u d^{\circ} X_u = \mathbb{P}\text{-}\lim_{\varepsilon \to 0} \int_0^T \frac{Y_{u+\varepsilon} + Y_u}{2} \frac{X_{u+\varepsilon} - X_u}{\varepsilon} du, \tag{2.4}$$

provided the limit exists and with the convention that $Y_t = Y_T$ and $X_t = X_T$ when $t > T$.

**Theorem 2.3.** *([29], page793) The symmetric integral $\int_0^T f(B^H_u) d^{\circ} B^H_u$ exists for any $f : \mathbb{R} \to \mathbb{R}$ of class $C^5$ if and only if $H \in (\frac{1}{6}, 1)$. In this case, we have, for any primitive $F$ of $f$:*

$$F(B^H_T) = F(0) + \int_0^T f(B^H_u) d^{\circ} B^H_u.$$

When $H \leq 1/6$, one can consider the so-called m-order Newton-Côtes integral:

**Definition 2.1.8.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, let $X$, $Y$ be two continuous processes on $[0,T]$ and let $m \geq 1$ be an integer. The *m-order Newton-Côtes integral* (in the sense of Russo-Vallois) of $f(Y)$ with respect to $X$ is defined by

$$\int_0^T f(Y_u) d^{NC,m} X_u = \mathbb{P}\text{-}\lim_{\varepsilon \to 0} - \int_0^T \left( \int_0^1 f(Y_s + \beta(Y_{s+\varepsilon} - Y_s)) \mu_m(d\beta) \right) \frac{X_{u+\varepsilon} - X_u}{\varepsilon} du,$$

provided the limit exists and with the same convention above with $\mu_1 = \frac{1}{2}(\delta_0 + \delta_1)$ and, for $m \geq 2$,

$$\mu_m = \sum_{j=0}^{2(m-1)} \left( \int_0^1 \prod_{j \neq k} \frac{2(m-1)u - k}{j - k} du \right) \delta_{\frac{j}{(2m-2)}},$$

$\delta$ being the Dirac measure.

**Theorem 2.4.** *([29], page793) Let $m \geq 1$ be an integer. The m-order Newton-Côtes integral $\int_0^T f(B_u^H) d^{NC,m} B_u^H$ exists for any $f : \mathbb{R} \to \mathbb{R}$ of class $C^{4m+1}$ if and only if $H \in (\frac{1}{4m+2}, 1)$. In this case, we have, for any primitive $F$ of $f$:*

$$F(B_T^H) = F(0) + \int_0^T f(B_u^H) d^{NC,m} B_u^H.$$

### 2.1.5 Skorohod Integral

In this section we focus on the Skorohod integral. This stochastic integral, introduced for the first time by A. Skorohod in 1975, may be regarded as an extension of the Itô integral to integrands that are not necessarily $\mathbb{F}$-adapted. The Skorohod integral is also connected to the Malliavin derivative, which is introduced with full detail in [20, Chap. 3].

Let $u = u(t, \omega)$, $t \in [0, T]$, $\omega \in \Omega$, be a measurable stochastic process such that, for all $t \in [0, T]$, $u(t)$ is a $\mathcal{F}_T$ -measurable random variable and

$$\mathbb{E}[u^2(t)] < \infty.$$

Then, for each $t \in [0, T]$, we can apply the Wiener-Itô chaos expansion to the random variable $u(t) = u(t, \omega)$, $\omega \in \Omega$, and thus there exist symmetric functions $f_{n,t} = f_{n,t}(t_1, ..., t_n)$, $(t_1, ..., t_n) \in [0, T]^n$, in $\tilde{L}^2([0, T]^n)$, $n = 1, 2, ...$, such that

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}),$$

where

$$I_n(f) = \int_{[0,T]^n} f(t_1, ..., t_n) dW(t_1)...dW(t(n)),$$

$(W_t)_{t \in [0,T]}$ is a Wiener process and $f \in \tilde{L}^2([0, T]^n)$, and the convergence takes place in $L^2(\mathbb{P})$. Moreover, we have the isometry

$$\parallel u \parallel^2_{L^2(\mathbb{P})} = \sum_{n=0}^{\infty} n! \parallel f_n \parallel^2_{L^2([0,T]^n)} \tag{2.5}$$

for more details see [20]. Note that the functions $f_{n,t}$, $n = 1, 2, ...$, depend on the parameter $t \in [0, T]$, and so we can write

$$f_n(t_1, ..., t_n, t_{n+1}) = f_n(t_1, ..., t_n, t) := f_{n,t}(t_1, ..., t_n)$$

and we may regard $f_n$ as a function of $n+1$ variables. Since this function is symmetric with respect to its first $n$ variables, its symmetrization $\tilde{f}_n$ is given by

$$\tilde{f}_n(t_1, ..., t_{n+1}) = \frac{1}{n+1} \left[ f_n(t_1, ..., t_{n+1}) + f_n(t_2, ..., t_{n+1}, t_1) + ... + f_n(t_1, ...., t_{n-1}, t_{n+1}, t_n) \right] \tag{2.6}$$

**Definition 2.1.9.** Let $u(t)$, $t \in [0,T]$, be a measurable stochastic process such that for all $t \in [0,T]$ the random variable $u(t)$ is $\mathcal{F}_T$ -measurable and $\mathbb{E}[u^2(t)] < \infty$. Let its Wiener-Itô chaos expansion be

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}) = \sum_{n=0}^{\infty} I_n(f_n(.,t)).$$

Then we define the Skorohod integral of $u$ by

$$\delta(u) = \int_0^T u(t)\delta W(t) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$$

when the sum convergent in $L^2(\mathbb{P})$. Here $\tilde{f}_n$, $n = 1, 2, ...$, are the symmetric functions (2.6) derived from $f_n(.,t)$, $n = 1, 2, ....$ We say that $u$ is *Skorohod integrable*, and we write $u \in Dom(\delta)$ if the series $\delta(u)$ converges in $L^2(\mathbb{P})$.

**Remark 2.1.3.** By (2.5) a stochastic process $u$ belongs to $Dom(\delta)$ if and only if

$$\mathbb{E}[\delta(u)^2] = \sum_{n=0}^{\infty}(n+1)! \parallel f_n \parallel^2_{L^2([0,T]^{n+1})} < \infty.$$

#### 2.1.5.1 The Skorohod Integral for fBm

The stochastic Integrals w.r.t fBm were defined mostly for deterministic or linear integrands, but in other cases it was much more complicated to establish such integral, since the path regularity of the fBM varies with the Hurst parameter $H$. In particular, if $H > \frac{1}{2}$ , then the paths of $B^H$ are essentially $\alpha$-Hölder continuous for all $\alpha < H$, hence a pathwise stochastic integral approach is quite effective likewise Young (see [12]). In the general case, especially when $H < \frac{1}{2}$ , the path of fBm becomes rather "rough" and the pathwise approach for stochastic integrals, therefore other definitions of stochastic integrals have been introduced. Most notable is the divergence-type integration (or Skorohod integral), which is based on the idea of Malliavin calculus (see for example [20, 21, 22]), for this case we briefly introduce Malliavin derivative with respect to certain Gaussian processes; in particular, for fractional Brownian motion.
Let $W$ be a standard Brownian motion and assume $G = (G_t)_{t\in[0,T]}$ is a continuous centred Gaussian process of the form

$$G_t = \int_0^t K(t,s)dW_s \tag{2.7}$$

where the kernel $K$ satisfies $\sup\limits_{t\in[0,T]} \int_0^t K(t,s)^2 ds < \infty$. In particular, the fractional Brownian motion is of this form by representation (1.4). First we recall some definitions.

**Definition 2.1.10.** We denote by $\mathcal{E}_G$ the set of simple random variables of the form

$$F = \sum_{k=1}^{n} a_k G_{t_k}$$

where $n \in \mathbb{N}$, $a_k \in \mathbb{R}$ and $t_k \in [0, T]$ for $k = 1, ..., n$.

**Definition 2.1.11.** The Gaussian space $\mathcal{H}_1$ associated to $G$ is the closure of $\mathcal{E}_G$ in $L^2(\Omega)$.

**Definition 2.1.12.** *The reproducing Hilbert space $\mathcal{H}_G$ of $G$ is the closure of $\mathcal{E}_G$ with respect to the inner product*

$$< \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} >_{\mathcal{H}} = R_G(t, s).$$

In what follows we will drop G in the notation.

The mapping $\mathbb{1}_{[0,t]} \to G_t$ can be extended to an isometry between the Hilbert space $\mathcal{H}$ and the Gaussian space $\mathcal{H}_1$. The image of $\varphi \in \mathcal{H}$ in this isometry is denoted by $G(\varphi)$. In particular, we have $G(\mathbb{1}_{[0,t]}) = G_t$.

**Definition 2.1.13.** Denote by $\mathcal{S}$ the space of all smooth random variables of the form

$$F = f(G(\varphi_1), ..., G(\varphi_n)), \qquad \varphi_1, ..., \varphi_n \in \mathcal{H},$$

where $f \in C_b^\infty(\mathbb{R}^n)$ i.e. $f$ and all its derivatives are bounded. The *Malliavin derivative* $D = D(G)$ of $F$ is an element of $L^2(\Omega, \mathcal{H})$ defined by

$$DF = \sum_{i=1}^{n} \partial_i f(G(\varphi_1), ..., G(\varphi_n))\varphi_i.$$

In particular, $DG_t = \mathbb{1}_{[0,t]}$.

**Definition 2.1.14.** We denote $\mathbb{D}_G^{1,2} = \mathbb{D}^{1,2}$ be the Hilbert space of all square integrable Malliavin derivative random variables defined as the closure of $\mathcal{S}$ with respect to norm

$$\| F \|_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}(\| DF \|_{\mathcal{H}}^2).$$

Now we are ready to define the *divergence operator* $\delta$ as the adjoint operator of the Malliavin derivative $D$.

**Definition 2.1.15.** The domain Dom$\delta$ of the operator $\delta$ is the set of random variables $u \in L^2(\Omega, \mathcal{H})$ satisfying

$$|\mathbb{E}(< DF, u >_{\mathcal{H}})| \le c_u \| F \|_{L^2}$$

for any $F \in \mathbb{D}^{1,2}$ and some constant $c_u$ depending only on $u$. For $u \in$ Dom $\delta$ the divergence operator $\delta(u)$ is a square integrable random variable defined by the duality relation

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(< DF, u >_{\mathcal{H}}), \qquad \forall F \in \mathbb{D}^{1,2}$$

for any $F \in \mathbb{D}^{1,2}$.

We use the notation

$$\delta(u) = \int_0^T u_s \delta G_s.$$

Recall now the special form of $G$ given by (2.7) which is clearly the fractional Brownian motion, and define a linear operator $K^*$ from $\mathcal{E}$ to $L^2[0,T]$ by

$$(K^*\varphi)(s) = \varphi(s)K(T,s) + \int_s^T [\varphi(t) - \varphi(s)]K(dt,s).$$

With the help of this operator according to [22], the Hilbert space $\mathcal{H}$ generated by $G$ can be represented as $\mathcal{H} = (K^*)^{-1}(L^2[0,T])$. Furthermore, $\mathbb{D}^{1,2}_G(\mathcal{H}) = (K^*)^{-1}(\mathbb{D}^{1,2}_W(L^2[0,T]))$. Moreover, we can represent $\delta^{(G)}$ with $\delta^{(W)}$ by the relation

$$\int_0^t u_s \delta G_s = \int_0^t (Ku)_s \delta W_s$$

provided that $Ku \in$ Dom $\delta^{(W)}$.

### 2.1.6 Itô's Formula for fBm

In this section we will show the Itô formula for indefinite Skorohod integral.

**Theorem 2.5.** *[9] Let $F$ be a function of class $C^2(\mathbb{R})$. For each $t \in [0,T]$ the following formula holds*

$$f(B^H(t)) = f(0) + \int_0^t f'(B^H(s))\delta B^H(s) + H\int_0^t f''(B^H(s))s^{2H-1}ds$$

## 2.2 Stochastic Integration with respect to the Sub fractional Brownian motion

As we have seen in chapter 1, from the fact that SfBm is not a semimartingale nor a Markov process, Itô's classical calculus are not available for such process, this is why another approach was introduced by several authors: Tudor [14] and Ruiz de Chavez 2008, Tudor [7] 2010 in order to define a stochastic integration w.r.t the sub fractional Brownian motion.

These results were recently proved, we will introduce only two results in this section that are: construction of the Wiener integral w.r.t sfBm, and decomposition result for sfBm.

### 2.2.1 The Wiener integral w.r.t. SfBm

We need first to give a representation of the sub fractional Brownian motion. For $k \in (-\frac{1}{2}, \frac{1}{2})$, $k \neq 0$, we consider a sfBm $(S_t^k)_{t\in[0,T]}$.
Let $f : [0,T] \to \mathbb{R}$ be a measurable application and $\alpha, \sigma, \eta \in \mathbb{R}$. We define the Erdely-Kober-type fractional integral

$$(I^{\alpha}_{T-,\sigma,\eta} f)(s) = \frac{\sigma s^{\sigma\eta}}{\Gamma(\alpha)} \int_s^T \frac{t^{\sigma(1-\alpha-\eta)-1} f(t)}{(t^\sigma - s^\sigma)^{1-\alpha}} dt, s \in [0,T], \alpha > 0.$$

We introduce the following kernel

$$n(t,s) = \frac{\sqrt{\pi}}{2^k} I^k_{T-,2,\frac{1-k}{2}} (u^k \mathbb{1}_{0,t)})(s).$$

We fix a Brownian motion $(W_t)_{t\geq 0}$

**Theorem 2.6.** *[14] We have the Wiener integral representation*

$$S_t^k \stackrel{d}{=} c_k \int_0^1 n(t,s) dW_s, \;\; t \in [0,T], \tag{2.8}$$

$$c_k^2 = \frac{\Gamma(2k+2) \sin \pi (k + \frac{1}{2})}{\pi}$$

Let $S^k$ is a sfBm given pathwise by the right-hand side of (2.8). Denote $\mathcal{E}_T$ the family of elementary functions $f : [0,T] \to \mathbb{R}$,

$$f = \sum_{j=1}^{N-1} a_j \mathbb{1}_{[t_j, t_{j+1})}, \;\; 0 = t_0 < t_1 < ... < t_N = T, \;\; a_j \in \mathbb{R}.$$

For $f$ as above we define the Wiener integral $\int_0^T f(t) dS_t^k$ in the natural way by

$$\int_0^T f(t) dS_t^k = \sum_{j=1}^{N-1} a_j (S^k_{t_{j+1}} - S^k_{t_j}).$$

We endow $\mathcal{E}_T$ with the inner product

$$< f, g >_{\Lambda^{sf}_{k,T}} = \mathbb{E}\left[ \int_0^T f(t) dS_t^k \int_0^T g(t) dS_t^k \right]$$

(we identify the elements $f$ and $g$ when $< f-g, f-g >_{\Lambda^{sf}_{k,T}} = 0$). For $f \in \mathcal{E}_T$.

**Definition 2.2.1.** The completion of $(\mathcal{E}_T, < .,. >_{\Lambda^{sf}_{k,T}})$ is called *the domain of the Wiener integral* and we denote it by $(\Lambda^{sf}_{k,T}, < .,. >_{\Lambda^{sf}_{k,T}})$.
For $f \in \Lambda^{sf}_{k,T}$ we define the *Wiener integral* of $f$ with respect to $S^k$ by

$$\int_0^\infty f(t) dS_t^k = L^2(\Omega, \mathcal{F}, \mathbb{P})\text{-} \lim_{n\to\infty} \int_0^\infty f_n(t) dS_t^k,$$

where $(f_n)_n \in \mathcal{E}_T$ is such that $< f_n - f, f_n - f >_{\Lambda^{sf}_{k,T}} \to 0$.

### 2.2.2 Decomposition of SfBm

Denote

$$\mid \Lambda \mid_{X^k} = \left\{ f : [0,T] \to \mathbb{R} : \int_0^T \int_0^T \mid f(s)f(t) \mid (s+t)^{2k-1} ds dt < \infty \right\}$$

we define the process $(X_t^k)_{t\in[0,T]}$ as the Wiener integral

$$X_t^k = \int_0^\infty (1 - e^{-\theta t})\theta^{-k-1} dW_\theta.$$

**Remark 2.2.1.** *The centered Gaussian process* $(X_t^k)_{t\in[0,T]})$ *has the covariance*

$$C_{X^k}(s,t) = -\frac{\Gamma(1-2k)}{k(2k+1)} K(s,t)$$

*and the representation*

$$X_t^k = \int_0^t Y_s^k ds$$

*with*

$$Y_t^k = \int_0^1 e^{-\theta t}\theta^{-k} dW_\theta.$$

**Lemma 2.2.1.** *[7] We have the inclusion* $\mid \Lambda \mid_{X^k} \subset \Lambda_{X^k}$ *and the relation*

$$\parallel f \parallel^2_{\Lambda_{X^k}} = \Gamma(1-2k) \int_0^T \int_0^T f(s)f(t)(s+t)^{2k-1} ds dt, \quad f \in \mid \Lambda \mid_{X^k}.$$

*Moreover, if* $f \in L^1([0,T], t^{k-\frac{1}{2}}dt)$ *then* $f \in \mid \Lambda \mid_{X^k}$ *and*

$$\int_0^T f(t) dX_t^k = \int_0^T f(t) Y_t^k dt.$$

The main result is:

**Theorem 2.7.** *[7] Let* $k \in (-\frac{1}{2}, 0)$ *and let* $(B_t^k)_{t\in[0,T]}$ *be a fBm independent of the Bm* $(W_t)_{t\in[0,T]}$*. Then the process*

$$S_t^k = \sqrt{-\frac{k(2k+1)}{\Gamma(1-2k)}} X_t^k + B_t^k, \ \ t \in [0,T]$$

*is a sfBm. In particular,*

$$\Lambda_{X^k} \cap \Lambda_{B^k} = \Lambda_{S^k}.$$

*Moreover, if*

$$f \in I_{T-}^{-k}\left(L^2([0,T])\right) \cap L^1\left([0,T], t^{k-\frac{1}{2}}dt\right),$$

*then* $f \in \Lambda_{S^k}$ *and*

$$\int_0^T f(t) dS_t^k = \sqrt{-\frac{k(2k+1)}{\Gamma(1-2k)}} \int_0^T f(t) Y_t^k dt + \int_0^T f(t) dB_t^k.$$

## 2.3 Itô's Formula for SfBm

Itô's lemma is an identity used in Itô calculus to find the differential of a time-dependent function of a stochastic process. The lemma is widely employed in mathematical finance, and its best known application is in the derivation of the Black-Scholes equation for option values.
In this section we introduce Itô's formula for the SfBm in two cases: the first one when the Hurst parameter $H > 1/2$ was proved by Litan, Guangjun, Kun in 2012 [8], and the seconde case when $H < 1/2$ was done by Guangjun Shena, Chao Chenb in 2011 [19]

### 2.3.1 The case when $H > \frac{1}{2}$

**Theorem 2.8.** *[8] Let $F \in C^2(\mathbb{R})$ and $H \in (\frac{1}{2}, 1)$. Then*

$$F(S_t^H) = F(0) + \int_0^t F'(S_s^H) dS_s^H + H(2 - 2^{2H-1}) \int_0^t F''(S_s^H) s^{2H-1} ds.$$

### 2.3.2 The case when $H < \frac{1}{2}$

In this part, we establish versions of Itô's formula for any value of the Hurst parameter $H \in (0, \frac{1}{2})$. The basic result is the next theorem.

**Theorem 2.9.** *[19] Let $F \in C^\infty(\mathbb{R})$. Then for all $t \in [0, T]$, $F'(S_s^H)\mathbb{1}[0,t](s) \in Dom\delta$, and*

$$F(S_t^H) = F(0) + \int_0^t F'(S_s^H) \delta S_s^H + (2 - 2^{2H})(H + \frac{1}{2}) \int_0^t F''(S_s^H) s^{2H} ds.$$

# Chapter 3

# Stochastic Differential Equations

## 3.1 Preliminaries

Differential equations are used to describe the evolution of a system. Stochastic Differential Equations (SDEs) arise when a random noise is introduced into ordinary differential equations (ODEs). For all the proofs of this section we refer to [1]

Let $(B_t)_{t\geq 0}$, be a Brownian motion process. An equation of the form

$$dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dB(t), \tag{3.1}$$

where functions $\mu(x,t)$ and $\sigma(x,t)$ are given and $X(t)$ is the unknown process, is called a stochastic differential equation (SDE) driven by Brownian motion. The functions $\mu(x,t)$ and $\sigma(x,t)$ are called the coefficients.

**Strong solutions to SDEs**

**Definition 3.1.1.** A process X(t) is called a *strong solution* of the SDE 3.1 if for all $t > 0$ the integrals $\int_0^t \mu(X(s),s)ds$ and $\int_0^t \sigma(X(s),s)dB(s)$ exist, with the second being an Itô integral, and

$$X(t) = X(0) + \int_0^t \mu(X(s),s)ds + \int_0^t \sigma(X(s),s)dB(s) \tag{3.2}$$

**Remark 3.1.1.**

1. A strong solution is some function $F(t,(B_s)_{s\leq t})$ of the given Brownian motion $B(t)$.
2. When $\sigma = 0$, the SDE becomes an ordinary differential equation (ODE).
3. Another interpretation of (3.1), called the weak solution, is a solution in distribution which will be given later.

**Weak Solutions to SDEs**

The concept of weak solutions allows us to give a meaning to an SDE when strong solutions do not exist. Weak solutions are solutions in distribution, they can be realized (defined) on some other probability space and exist under less stringent conditions on the coefficients of the SDE.

**Definition 3.1.2.** If there exist a probability space with a filtration, a Brownian motion $\hat{B}(t)$ and a process $\hat{X}(t)$ adapted to that filtration, such that: $\hat{X}(0)$ has the given distribution, for all $t$ the integrals below are defined, and $\hat{X}(t)$ satisfies

$$\hat{X}(t) = \hat{X}(0) + \int_0^t \mu(\hat{X}(u), u)du + \int_0^t \sigma(\hat{X}(u), u)d\hat{B}(u),$$

then $\hat{X}(t)$ is called a *weak solution* to the SDE 3.1

**Definition 3.1.3.** A weak solution is called *unique* if whenever $X(t)$ and $X'(t)$ are two solutions (perhaps on different probability spaces) such that the distributions of $X(0)$ and $X'(0)$ are the same, then all finite-dimensional distributions of $X(t)$ and $X'(t)$ are the same.

### 3.1.1 Existence and Uniqueness of Weak Solutions

**Theorem 3.1.** *If for each $t > 0$, functions $\mu(x,t)$ and $\sigma(x,t)$ are bounded and continuous then the SDE (3.1) has at least one weak solution starting at time $s$ at point $x$, for all $s$, and $x$.*
*In addition if their partial derivatives with respect to $x$ up to order two are also bounded and continuous, then the SDE (3.1) has a unique weak solution starting at time $s$ at point $x$. Moreover this solution has the strong Markov property.*

**Theorem 3.2.** *If $\sigma(x,t)$ is positive and continuous and for any $T > 0$ there is $K_T$ such that for all $x \in \mathbb{R}$*

$$| \mu(x,t) | + | \sigma(x,t) | \leq K_T(1 + | x |)$$

*then there exists a unique weak solution to SDE (3.1) starting at any point $x \in \mathbb{R}$ at any time $s \geq 0$, moreover it has the strong Markov property.*

### 3.1.2 Existence and Uniqueness of Strong Solutions

Let $X(t)$ satisfies the SDE (3.1)

**Theorem 3.3.** ***(Existence and Uniqueness)*** If the following conditions are satisfied

1. Coefficients are locally Lipschitz in $x$ uniformly in $t$, that is, for every $T$ and $N$, there is a constant $K$ depending only on $T$ and $N$ such that for all $\mid x \mid, \mid y \mid \leq N$ and all $0 \leq t \leq T$

$$\mid \mu(x,t) - \mu(y,t) \mid + \mid \sigma(x,t) - \sigma(y,t) \mid < K \mid x - y \mid$$

2. Coefficients satisfy the linear growth condition

$$\mid \mu(x,t) \mid + \mid \sigma(x,t) \mid \leq K(1 + \mid x \mid)$$

3. $X(0)$ is independent of $(B_t)_{0 \leq t \leq T}$, and $\mathbb{E}(X^2(0)) < \infty$

Then there exists a *unique strong solution* $X(t)$ of the SDE (3.1) and it has continuous paths, moreover

$$\mathbb{E}(\sup_{0 \leq t \leq T} X^2(t)) < C(1 + \mathbb{E}(X^2(0)))$$

where constant C depends only on K and T .

**Theorem 3.4.** ***(Yamada-Watanabe)*** *Suppose that $\mu(x)$ satisfies the Lipschitz condition and $\sigma(x)$ satisfies a Hölder condition of order $\alpha > 1/2$, that is, there is a constant $K$ such that*

$$\mid \sigma(x) - \sigma(y) \mid < K \mid x - y \mid^{\alpha}$$

*Then the strong solution exists and is unique.*

## 3.2 Stochastic Differential Equations driven by Fractional Brownian motion

In this section, we study the well-posedness of a class of stochastic differential equations driven by fractional Brownian motions with arbitrary Hurst parameter $H \in (0,1)$ we consider the following SDE:

$$dX_t = \sigma(X_t)dB_t^H + b(X_t)dt, \qquad t \in [0,T] \tag{3.3}$$

allowing to study this SDE under which conditions on the coefficients, using the previous integrals to give the existence and uniqueness of solution for each case.

### 3.2.1 SDEs in the sense of Russo-Vallois integral

The mathematician I Nourdin describes a theory to study the SDE (3.3) in 2007 [13], he gaves conditions that insure the existence and uniqueness, using the integral of Russo-Vallois in order to make sense to $\int_0^t (\sigma(X_s)dB_s^H$ but just considering integrands of the form $f(B^H)$ with $f$ regular enough for this reason he choose several definitions of solution for each case, we will mention them in this subsection.
In the sequel, we put $n_H = \inf\{n \geq 1 : H > \frac{1}{4n+2}\}$.

**Definition 3.2.1.** *Assume that $\sigma \in C^{4n_H+1}$.*

- *Let $\mathcal{C}_1$ be the class of processes $X : [0,T] \times \Omega \to \mathbb{R}$ verifying that there exist $f : \mathbb{R} \to \mathbb{R} \in C^{4n_H+1}$ such that, a.s.,$\forall t \in [0,T], X_t = f(B_t^H)$.*
- *A process $X : [0,T] \in \mathbb{R}$ is a solution to (3.3) if:*

  **i)** $X \in C_1$,

  **ii)** $\forall t \in [0,T], X_t = x_0 + \int_0^t \sigma(X_s) d^{NC} B_s^H + \int_0^t b(X_s) ds$.

**Theorem 3.5.** *Let $\sigma \in C^{4n_H+1}$ be a Lipschitz function, $b$ be a continuous function and $x_0$ be a real. Then the equation (3.3) admits a solution $X$ in the sense of Definition 3.2.1 if and only if $b$ vanishes on $\mathcal{S}(\mathbb{R})$, where $\mathcal{S}$ is the unique solution to $\mathcal{S}' = \sigma \circ \mathcal{S}$ with initial value $\mathcal{S}(0) = x_0$. In this case, $X$ is unique and is given by $X_t = \mathcal{S}(B_t^H)$.*

**Proof.** Assume that $X_t = f(B_t^H)$ is a solution to (3.3) in the sense of Definition 3.2.1 Then, we have

$$f(B_t^H) = x_0 + \int_0^t \sigma \circ f(B_s^H) d^{NC} B_s^H + \int_0^t b \circ f(B_s^H) ds = G(B_t^H) + \int_0^t b \circ f(B_s^H) ds, \tag{3.4}$$

where $G$ is the primitive of $\sigma \circ f$ verifying $G(0) = x_0$. Put $h = f - G$ and denote by $\Omega^*$ the set of $\omega \in \Omega$ such that $t \to B_t^H$ is derivable in at least one point $t_0 \geq 0$ (it is well-known that. If $h'(B_{t_0}^H(\omega)) \neq 0$ for one $t_0 \in [0,T]$ and one $\omega \in \Omega$ then $h$ is strictly monotonous in a neighborhood of $B_{t_0}^H(\omega)$ and, for $|t - t_0|$ sufficiently small, we have

$$B_t^H(\omega) = h^{-1}\left(\int_0^t b(X_s(\omega)) ds\right)$$

and, consequently, $\omega \in \Omega^*$. Then, a.s., $h'(B_t^H) = 0$ for all $t \in [0,T]$ and $h \equiv 0$. By uniqueness, we deduce $f = \mathcal{S}$. Thus, if (3.3) admits a solution $X$ in the sense of Definition 3.2.1, we have necessarily $X_t = \mathcal{S}(B_t^H)$. Thanks to (3.4), we then have

$$b \circ \mathcal{S}(B_t^H) = 0 \qquad \text{for all } t \in [0,T] \quad a.s.$$

and then $b$ vanishes on $\mathcal{S}(\mathbb{R})$.

Now another definition where the integrand is of the forme $f(B^H + A)$ with $A \in \mathcal{A}$ and $\mathcal{A}$ the set of processes $A : [0,T] \to \mathbb{R}$ having $C^1$-trajectories and verifying that

$$\mathbb{E}\left(e^{\lambda \int_0^T A_s^2 ds}\right) < \infty \qquad \text{for some } \lambda > 1.$$

**Definition 3.2.2.** *Assume that $\sigma \in C^{4n_H+1}$.*

- *Let $\mathcal{C}_2$ be the class of processes $X : [0,T] \times \Omega \to \mathbb{R}$ verifying that there exist a function $f : \mathbb{R} \to \mathbb{R} \in C^{4n_H+1}$ and a process $A \in \mathcal{A}$ such that $A_0 = 0$ and, a.s., $\forall t \in [0,T]$, $X_t = f(B_t^H + A_t)$.*

- *A process* $X : [0,T] \times \Omega \to \mathbb{R}$ *is a solution to* (3.3) *if:*

  **i)** $X \in \mathcal{C}_2$,

  **ii)** $\forall t \in [0,T], \quad X_t = x_0 + \int_0^t \sigma(X_s) d^{NC} B_s^H + \int_0^t b(X_s) ds.$

**Theorem 3.6.** *Let* $\sigma \in C^{4n_H+1}$ *be a Lipschitz function,* $b$ *be a continuous function and* $x_0$ *be a real.*

- *If* $\sigma(x_0) = 0$ *then* (3.3) *admits a solution* $X$ *in the sense of Definition* 3.2.2 *if and only if* $b(x_0) = 0$*. In this case,* $X$ *is unique and is given by* $X_t \equiv x_0$*.*

- *If* $\sigma(x_0) \neq 0$*, then* (3.3) *admits a solution* $X$*. If moreover* $\inf_{\mathbb{R}} |\sigma| > 0$ *and* $b \in Lip$ *then* $X$ *is unique.*

**Proof** Assume that $X = f(B^H + A)$ is a solution to (3.3) in the sense of Definition 3.2.2. Then, we have

$$f(B_t^H + A_t) = G(B_t^H + A_t) - \int_0^t \sigma(X_s) A_s ds + \int_0^t b(X_s) ds \tag{3.5}$$

where $G$ is the primitive of $\sigma \circ f$ verifying $G(0) = x_0$. As in the proof of the previous theorem, we obtain that $f = \mathcal{S}$ where $\mathcal{S}$ is defined by $\mathcal{S}' = \sigma \circ \mathcal{S}$ with initial value $\mathcal{S}(0) = x_0$. Thanks to (3.5), we deduce that, a.s., $b \circ S(B_t^H + A_t) = A_t' \sigma \circ \mathcal{S}(B_t^H + A_t)$ for all $t \in [0,T]$. Consequently:

- If $\sigma(x_0) = 0$ then $\mathcal{S} \equiv x_0$ and $b(x_0) = 0$.

- If $\sigma(x_0) \neq 0$ then $\mathcal{S}$ is strictly monotonous and the ordinary integral equation $A_t = \int_0^t \frac{b \circ \mathcal{S}}{\mathcal{S}'}(B_s^H + A_s) ds$ admits a maximal (in fact, global since we know already that $A$ is defined on $[0,T]$) solution thanks to Peano theorem. If moreover $\inf_{\mathbb{R}} |\sigma| > 0$ and $b \in$ Lip then $\frac{b \circ \mathcal{S}}{\mathcal{S}'} = \frac{b \circ \mathcal{S}}{\sigma \circ \mathcal{S}} \in$ Lip and $A$ is uniquely determined.

Finally, we can introduce a last definition for solution to (3.3):

**Definition 3.2.3.** *Assume that* $\sigma \in C^{2m_H}$*, we define* $m_H = \inf\{m \geq 1 : \quad H > 1/(2m+1)\}$*.*

- *Let* $\mathcal{C}_3$ *be the class of processes* $X : [0,T] \times \Omega \to \mathbb{R}$ *verifying that there exist a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *of class* $C^{2m_H}$ *and a process* $A : [0,T] \times \Omega \to \mathbb{R}$ *having* $C^1$*-trajectories such that* $A_0 = 0$ *and, a.s.,* $\forall t \in [0,T], \quad X_t = f(B_t^H, A_t)$*.*

- *A process* $X : [0,T] \times \Omega \to \mathbb{R}$ *is a solution to* (3.3) *if:*

  **i)** $X \in \mathcal{C}_3$,

  **ii)** $\forall t \in [0,T], \quad X_t = x_0 + \int_0^t \sigma(X_s) d^{NC} B_s^H + \int_0^t b(X_s) ds.$

**Theorem 3.7.** *Let $\sigma \in C_b^2$ , $b$ be a Lipschitz function and $x_0$ be a real. Then the equation (3.3) admits a solution $X$ in the sense of Definition 3.2.3. Moreover, if $\sigma$ is analytic, then $X$ is the unique solution of the form $f(B^H, A)$ with $f$ analytic (resp. of class $C^1$) in the first (resp. second) variable and $A$ a process having $C^1$-trajectories and verifying $A_0 = 0$.*

**Proof.** We will concentrate on the uniqueness. Assume that $X = f(B^H, A)$ is a solution to (3.3) in the sense of Definition 3.2.3. On the one hand, we have

$$\begin{aligned} X_t &= x_0 + \int_0^t \sigma(X_s) d^{NC} B_s^H + \int_0^t b(X_s) ds && (3.6)\\ &= x_0 + \int_0^t \sigma \circ f(B_s^H, A_s) d^{NC} B_s^H + \int_0^t b \circ f(B_s^H, A_s) ds. && (3.7) \end{aligned}$$

On the other hand, using the change of variables formula, we can write

$$X_t = x_0 + \int_0^t f'(B_s^H, A_s) d^{NC} B_s^H + \int_0^t f'(B_s^H, A_s) A_s' ds. \qquad (3.8)$$

Using (3.7) and (3.8), we deduce that $t \to \int_0^t \varphi'(B_s^H, A_s) d^{NC} B_s^H$ has $C^1$-trajectories where $\varphi' = f' - \sigma \circ f$. As in the proof of Theorem 3.6, we show that, a.s.,

$$\forall t \in ]0, T[, \varphi'(B_t^H, A_t) = 0.$$

Similarly, we can obtain that, a.s.,

$$\forall k \in \mathbb{N}, \; \forall t \in ]0, T[, \; \frac{\partial^k \varphi}{\partial b^k}(B_t^H, A^t) = 0.$$

If $\sigma$ and $f(., y)$ are analytic, then $\varphi(., y)$ is analytic and

$$\forall t \in ]0, T[, \; \forall x \in \mathbb{R}, \; \varphi(x, A_t) = f'(x, A_t) - \sigma \circ f(x, A_t) = 0.$$

By uniqueness, we deduce

$$\forall t \in [0, T], \; \forall x \in \mathbb{R}, \; f(x, A_t) = u(x, A_t),$$

where $u$ is the unique solution to $u' = \sigma(u)$ with initial value $u(0, y) = y$ for any $y \in \mathbb{R}$. In particular, we obtain a.s.

$$\forall t \in [0, T], X_t = f(B_t^H, A_t) = u(B_t^H, A_t).$$

Identity (3.7) can then be rewritten as:

$$X_t = x_0 + \int_0^t \sigma \circ u(B_s^H, A_s) d^{NC} B_s^H + \int_0^t b \circ u(B_s^H, A_s) ds,$$

while change of variables formula yields:

$$X_t = x_0 + \int_0^t u'b(B_s^H, A_s) d^{NC} B_s^H + \int_0^t u'(B_s^H, A_s) A_s' ds.$$

Since $u' = \sigma \circ u$, we obtain a.s.:

$$\forall t \in [0,T], b \circ u(B_t^H, A_t) = u'(B_t^H, A_t)A_t'.$$

But we have existence and uniqueness in the last equation. Then the proof of Theorem is done.

### 3.2.2 SDEs in the sense of Skorohod integral

The well-posedness of a class of stochastic differential equations driven by fractional Brownian motions with arbitrary Hurst parameter $H \in (0,1)$ in the general case where the coefficients are allowed to be random were studied by the Chinese mathematicians YU-JUAN JIEN and JIN MA in 2009, proving the result using the anticipating Girsanov transformation for the fBm which they establish (for more details see [21]).

we assume that all processes are defined on a finite duration $I = [0,T]$. Let $W \overset{not}{=} C^0(I,\mathbb{R})$ be the Banach space of continuous functions defined on $I$ , null at $t = 0$ and equipped with the sup-norm. Let $\mathcal{F} \overset{not}{=} \mathcal{B}(W)$ be the topological $\sigma$-field on $W$ and $\mu_H$ the unique probability measure on $W$ under which the canonical process $B_t^H(\omega) \overset{not}{=} \omega_t$, $t \in I$, is an fBm. $(W, \mathcal{F}, \mu_H)$ then form a canonical space.

We define we define $\mathbb{D}^{1,\infty}(\mathcal{X})$ where $\mathcal{X}$ is a separable Hilbert space, to be the space of all $G \in \mathbb{D}^{1,2}(\mathcal{X})$ (see definition 2.1.14) such that

$$\|G\|_{1,\infty} = \||G|_{\mathcal{X}}\|_\infty \vee \||DG|_{\mathcal{X}}\|_\infty < \infty.$$

with $DG$ is the Malliavin derivative (2.1.13). For later use we denote the space $\mathbb{L}^{1,\infty} = L^2(I, \mathbb{D}^{1,\infty})$, and define the operators $T$ and $A$ for all $\omega \in W$ and $v \leq t \in I$ by

$$(T_t\omega)_s = \omega_s + \int_0^{t\wedge s} K^H(s,r)\sigma_r(T_r\omega)dr,$$

$$(A_{v,t}\omega)_s = \omega_s - \int_{v\wedge s}^{t\wedge s} K^H(s,r)\sigma_r(A_{r,t}\omega)dr$$

such that $T(A\omega) = A(T\omega) = \omega$.

In the sequel, lets consider the stochastic differential equation in the Skorohod sense (see section 2.1.5.1):

$$X_t = X_0 + \int_0^t \sigma_s X_s dB_s^H + \int_0^t b(s, X_s)ds, \qquad t \in [0,T] \tag{3.9}$$

where $X_0 \in L^p(W)$ for some $p \geq 2$, $\sigma \in \mathbb{L}^{1,\infty}$ and $b : I \times \mathbb{R} \times X \to \mathbb{R}$ is a measurable function satisfying the following conditions for $\mu$-a.e. $\omega \in W$:

(**H1**).There exist an integrable function $\gamma_t \geq 0$ on $I$ and a constant $M > 0$ such that

**i)** $\int_0^1 \gamma_t dt \leq M$ and $|b(t,0,\omega)| \leq M$ for any $t \in I$ ;

**ii)** $|b(t,x,\omega) - b(t,y,\omega)| \leq \gamma_t |x-y|$ for all $x,y \in \mathbb{R}$, $t \in I$.

now consider the following ordinary differential equation for any fixed $\omega \in W$:

$$Z_t(\omega,x) = x + \int_0^t L_s^{-1}(T_s\omega) b(s, L_s(T_s\omega) Z_s(\omega,x), T_s\omega) ds, \qquad x \in \mathbb{R},\ t \in I. \quad (3.10)$$

Where $L_t$ denotes the density of $T_t$. It is known from ODE theory that under Assumption (**H1**), the unique solution $Z_t(\omega,x)$, $t \geq 0$, depends continuously on the initial state $x$. Thus, the mapping $(t,\omega) \to Z_t(\omega, X_0(\omega))$ defines a measurable process. Let us now set

$$X_t = L_t Z_t(A_t, X_0(A_t)), \qquad t \in I. \quad (3.11)$$

We need the following lemma to proof the next theorem.

**Lemma 3.2.1.** *[21] Suppose that $F = \{F_t, t \in I\} \in \mathbb{L}^{\mathcal{S}}$ and the mapping $t \to F_t(.)$ is differentiable. Then $\{F_t(T_t), t \in I\}$ is differentiable with respect to $t$ and it holds that*

$$\frac{d}{dt}[F_t(T_t)] = \left(\frac{d}{dt}F_t\right)(T_t) + \sigma_t(T_t)(D_tF_t)(T_t), \qquad \mu - a.e. \quad (3.12)$$

*For any $G \in \mathcal{S}$, the mapping $t \in G(A_t)$ is differentiable and it holds that*

$$\frac{d}{dt}G(A_t) = -\sigma_t D_t[G(A_t)], \qquad \mu - a.e. \quad (3.13)$$

The main result is the following theorem.

**Theorem 3.8.** *[21] The process $\{X_t, t \in I\}$ in (3.11) satisfies $\mathbb{1}_{[0,t]}\sigma X \in Dom(\delta)$ for all $t \in I$ and $X \in L^2(W, L^2(I))$ is the unique solution of the SDE (3.9).*

***Proof.*** Existence. We will show that $\mathbb{1}_{[0,\tau]}\sigma X \in \mathrm{Dom}(\delta)$ for $\tau \in I$ and that SDE (3.9) holds. To this end, let $G \in \mathcal{S}$ and denote $Z_t(., X_0(.))$ by $Z_t(X_0)$. Using (3.11), we have

$$\begin{aligned} \mathbb{E}\left\{G\int_0^1 \mathbb{1}_{[0,\tau]}(t)\sigma_t X_t dB_t\right\} &= \mathbb{E}\left\{\int_0^\tau \sigma_t X_t D_t G dt\right\} \\ &= \mathbb{E}\left\{\int_0^\tau \sigma_t L_t Z_t(A_t, X_0(A_t)) D_t G dt\right\} \quad (3.14) \\ &= \mathbb{E}\left\{\int_0^\tau \sigma_t(T_t) Z_t(X_0) D_t G(T_t) dt\right\}. \end{aligned}$$

Applying Lemma 3.37, (3.12) and integration by parts, (3.14) becomes

$$\begin{aligned}\mathbb{E}\left\{\int_0^\tau \sigma_t(T_t)Z_t(X_0)D_tG(T_t)dt\right\} &= \mathbb{E}\left\{\int_0^\tau Z_t(X_0)\frac{d}{dt}G(T_t)dt\right\} \\ &= \mathbb{E}\left\{Z_\tau(X_0)G(T_t) - Z_0(X_0)G\right. \\ &\quad \left. - \int_0^\tau Z_t(X_0)\left(\frac{d}{dt}Z_t(X_0)\right)G(T_t)dt\right\}\end{aligned} \tag{3.15}$$

Next, using ODE (3.10) as well as the fact that $L_t^{-1}(T_t) = \mathcal{L}_t$ is the density of $A_t$, (3.15) yields that

$$\mathbb{E}\left\{Z_\tau(X_0)G(T_\tau) - Z_0(X_0)G - \int_0^\tau L_t^{-1}(T_t)b(t, L_t(T_t)Z_t(X_0), T_t)G(T_t)dt\right\}$$

$$= \mathbb{E}\{L_\tau Z_\tau(A_\tau, X_0(A_\tau))G\} - \mathbb{E}\{Z_0(X_0)G\} - \mathbb{E}\left\{\int_0^\tau b(t, L_tZ_t(A_t, X_0(A_t)))Gdt\right\}$$

$$\mathbb{E}\{X_\tau G\} - \mathbb{E}\{X_0 G\} - \mathbb{E}\left\{\int_0^\tau b(t, X_t)Gdt\right\} = \mathbb{E}\left\{G\left(X_\tau - X_0 - \int_0^\tau b(t, X_t)dt\right)\right\}$$

This, together with (3.14), leads to the fact that for any $G \in \mathcal{S}$,

$$\mathbb{E}\left\{G\int_0^1 \mathbb{1}_{[0,\tau]}(t)\sigma_t X_t dB_t\right\} = \mathbb{E}\left\{G\left(X_\tau - X_0 - \int_0^\tau b(t, X_t)dt\right)\right\}$$

Since $X_\tau - X_0 - \int_0^\tau b(t, X_t)dt$ is square-integrable, we deduce that $\{\mathbb{1}_{[0,\tau]}\sigma X,\ \tau \in I\}$ belong
to $\mathrm{Dom}(\delta)$ and $X$ satisfies (3.9).

Uniqueness. Let $Y \in L^2(W; L^2(I))$, where $\mathbb{1}_{[0,t]}\sigma Y \in \mathrm{Dom}(\delta)$ for all $t \in I$, be any solution of equation (3.9), that is,

$$Y_t = X_0 + \int_0^t \sigma_s Y_s dB_s^H + \int_0^t b(s, Y_s)ds, \qquad t \in I. \tag{3.16}$$

We consider a fixed $t \in I$ and a random variable $G \in \mathcal{S}$. Multiplying both sides of (3.16) by $G(A_t)$ and taking expectations, it becomes

$$\mathbb{E}\{Y_tG(A_t)\} = \mathbb{E}\{Y_0G(A_t)\} + \mathbb{E}\left\{\int_0^t D_s[G(A_t)]\sigma_s Y_s ds\right\} + \mathbb{E}\left\{\int_0^t b(s, Y_s)G(A_t)ds\right\}.$$

Since $G(A_t) = G(A_s) - \int_s^t \sigma_r D_r[G(A_r)]dr$ for any $s \in [0, t]$ by Lemma 3.37, (3.13), we obtain

$$\begin{aligned}
\mathbb{E}\{Y_t G(A_t)\} &= \mathbb{E}\{Y_0 G\} - \mathbb{E}\left\{Y_0 \int_0^t \sigma_r D_r[G(A_r)]dr\right\} \\
&+\mathbb{E}\left\{\int_0^t D_s[G(A_t)]\sigma_s Y_s ds\right\} - \mathbb{E}\left\{\int_0^t D_s\left[\int_s^t \sigma_r D_r[G(A_r)]dr\right]\sigma_s Y_s ds\right\} \\
&+\mathbb{E}\left\{\int_0^t b(s,Y_s)G(A_s)ds\right\} - \mathbb{E}\left\{\int_0^t b(s,Y_s)\int_s^t \sigma_r D_r[G(A_r)]drds\right\} \quad (3.17)\\
&= \mathbb{E}\{Y_0 G\} + \mathbb{E}\left\{\int_0^t D_s[G(A_s)]\sigma_s Y_s ds\right\} + \mathbb{E}\left\{\int_0^t b(s,Y_s)G(A_s)ds\right\} \\
&-\mathbb{E}\left\{\int_0^t \sigma_r D_r[G(A_r)]Y_0 dr\right\} - \mathbb{E}\left\{\int_0^t\int_0^r D_s\left[\sigma_r D_r[G(A_r)]\right]\sigma_s Y_s dsdr\right\} \\
&-\mathbb{E}\left\{\int_0^t \sigma_r D_r[G(A_r)]\int_0^r b(s,Y_s)dsdr\right\}.
\end{aligned}$$

Here, the last equality is due to Fubini's theorem. Now, by definition of the Skorohod integral,

$$\mathbb{E}\left\{\int_0^t\int_0^r D_s\left[\sigma_r D_r[G(A_r)]\right]\sigma_s Y_s dsdr\right\} = \mathbb{E}\left\{\int_0^t \sigma_r D_r[G(A_r)]\int_0^r \sigma_s Y_s dB_s^H dr\right\}.$$

Note that because the density of $A_t$ is $L_t = L_t^{-1}(T_t)$ and $Y$ satisfies (3.16), (3.17) can be rewritten as

$$\begin{aligned}
\mathbb{E}\{L_t^{-1}(T_t)Y_t(T_t)G\} &= \mathbb{E}\{Y_0 G\} + \mathbb{E}\left\{\int_0^t D_s[G(A_s)]\sigma_s Y_s ds\right\} \\
&+\mathbb{E}\left\{\int_0^t b(s,Y_s)G(A_s)ds\right\} - \mathbb{E}\left\{\int_0^t \sigma_r D_r[G(A_r)]Y_r dr\right\} \\
&= \mathbb{E}\{Y_0 G\} + \mathbb{E}\left\{\int_0^t b(s,Y_s)G(A_s)ds\right\} \\
&= \mathbb{E}\{Y_0 G\} + \mathbb{E}\{\int_0^t L_s^{-1}(T_s)b(s,Y_s(T_s))Gds\}
\end{aligned}$$

Since the smooth random variable $G$ is arbitrary, we have

$$\begin{aligned}
L_t^{-1}(T_t)Y_t(T_t) &= Y_0 + \int_0^t L_s^{-1}(T_s)b(s,Y_s(T_s))ds \\
&= Y_0 + \int_0^t L_s^{-1}(T_s)b(s,L_s(T_s)L_s^{-1}(T_s)Y_s(T_s))ds, \qquad \mu - a.e.
\end{aligned}$$

That is, $L_t^{-1}(T_t)Y_t(T_t)$ is a solution of equation (3.10). By the uniqueness of the ODE, we must have $L_t^{-1}(T_t)Y_t(T_t) = Z_t(.,Y_t)$. Consequently,

$$Y_t = L_t Z_t(A_t, Y_0(A_t)) = X_t, \qquad \mu - a.e.,$$

which is the unique solution of SDE (3.9). This completes the proof.

### 3.2.3 Stochastic differential equations driven by Hölder paths

**Theorem 3.9.** *[39] Let $g \in C^\gamma([0,T])$ where $\frac{1}{2} < \gamma \leq 1$. Let $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ be two functions such that:*

**i)** *$b$ and $\sigma$ are globally Lipschitz continuous;*

**ii)** *$\sigma$ is continuously differentiable with a globally Lipschitz derivative.*

*For every $x_0 \in \mathbb{R}$, the ordinary differential equation*

$$dx(t) = b(x(t))dt + \sigma(x(t))dg(t), \tag{3.18}$$

*has a unique solution in $C^\gamma([0,T])$.*

The basic idea is to generalize this result from the ordinary case to the stochastic one, i.e solving a stochastic differential equations using integration path by path, since the fBm is a continuous process whose path are $\gamma$-Hölderian for every $\gamma < H$. Therefore if $H > 1/2$ the SDE (3.3) admits a unique solution moreover it has the same Hölder property, under the assumption i) and ii).

### 3.2.4 Numerical Solution

Stochastic differential equations which admit an explicit solution are the exception from the rule. Therefore numerical techniques for the approximation of the solution to a stochastic differential equation are called for. In what follows, such an approximation is called a numerical solution.

Numerical solutions are needed for different aims. One purpose is to visualize a variety of sample paths of the solution. A collection of such paths is sometimes called a scenario. It gives an impression of the possible sample path behavior. In this sense, we can get some kind of "prediction" of the stochastic process at future instants of time. But a scenario has to be interpreted with care. In real life we never know the fractional Brownian sample path driving the stochastic differential equation, and the simulation of a couple of such paths is not representative for the general picture.

A second objective (perhaps the most important one) is to achieve reasonable approximations to the distributional characteristics of the solution to a stochastic differential equation. They include expectations, variances, covariance and higher-order moments. This is indeed an important matter since only in a few cases one is able to give explicit formulae for these quantities, and even then they frequently involve special functions which have to be approximated numerically. Numerical solutions allow us to simulate as many sample paths as we want; they constitute the basis for Monte-Carlo techniques to obtain the distributional characteristics. For the purpose of illustration we restrict ourselves to the numerical solution of the stochastic differential equation driven by fractional Brownian motion. We also assume that the coefficients functions verify the assumption that guarantee the existence and uniqueness of solution.

In this section we are interested in some approximation schemes of Euler and Milstien associated to stochastic differential equation driven fractional Brownian motion, in the sense of specified integral: the Russo-vallois integral 2.4, and of Young integral 2.1.2 when the integrator has Hölder paths the case of fBm. We will define these schemes in order to do some simulation.

We consider the Euler scheme associated to SDE (3.3) with step $\frac{1}{n}$, when the integral with respect to fBm is in the sense of Russo-Vallois symmetric integral 2.4 and with $H > 1/6$ and $k = 0, 1.., n-1$.

$$\begin{cases} \widehat{X}_0^{(n)} = x_0, \\ \widehat{X}_{(k+1)/n}^{(n)} = \widehat{X}_{k/n}^{(n)} + \frac{1}{2}\left(\sigma(\widehat{X}_{k/n}^{(n)}) + \sigma(\widehat{X}_{(k+1)/n}^{(n)})\right)\left(B_{(k+1)/n}^H - B_{k/n}^H\right) + \frac{1}{n} b(\widehat{X}_{k/n}^{(n)}). \end{cases} \tag{3.19}$$

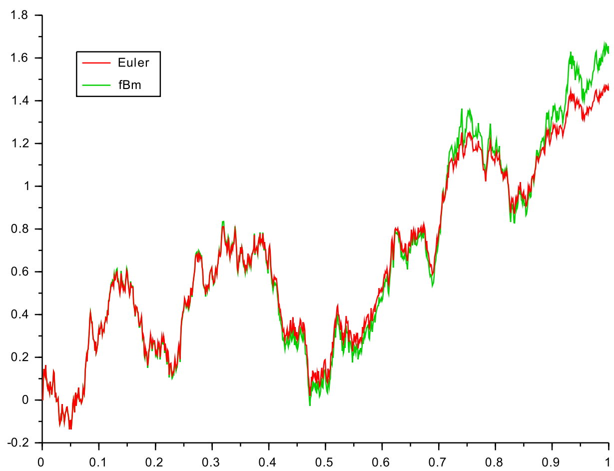


Figure 3.1: The equidistant Euler scheme (3.19): numerical solution and exact solution to the SDE $dX_t = 0.02X_t dt + cos(X_t)dB_t^H$

Next we consider the approximation schemes associated to stochastic differential equations driven by $\alpha$-Hölderian functions the case when these functions are the paths of fBm.

$$\begin{cases} \widehat{X}_0^{(n)} = x_0, \\ \widehat{X}_t^{(n)} = \widehat{X}_{k/n}^{(n)} + \sigma(\widehat{X}_{k/n}^{(n)}) \left(g_t - g_{k/n}\right) + (t - \frac{k}{n}) b(\widehat{X}_{k/n}^{(n)}) \quad t \in [k/n, (k+1)/n]. \end{cases} \tag{3.20}$$

$$\begin{cases} \widehat{X}_0^{(n)} = x_0, \\ \widehat{X}_t^{(n)} = \widehat{X}_{k/n}^{(n)} + \sum_{j=1}^{2m} \frac{1}{j!} P_j(\sigma, \sigma', ..., \sigma^{(j-1)})(\widehat{X}_{k/n}^{(n)}) \left(g_t - g_{k/n}\right)^j + (t - \frac{k}{n}) b(\widehat{X}_{k/n}^{(n)}) \\ t \in [k/n, (k+1)/n]. \end{cases} \tag{3.21}$$

Where $P_j$ are polynomial functions defined by:

$$g' = f \circ g \Rightarrow \forall j \in \mathbb{N}^*, g^{(j)} = P_j(f, f', ..., f^{(j-1)}) \circ g \quad \text{with} \quad P_j \in \mathbb{R}[X_0, ..., X_{j-1}].$$

For example, we have:

$$g' = f \circ g \Rightarrow P_1 = X_0 \in \mathbb{R}[X_0] \quad \text{and} \quad g'' = g' \times (f' \circ g) = (f f') \circ g \Rightarrow P_2 = X_0 X^1 \in \mathbb{R}[X_0, X_1]...ect.$$

The first scheme (3.20) is the associated Euler scheme to SDE (3.18) when $\alpha > \frac{1}{2}$, and the second (3.21) is the Milstein scheme when $\alpha \leq \frac{1}{2}$. With $H > \frac{1}{2m+1}, m \in \mathbb{N}^*$.

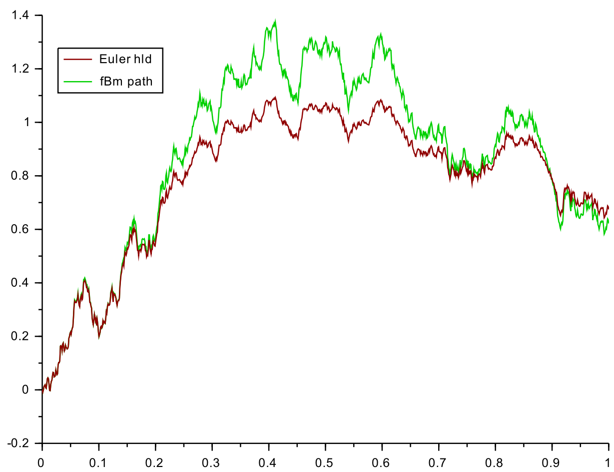


Figure 3.2: The equidistant Euler scheme (3.20): numerical solution and exact solution to the SDE $dX_t = 0.02 X_t dt + cos(X_t) dB_t^H$, $H = 0.6$

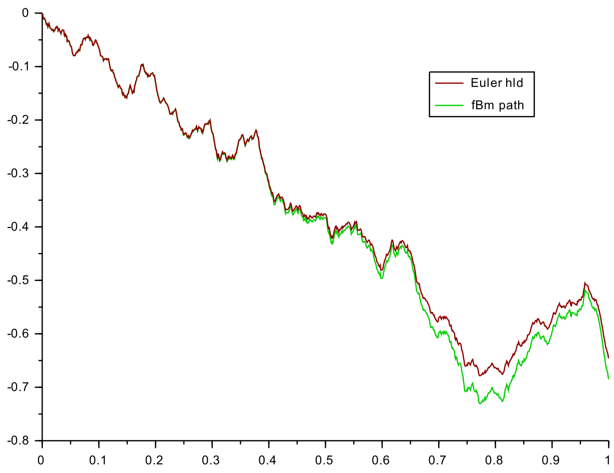


Figure 3.3: The equidistant Euler scheme (3.20): numerical solution and exact solution to the SDE $dX_t = 0.02X_t dt + cos(X_t)dB_t^H,\ H = 0.8$

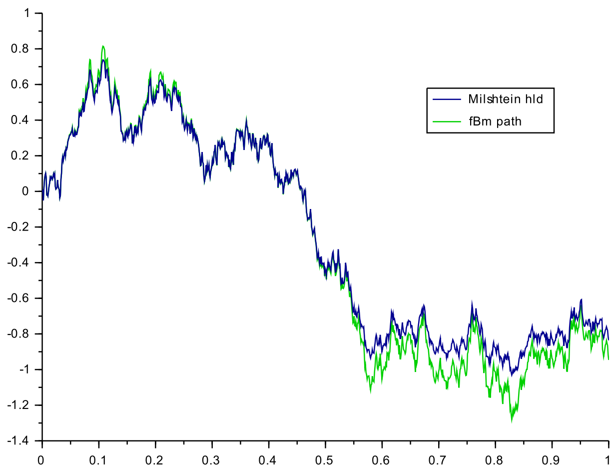


Figure 3.4: The equidistant Milstien scheme (3.21): numerical solution and exact solution to the SDE $dX_t = 0.02X_t dt + cos(X_t)dB_t^H,\ H = 0.5$

### Comparison between the previous schemes

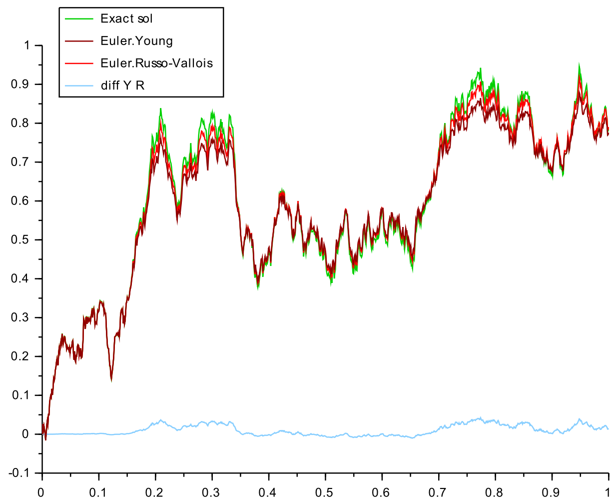


Figure 3.5: The equidistant Euler scheme (3.19) and the Euler scheme (3.20) with $H = 0.6$

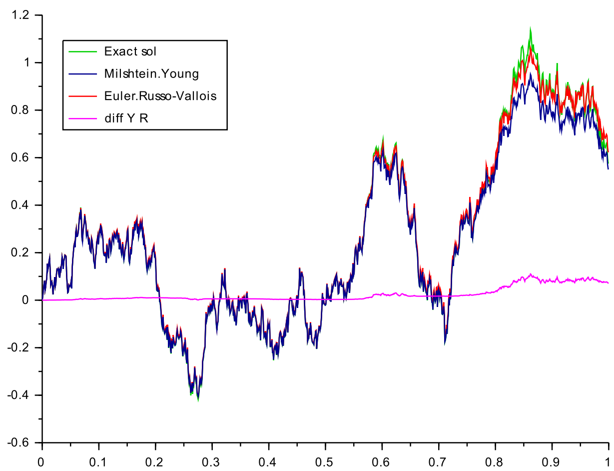


Figure 3.6: The equidistant Milshtein scheme (3.21) and the Euler scheme (3.19) with $H = 0.5$

As we can see in 3.5 that the difference between the schemes (3.19) and (3.20) is null the same as in 3.6 between the schemes (3.21) and (3.19).

## 3.3 Stochastic Differential equations Driven by Sub-fractional Brownian motion

Stochastic differential equations driven by sub-fractional Brownian motion has been considered only by Mendy in 2010 (we could not obtain this paper). Zhi Li, Guoli Zhou and Jiaowan Luo in 2015 have investigate the existence and uniqueness of mild solutions to the stochastic delay differential equation [35] and study its long-time behavior as well which we based on in this section to investigate the existence and uniqueness of solution to stochastic delay evolution equations perturbed by a Sub-fractional Brownian motion with index $H > \frac{1}{2}$, but only mild and weak solution.

Let $\mathcal{H}_{S^H}$ be the canonical Hilbert space associated to the sub-fBm $S^H$. That is the closure of the linear space $\mathcal{E}$ of $\mathbb{R}$-valued step function on $[0, T]$ with respect to the scalar product

$$< \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} >_{\mathcal{H}_{S^H}} = C_H(t, s).$$

We have that the covariance of sub-fractional Brownian motion can be written as

$$\mathbb{E}[S^H(t)S^H(s)] = \int_0^t \int_0^s \phi_H(u, v)dudv = C_H(s, t), \tag{3.22}$$

where $\phi_H(u, v) = H(2H-1)\left[| u - v |^{2H-2} - (u + v)^{2H-2}\right]$. Equation (3.22) implies that

$$< \varphi, \psi >_{\mathcal{H}_{S^H}} = \int_0^t \int_0^t \varphi_u \psi_v \phi_H(u, v)dudv \tag{3.23}$$

for any pair step functions $\varphi$ and $\psi$ on $[0, T]$. Consider the kernel

$$n_H(t, s) = \frac{2^{1-H\sqrt{\pi}}}{\Gamma(H - \frac{1}{2})} s^{\frac{3}{2}-H} \left( \int_s^t (x^2 - s^2)^{H-\frac{3}{2}} dx \right) \mathbb{1}_{[0,t]}(s). \tag{3.24}$$

By Dzhaparidze and Van Zanten [36], we have

$$C_H(t, s) = c_H^2 \int_0^{s\wedge t} n_H(t, u)n_H(s, u)du, \tag{3.25}$$

where

$$c_H^2 = \frac{\Gamma(1 + 2H)\sin(\pi H)}{\pi}$$

Property (3.25) implies that $C_H(s, t)$ is non-negative definite. Consider the linear operator $n_H^*$ from $\mathcal{E}$ to $L^2([0, T])$ defined by

$$n_H^*(\varphi)(s) = c_H \int_s^r \varphi_r \frac{\partial n_H}{\partial r}(r, s)dr.$$

Using (3.23) and (3.25) we have

$$
\begin{aligned}
< n_H^*\varphi, n_H^*\psi >_{L^2([0,T])} &= c_H^2 \int_0^T \left( \int_s^T \varphi_r \frac{\partial n_H}{\partial r}(r,s)dr \right) \left( \int_s^T \psi_u \frac{\partial n_H}{\partial u}(u,s)du \right) ds \\
&= c_H^2 \int_0^T \int_0^T \left( \int_0^{r\wedge u} \frac{\partial n_H}{\partial r}(r,s)\frac{\partial n_H}{\partial u}(u,s)ds \right) \varphi_r \psi_u dr du \\
&= c_H^2 \int_0^T \int_0^T \frac{\partial^2 n_H}{\partial r \partial u}(r,u) ds \varphi_r \psi_u dr du \\
&= H(2H-1) \int_0^T \int_0^T \left[ \mid u-r \mid^{2H-2} - (u+r)^{2H-2} \right] \varphi_r \psi_u dr du \\
&= < \varphi, \psi >_{\mathcal{H}_{S^H}} \qquad (3.26)
\end{aligned}
$$

As a consequence, the operator $n_H^*$ provides an isometry between the Hilbert space $\mathcal{H}_{S^H}$ and $L^2([0,T])$. Hence, the process $W$ defined by

$$W(t) := S^H \left( (n_H^*)^{-1} (\mathbb{1}_{[0,t]}) \right)$$

is a Wiener process, and $S^H$ has the following Wiener integral representation:

$$S^H(t) = c_H \int_0^t n_H(t,s) dW(s)$$

because $(n_H^*)(\mathbb{1}_{[0,t]})(s) = c_H n_H(t,s)$. By Dzhaparidze and Van Zanten [36], we have

$$W(t) = \int_0^t \psi_H(t,s) dS^H(s),$$

where

$$\psi_H(t,s) = \frac{s^{H-1/2}}{\Gamma(3/2-H)} \left[ t^{H-3/2}(t^2-s^2)^{1/2-H} - \left(H - \frac{3}{2}\right) \int_s^t (x^2-s^2)^{1/2-H} x^{H-3/2} dx \right] \mathbb{1}_{[0,t]}(s).$$

In addition, for any $\varphi \in \mathcal{H}_{S^H}$ ,

$$\int_0^T \varphi(s) dS^H(s) = \int_0^T (n_H^*\varphi)(t) dW(t)$$

if and only if $n_H^*\varphi \in L^2([0,T])$. Also denoting $L^2_{\mathcal{H}_{S^H}}([0,T]) = \{\varphi \in \mathcal{H}_{S^H}, n_H^* \in L^2([0,T])\}$. Since $H > \frac{1}{2}$, we have by (3.26),

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset L^2_{\mathcal{H}_{S^H}}([0,T]). \qquad (3.27)$$

We are interested in considering a sub-fBm with values in Hilbert space and giving the definition of the corresponding stochastic integral.
Let $(U, \|.\|_U, < . >_U)$ and $(K, \|.\|_K, < . >_K)$ be two separable Hilbert spaces. Let $L(K,U)$ denote the space of all bounded linear operators from $K$ to $U$. Let $Q \in L(K,K)$ be a non-negative self-adjoint operator. Denote by $L_Q^0(K,U)$ the space of all $\xi \in L(K,U)$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$\|\xi\|^2_{L^0_Q(K,U)} = \|\xi Q^{\frac{1}{2}}\|_{\mathcal{H}_{SH}} = tr(\xi Q\xi^*).$$

Then $\xi$ is called a $Q$-Hilbert-Schmidt operator from $K$ to $U$.

Let $\{S^H_n(t)\}_{n\in\mathbb{N}}$ be a sequence of one-dimensional standard sub-fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. When one considers the following series:

$$\sum_{n=1}^{\infty} S^H_n(t)e_n, \qquad t \geq 0,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in $K$, this series does not necessarily converge in the space $K$. Thus we consider a $K$-valued stochastic process $S^H_Q(t)$ given formally by the following series:

$$S^H_Q(t) = \sum_{n=1}^{\infty} S^H_n(t)Q^{\frac{1}{2}}e_n, \qquad t \geq 0.$$

If $Q$ is a non-negative self-adjoint trace class operator, then this series converges in the space $K$, that is, we have $S^H_Q(t) \in L^2(\Omega, K)$. Then we say that the above $S^H_Q(t)$ is a $K$-valued $Q$-cylindrical sub-fractional Brownian motion with covariance operator $Q$. For example, if $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \sigma_n e_n$, assuming that $Q$ is a nuclear operator in $K$ (that is,$\sum_{n=1}^{\infty}\sigma_n < \infty$), then the stochastic process

$$S^H_Q(t) = \sum_{n=1}^{\infty} S^H_n(t)Q^{\frac{1}{2}}e_n = \sum_{n=1}^{\infty}\sqrt{\sigma_n}S^H_n(t)e_n, \qquad t \geq 0,$$

is well defined as a $K$-valued $Q$-cylindrical sub-fractional Brownian motion.

Let $\varphi : [0,T] \to L^0_Q(K,U)$ such that

$$\sum_{n=1}^{\infty}\left\|n^*_H(\varphi Q^{\frac{1}{2}}e_n)\right\|_{L^2([0,T];U)} < \infty \tag{3.28}$$

**Definition 3.3.1.** *Let $\varphi : [0,T] \to L^0_Q(K,U)$ satisfy (3.28). Then its stochastic integral with respect to the sub-fBm $S^H_Q$ is defined, for $t \geq 0$, as follows:*

$$\int_0^t \varphi(s)dS^H_Q(s) = \sum_{n=1}^{\infty}\int_0^t \varphi(s)Q^{\frac{1}{2}}e_n dS^H_n(s) \tag{3.29}$$

$$= \sum_{n=1}^{\infty}\int_0^t (n^*_H(\varphi Q^{\frac{1}{2}}e_n))(s)dW(s). \tag{3.30}$$

Notice that if

$$\sum_{n=1}^{\infty}\left\|\varphi Q^{\frac{1}{2}}e_n\right\|_{L^{\frac{1}{H}}([0,T];U)} < \infty, \tag{3.31}$$

then in particular (3.28) holds, which follows immediately from (3.27).

**Lemma 3.3.1.** *[35] For any $\varphi : [0,T] \to L_Q^0(K,U)$ such that (3.31) holds, and for any $\alpha, \beta \in [0,T]$ with $\alpha > \beta$,*

$$\mathbb{E}\left\|\int_\alpha^\beta \varphi(s)dS_Q^H(s)\right\|_U^2 \leq C_H(\alpha-\beta)^{2H-1}\sum_{n=1}^{\infty}\int_\alpha^\beta \left\|\varphi(s)Q^{1/2}e_n\right\|_U^2 ds.$$

*If, in addition,*

$$\sum_{n=1}^{\infty}\left\|\varphi(s)Q^{1/2}e_n\right\|_U^2 \text{ is uniformly convergent for } t \in [0,T],$$

*then*

$$\mathbb{E}\left\|\int_\alpha^\beta \varphi(s)dS_Q^H(s)\right\|_U^2 \leq C_H(\alpha-\beta)^{2H-1}\int_\alpha^\beta \|\varphi(s)\|_{L_Q^0(K,U)}^2 ds. \tag{3.32}$$

### 3.3.1 Existence and uniqueness of mild solution

We denote by $C(a,b;L^2(\Omega;U)) = C(a,b;L^2(\Omega,\mathcal{F},\mathbb{P};U))$ the Banach space of all continuous functions from $[a,b]$ into $L^2(\Omega;U)$ equipped with sup norm. Let us consider two fixed real numbers $r \geq 0$ and $T > 0$. If $x \in C(-r,T;L^2(\Omega;U))$ for each $t \in [0,T]$ we denote $x_t \in C(-r,0;L^2(\Omega;U))$ the function defined by $x_t(\theta) = x(t+\theta)$, for $\theta \in [-r,0]$.
In this section we consider the existence and uniqueness of mild solutions to the following stochastic evolution equation with delays:

$$\begin{cases} dX(t) = (AX(t) + f(t,X_t))dt + g(t)dS_Q^H(t), & t \in [0,T], \\ X(t) = \varphi(t), & t \in [-r,0] \end{cases} \tag{3.33}$$

where $S_Q^H(t)$ is the sub-fractional Brownian motion which was introduced previously, the initial data $\varphi \in C(-r,0;L^2(\Omega;U))$, and $A : \mathrm{Dom}(A) \subset U \to U$is the infinitesimal generator of a strongly continuous semigroup $S(.)$ on $U$, that is, for $t \geq 0$, we have

$$\|S(t)\|_U \leq Me^{\rho t}, \qquad M \geq 1, \rho \in \mathbb{R}.$$

$f : [0,T] \times C(-r,0;U) \to U$ is a family of nonlinear operators defined for almost every $t$

which satisfy:

$(f.1)$ The mapping $t \in [0,T] \to f(t,\xi) \in U$ is Lebesgue measurable for all $\xi \in C(-r,0;L^2(\Omega;U))$.

$(f.2)$ There exists a constant $C > 0$ such that for any $x, y \in C(-r,T;U)$ and $t \in [0,T]$,

$$\int_0^t \|f(s,x_s) - f(s,y_s)\|_U^2 ds \leq C\int_{-r}^t \|x(s)-y(s)\|_U^2 ds.$$

$(f.3)$ $\int_0^T \|f(s,0)\|_U^2 \, ds < \infty$.

Moreover, for $g : [0,T] \to L_Q^0(K,U)$ we assume the following conditions: for the complete orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in $K$, we have:

$(g.1)$ $\sum_{n=1}^{\infty} \left\|gQ^{1/2}e_n\right\|_{L^2([0,T];U)} < \infty$.

$(g.2)$ $\sum_{n=1}^{\infty} \left\|g(t)Q^{1/2}e_n\right\|_U$ is uniformly convergent for $t \in [0,T]$.

**Definition 3.3.2.** A $U$-valued process $X(t)$ is called a *mild solution* of (3.33) if $X \in C(-r,T;L^2(\Omega;U))$, $X(t) = \varphi(t)$, for $t \in [-r,0]$ and for $r \in [0,T]$, satisfies

$$X(t) = S(t)\varphi(0)+\int_0^t S(t-s)f(s,X_s)ds+\int_0^t S(t-s)g(s)dS_Q^H(t) \qquad \mathbb{P}-a.s. \quad (3.34)$$

Notice that, thanks to $(g.1)$ and the fact that $H > 1/2$, (3.31) holds, which implies that the stochastic integral in (3.34) is well defined since $S(.)$ is a strongly continuous semigroup. Moreover, $(g.1)$ together with $(g.2)$ immediately imply that, for every $t \in [0,T]$,

$$\int_0^t \|g(s)\|_{L_Q^0(K,U)}^2 \, ds < \infty.$$

**Theorem 3.10.** *Under the assumptions on $A$ and conditions $(f.1)$-$(f.3)$ and $(g.1)$-$(g.2)$, for every $\varphi \in C(-r,0;L^2(\Omega,U))$ there exists a unique mild solution $X$ to (3.33).*

**Proof.** We can assume that $\rho > 0$, otherwise we can take $\rho_0 > 0$ such that, for $t \geq 0$, $\|S(t)\|_U \leq Me^{\rho_0 t}$.
We start the proof by checking the uniqueness of solutions. Assume that $X, Y$ are two mild solutions of (3.33). Then

$$\begin{aligned}
\mathbb{E}\|X(t)-Y(t)\|_U^2 &\leq t\mathbb{E}\int_0^t \|S(t-s)(f(s,X_s)-f(s,Y_s)\|_U^2 \, ds \\
&\leq tM^2e^{2\rho t}\mathbb{E}\int_0^t \|f(s,X_s)-f(s,Y_s)\|_U^2 \, ds \\
&\leq tM^2e^{2\rho t}C\int_0^t \mathbb{E}\|X(t)-Y(t)\|_U^2 \, ds \\
&\leq tM^2e^{2\rho t}C\int_0^t \sup_{0\leq\tau\leq s} \mathbb{E}\|X(\tau)-Y(\tau)\|_U^2 \, ds
\end{aligned}$$

and therefore, since $X = Y$ over the interval $[-r,0]$, by taking the supremum in the above inequality,

$$\sup_{0\leq\theta\leq t} \mathbb{E}\|X(\theta)-Y(\theta)\|_U^2 \leq TM^2e^{2\rho t}C\int_0^t \sup_{0\leq\tau\leq s} \mathbb{E}\|X(\tau)-Y(\tau)\|_U^2 \, ds$$

The Gronwall's lemma implies now the uniqueness result.
Now we prove the existence of solutions to problem (3.33). First of all, we check that the well-defined stochastic integral possesses the repaired regularity. To this end, let us consider $\sigma > 0$ small enough. We have

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^{t+\sigma} S(t+\sigma-s)g(s)dS_Q^H(s) - \int_0^t S(t-s)g(s)dS_Q^H(s)\right\|_U^2 \\
&\leq 2\,\mathbb{E}\left\|\int_0^{t+\sigma}(S(t+\sigma-s)-S(t-s))g(s)dS_Q^H(s)\right\|_U^2 \\
&\quad +2\,\mathbb{E}\left\|\int_0^{t+\sigma} S(t-s)g(s)dS_Q^H(s)\right\|_U^2 \\
&= J_1 + J_2
\end{aligned}$$

Applying inequality (3.32) to $J_1$, we obtain

$$\begin{aligned}
J_1 &\leq 2C_H t^{2H-1}\int_0^t \|S(t-s)(S(\sigma)-Id)g(s)\|^2_{L_Q^0(K,U)}\,ds \\
&\leq C_H t^{2H-1}M^2e^{2\rho t}\int_0^t \|(S(\sigma)-Id)g(s)\|^2_{L_Q^0(K,U)}\,ds \to 0
\end{aligned}$$

when $\sigma \to 0$ thanks to the Lebesgue majoring theorem, since, for every $s$ fixed,

$$S(\sigma)g(s) \to g(s), \qquad \|S(\sigma)g(s)\|_{L_Q^0(K,U)} \leq Me^{\rho\sigma}\,\|g(s)\|^2_{L_Q^0(K,U)}\,.$$

Applying now (3.32) to $J_2$, we have

$$J_2 \leq C_H\sigma^{2H-1}M^2e^{2\rho\sigma}\int_t^{t-\sigma}\|g(s)\|^2_{L_Q^0(K,U)}\,ds \to 0$$

when $\sigma \to 0$. Therefore the stochastic integral belongs to the space $C(-r,T;L^2(\Omega;U))$.

We denote $X_0 = 0$ and define by recurrence a sequence $\{X_n\}_{n\in\mathbb{N}}$ of processes as

$$\begin{cases} X^n(t) = S(t)\varphi(0) + \displaystyle\int_0^t S(t-s)f(s,X_s^{n-1})ds + \int_0^t S(t-s)g(s)dS_Q^H(s), t\in[0,T]; \\ X^n(t) = \varphi(t),\ t\in[-r,0]. \end{cases} \tag{3.35}$$

The sequence (3.35) is well defined, since $X^0 = 0 \in C(-r,T;L^2(\Omega;U))$ and given $X^{n-1} \in C(-r,T;L^2(\Omega;U))$, let us check that $X^n \in C(-r,T;L^2(\Omega;U))$ as well. To this end, let us consider $\sigma > 0$ sufficiently small. Then

$$\begin{aligned}\|X^n(t+\sigma)-X_n(t)\|_U^2 &\leq 2\left\|\int_0^t (S(t+\sigma-s)-S(t-s))\, f(s,X_s^{n-1})ds\right\|_U^2 \\ &+ 2\left\|\int_t^{t+\sigma} S(t+\sigma-s)f(s,X_s^{n-1})ds\right\|_U^2 \\ &= I_1+I_2\end{aligned}$$

On the one hand,

$$\mathbb{E}(I_1)\leq 2tM^2e^{2\rho t}\mathbb{E}\left(\int_0^t \left\|(S(\sigma)-Id)f(s,X_s^{n-1})\right\|_U^2 ds\right)\to 0$$

when $\sigma\to 0$ thanks to the Lebesgue majoring theorem, since, for each $s$ fixed,

$$S(\sigma)f(s,X_s^{n-1})\to f(s,X_s^{n-1}),\qquad \left\|S(\sigma)f(s,X_s^{n-1})\right\|_U\leq Me^{\rho\sigma}\left\|f(s,X_s^{n-1})\right\|_U^2$$

and

$$\mathbb{E}\left(\int_0^t \left\|f(s,X_s^{n-1})\right\|_U^2 ds\right)\leq C\mathbb{E}\left(\int_{-r}^t \left\|X^{n-1}(s))\right\|_U^2 ds\right)+\mathbb{E}\left(\int_0^t \|f(s,0)\|_U^2 ds\right)$$

due to conditions $(f.1)$ and $(f.3)$ and the fact that $X^{n-1}\in C(-r,T;L^2(\Omega;U))$. On the other hand,

$$I_2\leq 2\sigma M^2e^{2\rho\sigma}\int_t^{t+\sigma}\left\|f(s,X_s^{n-1})-f(s,0)\right\|_U^2 ds+2\sigma M^2e^{2\rho\sigma}\int_t^{t+\sigma}\|f(s,0)\|_U^2 ds$$

$$\leq 2\sigma M^2e^{2\rho\sigma}C\int_{-r}^{t+\sigma}\left\|X^{n-1}(s))\right\|_U^2 ds 2\sigma M^2e^{2\rho\sigma}\int_t^{t+\sigma}\|f(s,0)\|_U^2 ds$$

so that, when $\sigma\to 0$

$$\mathbb{E}(I_2)\leq 2\sigma M^2e^{2\rho\sigma}C\int_{-r}^{t+\sigma}\mathbb{E}\left(\left\|X^{n-1}(s)\right\|_U^2\right)ds+2\sigma M^2e^{2\rho\sigma}\int_t^{t+\sigma}\|f(s,0)\|_U^2 ds\to 0$$

Next, we want to show that $\{X_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(-r,T;L^2(\Omega;U))$. Firstly, for $t\in[0,T]$ and $n\in\mathbb{N}$, since $X^n=X^{n-1}$ on $[-r,0]$, we have

$$\left\|X^{n+1}(t)-X^n(t)\right\|_U^2\leq tM^2e^{2\rho t}C\int_0^t \left\|X^{n+1}(s)-X^n(s)\right\|_U^2 ds$$

and this implies that

$$\mathbb{E}\left(\left\|X^{n+1}(t)-X^n(t)\right\|_U^2\right)\leq tM^2e^{2\rho t}C\int_0^t \sup_{0\leq\tau\leq s}\left\|X^{n+1}(\tau)-X^n(\tau)\right\|_U^2 ds$$

Defining

$$G^n(t) = \sup_{0\leq\theta\leq s} \left\| X^{n+1}(\theta) - X^n(\theta) \right\|_U^2$$

we obtain

$$G^n(t) \leq k \int_0^t G^{n-1}(s)ds, \qquad n \geq 2$$

for $k = TM^2 e^{2\rho T} C$. Consequently, by iteration we can obtain for all $t \in [0, T]$,

$$G^n(t) \leq \frac{k^{n-1}T^{n-1}}{(n-1)!} G^1(T), \qquad n \geq 2$$

Since $X^{n+1}(t) = X^n(t)$, $\forall t \in [-r, 0]$, the last estimate implies that $\{X^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(-r, T; L^2(\Omega; U))$.

Finally, we check that the limit $X$ of the sequence $\{X^n\}_{n\in\mathbb{N}}$ is a solution of (3.33). But this is straightforward, taking into account that $X^n$ is defined by (3.35) and that $f$ satisfies $(f.2)$, so that, in particular, when $n \to \infty$,

$$\mathbb{E} \left\| \int_0^t S(t-s)(f(s, X_s^{n-1}) - f(s, X_s))ds \right\|_U^2 \leq tM^2 e^{2\rho t} C \int_0^t \mathbb{E} \left\| X^{n+1}(s) - X^n(s) \right\|_U^2 ds \to 0$$

and therefore $X$ is the unique (mild) solution of (3.33).

## 3.3.2 Existence of weak solution

**Definition 3.3.3.** An $U$-valued process $X(t), t \in [-r, T]$ is called a *weak solution* of (3.33) if $X(t) = \varphi(t)$, for $t \in [-r, 0]$, and for all $\xi \in D(A^*)$ and all $r \in [0, T]$,

$$\langle X(t), \xi \rangle_U = \langle \varphi(0), \xi \rangle_U + \int_0^t \left( \langle X(s), A^*\xi \rangle_U + \langle f(s, X_s), \xi \rangle_U \right) ds$$
$$+ \int_0^t \langle g(s), \xi \rangle_U dS_Q^H(s) \qquad \mathbb{P} - a.s.$$

**Theorem 3.11.** *Under the assumptions of Theorem 3.10, the mild solution of (3.33) is also a weak solution.*

**Proof.** For each $\xi \in D(A^*)$ it follows that

$$\mathbb{E}[| \textstyle\int_0^t \langle X(s), A^*\xi \rangle_U ds - \int_0^t \langle S(s)\varphi(0), A^*\xi \rangle_U ds - \int_0^t \int_0^s \langle S(s-\tau) f(\tau, X_\tau), A^*\xi \rangle_U d\tau ds$$

$$- \textstyle\int_0^t \int_0^s \langle S(s-\tau) g(\tau), A^*\xi \rangle_U dS_Q^H(\tau) ds|]$$

$$\leq \textstyle\int_0^t \mathbb{E}[| \langle X(s), A^*\xi \rangle_U - \langle S(s)\varphi(0), A^*\xi \rangle_U - \int_0^s \langle S(s-\tau) f(\tau, X_\tau), A^*\xi \rangle_U d\tau$$

$$- \textstyle\int_0^s \langle S(s-\tau) g(\tau), A^*\xi \rangle_U dS_Q^H(\tau)|] ds$$

$= \int_0^t \mathbb{E}[|\langle X(s) - S(s)\varphi(0) - \int_0^s S(s-\tau)f(\tau, X_\tau)d\tau$

$- \int_0^s S(s-\tau)g(\tau)dS_Q^H(\tau)\rangle_U|]ds$

$= 0$

Thus, for a.e. $\omega \in \Omega$, we have

$$\begin{aligned}\int_0^t \langle X(s), A^*\xi\rangle_U ds &= \int_0^t \langle S(s)\varphi(0), A^*\xi\rangle_U ds + \int_0^t\int_0^s \langle S(s-\tau)f(\tau, X_\tau), A^*\xi\rangle_U d\tau ds \\ &+ \int_0^t\int_0^s \langle S(s-\tau)g(\tau), A^*\xi\rangle_U dS_Q^H(\tau)ds \qquad (3.36)\end{aligned}$$

Now we use the fact that, for $\xi \in D(A^*)$, $\frac{d}{dt}S^*(t)\xi = S^*(t)A^*\xi$. We can obtain

$$\int_0^t \langle S(s)\varphi(0), A^*\xi\rangle_U ds = \int_0^t \langle \varphi(0), S^*(s)A^*\xi\rangle_U ds = \langle S(t)\varphi(0) - \varphi(0), \xi\rangle_U ds$$

On the other hand, using Fubini's theorem we have

$$\begin{aligned}\int_0^t\int_\tau^s \langle S(s-\tau)f(\tau, X_\tau), A^*\xi\rangle_U d\tau ds &= \int_0^t\int_0^s \langle \mathbb{1}_{(0,s]}(\tau)f(\tau, X_\tau), S^*(s-\tau)A^*\xi\rangle_U ds d\tau \\ &= \int_0^t \langle S(t-\tau)f(\tau, X_\tau) - f(\tau, X_\tau), \xi\rangle_U d\tau\end{aligned}$$

Finally,

$$\begin{aligned}\int_0^t\int_0^s \langle S(s-\tau)g(\tau), A^*\xi\rangle_U dS_Q^H(\tau)ds &= \int_0^t\int_\tau^t \langle \mathbb{1}_{(0,s]}(\tau)g(\tau), S^*(s-\tau)A^*\xi\rangle_U ds dS_Q^H(\tau) \\ &= \int_0^t \langle g(\tau), S^*(s-\tau)\xi - \xi\rangle_U dS_Q^H(\tau) \\ &= \int_0^t \langle S(t-\tau)g(\tau), \xi\rangle_U dS_Q^H(\tau) - \int_0^t \langle g(\tau), \xi\rangle_U dS_Q^H(\tau)\end{aligned}$$

Therefore by (3.36) for a.e. $\omega \in \Omega$, it follows that

$$\begin{aligned}\int_0^t \langle AX(s), \xi\rangle_U ds &= \int_0^t \langle X(s), A^*\xi\rangle_U ds \\ &= \langle S(t)\varphi(0) - \varphi(0), \xi\rangle_U + \int_0^t \langle S(t-\tau)f(\tau, X_\tau) - f(\tau, X_\tau), \xi\rangle_U d\tau \\ &+ \int_0^t \langle S(t-\tau)g(\tau), \xi\rangle_U dS_Q^H(\tau) - \int_0^t \langle g(\tau), \xi\rangle_U dS_Q^H(\tau) \\ &= \langle X(t)\varphi(0) - \varphi(0), \xi\rangle_U + \int_0^t \left(\langle X(\tau), A^*\xi\rangle_U + \langle f(\tau, X_\tau), \xi\rangle_U\right) d\tau \\ &+ \int_0^t \langle g(\tau), \xi\rangle_U dS_Q^H(\tau)\end{aligned}$$

Consequently, it follows that almost surely

$$\langle X(t)\varphi(0),\xi\rangle_U = \langle\varphi(0),\xi\rangle + \int_0^t \left(\langle X(s),A^*\xi\rangle_U + \langle f(s,X_s),\xi\rangle_U\right) ds + \int_0^t \langle g(s),\xi\rangle_U dS_Q^H(s)$$

which means that $X(t)$ is the weak solution to (3.33).

The following theorem shows the exponential decay to zero in mean square, with an explicit exponential decay rate $\gamma$, we impose the following conditions :

**Condition 1:** The operator $A$ is a closed linear operator generating a strongly continuous semigroup $S(t)$, $t \geq 0$, on the separable Hilbert space $U$ and satisfies

$$\|S(t)\|_U \leq Me^{-\lambda t}, \quad \forall t \geq 0, \text{where } M \geq 1,\ \lambda > 0$$

.

**Condition 2:** There exists a constant $C \geq 0$ such that for any $x, y \in C(-r,T;U)$ and for all $t \geq 0$,

$$\int_0^t e^{ms}\|f(s,x_s) - f(s,y_s)\|_U^2 \leq C\int_{-r}^t e^{ms}\|x(s)-y(s)\|_U^2 ds \quad \text{for all } 0 \leq m \leq \lambda$$

and

$$\int_0^\infty e^{\lambda s}\|f(s,0)\|_U^2 ds < \infty.$$

**Condition 3:** In addition to assumptions $(g.1)$ and $(g.2)$, assume

$$\int_0^\infty e^{\lambda s}\|g(s)\|^2_{L_Q^0(K,U)} ds < \infty.$$

**Theorem 3.12.** *[35] In addition to Conditions 1-3, assume that the mild solution $X(t)$ of system (3.33) corresponding to initial function $\varphi \in C(-r,0;L^2(\Omega;U))$, exists for all $t \geq -r$, and that*

$$\lambda^2 > 6CM^2.$$

*Then there exists a constant $\gamma > 0$ such that*

$$\limsup_{t\to\infty}\left(\frac{1}{t}\right)\log \mathbb{E}\,\|X(t)\|_U^2 \leq -\gamma.$$

*In other words, every mild solution exponentially decays to zero in mean square.*

Next, we are interested to give an example that illustrate this result.

### Example

Let $K = L^2(0,\pi)$ and $e_n = \sqrt{(\frac{2}{\pi})}\sin(nx)$, $n \in \mathbb{N}$. Then $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in $K$. Let $U = L^2(0,\pi)$ and $A = \frac{\partial^2}{x2}$ with domain $D(A) = L_0^1(0,\pi) \cap L^2(0,\pi)$. Then it is well known that $Au = \sum_{n=1}^{\infty} n^2 \langle u, e_n\rangle_U e_n$ for any $u \in U$, and $A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t) : U \to U$, where $S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle_U e_n$ and for all $t \geq 0$, $\|S(t)\|_U \leq e^{-t}$. In order to define the operator $Q : K \to K$, we choose a sequence $\{\sigma_n\}_{n\geq 1} \subset \mathbb{R}^+$ and set $Qe_n = \sigma_n e_n$, and assume that $\mathrm{tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty$. Define the process $S_Q^H$ by

$$S_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} S_n^H(t) e_n$$

where $H \geq \frac{1}{2}$ and $\{S_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional sub-fractional Brownian motions mutually independent.

Then we consider the following stochastic evolution equation:

$$\begin{cases} du(t,x) = \left[\dfrac{\partial^2}{\partial x^2} u(t,x) + b(t)u(t,x(t-r))\right] dt + g(t) dS_Q^H(t), & t \in [0,T],\ x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \in [0,T], \\ u(t,x) = \varphi(t,x), & t \in [-\tau, 0],\ x \in [0,\pi], \end{cases} \tag{3.37}$$

where $r > 0$ and $b, g : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions such that $g$ satisfies Condition 3 above and $b$ satisfies

$$\int_0^{\infty} e^{\lambda s} |b(s)|^2 ds < \infty.$$

Observe that the fact $\int_0^{\infty} e^{\lambda s}|b(s)|^2 ds < \infty$ implies that $b(t)$ is bounded for all $t \geq 0$. Denote by $b_0$ the smallest upper bound of the function $b$. Taking

$$f(t, \varphi_t)(\eta) = \sin(t)\varphi(\eta_t).$$

Thus, for any $x, y \in C(-r, T; U)$, and for all $t \geq 0$, one has

$$\int_0^t \|f(s,x_s) - f(s,y_s)\|_U^2 \leq b_0^2 \int_{-r}^t \|x(s) - y(s)\|_U^2 ds \quad \text{for all } 0 \leq m \leq \lambda$$

and

$$\int_0^t e^{ms}\|f(s,x_s) - f(s,y_s)\|_U^2 \leq b_0^2 \int_{-r}^t e^{ms}\|x(s) - y(s)\|_U^2 ds \quad \text{for all } 0 \leq m \leq \lambda$$

Then we can check that there exists a unique mild solution to (3.37) according to Theorem 3.12. If we assume, in addition, that

$$b_0^2 < \frac{1}{6},$$

then any mild solution to (3.37) decays exponentially to zero in mean square.

# Conclusion

The main goal of this dissertation was to introduce two processes which are much more irregular than the standard Brownian motion moreover they are not semimartingale, and to give stochastic calculus on this class of processes. These processes are the fractional and sub-fractional Brownian motion. We mentioned different types of stochastic integration the most useful in our knowledge, and announce the relevant Itô's formula. Then we study the dynamical system driven by these processes by giving conditions that insure the existence and uniqueness of solution. In addition to that we have involved numerical simulation to exhibit their behavior.

Of course as in many researches, we found some difficulties because of the fact that these processes are not semimartingale even nor Markovian so the classical Itô calculus is not useful, and also due to the property of long and short range dependence of the increments and the non stationary of those of the sub-fractional Brownian motion, it was difficult to use some of stochastic calculus results in order de study this kind of processes. For this last reason it was tough to simulate the sub-fractional Brownian motion.

The most useful models in practice are much more complicated than the fractional and sub-fractional Brownian motion. For example in finance a general class of process were introduced likewise Rosenblatt, Hermite and Volterra processes...ect. Where the most of related issues are still open, so we wish that this work has for perspective a general study on the fractional and sub-fractional Brownian motions in order to investigate new approaches that allow us to answer at least one of these questions.

# Appendix

### Simulation program of fractional Brownian motion

```
    clf();clear;
//donner les coefficients et la pas du temps
c1=0.01
c2=0.02
N=1000
T=1
t=0:(T/N):T
t=linspace(T/N,T,N-1)
//définir un vecteur de variable aléatoire iid
function [r]=randn()
    r=rand(1,"normal");
endfunction
 //donner les fonction
function [y]=sigm(x)
    //y=c1*(x^2)+c2*x+1
   // y=sin(x)
   y(1)=cos(x)
   y(2)=y(1)*-sin(x)
   y(3)=y(2)*-cos(x)
   y(4)=y(3)*sin(x)
endfunction

function [h]=sigme(x)
   h=c1*x^2+c1
   //h=c1*x+c2
//h=cos(x)
//h=1/2*(1/sqrt(x))
endfunction
   function [z]= b(x)
    z=c2*x
endfunction
```

```
//calculer la fonction de covariance
U=zeros(length(t),1)
for i=1:length(t)
   U(i,1)=randn()
end
 function y=g(x,H)
    y=zeros(length(x),length(x))
    for i=1:length(x)
    for j=1:i
     y(i,j)=1/2*((x(i))^(2*H)+(x(j))^(2*H)-(abs(x(i)-x(j)))^(2*H))
     y(j,i)=y(i,j)
    end
 end
endfunction

//H=0.2:0.2:0.6
//for i=1:3
  H=0.5
  cov=g(t,H)

//les accroissements du mBf avec methode de cholosky
  A=chol(cov)
B=A'*U
B=[0 B']
//tracer le graphe des trajectoires
plot2d([0 t],B,15)
//end

//simulation du solution numérique
//dx Russo-Vallois
//dk  young alpha>1/2
//dy young alpha<1/2

x=zeros(1,N)
y=zeros(1,N)
k=zeros(1,N)

for i=1:(N-1)
dz=sigm(x(i))+sigm(x(i+1))
```

```
dh=sigm(y(i))
ds=sigm(k(i))
dB=B(i+1)-B(i)
//dx=1/2*dz*dB+(1/N)*b(x(i))
dx=1/2*dz(1)*dB+(1/N)*b(x(i))
dy=dh(1)*dB+(1/2)*dh(2)*dB^2 +(1/N)*b(y(i))+dh(3)*(1/6)*dB^3+dh(4)*(1/24)*dB^4
dk=ds(1)*dB+(1/N)*b(k(i))
k(i+1)=k(i)+dk
x(i+1)=x(i)+dx
y(i+1)=y(i)+dy
end
plot2d([0 t],x,5)
z=x-k
l=x-y

plot2d([0 t],z,12)
plot2d([0 t],y,9)
plot2d([0 t],k,19)
plot2d([0 t],l,6)
//legends(['Euler' 'fBm'],[5,15])
//legends(['Milshtein hld' 'fBm path'],[9,15])
legends(['Exact sol' 'Euler.Young' 'Euler.Russo-Vallois' 'diff Y R'],
[15,19,5,12])
legends(['Exact sol' 'Milshtein.Young' 'Euler.Russo-Vallois' 'diff Y R'],
[15,9,5,6])
// sauvegarde du dessin sous le nom de fig.pdf
xs2pdf(gcf(),'fig');
```

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