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Statistique fonctionnelle de certaines caractéristiques sur les modèles de survie

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Contents

Abstract

The objective of this thesis is the nonparametric estimation core for conditional hazard functions and their derivatives are considered under different model randomly. Many techniques have been studied in the literature to treat these different situations all but only deals with the functional explanatory random variables.

The first part is dedicated to the convergence in quadratic mean and asymptotic normality of the estimator of the hazard function of a real random variable conditional on a functional variable

In the second part, we are interested in studying The maximum of the conditional hazard function is a parameter of great importance in seismicity studies, because it constitutes the maximum risk of occurrence of an earthquake in a given interval of time. using the kernel nonparametric estimates of the first derivative of the conditional hazard function, we establish uniform convergence properties and asymptotic normality of an estimate of the maximum in the context of strong mixing dependence.

Key words: Almost complete convergence; Asymptotic normality; Conditional hazard function; Functional data; Nonparametric estimation; Small ball probability; Strong mixing processes.

Résumé

L'objectif de cette thèse est l'estimation non paramétrique à noyau pour les fonctions de risque conditionnel et leurs dérivées sont considérées en vertu du différent modèle au hasard. De nombreuses techniques ont été étudiées dans la littérature pour traiter ces différentes situations mais toutes ne traitent que des variables aléatoires explicatives fonctionnelles.

La première partie est consacrée à la convergence en moyenne quadratique et la normalité asymptotique de l'estimateur de la fonction de risque d'une variable aléatoire réelle conditionnellement à une variable fonctionnelle.

Dans le deuxième partie, nous nous intéressons à étudier Le maximum de la fonction de hasard conditionnelle, est un paramètre d'une grande importance dans les études de sismicité, car il constitue le risque maximal de survenance d'un tremblement de terre dans un intervalle de temps donné. en utilisant les estimations non paramétriques du noyau de la première dérivé de fonction de hasard conditionnelle, nous établissons des propriétés de convergence uniformes et normalité asymptotique d'une estimation du maximum dans le contexte de la dépendance de mélange fort.

Mots clés: convergence presque complète; Normalité asymptotique; Fonction de hasard conditionnelle; Les données fonctionnelles; Estimation non paramétrique; probabilités de petites boules,; Processus de α-mélangeant,

The List Of Works

Publications

1. Tayeb Djebbouri, El Hadj Hamel and Abbes Rabhi. On conditional hazard function estimate for functional mixing data. New Trends in Mathematical Sciences. NTMSCI 3, No. 2, 79-95 (2015)

2. Abbes Rabhi, Yassine Hammou and Tayeb Djebbouri. Nonparametric estimation of the maximum of conditional hazard function under dependence conditions for functional data.Afrika Statistika Vol. 10, 2015, pages 726-743.

Communications

1. Participation aux séminaire hebdomadaire de LMSSA:

   - Djebbouri Tayeb, ***Introduction à L'analyse des durees de survie***, 01 juil 2012
   - Djebbouri Tayeb, ***Estimation non paramétrique de la fonction du hasard avec variables explicatives fonctionnelles***,16 Février 2013.
   - Djebbouri Tayeb, ***Estimation non paramétrique du risque maximal sous condition de dépendance***,15 Février 2014.
   - Djebbouri Tayeb, ***Une note sur l'estimation non paramétrique du maximum de la fonction de hasard conditionnelle***,07 Mars 2015.

2. Participation aux 1^{er} colloque International sur La Statistique et ses Application, 06 et 07 Mai 2015, ***Strong Uniform Consistency Rates of Some Characteristique of the Conditional Distrubution Estimator in The Functional Single-Index Model.***

Chapter 1

General Introduction

1.1 Functional statistics

The functional statistics has recently become an important area of research which knew very interesting development in the last few years in which come to mix and to be supplemented several approaches of the statistics which appear remote a priori. This way of the statistics studies data resulting from large samples and their functions are collected on a very fine grids, which can be comparable with curves or surfaces, for example functions of time or space. The need to consider this kind of data, maintaining usually met under the name of functional data in the literature, is before a whole practical need. Account held of the current capacities of measuring equipment and data-processing storage, them situations being able to provide such data are multiple and resulting from different fields :one can imagine for example curves of growth, of temperature, images observed by satellite...

The first works in which this idea of functional data is found are relatively old: Rao (1958) [50] and Tucker (1958) [58] consider principal components analysis and factor analysis for functional data and consider the functional data even explicitly as a particular type of data. Thereafter, one finds the work of Deville (1974) [16], Dauxois and Pousse (1976) [15], Besse and Ramsay (1986) [5]. The concrete Problematic terminology in statistics for functional variables referring to functional data seems to be resulting from the work of Ramsay, in 1982, in Psychometric, under the title

When the dated are functions [46]". This denomination seems to gather a significant number of statisticians who make statistics of curves, smoothing, decompositions of infinite-dimension space in basis of functions (by using the Riez theorem for Hilbert spaces and a little more complex theorems to build Schauder bases for certain Banach spaces), differential geometry,. . . For more details, see the monographs of Ramsay and Silverman (2002 and 2005) [48], [49].

1.2 Concrete problems in statistics for functional variable

The great strides which the functional statistics through its various fields of application make cation is found on the level as of many theoretical approaches developed for the study of functional random variables, the study of these various models is motivated at the beginning by practical problems. In this paragraph we wish to quote some fields in which appear the functional data, to give an idea of the type of problems which the functional statistics make it possible to solve. it is a non exhaustive list of situations where such data are met is not not possible, but of the precise examples of functional data will be approached in these fields.

- In biology, we find first of all the precursory work of Rao (1958 [50]) concerning a study of growth curves . More recently, another example is the study of the angle variations of the knee during the walking(step) and the movements of the knee during the effort under constraint (Antoniadis and Sapatinas, on 2007) [2]. Concerning the animal biology, studies of the laying(eggs) of Mediterranean flies were made by several authors (Chiou and Müller .and al (on 2007)) [11]. The data consist of curves giving for every fly the quantity of eggs laid according to time(weather).

- The chimiometrie is also a part of fields of study convenient to the use of methods of the functional statistics. More recently, Ferraty and Vieu(2002 [27]) were interested in the study of fat volume of meat pieces (variable of interest) being given the curves of waves lengthes absorbtion of these pieces of meat (explanatory variable).

- Applications connected to the environment were studied by several author which worked on a problem of pollution forecast. These data consist of measures of peaks of pollution by the ozone every day (variable of interest) given curves of pollutants as well as meteorological curves of the day before(watch) (explanatory variables).

- The climatology is a domain where the functional data appear naturally. A study of the phenomenon El Niño (The common(current) warmth of the ocean Paci? That) was so realized by Besse, Cardot and Stephenson (2000 [6]); Ramsay and Silverman (2005 [49]), Hall and Vial (2006 [29]).

- In linguistics, works were realized, in particular concerning the voice recognition. (Ferraty and Vieu (2003,2006) [26]). These works are strongly connected to the methods of classification when the explanatory variable is a curve. Briefly, the data are curves corresponding to recordings of phonemes pronounced by different individuals. We associate a label with every phoneme (variable of interest) and the purpose is to establish a classification of these curves by using as explanatory variable the registered curve.

- In the field of the graphology, the contribution of the techniques of the functional statistics found there also an application. Ramsay (2000 [47]) for example models the position of the pen (abscissas and ordered according to time) by means of differential equations.

- The applications in the economy are also relatively numerous. Recently Studies of Benko, Härdle and Kneip (2006 [4]), based in particular on a functional main in components analysis. This estimation method will be analyzed when we shall use it , even if we can already underline that the basic idea is, during the estimation of the operator of covariance, to estimate scalar products between curves observed instead of estimating curves themselves.

There exist other fields where the functional statistics were employed such as the signal processing sound or recorded by a radar, demographic studies , geology (Manté et al. (2007))[38],. . . and of the applications in fields as varied as criminology (how to model and compare the evolution crime of an individual in the course of time?) the paléo pathology (can one to say if an individual . . . .. of arthritis starting from the form of his ?) the study of results with school tests,. . .

Finely , one can have to study functional random variables even if one have real initial data or multivariate independent. It is thus the case when one wishes to compare or study functions that one be able to estimate to leave data. Among the typical examples of this kind of situation one can evoke comparison of different functions of density, functions of regressions, the study function representing the probability that an individual has to answer correctly with a test according to its "qualities" (Ramsay and Silverman (2002) [48],. . .

1.3 survival models

We can make go back up the survival data analysis in 1693 with the astronomer " Halley " who after a study of the statements of registry office of London gave the first life tables and taught the way to read it the survival probability of an individual. These analysis,are not refined until the 19th century, with the appearance of the following categorizations " exogenous variables " (sex, nationality, socio-professional groups). In this century, also appear the first modelings concerning the probability to die at certain age, the probability which will afterward be appointed under the term of " risk function ".
Finely , analysis of survival data begins to overflow the strict framework of the demographics to invest, in 20th century, in particular years which followed the second world war, we were interested in more the analysis of survival data for industrial applications (with the appearance of viability theory) by using models parametric

with exponential laws or of Weibull. It is only more recently, motivated by medical applications (pharmaceutical, biomedical), that appeared nonparametric methods (Kaplan-Meier on 1958) [32], for nonparametric estimation of survival function . Of the resultant value, they study the hope, the variance and the asymptotic properties. The semi-parametric aspect was introduced by Cox in 1972 [12]. This last model contains exogenous variables which are introduced, in risk function, by means of a component of parametric regression, the rest of this nonparametric risk function remaining indefinite.
The survival models form a class of statistical methods which aim at studying the number and the times distribution of appearance events . We can be interested in models where we consider only the time of events appearance , but we are generally interested more in models where the appearance risk on an event depends on co-variables. We so find the expression of regression model.

1.3.1 Survival data analysis

The analysis of survival data is the study of the arisen, in time, of one precise event for one or several groups of given individuals. This event, often called death(deaths), can be as well the death of an individual as the arisen of a disease, the answer to a treatment or the breakdown of a machine (generally it is a change of state .) every observation is defined by:

The origin date: it is the birth date of the subject, if we study the age of the subject when arises the event or date of putting in touch with an infectious agent, if we study the duration of incubation of an infectious disease. Every individual has a date of origin The measure different on the calendar, but which interests us is the extension since this date. The date of origin defined for every individual the time 0.

To allow the comparison of the survival durations between the individuals, one definition precise of the interest event is necessary. If it is the death caused by a disease, it should be made sure that each death is indeed due to the disease studied, and not with other causes.

Survival duration: It is defined as the time between the origination date and the occurring of the interest event.

The survival durations corresponding to positive random variables, of generally dis-symmetrical distribution, making difficult their description by the usual distribution laws.

The individuals or groups of individuals are likely to differ for one or several factors. These factors, called explanatory variable or co-variables can explain an important difference of the survival duration of the studied subjects. Their effects are analyzed by models of regression. They can be individual factors(sex, age, biological parameters relative to a disease, genetics parameters…), or related to therapeutic test (membership of the group of treatment or with the placebo group, medicamentous proportioning…).

The analysis of survival data is attached to the description of survival times and to see up to what point they depend on these explanatory variables. classical approaches in survival data analysis are of stochastic type, appearance time of an event is supposed to be the realization of a random process associated with a particular distribution.

Many work is devoted to survival data analysis : Kalbeisch and Prentice (1980) [31], Cox and Oakes (1984) [13], Klein and Moeschberger (1997) [33],…

1.3.2 Functions associated with the survival distributions

let T be a positive random variable corresponding to survival duration. The probability law of T can be characterized by several functions dependent between them.

Definition 1.3.1. *The probability density function, noted $f(t)$:*

$$f(t) = \lim_{\Delta t \to 0^+} \frac{\mathbb{P}(t \leq T \leq t + \Delta t)}{\Delta t}$$

$f(t)\Delta t + o(\Delta t)$is thus the probability of knowing the event of interest between t and $T + \Delta t$. The distribution function, noted $F(T)$, satisfy:

$$F(t) = \mathbb{P}(T \leq t) = \int_0^t f(u)du$$

$F(t)$, define the probability of knowing the event of interest between $[0, T]$, this function is monotonous and we have

$$F(0) = 0 \quad and \quad \lim_{t \to \infty} F(t) = 1$$

Definition 1.3.2. *The survival function, denoted $S(t)$, is defined as*

$$S(t) = \mathbb{P}(T > t) = 1 - F(t).$$

The survival function is the probability that the time of death is later than some specified time t. survival function $S(t)$ is monotonically decreasing, such that

$$S(0) = 1 \quad and \quad \lim_{t \to \infty} S(t) = 0.$$

It Also characterized the law of T .

Definition 1.3.3. *The risk function, or fate function, or the immediate risk of change of state noted $h(t)$, is defined as being the immediate probability that a duration T of "stay" in a state ends at the moment $t + \Delta t$ knowing that we were at the moment t there, i.e.:*

$$h(t) = \lim_{\Delta t \to 0^+} \frac{\mathbb{P}(t \leq T \leq t + \Delta t / T \geq t)}{\Delta t}$$

We show easily that

$$\begin{aligned} h(t) &= \frac{f(t)}{S(t)} \\ &= \frac{-dlog(s(t))}{d(t)} \end{aligned}$$

thus a $h(t)\Delta t$ represent, whenΔt is small, the probability "approached" for an individual to reach the event of interest before $t + \Delta t$, conditionally in the fact that it is still in the previous state just before t. This function is also called immediate risk at the moment t. We also notice that the function of risk characterizes the law of T (or $S(t)$).

Definition 1.3.4. *The function of accumulated risk, noted $H(t)$ defined by:*

$$H(t) = \int_0^t h(u)du$$

By manipulation of the previous definitions, we find easily the following relations:

$$\begin{aligned} f(t) &= -\frac{dS(t)}{dt} \\ S(t) &= \exp(-\textstyle\int_0^t h(u)du) \\ S(t) &= \exp(-H(t)) \\ f(t) &= h(t)\exp(-\textstyle\int_0^t h(u)du) \end{aligned}$$

Thus the accumulated risk function characterize the law of T (or $S(t)$).

Definition 1.3.5. *The duration averages survival function , noted $r(t)$ defined by:*

$$r(t) = \mathbb{E}(T - t/T > t)$$

We show that

$$r(t) = \frac{1}{S(t)} \int_t^\infty S(u)du \text{ and } S(t) = \frac{r(0)}{r(t)} e^{-\int_0^t \frac{1}{r(u)} du}$$

What allows to say also that the function of duration average of survival characterizes the law of T (or $S(t)$.)

The distribution of the duration of survival T can be described by one of the functions defined above. However one of the most interesting is the risk function $h(t)$ because it is a probabilistic description of the immediate future of the subject "still with risk" and reflect differences between the models often less visible through the distribution functions or survival functions. In epidemiology, it can in certain cases be interpreted in terms of incidence.

Note that if $h(t)$ is constant (it is noted λ), then

$$S(t) = \exp(-\int_0^t h(u)du) = e^{-\lambda t}$$

becomes the tail of a distribution of exponential law. That supposes that one can adopt the Markovian model in two states to estimate survival and the problem becomes purely parametric. But in general, $h(t)$ is not constant; what leaves one place to deal with the problem by using the functional statistics.

1.4 Estimation of the hazard function

The estimate of hazard function has a great interest in statistics. Indeed, it is used in the analysis of risk or for the study of the phenomena of survival. The unconditional hazard rate is defined as the instantaneous probability that the change of state is done in the infinitesimal moment which follows the moment present, noted t. More precisely, the hazard rate $h(t)$ is defined by:

$$h(t) = \lim_{\Delta t \to 0^+} \frac{\mathbb{P}(t \leq T \leq t + \Delta t / T \geq t)}{\Delta t} \qquad (t > 0)$$

It is not difficult to see that the hazard rate can be rewritten as the report of the density $f(.)$ which it is absolutely continuous with respect to Lebesgue measure and the survival function $S(.) = 1 - F(.)$ of T at the moment t; otherwise says:

$$h(t) = \frac{f(t)}{S(t)} \tag{1.1}$$

where the function of survival $S(t)$ is not other than the distribution function complement of the considered event. In fact it is the derivative of the probability that the duration is between t and Δt, knowing that one reached the period t. More practical he lies acts of an instantaneous rate of exit of the state at the date t. The survival curve has particular signification data given by:

$$S(t) = \exp(-\int_0^t h(u)du)$$

There exists a literature extended on the estimator of the nonparametric rate chance, in an approximate way and for the nonparametric case, two methods were proposed to estimate the hazard rate. The first approach replaces $f(t)$ and $S(t)$ in the expression of $h(t)$ by their estimators $\hat{f}(t)$ and $\hat{S}(t)$ respectively, which gives us the estimator of hazard rate by:

$$\hat{h}(t) = \frac{\hat{f}(t)}{\hat{S}(t)} \tag{1.2}$$

Nielsen and Linton (1995) call this kind of estimator by (external estimator). The estimator with external kernel of hazard rate of non censured data was introduced by Watson and Leadbetter (1964) and Munhy (1965). The second method is based on the relation between the cumulated chance and the rate of chance where the cumulated chance is defined by:

$$H(t) = \int_0^t h(u)du \tag{1.3}$$

Nielsen and Linton (1995) call this kind of estimators by (internal estimator). The relation between the cumulated chance and the rate of chance suggests that $h(T)$ can be obtained by smoothing $H(T)$ by using a kernel in other words:

$$h(t) = \int K_h(t-u)d\hat{H}u$$

where H is a width of window such as $h \to 0$ when $n \to \infty$. The internal estimator rate of chance for the data censured at summer also introduced by Watson and Leadbetter (1964). Ramlau-Hansen (1983), Yandell (1983), To tan and Wong (1983, 1984), Blum and susarla (1980), Fötdes and Retjö (1981) and Lo, Mack and Wang (1989) similar estimators in the presence of the censured data studied. Moreover, To tan and Wang (1984) like Sarda and Vieu (1996) use the selection of width of window for this kind of estimators. Until now, interest related to the rate of chance generally will depend on certain covariance, for example, it time of survival of a patient will be has affected by several characteristics the such age and the kind. The rate of conditional chance of t knowing $Z = z$ is defined by:

$$h^z(t) = \lim_{\Delta t \to 0} \frac{\mathbb{P}(T \leq t + \Delta t / T > t, Z = z)}{\Delta t}$$

Thus the conditional function of chance T knowing $Z = z$ is defined by:

$$\hat{h}^z(t) = \frac{\hat{f}^z(t)}{\hat{S}^z(t)}$$

such that F^z (resp f^z) is the conditional distribution (resp. conditional density) T knowing $Z = z$ that it is supposed that it is absolutely continuous compared to measurement of Lebesgue on R.

1.4.1 Tools

Proposition 1.4.1. *Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ two real continues random variables . If X_n converge almost completely to 0 and if there exists $\exists\delta > 0$ such that $\sum_{i=1}^{\infty} \mathbb{P}\{Y_n < \delta\} < \infty$. Then, the sequence $(X_n/Y_n)_{n\in\mathbb{N}}$ converges almost completely to 0.*

Lemma 1.4.2. *"Bernstein's exponential inequality " Let $X_1, \ldots, X_n$ of the centered, independent and of the same real random variables law (i.i.d) defined on the probability space , such that there exist two positive real θ_1 and θ_2 satisfy $X_1 < \theta_1$ and $\mathbb{E}X_1^2 < \theta_2$ then , for any $\varepsilon \in]0, \frac{\theta_1}{\theta_2}[$ we get :*

$$\mathbb{P}\left(n^{-1}\left|\sum_{i=1}^{\infty} X_i\right| \geq \varepsilon\right) \leq 2\exp\left(\frac{-n\varepsilon^2}{4\theta_2}\right)$$

Lemma 1.4.3. *" Fuk-Nagaev type Inequality under algebraic mixing " Let* $\{\Delta_i, i \in \mathbb{N}\}$ *be a family of random variables valued in* $\mathbb{R}$ *, of algebraically mixing decreasing coefficient. One pose*

$$s_n^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |cov(\Delta_i, \Delta_j)|,$$

if $\forall i, \|\Delta_i\|_\infty < \infty$*, then for all* $\varepsilon > 0$ *and any* $r > 1$*, we have:*

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} \Delta_i\right| > 4\varepsilon\right) \leq 4\left(1 + \frac{\varepsilon^2}{rs_n^2}\right)^{\frac{-r}{2}} + 2ncr^{-1}\left(\frac{2r}{\varepsilon}\right)^{a+1}$$

Lemma 1.4.4. *"Inequality of covariance for limited variables" Let* $\{\Delta_i, i \in \mathbb{N}\}$ *be a family of strong mixing random variables valued in* $\mathbb{R}$ *such that* $\forall i, \|\Delta_i\|_\infty < \infty$*, then, for any* $i \neq j$

$$|cov(\Delta_i, \Delta_j)| \leq 4\|\Delta_i\|_\infty \|\Delta_j\|_\infty \alpha(|i - j|).$$

1.5 Structure of this thesis

This thesis is presented in five chapters.
The first chapter is devoted to an overview of the functional model, definitions and technical tools that we will use to get and build our estimator and the convergence rates. In particular, we recall the definition of survival models, discussed the problems addressed by the functional statistics, definitions of the survival function, chance and the conditional hazard function, the exponential inequality of Bernstein and the Fuc-Nagaev ...

In the second chapter, we are interested in a non-parametric model for functional random variables. A kernel estimator for the conditional hazard function with Complete data under less restrictive terms and conditions, we establish the almost complete convergence with precision in the case $\alpha-$ mixing is constructed. These asymptotic properties are closely related to the phenomenon of concentration of the probability measure of the explanatory variable on small balls.

In the third chapter, we discuss the convergence in quadratic mean and asymptotic normality of the estimator of the hazard function of a real random variable conditional on a functional variable in the context of addiction ($\alpha-$ mixing) and complete data.

The fourth chapter is devoted to the study of uniform convergence, properties and asymptotic normality of estimates of the maximum of the conditional hazard function in the context of addiction (α-mixing), using estimates not kernel parameter of the first derivative of the conditional random function.

The last chapter of this thesis is devoted to some comments and discussions on the many open questions that result.

All chapters of this thesis are the subject of communications or publications.

Chapter 2

Estimation conditional hazard function

2.1 Introduction

This chapter is devoted to the problem of the estimate of the conditional hazard function , of a real random variable Y knowing a random variable X valued in a functional space (semi-metric functional probabilistic space) with complete data, i.e one observes all the event. As in any problem of not-parametric estimate, the dimension of space $\mathcal{F}$ plays an important role in the concentration properties of the variable X. The estimate is made using kernel method .
The present chapter is divided into two sections. The first one is devoted to the presentation of the model and the construction of conditional hazard function estimator. In the second section, we are interested to the almost complete convergence of the estimator built in the case where the observations are α-mixing

2.2 Nonparametric Model

Let (X, Y) a couple of random variable valued in $\mathcal{F} \times \mathbb{R}$ where $\mathcal{F}$ is a semi-metric space provided with a semi-metric $d(.;.)$. This section is devoted to the general problem of the estimate of a conditional hazard function for a real random variable Y

knowing a random variable X valued in a functional space (semi-metric functional probabilistic space,) where X and Y are defined on the same probabilistic spaces $(\Omega, \mathcal{A}, \mathbb{P})$. In addition, to be able to extend to the case depending the results got in the independent case. We will adopt certain assumptions on process $(X_i; Y_i)_{i\in\mathbb{N}}$. Are:

$$\begin{array}{rcl} Y_i : (\Omega, \mathcal{A}, \mathbb{P}) & \rightarrow & (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \\ X_i : (\Omega, \mathcal{A}, \mathbb{P}) & \rightarrow & (\mathcal{F}, \mathfrak{F}) \end{array}$$

where$\mathcal{F}$ are provided with semi-metric a $d_i; i \in \mathbb{N}$, one proposes to estimate the conditional hazard function of Y knowing $X = x$. One indicates by F^x the conditional distribution function of Y knowing $X = x$, it is supposed that F^x is absolutely continuous w.r.t Lebesgue measure of density f^x.
Given $(X_1, Y_1), \ldots, (X_N; Y_N)$ a sequence of the observations of same law that (X, Y) the estimator of conditional distribution function F^x by the kernel method (noted $\widehat{F}^x$), defined by:

$$\widehat{F}^x(y) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1} d(x, X_i)\right) H\left(h_H^{-1}(y - Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1} d(x, X_i)\right)} \quad \forall y \in \mathbb{R}.$$

K is a kernel, H is a distribution function and $h_K = h_{K,N}$ (resp. $h_H = h_{H,N}$) is a sequence of positive real. One pose

$$K_i(x) = K(h_K^{-1} d(x, X_i)) \text{ et } H_i(y) = H(h_H^{-1}(y - Y_i))$$

What enables us to express $\widehat{F}^x(y)$ by:

$$\widehat{F}^x(y) = \frac{\widehat{F}_N^x(y)}{\widehat{F}_D^x}$$

with

$$\widehat{F}_N^x(y) = \frac{1}{n\mathbb{E}K_1} \sum_{i=1}^{n} K_i H_i(y) \text{ and } \widehat{F}_D^x = \frac{1}{n\mathbb{E}K_1} \sum_{i=1}^{n} K_i$$

From this estimator, one deduces an estimator for the conditional density, noted $\widehat{f}^X$, defined by:

$$\widehat{f}^x(y) = \frac{h_H^{-1} \sum_{i=1}^{n} K\left(h_K^{-1} d(x, X_i)\right) H'\left(h_H^{-1}(y - Y_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1} d(x, X_i)\right)} \quad \forall y \in \mathbb{R}.$$

Which is written as

$$\widehat{f}^x(y) = \frac{\widehat{f}_N^x(y)}{\widehat{F}_D^x}$$

where

$$\widehat{f}_N^x(y) = \frac{1}{n h_H \mathbb{E} K_1} \sum_{i=1}^{n} K_i H_i^{(1)}(y)$$

The conditional hazard rate of Y knowing $X = x$ is defined by

$$h^x(y) = \lim_{\Delta \to \infty} \frac{\mathbb{P}(Y \leq y + \Delta y / Y > y, X = x)}{\Delta y} \quad y > 0$$

Now the hazard rate can be written as the rate of the conditional density $f^x(.)$ and the survival function $S^x(.) = 1 - F^x(.)$ of y, i.e.:

$$h^x(y) = \frac{f^x(y)}{S^x(y)}$$

Thus conditional hazard function Y knowing $X = x$ is defined by:

$$\forall X \in \mathcal{F}, \forall Y \in \mathbb{R} \quad h^x(y) = \frac{f^x(y)}{1 - F^x(y)}$$

The main aim of this chapter is to give the speed of convergence of our estimator defined by: $\widehat{h}^x(y) = \dfrac{\widehat{f}^x(y)}{1 - \widehat{F}^x(y)}$ to $h^x(y) = \dfrac{f^x(y)}{1 - F^x(y)}$ In the case where the observations are α-mixing.

2.3 Dependent Case

The object of this section is to study a model of conditional hasard in which the explanatory variable X is not necessarily real or multidimensional but only supposed valued in an abstracts space $\mathcal{F}$ provided with semi-metric a d. As in any problem of not-parametric estimate, the dimension of space $\mathcal{F}$ plays an important role in the concentration properties of the variable X. Thus, when this dimension is not necessarily finite , functions probability of small balls defined by:

$$\phi_x(h) = \mathbb{P}(X \in B(x,h)) = \mathbb{P}(X \in \{x' \in \mathcal{F}/d(x,x') < h\})$$

involved directly in the asymptotic behavior of any non-parametric functional estimator.

2.3.1 General notations and assumptions

All along our study, when no confusion is possible, one note A and/or A' a generic constant of $\mathbb{R}^{*+}$ A point x is fixed in $\mathcal{F}$ which one notes N_x a neighborhood of x, S will be a fixed compact subset of $\mathbb{R}^+$ and we put $B(x,h) = \{x' \in \mathcal{F}/d(x,x') < h\}$ the ball of center x and of radius h.
We introduce The following assumptions:

(H1) $\forall x \in \mathcal{F}, \forall h > 0, \mathbb{P}(X \in B(x,h)) = \phi_x(h) > 0$

(H2) $\forall y \in S, F^x(y) < 1, \forall (y_1,y_2) \in S \times S, \forall (x_1,x_2) \in N_x \times N_x,$
$|F^{x_1}(y_1) - F^{x_2}(y_2)| \leq A_x(d(x_1,x_2)^{b_1} + |y_1-y_2|^{b_2}),\ b_1 > 0, b_2 > 0,$

(H3) $\forall (y_1,y_2) \in S \times S, \forall (x_1,x_2) \in N_x \times N_x$
$|f^{x_1}(y_1) - f^{x_2}(y_2)| \leq A_x(d(x_1,x_2)^{b_1} + |y_1-y_2|^{b_2}),\ b_1 > 0, b_2 > 0,$

(H4) $\forall (y_1,y_2) \in \mathbb{R}^2, |H^{(j)}(y_1) - H^{(j)}(y_2)| \leq A|y_1-y_2|$
$\int |t|^{b_2} H^{(1)}(t)dt < 1$ and $\exists \nu > 0, \forall j' \leq j+1,\ \lim_{y\to\infty} |y|^{1+\nu}|H^{(j')}(y)| = 0$ for some $j = 0,1$

(H5) K a kernel of compact support $(0,1)$ satisfying $0 < A_1 < K(t) < A_2 < 1$

(H6) $\lim_{n\to\infty} h_K = 0$ and $\lim_{n\to\infty} \dfrac{log\ n}{nh_H^j\phi_x(h_K)} = 0,\ \forall j = 0,1$

(H7) $\lim_{n\to\infty} h_H = 0$ and $\lim_{n\to\infty} n^\alpha h_H = 0, \quad \forall\alpha > 0.$

(H8) the sequence $(X_i; Y_i)_{i=1,\dots,n}$ is α-mixing whose coefficient of mixture checks:

$$\exists a > \frac{5+\sqrt{17}}{2},\ c \in \mathbb{R} \quad such\ that \quad \alpha(n) \leq cn^{-a}$$

(H9) $\dfrac{\sup_{i\neq j}\mathbb{P}((X_i, X_j) \in B(x,h) \times b(x,h))}{\mathbb{P}X_i \in B(x,h)} = \mathcal{O}((n^{-1}\phi_x(h))^{1/a})$

(H10) $\exists\eta > 0,\ An^{\frac{3-a}{a+1}+\eta} \leq h_h\phi_x(h_K)$ and $\phi_x(h_K) \leq A'n^{\frac{1}{1-a}}.$

2.3.2 Asymptotic Properties

Theorem 2.3.1. *Under the assumptions* $(H1)-(H10)$ *we have:*

$$\sup_{y\in S}|\widehat{h}^x(y) - h^x(y)| = \mathcal{O}(h_K^{b_1}) + \mathcal{O}(h_H^{b_2}) + \mathcal{O}\left(\sqrt{\frac{log\ n}{nh_H\phi_x(h_K)}}\right) \tag{2.1}$$

where $\phi_x(h_K)$ are the concentration of probability measure of the functional variable X in the ball of center x and of radius h_K.

Proof of theorem (2.3.1):

One can write $\widehat{h}^x(y) - h^x(y)$ as

$$\begin{aligned}
\widehat{h}^x(y) - h^x(y) &= \frac{\widehat{f}^x(y)}{1-\widehat{F}^x(y)} - \frac{f^x(y)}{1-F^x(y)} \\
&= \frac{\widehat{f}^x(y) - \widehat{f}^x(y)F^x(y) - f^x(y) + f^x(y)\widehat{F}^x(y)}{(1-\widehat{F}^x(y))(1-F^x(y))} \\
&= \frac{1}{1-\widehat{F}^x(y)}\left[(\widehat{f}^x(y) - f^x(y)) + \frac{f^x(y)}{1-F^x(y)}(\widehat{F}^x(y) - F^x(y))\right]
\end{aligned} \tag{2.2}$$

Valid for all $y \in \mathcal{S}$. Which for a constant $C < \infty$, this leads

$$\sup_{y\in S} |\widehat{h}^x(y) - h^x(y)| \quad \leq \quad C \frac{\{\sup_{y\in S} |\widehat{f}^x(y) - f^x(y)| + \sup_{y\in S} |\widehat{F}^x(y) - F^x(y)|\}}{\inf_{y\in S} |1 - \widehat{F}^x(y)|} \tag{2.3}$$

According to the previous decomposition, it's sufficient to show that:

$$\sup_{y\in S} |\widehat{F}^x(y) - F^x(y)| = \mathcal{O}(h_K^{b_1}) + \mathcal{O}(h_H^{b_2}) + \mathcal{O}\left(\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right) \quad a.co \tag{2.4}$$

$$\sup_{y\in S} |\widehat{f}^x(y) - f^x(y)| = \mathcal{O}(h_K^{b_1}) + \mathcal{O}(h_H^{b_2}) + \mathcal{O}\left(\sqrt{\frac{log\ n}{nh_H\phi_x(h_K)}}\right) \quad a.co \tag{2.5}$$

$$\exists \delta > 0 \text{ such that } \sum_{j=0}^{\infty} \mathbb{P}\left\{\inf_{y\in S} |1 - \widehat{F}^x(y)| < \delta\right\} < \infty. \tag{2.6}$$

It is noticed that

$$\begin{aligned} \widehat{F}^x(y) - F^x(y) &= \frac{1}{\widehat{F}_D^x}\left\{\left(\widehat{F}_N^x(y) - \mathbb{E}\widehat{F}_N^x(y)\right) - \left(F^x(y) - \mathbb{E}\widehat{F}_N^x(y)\right)\right\} \\ &+ \frac{F^x(y)}{\widehat{F}_D^x}\left\{\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right\} \end{aligned} \tag{2.7}$$

$$\begin{aligned} \widehat{f}^x(y) - f^x(y) &= \frac{1}{\widehat{f}_D^x}\left\{\left(\widehat{f}_N^x(y) - \mathbb{E}\widehat{f}_N^x(y)\right) - \left(f^x(y) - \mathbb{E}\widehat{f}_N^x(y)\right)\right\} \\ &+ \frac{f^x(y)}{\widehat{F}_D^x}\left\{\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x.\right\} \end{aligned} \tag{2.8}$$

□

What enables us to conclude that the proof of the theorem is based on the results below.

Lemma 2.3.2. *Under the assumptions of the theorem (2.3.1) we have:*

$$\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x = \mathcal{O}\left(\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right) \quad p.co \tag{2.9}$$

proof of lemma (2.3.2)

our objective is to show;

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left| \widehat{f}_D^x - \mathbb{E}\widehat{f}_D^x \right| > \epsilon \sqrt{\frac{\log n}{n\phi_x(h_K)}} \right) \leq \infty \tag{2.10}$$

we have

$$\widehat{f}_D^x - \mathbb{E}\widehat{f}_D^x = \frac{1}{n\mathbb{E}K_1} \sum_{i=1}^{n} \Delta_i$$

such that $\Delta_i = K_i - \mathbb{E}K_i$ It is enough to apply the inequality of Fuc-Nagaev. For that, we must initially calculate asymptotically s_n^2 defined by:

$$s_n^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |cov(\Delta_i, \Delta_j)| = s_n^{*^2} + \sum_{i=1}^{n} var(\Delta_i) \tag{2.11}$$

such that

$$s_n^{*^2} = \sum_{i=1}^{n}\sum_{i\neq j} |cov(\Delta_i, \Delta_j)|$$

thus for all $i \neq j$ we have

$$cov(\Delta_i, \Delta_j) = \mathbb{E}(\Delta_i\Delta_j) - \mathbb{E}(\Delta_i)\mathbb{E}(\Delta_j)$$

Thus by definition one finds

$$\begin{aligned} |cov(\Delta_i, \Delta_j)| &\leq A\mathbb{E}(I_{B(x,h_K)\times B(x,h_K)}(X_i, X_j)) + A\mathbb{E}(I_{B(x,h_K)}(X_i))\mathbb{E}(I_{B(x,h_K)}(X_j)) \\ &\leq A\mathbb{P}((X_i, X_j) \in B(x, h_K) \times B(x, h_K)) + A\mathbb{P}(X_i \in B(x, h_K))\mathbb{P}(X_j \in B(x, h_K)) \\ &\leq A'\phi_x(h_k)\left((n^{-1}\phi_x(h))^{1/a} + \phi_x(h_k)\right) \end{aligned} \tag{2.12}$$

By using the techniques of Masry [?] and defines the sets $S1, S2$,

$$S1 = \{(i, j) \quad such\ that \quad 1 \leq j - i \leq m_n\};$$

$$S2 = \{(i, j) \quad such\ that \quad m_n + 1 \leq j - i \leq n - 1\}.$$

where $(m_N)_N$ are arbitrary sequences of positive integer checking $m_N \to \infty$. Thus for n rather large one obtains

$$s_n^{*^2} = \sum_{S1} |cov(\Delta_i, \Delta_j)| + \sum_{S2} |cov(\Delta_i, \Delta_j)|$$

According to the definition of $S1$ and (2.12) one deduce that

$$\sum_{S1} |cov(\Delta_i, \Delta_j)| \leq A' n m_n \phi_x(h_k)(n^{-1}\phi_x(h))^{1/a}$$

It results according to the covariance inequality for limited variable(Lemma 1.4.4) one obtains:

$$\sum_{S2} |cov(\Delta_i, \Delta_j)| \leq A n^2 \alpha(m_n) \leq A' n^2 m_n^{-a}$$

Taking $m_n = \left(\frac{n}{\phi_x(h_k)}\right)^{1/a}$, it results that

$$s_n^{*^2} = \mathcal{O}(n\phi_x(h_k)).$$

In the second time, one has, for all $i = 1, \ldots, n$

$$\sum_{i=1}^{n} var(\Delta_i) = \sum_{i=1}^{n} \mathbb{E}(\Delta_i^2) - (\mathbb{E}(\Delta_i))^2.$$

One shows by the same method to use in the calculation of the $cov(\Delta_I, \Delta_J)$ that $cov(\Delta_i, \Delta_j)$ that

$$cov(\Delta_i, \Delta_j) \leq A' \phi_x(h_k).$$

and consequently

$$\sum_{i=1}^{n} var(\Delta_i) \leq \mathcal{O}(n\phi_x(h_k)) \tag{2.13}$$

Finally, according to these result one finds

$$s_n^2 = \mathcal{O}(n\phi_x(h_k)) \tag{2.14}$$

and one has to complete asymptotically to calculate s_n^2.
The Fuk-Nagaev inequality on the variable Δ_I involves for $\epsilon > 0$ and $r > 1$,

$$\begin{aligned}
\mathbb{P}\left(\left|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right| > \epsilon\right) &= \mathbb{P}\left(\left|\sum_{i=1}^n \Delta_i\right| > \epsilon n \mathbb{E}K_1\right) \\
&\leq 4\left(1 + \frac{\epsilon^2 n^2 \mathbb{E}^2 K_1}{16 r s_n^2}\right)^{\dfrac{-r}{2}} + 2ncr^{-1}\left(\frac{8r}{\epsilon n \mathbb{E}K_1}\right)^{a+1}
\end{aligned}$$

Thus one arrives at

$$\begin{aligned}
\mathbb{P}\left(\left|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right| > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) &\leq 4\left(1 + \frac{\epsilon^2 n^2 \mathbb{E}^2 K_1 \dfrac{\log n}{n\phi_x(h_K)}}{16 r s_n^2}\right)^{\dfrac{-r}{2}} + \\
&\quad 2ncr^{-1}\left(\frac{8r}{\epsilon n \mathbb{E}K_1}\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right)^{a+1} \\
&\leq 4\left(1 + \frac{\epsilon^2 n \log n \; n\phi_x(h_K)}{16 r s_n^2}\right)^{\dfrac{-r}{2}} + \\
&\quad Anr^a \epsilon^{-(a+1)} (n \log n \; n\phi_x(h_K))^{\dfrac{-(a+1)}{2}} \\
&\leq An^{1-\dfrac{(a+1)}{2}} r^a \epsilon^{-(a+1)} (n \log n \; n\phi_x(h_K))^{\dfrac{-(a+1)}{2}} + \\
&\quad 4\left(1 + \frac{\epsilon^2 \log n}{16r}\right)^{\dfrac{-r}{2}} \\
&\leq An^{1-\dfrac{(a+1)}{2}} r^a \epsilon^{-(a+1)} (\log n)^{1-\dfrac{(a+1)}{2}} \phi_x(h_K)^{1-\dfrac{(a+1)}{2}} + \\
&\quad Ae^{\dfrac{-r}{2}} \log\left(1 + \frac{\epsilon^2 \log n}{16r}\right)
\end{aligned}$$

One can always choose r in the form $r = C(\log n)^2$, where C is a constant.

$$\mathbb{P}\left(\left|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right| > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) \leq An^{\frac{-\epsilon^2}{32}} + A(\log n)^{2a-\frac{(a+1)}{2}} n^{1-\frac{(a+1)}{2}} \phi_x(h_K)^{\frac{-(a+1)}{2}}$$

Thanks to the left inequality in (H10)

$$\begin{aligned}
\mathbb{P}\left(\left|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right| > \epsilon\sqrt{\tfrac{\log n}{n\phi_x(h_K)}}\right) &\leq An^{\frac{-\epsilon^2}{32}} + A(\log n)^{2a-\frac{(a+1)}{2}} n^{1-\frac{(a+1)}{2}} n^{-\frac{(a+1)}{2}(\frac{3-a}{a+1}+\eta)} \\
&\leq An^{\frac{-\epsilon^2}{32}} + An^{2a-\frac{(a+1)}{2}} n^{1-\frac{(a+1)}{2}} n^{-\frac{(a+1)}{2}(\frac{3-a}{a+1}+\eta)} \\
&\leq An^{\frac{-\epsilon^2}{32}} + An^{-1-(\frac{(1-a)}{2}+\frac{(a+1)}{2}\eta)}
\end{aligned}$$

for ϵ sufficiently large and $\nu > 0$ one will lead,

$$\mathbb{P}\left(\left|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right| > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) \leq A'n^{-1-\nu} \tag{2.15}$$

finally,

$$\mathbb{P}\left(\left|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right| > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) \leq \sum_{n=1}^{\infty} A'n^{-1-\nu} < \infty. \tag{2.16}$$

□

Corollary 2.3.3. *Under the assumptions of the theorem (2.3.1), we have :*

$$\sum_{i=1}^{n} \mathbb{P}\left(\widehat{F}_D^x < 1/2\right) < \infty$$

Proof of corollary (2.3.3):

one has

$$\left\{|\widehat{F}_D^x| < 1/2\right\} \subseteq \left\{|\widehat{F}_D^x - 1| < 1/2\right\}$$

consequently

$$\begin{aligned}
\mathbb{P}\left\{|\widehat{F}_D^x| < 1/2\right\} &\leq \mathbb{P}\left\{|\widehat{F}_D^x - 1| < 1/2\right\} \\
&\leq \mathbb{P}\left\{|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x| < 1/2\right\}
\end{aligned}$$

because $\mathbb{E}\widehat{F}_D^X = 1$ we apply the result of the lemma (2.3.2), we show that

$$\sum_{i=1}^{n} \mathbb{P}\left(\widehat{F}_D^x < 1/2\right) < \infty$$

□

Lemma 2.3.4. *Under the assumptions* $(H1) - (H6)$*, we have*

$$\frac{1}{\widehat{F}_D^x} \sup_{y \in S} |F^x(y) - \mathbb{E}\widehat{F}_N^x(y)| = \mathcal{O}(h_K^{b_1}) + \mathcal{O}(h_H^{b_2}) \tag{2.17}$$

$$\frac{1}{\widehat{F}_D^x} \sup_{y \in S} |f^x(y) - \mathbb{E}\widehat{f}_N^x(y)| = \mathcal{O}(h_K^{b_1}) + \mathcal{O}(h_H^{b_2}) \tag{2.18}$$

Proof of lemma (2.3.4):

We obtain successively

$$\begin{aligned} \mathbb{E}\widehat{F}_N^x(y) - F^x(y) &= \tfrac{1}{n\mathbb{E}(K_1)} \sum_{i=1}^{\infty} \mathbb{E}(K_i) H_i(y) - F^x(y) \\ &= \tfrac{1}{\mathbb{E}(K_1)} \left[\mathbb{E}K_1 H_1\left(\tfrac{y-Y_i}{h_H}\right) F^x(y)\right] \\ &= \tfrac{1}{\mathbb{E}(K_1)} \mathbb{E}\left(K_1 \left[\mathbb{E}\left(H_1(h_H^{-1}(y-Y_i)/X)\right) - F^x(y)\right]\right) \end{aligned} \tag{2.19}$$

we have

$$\begin{aligned} \mathbb{E}\left(H_1(h_H^{-1}(y-Y_i)/X)\right) &= \int_{\mathbb{R}} H\left(\frac{y-u}{h_H}\right) f^x(u)du \\ &= \int_{\mathbb{R}} H^{(1)}(t) F^x(y - h_H t)dt \end{aligned}$$

in addition one has

$$\begin{aligned} |\mathbb{E}\left(H_1(h_H^{-1}(y-Y_i)/X)\right) - F^x(y)| &= \left|\int_{\mathbb{R}} H\left(\frac{y-u}{h_H}\right) f^x(u)du - F^x(y)\right| \\ &= \int_{\mathbb{R}} H^{(1)}(t)|F^x(y - h_H t) - F^x(y)|dt \end{aligned}$$

Thus, thanks to the assumption (H2) one obtains

$$|\mathbb{E}\left(H_1(h_H^{-1}(y-Y_i)/X)\right) - F^x(y)| \le A_x \int_{\mathbb{R}} H^{(1)}(h_k^{b_1} + |t|^{b_2} h_H^{b_2})dt \tag{2.20}$$

This inequality is uniform in y, while replacing in the equation (2.19) and by simplifying the term $\mathbb{E}(K_1)$ one finds

$$\mathbb{E}\widehat{F}_N^x(y) - F^x(y) \leq A_x \left(h_k^{b_1} \int_{\mathbb{R}} H^{(1)}(t) + h_H^{b_2} \int_{\mathbb{R}} |t|^{b_2} H^{(1)}(t)dt \right)$$

Finally, the assumption (H4) and the corollary (2.3.3) involve the proof of the equation (2.17).

It remains us to show the equation (2.18), indeed

$$\begin{aligned} \mathbb{E}\widehat{f}_N^x(y) - f^x(y) &= \tfrac{1}{h_H \mathbb{E}(K_1)} \left[\mathbb{E} K_1 H_1^{(1)} \left(\tfrac{y-Y_i}{h_H} \right) - h_H f^x(y) \right] \\ &= \tfrac{1}{h_H \mathbb{E}(K_1)} \mathbb{E} \left(K_1 \left[\mathbb{E} \left(H_1^{(1)}(h_H^{-1}(y - Y_i)/X) \right) - h_H f^x(y) \right] \right) \end{aligned}$$

moreover

$$\begin{aligned} \mathbb{E} \left(H_1^{(1)}(h_H^{-1}(y - Y_i)/X) \right) &= \int_{\mathbb{R}} H^{(1)} \left(\frac{y-u}{h_H} \right) f^x(u)du \\ &= h_H \int_{\mathbb{R}} H^{(1)}(t) f^x(y - h_H t)dt. \end{aligned}$$

And consequently

$$|\mathbb{E} \left(H_1^{(1)}(h_H^{-1}(y - Y_i)/X) \right) - h_H f^x(y)| \leq h_H \int_{\mathbb{R}} H^{(1)}(t)|f^x(y - h_H t) - f^x(y)|dt$$

the assumption $(H3)$ involves that

$$|\mathbb{E} \left(H_1^{(1)}(h_H^{-1}(y - Y_i)/X) \right) - h_H f^x(y)| \leq A_x h_H \int_{\mathbb{R}} H^{(1)}(h_k^{b_1} + |t|^{b_2} h_H^{b_2})dt$$

the assumption $(H4)$ and the corollary (2.3.3) involve the proof of the equation (2.18). It completes the proof of the lemma (2.3.4). □

Lemma 2.3.5. *Under the assumptions (H1) - (H7) one has:*

$$\frac{1}{\widehat{F}_D^x} \sup_{y \in S} |\widehat{F}_N^x(y) - \mathbb{E}\widehat{F}_N^x(y)| = \mathcal{O}\left(\sqrt{\frac{\log n}{n\phi_x(h_K)}} \right) \quad p.co \tag{2.21}$$

$$\frac{1}{\widehat{F}_D^x} \sup_{y \in S} |\widehat{f}_N^x(y) - \mathbb{E}\widehat{f}_N^x(y)| = \mathcal{O}\left(\sqrt{\frac{\log n}{nh_H\phi_x(h_K)}} \right) \quad p.co \tag{2.22}$$

Proof of lemma (2.3.5):

The idea of the proof is to cover the compact S by intervals S_K with equal lengths. However, the compactness of S implies that one can extract from this covering a finished covering of which the number of the intervals will be noted S_N. In other words, $S \subset \bigcup_{k=1}^{S_n} S_k$ where $S_k = (m_k - l_n, m_k + l_n)$

Let us put $m_y = \arg \min_{k \in 1,\dots,S_n} |y - m_k|$ by adding and subtracting the term $\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y)$ and applying the trigonometrical inequality. It is shown that:

$$|\widehat{F}_N^x(y) - \mathbb{E}\widehat{F}_N^x(y)| \leq |\widehat{F}_N^x(y) - \widehat{F}_N^x(m_y)| + |\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y)| + |\mathbb{E}\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(y)|$$

Thus

$$\frac{1}{\widehat{F}_D^x} \sup_{y \in S} |\widehat{F}_N^x(y) - \mathbb{E}\widehat{F}_N^x(y)| \leq \underbrace{\frac{1}{\widehat{F}_D^x} \sup_{y \in S} |\widehat{F}_N^x(y) - \widehat{F}_N^x(m_y)|}_{T_1} + \underbrace{\frac{1}{\widehat{F}_D^x} \sup_{y \in S} |\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y)|}_{T_2} + \underbrace{\frac{1}{\widehat{F}_D^x} \sup_{y \in S} |\mathbb{E}\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(y)|}_{T_3} \tag{2.23}$$

- **Concerning** (T_1) The assumption (H4) involves

$$\begin{aligned}
\frac{1}{\widehat{F}_D^x} \sup_{y \in S} |\widehat{F}_N^x(y) - \widehat{F}_N^x(m_y)| &\leq \frac{1}{\widehat{F}_D^x} \sup_{y \in S} \frac{1}{n\mathbb{E}K_1} \sum_{i=1}^{n} |H_i(y) - H_i(m_y)| K_i \\
&\leq \frac{1}{\widehat{F}_D^x} \sup_{y \in S} \frac{A|y - m_y|}{h_H} \left(\frac{1}{n\mathbb{E}K_1} \sum_{i=1}^{n} K_i \right) \\
&\leq \frac{1}{\widehat{F}_D^x} \sup_{y \in S} \frac{A|y - m_y|}{h_H} \widehat{F}_D^x \\
&\leq A \frac{l_n}{h_H}.
\end{aligned} \tag{2.24}$$

By taking $l_N = n^{-\alpha-1/2}$ and one shows that

$$\frac{l_n}{h_H} = \mathcal{O}\left(\sqrt{\frac{log\ n}{n\phi_x(h_K)}} \right) = \mathcal{O}\left(\sqrt{log\ n(n\phi_x(h_K))^{-1}} \right)$$

Indeed

$$\lim_{n\to+\infty}\frac{l_n}{h_H}\left(\sqrt{\frac{n\phi_x(h_K)}{log\ n}}\right)=\lim_{n\to+\infty}\frac{1}{h_H n^{\alpha}}\left(\sqrt{\frac{n\phi_x(h_K)}{log\ n}}\right)$$

According to the assumption (H7) one has:

$$\lim_{n\to+\infty}\frac{1}{h_H n^{\alpha}}\left(\sqrt{\frac{n\phi_x(h_K)}{log\ n}}\right)=0$$

and shows that

$$\frac{l_n}{h_H}=\mathcal{O}\left(\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right)=\mathcal{O}\left(\sqrt{log\ n(n\phi_x(h_K))^{-1}}\right)$$

in addition we have

$$\forall\eta>0,\ \exists N_\eta>0\quad pour\ n>N_\eta,\ \frac{l_n}{h_H}\left(\sqrt{\frac{n\phi_x(h_K)}{log\ n}}\right)\leq\eta$$

thus for

$$\frac{\eta}{3},\ \exists N_0,\quad pour\ n>N_0,\quad \frac{l_n}{h_H}\left(\sqrt{\frac{n\phi_x(h_K)}{log\ n}}\right)\leq\frac{\eta}{3}$$

and according to the result(2.24) ($\leq A\frac{l_n}{h_H}$) it is deduced that:

$$\frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\widehat{F}_N^x(y)-\widehat{F}_N^x(m_y)|\leq\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}},$$

and it results that, for $n>N_0$

$$\mathbb{P}\left(\frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\widehat{F}_N^x(y)-\widehat{F}_N^x(m_y)|>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right)=0 \tag{2.25}$$

Thus, we can write

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}\left(T_1>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right)\ &\leq\ \sum_{n=1}^{N_0}\mathbb{P}\left(T_1>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right)\\&+\ \sum_{n=N_0+1}^{\infty}\mathbb{P}\left(T_1>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right)\end{aligned}$$

the first term of the right member is finite , and the second is null according to the result (2.25). From where

$$\sum_{n=1}^{\infty} \mathbb{P}\left(T_1 > \frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right) < \infty \tag{2.26}$$

• **Concerning** (T_2)

we have,

$$\mathbb{P}\left(\sup_{y\in S}|\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y)| > \frac{\epsilon}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right) \leq$$

$$\frac{A}{l_n}\max_{m_k\in(m_1,\ldots,m_{S_n})}\mathbb{P}\left(|\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y)| > \frac{\epsilon}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right)$$

like

$$\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y) = \frac{1}{n\mathbb{E}K_1}\sum_{i=1}^{n}\underbrace{H_i(m_y)K_i - \mathbb{E}(H_i(m_y)K_i)}_{\Lambda_i^*}$$

Which requires the calculation of $s_N^{'2}$ where

$$s_n^{'2} = \sum_{i=1}^{n}\sum_{j=1}^{n}|cov(\Lambda_i^*, \Lambda_j^*|$$

By using the same argument used in s_N^2 and by taking $m_N = \frac{1}{\phi_X(h_K)}$, one show that

$$s_n^{'2} = \mathcal{O}(n\phi_x(h_K)) + \mathcal{O}(n\phi_x(h_K))$$

The inequality of Fuk-Nagaev on the variable Λ_I^* involve for $\epsilon > 0$ and $r > 1$

$$\begin{aligned}\mathbb{P}\left(|\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y)| > \epsilon\right) &= \mathbb{P}\left(\left|\sum_{i=1}^{n}\Lambda_i^*\right| > \epsilon n\mathbb{E}K_1\right)\\ &\leq 4\left(1+\tfrac{\epsilon^2 n^2\mathbb{E}^2K_1}{16rs_n^2}\right)^{\frac{-r}{2}} + 2ncr^{-1}\left(\tfrac{8r}{\epsilon n\mathbb{E}K_1}\right)^{a+1}\end{aligned}$$

Thus one arrives at

$$\begin{aligned}
\mathbb{P}\left(\left|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right| > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) &\leq 4\left(1+\frac{\epsilon^2 n^2 \mathbb{E}^2 K_1 \frac{\log n}{n\phi_x(h_K)}}{16 r s_n'^2}\right)^{\frac{-r}{2}} + 2ncr^{-1}\left(\frac{8r}{\epsilon n \mathbb{E}K_1}\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right)^{a+1} \\
&\leq 4\left(1+\frac{\epsilon^2 n \log n \ n\phi_x(h_K)}{16 r s_n'^2}\right)^{\frac{-r}{2}} + \\
&\quad Anr^a \epsilon^{-(a+1)}(n\log n \ n\phi_x(h_K))^{\frac{-(a+1)}{2}} \\
&\leq An^{1-\frac{(a+1)}{2}} r^a \epsilon^{-(a+1)}(n\log n \ n\phi_x(h_K))^{\frac{-(a+1)}{2}} + \\
&\quad 4\left(1+\frac{\epsilon^2 \log n}{16r}\right)^{\frac{-r}{2}} \\
&\leq An^{1-\frac{(a+1)}{2}} r^a \epsilon^{-(a+1)}(\log n)^{1-\frac{(a+1)}{2}} \phi_x(h_K)^{1-\frac{(a+1)}{2}} + \\
&\quad Ae^{\frac{-r}{2}} \log\left(1+\frac{\epsilon^2 \log n}{16r}\right)
\end{aligned}$$

We can always choose r in $r = C(\log N)^2$, where C is a constant. what gives

$$\mathbb{P}\left(\left|\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y)\right| > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) \leq An^{\frac{-\epsilon^2}{32}} + A(\log n)^{2a-\frac{(a+1)}{2}} n^{1-\frac{(a+1)}{2}} \phi_x(h_K)^{\frac{-(a+1)}{2}}$$

Thanks to the left inequality in (H10)

$$\mathbb{P}\left(\left|\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y)\right| > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) \leq An^{\frac{-\epsilon^2}{32}} + A(\log n)^{2a-\frac{(a+1)}{2}} n^{1-\frac{(a+1)}{2}} n^{-\frac{(a+1)}{2}(\frac{3-a}{a+1}+\eta)}$$

thus one has

$$\mathbb{P}\left(\sup_{y\in S}|\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y)| > \frac{\epsilon}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right) \leq Al_n^{-1}\left(n^{\frac{-\epsilon^2}{32}} + n^{-1-\frac{(a+1)}{2}\eta}\right)$$

we apply the corollary (2.3.3), under a suitable choice of ϵ we show that

$$\sum_{i=1}^{n}\mathbb{P}\left(\sup_{y\in S}\left|\widehat{F}_N^x(m_y)-\mathbb{E}\widehat{F}_N^x(m_y)\right|>\frac{\epsilon}{3}\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right)<+\infty$$

• **Concerning** (T_3) we have

$$\frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\mathbb{E}\widehat{F}_N^x(m_y)-\mathbb{E}\widehat{F}_N^x(y)|\leq\frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\widehat{F}_N^x(m_y)-\widehat{F}_N^x(y)|$$

and according to the result (2.24)($\leq A\frac{l_n}{h_H}$) we have:

$$\frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\mathbb{E}\widehat{F}_N^x(m_y)-\mathbb{E}\widehat{F}_N^x(y)|\leq A\frac{l_n}{h_H}$$

we have

$$\left\{T_3>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right\}\subseteq\left\{T_1>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right\}$$

which implies that

$$\mathbb{P}\left\{T_3>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right\}\leq\mathbb{P}\left\{T_1>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right\}$$

and consequently

$$\sum_{n=1}^{\infty}\mathbb{P}\left(T_3>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right)\leq\sum_{n=1}^{\infty}\mathbb{P}\left(T_1>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right)$$

and finally thanks to (2.26) we would have

$$\sum_{n=1}^{\infty}\mathbb{P}\left(T_3>\frac{\eta}{3}\sqrt{\frac{log\ n}{n\phi_x(h_K)}}\right)<\infty \tag{2.27}$$

What proves the equation (2.21) of the lemma (2.3.5). It remains us, now, the equation (2.22), notice that:

$$\frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\widehat{f}_N^x(y)-\mathbb{E}\widehat{f}_N^x(y)| \leq \underbrace{\frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\widehat{f}_N^x(y)-\widehat{f}_N^x(m_y)|}_{F_1}+\underbrace{\frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\widehat{f}_N^x(m_y)-\mathbb{E}\widehat{f}_N^x(m_y)|}_{F_2}+\underbrace{\frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\mathbb{E}\widehat{f}_N^x(m_y)-\mathbb{E}\widehat{f}_N^x(y)|}_{F_3} \tag{2.28}$$

• **Concerning** F_1 and F_3 we use the same arguments employed in the demonstration of T_1 and T_3, one replace H by H^1 we show that

$$\frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\widehat{f}_N^x(y)-\widehat{f}_N^x(m_y)| \leq A\frac{l_n}{h_H^2} \text{ and } \frac{1}{\widehat{F}_D^x}\sup_{y\in S}|\mathbb{E}\widehat{f}_N^x(m_y)-\mathbb{E}\widehat{f}_N^x(y)| \leq A\frac{l_n}{h_H^2}$$

Now, we choose l_n in the form $l_n = n^{-\frac{3\alpha}{2}-\frac{1}{2}}$ and according to (H7), we deduces that:

$$\frac{l_n}{h_H^2} = \mathcal{O}\left(\sqrt{\frac{log\ n}{nh_H\phi_x(h_K)}}\right)$$

• **Concerning** F_2

one has,

$$\mathbb{P}\left(\sup_{y\in S}|\widehat{f}_N^x(m_y)-\mathbb{E}\widehat{f}_N^x(m_y)| > \frac{\epsilon}{3}\sqrt{\frac{log\ n}{nh_h\phi_x(h_K)}}\right) \leq$$

$$\frac{A}{l_n}\max_{m_k\in(m_1,\ldots,m_{S_n})}\mathbb{P}\left(|\widehat{f}_N^x(m_y)-\mathbb{E}\widehat{f}_N^x(m_y)| > \frac{\epsilon}{3}\sqrt{\frac{log\ n}{nh_h\phi_x(h_K)}}\right)$$

we have also

$$\widehat{f}_N^x(m_y)-\mathbb{E}\widehat{f}_N^x(m_y) = \frac{1}{nh_h\mathbb{E}K_1}\sum_{i=1}^{n}\underbrace{H_i^{(1)}(m_y)K_i-\mathbb{E}(H_i^{(1)}(m_y)K_i)}_{\Gamma_i^*}$$

Which requires the calculation of $s_N^{\prime 2}$ where

$$s_n^{\prime 2} = \sum_{i=1}^{n}\sum_{j=1}^{n}|cov(\Gamma_i^*,\Gamma_j^*|$$

By using the same method that in s_N^2 and by taking $m_N = \frac{1}{h_H \phi_X(h_K)}$, one show that

$$s_n^{'2} = \mathcal{O}(nh_H\phi_x(h_K))$$

The inequality of Fuk-Nagaev gives

$$\mathbb{P}\left(\sup_{y\in S} |\widehat{f}_N^x(m_y) - \mathbb{E}\widehat{f}_N^x(m_y)| > \frac{\epsilon}{3}\sqrt{\frac{log\ n}{nh_h\phi_x(h_K)}}\right) < A_1 + A_2.$$

with $A_1 = Ae\frac{-r}{2}\log\left(1+\frac{\epsilon^2\log n}{16r}\right)$

and $A_2 = An^{1-\frac{(a+1)}{2}} r^a \epsilon^{-(a+1)} (h_H \log n)^{1-\frac{(a+1)}{2}} \phi_x(h_K)^{1-\frac{(a+1)}{2}}$

We apply the assumption (H10) and the choice of $r = C(logn)^2$ and $l_n = n^{-\frac{3}{2}\alpha+\frac{1}{2}}$ one shows that there exists $\nu > 0$ for η rather large, one has

$$\frac{1}{l_n}(A_1 + A_2) \leq An^{-1-\nu}$$

according to the corollary (2.3.3), we deduced that

$$\mathbb{P}\left(\sup_{y\in S} |\widehat{f}_N^x(m_y) - \mathbb{E}\widehat{f}_N^x(m_y)| > \frac{\epsilon}{3}\sqrt{\frac{log\ n}{nh_h\phi_x(h_K)}}\right) \leq An^{-1-\nu}$$

□

Lemma 2.3.6. *Under the conditions of the theorem (2.3.1) we have*

$$\exists \delta > 0, \quad such\ that\ \sum_{n=1}^{\infty} \mathbb{P}\{\inf_{y\in S} |1 - \widehat{F}^x(y)| < \delta\} < \infty$$

proof of lemma (2.3.6)

by equation (2.4), we have the almost complete convergence of $\widehat{F}^x(y)$ to $F^x(y)$

$$\widehat{F}^x(y) \xrightarrow{a.co} F^x(y).$$

Which implies that

$$\sum_{n=1}^{\infty} \mathbb{P}\{\inf_{y\in S} |\widehat{F}^x(y) - F^x(y)| > \epsilon\} < \infty$$

In addition, we would have by the assumption $F^X < 1$ i.e.

$$1 - \widehat{F}^x \geq F^x - \widehat{F}^x$$

thus

$$\inf_{y\in S} |1 - \widehat{F}^x(y)| \leq (1 - \sup_{y\in S} F^x(y))/2 \implies \sup_{y\in S} |\widehat{F}^x(y) - F^x(y)| \geq (1 - \sup_{y\in S} F^x(y))/2$$

In terms of probability is obtained

$$\mathbb{P}\{\inf_{y\in S} |1 - \widehat{F}^x(y)| < \delta\} \leq \mathbb{P}\{\sup_{y\in S} |\widehat{F}^x(y) - F^x(y)| \geq (1 - \sup_{y\in S} F^x(y))/2\} < \infty$$

Finally it suffices to take $\delta = 1 - \sup_{y\in S} F^x(y)/2$ and apply the results (2.4)to finish the proof of the lemma $\square$

Chapter 3

On conditional hazard function estimate for functional mixing data

This chapter[17] is the subject of a publication in Journal of New Trends in Mathematical Sciences.

3.1 Introduction

Statistical problems involved in the modelization of functional data have received an increasing interest in the few past decade. The infatuation for this topic is linked with many fields of applications in which the data are collected in the functional order. Under this hypothesis, the statistical analysis focuses on a framework of infinite dimension for the data under study. This type of data appears in many fields of applied statistics: environmetrics [14], chemometrics [3], meteorological sciences [6], etc. This field of modern statistics has received much attention recently, it has been popularized in the book of Ramsay and Silverman [49].

The nonparametric estimation of the hazard and/or the conditional hazard function is quite important in a variety of fields such as medicine, reliability, survival analysis or in seismology. The hazard estimate was introduced by Watson and Leadbetter [60], after that considerable results have been given, see for example, Ahmad [16], Singpurwalla and Wong [55], and we can also cite Quintela [41] for a survey, Roussas [54] (for previous works), Li and Tran [35] (for recent advances and references).

When hazard rate estimation is performed with multiple variables, the result is an estimate of the conditional hazard rate for the first variable, given the levels of the remaining variables. Many references, practical examples and simulations in the case of non-parametric estimation using local linear approximations can be found in Spierdijk [56].

From a theoretical point of view, a sample of functional data can be involved in many different statistical problems, such as for instance: classification and principal components analysis (PCA)[5, 51] or longitudinal studies, regression and prediction [3, 10].

The literature is strictly not limited in the case where the data is of functional nature (a curve). The first result in this context, was given by Ferraty *et al* . [25], authors established the almost complete convergence of the kernel estimate of the conditional hazard function in the i.i.d. case and under α-mixing condition, and recently Rabhi *et al.* [43] studied the mean quadratic convergence in the i.i.d. case of this estimate. More recently Mahiddine *et al.* [37] give the uniform version of the almost complete convergence rate in the i.i.d. case.

The recent monograph by Ferraty and Vieu [26] summarizes many of their contributions to the non-parametric estimation with functional data; among other properties, consistency of the conditional density, conditional distribution and regression estimates are established in the i.i.d. case as well as under dependence conditions (strong mixing). Almost complete rates of convergence are also obtained, and the different techniques are applied to various examples of functional data samples. Related work can be found in the paper of Masry [39], where the asymptotic normality of the functional non-parametric regression estimate is proven, considering strong mixing dependence conditions for the sample data. For automatic smoothing parameter selection in the regression setting, see Rachdi and Vieu [45].
The main aim of this chapter, is to study, under general conditions, the asymptotic proprieties of the functional data kernel estimate of the conditional hazard function introduced by Ferraty *et al.* [25]. More precisely, we establish the asymptotic normality of the construct estimator. We point out that our asymptotic results are useful in some statistical problems such as the choice of the smoothing parameters. The present work extended to dependent case the result of Rabhi *et al.* [43] given in i.i.d. case functional. Note that, one of the main difficulties, when dealing with functional variables, relies on the difficulty for choosing some appropriate measure of reference in infinite dimensional spaces. The fundamental feature of our approach is to build estimates and to derive their asymptotic properties without any notion of density for the functional variable X. This approach allows us to avoid the use of a reference measure in such functional spaces. In each of the above described sections, we will give general asymptotic results without assuming existence of such a density, and each of these results will be discussed in relation with earlier literature existing in the usual finite dimensional case.
Our chapter presents some asymptotic properties related with the non-parametric estimation of the conditional hazard function. In a functional data setting, the conditioning variable is allowed to take its values in some abstract semi-metric space. In this case, Ferraty *et al.* [25] define non-parametric estimators of the conditional density and the conditional distribution. They give the rates of convergence (in an almost complete sense) to the corresponding functions, in an a dependence (α-mixing)

context. In Rabhi *et al.* [43], the same properties are shown in an i.i.d. context in the data sample. We extend their results to dependent case by calculating the bias and variance of these estimates, and establishing their asymptotic normality, considering a particular type of kernel for the functional part of the estimates. Because the hazard function estimator is naturally constructed using these two last estimators, the same type of properties is easily derived for it. Our results are valid in a real (one- and multi-dimensional) context.
The chapter is organized as follows: In the next section we present our model. In section 3 we present notations and hypotheses, Section 4 is dedicated for our main results. Section 5 is devoted to some discuss on the applicability of our asymptotic result in some statistical problems.

3.2 The model

Consider $Z_i = (X_i, Y_i)$, $i \in \mathbb{N}$ be a $\mathcal{F} \times \mathbb{R}$-valued measurable strictly stationary process, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $(\mathcal{F}, d)$ is a semi-metric space. In the following x will be a fixed point in $\mathcal{F}$ and N_x will denote a fixed neighborhood of x. We assume that the regular version of the conditional probability of Y given X exists. Moreover, we suppose that, for all $x \in N_x$ the conditional distribution function of Y given $X = x$, $F^x(\cdot)$, is 3-times continuously differentiable and we denote by f^x its conditional density with respect to (w.r.t.) Lebesgue's measure over $\mathbb{R}$. In this chapter, we consider the problem of the nonparametric estimation of the conditional hazard function defined, for all $y \in \mathbb{R}$ such that $F^x(y) < 1$, by

$$h^x(y) = \frac{f^x(y)}{1 - F^x(y)}.$$

In our spatial context, we estimate this function by

$$\widehat{h}^x(y) = \frac{\widehat{f}^x(y)}{1 - \widehat{F}^x(y)}$$

where

$$\widehat{F}^x(y) = \frac{\sum_{i=1}^n K(h_K^{-1} d(x, X_i)) H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^n K(h_K^{-1} d(x, X_i))}, \qquad \forall y \in \mathbb{R}$$

and

$$\widehat{f}^x(y) = \frac{h_H^{-1}\sum_{i=1}^n K(h_K^{-1}d(x,X_i))H'(h_H^{-1}(y-Y_i))}{\sum_{i=1}^n K(h_K^{-1}d(x,X_i))}, \qquad \forall y \in \mathbb{R}$$

with K is the kernel, H is a given continuously differentiable distribution function, $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) is a sequence of positive real numbers and H' is the derivative of H. Furthermore, the estimator $\widehat{h}^x(y)$ can we written as

$$\widehat{h}^x(y) = \frac{\widehat{f}_N^x(y)}{\widehat{F}_D^x - \widehat{F}_N^x(y)} \tag{3.1}$$

where

$$\begin{aligned}
\widehat{F}_D^x &:= \frac{1}{n\mathbb{E}[K_1]}\sum_{i=1}^n K(h_K^{-1}d(x,X_i)),\ K_1 = K(h_K^{-1}d(x,X_1))\\
\widehat{F}_N^x(y) &:= \frac{1}{n\mathbb{E}[K_1]}\sum_{i=1}^n K(h_K^{-1}d(x,X_i))H(h_H^{-1}(y-Y_i))\\
\widehat{f}_N^x(y) &:= \frac{1}{nh_H\mathbb{E}[K_1]}\sum_{i=1}^n K(h_K^{-1}d(x,X_i))H'(h_H^{-1}(y-Y_i)).
\end{aligned}$$

Our main purpose is to study the L^2- consistency and the asymptotic normality of the nonparametric estimate $\widehat{h}^x$ of h^x when the random filed $(Z_i, i \in \mathbb{N})$ satisfies the following mixing condition.

3.3 Notations and hypotheses

All along the chapter, when no confusion is possible, we will denote by C and C' some strictly positive generic constants. In order to establish our asymptotic results we need the following hypotheses:

(L0) $\forall r > 0, \mathbb{P}(X \in B(x,r)) =: \phi_x(r) > 0$, where $B(x,r) = \{x' \in \mathcal{F}/d(x,x') < r\}$.

(L1) $(X_i, Y_i)_{i\in\mathbb{N}}$ is an α-mixing sequence whose the coefficients of mixture verify:

$$\exists a > 0,\ \exists c > 0:\ \forall n \in \mathbb{N},\ \alpha(n) \leq cn^{-a}.$$

(L2) $0 < \sup_{i \neq j} \mathbb{P}\left((X_i, X_j) \in B(x,h) \times B(x,h)\right) \; = \; \mathcal{O}\left(\frac{(\phi_x(h))^{(a+1)/a}}{n^{1/a}}\right).$

Note that (L0) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* X, whereas (L2) concerns the behavior of the joint distribution of the pairs (X_i, X_j). In fact, this hypothesis is equivalent to suppose that, for n large enough

$$\sup_{i \neq j} \frac{\mathbb{P}\left((X_i, X_j) \in B(x,h) \times B(x,h)\right)}{\mathbb{P}\left(X \in B(x,h)\right)} \leq C \left(\frac{\phi_x(h)}{n}\right)^{1/a}.$$

(L3) For $l \in \{0, 2\}$, the functions $\Psi_l(s) = \mathbb{E}\left[\frac{\partial^l F^x(y)}{\partial y^l} - \frac{\partial^l F^X(y)}{\partial y^l} \middle| d(x,X) = s\right]$ and

$\Phi_l(s) = \mathbb{E}\left[\frac{\partial^l f^x(y)}{\partial y^l} - \frac{\partial^l f^X(y)}{\partial y^l} \middle| d(x,X) = s\right]$ are derivable at $s = 0$.

(L4) The bandwidth h_K satisfies:

$$h_K \downarrow 0, \quad \forall t \in [0,1] \quad \lim_{h_K \to 0} \frac{\phi_x(t h_K)}{\phi_x(h_K)} = \beta_x(t) \quad \text{and} \quad n h_H \phi_x(h_K) \to \infty \quad \text{as } n \to \infty.$$

(L5) The kernel K from $\mathbb{R}$ into $\mathbb{R}^+$ is a differentiable function supported on $[0,1]$. Its derivative K' exists and is such that there exist two constants C and C' with $-\infty < C < K'(t) < C' < 0$ for $0 \leq t \leq 1$.

(L6) H has even bounded derivative function supported on $[0,1]$ that verifies

$$\int_{\mathbb{R}} |t|^{b_2} H'(t) dt < \infty.$$

(N1) There exist sequences of integers (u_n) and (v_n) increasing to infinity such that $(u_n + v_n) \leq n$, satisfying

(i) $v_n = o((n h_H \phi_x(h_K))^{1/2})$ and $\left(\frac{n}{h_H \phi_x(h_K)}\right)^{1/2} \alpha(v_n) \to 0$ as $n \to 0$,

(ii) $q_n v_n = o((n h_H \phi_x(h_K))^{1/2})$ and $q_n \left(\frac{n}{h_H \phi_x(h_K)}\right)^{1/2} \alpha(v_n) \to 0$ as $n \to 0$

where q_n is the largest integer such that $q_n(u_n + v_n) \leq n$.

3.3.1 Remarks on the assumptions

Remark 3.3.1. *Assumption (L0) plays an important role in our methodology. It is known as (for small h) the "concentration hypothesis acting on the distribution of X" in infinite-dimensional spaces. This assumption is not at all restrictive and overcomes the problem of the non-existence of the probability density function. In many examples, around zero the small ball probability*$\phi_x(h)$ *can be written approximately as the product of two independent functions* $\psi(z)$ *and* $\varphi(h)$ *as* $\phi_z(h) = \psi(z)\varphi(h) + o(\varphi(h))$. *This idea was adopted by Masry [39] who reformulated the Gasser* et al. *[28] one. The increasing property of* $\phi_x(.)$ *implies that* $\zeta_h^x(.)$ *is bounded and then integrable (all the more so* $\zeta_0^x(.)$ *is integrable).*
Without the differentiability of $\phi_x(.)$, *this assumption has been used by many authors where* $\psi(.)$ *is interpreted as a probability density, while* $\varphi(.)$ *may be interpreted as a volume parameter. In the case of finite-dimensional spaces, that is* $\mathcal{S} = \mathbb{R}^d$, *it can be seen that* $\phi_x(h) = C(d)h^d\psi(x) + oh^d)$, *where* $C(d)$ *is the volume of the unit ball in* $\mathbb{R}^d$. *Furthermore, in infinite dimensions, there exist many examples fulfilling the decomposition mentioned above. We quote the following (which can be found in Ferraty* et al. *[23]):*

1. $\phi_x(h) \approx \psi(h)h^\gamma$ *for som* $\gamma > 0$.

2. $\phi_x(h) \approx \psi(h)h^\gamma \exp\{C/h^p\}$ *for som* $\gamma > 0$ *and* $p > 0$.

3. $\phi_x(h) \approx \psi(h)/|\ln h|$.

The function $\beta_h^x(.)$ *which intervenes in Assumption (H4) is increasing for all fixed h. Its pointwise limit* $\beta_0^x(.)$ *also plays a determinant role. It intervenes in all asymptotic properties, in particular in the asymptotic variance term. With simple algebra, it is possible to specify this function (with* $\beta_0(u) := \beta_0^x(u)$ *in the above examples by:*

1. $\beta_0(u) = u^\gamma$,

2. $\beta_0(u) = \delta_1(u)$ *where* $\delta_1(.)$ *is Dirac function,*

3. $\beta_0(u) = \mathbf{1}_{]0,1]}(u)$.

Assumption (L2) is classical and permits to make the variance term negligible.

Remark 3.3.2. *Assumptions (L3) is a regularity condition which characterize the functional space of our model and is needed to evaluate the bias.*

Remark 3.3.3. *Assumptions (L5) and (L6) are classical in functional estimation for finite or infinite dimension spaces.*

3.4 Main results

3.4.1 Mean squared convergence

In this part we establish the L^2-consistency of $\widehat{h}^x(y)$.

Theorem 3.4.1. *Under assumptions (L0)-(L6), we have*

$$\mathbb{E}\left[\widehat{h}^x(y)-h^x(y)\right]^2 = B_n^2(x,y)+\frac{\sigma_h^2(x,y)}{nh_H\phi_x(h_K)}+o(h_H^4)+o(h_K)+o\left(\frac{1}{n\,h_H\phi_x(h_K)}\right),$$

where

$$B_n(x,y)=\frac{(B_H^f-h^x(y)B_H^F)h_H^2+(B_K^f-h^x(y)B_K^F)h_K}{1-F^x(y)}$$

with

$$\begin{aligned}
B_H^f(x,y) &= \frac{1}{2}\frac{\partial^2 f^x(y)}{\partial y^2}\int t^2H'(t)dt\\
B_K^f(x,y) &= h_K\Phi_0'(0)\frac{\left(K(1)-\int_0^1(sK(s))'\beta_x(s)ds\right)}{\left(K(1)-\int_0^1 K'(s)\beta_x(s)ds\right)}\\
B_H^F(x,y) &= \frac{1}{2}\frac{\partial^2 F^x(y)}{\partial y^2}\int t^2H'(t)dt\\
B_K^F(x,y) &= h_K\Psi_0'(0)\frac{\left(K(1)-\int_0^1(sK(s))'\beta_x(s)ds\right)}{\left(K(1)-\int_0^1 K'(s)\beta_x(s)ds\right)}.
\end{aligned}$$

and

$$\sigma_h^2(x,y)=\frac{\beta_2 h^x(y)}{(\beta_1^2(1-F^x(y))}\ \left(\textit{with } \beta_j=K^j(1)-\int_0^1(K^j)'(s)\beta_x(s)ds,\ \textit{for},\ j=1,\,2\right).$$

Proof of Theorem (3.4.1)

By using the decomposition

$$\begin{aligned}
\widehat{h}^x(y) - h^x(y) &= \frac{\widehat{f}^x(y)}{1-\widehat{F}^x(y)} - \frac{f^x(y)}{1-F^x(y)} \\
&= \frac{1}{1-\widehat{F}^x(y)}\left[(\widehat{f}^x(y) - f^x(y)) + \frac{f^x(y)}{1-F^x(y)}(\widehat{F}^x(y) - F^x(y))\right] \\
&\leq \frac{1}{1-\widehat{F}^x(y)}\left[(\widehat{f}^x(y) - f^x(y)) + \frac{\tau}{\beta}(\widehat{F}^x(y) - F^x(y))\right] \\
&\leq \left[(\widehat{f}^x(y) - f^x(y)) + \frac{\tau}{\beta}(\widehat{F}^x(y) - F^x(y))\right]
\end{aligned} \tag{3.2}$$

Therefore

$$\mathbb{E}\left[\widehat{h}^x(y) - h^x(y)\right]^2 \leq \mathbb{E}\left[(\widehat{f}^x(y) - f^x(y)) + \frac{\tau}{\beta}(\widehat{F}^x(y) - F^x(y))\right]^2 \tag{3.3}$$

We show that the proof of Theorem (3.4.1) can be deduced from the following intermediate results: □

Lemma 3.4.2. *Under the hypotheses of Theorem (3.4.1), we have*

$$\mathbb{E}\left[\widehat{f}_N^x(y)\right] - f^x(y) = B_H^f(x,y)h_H^2 + B_K^f(x,y)h_K + o(h_H^2) + o(h_K)$$

and

$$\mathbb{E}\left[\widehat{F}_N^x(y)\right] - F^x(y) = B_H^F(x,y)h_H^2 + B_K^F(x,y)h_K + o(h_H^2) + o(h_K).$$

Remark 3.4.3. *Observe that, the result of this lemma permits to write*

$$\left[\mathbb{E}\widehat{F}_N^x(y) - F^x(y)\right] = \mathcal{O}(h_H^2) + \mathcal{O}(h_K)$$

and

$$\left[\mathbb{E}\widehat{f}_N^x(y) - f^x(y)\right] = \mathcal{O}(h_H^2) + \mathcal{O}(h_K).$$

Lemma 3.4.4. *Under the hypotheses of Theorem (3.4.1), we have*

$$Var\left[\widehat{f}_N^x(y)\right] = \frac{\sigma_f^2(x,y)}{nh_H\phi_x(h_K)} + \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right),$$

$$Var\left[\widehat{F}_N^x(y)\right] = \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right)$$

and

$$Var\left[\widehat{F}_D^x\right] = \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right).$$

where $\sigma_f^2(x,y) := f^x(y)\int H'^2(t)dt$.

Lemma 3.4.5. *Under the hypotheses of Theorem (3.4.1), we have*

$$Cov(\widehat{f}_N^x(y), \widehat{F}_D^x) = \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right),$$

$$Cov(\widehat{f}_N^x(y), \widehat{F}_N^x(y)) = \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right)$$

and

$$Cov(\widehat{f}_D^x, \widehat{F}_N^x(y)) = \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right).$$

Remark 3.4.6. *It is clear that, the results of Lemmas (3.4.4 and 3.4.5) allows to write*

$$Var\left[\widehat{F}_D^x - \widehat{F}_N^x\right] = \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right)$$

3.4.2 Asymptotic normality

This section contains results on the asymptotic normality of $\widehat{h}^x(y)$.

Theorem 3.4.7. *Assume that (L0)-(L6) and (N1) hold, and if the following inequalities*

$$\exists \eta > 0,\ C,\ C' > 0 \text{ such that } C\, n^{\frac{3-a}{a+1}+\eta} \leq h_H\, \phi_x(h_K) \text{ and } \phi_x(h_K) \leq C' n^{\frac{1}{1-a}} \quad (3.4)$$

are verified with $a > (5+\sqrt{17})/2$*, then we have for any* $x \in \mathcal{A}$*,*

$$\left(\frac{nh_H\phi_x(h_K)}{\sigma_h^2(x,y)}\right)^{1/2}\left(\widehat{h}^x(y) - h^x(y) - B_n(x,y)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty.$$

where

$$\mathcal{A} = \{x \in \mathcal{F},\ f^x(y)(1 - F^x(y)) \neq 0\}$$

and $\xrightarrow{\mathcal{D}}$ *means the convergence in distribution.*

Evidently, if one imposes some additional assumptions on the function $\phi_x(\cdot)$ and the bandwidth parameters (h_K and h_H) our asymptotic normality can be improved by removing the bias term $B_n(x,y)$.

Corollary 3.4.8. *Under the hypotheses of Theorem (3.4.7) and if the bandwidth parameters (*h_K *and* h_H*) and if the function* $\phi_x(h_K)$ *satisfies:*

$$\lim_{n\to\infty} (h_H^2 + h_K)\sqrt{n\phi_x(h_K)} = 0$$

we have

$$\left(\frac{nh_H\phi_x(h_K)}{\sigma_h^2(x,y)}\right)^{1/2}\left(\widehat{h}^x(y) - h^x(y)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty.$$

Proof of Theorem and Corollary Consider the decomposition

$$\begin{aligned}
\widehat{h}^x(y) - h^x(y) \quad &= \quad \frac{1}{\widehat{F}_D^x - \widehat{F}_N^x(y)}\left(\widehat{f}_N^x(y) - E\widehat{f}_N^x(y)\right) \\
&+ \frac{1}{\widehat{F}_D^x - \widehat{F}_N^x(y)}\left\{h^x(y)\left(\mathbb{E}\widehat{F}_N^x(y) - F^x(y)\right) + \left(\mathbb{E}\widehat{f}_N^x(y) - f^x(y)\right)\right\} \\
&+ \frac{h^x(y)}{\widehat{F}_D^x - \widehat{F}_N^x(y)}\left\{1 - \mathbb{E}\widehat{F}_N^x(y) - \left(\widehat{F}_D^x - \widehat{F}_N^x(y)\right)\right\} \qquad (3.5)
\end{aligned}$$

Therefore, Theorem (3.4.7) and Corollary (3.4.8) are a consequence of Lemma (3.4.2), remark (3.4.3) and the following results. □

Lemma 3.4.9. *Under the hypotheses of Theorem (3.4.7)*

$$\left(\frac{nh_H\phi_x(h_K)}{\sigma_f^2(x,y)}\right)^{1/2}\left(\widehat{f}_N^x(y)-\mathbb{E}\left[\widehat{f}_N^x(y)\right]\right)\to\mathcal{N}(0,1).$$

Lemma 3.4.10. *Under the hypotheses of Theorem 3.4.7*

$$\widehat{F}_D^x-\widehat{F}_N^x(y)\to 1-F^x(y) \qquad \textit{in probability}$$

and

$$\left(\frac{nh_H\phi_x(h_K)}{\sigma_h^2(x,y)}\right)^{1/2}\left(\widehat{F}_D^x-\widehat{F}_N^x(y)-1+\mathbb{E}[\widehat{F}_N^x(y)]\right)=\mathcal{O}_{\mathbb{P}}(1).$$

3.5 Appendix

In the following, we will denote $\forall i$

$$K_i=K(h_H^{-1}d(x,X_i)),\quad H_i=H(h_H^{-1}(y-Y_i)\quad \text{and}\quad H_i'=H'(h_H^{-1}(y-Y_i).$$

Proof of Lemma (3.4.2)

Firstly, for $\mathbb{E}[\widehat{f}_N^x(y)]$, we start by writing

$$\mathbb{E}[\widehat{f}_N^x(y)]=\frac{1}{\mathbb{E}[K_1]}\mathbb{E}\left[K_1\mathbb{E}[h_H^{-1}H_1'|X]\right]\ \text{with}\ h_H^{-1}\mathbb{E}\left[H_1'|X\right]=\int_{\mathbb{R}}H'(t)f^x(y-h_Ht)dt.$$

The latter can be re-written, by using a Taylor expansion under (L3), as follows

$$h_H^{-1}\mathbb{E}[H_1'|X]=f^x(y)+\frac{h_H^2}{2}\left(\int t^2H'(t)dt\right)\frac{\partial^2 f^x(y)}{\partial^2 y}+o(h_H^2).$$

Thus, we get

$$\mathbb{E}\left[\widehat{f}_N^x(y)\right]=\frac{1}{\mathbb{E}[K_1]}\left(\mathbb{E}\left[K_1f^x(y)\right]+\left(\int t^2H'(t)dt\right)\mathbb{E}\left[K_1\frac{\partial^2 f^x(y)}{\partial^2 y}\right]+o(h_H^2)\right).$$

Let $\psi_l(\cdot,y) := \frac{\partial^l f^{\cdot}(y)}{\partial^l y}$: for $l \in \{0,2\}$, since $\Phi_l(0)=0$, we have

$$\begin{aligned}\mathbb{E}\left[K_1\psi_l(X,y)\right] &= \psi_l(x,y)\mathbb{E}[K_1] + \mathbb{E}\left[K_1\left(\psi_l(X,y)-\psi_l(x,y)\right)\right] \\ &= \psi_l(x,y)\mathbb{E}[K_1] + \mathbb{E}\left[K_1\left(\Phi_l(d(x,X)\right)\right] \\ &= \psi_l(x,y)\mathbb{E}[K_1] + \Phi_l'(0)\mathbb{E}\left[d(x,X)K_1\right] + o(\mathbb{E}\left[d(x,X)K_1\right]).\end{aligned}$$

So,

$$\begin{aligned}\mathbb{E}\left[\widehat{f}_N^x(y)\right] \quad &= \quad f^x(y) + \frac{h_H^2}{2}\frac{\partial^2 f^x(y)}{\partial y^2}\int t^2 H'(t)dt + o\left(h_H^2\frac{\mathbb{E}\left[d(x,X)K_1\right]}{\mathbb{E}[K_1]}\right) \\ &\quad + \Phi_0'(0)\frac{E\left[d(x,X)K_1\right]}{\mathbb{E}[K_1]} + o\left(\frac{\mathbb{E}\left[d(x,X)K_1\right]}{E[K_1]}\right).\end{aligned}$$

Similarly to Ferraty *et al.* [23] we show that

$$\frac{1}{\phi_x(h_K)}\mathbb{E}\left[d(x,X)K_1\right] = h_K\left(K(1) - \int_0^1 (sK(s))'\beta_x(s)ds + o(1)\right)$$

and

$$\frac{1}{\phi_x(h_K)}\mathbb{E}\left[K_1\right] = K(1) - \int_0^1 K'(s)\beta_x(s)ds + o(1).$$

Hence,

$$\begin{aligned}\mathbb{E}\left[\widehat{f}_N^x(y)\right] \quad &= \quad f^x(y) + \frac{h_H^2}{2}\frac{\partial^2 f^x(y)}{\partial y^2}\int t^2 H'(t)dt \\ &\quad + h_K\Phi_0'(0)\frac{\left(K(1) - \int_0^1 (sK(s))'\beta_x(s)ds\right)}{\left(K(1) - \int_0^1 K'(s)\beta_x(s)ds\right)} + o(h_H^2) + o(h_K).\end{aligned}$$

Secondly, concerning $\mathbb{E}[\widehat{F}_N^x(y)]$, we write by an integration by part

$$\mathbb{E}[\widehat{F}_N^x(y)] = \frac{1}{\mathbb{E}[K_1]}\mathbb{E}\left[K_1 E[H_1|X]\right] \text{ with } \mathbb{E}\left[H_1|X\right] = \int_{\mathbb{R}} H'(t)F^x(y-h_H t)dt.$$

The same steps used in studying $\mathbb{E}[\widehat{f}_N^x(y)]$ can be followed to prove that

$$\begin{aligned}\mathbb{E}\left[\widehat{F}_N^x(y)\right] &= F^x(y) + \frac{h_H^2}{2}\frac{\partial^2 F^x(y)}{\partial y^2}\int t^2 H'(t)dt \\ &+h_K\Psi_0'(0)\frac{\left(K(1)-\int_0^1 (sK(s))'\beta_x(s)ds\right)}{\left(K(1)-\int_0^1 K'(s)\beta_x(s)ds\right)} + o(h_H^2) + o(h_K).\end{aligned}$$

□

Proof of Lemma (3.4.4) For the first quantity $Var[\widehat{f}_N^x(y)]$, we have

$$s_n^2 = Var[\widehat{f}_N^x(y)] = \frac{1}{(nh_H\mathbb{E}\left[K_1(x)\right])^2}Var\left[\sum_{i=1}\Gamma_i(x)\right]$$

where

$$\Gamma_i(x) = K_i(x)H_i'(y) - \mathbb{E}\left[K_i(x)H_i'(y)\right].$$

Thus

$$\begin{aligned}Var[\widehat{f}_N^x(y)] &= \frac{1}{(nh_H\mathbb{E}\left[K_1\right])^2}\underbrace{\sum_{i\neq j}Cov\left(\Gamma_i(x),\Gamma_j(x)\right)}_{s_n^{cov}} + \underbrace{\sum_{i=1}^{n}Var\left(\Gamma_i(x)\right)}_{s_n^{var}} \\ &= \frac{1}{n(h_H\mathbb{E}\left[K_1\right])^2}Var\left[\Gamma_1\right] + \frac{1}{(nh_H\mathbb{E}\left[K_1\right])^2}\sum_{i\neq j}Cov(\Gamma_i,\Gamma_j).\end{aligned}$$

Let us calculate the quantity $Var\left[\Gamma_1(x)\right]$. We have:

$$\begin{aligned}Var\left[\Gamma_1(x)\right] &= \mathbb{E}\left[K_1^2(x)H_1'^2(y)\right] - \left(\mathbb{E}\left[K_1(x)H_1'(y)\right]\right)^2 \\ &= \mathbb{E}\left[K_1^2(x)\right]\frac{\mathbb{E}\left[K_1^2(x)H_1'^2(y)\right]}{\mathbb{E}\left[K_1^2(x)\right]} - \left(\mathbb{E}\left[K_1(x)\right]\right)^2\left(\frac{\mathbb{E}\left[K_1(x)H_1'(y)\right]}{\mathbb{E}\left[K_1(x)\right]}\right)^2.\end{aligned}$$

So, by using the same arguments as those used in pervious lemma we get

$$\frac{1}{\phi_x(h_K)}\mathbb{E}\left[K_1^2(x)\right] = K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds + o(1)$$

$$\frac{\mathbb{E}\left[K_1^2(x)H_1'^2(y)\right]}{\mathbb{E}\left[K_1^2(x)\right]} = h_H f^x(y)\int H'^2(t)dt + o(h_H)$$

$$\frac{\mathbb{E}[K_1(x)H_1'(y)]}{\mathbb{E}\left[K_1(x)\right]} = h_H f^x(y) + o(h_H)$$

which implies that

$$Var\left[\Gamma_i(x)\right] = h_H\phi_x(h_K)f^x(y)\int H'^2(t)dt\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s))ds\right) + o\left(h_H\phi_x(h_K)\right). \tag{3.6}$$

Now, let us focus on the covariance term. To do that, we need to calculate the asymptotic behavior of quantity defined as

$$\sum_{i\neq j}\left|Cov(\Gamma_i(x),\Gamma_j(x))\right| = \sum_{1\leq|i-j|\leq c_n}\left|Cov(\Gamma_i(x),\Gamma_j(x))\right| = J_{1,n} + J_{2,n}.$$

with $c_n \to \infty$, as $n \to \infty$.
For all (i, j) we write

$$Cov\left(\Gamma_i(x),\Gamma_j(x)\right) = \mathbb{E}\left[K_i(x)K_j(x)H_i'(y)H_j'(y)\right] - \left(\mathbb{E}\left[K_i(x)H_i'(y)\right]\right)^2$$

and we use the fact that

$$\mathbb{E}\left[H_i'(y)H_j'(y)|(X_i,X_j)\right] = \mathcal{O}(h_H^2);\ \forall\, i\neq j,\ \ \mathbb{E}\left[H_i'(y)|X_i\right] = \mathcal{O}(h_H);\ \forall\, i.$$

For $J_{1,n}$: by means of the integral realized above and under (L2) and (L5), we get

$$\mathbb{E}\left[K_iK_jH_i'H_j'\right] \leq Ch_H^2\mathbb{P}\left[(X_i,X_j)\in B(x,h_K)\times B(x,h_K)\right]$$

and

$$\mathbb{E}\left[K_i(x)H_i'(y)\right] \leq Ch_H\mathbb{P}\left(X_i\in B(x,h_K)\right).$$

It follows that, the hypothesis (L0), (L2) and (L5), imply

$$Cov\left(\Gamma_i(x), \Gamma_j(x)\right) \leq C h_H^2 \phi_x(h_K) \left(\phi_x(h_K) + \left(\frac{\phi_x(h_K)}{n}\right)^{1/a}\right)$$

So

$$J_{1,n} \leq C \left(n c_n h_H^2 \left(\frac{\phi_x(h_K)}{n}\right)^{1/a} \phi_x(h_K)\right).$$

Hence

$$J_{1,n} = \mathcal{O}\left(n c_n h_H^2 \left(\frac{\phi_x(h_K)}{n}\right)^{1/a} \phi_x(h_K)\right).$$

On the other hand, these covariances can be controlled by mean of the usual Davydov-Rios's covariance inequality for mixing processes (see Rio 2000, formula 1.12a). Together with (L1), this inequality leads to:

$$\forall i \neq j,\ \left|Cov(D_i(x), D_j(x))\right| \ \leq \ C\,|i-j|^{-a}.$$

By the fact, $\displaystyle\sum_{k \geq c_n+1} k^{-a} \leq \int_{c_n}^{\infty} t^{-a} dt = \frac{c_n^{-a+1}}{a-1}$, we get by applying (L1),

$$J_{2,n} \leq \sum_{|i-j| \geq c_n+1} |i-j|^{-a} \leq \frac{n c_n^{-a+1}}{a-1}$$

Thus, by using the following classical technique (see Bosq, 1998 [8]), we can write

$$s_n^{cov} \ = \ \sum_{0<|i-j| \leq u_n} \left|Cov(\Gamma_i(x), \Gamma_j(x))\right| + \sum_{|i-j|>u_n} \left|Cov(\Gamma_i(x), \Gamma_j(x))\right|.$$

Thus

$$s_n^{cov} \ \leq \ Cn \left(c_n h_H^2 \left(\frac{\phi_x(h_K)}{n}\right)^{1/a} \phi_x(h_K) + \frac{c_n^{-a+1}}{a-1}\right)$$

Choosing $c_n = h_H^{-2} \left(\frac{\phi_x(h_K)}{n}\right)^{-1/a}$, and owing to the right inequality in (3.4), we can deduce

$$s_n^{cov} \ = \ o\left(n h_H \phi_x(h_K)\right). \tag{3.7}$$

Finally,

$$\begin{aligned} s_n^2 &= o\left(nh_H\phi_x(h_K)\right) + \mathcal{O}\left(nh_H\phi_x(h_K)\right) \\ &= \mathcal{O}\left(nh_H\phi_x(h_K)\right) \end{aligned}$$

In conclusion, we have

$$Var[\widehat{f}_N^x(y)] = \frac{f^x(y)}{nh_H\phi_x(h_K)}\left(\int H'^2(t)dt\right)\left(\frac{\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right)}{\left(K(1) - \int_0^1 K'(s)\beta_x(s)ds\right)^2}\right) + o\left(\frac{1}{nh_H\phi_x(h_K)}\right) \tag{3.8}$$

Now, for $\widehat{F}_N^x(y)$, (resp. $\widehat{F}_D^x$) we replace $H_i'(y)$ by $H_i(y)$ (resp. by 1) and we follow the same ideas, under the fact that $H \leq 1$

$$Var[\widehat{F}_N^x(y)] = \frac{F^x(y)}{n\phi_x(h_K)}\left(\int H'^2(t)dt\right)\left(\frac{\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right)}{\left(K(1) - \int_0^1 K'(s)\beta_x(s)ds\right)^2}\right) + o\left(\frac{1}{n\phi_x(h_K)}\right).$$

and

$$Var[\widehat{F}_D^x] = \frac{1}{n\phi_x(h_K)}\left(\frac{\left(K^2(1) - \int_0^1 (K^2(s))'\beta_x(s)ds\right)}{\left(K(1) - \int_0^1 K'(s)\beta_x(s)ds\right)^2}\right) + o\left(\frac{1}{n\phi_x(h_K)}\right).$$

This yields the proof.

□

Proof of Lemma (3.4.5)

The proof of this lemma follows the same steps as the previous Lemma. For this, we keep the same notation and we write

$$\begin{aligned} Cov(\widehat{f}_N^x(y), \widehat{F}_N^x(y)) &= \frac{1}{nh_H(\mathbb{E}\left[K_1(x)\right])^2} Cov\left(\Gamma_1(x), \Delta_1(x)\right) \\ &+ \frac{1}{n^2h_H(\mathbb{E}\left[K_1(x)\right])^2} \sum_{i \neq j} Cov(\Gamma_i(x), \Delta_j(x)) \end{aligned}$$

where

$$\Delta_i(x) = Ki(x)H_i(y) - \mathbb{E}\left[Ki(x)H_i(y)\right].$$

For the first term, we have under (L4)

$$\begin{aligned} Cov\left(\Gamma_1(x), \Delta_1(x)\right) &= \mathbb{E}[K_1^2(x)H_1(y)H_1'(y)] - \mathbb{E}[K_1(x)H_1(y)]\mathbb{E}[K_1(x)H_1'(y)] \\ &= \mathcal{O}(h_H\phi_x(h_K)) + \mathcal{O}(h_H\phi_x^2(h_K)) \\ &= \mathcal{O}(h_H\phi_x(h_K)). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{nh_H(\mathbb{E}\left[K_1(x)\right])^2} Cov\left(\Gamma_1(x), \Delta_1(x)\right) &= \mathcal{O}\left(\frac{1}{n\phi_x(h_K)}\right) \\ &= \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right). \end{aligned} \tag{3.9}$$

So, by using similar arguments as those invoked in the proof of Lemma (4.3.5), and we use once again the boundedness of K and H, and the fact that (L1) and (L6) imply that

$$\mathbb{E}\left(H_i'(y)|X_i\right) = \mathcal{O}(h_H).$$

Moreover, the right part of (L7b) implies that

$$Cov\left(\Gamma_i(x), \Delta_j(x)\right) = \mathcal{O}\left(h_H\phi_x(h_K)\left(\frac{\phi_x(h_K)}{n}\right)^{1/a} + \phi_x(h_K)\right),$$

Meanwhile, using the Davydov-Rio's inequality in Rio (2000) for mixing processes leads to

$$|Cov\left(\Gamma_i(x), \Delta_j(x)\right)| \le C\alpha\left(|i-j|\right) \le C|i-j|^{-a},$$

we deduce easily that for any $c_n > 0$:

$$\begin{aligned} \sum_{i \neq j} Cov\left(\Gamma_i(x), \Delta_j(x)\right) &= \mathcal{O}\left(n\, c_n\, h_H\phi_x(h_K)\left(\frac{\phi_x(h_K)}{n}\right)^{1/a} + \phi_x(h_K)\right) \\ &+ \mathcal{O}\left(n\, h_H\, c_n^{-a}\right). \end{aligned}$$

It suffices now to take $c_n = h_H^{-1}\left(\frac{\phi_x(h_K)}{n}\right)^{-1/a}$ to get the following expression for the sum of the covariances:

$$\sum_{i\neq j} Cov\left(\Gamma_i(x), \Delta_j(x)\right) \;=\; o\left(n\,\phi_x(h_K)\right). \tag{3.10}$$

From (3.9) and (3.10) we deduce that

$$Cov(\widehat{f}_N^x(y), \widehat{F}_N^x(y)) = \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right).$$

The same arguments can be used to shows that

$$Cov(\widehat{f}_N^x(y), \widehat{F}_D^x) = \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right) \quad \text{and} \quad Cov(\widehat{F}_N^x(y), \widehat{F}_D^x) = \mathcal{O}\left(\frac{1}{nh_H\phi_x(h_K)}\right).$$

□

Proof of Lemma (3.4.9)
Let

$$S_n = \sum_{i=1}^{n} \Lambda_i(x)$$

where

$$\Lambda_i(x) := \frac{\sqrt{h_H\phi_x(h_K)}}{h_H\mathbb{E}[K_1(x)]}\Gamma_i(x). \tag{3.11}$$

Obviously, we have

$$\sqrt{nh_H\phi_x(h_K)}\,[\sigma_f(x,y)]^{-1}\left(\widehat{f}_N^x(y) - E\widehat{f}_N^x(y)\right) = \left(n(\sigma_f(x,y))^2\right)^{-1/2} S_n.$$

Thus, the asymptotic normality of $\left(n(\sigma_f(x,y))^2\right)^{-1/2} S_n$, is sufficient to show the proof of this Lemma. This last is shown by the blocking method, where the random variables Λ_i are grouped into blocks of different sizes defined.

We consider the classical big- and small-block decomposition. We split the set $\{1, 2, \ldots, n\}$ into $2k_n + 1$ subsets with large blocks of size u_n and small blocks of size v_n and put

$$k_n \;:=\; \Big[\frac{n}{u_n + v_n}\Big].$$

Assumption (N1)(ii) allows us to define the large block size by

$$u_n \;=:\; \Big[\Big(\frac{nh_H\phi_x(h_K)}{q_n}\Big)^{1/2}\Big].$$

Using Assumption (N1) and simple algebra allows us to prove that

$$\frac{v_n}{u_n} \to 0, \quad \frac{u_n}{n} \to 0, \quad \frac{u_n}{\sqrt{nh_H\phi_x(h_K)}} \to 0, \quad \text{and} \quad \frac{n}{u_n}\alpha(v_n) \to 0 \tag{3.12}$$

Now, let Υ_j, Υ_j' and Υ_j'' be defined as follows:

$$\Upsilon_j \;=\; \sum_{i=j(u+v)+1}^{j(u+v)+u} \Lambda_i(x), \quad 0 \le j \le k+1,$$

$$\Upsilon_j' \;=\; \sum_{i=j(u+v)+u+1}^{(j+1)(u+v)+u} \Lambda_i(x), \quad 0 \le j \le k+1,$$

$$\Upsilon_j'' \;=\; \sum_{i=k(u+v)+1}^{n} \Lambda_i(x), \quad 0 \le j \le k+1.$$

Clearly, we can write

$$S_n \;=\; \sum_{j=0}^{k-1} \Upsilon_j + \sum_{j=0}^{k-1} \Upsilon_j' + \Upsilon_k'' r \;=:\; S_n' + S_n'' + S_n'''.$$

We prove that

$$(i)\ \frac{1}{n}\mathbb{E}(S_n'')^2 \longrightarrow 0, \quad (ii)\ \frac{1}{n}\mathbb{E}(S_n''')^2 \longrightarrow 0, \tag{3.13}$$

$$\left|\mathbb{E}\left\{\exp\left(itn^{-1/2}S_n'\right)\right\} - \prod_{j=0}^{k-1}\mathbb{E}\left\{\exp\left(itn^{-1/2}\Upsilon_j\right)\right\}\right| \longrightarrow 0, \tag{3.14}$$

$$\frac{1}{n}\sum_{j=0}^{k-1}\mathbb{E}\left(\Upsilon_j^2\right) \longrightarrow \sigma_f^2(x,y), \tag{3.15}$$

$$\frac{1}{n}\sum_{j=0}^{k-1}\mathbb{E}\left(\Upsilon_j^2\mathbf{1}_{\{|\Upsilon_j|>\varepsilon\sqrt{n\sigma_f^2(x,y)}\}}\right) \longrightarrow 0 \tag{3.16}$$

for every $\varepsilon > 0$.

Expression (3.13) show that the terms S_n'' and S_n''' are negligible, while Equations (3.14) and (3.15) show that the Υ_j are asymptotically independent, verifying that the sum of their variances tends to $\sigma_f^2(x,y)$. Expression (3.16) is the Lindeberg-Feller's condition for a sum of independent terms. Asymptotic normality of S_n is a consequence of Equations (3.13)-(3.16).

- **Proof of (3.13)** Because $\mathbb{E}(\Lambda_j) = 0,\ \forall j$, we have that

$$\mathbb{E}(S_n'')^2 = Var\left(\sum_{j=0}^{k-1}\Upsilon_j'\right) = \sum_{j=0}^{k-1}Var\left(\Upsilon_j'\right) + \sum_{0\le i<j\le k-1}Cov\left(\Upsilon_i',\Upsilon_j'\right) := \Pi_1 + \Pi_2.$$

By the second-order stationarity we get

$$\begin{aligned} Var\left(\Upsilon_j'\right) &= Var\left(\sum_{i=j(u_n+v_n)+u_n+1}^{(j+1)(u_n+v_n)}\Lambda_i(x)\right) \\ &= v_n Var(\Lambda_1(x)) + \sum_{i\neq j}^{v_n} Cov\left(\Lambda_i(x),\Lambda_j(x)\right). \end{aligned}$$

Then

$$\begin{aligned}
\frac{\Pi_1}{n} &= \frac{kv_n}{n} Var(\Lambda_1(x)) + \frac{1}{n}\sum_{j=0}^{k-1}\sum_{i\neq j}^{v_n} Cov\left(\Lambda_i(x), \Lambda_j(x)\right) \\
&\leq \frac{kv_n}{n}\left\{\frac{\phi_x(h_K)}{h_H \mathbb{E}^2 K_1(x)} Var\left(\Gamma_1(x)\right)\right\} + \frac{1}{n}\sum_{i\neq j}^{n}\Big|Cov\left(\Lambda_i(x), \Lambda_j(x)\right)\Big| \\
&\leq \frac{kv_n}{n}\left\{\frac{1}{h_H \phi_x(h_K)} Var\left(\Lambda_1(x)\right)\right\} + \frac{1}{n}\sum_{i\neq j}^{n}\Big|Cov\left(\Lambda_i(x), \Lambda_j(x)\right)\Big|.
\end{aligned}$$

Simple algebra gives us

$$\frac{kv_n}{n} \cong \left(\frac{n}{u_n + v_n}\right)\frac{v_n}{n} \cong \frac{v_n}{u_n + v_n} \cong \frac{v_n}{u_n} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Using Equation (3.7) we have

$$\lim_{n\to\infty} \frac{\Pi_1}{n} = 0. \tag{3.17}$$

Now, let us turn to Π_2/n. We have

$$\begin{aligned}
\frac{\Pi_2}{n} &= \frac{1}{n}\sum_{i=0_{i\neq j}}^{k-1}\sum_{j=0}^{k-1} Cov\left(\Upsilon_i(x), \Upsilon_j(x)\right) \\
&= \frac{1}{n}\sum_{i=0_{i\neq j}}^{k-1}\sum_{j=0}^{k-1}\sum_{l_1=1}^{v_n}\sum_{l_2}^{v_n} Cov\left(\Lambda_{m_j+l_1}, \Lambda_{m_j+l_2}\right),
\end{aligned}$$

with $m_i = i(u_n + v_n) + v_n$. As $i \neq j$, we have $|m_i - m_j + l_1 - l_2| \geq u_n$. It follows that

$$\frac{\Pi_2}{n} \leq \frac{1}{n}\sum_{i=1}^{n}\sum_{|i-j|\geq u_n}\sum_{j=1}^{n} Cov\left(\Lambda_i(x), \Lambda_j(x)\right),$$

then

$$\lim_{n\to\infty} \frac{\Pi_2}{n} = 0. \tag{3.18}$$

By Equations (3.17) and (3.18) we get Part(i) of the Equation(3.13).

We turn to (ii), we have

$$\begin{aligned}\frac{1}{n}\mathbb{E}\left(S_n'''\right)^2 &= \frac{1}{n}Var\left(\Upsilon_k''\right)\\ &= \frac{\vartheta_n}{n}Var\left(\Lambda_1(x)\right)+\frac{1}{n}\sum_{i=1}^{\vartheta_n}\sum_{i\neq j}\sum_{j=1}^{\vartheta_n}Cov\left(\Lambda_i(x),\Lambda_j(x)\right),\end{aligned}$$

where $\vartheta_n = n - k_n(u_n + v_n)$; by the definition of k_n, we have $\vartheta_n \leq u_n + v_n$. Then

$$\frac{1}{n}\mathbb{E}\left(S_n'''\right)^2 \leq \frac{u_n+v_n}{n}Var\left(\Lambda_1(x)\right)+\frac{1}{n}\sum_{i=1}^{\vartheta_n}\sum_{i\neq j}\sum_{j=1}^{\vartheta_n}Cov\left(\Lambda_i(x),\Lambda_j(x)\right)$$

and by the definition of u_n and v_n we achieve the proof of (ii) of Equation (3.13).

- **Proof of (3.14)** We make use of Volkonskii and Rozanov's lemma (see the appendix in Masry, 2005) and the fact that the process $(X_i, X_j)$is strong mixing. Note that Υ_a is $\mathcal{F}_{i_a}^{j_a}$-mesurable with $i_a = a(u_n+v_n)+1$ and $j_a = a(u_n+v_n)+u_n$; hence, with $V_j = \exp\left(itn^{-1/2}\Upsilon_j\right)$ we have

$$\left|\mathbb{E}\left\{\exp\left(itn^{-1/2}S_n'\right)\right\}-\prod_{j=0}^{k-1}\mathbb{E}\left\{\exp\left(itn^{-1/2}\Upsilon_j\right)\right\}\right| \leq 16k_n\alpha(v_n+1) \cong \frac{n}{v_n}\alpha(v_n+1)$$

which goes to zero by the last part of Equation (3.12). Now we establish Equation (3.15).

- **Proof of (3.15)** Note that $Var(S_n') \longrightarrow \sigma_f^2(x,y)$ by Equation (3.13) and since $Var(S_n') \longrightarrow \sigma_f^2(x,y)$ (by the definition of the Λ_i and Equation (3.8)). Then because

$$\mathbb{E}\left(S_n'\right)^2 = Var\left(S_n'\right) = \sum_{j=0}^{k-1}Var\left(\Upsilon_j\right)+\sum_{i=0}^{k-1}\sum_{i\neq j}\sum_{j=0}^{k-1}Cov\left(\Upsilon_i,\Upsilon_j\right),$$

all we have to prove is that the double sum of covariances in the last equation tends to zero. Using the same arguments as those previously used for Π_2 in the proof of first term of Equation (3.13)we obtain by replacing v_n by u_n we get

$$\frac{1}{n}\sum_{j=0}^{k-1}\mathbb{E}\left(\Upsilon_j^2 5\right) = \frac{ku_n}{n}Var\left(\Lambda_1\right) + \mathcal{O}(1).$$

As $Var\left(\Lambda_1\right) \longrightarrow \sigma_f^2(x,y)$ and $ku_n/n \longrightarrow 1$, we get the result.

Finally, we prove Equation (3.16).

- **Proof of (3.16)** Recall that

$$\Upsilon_j = \sum_{i=j(u_n+v_n)+1}^{j(u_n+v_n)+u_n}\Lambda_i.$$

Making use Assumptions (L5) and (L6), we have

$$\left|\Lambda_i\right| \leq C\left(h_H\phi_x(h_K)\right)^{-1/2}$$

thus

$$\left|\Upsilon_j\right| \leq Cu_n\left(h_H\phi_x(h_K)\right)^{-1/2},$$

which goes to zero as n goes to infinity by Equation (3.12). Then for n large enough, the set $\left\{|\Upsilon_j| > \varepsilon\left(n\sigma_f^2(x,y)\right)^{-1/2}\right\}$ becomes empty, this completes the proof and therefore that of the asymptotic normality of $\left(n(\sigma_f(x,y))^2\right)^{-1/2}S_n$,

□

Proof of Lemma (3.4.10):

It is clear that, the result of Lemma (3.4.2) and Lemma (3.4.4) permits us

$$\mathbb{E}\left(\widehat{F}_D^x - \widehat{F}_N^x - 1 + F^x(y)\right) \longrightarrow 0$$

and

$$Var\left(\widehat{F}_D^x - \widehat{F}_N^x - 1 + F^x(y)\right) \longrightarrow 0$$

then

$$\widehat{F}_D^x - \widehat{F}_N^x - 1 + F^x(y) \xrightarrow{\mathbb{P}} 0.$$

Moreover, the asymptotic variance of $\widehat{F}_D^x - \widehat{F}_N^x$ given in remark (3.4.6) allows to obtain

$$\frac{nh_H \phi_x(h_K)}{\sigma_h(x,y)^2} Var\left(\widehat{F}_D^x - \widehat{F}_N^x - 1 + \mathbb{E}\left(\widehat{F}_N^x(y)\right)\right) \longrightarrow 0.$$

By combining result with the fact that

$$\mathbb{E}\left(\widehat{F}_D^x - \widehat{F}_N^x - 1 + \mathbb{E}\left(\widehat{F}_N^x(y)\right)\right) = 0$$

we obtain the claimed result. □

Chapter 4

Nonparametric estimation of the maximum of conditional hazard function under dependence conditions for functional data

This chapter[44] is the subject of a publication in Journal of Afrika Statistika

4.1 Introduction

The statistical analysis of functional data studies the experiments whose results are generally the curves. Under this supposition, the statistical analysis focuses on a framework of infinite dimension for the data under study. This field of modern statistics has received much attention in the last 20 years, and it has been popularised in the book of Ramsay and Silverman [49]. This type of data appears in many fields of applied statistics: environmetrics (Damon and Guillas,[14]), chemometrics (Benhenni et al., [3]), meteorological sciences (Besse et al.,[6]), etc.

From a theoretical point of view, a sample of functional data can be involved in many different statistical problems, such as: classification and principal components analysis (PCA) (1986,1991) or longitudinal studies, regression and prediction (Benhenni et al., [3]; Cardo et al., [10]). The recent monograph by Ferraty and Vieu [26] summarizes many of their contributions to the non-parametric estimation with functional data; among other properties, consistency of the conditional density, conditional distribution and regression estimates are established in the i.i.d. case under dependence conditions (strong mixing). Almost complete rates of convergence are also obtained, and different techniques are applied to several examples of functional data samples. Related work can be seen in the paper of Masry [39], where the asymptotic normality of the functional nonparametric regression estimate is proven, considering strong mixing dependence conditions for the sample data. For automatic smoothing parameter selection in the regression setting, see Rachdi and Vieu [45].

4.1.1 Hazard and conditional hazard

The estimation of the hazard function is a problem of considerable interest, especially to inventory theorists, medical researchers, logistics planners, reliability engineers and seismologists. The non-parametric estimation of the hazard function has been extensively discussed in the literature. Beginning with Watson and Leadbetter (1964), there are many papers on these topics: Ahmad [16], Singpurwalla and Wong [55], etc.We can cite Quintela [41] for a survey.

The literature on the estimation of the hazard function is very abundant, when observations are vectorial. Cite, for instance, Watson and Leadbetter (1964), Roussas [54], Lecoutre and Ould-Saïd [34], Estèvez et al. [19] and Quintela-del-Rio [40] for recent references. In all these works the authors consider independent observations or dependent data from time series. The first results on the nonparametric estimation of this model, in functional statistics were obtained by Ferraty et al. [25]. They studied the almost complete convergence of a kernel estimator for hazard function of a real random variable dependent on a functional predictor. Asymptotic normality of the latter estimator was obtained, in the case of α- mixing, by Quintela-del-Rio [42]. We refer to Ferraty *et al.* [22] and Mahhiddine *et al.* [37] for uniform almost complete convergence of the functional component of this nonparametric model. When hazard rate estimation is performed with multiple variables, the result is an estimate of the conditional hazard rate for the first variable, given the levels of the remaining variables. Many references, practical examples and simulations in the case of nonparametric estimation using local linear approximations can be found in Spierdijk [56].

Our chapter presents some asymptotic properties related with the non-parametric estimation of the maximum of the conditional hazard function. In a functional data setting, the conditioning variable is allowed to take its values in some abstract semi-metric space. In this case, Ferraty *et al.* [24] define non-parametric estimators of the conditional density and the conditional distribution. They give the rates of convergence (in an almost complete sense) to the corresponding functions, in a dependence (α-mixing) context. We extend their results by calculating the maximum of the conditional hazard function of these estimates, and establishing their asymptotic normality, considering a particular type of kernel for the functional part of the estimates. Because the hazard function estimator is naturally constructed using these two last estimators, the same type of properties is easily derived for it. Our results are valid in a real (one- and multi-dimensional) context.

If X is a random variable associated to a lifetime (ie, a random variable with values in $\mathbb{R}^+$, the hazard rate of X (sometimes called hazard function, failure or survival rate) is defined at point x as the instantaneous probability that life ends at time x. Specifically, we have:

$$h(x) = \lim_{dx \to 0} \frac{\mathbb{P}\left(X \leq x + dx | X \geq x\right)}{dx}. \qquad (x > 0)$$

When X has a density f with respect to the measure of Lebesgue, it is easy to see that the hazard rate can be written as follows:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}, \text{ for all } x \text{ such that } F(x) < 1,$$

where F denotes the distribution function of X and $S = 1 - F$ the survival function of X.

In many practical situations, we may have an explanatory variable Z and the main issue is to estimate the conditional random rate defined as

$$h^z(x) = \lim_{dx \to 0} \frac{\mathbb{P}\left(X \leq x + dx | X > x, \quad Z\right)}{dx}, \text{ for } x > 0$$

which can be written naturally as follows:

$$h^z(x) = \frac{f^z(x)}{S^z(x)} = \frac{f^z(x)}{1 - F^z(x)}, \text{ once } F^z(x) < 1. \tag{4.1}$$

Study of functions h and h^z is of obvious interest in many fields of science (biology, medicine, reliability , seismology, econometrics, ...) and many authors are interested in construction of nonparametric estimators of h.

In this chapter we propose an estimate of the maximum risk, through the nonparametric estimation of the conditional hazard function.

The layout of the chapter is as follows. Section 4.2 describes the non-parametric functional setting: the structure of the functional data and the mixing conditions, the conditional density, distribution and hazard operators, and the corresponding nonparametric kernel estimators. Section 4.3 states the almost complete convergence[1] (with rates of convergence[2]) for nonparametric estimates of the derivative of the conditional hazard and the maximum risk. In Section 4.4, we calculate the variance of the conditional density, distribution and hazard estimates, the asymptotic normality of the three estimators considered is developed in this Section. Finally, Section 4.5 includes some proofs of technical Lemmas.

4.2 Nonparametric estimation with dependent functional data

Let $\{(Z_i, X_i),\ i = 1, \dots, n\}$ be a sample of n random pairs, each one distributed as (Z, X), where the variable Z is of functional nature and X is scalar. Formally, we will consider that Z is a random variable valued in some semi-metric functional space $\mathcal{F}$, and we will denote by $d(.,.)$ the associated semi-metric. The conditional cumulative distribution of X given Z is defined for any $x \in \mathbb{R}$ and any $z \in \mathcal{F}$ by

$$F^z(x) = \mathbb{P}(X \leq x | Z = z),$$

while the conditional density, denoted by $f^z(x)$ is defined as the density of this distribution with respect to the Lebesgue measure on $\mathbb{R}$. The conditional hazard is defined as in the non-infinite case (4.1).

In a general functional setting, f, F and h are not standard mathematical objects. Because they are defined on infinite dimensional spaces, the term operators may be a more adjusted in terminology.

[1]Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to converge almost completely to some variable T, if for any $\epsilon > 0$, we have $\sum_n \mathbb{P}(|T_n - T| > \epsilon) < \infty$. This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre, (1987)).

[2]Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to be of order of complete convergence u_n, if there exists some $\epsilon > 0$ for which $\sum_n \mathbb{P}(|T_n| > \epsilon u_n) < \infty$. This is denoted by $T_n = \mathcal{O}(u_n)$, *a.co.* (or equivalently by $T_n = \mathcal{O}_{a.co.}(u_n)$).

4.2.1 Dependance structure

We assume the sample data $(Z_i, X_i)_{1\leq i\leq n}$ to be dependent and to satisfy the strong mixing condition (α-mixing), introduced by Rosenblatt [53], defined as:
let $\mathbb{N}^*$ denotes the set of positive integers, and for any i and j in $\mathbb{N}^* \cup \infty$, $(i \leq j)$, define $\mathcal{F}_i^j$ to be σ algebra spanned by the variables $(z_i, x_i)\cdots(z_j, x_j)$. The sequence (Z_i, X_i) is said to be α mixing if there exist mixing coefficients $\alpha(k)$ such that $|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq \alpha(k)$, for any sets A and B, that are, respectively, $\mathcal{F}_i^m$ measurable $\mathcal{F}_{m+k}^{\infty}$ measurable (k, m positive integers), and $\alpha(k) \downarrow 0$.

This is the weakest condition used in studies of dependent samples (for example, the ARMA process, generated by a continuous white noise verifies it). The reader can see Doukhan [18] for a more complete discussion of the strong mixing condition.

4.2.2 The functional kernel estimates

Following in Ferraty *et al.* [25], the conditional density operator $f^z(.)$ is defined by using kernel smoothing methods

$$\widehat{f}^z(x) = \frac{\sum_{i=1}^{n} h_H^{-1} K\left(h_K^{-1} d(z, Z_i)\right) H'\left(h_H^{-1}(x - X_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right)},$$

where K and H' are kernel functions and h_H and h_K are sequences of smoothing parameters. The conditional distribution operator $F^z(.)$ can be estimated by

$$\widehat{F}^z(x) = \frac{\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right) H\left(h_H^{-1}(x - X_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right)},$$

with the function $H(.)$ defined by $H(x) = \int_{-\infty}^{x} H'(t)dt$. Consequently, the conditional hazard operator is defined in a natural way by

$$\widehat{h}^z(x) = \frac{\widehat{f}^z(x)}{1 - \widehat{F}^z(x)}.$$

For $z \in \mathcal{F}$, we denote by $h^z(.)$ the conditional hazard function of X_1 given $Z_1 = z$. We assume that $h^z(.)$ is unique maximum and its high risk point is denoted by $\theta(z) := \theta$, which is defined by

$$h^z(\theta(z)) := h^z(\theta) = \max_{x \in \mathcal{S}} h^z(x) \tag{4.2}$$

A kernel estimator of θ is defined as the random variable $\widehat{\theta}(z) := \widehat{\theta}$ which maximizes a kernel estimator $\widehat{h}^z(.)$, that is,

$$\widehat{h}^z(\widehat{\theta}(z)) := \widehat{h}^z(\widehat{\theta}) = \max_{x \in \mathcal{S}} \widehat{h}^z(x) \tag{4.3}$$

where h^z and $\widehat{h}^z$ are defined above.
Note that the estimate $\widehat{\theta}$ is note necessarily unique and our results are valid for any choice satisfying (4.3). We point out that we can specify our choice by taking

$$\widehat{\theta}(z) = \inf \left\{ t \in \mathcal{S} \text{ such that } \widehat{h}^z(t) = \max_{x \in \mathcal{S}} \widehat{h}^z(x) \right\}$$

As in any non-parametric functional data problem, the behavior of the estimates is controlled by the concentration properties of the functional variable Z.

$$\phi_z(h) = \mathbb{P}(Z \in B(z, h)),$$

where $B(z, h)$ being the ball of center z and radius h, namely
$B(z, h) = \{f \in \mathcal{F}, d(z, f) < h\}$ (for more details, see Ferraty and Vieu [26], Chapter 6).
In the following, z will be a fixed point in $\mathcal{F}$, $\mathcal{N}_z$ will denote a fixed neighborhood of z, $\mathcal{S}$ will be a fixed compact subset of $\mathbb{R}^+$. We will led to the hypothesis below concerning the function of concentration ϕ_z

(M0) $\forall h > 0,\ 0 < \mathbb{P}\left(Z \in B(z,h)\right) = \phi_z(h)$ and $\lim\limits_{h\to 0} \phi_z(h) = 0$

(M1) $(Z_i, X_i)_{i\in\mathbb{N}}$ is an α-mixing sequence whose the coefficients of mixture verify:

$$\exists a > 0,\ \exists c > 0 :\ \forall n \in \mathbb{N},\ \alpha(n) \leq c n^{-a}.$$

(M2) $0 < \max\limits_{i\neq j} \psi_{i,j}(h) = \sup\limits_{i\neq j} \mathbb{P}\left((Z_i, Z_j) \in B(z,h) \times B(z,h)\right) = \mathcal{O}\left(\dfrac{(\phi_z(h))^{(a+1)/a}}{n^{1/a}}\right).$

Note that (MO) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* of Z, whereas (M2) concerns the behavior of the joint distribution of the pairs (Z_i, Z_j). In fact, this hypothesis is equivalent to assume that, for n large enough

$$\sup_{i\neq j} \frac{\mathbb{P}\left((Z_i, Z_j) \in B(z,h) \times B(z,h)\right)}{\mathbb{P}\left(Z \in B(z,h)\right)} \leq C \left(\frac{\phi_z(h)}{n}\right)^{1/a}.$$

This is one way to control the local asymptotic ratio between the joint distribution and its margin. Remark that the upper bound increases with a. In other words, more the dependence is strong, more restrictive is (M2). The hypothesis (M1) specifies the asymptotic behavior of the α-mixing coefficients.

Our nonparametric models will be quite general in the sense that we will just need the following simple assumption for the marginal distribution of Z, and let us introduce the technical hypothesis necessary for the results to be presented. The non-parametric model is defined by assuming that

$$\text{(M3)} \begin{cases} \forall (x_1, x_2) \in \mathcal{S}^2, \forall (z_1, z_2) \in \mathcal{N}_z^2, \text{ for some } b_1 > 0,\ b_2 > 0 \\ |F^{z_1}(x_1) - F^{z_2}(x_2)| \leq C_z(d(z_1, z_2)^{b_1} + |x_1 - x_2|^{b_2}), \end{cases}$$

$$\text{(M4)} \begin{cases} \forall (x_1, x_2) \in \mathcal{S}^2, \forall (z_1, z_2) \in \mathcal{N}_z^2, \text{ for some } j = 0, 1,\ \nu > 0,\ \beta > 0 \\ |f^{z_1\,(j)}(x_1) - f^{z_2\,(j)}(x_2)| \leq C_z(d(z_1, z_2)^{\nu} + |x_1 - x_2|^{\beta}), \end{cases}$$

(M5) $\exists \gamma < \infty, f'^{z}(x) \leq \gamma,\ \ \forall (z, x) \in \mathcal{F} \times \mathcal{S},$

(M6) $\exists \tau > 0, F^z(x) \leq 1 - \tau, \ \forall (z, x) \in \mathcal{F} \times \mathcal{S}$.

(M7) H' is twice differentiable such that

$$\begin{cases} \text{(M7a) } \forall (t_1, t_2) \in \mathbb{R}^2; \ |H^{(j)}(t_1) - H^{(j)}(t_2)| \leq C|t_1 - t_2|, \ \text{ for } j = 0, 1, 2 \\ \text{and } H^{(j)} \text{are bounded for } j = 0, 1, 2 \\ \text{(M7b) } \displaystyle\int_{\mathbb{R}} t^2 H'^2(t) dt < \infty, \\ \text{(M7c) } \displaystyle\int_{\mathbb{R}} |t|^{\beta} (H''(t))^2 dt < \infty \end{cases}$$

(M8) The kernel K is positive bounded function supported on $[0, 1]$ and it is of class $\mathcal{C}^1$ on $(0, 1)$ such that $\exists C_1, \ C_2, \ -\infty < C_1 < K'(t) < C_2 < 0$ for $0 < t < 1$

(M9) There exists a function $\zeta_0^z(.)$ such that for all $t \in [0, 1]$ $\displaystyle\lim_{h \to 0} \frac{\phi_z(th)}{\phi_z(h)} = \zeta_0^z(t)$.

(M10) The bandwidth h_H and h_K, small ball probability $\phi_z(h)$ and arithmetical α mixing coefficient with order $a > 3$ satisfying

$$\begin{cases} \text{(M10a)} \exists C > 0, \ h_H^{2j+1} \phi_z(h_K) \geq \frac{C}{n^{2/(a+1)}}, \ \text{ for } j = 0, 1 \\ \text{(M10b) } \left(\frac{\phi_z(h_K)}{n} \right)^{1/a} + \phi_z(h_K) = o\left(\frac{1}{n^{2/(a+1)}} \right), \ \text{ for } j = 0, 1 \\ \text{(M10c) } \displaystyle\lim_{n \to \infty} h_K = 0, \ \lim_{n \to \infty} h_H = 0, \ \text{ and } \lim_{n \to \infty} \frac{\log n}{n h_H^{2j+1} \phi_z(h_K)} = 0, \ j = 0, 1; \end{cases}$$

Remark 4.2.1. *Assumptions (M3) and (M4) are the only conditions involving the conditional probability and the conditional probability density of Z given X. It means that $F(\Delta|\Delta)$ and $f(.|.)$ and its derivatives satisfy the Hölder condition with respect to each variable. Therefore, the concentration condition (MO) plays an important role. Here we point out that our assumptions are very usual in the estimation problem for functional regressors (see, e.g., Ferraty et al. [24]).*

Remark 4.2.2. *Assumptions (M7), (M8) and (M10) are classical in functional estimation for finite or infinite dimension spaces.*

4.3 Nonparametric estimate of the maximum of the conditional hazard function

Let us assume that there exists a compact $\mathcal{S}$ with a unique maximum θ of h^z on $\mathcal{S}$. We will suppose that h^z is sufficiently smooth (at least of class $\mathcal{C}^2$) and verifies that $h'^z(\theta) = 0$ and $h''^z(\theta) < 0$.

Furthermore, we assume that $\theta \in \mathcal{S}^\circ$, where $\mathcal{S}^\circ$ denotes the interior of $\mathcal{S}$, and that θ satisfies the uniqueness condition, that is; for any $\varepsilon > 0$ and $\mu(z)$, there exists $\xi > 0$ such that $|\theta(z) - \mu(z)| \geq \varepsilon$ implies that $|h^z(\theta(z)) - h^z(\mu(z))| \geq \xi$.

We can write an estimator of the first derivative of the hazard function through the first derivative of the estimator. Our maximum estimate is defined by assuming that there is some unique $\widehat{\theta}$ on $\mathcal{S}^\circ$.

It is therefore natural to try to construct an estimator of the derivative of the function h^z on the basis of these ideas. To estimate the conditional distribution function and the conditional density function in the presence of functional conditional random variable Z.

The kernel estimator of the derivative of the function conditional random functional h^z can therefore be constructed as follows:

$$\widehat{h'}^z(x) = \frac{\widehat{f'}^z(x)}{1 - \widehat{F}^z(x)} + (\widehat{h}^z(x))^2, \tag{4.4}$$

the estimator of the derivative of the conditional density is given in the following formula:

$$\widehat{f'}^z(x) = \frac{\sum_{i=1}^{n} h_H^{-2} K(h_K^{-1} d(z, Z_i)) H''(h_H^{-1}(x - X_i))}{\sum_{i=1}^{n} K(h_K^{-1} d(z, Z_i))} \tag{4.5}$$

Later, we need assumptions on the parameters of the estimator, ie on K, H, H', H'', h_H and h_K are little restrictive. Indeed, on one hand, they are not specific to the problem estimate of h^z (but inherent problems of F^z, f^z and f'^z estimation), and secondly they consist with the assumptions usually made under functional variables.

We state the almost complete convergence (withe rates of convergence) of the maximum estimate by the following results:

Theorem 4.3.1. *Under assumption (MO)-(M8) we have*

$$\widehat{\theta} - \theta \to 0 \quad a.co. \tag{4.6}$$

Remark 4.3.2. *The hypothesis of uniqueness is only established for the sake of clarity. Following our proofs, if several local estimated maxima exist, the asymptotic results remain valid for each of them.*

Proof of theorem (4.3.1). Because $h'^z(\cdot)$ is continuous, we have, for all $\epsilon > 0$. $\exists\, \eta(\epsilon) > 0$ such that

$$|x - \theta| > \epsilon \Rightarrow |h'^z(x) - h'^z(\theta)| > \eta(\epsilon).$$

Therefore,

$$\mathbb{P}\{|\widehat{\theta} - \theta| \geq \epsilon\} \leq \mathbb{P}\{|h'^z(\widehat{\theta}) - h'^z(\theta)| \geq \eta(\epsilon)\}.$$

We also have

$$|h'^z(\widehat{\theta}) - h'^z(\theta)| \leq |h'^z(\widehat{\theta}) - \widehat{h}'^z(\widehat{\theta})| + |\widehat{h}'^z(\widehat{\theta}) - h'^z(\theta)| \leq \sup_{x \in \mathcal{S}} |\widehat{h}'^z(x) - h'^z(x)|, \tag{4.7}$$

because $\widehat{h}'^z(\widehat{\theta}) = h'^z(\theta) = 0$.

Then, uniform convergence of h'^z will imply the uniform convergence of $\widehat{\theta}$. That is why, we have the following lemma.

Lemma 4.3.3. *Under assumptions of Theorem (4.3.1), we have*

$$\sup_{x \in \mathcal{S}} |\widehat{h}'^z(x) - h'^z(x)| \to 0 \quad a.co. \tag{4.8}$$

□

The proof of this lemma will be given in Appendix.

Theorem 4.3.4. *Under assumption (M1)-(M8) and (M10c) we have*

$$\sup_{x\in\mathcal{S}} |\widehat{\theta}-\theta| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nh_H^3\phi_z(h_K)}}\right) \tag{4.9}$$

Proof of theorem (4.3.4). By using Taylor expansion of the function h'^z at the point $\widehat{\theta}$, we obtain

$$h'^z(\widehat{\theta}) = h'^z(\theta)+(\widehat{\theta}-\theta)h''^z(\theta^*), \tag{4.10}$$

with θ^* a point between θ and $\widehat{\theta}$. Now, because $h'^z(\theta)=\widehat{h}'^z(\widehat{\theta})$

$$|\widehat{\theta}-\theta| \leq \frac{1}{h''^z(\theta_n^*)}\sup_{x\in\mathcal{S}} |\widehat{h}'^z(x)-h'^z(x)| \tag{4.11}$$

The proof of Theorem will be completed showing the following lemma.

Lemma 4.3.5. *Under the assumptions of Theorem (4.3.4), we have*

$$\sup_{x\in\mathcal{S}} |\widehat{h}'^z(x)-h'^z(x)| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nh_H^3\phi_z(h_K)}}\right) \tag{4.12}$$

The proof of lemma will be given in the Appendix.

□

4.4 Asymptotic normality

To obtain the asymptotic normality of the conditional estimates, we have to add the following assumptions:

(H7d) $\displaystyle\int_{\mathbb{R}} (H''(t))^2 dt < \infty,$

(H11) $0 = \widehat{h'}^z(\widehat{\theta}) < |\widehat{h'}^z(x)|), \forall x \in \mathcal{S},\ x \neq \widehat{\theta}$

The following result gives the asymptotic normality of the maximum of the conditional hazard function. Let

$$\mathcal{A} = \{(z,x) : (z,x) \in \mathcal{S} \times \mathbb{R},\ a_2^x F^z(x)\,(1 - F^z(x)) \neq 0\}$$

Theorem 4.4.1. *Under conditions (MO)-(H11) we have* $(\theta \in \mathcal{S}/f^z(\theta), 1-F^z(\theta) > 0)$

$$\left(nh_H^3\phi_z(h_K)\right)^{1/2}\left(\widehat{h}'^z(\theta) - h'^z(\theta)\right) \xrightarrow{\mathcal{D}} N\left(0, \sigma^2_{h'(\theta)}\right)$$

where $\rightarrow^{\mathcal{D}}$ *denotes the convergence in distribution,*

$$a_l^x = K^l(1) - \int_0^1 \left(K^l(u)\right)' \zeta_0^x(u)du \quad \textit{for } l = 1, 2$$

and

$$\sigma^2_{h'(\theta)} = \frac{a_2^x h^z(\theta)}{(a_1^x)^2\,(1 - F^z(\theta))} \int (H''(t))^2 dt.$$

Proof of theorem (4.4.1). Using again (4.17), and the fact that

$$\frac{(1 - F^z(x))}{(1 - \widehat{F}^z(x))\,(1 - F^z(x))} \longrightarrow \frac{1}{1 - F^z(x)}$$

and

$$\frac{f'^z(x)}{\left(1 - \widehat{F}^z(x)\right)(1 - F^z(x))} \longrightarrow \frac{f'^z(x)}{(1 - F^z(x))^2}$$

The asymptotic normality of $(nh_H^3\phi_z(h_K))^{1/2}\left(\widehat{h'}^z(\theta) - h'^z(\theta)\right)$ can be deduced from both following lemmas,

Lemma 4.4.2. *Under Assumptions (MO)-(M3) and (M7)-(M9), we have*

$$(n\phi_z(h_K))^{1/2}\left(\widehat{F}^z(x) - F^z(x)\right) \xrightarrow{\mathcal{D}} N\left(0, \sigma^2_{F^z(x)}\right) \tag{4.13}$$

where

$$\sigma^2_{F^z(x)} = \frac{a_2^x F^z(x)\,(1 - F^z(x))}{(a_1^x)^2}$$

Lemma 4.4.3. *Under Assumptions (MO)-(M4) and (M6)-(M10), we have*

$$(nh_H\phi_z(h_K))^{1/2}\left(\widehat{h}^z(x)-h^z(x)\right)\xrightarrow{\mathcal{D}}N\left(0,\sigma^2_{h^z(x)}\right) \tag{4.14}$$

where

$$\sigma^2_{h^z(x)}=\frac{a_2^x h^z(x)}{(a_1^x)^2\left(1-F^z(x)\right)}\int_{\mathbb{R}}(H'(t))^2dt$$

Lemma 4.4.4. *Under Assumptions of Theorem (4.4.1), we have*

$$\left(nh_H^3\phi_z(h_K)\right)^{1/2}\left(\widehat{f'}^z(x)-f'^z(x)\right)\xrightarrow{\mathcal{D}}N\left(0,\sigma^2_{f'^z(x)}\right) \tag{4.15}$$

where

$$\sigma^2_{f'^z(x)}=\frac{a_2^x f^z(x)}{(a_1^x)^2}\int_{\mathbb{R}}(H''(t))^2dt$$

The proofs of Lemma (4.4.2) can be seen in Ezzahrioui and Ould-Saïd [20].

□

Finally, by this last result and (4.10), the following theorem follows:

Theorem 4.4.5. *Under conditions (M1)-(M11) we have* $(\theta\in\mathcal{S}/f^z(\theta),1-F^z(\theta)>0)$

$$\left(nh_H^3\phi_z(h_K)\right)^{1/2}\left(\widehat{\theta}-\theta\right)\xrightarrow{\mathcal{D}}N\left(0,\frac{\sigma^2_{h'(\theta)}}{(h''^z(\theta))^2}\right)$$

with

$$\sigma^2_{h'(\theta)}=\frac{a_2^x h^z(\theta)}{(a_1^x)^2\left(1-F^z(\theta)\right)}\int(H''(t))^2dt.$$

4.5 Proofs of technical lemmas

***Proof of lemma* (4.3.3)** *and* ***lemma* (4.3.5)**. Let

$$\widehat{h}'^z(x)=\frac{\widehat{f}'^z(x)}{1-\widehat{F}^z(x)}+(\widehat{h}^z(x))^2, \tag{4.16}$$

with

$$\widehat{h'}^z(x) - h'^z(x) = \underbrace{\left\{ \left(\widehat{h}^z(x)\right)^2 - (h^z(x))^2 \right\}}_{\Gamma_1} + \underbrace{\left\{ \frac{\widehat{f'}^z(x)}{1-\widehat{F}^z(x)} - \frac{f'^z(x)}{1-F^z(x)} \right\}}_{\Gamma_2} \tag{4.17}$$

for the first term of (4.17) we can write

$$\left| \left(\widehat{h}^z(x)\right)^2 - (h^z(x))^2 \right| \leq \left|\widehat{h}^z(x) - h^z(x)\right| . \left|\widehat{h}^z(x) + h^z(x)\right| \tag{4.18}$$

because the estimator $\widehat{h}^z(\cdot)$ converge a.co. to $h^Z(\cdot)$ we have

$$\sup_{x\in\mathcal{S}} \left| \left(\widehat{h}^z(x)\right)^2 - (h^z(x))^2 \right| \leq 2\left|h^z(\theta)\right| \sup_{x\in\mathcal{S}} \left|\widehat{h}^z(x) - h^z(x)\right| \tag{4.19}$$

for the second term of (4.17) we have

$$\begin{aligned} \frac{\widehat{f'}^z(x)}{1-\widehat{F}^z(x)} - \frac{f'^z(x)}{1-F^z(x)} &= \frac{1}{(1-\widehat{F}^z(x))(1-F^z(x))} \Big\{ \left(\widehat{f'}^z(x) - f'^z(x)\right) \\ &\quad + f'^z(x)\left(\widehat{F}^z(x) - F^z(x)\right) - F^Z(x)\left(\widehat{f'}^z(x) - f'^z(x)\right) \Big\}, \\ &= \frac{1}{(1-\widehat{F}^z(x))(1-F^z(x))} \Big\{ (1-F^z(x))\left(\widehat{f'}^z(x) - f'^z(x)\right) \\ &\quad + f'^z(x)\left(\widehat{F}^z(x) - F^z(x)\right) \Big\} \end{aligned}$$

Valid for all $x \in \mathcal{S}$. Which for a constant $C < \infty$, this leads

$$\sup_{x\in\mathcal{S}} \left| \frac{\widehat{f'}^z(x)}{1-\widehat{F}^z(x)} - \frac{f'^z(x)}{1-F^z(x)} \right| \leq$$

$$C \frac{\left\{ \sup_{x\in\mathcal{S}} \left|\widehat{f'}^z(x) - f'^z(x)\right| + \sup_{x\in\mathcal{S}} \left|\widehat{F}^z(x) - F^z(x)\right| \right\}}{\inf_{x\in\mathcal{S}} \left|1 - \widehat{F}^z(x)\right|} \tag{4.20}$$

Therefore, the conclusion of the lemma follows from the following results:

$$\sup_{x\in\mathcal{S}}|\widehat{F}^z(x)-F^z(x)| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_z(h_K)}}\right) \tag{4.21}$$

$$\sup_{x\in\mathcal{S}}|\widehat{f}'^z(x)-f'^z(x)| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nh_H^3\phi_z(h_K)}}\right) \tag{4.22}$$

$$\sup_{x\in\mathcal{S}}|\widehat{h}^z(x)-h^z(x)| = \mathcal{O}\left(h_K^{b_1}+h_H^{b_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nh_H\phi_z(h_K)}}\right) \tag{4.23}$$

$$\exists\delta>0 \ \text{ such that } \ \sum_1^\infty \mathbb{P}\left\{\inf_{y\in\mathcal{S}}|1-\widehat{F}^z(x)|<\delta\right\}<\infty \tag{4.24}$$

The proofs of (4.22) appear in Ferraty *et al.* [24], and (4.21),(4.23)(4.24) is proven in chapter 2.

□

***Proof of lemma* (4.4.3)**. We can write for all $x\in\mathcal{S}$

$$\begin{aligned}
\widehat{h}^z(x)-h^z(x) &= \frac{\widehat{f}^z(x)}{1-\widehat{F}^z(x)}-\frac{f^z(x)}{1-F^z(x)}\\
&= \frac{1}{\widehat{D}^z(x)}\Big\{\left(\widehat{f}^z(x)-f^z(x)\right)+f^z(x)\left(\widehat{F}^z(x)-F^z(x)\right)\\
&\qquad -F^z(x)\left(\widehat{f}^z(x)-f^z(x)\right)\Big\},\\
&= \frac{1}{\widehat{D}^z(x)}\Big\{(1-F^z(x))\left(\widehat{f}^z(x)-f^z(x)\right)\\
&\qquad +f^z(x)\left(\widehat{F}^z(x)-F^z(x)\right)\Big\}
\end{aligned} \tag{4.25}$$

with $\widehat{D}^z(x)=(1-F^z(x))\left(1-\widehat{F}^z(x)\right)$.

As a direct consequence of the Lemma (4.4.2), the result (4.26) (see Ezzahrioui and Ould-Saïd [21]) and the expression (4.25), permit us to obtain the asymptotic normality for the conditional hazard estimator.

$$(nh_H\phi_z(h_K))^{1/2}\left(\widehat{f}^z(x)-f^z(x)\right)\xrightarrow{\mathcal{D}} N\left(0,\sigma^2_{f^z(x)}\right) \tag{4.26}$$

where

$$\sigma^2_{f^z(x)}=\frac{a_2^x f^z(x)}{(a_1^x)^2}\int_{\mathbb{R}}(H'(t))^2dt$$

□

***Proof of lemma* (4.4.4)**. For $i=1,\ldots,n$, we consider the quantities $K_i = K\left(h_K^{-1}d(z,Z_i)\right)$, $H_i''(x)=H''\left(h_H^{-1}(x-X_i)\right)$ and let $\widehat{f'}^z_N(x)$ (resp. $\widehat{F}^z_D$) be defined as

$$\widehat{f'}^z_N(x) = \frac{h_H^{-2}}{n\,\mathbb{E}K_1}\sum_{i=1}^n K_iH_i''(x) \qquad (\text{resp. } \widehat{F}^z_D = \frac{1}{n\,\mathbb{E}K_1}\sum_{i=1}^n K_i).$$

This proof is based on the following decomposition

$$\begin{aligned}\widehat{f'}^z(x)-f'^z(x) &= \frac{1}{\widehat{F}^z_D}\left\{\left(\widehat{f'}^z_N(x)-\mathbb{E}\widehat{f'}^z_N(x)\right)-\left(f'^z(x)-\mathbb{E}\widehat{f'}^z_N(x)\right)\right\}+\\ &\quad \frac{f'^z(x)}{\widehat{F}^z_D}\left\{\mathbb{E}\widehat{F}^z_D-\widehat{F}^z_D\right\}\end{aligned} \tag{4.27}$$

and on the following intermediate results.

$$\sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{f'}^z_N(x)-\mathbb{E}\widehat{f'}^z_N(x)\right)\xrightarrow{\mathcal{D}} N\left(0,\sigma^2_{f'^z(x)}\right) \tag{4.28}$$

where $\sigma^2_{f'^z}(x)$ is defined as in Lemma (4.4.4).

$$\lim_{n\to\infty}\sqrt{nh_H^3\phi_z(h_K)}\left(\mathbb{E}\widehat{f'}^z_N(x)-f'^z(x)\right)=0 \tag{4.29}$$

$$\sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{F}^z_D(x)-1\right)\xrightarrow{\mathbb{P}}0, \quad \text{as} \quad n\to\infty. \tag{4.30}$$

- Concerning (4.28). By the definition of $\widehat{f'}_N^z(x)$, it follows that

$$\sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{f'}_N^z(x)-\mathbb{E}\widehat{f'}_N^z(x)\right)=\sum_{i=1}^n\frac{\sqrt{\phi_z(h_K)}}{\sqrt{nh_H}\mathbb{E}K_1}\left(K_iH_i''-\mathbb{E}K_iH_i''\right)=\sum_{i=1}^n\Delta_i,$$

which leads

$$\sum_{i=1}^n\mathbb{E}\Delta_i^2=\frac{\phi_z(h_K)}{h_H\mathbb{E}^2K_1}\mathbb{E}K_1^2(H_1'')^2-\frac{\phi_z(h_K)}{h_H\mathbb{E}^2K_1}\left(\mathbb{E}K_1H_1''\right)^2=\Pi_{1n}-\Pi_{2n}. \qquad (4.31)$$

As for Π_{1n}, by the property of conditional expectation, we get

$$\Pi_{1n}=\frac{\phi_z(h_K)}{\mathbb{E}^2K_1}\mathbb{E}\left\{K_1^2\int H''^2(t)\left(f'^z(x-th_H)-f'^Z(x)+f'^z(x)\right)dt\right\}.$$

Meanwhile, by (MO), (M4), (M8) and (M9), it follows that:

$$\frac{\phi_z(h_K)\mathbb{E}K_1^2}{\mathbb{E}^2K_1}\underset{n\to\infty}{\longrightarrow}\frac{a_2^x}{(a_1^x)^2},$$

which leads

$$\Pi_{1n}\underset{n\to\infty}{\longrightarrow}\frac{a_2^xf^Z(x)}{(a_1^x)^2}\int(H''(t))^2dt, \qquad (4.32)$$

Regarding Π_{2n}, by (MO), (M4) and (M7), we obtain

$$\Pi_{2n}\underset{n\to\infty}{\longrightarrow}0. \qquad (4.33)$$

This result, combined with (4.31) and (4.32), allows us to get

$$\lim_{n\to\infty}\sum_{i=1}^n\mathbb{E}\Delta_i^2=\sigma^2_{f'^Z(x)} \qquad (4.34)$$

Secondly, by the boundedness of H'', we have

$$\begin{aligned}\mathbb{E}\left(|\Delta_i\Delta_j|\right) &\leq \frac{C\phi_z(h_K)}{n\mathbb{E}^2K_1}\left(K_iK_j+\mathbb{E}K_iK_j\right)\\ &\leq \frac{C}{nh_H}\left\{\left(\frac{\phi_z(h_K)}{n}\right)^{1/a}+\phi_z(x)(h_K)\right\}, \quad \forall i\neq j.\end{aligned}$$

Then, taking

$$\delta_n = \max_{1\leq i\neq j\leq n} \{\mathbb{E}\left(\mid \Delta_i\Delta_j|\right)\} = \frac{C}{nh_H}\left(\left(\frac{\phi_z(h_K)}{n}\right)^{1/a} + \phi_z(x)(h_K)\right).$$

leads

$$nm_n\delta_n = \frac{Cm_n}{h_H}\left(\left(\frac{\phi_z(h_K)}{n}\right)^{1/a} + \phi_z(x)(h_K)\right). \tag{4.35}$$

Similarly, the boundedness of H'' and K allows us to take

$$C_i = \mathcal{O}\left(\frac{1}{\sqrt{nh_H^3\phi_z(h_K)}}\right),$$

which implies that

$$\left(\sum_{j=m_n+1}^{\infty}\alpha(j)\right)\sum_{i=1}^{n}C_i^2 \leq \frac{C}{h_H\phi_z(h_K)}\int_{t\geq m_n} t^{-a}dt = \frac{C}{h_H\phi_z(h_K)}\frac{m_n^{-a+1}}{a-1}. \tag{4.36}$$

Then, the sum of the right side of (4.35) and (4.36) is of type $Am_n + Bm_n^{-a+1}$, by talking $m_n = (A/B)^{-1/a} = \{(a-1)\phi_z(h_K)((\frac{\phi_z(h_K)}{n})^{1/a} + \phi_z(h_K))\}^{-1/a} \to \infty$, it is clear that, under conditions (H10a) and (H10b), combining (4.35) and (4.36) allows us to get

$$nm_n\delta_n = o(1), \tag{4.37}$$

and

$$\left(\sum_{j=m_n+1}^{\infty}\alpha(j)\right)\sum_{i=1}^{n}C_i^2 = o(1), \tag{4.38}$$

respectively. Finally, by choosing $\varrho_n = \sqrt{\frac{nh_H^3\phi_z(h_K)}{\log n}}$, under (H10a) again and $a > 3$, we have

$$\frac{\varrho_n}{\sqrt{n}} = o(1) \tag{4.39}$$

and

$$\begin{aligned}\frac{n}{\varrho_n}\alpha(\varepsilon\varrho_n) &\leq C\frac{(\log n)^{(a+1)/2}}{n^{(a-1)/2}(h_H^3\phi_z(h_K))^{(a+1)/2}} \\ &\leq C\frac{(\log n)^{(a+1)/2}}{n^{(a-3)/2}} \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Therefore, combining (4.33)-(4.39) with Corollary 2.2 in Liebscher [36], (4.28) is valid.

- Concerning (4.29). The proof is completed along the same steps as that of Π_{1n}. We omit it here.

- Concerning (4.30). The idea is similar to that given by Ferraty *et al.* [24].

  By definition of $\widehat{F}_D^z(x)$, we have

$$\sqrt{nh_H^3\phi_z(h_K)}(\widehat{F}_D^z(x)-1) = \Omega_n - \mathbb{E}\Omega_n,$$

where $\Omega_n = \frac{\sqrt{nh_H^3\phi_z(h_K)}\sum_{i=1}^n K_i}{n\mathbb{E}K_1}$. In order to prove (4.30), similar to Ferraty *et al.* (2005), we only need to proov $Var\ \Omega_n \to 0$, as $n\to\infty$. In fact, since

$$\begin{aligned}Var\ \Omega_n &= \frac{nh_H^3\phi_z(h_K)}{n\mathbb{E}^2K_1}\left(nVarK_1 + \sum_{1\leq i}\sum_{j\leq n} cov(K_i,K_j)\right) \\ &\leq \frac{nh_H^3\phi_z(h_K)}{\mathbb{E}^2K_1}\mathbb{E}K_1^2 + \frac{nh_H^3\phi_z(h_K)}{n\mathbb{E}^2K_1}\sum\sum_{0\leq|i-j|\leq v_n} cov(K_i,K_j) \\ &\quad + \frac{nh_H^3\phi_z(h_K)}{n\mathbb{E}^2K_1}\sum\sum_{0\leq|i-j|\geq v_n} cov(K_i,K_j) \\ &= \Psi_1+\Psi_2+\Psi_3,\end{aligned}$$

then, using the boundedness of function K allows us to get that:

$$\Psi_1 \leq Ch_H^3\phi_z(h_K) \to 0, \quad \text{as } n\to\infty.$$

Meanwhile, by (MO) and (M1), it follows that

$$\Psi_2 \leq v_n h_H^3 \left\{ \left(\frac{\phi_z(h_K)}{n} \right)^{1/a} + \phi_z(h_K) \right\}. \tag{4.40}$$

Finally, using the Davydov-Rio's inequality in Rio [52] for mixing processes leads to

$$|cov(K_i, K_j| \leq C\alpha(|i-j|),$$

for all $i \neq j$. Then, we have

$$\begin{aligned} \Psi_3 &\leq \frac{h_H^3 \phi_z(h_K)}{n\mathbb{E}^2 K_1} n^2 C\alpha(|i-j|) \\ &\leq C\frac{h_H^3 \phi_z(h_K)}{n\mathbb{E}^2 K_1} n^2 v_n^{-a+1} \\ &\leq Ch_H^3 n v_n^{-a+1}. \end{aligned} \tag{4.41}$$

Since the right side of (4.40) and (4.41) is also of type $Av_n + Bv_n^{-a+1}$,
by choosing $v_n = [n^{-1}((\frac{\phi_z(h_K)}{n})^{1/a} + \phi_z(h_K))]^{-1/a} \to \infty$
and simple calculations, we get that $\Psi_2 \to 0$ and $\Psi_3 \to 0$ as $n \to \infty$, respectively.
Therefore, the proof of this result is completed.

Therefore, the proof of this lemma is completed.

□

Chapter 5

General Conclusion and prospects

5.1 General conclusion

We were interested specifically in this thesis to a non-parametric model that treats the case of functional variables in which "response" variable is true while the explanatory variable is functional. The objective was the estimation of the derivative of the conditional hazard function by means of the conditional distribution function and its derivative by the kernel method. The case in question deals with complete data.
The richness of this functional statistical research area offers many perspectives both theoretically and practically. In the following, we will comment on some results already obtained, with the major concern of focusing on all open issues some of which are under development.

5.2 prospects

The work developed in this thesis offers many prospects in the short and long term. Regarding the short-term prospects:
• one can consider while adapting the tools developed by Niang and Rhomari (2003) to study the convergence standard L^p of our estimators in the case dependent and ergodic case.
•Another possible prospect is to obtain convergence rates and the formula of the

smoothing parameter using the integrated square error and square error.

• Other issues can be addressed such as the long-term by conditioning p functional variables or a linear combination of these functional variables p.

• The work on the estimation of conditional quantiles and the conditional hazard function for functional explanatory variable opens several perspectives. For example, another estimator may consider using a different method than the estimate by the kernel method as Fourier techniques: Fourier series decomposition, wavelet series decomposition, series decomposition of polynomials .. .,

• The search for optimal convergence rates are all interesting topics in the field

• On another front, regarding the mixing hypothesis, it is also possible to obtain convergence rates for ergodic data, making ergodic hypothesis with a neighboring spectral gap of the unit

• In the latter study truncated data and censored at a time can be interesting.

• estimation with spatial functional data can be approached in several ways.

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