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par

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Dedication

This thesis is dedicated to

My mother, whose sincerely raised me with their caring and you have actively supported me in my determination to find and realize my potential, a very special thank for my Grandmother offered me unconditional love.

My husband Boualem, I am profoundly and eternally indebted to him for their love and encouragement throughout. Without forgetting special Dedicate to my son Mohamed which is my darling

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The List Of Works

Publication

1. Soumia Dani, Abdeldjebbar Kandouci. Conditional Full Support and no arbitrage. in Journal of General Mathematical Notes (2016).
2. Soumia Dani, Abdeldjebbar Kandouci. Study of the absence of arbitrage opportunities without calculating the risk-neutral probability. in Journal of Acta Univesitatis Sapientiae Mathematica in Vol. 8, No. 2,2016.
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Work carried out during these years

- Dani Soumia, Communication in July 01,2012 **Introduction aux modèles financiers**. Within weekly seminar of LMSSA
- Participation in the 8th Meeting of Mathematical Analysis and its Applications.The Mathematics Faculty of USTHB.Algiers, 26th-29th November 2012.
- Dani Soumia, Presentation in January 26,2013 **Modélisation des marchés financiers en temps continuous (Arbitrage)**. in the presence of Professor Stéphane Crépey. Within weekly seminar of LMSSA.

- Dani Soumia, Presentation in May 29,2014 **Stochastic Integral and conditional Full Support**. Within International Workshop On the Stochastic Calculus and Its Applications.
- Dani Soumia, Presentation in June 18,2014. **Conditional Full Support and no arbitrage**. Rouen University. Mathematics Laboratory Raphael Salem "France".

Introduction

Stochastic Portfolio Theory (SPT), as we currently think of it, began in 1995 with the manuscript "On the diversity of equity markets", which eventually appeared as Fernholz (1999) in the Journal of Mathematical Economics.
Stochastic portfolio theory is a relatively new branch of mathematical finance. It was introduced and studied by Fernholz [10, 9], and then further developed by Fernholz, Karatzas and Kardaras [11]. It provides a framework for analysing portfolio performance under an angle which is different from the usual one.

One of the most important notions here is absence of arbitrage (riskless profit) is the cornerstone of mode in mathematical finance. At the technical level, there are various formulations of arbitrage but basic economic considerations forbid that such opportunities persist in a liquid market.

In modern financial practice, asset prices are modelled by means of stochastic processes. Continuous-time stochastic calculus thus plays a central role in financial modelling. The approach has its roots in the foundational work of Black, Scholes and Merton. Asset prices are further assumed to be rationalizable, that is, determined by the equality of supply and demand in some market. This approach has its roots in the work of Arrow, Debreu and McKenzie on general equilibrium.

In mathematical finance there has lately been considerable interest in pushing beyond the by now classical setting of asset prices modelled by semimartingales,cf. e.g. Delbaen and Schachermayer (2006) and the references therein. New approaches

that either restrict the class of allowed trading strategies or introduce transaction costs allow for instance for the use of condition namely conditional full support.
The CFS property was first introduced by Guasoni, Résonyi, and Schachermayer, where it was proved that the fractional Brownian motion with arbitrary Hurst parameter has a desired property. This later was generalized by Cherny (2008) [4] who proved that any Brownian moving average satisfies the conditional full support condition. Then, the (CSF) property was established for Gaussian processes with stationary increments by Gasbarra et al. (2011) [12]. In 2013 Attila Herczegh et al. provides a new result on conditional full support in higher dimensions [1].

Let's note that, by the main result of Guasoni et al [13]. asserts that if a continuous price process has CFS, then for any $\varepsilon > 0$ there exists a socalled ε-consistent price system, which is a martingale (after an equivalent change of measure).

The existence of ε-consistent price systems for all $\varepsilon > 0$ implies that the price process does not admit arbitrage opportunities under arbitrary small transaction costs since any arbitrage strategy would generate arbitrage also in the consistent price system, which is a contradiction because of the martingale property.

Consistent price systems can be seen as generalizations of equivalent martingale measures (EMM's), since if a price process admits an EMM, then the price process itself qualifies as a trivial ε-consistent price system for any $\varepsilon > 0$.

However, CFS is worth studying even when it comes to price processes that admit EMM's, since it enables the construction of specific consistent price systems that are useful in solving superreplication problems under proportional transaction costs. This is manifested by the "face-lifting" result in [13].

Aside from having these applications in mathematical finance, CFS is an interesting fundamental property from a purely mathematical point of view. In particular, research on the CFS property can be seen as a natural continuation to the classical studies of the supports of the laws of continuous Gaussian processes, by Kallianpur [21], and diffusions, initiated by Stroock and Varadhan [41] and continued by several other authors (see e.g. [28] and the references therein).

This thesis consists three chapters. In the first chapter we focus on Stochastic Calculus for finance. We devoted to a brief summary of the theory of stochastic and fractional calculus. In this chapter we will give definitions and properties of the needed theory. We briefly recall some basic properties of the Brownian motion and the fractional Brownian motion, then we discuss integration with respect to Wiener processes (resp fBm).

The aim of the second chapter is to provide an introduction to the mathematical methods used in continuous-time modeling of financial markets. It will focus on problems of pricing of options by arbitrage. The goal is not to provide a comprehensive presentation of the theory but rather to emphasize the major ideas and techniques. In this chapter we are looking the description of asset model in continuous time(the model of Black Scholes), self-financing portfolios, and arbitrage.

In the chapter 3, we study a simple condition on asset prices, namely conditional full support, which generates a large class of consistent price systems which links the problems of no-arbitrage. In fact, all natural examples (which we can think of) enjoy this property. We study the problems of no-arbitrage for asset prices driven by a continuous process and with constant proportional transaction costs and we studied the relation between the condition CFS and stochastic integral. In the last section of this chapter we give a set of conditions to provide our main results on conditional full support for the processes the Ornstein Uhlenbeck, stochastic integral sach that the Brownian Bridge is the integrator and Fractional brownian motion and build the absence of arbitrage opportunities without calculating the risk-neutral probability.

Chapter 1

Preliminary Background

In this chapter the basic concepts and results concerning stochastic calculus of continuous stochastic processes are given. We omit some introductory facts from probability theory. For more detail we refer the reader to [3, 17, 22, 23]. We first start with stochastic process, Wiener process and fractional Brownian motion.

1.1 Basic definitions

In this section the basic notations of the theory of stochastic calculus are considered. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_s\}$ satisfying the *usual conditions*:

- $\mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t$ for all $s \geq 0$;
- All $A \in \mathcal{F}$ with $\mathbf{P}(A) = 0$ are contained in $\mathcal{F}_t$.

A family $(X(t), t \geq 0)$ of $\mathbb{R}^d$-valued random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ is called a *stochastic process*, this process is adapted if all $(X(t), t \geq 0)$ are $\mathcal{F}_t$-measurable. Denoting $\mathcal{B}$, the Borel σ-field on $[0, \infty)$. The process X is measurable if $(t, \omega) \mapsto X(t, \omega)$ is a $\mathcal{B} \bigotimes \mathcal{F}$-measurable mapping. We say that $(X(t), t \geq 0)$ is *continuous* if the *trajectories* $t \mapsto X(t, \omega)$ are continuous for all $\omega \in \Omega$ except on a negligible set.

1.1.1 Continuous-Time Martingales

The concept of a martingale is fundamental to modern probability and is one of the key tools needed to study mathematical finance.

Let $\{X_t, t \geq 0\}$ be a continuous-time stochastic process. Recall that this implies that there are uncountably many random variables, one for each value of the time index t. For $t \geq 0$, let $\mathcal{F}_t$ denote the information contained in the process up to (and including) time t. Formally, let

$$\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t).$$

We call $\mathcal{F}_t, t \geq 0$ a filtration, and we say that X_t is adapted if $X_t \in \mathcal{F}_t$. Notice that if $s \leq t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$ so that $X_s \in \mathcal{F}_t$ as well.

Definition 1.1.1. *A collection $\{X_t, t \geq 0\}$ of random variables is said to be a martingale with respect to the filtration $\mathcal{F}_t, t \geq 0$ if*

(i) *X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$,*

(ii) *$E[|X_t|]$ is finite for all $t \geq 0$, and*

(iii) *$E[X_t|\mathcal{F}_s] = X_s$ for all $0 \leq s < t$.*

Note that in the third part of the definition, the present time t must be strictly larger than the past time s.

Theorem 1.1.2. *[27]*
Let $\{X_t, t \geq 0\}$ be a stochastic process and consider the filtration $\{\mathcal{F}_t, t \geq 0\}$ where $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$. Let Y be a random variable, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a function. Suppose that $0 \leq t_1 < t_2 < \ldots < t_n$ are n times, and let s be such that $0 \leq s < t_1$. (Note that if $t_1 = 0$, then $s = 0$.) It then follows that

(a) *$E(g(X_{t_1}, \ldots, X_{t_n})Y|\mathcal{F}_s) = g(X_{t_1}), \ldots, X_{t_n})E(Y|\mathcal{F}_s)$ (taking out what is known)*

(b) *$E(Y|\mathcal{F}_s) = E(Y)$ if Y is independent of $\mathcal{F}_s$, and*

(c) *$E(E(Y|\mathcal{F}_s)) = E(Y)$.*

1.2 Brownian motion

In what follows, we will state a number of important facts regarding Brownian motion.
Historically:

- **1828**: Robert Brown, botanist observes the movement of pollen suspended in water.
- **1877**: Delsaux explains that this is due to the irregular motion of pollen with shock water molecules (constant changes of direction),
- **1900**: Louis Bachelier in his dissertation "Theory of Speculation" models the course the stock market as a process with independent increments and Gaussian (problem: the price of the asset, Gaussian process can be negative)
- **1905**: Einstein determines the density of the BM and binds to PDEs. the Schmolushowski described as random walk limit.
- **1923**: Rigorous Study of BM by Wiener, among others demonstration of existence.

Definition 1.2.1. *A Brownian motion process is a stochastic process $B_t, t \geq 0$, which satisfy:*

1. *The process starts at the origin, $B_0 = 0$;*
2. *B_t has stationary, independent increments;*
3. *The process B_t is continuous in t;*
4. *The increments $B_t - B_s$ are normally distributed with mean zero and variance $|t-s|$,*
$$B_t - B_s \sim N(0, |t-s|).$$

The process $X_t = x + B_t$ has all the properties of a Brownian motion that starts at x. Since $B_t - B_s$ is stationary, its distribution function depends only on the time interval $t - s$,

$$\mathbf{P}(B_{t+s} - B_s \le a) = \mathbf{P}(B_t - B_0 \le a) = \mathbf{P}(B_t \le a).$$

It is worth noting that even if B_t is continuous, it is nowhere differentiable. From condition 4 we get that B_t is normally distributed with mean $E[B_t] = 0$
and $Var[B_t] = t$

$$B_t \sim N(0, t).$$

This implies also that the second moment is $E[B_t^2] = t$. Let $0 < s < t$. Since the increments are independent, we can write

$$E[B_sB_t] = E[(B_s - B_0)(B_t - B_s) + B_s^2] = E[B_s - B_0]E[B_t - B_s] + E[B_s^2] = s.$$

Consequently, B_s and B_t are not independent.

Condition 4 has also a physical explanation. A pollen grain suspended in water is kicked by a very large numbers of water molecules. The influence of each molecule on the grain is independent of the other molecules. These effects are average out into a resultant increment of the grain coordinate.

Proposition 1.2.2. *A Brownian motion process B_t is a martingale with respect to the information set $\mathcal{F}_t = \sigma(B_s; s \le t)$.*

Proof. The integrability of B_t follows from Jensen's inequality

$$E[|B_t|]^2 \le E[B_t^2] = Var(B_t) = |t| < \infty$$

B_t is obviously $\mathcal{F}_t$-measurable. Let $s < t$ and write $B_t = B_s + (B_t - B_s)$. Then

$$\begin{aligned} E[B_t|F_s] &= E[B_s + (B_t - B_s)|\mathcal{F}_s] \\ &= E[B_s|\mathcal{F}_s] + E[B_t - B_s|\mathcal{F}_s] \\ &= B_s + E[B_t - B_s] = B_s + E[B_{t-s} - B_0] = B_s, \end{aligned}$$

where we used that B_s is $\mathcal{F}_s$-predictable (from where $E[B_s|\mathcal{F}_s] = B_s$) and that the increment $B_t - B_s$ is independent of previous values of B_t contained in the information set $\mathcal{F}_t = \sigma(B_s; s \le t)$. □

A process with similar properties as the Brownian motion was introduced by Wiener.

Definition 1.2.3. *A Wiener process W_t is a process adapted to a filtration $\mathcal{F}_t$ such that*

1. *The process starts at the origin, $W_0 = 0$;*
2. *W_t is an $\mathcal{F}_t$-martingale with $E[W_t^2] < \infty$ for all $t \geq 0$ and*
$$E[(W_t - W_s)^2] = t - s, s \leq t;$$
3. *The process W_t is continuous in t.*

Theorem 1.2.4. *[32]* ***(Lévy)***
A Wiener process is a Brownian motion process.

Proposition 1.2.5. *If W_t is a Wiener process with respect to $\mathcal{F}_t$, then $Y_t = W_t^2 - t$ is a martingale.*

Proof. Y_t is integrable since

$$E[|Y_t|] \leq E[W_t^2 + t] = 2t < \infty, t > 0.$$

Let $s < t$. Using that the increments $W_t - W_s$ and $(W_t - W_s)^2$ are independent of the information set $\mathcal{F}_s$ and we have

$$\begin{aligned}
E[W_t^2|F_s] &= E[(W_s + W_t - W_s)^2|\mathcal{F}_s] \\
&= E[W_s^2 + 2W_s(W_t - W_s) + (W_t - W_s)^2|\mathcal{F}_s] \\
&= E[W_s^2|\mathcal{F}_s] + E[2W_s(W_t - W_s)|\mathcal{F}_s] + E[(W_t - W_s)^2|\mathcal{F}_s] \\
&= W_s^2 + 2W_s E[W_t - W_s|\mathcal{F}_s] + E[(W_t - W_s)^2|\mathcal{F}_s] \\
&= W_s^2 + 2W_s E[W_t - W_s] + E[(W_t - W_s)^2] \\
&= W_s^2 + t - s,
\end{aligned}$$

and hence $E[W_t^2 - t|\mathcal{F}_s] = W_s^2 - s$, for $s < t$. □

Proposition 1.2.6. *Let* $0 \le s \le t$. *Then*

1. $Cov(W_s, W_t) = s$;

2. $Corr(W_s, W_t) = \sqrt{\frac{s}{t}}$.

Proof. 1. Using the properties of covariance

$$\begin{aligned} Cov(W_s, W_t) &= Cov(W_s, W_s + W_t - W_s) \\ &= Cov(W_s, W_s) + cov(W_s, W_t - W_s) \\ &= Var(W_s) + E[W_s(W_t - W_s)] - E[W_s]E[W_t - W_s] \\ &= s + E[W_s]E[W_t - W_s] \\ &= s, \end{aligned}$$

since $E[W_s] = 0$. We can also arrive at the same result starting from the formula

$$Cov(W_s, W_t) = E[W_s W_t] - E[W_s]E[W_t] = E[W_s W_t].$$

Using that conditional expectations have the same expectation, factoring the predictable part out, and using that W_t is a martingale, we have

$$\begin{aligned} E(W_s W_t) &= E[E[W_s W_t|\mathcal{F}_s]] = E[W_s E[W_t|\mathcal{F}_s]] \\ &= E[W_s W_s] = E[W_s^2] = s, \end{aligned}$$

so $Cov(W_s, W_t) = s$.

2. The correlation formula yields $Corr(W_s, W_t) = \frac{Cov(W_s, W_t)}{\sigma(W_t)\,\sigma(W_s)} = \frac{s}{\sqrt{s}\sqrt{t}} = \sqrt{\frac{s}{t}}$

□

1.2.1 Brownian Bridge

The process $(X_t = W_t - tW_1)_{0\leq t\leq 1}$ is called the Brownian bridge fixed at both 0 and 1. Since we can also write

$$\begin{aligned} X_t &= W_t - tW_t - tW_1 + tW_t \\ &= (1-t)(W_t - W_0) - t(W_1 - W_t), \end{aligned}$$

using that the increments $W_t - W_0$ and $W_1 - W_t$ are independent for all $0 < t < 1$ and normally distributed, with

$$W_t - W_0 \sim N(0,t), \qquad W_1 - W_t \sim N(0, 1-t),$$

it follows that X_t is normally distributed with

$$\begin{aligned} E[X_t] &= (1-t)E[(W_t - W_0)] - tE[(W_1 - W_t)] = 0 \\ Var[X_t] &= (1-t)^2 Var[(W_t - W_0)] + t^2 Var[(W_1 - W_t)] \\ &= (1-t)^2(t-0) + t^2(1-t) \\ &= t(1-t). \end{aligned}$$

This can be also stated by saying that the Brownian bridge tied at 0 and 1 is a Gaussian process with mean 0 and variance $t(1-t)$, so $X_t \sim N(0, t(1-t))$.

1.3 Fractional Brownian Motion

The name fractional Brownian motion (fBm) was given by Mandelbrot and Van Ness in [26]. However, Kolmogorov studied it first within the Hilbert Space framework. The fBm is a H-ss process that has stationary increments. In fact, it is the unique Gaussian H-sssi process. It has very interesting properties for many applications. One of them is its "memory". It can then be applied in telecommunications as well as in finance. This is the process for which we shall study the Wiener integration with respect to it. Our presentation is based on [8], [38] and [26].

Definition 1.3.1. *For $H \in (0,1)$, a standard fractional Brownian motion of Hurst parameter H is a centered and continuous Gaussian process, denoted by (B_t^H), with covariance function*

$$E(B_t^{(H)}B_s^{(H)}) = \frac{1}{2}(\mid t \mid^{2H} + \mid s \mid^{2H} - \mid t-s \mid^{2H}) := R_H(t,s)$$

1.3.1 Selfsimilarity

There is an other classic definition of the fBm using selfsimilar properties, which we give as a theorem.

Theorem 1.3.2. *[18] For $H \in (0,1)$, the fBm $(B_t^{(H)})$ is a Gaussian H-ss process.*

1.3.2 Hölder continuity

We have seen that a Brownian motion is locally Hölder continuous of order strictly less than 1/2. Hence we have the following proposition which generalize this result to the fBm.

Proposition 1.3.3. *[18]. Let $H \in (0,1)$. The fBm B^H admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*

Proof. It follows from the Kolmogorov's continuity criterion and the fact that for any $\alpha > 0$, we have

$$\mathbb{E}\left(\left|B_t^H - B_s^H\right|^\alpha\right) = \mathbb{E}\left(\left|B_1^H\right|^\alpha\right)|t-s|^{\alpha H}. \qquad \square$$

1.3.3 Differentiability

As in the Brownian case, the fBm is a.s., nowhere differentiable. Effectively, we have the following proposition.

Proposition 1.3.4. *Let $H \in (0,1)$. The fBm sample path $B^H(\cdot)$ is not differentiable. Indeed, for every $t_0 \in [0,\infty)$*

$$\lim_{t \to t_0} \sup \left| \frac{B^H(t) - B^H(t_0)}{t - t_0} \right| = \infty,$$

with probability one.

Proof. We refer the reader to [18] □

1.3.4 The fBm is not a semimartingale for $H \neq \frac{1}{2}$

This is a crucial result of this section. Indeed, the fact that the fBm is not a semimartingale implies that we are not able to integrate with respect to it as we usually do in the classical stochastic calculus. Effectively, the most general class of integrators are semimartingales.
Let us now prove this result (fBm is not a semimartingale).
Proof. In fact, it is sufficient to compute p-variation of B^H. More precisely, we asserts that the index of $p-$variation of a fBm is $\frac{1}{H}$. Indeed, let us consider for fixed $p > 0$,

$$Y_{n,p} := \sum_{i=1}^{n} \left| B^H_{\frac{i}{n}} - B^H_{\frac{i-1}{n}} \right|^p n^{p(H-1)}.$$

Since B^H has the self-similarity property, the sequence $Y_{n,p}$ has the same distribution as

$$\tilde{Y}_{n,p} := \sum_{i=1}^{n} \left| B^H_i - B^H_{i-1} \right|^p \frac{1}{n}.$$

By the Ergodic theorem (see, [3]) the sequence $\tilde{Y}_{n,p}$ converges almost surely and in L^1 to $\mathbb{E}\big[|B^H(1)|^p\big]$ as n tends to infinity; hence, it converges also in probability to $\mathbb{E}\big[|B^H(1)|^p\big]$. It follows that

$$V_{n,p} := \sum_{i=1}^{n} |B^H_{\frac{i}{n}} - B^H_{\frac{i-1}{n}}|^p \xrightarrow{\mathbb{P}} \begin{cases} 0, & if \quad pH > 1 \\ \infty & if \quad pH < 1 \end{cases} \quad as \quad n \to \infty. \tag{1.1}$$

Then we showed that the index of $p-$variation is $\frac{1}{H}$. However, for a semimartingale, the index must be either in $[0,1]$ either equal to 2, i.e., $\frac{1}{H} \in [0,1] \cup \{2\}$. But since $H \in (0,1)$, $H^{-1} \notin [0,1]$. Therefore, the fBm is a semimartingale only for $H = \frac{1}{2}$.

□

1.3.5 Representions of FBm on a finite interval

There are also representations of the fBm as a Wiener integral but defined on an interval, e.g. commonly taken as [0, T]. We shall still use fractional analysis. As for the representation on the real line we would like to have for a one-sided fBm $(B_t^{(H)})_{0\leq t\leq T}$ a general formula

$$B_t^{(H)} = \int_0^t K_H(t,s)dB_s, \quad t \in [0,T] \tag{1.2}$$

where $(B_t)_{0\leq t\leq T}$ is a one-sided standard Brownian motion.

Lévy-Hida Representation Note that the fractional Brownian motion is a particular case of Volterra processes. Following Decreusfond and Üstünel in [?] we have this kernel

$$K_H(t,s) = \frac{(t-s)_+^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F\left(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, 1-\frac{t}{s}\right), \quad 0 < s < t < \infty$$

where F is the Gauss hypergeometric function. Remark that, generally, the covariance $R_H(t,s)$ of B^H is given by

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du.$$

Indeed, by (1.2), it follows that

$$R_H(t,s) = E(B_t^{(H)}B_s^{(H)}) = E\left(\int_0^{t\wedge s} K_H(t,u)K_H(s,u)dB_u\right) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du.$$

- **Case** $H \in \left(\frac{1}{2}, 1\right)$

Proposition 1.3.5. *[18] For the case $H \in \left(\frac{1}{2}, 1\right)$, the kernel K_H can be written*

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t \mid u-s \mid^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad t > s,$$

where

$$c_H = \left(\frac{H(2H-1)}{\mathbf{B}(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}}$$

where $\mathbf{B}$ *the Bêta function, i.e.* $\mathbf{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$.

Corollary 1.3.6. *[18] Besides, we have*

$$R_H(t,s) = (\varpi_1(H))^2 \int_0^T \left(r^{\frac{1}{2}-H} (I_{T-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} \mathbf{1}_{[0,t)}(u))(r) \right) \left(r^{\frac{1}{2}-H} (I_{T-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} \mathbf{1}_{[0,s)}(u))(r) \right) dr$$

with $\varpi_1(H) = \left(\frac{\Gamma(H-\frac{1}{2})^2 H(2H-1)}{\mathbf{B}(2-2H, H6\frac{1}{2})} \right)^{\frac{1}{2}}$

Theorem 1.3.7. *[18] The representation of a fbm for $H \in \left(\frac{1}{2}, 1\right)$ over a finite interval is*

$$B_t^{(H)} = \int_0^t K_H(t,s) dW_s, \qquad s,t \in [0,T],$$

where $(W_t)_{t\in[0,T]}$ is a particular Wiener process.

- **Case** $H \in \left(0, \frac{1}{2}\right)$

Proposition 1.3.8. *[18] For the case $H \in \left(0, \frac{1}{2}\right)$ we have that the kernel is given by*

$$K_H(t,s) = b_H \left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{1}{2}} du \right),$$

where

$$b_H = \left(\frac{2H}{(1-2H)\mathbf{B}(1-2H, H+\frac{1}{2})} \right)^{\frac{1}{2}}$$

1.4 Stochastic Integrals with respect to Brownian Motion

We want to give a meaning to random variable:

$$\int_0^T \theta_s dW_s$$

When integrating a function g with respect to a differentiable function f, if g is regular, its integral is defined as:

$$\int_0^T g(s)df(s) = \int_0^T g(s)f'(s)ds$$

If f is not differentiable but merely of bounded variation, it is still out defining the integral by:

$$\int_0^T g(s)df(s) = \lim_{\pi_n \to 0} \sum_{i=1}^{n-1} g(t_i)(f(t_{i+1}) - f(t_i)) \quad \pi_n = \max(t_{i-1}^n - t_i^n)$$

this integral is called Stieltjes integral.

In our case, the Brownian motion is not bounded variation, so we can not define this limit path by path.
On the other hand, as it has a finite quadratic variation, it is natural to define the integral with respect to Brownian motion as a limit in L^2 (convergence in the sense of $\|.\|_2$) of this random variable.

$$\int_0^T \theta_s dW_s = \lim_{\Pi_n \to 0} \sum_{i=1}^{n-1} \theta_{t_i}(W_{t_{i+1}} - W_{t_i}).$$

The convergence in the sense of convergence of random variables in $L^2(\Omega)$. For that we will therefore imposed the process θ to be in $L^2(\Omega, [0,T])$.

Will also be imposed θ $\mathcal{F}$-adapted [1] in order that θ_{t_i} is independent of $W_{t_{i+1}} - W_{t_i}$. If there was not, we could the same define an integral with respect to Brownian motion but it would be very different because the quadratic variation of the Brownian motion is nonzero. A simple example such, as approximations of $\int_0^T W_t dW_t$, we have among others the choice between the two following approximations :

$$\sum_{i=0}^{n-1} W_{t_i}[W_{t_{i+1}} - W_{t_i}] \quad ou \quad \sum_{i=0}^{n-1} W_{t_{i+1}}[W_{t_{i+1}} - W_{t_i}].$$

The gap between the two integrals is then equal to :

$$\sum_{i=0}^{n-1}[W_{t_{i+1}} - W_{t_i}]^2 \overset{L^2}{\rightarrow} T.$$

We will build the stochastic integral or the approximation is made at the leftmost point in order that the integrated is independent of integrating.
This is integral in the sense of Ito. At last, for technical reasons, we will request the regulatory of the processes that we handle. We assume that the stochastic process $(\theta_t)_{0\leq s\leq T}$ is left continuous and is right limited and shall the French abbreviation (càdlàg).

Finally, we'll build the stochastic integral on the set

$$L^2_{\mathcal{F}}(\Omega, [0,T]) = \left\{(\theta_t)_{0\leq t\leq T}, \quad processes\ càdlàg\ \mathcal{F}-adapted\ s.t\ E\left[\left(\int_0^T \theta_s^2 ds\right)\right] < \infty\right\}.$$

Firstly build the stochastic integral on the set of the elementary processes.

Definition 1.4.1. *A process $(\theta_t)_{0\leq t\leq T}$ is called the elementary process if there is a subdivision $0 = t_0 \leq t_1 \leq \ldots \leq t_n = T$ and a discrete process $(\theta_i)_{0\leq i\leq n-1}$ such that*

[1] A process θ is said $\mathcal{F}$-adapted if for all t, the variable random θ_t is $\mathcal{F}_t$-measurable.

all θ_i is $\mathcal{F}_{t_i}$-adapted and in $L^2(\Omega)$ such that:

$$\theta_t(\omega) = \sum_{i=0}^{n-1} \theta_i(\omega) 1_{]t_i,t_{i+1}]}(t)$$

we note ε the set of elementary process which is a subspace of $L^2_{\mathcal{F}}(\Omega, [0,T])$.

Definition 1.4.2. *With the same notations, the stochastic integral between 0 and $t \le T$ of an elementary process $\theta \in \varepsilon$ is the random variable defined by:*

$$\int_0^T \theta_s dW_s := \sum_{i=0}^{k} \theta_i (W_{t_{i+1}} - W_{t_i}) + \theta_i (W_t - W_{t_k}) \quad on \ ]t_k, t_{k+1}],$$

Proposition 1.4.3. *[36]**Properties of stochastic integral on $\mathcal{E}$***
On the set of elementary processes $\mathcal{E}$, the stochastic integral satisfies the properties:

1. $\theta \mapsto \int_0^t \theta_s dW_s$ *is linear,*
2. $t \mapsto \int_0^t \theta_s dW_s$ *is a.s continuous,*
3. $(\int_0^t \theta_s dW_s)_{0 \le t \le T}$ *is a process $\mathcal{F}$-adapted,*
4. $E[\int_0^t \theta_s dW_s] = 0$ *and* $Var(\int_0^t \theta_s dW_s) = E[\int_0^t \theta_s^2 ds]$,
5. *Isometric property :*

$$E\Big[\Big(\int_0^t \theta_s dW_s\Big)^2\Big] = E\Big[\int_0^t \theta_s^2 ds\Big],$$

6. *More generally, we have:*

$$E\Big[\int_s^t \theta_u dW_u | \mathcal{F}_s\Big] = 0 \ et \ E\Big[\Big(\int_s^t \theta_v dW_v\Big)^2 | \mathcal{F}_s\Big] = E\Big[\int_s^t \theta_v^2 dv | \mathcal{F}_s\Big].$$

7. *We even have the more general result:*

$$E\Big[\Big(\int_s^t \theta_v dW_v\Big)\Big(\int_s^u \phi_v dW_v | \mathcal{F}_s\Big) = E\Big[\int_s^{t\wedge u} \theta_v \phi_v dv | \mathcal{F}_s\Big].$$

8. $\left(\int_0^t \theta_s dB_s\right)_{0\leq t\leq T}$ *is a* $\mathcal{F}$*-martingale.*

9. *The process* $\left((\int_0^t \theta_s dW_s)^2 - \int_0^t \theta_s^2 ds\right)_{0\leq t\leq T}$ *is a* $\mathcal{F}_t$*-martingale.*

10. *The quadratic variation of the stochastic integral is given by:*

$$\left\langle \int_0^t \theta_s dW_s \right\rangle = \int_0^t \theta_s^2 ds.$$

11. *The quadratic covariation between two stochastic integrals is given by:*

$$\left\langle \int_0^t \theta_s dW_s, \int_0^u \phi_s dW_s \right\rangle = \int_0^{t\wedge u} \theta_s \phi_s ds.$$

Finally, the stochastic integral of an element of $\mathcal{E}$ is a continuous martingale square integrable. We denote $\mathcal{M}^2([0,T])$ the set of the continuous martingale square integrable:

$$\mathcal{M}^2([0,T]) := \{M\mathcal{F}_t - martingales\; s.t\; E[M_t^2] < \infty \;\forall t \in [0,T]\}.$$

The stochastic integral is a function of $\mathcal{E} \times [0,T]$ in $\mathcal{M}^2([0,T])$.
Will now, as announced, extend the definition of the stochastic integral to the process adapted having a moment in order 2, i.e. to :

$$L^2_{\mathcal{F}}(\Omega, [0,T]) = \left\{(\theta_t)_{0\leq t\leq T},\; process\; càdlàg\; \mathcal{F} - adapted\; s.t\; E\left[\left(\int_0^T \theta_s^2 ds\right)\right] < \infty\right\}$$

1.4.1 Itô's Formula

Here is the tool to calculate the stochastic integrals without going approximating.

Theorem 1.4.4. *[36] Any function* $f \in \mathcal{C}^2(\mathbb{R})$ *to second derivative bounded verifies a.s:*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \quad \forall t \leq T.$$

The infinitesimal notation of this relationship is:

$$df(B_s) = f'(B_s)dB_s + \frac{1}{2}f''(B_s)ds.$$

Theorem 1.4.5. *Any function $f \in \mathcal{C}^2(\mathbb{R})$ verifies a.s*

$$f(B_T) = f(B_0) + \int_0^T f^{'}(B_s)dB_s + \int_0^T f^{''}(B_s)ds.$$

1.4.2 Itô's Process

Introduce a new class of processes by which we can still define a stochastic integral.

Definition 1.4.6. *A Itô's process is a process of the form*

$$X_t = X_0 + \int_0^t \varphi_s ds + \int_0^t \theta_s dB_s, \tag{1.3}$$

with X_0 $\mathcal{F}_0$-measurable, θ and φ two processes $\mathcal{F}$-adapted verifying the integrability conditions

$$\int_0^T |\theta_s|^2 ds < \infty \;\; a.s \quad and \quad \int_0^T |\varphi_s| ds < \infty \;\; a.s,$$

We note infinitesimal way:

$$dX_t = \varphi_s ds + \theta_s dB_s.$$

The study that we conducted until now requires integrability conditions stronger on process θ and φ. We will need to impose the following integrability conditions (IC)

$$E\Big[\int_0^T |\theta_s|^2 ds\Big] < \infty \quad and \quad E\Big[\int_0^T |\varphi_s|^2 ds\Big] < \infty$$

Definition 1.4.7. *The concept of stochastic integral with respect to Itô's process is defined in the following natural way.*
For ϕ element of $L^2_{\mathcal{F}}(\Omega, [0,T])$, satisfying good integrability conditions, we define:

$$\int_0^t \phi_s dX_s := \int_0^t \phi_s \theta_s dB_s + \int_0^t \phi_s \varphi_s ds.$$

Itô's formula generalizes to Itô's process .

Theorem 1.4.8. *[36] Let f a function $\mathcal{C}^2$, so we have:*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d < X >_s \\ &= f(X_0) + \int_0^t f'(X_s)\varphi_s ds + \int_0^t f'(X_s)\theta_s dW_s + \frac{1}{2}\int_0^t f''(X_s)\theta_s^2 ds. \end{aligned}$$

The infinitesimal notation of this relationship is:

$$df(X_t) = f'(X_s)dX_s + \frac{1}{2}f''(X_s)d < X >_s$$

Example 1.4.9. *Suppose a stock price, S_t, satisfies the SDE*

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t.$$

Then we can use the substitution, $Y_t = \log(S_t)$ and Itô's Lemma applied to the function $f(x) := \log(x)$ to obtain

$$S_t = S_0 \exp\left(\int_0^t (\mu_s - \sigma_s^2/2)ds + \int_0^t \sigma_s dW_s\right). \tag{1.4}$$

Note that S_t does not appear on the right-hand-side of (1.4) so that we have indeed solved the SDE. When $\mu_s = \mu$ and $\sigma_s \sigma$ are constants we obtain

$$S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma W_t) \tag{1.5}$$

so that $\log(S_t) \sim N((\mu - \sigma^2/2)t, \sigma^2 t)$

Example 1.4.10. *(**Ornstein-Uhlenbeck Process**)*
Let S_t be a security price and suppose $X_t = \log(S_t)$ satisfies the SDE

$$dX_t = [-\gamma(X_t - \mu t) + \mu]dt + \sigma dW_t.$$

Then we can apply Itô's Lemma to $Y_t := \exp(\gamma t)X_t$ *to obtain*

$$\begin{aligned} dY_t &= \exp(\gamma t)dX_t + X_t d(\exp(\gamma t)) \\ &= \exp(\gamma t)([-\gamma(X_t - \mu t) + \mu]dt + \sigma dW_t) + X_t \gamma \exp(\gamma t)dt \\ &= \exp(\gamma t)([\gamma \mu t + \mu]dt + \sigma dW_t) \end{aligned}$$

so that

$$Y_t = Y_0 + \mu \int_0^t e^{\gamma t}(\gamma s + 1)ds + \sigma \int_0^t e^{\gamma s} dW_s. \quad (1.6)$$

or alternatively(after simplifying the Riemann integral in (1.6))

$$X_t = X_0 e^{-\gamma t} + \mu t + \sigma e^{-\gamma t} \int_0^t e^{\gamma s} dW_s. \quad (1.7)$$

Once again, note that X_t *does not appear on the right-hand-side of (1.7) so that we have indeed solved the SDE. We also obtain* $E[X_t] = X_0 e^{-\gamma t} + \mu t$ *and*

$$\begin{aligned} Var(X_t) = Var\Big(\sigma e^{-\gamma t} \textstyle\int_0^t e^{\gamma s} dW_s &= \sigma^2 \exp(-2\gamma t) E\left[\left(\int_0^t e^{\gamma s} dW_s\right)^2\right] \\ &= \sigma^2 \exp(-2\gamma t) \int_0^t e^{2\gamma s} ds \qquad \text{(by Itô's isometry)} \\ &= \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}). \end{aligned}$$

$$(1.8)$$

1.5 Stochastic Integration with respect to Fractional Brownian Motion

To construct the integral with respect to fractional Brownian motion ($H \in (0,1)$, we use the generalized (fractional) Stieltjes integral (see [31]-[43]).
Consider two nonrandom functions f and g, defined on some interval $[a,b] \subset \mathbb{R}$. Suppose that the limits $f(u+) := \lim_{\delta\downarrow 0} f(u+\delta)$ and $g(u-) := \lim_{\delta\downarrow 0} g(u-\delta), a \leq u \leq b$ exist. Put $f_{a+}(x) := (f(x) - f(a+))\mathbf{1}_{(a,b)}(x)$, $g_{b-}(x) := (g(b-) - g(x))\mathbf{1}_{(a,b)}(x)$.

1.5.1 Generalized Stieltjes integral

Let $\alpha \in (0, \frac{1}{2})$. For any measurable function $f : [0,T] \to \mathbb{R}$ we introduce the following notation

$$\|f\|_\alpha := |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds. \tag{1.9}$$

Denote by $W^{\alpha,\infty}$ the space of measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t\in[0,T]} \|f(t)\|_\alpha < \infty. \tag{1.10}$$

An equivalent norm can be defined by

$$\|f\|_{\alpha,\mu} := \sup_{t\in[0,T]} e^{-\mu t} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds \right), \qquad \mu \geq 0. \tag{1.11}$$

Note that for any ϵ, $(0 < \epsilon < \alpha)$, we have the inclusions

$$\mathcal{C}^{\alpha+\epsilon}([0,T];\mathbb{R}) \subset W^{\alpha,\infty}([0,T];\mathbb{R}) \subset \mathcal{C}^{\alpha-\epsilon}([0,T];\mathbb{R}).$$

In particular, both the fractional Brownian motion B^H, with $H > \frac{1}{2}$, and the standard Brownian motion W, have their trajectories in $W^{\alpha,\infty}$. We refer the reader to ([15], [29]) for further details on this topics. We denote by $W_T^{1-\alpha,\infty}([0,T];\mathbb{R})$ the space of continuous functions $g : [0,T] \to \mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0<s<t<T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty.$$

Clearly, for all $\epsilon > 0$ we have

$$\mathcal{C}^{1-\alpha+\epsilon}([0,T];\mathbb{R}) \subset W_T^{1-\alpha,\infty}([0,T];\mathbb{R}) \subset \mathcal{C}^{1-\alpha}([0,T];\mathbb{R}).$$

Denoting

$$\Lambda_\alpha(g;[0,T]) = \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |(D_{t^-}^{1-\alpha} g_{t-})(s)|,$$

where $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ is the Euler function and

$$(D_{t^-}^{1-\alpha} g_{t-})(s) = \frac{e^{i\pi(1-\alpha)}}{\Gamma(\alpha)} \left(\frac{g(s)-g(t)}{(t-s)^{1-\alpha}} + (1-\alpha)\int_s^t \frac{g(s)-g(y)}{(y-s)^{2-\alpha}} dy \right) \mathbf{1}_{(0,t)}(s).$$

We also define the space $W^{\alpha,1}([0,T];\mathbb{R})$ of measurable functions f on $[0,T]$ such that

$$\|f\|_{\alpha,1;[0,T]} = \int_0^T \left[\frac{|f(t)|}{t^\alpha} + \int_0^t \frac{|f(t)-f(y)|}{(t-y)^{\alpha+1}} dy \right] dt < \infty.$$

We have $W^{\alpha,\infty}([0,T];\mathbb{R}) \subset W^{\alpha,1}([0,T];\mathbb{R})$ and $\|f\|_{\alpha,1;[0,T]} \leq \left(T + \frac{T^{1-\alpha}}{1-\alpha} \right) \|f\|_{\alpha,\infty;[0,T]}$.

In [43], Zähle introduced the generalized Stieltjes integral as follows.

Definition 1.5.1. *Suppose that* $f_{a+} \in I_{a+}^\alpha(L_p[a,b])$ *and* $g_{b-} \in I_{b-}^{1-\alpha}(L_q[a,b])$ *for some* $p \geq 1$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 1$, $0 \leq \alpha \leq 1$.

Under these assumptions, the generalized (fractional) Stieltjes integral $\int_0^T f(x)dg(x)$ *is defined in terms of the fractional derivative operators*

$$(D_{a+}^\alpha f_{a+})(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_{a+}(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f_{a+}(t)-f_{a+}(y)}{(t-y)^{\alpha+1}} dy \right) \mathbf{1}_{(a,b)}(t),$$

and

$$(D_{b-}^{1-\alpha} g_{b-})(t) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left(\frac{g_{b-}(t)}{(b-t)^{1-\alpha}} + (1-\alpha)\int_t^b \frac{g_{b-}(t)-g_{b-}(y)}{(y-t)^{2+\alpha}} dy \right) \mathbf{1}_{(a,b)}(t),$$

as

$$\int_a^b f(t)dg(t) := (-1)^\alpha \int_a^b (D_{a+}^\alpha f)(t)(D_{b-}^{1-\alpha} g_{b-})(t)dt. \tag{1.12}$$

The following proposition gives an estimation of the generalized Stieltjes integral.

Proposition 1.5.2. ([29]). *Fix* $0 < \alpha < \frac{1}{2}$. *Given two functions* $g \in W_T^{1-\alpha,\infty}(0,T)$ *and* $f \in W^{\alpha,1}(0,T)$ *we set*

$$G_s^t(f) = \int_s^t f_r dg_r.$$

Then for all $s < t \leq T$ *we have*

$$\begin{aligned} \left|\int_s^t f_r dg_r\right| &\leq \sup_{s\leq r<\tau\leq t} |(D_{\tau-}^{1-\alpha} g_{\tau-})(r)| \int_s^t |(D_{\tau-}^{\alpha} g_{s+})(\tau)| d\tau \\ &\leq \Lambda_\alpha(g;[s,t])\|f\|_{\alpha,1;[0,T]} \\ &\leq c_{\alpha,T}\Lambda_\alpha(g;[s,t])\|f\|_{\alpha,\infty}, \end{aligned} \tag{1.13}$$

$c_{\alpha,T} = \left(T + \frac{T^{1-\alpha}}{1-\alpha}\right)$.

As follows from [39], for any $1 - H < \alpha < 1$ there is exists a fractional derivative $D_{b-}^{1-\alpha} B_{b-}^H(t) \in L_\infty[a,b]$. Therefore, for $f \in I_{a+}^\alpha(L_1[a,b])$ we can define the integral w.r.t. the fBm according to (1.12).

Definition 1.5.3. ([30]). *The integral with respect to the fBm is defined as*

$$\int_a^b f dB^H := (-1)^\alpha \int_a^b (D_{a+}^\alpha f)(t)(D_{b-}^{1-\alpha} B_{b-}^H)(t) dt. \tag{1.14}$$

Chapter 2

Modelisation of Financial Markets

The purpose of this chapter is to provide an introduction to mathematical methods used in modeling continuous time financial markets.

2.1 Introduction to financial markets

A major revolution took place for thirty years in the financial markets, following a strong political deregulation.
This new financial landscape was born including imbalances and uncertainties about international economic relations since the early 1970. The development of inflation and the volatility of interest rates have affected investors' expectations. On the other hand, the internationalization of capital, technological advances in computing and communication have changed the relationships between different financial centers: New York, London ... it is now possible at any moment to intervene in all markets.
in France, the reforms started in mid-1984 as the goal, the deregulation of markets and the creation of a single capital market, the modernization of financial markets.
A major element of this policy was the creation of two very active financial markets, and with high liquidity, on which will be negotiated new financial instruments:

- MATIF Forward market or International of France, created in 1985, which are traded in forward.

- MONEP or Negotiable Options Market Paris, created in 1987, is a very active market organized options.

Users of these new cash instruments form a very wide range: industrial and commercial business, insurance company, banks .. These new instruments come to the rescue of investors to offset instability in market parameters such as interest rate, exchange rates ...

2.1.1 The derivative

For thirty years, we are witnessing a major revolution in the financial markets.
This financial activity is developed through many instruments such as the circulation of currency expressed in different currencies, loans operations and well on the actions issued by companies that reflect their capitalization.
The great variability and sometimes even the instability of these parameters (of interest rates, exchange rates ...) or these stocks led naturally a demand for risk transfer from certain market participants.
Banks have therefore proposed and created a number of new financial products, called **derivatives**, to meet this demand.

A derivative is a financial instrument that is bought or sold and that derives its value of those other basic financial assets. These assets are called underlying assets or derivative product support.
The assets underlying classics are negociated in different markets:

- equity markets
- exchange market: purchase / sale of foreign currency
- commodities market: oil, metals ...
- Energy Market: electricity, gas ...
- market interest rates

These derivatives allow to protect against a determined risk: down stock prices, interest rate risk . . . .

The most commonly traded derivatives are forward and options. Mainly we will focus on the problems of the options that has been the engine of the theory and shows remarkably mathematical applications in finance.

2.1.2 Forward

A forward contract is an agreement/contract between two parties that gives the right and the obligation to one of the parties to purchase (or sell) a specified commodity (or financial instrument) at a specified price at a specific future time. No money changes hands at the outset of the agreement.

- The terms of trade are permanently fixed to the date the contract is established, but the exchange of money takes place only at maturity. these contracts can cover both tons of petroleum, financial instruments, or any other property, the quality or quantity are clearly specified
- There is a risk of the counterparty with whom the contract was forged not meet its obligations. it is the risk of non-performance. its elimination led financial markets to adopt operating rules for these slightly different contracts. we talk on **future contracts**.
- Forwards are symmetrical, ie a priori each counterparty is as likely as the other to win or lose money without the future
- For stakeholders, the interest in forwards is to know the courtyard of an operation in the future. It is in this case a hedge.
- Any operation in the future may be implemented for speculative purposes. An operator who expects a certain kind of movement can buy a contract hoping for gain.

- As stressed Aftalion and poncet, these markets play an important role in terms of information dissemination. Futures prices reflect in some sense the forecast participants markets, although we will see that arbitrage arguments compel significantly.
- Another risk is permanently present on the futures markets: is liquidity risk. A stakeholder who would like to exchange his contract prior to maturity dates can not find matching quickly.
  Organized markets have ssay to establish operating rules that limit both counterparty risk and liquidity.
- Derivatives also allow for a link between different markets, (exchange rate, stock) so that all the available prices form a coherent.
  Indeed, combinations of several operations in different markets can help earn money for sure without losing any risk: We realize what is called arbitrage.

The presence of many very competent professionals in the trading rooms led by the law of supply and demand to price adjustments that reduce these arbitrage opportunities.

Another well known and most fundamental example is an option on a stock.

2.1.3 The options

An option is a contract giving the holder the right, not the obligation, to buy or sell a certain amount of an asset or to a date (Maturity) and fixed at an agreed price in advance.
The parameters of an option are:

- **The maturity** of the option which limits its exercise period.
- **The strike price** is the price fixed in advance which is the transaction if the option is exercised.
- **The premium** is the contract price paid by the buyer to the seller of the option.

European Call and Put Options

The buyer of the stock may seek protection from a market crash by purchasing a contract that allows him to sell his asset at time T at a guaranteed price K fixed at time t. This contract is called a put option with strike price K and exercise date T.

Definition 2.1.1. *A (European) put option is a contract that gives its holder the right (but not the obligation) to sell a quantity of assets at a predefined price K called the strike price (or exercise price) and at a predefined date T called the maturity.*

The major problem we face in the options is the calculation of the premium.
To formalize the ideas, consider the case of a European call option on a share whose price at time t is S_T, maturity T and strike price K.
Then S_T falls down below the level K, exercising the contract will give the holder of the option a gain equal to $K - S_T$ in comparison to those who did not subscribe the option and sell the asset at the market price S_T . In turn, the issuer of the option will register a loss also equal to $K - S_T$.

If S_T is above K then the holder of the option will not exercise the option as he may choose to sell at the price S_T. In this case the profit derived from the option is 0.
In general, the payoff of a (so called European) put option will be of the form

$$\phi(S_T) = (K - S_T)^+ = \begin{cases} K - S_T, & S_T \leq K, \\ 0, & S_T \geq K. \end{cases}$$

On the other hand, if the trader aims at buying some stock or commodity, his interest will be in prices not going up and he might want to purchase a call option, which is a contract allowing him to buy the considered asset at time T at a price not higher than a level K fixed at time t. Here, in the event that S_T goes above K, the buyer of the option will register a potential gain equal to $S_T - K$ in comparison to an agent who did not subscribe to the call option.

Definition 2.1.2. *A (European) call option is a contract that gives its holder the right (but not the obligation) to buy a quantity of assets at a predefined price K*

called the strike and at a predefined date T called the maturity.

In general, a (European) call option is an option with payoff function

$$\phi(S_T) = (S_T - K)^+ = \begin{cases} S_T - K, & S_T > K, \\ 0, & S_T \leq K. \end{cases}$$

Remark 2.1.3. *The adjective European is to be contrasted with American. While a European option can be exercised only on the expiry date, an American option can be exercised at any time between the start date and the expiry date. In Chapter 18 of Higham [7], it is shown that American call options have the same value as European call options. American put options, however, are more complicated.*

In market practice, options are often divided into a certain number n of warrants, the (possibly fractional) quantity n being called the entitlement ratio.
In order for an option contract to be fair, the buyer of the option should pay a fee (similar to an insurance fee) at the signature of the contract. The computation of this fee is an important issue, which is known as option pricing.
The second important issue is that of hedging, i.e. how to manage a given portfolio in such a way that it contains the required random payoff $(K - S_T)^+$ (for a put option) or $(S_T - K)^+$ (for a call option) at the maturity date T.

The answer to these two questions are closely related, is based course a minimum of assumptions that must be:
a modeling assumption of the markets and especially the prices of financial assets and an assumption of no arbitrage, which essentially says that is not possible to make money without taking risks.

2.2 Arbitrage

One of the key principles on which option valuation theory rests is no arbitrage.

There is never an opportunity to make a risk-free profit that gives a greater return than that provided by the interest from a bank deposit.

Note that this assumption applies only to risk-free profit, it is not relevant to portfolios that have a good chance of making a greater return than a bank deposit.

To justify the no arbitrage assumption, suppose it were possible to put together a portfolio that gave a guaranteed improvement on the bank's interest rate.
Sensible investors would simply borrow money from the bank and spend it on the portfolio, thereby locking in to a guaranteed risk-free profit.
The forces of supply and demand would then cause the yield from the portfolio to drop, or the interest rate to increase, or both, until parity was restored.
Further justification for this assumption is provided by the existence of arbitrageurs who scour the markets seeking to exploit any opportunities for risk-free profits beyond the interest rate level.

2.3 Stochastic models

To calculate the price of a derivative, we need a stochastic model to describe the uncertain evolution of or underlying securities. A stochastic model should reflect the observations of price history as well as possible from a statistical point of view. Meanwhile, the stochastic model must fit into a mathematical framework that enables efficient analysis of option prices. A "good" model must capture both the statistical properties of the dynamics of prices and their effective integration into the theory of stochastic analysis. The famous Black-Scholes model is a compromise between these two requirements and in many cases gives explicit formulas of option prices. In this introductory chapter to financial mathematics, we will study mainly this model.
There has been in recent years a major fad for developing more general models that the Black-Scholes model to better "stick" to the historical price observations. These are called stochastic volatility models with jumps ... Whatever the model used, it must then determine the game in the model parameters from the observation of the

underlying price and the same price of traded options : it is the problem of estimation and model calibration.

2.4 Valuation of financial markets by arbitrage

This section introduces the basic principles of financial market models in continuous time that the Black-Scholes model is a standard reference.
Before addressing this model, it is useful to give a brief overview of its history.

2.4.1 A little history

The origins of the mathematization of modern finance back to the thesis Louis Bachelier called "Théorie de la spéculation" sustained at the Sorbonne in 1900.
This work marked the birth of a share of stochastic continuous time processes in probability, and secondly that of continuous-time strategies for hedging in finance.
The mathematical side, his thesis that greatly influence research of A. N. Kolmogorov on continuous time processes in the 1920 and those K.Itô - the inventor of stochastic calculus - in 1950.
In contrast, as regards finance, approach Bachelier was forgotten for almost three quarters of a century, until 1973 with the release of Black works, Scholes and Merton.
Let's go back to that time in the 1970 to better understand the context. That's when the political will emerges to deregulate the financial markets, making them volatile interest rates and fluctuating exchange rates. In this deregulated environment, industrial companies and business are subject to increased risks related to such extreme variability of exchange rates: this is uncomfortable, especially when revenues and expenses are denominated in different currencies (dollar and euro say).
To provide businesses with tools adapted to these problems and more generally to allow insurance companies and banks to cover these new risks, new organized markets were created, allowing stakeholders to massively exchanging insurance products.

2.4.2 Continuous time model

The continuous time models are models or agents are allowed to negotiate continuously on the market and where we must model the evolution of asset prices as continuous time processes.
We make the assumptions made on the financial markets: there is no friction markets, ie no transaction costs for the sale and purchase of shares, no restrictions on short sales, assets are infinitely divisible and at any time there are buyers and sellers for all securities market.

The uncertainty in the financial markets is modeled by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ provided with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ where

- Ω represents all the states of the world
- the tribe $\mathcal{F}$ represents the global information structure available on the market
- $(\mathcal{F}_t)$ is an increasing filtration describing the information available to market agents at time t, $\mathcal{F}_t \subset \mathcal{F}$
- probability $\mathbf{P}$ which gives the a priori probabilities of events considered. It is the historical or objective probability.

We distinguish the basic securities, stocks, bonds, ... which are the elements constituent portfolios and derivatives, options, futures contracts that are the subject of the problem of valuation and coverage.
The basic tracks
There are $d+1$ basic assets on the market, denoted $S^0, S^1, \ldots, S^d$, can be traded on any date $t \geq 0$.
$S_t^i(\omega)$ is the price of the asset i at time t in the state of the world $\omega \in \Omega$. it is assumed that the price process $(S_t^i)_{t\geq 0}$ are continuous in time, i.e. for almost all ω the application $t \to S_t^i(\omega)$ is continuous. We note $X = (S^0, S^1, \ldots, S^d)$ the price process of $d+1$ assets.

The asset S_0 is cash, i.e. the financial product that describes the value of 1 euro capitalized on a daily basis to the bank. It is considered without risk because its performance in a time interval $[t, t + dt]$ is known at date t of the operation.
We denote r the interest rate per unit of time, assumed constant, for placement between t and $t + dt$ in the bank.
The evolution of the cash assets S^0 is:

$$dS_t^0 = rS_t^0 dt, \qquad S_0^0 = 1. \tag{2.1}$$

In other words, 1 euro capitalized in the bank reports:

$$S_t^0 = e^{rt}$$

euros at time t.
Assets $S = (S^1, ..., S^d)$ usually represent the prices of risky assets such as stocks, bonds . . . Our reference model is given by the famous model of Black-Scholes-Merton for $d = 1$ risky asset:

$$dS_t = S_t(bdt + \sigma dW_t) \tag{2.2}$$

where W is a Brownian motion with respect to $\mathbb{F}$ its filtration, and b, σ are constants. This model was introduced by Black, Scholes and Merton in 1973 (Merton and Scholes received the Nobel Prize in 1997 for this work; Black died before). There is actually an explicit formula for the price given by:

$$S_t = S_0 \mathrm{exp}\left(\sigma \mathrm{W_t} + \left(\mathrm{b} - \frac{\sigma^2}{2}\right)\mathrm{t}\right) \tag{2.3}$$

2.4.3 Self-financing portfolio

We model the concept of dynamic portfolio management. Consider an agent who can invest in basic market assets. In a continuous time model, a self-financing portfolio strategy (in assets $X = (S^0, \ldots, S^d)$) is the data of a process adapted $\phi = (\phi^0, \varphi)$ as the stochastic integral $\int \phi dX$ exists and whose portfolio value is characterized by:

$$V_t(\phi) = \phi_t X_t = V_0(\phi) + \int_0^t \phi_u dX_u$$

We also write the dynamics of the value of a self-financing portfolio as differential:

$$\begin{aligned} dV_t(\phi) &= = \phi_t^0 dS_t^0 + \varphi_t dS_t = (V_t(\phi) - \varphi_t S_t) r dt + \varphi_t dS_t \\ &= rV_t(\phi)dt + \varphi_t(-rS_t dt + dS_t). \end{aligned} \tag{2.4}$$

Actualization by cash

We examine the condition of self-financing when updated by the cash. we note $\widetilde{S}_t^i = S_t^i / S_t^0 = e^{-rt} S_t^i$, $i = 1, \dots, d$, the discounted price (compared to cash) of risky assets, and $\widetilde{V}_t(\phi) = V_t(\phi)/S_t^0 = e^{-rt} V_t(\phi)$ discounted wealth.
So by the Ito formula and (3.3.5), the dynamics of the value of a self-financing portfolio:

$$\begin{aligned} d\widetilde{V}_t(\phi) &= -re^{-rt} V_t(\phi) dt + e^{-rt} dV_t(\phi) = e^{-rt}[-rV_t(\phi)dt + dV_t(\phi)] \\ &= e^{-rt}\varphi_t(-rS_t dt + dS_t) \\ &= \varphi_t d\widetilde{S}_t, \end{aligned}$$

what writes:

$$\widetilde{V}_t(\phi) = \widetilde{V}_0(\phi) + \int_0^t \varphi_u d\widetilde{S}_u \tag{2.5}$$

2.4.4 Arbitrage and risk-neutral probability

The assumption of absence of arbitrage opportunities is a crucial condition in the theory of the valuation of derivatives. We formalize this concept with the following definition.

Definition 2.4.1. *An arbitrage opportunity on $[0, T]$ is a self-financing portfolio strategy ϕ whose value $V(\phi)$ satisfies:*

(i) $V_0(\phi) = 0$,

(ii) $V_T(\phi) \geq 0$ *and* $\mathbf{P}[V_T(\phi)] > 0$.

Thus, arbitration is the dynamic management of a self-financing portfolio allowing from zero capital, to create a profit without risk..
In continuous time models, we will be forced to make additional assumptions integrability on portfolio strategies to ensure the absence of arbitrage opportunities. Indeed, there are strategies that are arbitrage opportunities as shown in the following example.

Example 2.4.2. *Let a market with a risk-free asset $S^0 = 1$ and a risky asset $S_t = W_t$ brownian motion. Let $x > 0$ and τ_x the time to stop corresponding to the first time where $S = W$ touch x. Consider the risky asset strategy $\varphi = 1_{]0,\tau_x]}$. So starting from zero initial wealth $V_0 = 0$, the value of this portfolio is $V_t = \int_0^t 1_{]0,\tau_x]}(u)dW_u = W_{t\wedge\tau_x}$ and tends to a positive wealth $W_{\tau x} = x > 0$ for an infinite horizon.*

We subsequently propose the integrability conditions on strategies to exclude such pathologies. We call admissible strategies such strategies. This set of admissible strategies must be rich enough to allow assessment and coverage of many derivatives and not too big to avoid arbitrage opportunities.
The condition of no arbitrage opportunity also imposes conditions on prices. In a model in continuous time, this condition implies the existence of a probability $\mathbf{Q}$, called risk-neutral, equivalent to the objective probability as the price of discounted assets is a martingale.
Generally, it is then introduced the following definition.

Definition 2.4.3. *A $\mathbf{Q}$ probability is called risk-neutral probability or probability martingale if $\mathbf{Q}$ is equivalent to $\mathbf{P}$ and if the discounted price $\widetilde{S}_t = e^{-rt}S_t$ is a martingale under $\mathbf{Q}$.*

The condition of equivalence between $\mathbf{Q}$ and $\mathbf{P}$ means that for every event $A \subset \Omega$, if $\mathbf{P}(A) > 0$ then $\mathbf{Q}(A) > 0$ and vice versa. In other words, that predicted a positive probability, $\mathbf{Q}$ also predicted and the converse is true.
The name comes naturally martingale property of the discounted price and the word risk neutral is that the return on assets is equal to the interest rate r of $\mathbf{Q}$.

Chapter 3

Conditional Full Support and applications to finance

3.1 Introduction

In this section, we study a simple condition on asset prices, namely conditional full support, which generates a large class of consistent price systems. In fact, all natural examples (which we can think of) enjoy this property. We study the problems of no-arbitrage for asset prices driven by a continuous process and with constant proportional transaction costs.
The conditional full support (CFS) introduced by Guasoni, Rasonyi, and Schachermayer [13], in connection to mathematical finance, via. pricing models with transaction costs. Their main result asserts that if a continuous price process has CFS, then for any $\varepsilon > 0$ there exists a socalled ε-consistent price system, which is a martingale (after an equivalent change of measure).

The existence of ε-consistent price systems for all $\varepsilon > 0$ implies that the price process does not admit arbitrage opportunities under arbitrary small transaction costs, since any arbitrage strategy would generate arbitrage also in the consistent price system, which is a contradiction because of the martingale property.

Consistent price systems can be seen as generalizations of equivalent martingale measures (EMM's), since if a price process admits an EMM, then the price process itself qualifies as a trivial ε-consistent price system for any $\varepsilon > 0$.

Before starting the relation between the CFS and no arbitrage, let us introduce the conditional full support condition which prescribes that from any given time on, the asset price path can continue arbitrarily close to any given path with positive conditional probability.
To this end, recall first that when E be a separable metric space and $\mu : \mathbb{B}(E) \to [0, 1]$ a Borel probability measure.

Definition 3.1.1. *We say that the support of μ is the (unique) minimal closed set $A \subset E$ such that $\mu(A) = 1$. We denote this set by supp(μ).*

Let$(X_t)_{t\in[0,T]}$ be a continuous process taking values in an open interval $I \subset \mathbb{R}$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. and let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a filtration on this space. Moreover, let $C_x([u, v], I)$ be the space of functions $f \in C([u, v], I)$ such that $f(u) = x \in I$. As usual, we equip the spaces $C([u, v], I)$ and $C_x([u, v], I), x \in I$ with the uniform topologies.

Definition 3.1.2. *We say that the process X has conditional full support (CFS) with respect to the filtration $\mathbb{F}$, or briefly $\mathbb{F}$-CFS, if*

1. *X is adapted to $\mathbb{F}$*
2. *for all $t \in [0, T]$ and P-almost all $\omega \in \Omega$,*

$$supp(Law[(X_u)_{u\in[t,T]}|\mathcal{F}_t](\omega)) = C_{x_t(\omega)}([t, T], I) \tag{3.1}$$

3.2 Basic results on the conditional full support property

Since CFS is a very recent concept, in the absence of any comprehensive account, it is instructive to present a few basic results that can be used to establish the property.

We will consider processes and their CFS in the largest possible state space $\mathbb{R}$.

Notations and conventions

Let $\mathbb{T} \subset [0,\infty)$ be a left-closed interval and $(X_t)_{t\in T}$ a generic stochastic process on $(\Omega, \mathcal{F}, \mathbf{P})$.

For any $t \in \mathbb{T}$, we write $\widetilde{X}^t := (X_s - X_t)_{s\geq t}$.

We denote by $\widetilde{\mathbb{F}}^X = (\widetilde{\mathcal{F}}_t^X)_{t\in\mathbb{T}}$ the raw natural filtration of X and by $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\in\mathbb{T}}$ its usual augmentation (the minimal right-continuous augmentation of $\widetilde{\mathbb{F}}^X$ such that $\mathcal{F}_{\min}\mathbb{T}^X$ contains all P-null sets in $\widetilde{\mathcal{F}}_t^X$ for all $t \in \mathbb{T}$).

As usual, $\|.\|_\infty$ denotes the sup-norm, and for any $f, g \in C(\mathbb{T}) := C(\mathbb{T}, \mathbb{R})$ and $r > 0$, write $B(g,r) := \{h \in C(\mathbb{T}) : \|h - g\|_\infty < r\}$ and $I(f,g,r) := 1_{B(g,r)}(f)$.

Finally, $\mathbb{R}_+ := (0,\infty)$, $\mathbb{Q}_+ := \mathbb{Q} \cap \mathbb{R}_+$, and λ stands for the Lebesgue measure on $\mathbb{R}$.

Remark 3.2.1. *If $I \subset \mathbb{R}$ is an open interval and $f : \mathbb{R} \to \mathrm{I}$ is a homeomorphism then $g \longmapsto f \circ g$ is a homeomorphism between $\mathcal{C}_x([0,T])$ and $\mathcal{C}_{f(x)}([0,T])$. Hence, for $f(X)$, understood as a process in I, we have*

$$f(X) \; has \; \mathbb{F} - CFS \Longleftrightarrow X \; has \; \mathbb{F} - CFS \tag{3.2}$$

We begin with an alternative "small-ball" characterization of CFS, which is more tractable than the original definition.

Lemma 3.2.2. *[33]((**Small-ball probabilities**)*
Let $(X_t)_{t\in[0,T]}$ be a continuous process, adapted to filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. Then, X has $\mathbb{F}$-CFS if and only if

$$E[I(\widetilde{X}^t, f, \varepsilon)|\mathcal{F}_t] > 0 \qquad a.s \tag{3.3}$$

for all $t \in [0,T)$, $f \in \mathcal{C}_0([t,T])$, and $\varepsilon > 0$.

Lemma 3.2.3. *[33](**Positivity**)*
Let $\mathcal{G}$ and $\mathcal{H}$ be σ-algebras such that $\mathcal{G} \subset \mathcal{H}$, and $Y \in L^1$ such that $Y \geq 0$.
If $E[Y|\mathcal{H}] > 0$ a.s., then $E[Y|\mathcal{G}] > 0$ a.s.

Corollary 3.2.4. *[33](**Smaller filtration**)*
Let $(X_t)_{t\in[0,T]}$ *be a continuous process, adapted to filtrations* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ *and* $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ *that satisfy* $\mathcal{G}_t \subset \mathcal{H}_t$ *for all* $t \in [0,T)$*. Then, if* X *has* $\mathcal{F}$*-CFS, then it has also* $\mathcal{G}$*-CFS.*

Lemma 3.2.5. *[33](**Usual augmentation**)*
Let $(X_t)_{t\in[0,T]}$ *be a continuous process, adapted to filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$*. Then,* X *has* $\mathcal{F}$*-CFS if and only if it has CFS with respect to the usual augmentation of* $\mathcal{F}$*.*

Lemma 3.2.6. *[33]**Law invariance***
Let $(X_t)_{t\in[0,T]}$ *and* $(Y_t)_{t\in[0,T]}$ *be a continuous processes (possibly defined on distinct probability spaces) such that* $X =^{law} Y$*. Then,* X *has* $\mathbb{F}^X$*-CFS if and only if* Y *has* $\mathbb{F}^Y$*-CFS.*

3.3 Consistent Price System and Conditional Full support

In markets with transaction costs, consistent price systems (CPS) play the same role as martingale measures in frictionless markets. Guasoni, Rasonyi, and Schachermayer [13] prove that if a continuous price process has conditional full support, then it admits consistent price systems for arbitrarily small transaction costs. This result applies to a large class of Markovian and non-Markovian models, including fractional Brownian motion.
The first main result of [13] shows that the condition of CFS implies the existence of consistent price systems.

Theorem 3.3.1. *Let* X_t *be an* $\mathbb{R}^d_+$*-valued, continuous adapted process satisfying (CFS); then* X *admits an* ε*-consistent pricing system for all* $\varepsilon > 0$

The proof of this theorem is quite intuitive, at least in dimension $d = 1$: Guasoni, Rasonyi, and Schachermayer [13] show that any continuous price process satisfying the conditional full support condition (CFS) is arbitrarily close to the archetypal

model of a "random walk with retirement", where martingale measures are characterized in terms of "retirement probabilities".
Moreover it help to explain the difference between transaction costs and frictionless markets as regards option pricing and hedging.

3.3.1 One asset with proportional transaction costs

we present now the proof of Theorems 3.3.1 for a market with one asset and with proportional transaction costs.

Guasoni, Rasonyi, and Schachermayer [13] assume that the bid and ask prices are given by $(1+\varepsilon)^{-1}S_t$ and $(1+\varepsilon)S_t$, respectively, where (S_t) is a continuous adapted process with strictly positive trajectories and $\varepsilon > 0$ is fixed.
They begin with the definition of a CPS:

Definition 3.3.2. *Let $\varepsilon > 0$. An ε-consistent price system is a pair $(\widetilde{S}, \widetilde{\mathbf{P}})$, of a probability $\widetilde{\mathbf{P}}$ equivalent to $\mathbf{P}$ and a $\widetilde{\mathbf{P}}$-martingale $\widetilde{S}$ (adapted to $\mathcal{F}_t$) such that*

$$\frac{1}{1+\varepsilon} \leq \frac{\widetilde{S}_t^i}{S_t^i} \leq 1+\varepsilon, \quad almost\ surely\ for\ all\ t \in [0,T]\ and\ 1 \leq i \leq d.$$

The previous definition show that constructing consistent price systems is a key to solve the no-arbitrage problems under transaction costs by duality methods. They begin by introducing the basic model of Random Walk with Retirement, which allows a large class of consistent price systems to be produced. In the following, they employ this construction first to show the existence of consistent price systems (Theorem 3.3.1).
Random walk with retirement. Consider a discrete-time filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_n), \mathbf{P})$ such that $\mathcal{G}_0$ is trivial and $\bigvee_n \mathcal{G}_n = \mathcal{G}$.

Definition 3.3.3. *A Random Walk with Retirement is a process (X_n), adapted to $(\mathcal{G}_n)$, of the form*

$$X_n = X_0(1+\varepsilon)^{\sum_{i=1}^n R_i}, \quad n \geq 1,$$

where $\varepsilon > 0, X_0 \in \mathbb{R}_{++}$ *and the process* $\{R_n\}_{n\geq 1}$ *has values in* $\{-1, 0, +1\}$ *and satisfies:*

(i) $\mathbf{P}(R_m = 0 \ for \ all \ m \geq n | R_n = 0) = 1 \ for \ n \geq 1;$

(ii) $\mathbf{P}(R_n = x|\mathcal{G}_{n-1}) > 0$ *on* $\{R_{n-1} \neq 0\}$ *for all* $x \in \{-1, 0, +1\}$ *and* $n \geq 1$*(they set* $R_0 \neq 0 := \Omega$ *as a convention);*

(iii) $\mathbf{P}(R_n \neq 0 \ for \ all \ n \geq 1) = 0.$

In plain English, a Random Walk with Retirement is just a random walk on the geometric grid $(X_0(1+\varepsilon)^k)_{k\in\mathbb{Z}}$, starting at X_0 and "retiring" at the a.s. finite stopping time $\rho = min\{n \geq 1 : R_n = 0\}$. Note that the filtration $(\mathcal{G}_n)$ is, in general, larger than the one generated by X.
In the following lemma, they describe the general form of a probability measure $\mathbf{Q} \ll \mathbf{P}$ such that X is a $\mathbf{Q}$-martingale. The martingale condition determines the relative weights of probabilities of upward and downward movements. By contrast, at each time, they may choose arbitrarily the conditional probability of retirement, denoted by α.

Lemma 3.3.4. *[13] Let* (X_n) *be a Random Walk with Retirement, and* $(\alpha_n, n \geq 1)$ *a predictable (i.e.,* α_n *is* $\mathcal{G}_{n-1}$*-measurable) process with values in* $[0, 1]$ *If* α *satisfies*

$$\lim_{n\to\infty} E\left[\prod_{i=1}^{n}(1-\alpha_i)\right] = 0$$

then there exists a (unique) probability $\mathbf{Q}^\alpha$ *on* $\mathcal{G}$ *such that:*

(i) $\mathbf{Q}^\alpha$ *is absolutely continuous with respect to* $\mathcal{P}$*;*

(ii) X *is a* $\mathbf{Q}^\alpha$*-martingale;*

(iii) $\mathbf{Q}^\alpha(R_n = 0|\mathcal{G}_{n-1}) = \alpha_n$ *a.s. on* $\{R_{n-1} \neq 0\}$.

We have $\mathbf{Q} \sim \mathbf{P}$ *iff* $\alpha_n \in (0, 1)$ *a.s. for* $n \geq 1$.

The next lemma shows that, by choosing high probabilities of early retirement, one obtains an equivalent martingale measure with arbitrary integrability conditions. In particular, this implies the existence of equivalent martingale measures for which X is uniformly integrable.

Lemma 3.3.5. *[13] Let (X_n) be a Random Walk with Retirement. Then, for any function $f : \mathbb{R}_{++} \longmapsto \mathbb{R}$ and any $\varepsilon > 0$, there exists some $\mathbf{Q}^\alpha \sim \mathbf{P}$ as in Lemma 3.3.4 such that*

$$E_{\mathbf{Q}^\alpha}\left[sup_{n\geq 0} f(X_n)\right] < \infty.$$

Consistent price systems.

Now they employ the previous construction to prove the existence of consistent price systems. They construct an increasing sequence of stopping times at which the process S behaves like a "Random Walk with Retirement"; the conditional full support assumption is key to making this construction possible. Martingale measures for a Random Walk with Retirement are obtained by arbitrarily specifying the probability of retirement, as in Lemma 3.3.4 above. The $\mathbf{Q}$-martingale $\widetilde{S}$ is then defined as the continuous-time martingale determined by the terminal value of the random walk with retirement.

For technical reasons, we state a formally stronger version of the conditional full support condition in terms of stopping times.

Definition 3.3.6. *Let τ be a stopping time of the filtration $(\mathcal{F}_t, t \in [0,T])$.*
Let us define $S_t := S_T$ for $t > T$ and let $\mu^\tau(.,\omega)$ be (a regular version of) the $\mathcal{F}_t$-conditional law of the $C^+[0,T]$-valued random variable $(S_{\tau+t})$.
We say that the strong conditional full support condition (SCFS) holds if, for each $[0,T]$-valued stopping time τ and for almost all $\omega \in \{\tau < T\}$, the following is true: for each path $f \in C^+_{S_\tau(\omega)}[0, T - \tau(\omega)]$ and for any $\eta > 0$, the η-tube around f has positive $\mathcal{F}_\tau$-conditional probability, that is,

$$\mu^\tau(B_{f,\eta}(\omega), \omega) > 0,$$

where

$$B_{f,\eta} = \left\{ g \in C^{+}_{S_\tau(\omega)}[0,T] : \quad sup_{s\in[0,T-\tau(\omega)]}|f(s) - g(s)| < \eta \right\}.$$

In other words, this property means that for all τ*,*
(SCFS)

$$supp(\mathbf{P}(S|_{[\tau,T]}|\mathcal{F}_\tau]) = C^{+}_{S_\tau}[\tau, T] \quad a.s.,$$

that is, the conditional full support condition (CFS) also holds with respect to stopping times, while it was formulated in terms of deterministic times only in Definition 3.1.2 above.

The conditions (SCFS) and (CFS) are, in fact, equivalent. The precise formulation of this idea is somewhat technical, thus the proof of the next lemma is postponed to the [[13],Appendix].

Lemma 3.3.7. *The conditional full support condition (CFS) implies the strong conditional full support condition (SCFS), hence they are equivalent.*

We now present the proof of Theorem 3.3.1 in dimension one and under the above (SCFS) hypothesis. In this case, the arguments are hopefully transparent and intuitive.

Proof. Guasoni, Rasonyi, and Schachermayer [13] may suppose that $\varepsilon \in (0, 1)$. For any such ε, they associate to the process (S_t) a "random walk with retirement" as follows. They define the increasing sequence of stopping times

$$\tau_0 = 0, \tau_{n+1} = inf\left\{ t \geq \tau_n : \frac{S_t}{S_{\tau_n}} \notin ((1+\varepsilon)^{-1}, 1+\varepsilon) \right\} \wedge T.$$

For $n \geq 1$, they set

$$R_n = \begin{cases} sign(S_{\tau_n} - S_{\tau_{n-1}}), & if \tau_n < T \\ 0, & if \tau_n = T \end{cases} \tag{3.4}$$

Recall from the previous section the Random Walk with Retirement $(X_n, n \geq 0)$

$$X_n = X_0(1+\varepsilon)^{\sum_{i=1}^{n} R_i}$$

adapted to the filtration $(\mathcal{G}_n)$ where $\mathcal{G}_n = \mathcal{F}_{\tau_n}$. To check the properties in Definition 3.3.3, observe that (i) is trivial, while (iii) follows from the continuity of paths. Furthermore, the (CFS) condition implies (ii) by the following Lemma

Lemma 3.3.8. *[13] Let S be an $\mathbb{R}_{++}$-valued continuous process satisfying (CFS) and let R_n be defined by (3.4). Then, $\mathbf{P}(R_{n+1} = z|\mathcal{F}_{\tau_n}) > 0$ a.s. on $\tau_n < T$ for $z = -1, 0, +1$ and $n \geq 0$.*

By Lemma 3.3.5, there exists some $\mathbf{Q}^\alpha \sim \mathbf{P}$ on $\mathcal{F} = \mathcal{G} := \bigvee_n \mathcal{G}$ such that

$$E_{\mathbf{Q}^\alpha}\left[sup_{n\geq 0} X_n\right] < \infty.$$

Thus, X is a uniformly integrable $(\mathbf{Q}^\alpha, (\mathcal{G}_n))$-martingale and is closed by its terminal value X_∞. Define

$$\widetilde{S}_t := E_{\mathbf{Q}^\alpha}[X_\infty|\mathcal{F}_t], \quad t \in [0, T].$$

Fix $0 \leq t \leq T$, define the random times $\sigma = max\{\tau_n : \tau_n \leq t\}$ and
$\tau = min\{\tau_n : \tau_n > t\}$ and observe that τ is a stopping time. We have, by definition,

$$1 + \varepsilon^{-1} \leq \frac{S_t}{S_\sigma}, \frac{S_\tau}{S_\sigma} \leq 1 + \varepsilon, \quad \textit{almost surely for all } t \in [0, T]$$

and they therefore obtain

$$1 + \varepsilon^{-2} \leq \frac{S_\tau}{S_t} \leq (1 + \varepsilon)^2, \quad \textit{almost surely for all } t \in [0, T]$$

By construction, $\widetilde{S}_{\tau_n} = X_n$ and $S_{\tau_n} = X_n$ on $\tau_n < T$ for all $n \geq 0$. On $\tau_n = T$, we have the estimate

$$1 + \varepsilon^{-1} \leq \frac{\widetilde{S}_{\tau_n}}{S_{\tau_n}} \leq (1 + \varepsilon)$$

for all $n \geq 0$.
The optional sampling theorem then implies that

$$\frac{\widetilde{S}_t}{S_t} = \frac{E_{\mathbf{Q}^\alpha}[\widetilde{S}_\tau|\mathcal{F}_t]}{S_t} = E_{\mathbf{Q}^\alpha}\left[\frac{\widetilde{S}_\tau}{S_\tau}\frac{S_\tau}{S_t}|\mathcal{F}_t\right]$$

and therefore that

$$1+\varepsilon^{-3} \leq \frac{\widetilde{S}_t}{S_t} \leq (1+\varepsilon)^3, \quad almost\ surely\ for\ all\ t \in [0,T],$$

which completes the proof, up to the passage to a smaller ε. □

3.3.2 Previous results

To illustrate the scope of our results, Guasoni, Rasonyi, and Schachermayer [13] present some important classes of models where the conditions of Theorem 3.3.1 can be checked.

Fractional Brownian motion

We now turn to models based on fractional Brownian motion (FBM).
Models of asset prices based on fractional Brownian motion have long attracted the interest of researchers for their properties of long-range dependence [[5], [24], [25], [42]]. However, in a frictionless setting, it turns out that these models lead to arbitrage opportunities [[34], [37], [40]] and therefore cannot be meaningfully employed for studying optimal investment and derivatives pricing.
This situation is completely different as soon as arbitrarily small transaction costs are introduced. There then exist consistent price systems.
The next result improves on Proposition 5.1 of [14] and follows from a similar argument.

Proposition 3.3.9. *Let $S_t = exp\{\sigma X_t + f_t\}$, where X_t is FBM with parameter $0 < H < 1$ and f_t is a deterministic continuous function. $(S_t, t \in [0,T])$ then satisfies the conditional full support condition (CFS) with respect to its (right-continuous and saturated) natural filtration.*

Proof. Let us fix $v \in [0,T]$. It is enough to prove that the conditional *law* $\mathbf{P}(X|_{[v,T]}|\mathcal{F}_v)$ has full support on $C_{X_v}([v,T],\mathbb{R})$ almost surely. From the representation of Corollary 3.1 in [6], we know that, for some square-integrable kernel

$K_H(t,s)$, one has

$$X_t = \int_0^t K_H(t,s)dW_s, \tag{3.5}$$

for some Brownian motion (W_t) generating the same filtration as (X_t).
It is easily seen, by directly calculating the conditional joint characteristic function of finite-dimensional distributions of X, that for any $v \in [0,T]$, the process $(X_t, t \in [v,T])$ is Gaussian, conditionally on $\mathcal{F}_v$. Its conditional expectation and conditional covariance function are given by

$$c_t := E[X_t|\mathcal{F}_v] = \int_0^v K_H(t,s)dW_s \qquad t \geq v,$$

$$\widetilde{\Gamma}(t,s) := cov_{\mathcal{F}_v}(X_t, X_s) = \int_v^{t\wedge s} K_H(t,u)K_H(s,u)du, \qquad t,s \geq v.$$

Observe that $\widetilde{\Gamma}(t,s)$ does not depend on ω. Hence, for almost all ω, the law of $(X_t, t \in [v,T])$ conditional on $\mathcal{F}_v$ is equal to the law of $Y_t + c_t(\omega)$, where $(Y_t, t \in [v,T])$ is a centered Gaussian process with continuous paths on $[v,T]$ and with covariance function $\widetilde{\Gamma}$ Thus, recalling the kernel representation 3.5, it suffices to prove that the centered Gaussian process

$$Y_t := \int_v^t K_H(t,s)dW_s, \qquad t \in [v,T]$$

has full support on $C_0([v,T],\mathbb{R})$.
Theorem 3 in [21] states that the topological support of a continuous Gaussian process $(Y_t, t \in [v,T])$ is equal to the norm closure of its reproducing kernel Hilbert space, defined by

$$\mathbb{H} := \left\{ f \in C_0([v,T],\mathbb{R}) : f(t) = \int_v^t K_H(t,s)g(s)ds, \text{ for some } \mathrm{g} \in \mathrm{L}^2[\mathrm{v},\mathrm{T}] \right\}.$$

Thus, it is sufficient to show that $\mathbb{H}$ is norm-dense in $C_0([v,T],\mathbb{R})$.
To achieve this, we need to recall the Liouville fractional integral operator for any

$f \in L^1[a,b]$ and $\alpha > 0$

$$(I^{\alpha}_{a^+} f)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t f(s)(t-s)^{\alpha-1} ds, \qquad a \leq t \leq b,$$

and to introduce the kernel operator K_H,

$$(K_H f)(t) := \int_0^t K_H(t,s) f(s) ds, \qquad f \in L^2[0,T], \quad t \in [0,T].$$

(1) We first treat the case $H < \frac{1}{2}$ In this case, we have by [6], Theorem 2.1, that

$$(K_H f) = I^{2H}_{0^+}(s^{\frac{1}{2}-H} I^{\frac{1}{2}-H}_{0^+}(s^{H-\frac{1}{2}} f(s)))$$

For general v, The argument needs to be split into two steps.

- **Step 1**

Lemma 3.3.10. *If $f \in C_0[v,T]$, then $L_1 f \in C_0([v,T]$, where*

$$(L_1 f)(t) = (I^{\frac{1}{2}-H}_{0^+}(s^{H-\frac{1}{2}} f(s)))(t)$$

Morever, $L_1 : C_0[v,T] \longrightarrow C_0[v,T]$ is continuous and has dense range (with respect to the uniform norm).

Proof. Clearly, $L_1 f$ is a continuous function and $(L_1 f)(0) = 0$. The operator is continuous by the estimate

$$\|L_1 f - L_1 g\|_\infty \leq v^{H-1/2} \int_0^T (T-s)^{H-1/2} ds \|f - g\|_\infty.$$

Recall the identity for $a, b > 0$,

$$\int_0^t (t-u)^{a-1} u^{b-1} du = \mathcal{C}(a,b) t^{a+b-1},$$

where $\mathcal{C}(a,b) \neq 0$ is a constant. Defining, for a fixed $\alpha > 0$,

$$g(s) := 1_{[v,T]} \frac{(s-v)^{\alpha}}{s^{H-\frac{1}{2}}},$$

we obtain, for $t \in [v,T]$,

$$\begin{aligned}(L_1 g)(t) &= \int_v^t (t-s)^{-H-\frac{1}{2}} g(s) s^{H-\frac{1}{2}} ds = \int_v^t (t-s)^{-H-\frac{1}{2}} (s-v)^{\alpha} ds \\ &= \int_0^{t-v} u^{\alpha} (t-v-u)^{-H-\frac{1}{2}} du = \mathcal{C}(\alpha+1, \frac{1}{2} - H)(t-v)^{\alpha - H + \frac{1}{2}}.\end{aligned}$$

Varying α, we find that $(t-v)^n \in \operatorname{Im}(L_1)$ for $n \geq 1$ and the Stone-Weierstrass theorem guarantees that $\operatorname{Im}(L_1)$ is dense in $\mathcal{C}_0[v,T]$. □

• **Step 2**

Lemma 3.3.11. *If $f \in C_0[v,T]$, then $L_2 g \in C_0([v,T]$, where*

$$(L_2 f)(t) = (I_{O^+}^{2H}(s^{\frac{1}{2}-H} f(s)))(t)$$

and $L_2 : C_0[v,T] \longrightarrow C_0[v,T]$ is continuous and has dense range.

Proof. The same argument applies, but this time we use the estimation

$$\|L_1 f - L_1 g\|_{\infty} \leq T^{1/2-H} \int_0^T (T-s)^{2H-1} ds \|f-g\|_{\infty}.$$

and the function

$$g(s) := 1_{[v,T]} \frac{(s-v)^{\alpha}}{s^{\frac{1}{2}-H}},$$

□

Since the restriction of K_H to $C_0[v,T]$ is exactly $L_2 \circ L_1$, we may conclude that $K_H : C_0[v,T] \longrightarrow C_0[v,T]$ has dense range and, a fortiori, $\mathbb{H}$ is norm-dense in $C_0[v,T]$.

(2) In the case $H \geq \frac{1}{2}$, a similar representation holds

$$K_H f = I^1_{0+}(s^{H-\frac{1}{2}} I^{H-\frac{1}{2}}_{0+}(s^{\frac{1}{2}-H} f)),$$

and the same argument carries over.

□

And more generally all Brownian moving averages with non-vanishing kernels [4]. Moreover, Gaussian processes with stationary increments that satisfy a certain spectral density condition have CFS [12].

In the case of **continuous Markov processes**, showing CFS reduces to showing that the support of the (unconditional) law of the process is the largest possible, as pointed out in [13].
Moreover, it was shown in [13], that if continuous process X has CFS, then the Riemann integral process $\int_0^\cdot X_t dt$ has CFS, which allows (using iteration) the construction of processes that have CFS and arbitrarily smooth paths.

3.4 Conditional full support for stochastic integrals

We shall establish the CFS for processes of the form

$$Z_t := H_t + \int_0^t k_s dW_s, \qquad t \in [0, T]$$

where H is a continuous process, the integrator W is a Brownian motion, and the integrand k satisfies some varying assumptions (to be clarified below). We focus on three cases, each of which requires a separate treatment (see [33]).
First, we study the case:

1. **Independent integrands and Brownian integrators**

   Theorem 3.4.1. *[33] Let us define*

   $$Z_t := H_t + \int_0^t k_s dW_s, \qquad t \in [0, T]$$

Suppose that

- $(H_t)_{t\in[0,T]}$ *is a continuous process*
- $(k_t)_{t\in[0,T]}$ *is a measurable process s.t.* $\int_0^T K_s^2 ds < \infty$ *a.s,*
- $(W_t)_{t\in[0,T]}$ *is a standard Brownian motion independent of H and k.*

If we have

$$meas(t \in [0,T] : k_t = 0) = 0 \qquad \mathbf{P} - a.s \tag{3.6}$$

then Z has CFS.

Remark 3.4.2. *It follows from Fubini's theorem, that if* $k_t \neq 0$ *a.s. for all* $t \in [0,T]$*, then the condition 3.6 holds. Hence, in particular whenever* k_t *has continuous distribution for all* t*, the previous theorem applies.*

The proof of this Theorem requires some preparation. Specifically, we shall show that the Wiener integral of an almost-everywhere non-vanishing function has positive small-ball probabilities, using a time-change argument similar.

Lemma 3.4.3. *(**Wiener integrals**) Let* $h \in \mathcal{C}([0,T])$*,* $k \in L^2([0,T])$*,* $(W_t)_{t\in[0,T]}$ *Brownian motion, and define*

$$J_t := h(t) + \int_0^t k(s) dW_s, \qquad t \in [0,T].$$

If $k_t \neq 0$ *for a.a.* $t \in [0,T]$*, then for all* $\underline{t} \in [0,T]$*,* $f \in \mathcal{C}_0([\underline{t},T])$*, and* $\varepsilon > 0$ *we have*

$$\mathbf{P}\left[\sup_{\underline{t}\in[t,T]} |J_t - J_{\underline{t}} - f(t)| < \varepsilon\right] > 0.$$

Proof. Clearly, we may assume that $h = 0$. Let $\underline{t} \in [0,T]$, $f \in \mathcal{C}_0([\underline{t},T])$, and $\varepsilon > 0$. Denote

$$g(t,\underline{t}) := \int_{\underline{t}}^t d\langle J,J\rangle_u = \int_{\underline{t}}^t k(s)^2 ds, \qquad t \in [\underline{t},T],$$

and note that since $k(t) \neq 0$ for a.a. $t \in [0,T]$, g is a homeomorphism between $[\underline{t},T]$ and $[0,K]$, where $K := \int_{\underline{t}}^{T} k(s)^2 ds$. By the Dambis, Dubins-Schwarz theorem, there exists a Brownian motion $(B_s)_{s\in[0,K]}$ such that $J_t - J_{\underline{t}} = B_{g(t)}, t \in [\underline{t},T]$ a.s. Hence, we obtain

$$\begin{aligned}\sup\nolimits_{t\in[\underline{t},T]} |J_t - J_{\underline{t}} - f(t)| &= \sup_{t\in[\underline{t},T]} |B_{g(t)} - (f\circ g^{-1})(g(t))| \\ &= \sup_{s\in[0,K]} |B_u - (f\circ g^{-1})(u)| \qquad a.s.\end{aligned}$$

Since $f \circ g^{-1}$ is continuous, and since the Wiener measure is supported on $\mathcal{C}_0([0,K])$, we have

$$\mathbf{P}\left[\sup_{\underline{t}\in[t,T]} |J_t - J_{\underline{t}} - f(t)| < \varepsilon\right] = \mathbf{P}\left[\sup_{s\in[0,K]} |B_s - (f\circ g^{-1})(s) < \varepsilon\right] > 0$$

□

We shall now deduce the theorem from Lemma 4.3.1 using a suitable conditioning scheme.

Proof. of Theorem
(beginning)
Let $\underline{t} \in [0,T]$, $f \in \mathcal{C}_0([\underline{t},T])$, and $\varepsilon > 0$.
Further, let $(\Omega, \mathcal{F}, \mathbf{P}$ be the completed probabilility space that carries W, H, and k. By Lemma 4.2.1, it suces to show that

$$E[I(\widehat{Z}^{\underline{t}}, f, \varepsilon)|\widetilde{\mathcal{F}}^{Z}_{\underline{t}}] > 0 \qquad \mathbf{P} - a.s., \tag{3.7}$$

The proof of this assertion becomes more transparent when we work on an extension of the space $(\Omega, \mathcal{F}, \mathbf{P})$. Namely, we show an analogous property for a variant of Z, denoted by Z^*, in which the integrator is W up to time $\underline{t}$, but further Brownian increments of the integrator are defined on an auxiliary space.

Then, since Z and Z^* have the same distribution, it follows that (3.7) holds. We define the extended space by

$$\Omega^* := \Omega \times \mathcal{C}_0([0,T]), \quad \mathcal{F}^* := \overline{\mathcal{F} \otimes \mathcal{B}(\mathcal{C}_0([0,T]))}, \qquad \mathbf{P}^* := \overline{\mathbf{P} \otimes \nu},$$

where ν is the Wiener measure on $\mathcal{C}_0([0,T])$ and the bars denote completion. For any $\omega^* = (\omega, \omega^{'}) \in \Omega^*$, we define $B_t(\omega^*) := B_t(\omega^{'}) := \omega^{'}(t)$ and $W_t^*(\omega^*) := B_{t\vee\underline{t}}(\omega^{'}) - B_{\underline{t}}(\omega^{'}) + W_{t\wedge\underline{t}}(\omega)$ for all $t \in [0,T]$.

Moreover, we denote by $\mathbf{E}^*$ the expectation with respect to $\mathbf{P}^*$, by X the identity map on Ω, which can be seen as a random element in the measurable space $(\Omega, \mathcal{F})$, and by Z^* the process analogous to Z, with W^* as the integrator. Note that by joint measurability, we have $H_t(\omega) = \phi(t,\omega)$ and $k_t(\omega) = \psi(t,\omega)$ and ϕ and ψ $\mathcal{F}$-measurable functions from $[0,T] \times \Omega$ to $\mathbb{R}$.

For the conclusion of the proof we need the following auxiliary result, which asserts that "freezing" randomness on the original probability space Ω reduces Z^* to a Wiener integral with a drift.

Lemma 3.4.4. *[33](Freezing)*
For $\mathbf{P}$-a.a. $\omega \in \Omega$, we have

$$(\widehat{Z}_t^{*,\underline{t}}(\omega,.))_{t\in[\underline{t},T]} = \left(\phi(t,\omega) - \phi(\underline{t},\omega) + \int_{\underline{t}}^t \psi(s,\omega)dB_s\right)_{t\in[\underline{t},T]} \tag{3.8}$$

up to ν-indistinguishability, where the integral on the right hand side is a Wiener integral.

(conclusion)

Let us denote $\mathcal{G} := \mathcal{F} \otimes \mathcal{C}_0([0,T])$. We shall show that $E^*[I(\widehat{Z}^{*,\underline{t}}, f, \varepsilon)|\mathcal{G}] > 0$ $\mathbf{P}^*$-a.s., which by Lemma 4.2.2 implies that the same holds also with respect to $\widetilde{\mathcal{F}}_{\underline{t}}^{Z^*} \subset \mathcal{G}$, which in turn implies that (3.7) holds. We may compose

$$\widehat{Z}^{*,\underline{t}}(\omega,\omega^{'}) = \widehat{Z}^{*,\underline{t}}(X(\omega),B(\omega^{'})), \qquad (\omega,\omega^{'}) \in \Omega^{*}.$$

Moreover, by independence, ν is a version of the regular $\mathcal{G}$-conditional law of B on $C_0([0,T])$. By the disintegration theorem (Theorem 6.4 of [20]), we have $\mathbf{P}^*$-a.s.

$$\begin{aligned} E^*[I(\widehat{Z}^{*,\underline{t}},f,\varepsilon)|\mathcal{G}] &= E^*[1_{B(f,\varepsilon)}(\widehat{Z}^{*,\underline{t}}(X,B))|\mathcal{G}] \\ &= \int_{C_0([0,T])} 1_{B(f,\varepsilon)}(\widehat{Z}^{*,\underline{t}}(X,\omega^{'}))\nu(d\omega^{'}) := Y(X). \end{aligned}$$

By Lemma 4.3.2, for $\mathbf{P}$-a.a. $\omega \in \Omega$, $1_{B(f,\varepsilon)}(\widehat{Z}^{*,\underline{t}}(\omega,.)) = 1_{B(f,\varepsilon)}(J^{\omega})$ ν-a.s., where J^{ω} is the right hand side of 3.8. But for $\mathbf{P}$-a.a. $\omega \in \Omega$ the map $\psi(.,\omega)$ is a.e. non-vanishing, so it follows from Lemma 3.6 that for $\mathbf{P}$-a.a. $\omega \in \Omega$,

$$Y(X(\omega)) = \int_{\mathcal{C}_0([0,T])} 1_{B(f,\varepsilon)}(J^{\omega}(\omega^{'}))\nu(d\omega^{'}) > 0.$$

Hence, $Y > 0$ also $\mathbf{P}^*$-a.s., which concludes the proof. □

As an application of this result, we show that several popular stochastic volatility models have the CFS property.

Application to stochastic volatility model:

Let us consider price process $(P_t)_{t\in[0,T]}$ in $\mathbb{R}_+$ given by :

$$dP_t = P_t(f(t,V_t)dt + \rho g(t,V_t)dB_t + \sqrt{1-\rho^2}g(t,V_t)dW_t,$$

$P_0 = p_0 \in \mathbb{R}_+$ where

(a) $f,g \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$,

(b) (B,W) is a planar Brownian motion,

(c) $\rho \in (-1, 1)$,

(d) V is a (measurable) process in $\mathbb{R}^d$ s.t. $g(t, V_t) \neq 0$ a.s. for all $t \in [0, T]$,

(e) (B,V) is independent of W,

write using It$\hat{o}$'s formula:

$$logP_t = \underbrace{logP_0 + \int_0^t (f(s, V_s) - \frac{1}{2} g(s, V_s)^2) ds + \rho \int_0^t g(s, V_s) dB_s}_{=H_t}$$

$$\underbrace{+\sqrt{1-\rho^2} \int_0^t g(s, V_s) dW_s}_{=K_s}$$

Since W is independent of B and V, the previous Theorem implies that $logP$ has CFS, and from the next remark which it follows that P has CFS.

Next, we relax the assumption about independence, and consider the second case:

2. **Progressive integrands and Brownian integrators**

Remark 3.4.5. *The assumption about independence between W and (H,k) cannot be dispensed with in general without imposing additional conditions. Namely, if e.g.*

$$H_t = 1; k_t := e^{W_t - \frac{1}{2}t}; t \in [0, T]$$

then $Z = k = \xi(W)$, the Doléans exponential of W,
which is stricly positive and thus does not have CFS, if process is consider in $\mathbb{R}$.

Theorem 3.4.6. *[33]*

Suppose that

- $(X_t)_{t \in [0,T]}$ *and* $(W_t)_{t \in [0,T]}$ *are continuous process*

- *h and k are progressive* $[0,T] * C([0,T])^2 \longrightarrow \mathbb{R}$,
- ε *is a random variable.*
- *and* $\mathcal{F}_t = \sigma\{\varepsilon, X_s, W_s : s \in [0,t]\}, t \in [0,T]$

If W is an $\mathcal{F}_{t\in[0,T]}$ *− Brownian motion and*

- $E[e^{\lambda \int_0^T k_s^{-2} ds}] < \infty$ *for all* $\lambda > 0$
- $E[e^{2\int_0^T k_s^{-2} h_s^2 ds}] < \infty$ *and*
- $\int_0^T k_s^2 ds \leq \overline{K}$ *a.s for some constant* $\overline{K} \in (0,\infty)$

then the process

$$Z_t = \varepsilon + \int_0^t h_s ds + \int_0^t k_s dw_s, \qquad t \in [0,T]$$

has CFS.

3. **Independent integrands and general integrators**

Since Brownian motion has CFS, one might wonder if the preceeding results generalize to the case where the integrator is merely a continuous process with CFS. While the proofs of these results use quite heavily methods speciFIc to Brownian motion (martingales, time changes), in the case independent integrands of finite variation we are able to prove this conjecture.

Theorem 3.4.7. *[33] Suppose that*

- $(H_t)_{t\in[0,T]}$ *is a continuous process*
- $(k_t)_{t\in[0,T]}$ *is a process of finite variation, and*
- $X = (X_t)_{t\in[0,T]}$ *is a continuous process independent of H and k.*

Let us define

$$Z_t := H_t + \int_0^t k_s dX_s, \qquad t \in [0,T]$$

If X has CFS and

$$\inf_{t\in[0,T]} |k_t| > 0 \qquad \mathbf{P} - a.s$$

then Z has CFS.

3.5 Main results

The main aim of this section is to enjoy this property by thinking of the problems of no arbitrage for asset prices on a new financial models .
First for the Ornstein Uhlenbeck process , secondly for other financial model where the stochastic integration is w.r.t the Brownian Bridge [1] (resp w.r.t fractional Brownain motion [2])
We will use the theorem 3.3.1 to demonstrate the absence of arbitration without calculating the risk-neutral probability for the two models below.

3.5.1 Ornstein-Uhlenbeck Process driven by Brownian Motion

The (one-dimensional) Gaussian Ornstein-Uhlenbeck process $X = (X_t)_{t\geq 0}$ can be defined as the solution to the stochastic differential equation (SDE)

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t \qquad t > 0$$

Where we see

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)}\, dW_s. \qquad t \geq 0$$

[1]This application is the subject of a publication in Journal of Acta Univesitatis Sapientiae Mathematica in Vol. 8, No. 2,2016

[2]This application is the subject of article who submitted to publication in Journal of Math. Notes, October 2015

It is readily seen that X_t is normally distributed. We have

$$X_t = \underbrace{X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})}_{H_t} + \int_0^t \underbrace{\sigma e^{\theta(s-t)}}_{K_s} dW_s. \qquad t \geq 0 \tag{3.9}$$

to establish the property of CFS for this process, the conditions of theorem 3.1 will be applied.

The processes (H_s) and (K_s) in (3.9) satisfy

1. Process (H_s) is a continuous process,
2. (K_s) is a measurable process such that $\int_0^T K_s^2 ds < \infty$ a.s, and
3. (W_t) is a standard Brownian motion independent of H and K.

Consequently, the process (X_t) has the property of CFS and there is the consistent price systems which can be seen as generalization of equivalent martingale measures. This observation we basically say that this price process doesn't admit arbitrage opportunities under arbitrary small transaction, with it we guarantee no-arbitrage without calculating the risk-neutral probability.

3.5.2 Independent integrands and Brownian Bridge integrators.

To state our main result for the application of CFC in which the Brownian Bridge is the integrator, we need to recall some facts of Brownian bridge.

Let us start with a Brownian motion $B = (B_t, t \geq 0)$ and its natural filtration $\mathbb{F}^B$. Define a new filtration as $\mathbb{G} = (\mathcal{G}_t, t \geq 0)$ with $\mathcal{G}_t = \mathcal{F}_t^{(B_1)} = \mathcal{F}_t^B \vee \sigma(B_1)$. In this filtration, the process $(B_t, t \geq 0)$ is no longer a martingale. It is easy to be convinced of this by looking at the process $(E(B_1 \mid \mathcal{F}_t^{(B_1)}), t \leq 1)$: this process is identically equal to B_1, not to B_t, hence $(B_t; t \geq 0)$ is not a $\mathbb{G}$-martingale. However, $(B_t, t \geq 0)$

is a $\mathbb{G}$-semi-martingale, as follows from the next proposition 3.5.2.

In general, if $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$ is a filtration larger than $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$, i.e., $\mathcal{F}_t \subset \mathcal{H}_t, \forall t \geq 0$ (we shall write $\mathbb{F} \subset \mathbb{H}$), it is not true that an $\mathbb{F}$-martingale remains a martingale in the filtration $\mathbb{H}$. It is not even true that $\mathbb{F}$-martingales remain $\mathbb{H}$-semi-martingales.

Before giving this proposition, we recall the definition of Brownian bridge.

Definition 3.5.1. *The Brownian bridge $(b_t; 0 \leq t \leq 1)$ is defined as the conditioned process $(B_t; t \leq 1 | B_1 = 0)$.*

Note that $B_t = (B_t - tB_1) + tB_1$ where, from the Gaussian property, the process $(B_t - tB_1; t \leq 1)$ and the random variable B_1 are independent. Hence

$$(b_t; 0 \leq t \leq 1) =^{law} (B_t - tB_1; 0 \leq t \leq 1).$$

The Brownian bridge process is a Gaussian process, with zero mean and covariance function $s(1-t); s \leq t$. Moreover, it satisfies $b_0 = b_1 = 0$.

Proposition 3.5.2. *[36] Let $\mathcal{F}_t^{(B_1)} = \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon} \vee \sigma(B_1)$. The process*

$$\beta_t = B_t - \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds$$

is an $\mathbb{F}^{(B_1)}$-martingale, and an $\mathbb{F}^{(B_1)}$ Brownian motion. In other words,

$$B_t = \beta_t - \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds$$

is the decomposition of B as an $\mathbb{F}^{(B_1)}$-semi-martingale.

Example of application : The following example was studied by Monique Jeanblanc et al. [36], we will later introduce our approach to this application, this approach is based on the conditional full support property. M.Jeanblanc et al. study within the problem occurring in insider trading: existence of arbitrage using strategies adapted w.r.t. the large filtration.

Our approach is to prove the existence of no arbitrage in the case $0 \leq t < 1$ without calculating the dynamics of wealth and risk neutral probability.

Let

$$dS_t = S_t(\mu dt + \sigma db_t),$$

where μ and σ are constants and S_t defines the price of a risky asset. Assume that the riskless asset has a constant interest rate r.

The wealth of an agent is

$$dX_t = rX_t dt + \widehat{\pi}_t(dS_t - rS_t dt) = rX_t dt + \pi_t \sigma X_t(dW_t + \theta dt); \quad X_0 = x,$$

where $\theta = \frac{\mu - r}{\sigma}$ and $\pi = (\widehat{\pi} S_t / X_t)$ assumed to be an $\mathbb{F}^B$-adapted process.

Here, $\widehat{\pi}$ is the number of shares of the risky asset, and π the proportion of wealth invested in the risky asset. It follows that

$$\ln(X_T^{\pi,x}) = \ln x + \int_0^T (r - \frac{1}{2}\pi_s^2\sigma^2 + \theta\pi_s\sigma)ds + \int_0^T \sigma\pi_s dW_s$$

Then,

$$E(\ln(X_T^{\pi,x})) = \ln x + \int_0^T E\left(r - \frac{1}{2}\pi_s^2\sigma^2 + \theta\pi_s\sigma\right)ds$$

The solution of $\max E(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\theta}{\sigma}$ and

$$\sup E(\ln(X_T^{\pi,x}) = \ln x + T\left(r + \frac{1}{2}\theta^2\right)$$

Note that, if the coefficients r, μ and σ are $\mathbb{F}$-adapted, the same computation leads to

$$\sup E(\ln(X_T^{\pi,x}) = \ln x + \int_0^T E\left(r_t + \frac{1}{2}\theta_t^2\right)dt,$$

where $\theta_t = \frac{\mu_t - r_t}{\sigma_t}$.

We now enlarge the filtration with S_1.

In the enlarged filtration, setting, for $t < 1, \alpha_t = \frac{B_1 - B_t}{1-t}$, the dynamics of S are

$$dS_t = S_t((\mu + \sigma\alpha_t)dt + \sigma d\beta_t),$$

and the dynamics of the wealth are

$$dX_t = rX_t dt + \pi_t \sigma X_t (d\beta_t + \widetilde{\theta}_t dt), \quad X_0 = x$$

with $\widetilde{\theta}_t = \frac{\mu - r}{\sigma} + \alpha_t$.

The solution of $\max E(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\widetilde{\theta}_s}{\sigma}$.

Then, for $T < 1$,

$$\ln(X_T^{\pi,x,*}) = \ln x + \int_0^T (r + \frac{1}{2}\widetilde{\theta}_s^2)ds + \int_0^T \sigma\pi_s d\beta_s$$

$$E(\ln(X_T^{\pi,x,*})) = \ln x + \int_0^T (r + \frac{1}{2}(\theta^2 + E(\alpha_s^2) + 2\theta E(\alpha_s))ds = \ln x + (r + \frac{1}{2}\theta^2)T + \frac{1}{2}\int_0^T E(\alpha_s^2)ds,$$

where we have used the fact that $E(\alpha_t) = 0$ (if the coefficients r, μ and σ are $\mathbb{F}$-adapted, α is orthogonal to $\mathcal{F}_t$, hence $E(\alpha_t\theta_t) = 0$).

Let

$$\begin{aligned} V^{\mathbb{F}}(x) &= \max E(\ln(X_T^{\pi,x})); \pi \textit{ is } \mathbb{F} \textit{ admissible} \\ V^{\mathbb{G}}(x) &= \max E(\ln(X_T^{\pi,x})); \pi \textit{ is } \mathbb{G} \textit{ admissible} \end{aligned}$$

Then $V^{\mathbb{G}}(x) = V^{\mathbb{F}}(x) + \frac{1}{2}E\int_0^T \alpha_s^2 ds = V^{\mathbb{F}}(x) - \frac{1}{2}\ln(1-T)$.

If $T = 1$, the value function is infinite: there is an arbitrage opportunity and there exists no an e.m.m. such that the discounted price process $(e^{-rt}S_t, t \leq 1)$ is a

$\mathbb{G}$-martingale. However, for any $\epsilon \in]0;1]$, there exists a uniformly integrable $\mathbb{G}$-martingale L defined as

$$dL_t = \frac{\mu - r + \sigma\varsigma_t}{\sigma} L_t d\beta_t, t \leq 1-\epsilon, \quad L_0 = 1,$$

such that, setting $d\mathbb{Q} \mid_{\mathcal{G}_t} = L_t d\mathbb{P} \mid_{\mathcal{G}_t}$, the process $(e^{-rt}S_t; t \leq 1-\epsilon)$ is a $(\mathbb{Q}, \mathbb{G})$-martingale.

This is the main point in the theory of insider trading where the knowledge of the terminal value of the underlying asset creates an arbitrage opportunity and this is effective at time 1.

Our approach to this example : We consider the previous example. Let

$$dS_t = S_t(\mu dt + \sigma db_t),$$

The standard Brownian bridge $b(t)$ is a solution of the following stochastic equation.

$$\begin{aligned} db_t &= -\frac{b_t}{1-t}dt + dW_t; \qquad 0 \leq t < 1 \\ b_0 &= 0. \end{aligned} \tag{3.10}$$

The solution of the above equation is

$$b_t = (1-t)\int_0^t \frac{1}{1-s} dW_s,$$

We may now verify that S has CFS.

By positivity of S, Itô's formula yields

$$logS_t = logS_0 + \left\{ \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\left(1-t\right)\int_0^t \frac{1}{1-s} dW_s \right\}, \qquad 0 \leq t < 1.$$

We have

$$logS_t = \underbrace{logS_0 + \left(\mu - \frac{\sigma^2}{2}\right)t}_{=:H_t} + \int_0^t \underbrace{\sigma\left(1-t\right)\frac{1}{1-s}}_{=:K_s} dW_s, \qquad 0 \leq t < 1.$$

1. (H_t) is a continuous process,
2. $(K_s) = \sigma(1-t)\frac{1}{1-s}$ is a measurable process s.t. $\int_0^t K_s^2 ds < \infty$ a.s,
3. (W_t) is a standard Brownian motion independent of H and K,

which clearly satisfy the assumptions of theorem (3.1) and $logS_t$ has CFS, then S has CFS for $0 \leq t < 1$ and there is the consistent price systems and this is a martingale. Using it, we guarantee no-arbitrage without calculating the risk-neutral probability.

3.5.3 Conditional full support for Fractional Brownian Motion

We now present in this section an important class of model Fractional Brownian motion and proove a new result for etablish the CFS.

Theorem 3.5.3. *Let us consider the process*

$$S_t = R_t + \int_0^t \phi_s dB_s^H$$

where

- $(R_t)_{t\in[0,T]}$ *is a continuous adapted process,*
- $(\phi_t)_{t\in[0,T]}$ *is elementary predictable s.t.* $\int_0^T \phi_s^2 ds < \infty$*, and*
- $(B_t^H)_{t\in[0,T]}$ *is a fractional brownian motion independent of* R *and* ϕ*.*

If we have

$$\mathrm{meas(t \in [0, T] : \phi_t = 0) = 0 \quad P - a.s.},$$

then S has CFS.

Proof. We adapt the proof of proposition 3.3.9.
Let us denote

$$J(t) = \int_0^t \phi_s dB_s^H.$$

By considering the restriction of S on an interval $[v, \mu]$, $v < \mu < T$, it is enough to prove that the conditional law$\mathrm{P(J[v, T] \mid \mathcal{F}_v)}$ has full support on $C_{J_v}([v, \mu], \mathbb{R})$ almost surely. It is sufficient to prove this property on an interval where ϕ is constant with respect to time (and thus continuous). Thus, without loss of generality, we can take T small enough such that ϕ has the form $\phi(t) = \xi$ on $[v, T]$, where $\xi \neq 0$ and it is $\mathcal{F}_v$-measurable. It suffices to prove that

$$J(t) = \int_v^t \phi_s K_H(t, s) dB_s \quad s \in [v, T]$$

has full support on $C_0([v, T], \mathbb{R})$.
Theorem 3 in [21] states that the topological support of a continuous Gaussian process is equal to the norm closure of its reproducing kernel Hilbert space.
In our case, the support of $J(t)$ is

$$\mathbb{H} := \left\{ f \in C_0([v, T], \mathbb{R}) : f(t) = \int_v^t \phi(s) K_H(t, s) g(s) ds, \text{ for some } \mathrm{g \in L^2[v, T]} \right\}.$$

Thus, it is sufficient to show that $\mathbb{H}$ is norm-dense in $C_0([v, T], \mathbb{R})$.
To achieve this, we need to recall the Liouville fractional integral operator for any $f \in L^1[a, b]$ and $\alpha > 0$

$$(I_{a^+}^{\alpha} g)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t g(s)(t - s)^{\alpha - 1} ds, \qquad a \le t \le b,$$

and to introduce the kernel operator K_H,

$$(K_H g)(t) := \int_0^t K_H(t, s) g(s) ds, \qquad f \in L^2[0, T], \quad t \in [0, T].$$

(1) We first treat the case $H < \frac{1}{2}$

$$(K_H(g\phi))(t) := \xi \int_v^t K_H(t,s)g(s)ds, \qquad g \in L^2[0,T], \quad t \in [0,T].$$

In this case, we have

$$(K_H(g\phi)) = I_{0^+}^{2H}(s^{\frac{1}{2}-H} I_{0^+}^{\frac{1}{2}-H}(s^{H-\frac{1}{2}}(g\phi)(s)))$$

The argument needs to be split into two steps.

• **Step 1**

Lemma 3.5.4. *[13] If $g \in C_0[v,T]$, then $L_1 g \in C_0([v,T]$, where*

$$(L_1 g)(t) = (I_{0^+}^{\frac{1}{2}-H}(s^{H-\frac{1}{2}}(g)(s)))(t)$$

Morever, $L_1 : C_0[v,T] \longrightarrow C_0[v,T]$ is continuous and has dense range (with respect to the uniform norm).

We have $\varphi \in C_0[v,T]$, then $L_1\varphi\phi \in C_0([v,T])$, where

$$(L_1\varphi\phi)(t) = (I_{0^+}^{\frac{1}{2}-H}(s^{H-\frac{1}{2}}(g\phi)(s)))(t)$$

Recall the identity for $a, b > 0$,

$$\int_0^t (t-u)^{a-1}u^{b-1}du = \mathcal{C}(a,b)t^{a+b-1},$$

where $\mathcal{C}(a,b) \neq 0$ is a constant. Defining, for a fixed $\alpha > 0$,

$$\varphi(s) := \frac{(s-v)^\alpha}{\xi s^{H-\frac{1}{2}}},$$

we obtain, for $t \in [v,T]$,

$$\begin{aligned}(L_1\varphi\phi)(t) &= \frac{\xi}{\Gamma(\frac{1}{2}-H)}\int_v^t (t-s)^{-H-\frac{1}{2}}\varphi(s)s^{H-\frac{1}{2}}ds = \frac{1}{\Gamma(\frac{1}{2}-H)}\int_v^t (t-s)^{-H-\frac{1}{2}}(s-v)^\alpha ds \\ &\leq \int_0^{t-v} u^\alpha(t-v-u)^{-H-\frac{1}{2}}du = \mathcal{C}\left(\frac{1}{2}-H, \alpha+1\right)(t-v)^{\alpha-H+\frac{1}{2}}.\end{aligned}$$

Varying α, we find that $(t-v)^n \in \mathrm{Im}(L_1)$ for $n \geq 1$ and the Stone-Weierstrass theorem guarantees that $\mathrm{Im}(L_1)$ is dense in $\mathcal{C}_0[v,T]$.

- **Step 2**

Lemma 3.5.5. *[41] If $g \in C_0[v,T]$, then $L_2 g \in C_0([v,T]$, where*

$$(L_2 g)(t) = (I^{2H}_{O^+}(s^{\frac{1}{2}-H} g(s)))(t)$$

and $L_2 : C_0[v,T] \longrightarrow C_0[v,T]$ is continuous and has dense range.

We have $g \in C_0[v,T]$, then $L_1 g\phi \in C_0([v,T])$, where

$$(L_2 g\phi)(t) = (I^{2H}_{O^+}(s^{\frac{1}{2}-H}(g\phi)(s)))(t)$$

and $L_2 : C_0[v,T] \longrightarrow C_0[v,T]$ is continuous and has dense range.

Since the restriction of K_H to $C_0[v,T]$ is exactly $L_2 \circ L_1$, we may conclude that $K_H : C_0[v,T] \longrightarrow C_0[v,T]$ has dense range and, a fortiori, $\mathbb{H}$ is norm-dense in $C_0[v,T]$.

(2) In the case $H \geq \frac{1}{2}$, a similar representation holds

$$K_H(g\phi) = I^1_{0^+}(s^{H-\frac{1}{2}} I^{H-\frac{1}{2}}_{0^+}(s^{\frac{1}{2}-H}(g\phi))),$$

Also this argument needs to be split into two steps.

- **Step 1** We have $g \in C_0[v,T]$, then $L_3 g\phi \in C_0([v,T]$, where

$$(L_3(g\phi))(t) = \xi(I^1_{0^+}(s^{H-\frac{1}{2}} g(s)))(t)$$

Defining, for a fixed $\alpha > 0$,

$$g(s) := \frac{(s-v)^\alpha}{\xi s^{\frac{1}{2}-H}},$$

we obtain, for $t \in [v, T]$,

$$\begin{aligned} (L_3 g\phi)(t) &= \frac{\xi}{\Gamma(H-\frac{1}{2})}\int_v^t (t-s)^{H-\frac{3}{2}} g(s) s^{\frac{1}{2}-H} ds = \frac{1}{\Gamma(H-\frac{1}{2})}\int_v^t (t-s)^{H-\frac{3}{2}} (s-v)^{\alpha} ds \\ &\leq \int_0^{t-v} u^{\alpha}(t-v-u)^{H-\frac{3}{2}} du = \mathcal{C}(H-\frac{1}{2}, \alpha+1)(t-v)^{\alpha+H-\frac{1}{2}}. \end{aligned}$$

Varying α, we find that $(t-v)^n \in \mathrm{Im}(L_3)$ for $n \geq 1$ and the Stone-Weierstrass theorem guarantees that $\mathrm{Im}(L_3)$ is dense in $\mathcal{C}_0[v, T]$.

• **Step 2** We have $g \in C_0[v, T]$, then $L_4 g\phi \in C_0([v, T]$, where

$$(L_4(g\phi))(t) = \xi(I^1_{0+}(s^{H-\frac{1}{2}}(g\phi)(s)))(t)$$

and $\mathrm{Im}(L_4)$ is dense in $\mathcal{C}_0[v, T]$.

Since the restriction of K_H to $C_0[v, T]$ is exactly $L_4 \circ L_3$, we may conclude that $K_H : C_0[v, T] \longrightarrow C_0[v, T]$ has dense range and, a fortiori, $\mathbb{H}$ is norm-dense in $C_0[v, T]$.

□

Conclusion

The aim of this thesis is to provide an introduction to the mathematical methods used in continuous-time modeling of financial markets. It will focus on options valuation problems by arbitration.
The absence of arbitrage opportunity assumption is a crucial condition in the theory of the valuation of derivatives. which imposes conditions on prices. This condition AOA implies the existence of a probability, called risk-neutral probability, which is equivalent to the objective probability.

Our goal is not to provide a complete account of the theory of finance models but to insist on major ideas and techniques for the conditional full support(CFS).
Firstly we have introduce the basics notions on stochastic calculus in providing the appropriate mathematical tools description of the financial models methods of calculation of derivative asset prices. Secondly we study the conditional full support, which generates a large class of consistent price systems which guarantees the absence of arbitrage without calculating the risk neutral probability and finally we provide new applications of CFS in finance.

Our future work, is to give new applications for establish the CFS when the sub fractional brownian motion is the integrator, and to define a new conditions of CFS for Rosemblat processes, Dirichlet processes and Hermilte processes.

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