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Relative controllability of semilinear fractional stochastic control systems in Hilbert spaces

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Abstract. This paper is concerned with the relative controllability for a class of dynamical control systems described by semilinear fractional stochastic differential equations with nonlocal conditions in Hilbert space. Sufficient conditions for relative controllability results are obtained using Schaefer's fixed point theorem. An example is provided to show the application of our result.

1. Introduction

Fractional dynamical equations have played a central role in the modeling of anomalous relaxation and diffusion processes. The fact that fractional derivatives introduce a convolution integral with a power-law memory kernel makes the fractional differential equations important to describe memory effects in complex systems [11]. The increasing interest of fractional equations is motivated by their applications in various fields of science such as physics, fluid mechanics, viscoelasticity, heat conduction in materials with memory, chemistry and engineering [5, 10]. Hilfer [7, 8] showed that time fractional derivatives are equivalent to infinitesimal generators of generalized time fractional evolutions that arise in the transition from microscopic to macroscopic time scales. Also, it is shown that this transition from the ordinary time derivative to the fractional time derivative arises in different physical problems [9]. Further, many different applications of fractional calculus are presented in [10].

Controllability for nonlinear dynamical systems is not so uniform and connected as in the case of linear dynamical systems. Most of the results obtained and of the controllability criteria have a local character or concern only a very narrow class of dynamical systems. The main difficulty arising in the investigation of controllability for nonlinear dynamical systems is the lack of general methods for solving nonlinear differential or functional differential equations. Fixed point technique is the most powerful method to obtain the controllability results for nonlinear dynamical systems (see, for instance [2, 3]).

On the other hand, stochastic differential equations have attracted great interest due to their applications in various fields of science and engineering. There are many interesting results in the theory and applications of stochastic differential equations, (see [6, 18] and the references therein). In recent years, controllability problems for stochastic differential equations have become a field of increasing interest (see [12, 17, 19, 23, 24]

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and references therein). The extensions of deterministic controllability concepts to stochastic control systems have been discussed only in a limited number of publications. More precisely, there are less number of papers on the relative controllability of stochastic nonlinear systems [13, 26]. Klamka [12, 14] studied stochastic relative exact and approximate controllability problems for finite dimensional linear stationary dynamical systems with single time-variable point delay in the control by implementing the open mapping theorem. A set of necessary and sufficient conditions are established for the exact and approximate stochastic controllability of linear system with state delays in [16]. Shen et al. [26] investigated the relative controllability of the stochastic differential systems with delay in control. The authors derive a new sufficient conditions for the relative controllability and relative approximate controllability in finite and infinite dimensional spaces.

However, to the best of our knowledge, the relative controllability problem for semilinear fractional stochastic system in Hilbert spaces has not been investigated yet. Motivated by this consideration, in this paper we will study the relative controllability of semilinear fractional stochastic systems, which are natural generalizations of controllability concepts well known in the theory of infinite dimensional deterministic control systems. Specifically, we study the relative controllability of semilinear fractional control systems under the assumption that the associated linear system is relatively controllable. The paper is organized as follows. Some preliminary facts are recalled in Section 2. Section 3 is devoted to sufficient condition on the relative controllability of semilinear SDEs with nonlocal conditions in Hilbert spaces. In section 4, an example is discussed to illustrate the effectiveness of our results.

2. Preliminaries and basic properties

In this section, we provide definitions, lemmas and notations necessary to establish our main results. Throughout this paper, we use the following notations. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\mathcal{F}_t, t \in J = [0, T]$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). We consider three real separable spaces X, E and U, and Q-Wiener process on $(\Omega, \mathcal{F}_T, \mathbb{P})$ with a linear bounded covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}_{n\geq 1}$ on E, a bounded sequence of non-negative real numbers $\{\lambda_n\}$ such that $Qe_n = \lambda_n e_n$, $n = 1, 2, \ldots$ and a sequence $\{\beta_n\}_{n\geq 1}$ of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle \beta_n(t), \quad e \in E, t \in [0, T],$$

and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the sigma algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Q^{1/2}E; X)$ be the Banach space of all $\mathcal{F}_T$-measurable square integrable random variables with values in the Hilbert space X. Let $\mathbb{E}(.)$ denote the expectation with respect to the measure $\mathbb{P}$. Let $C([0, T]; L^2(\mathcal{F}, X))$ be the Banach space of continuous maps from $[0, T]$ into $L^2(\mathcal{F}, X)$ satisfying $\sup_{t\in J} \mathbb{E}\|x(t)\|^2 < \infty$. Let $H_2([0, T]; X)$ be the closed subspace of $C([0, T]; L^2(\mathcal{F}, X))$ consisting of all measurable and $\mathcal{F}_t$-adapted X-valued process $x \in C([0, T]; L^2(\mathcal{F}, X))$ endowed with the norm $\|x\|_{H_2} = (\sup_{t\in J} \mathbb{E}\|x(t)\|_X^2)^{1/2}$.

The purpose of this paper is to investigate the relative controllability for a class of semilinear stochastic fractional differential equation with nonlocal conditions of the form

$$\begin{aligned} {}^cD_t^\alpha x(t) + Ax(t) &= Bu(t) + f(t, x(t)) + \sigma(t, x(t))\frac{dw(t)}{dt}, \quad t \in J = [0, T], \\ x(0) + g(x) &= x_0, \end{aligned} \tag{1}$$

where $0 < \alpha < 1$; ${}^cD_t^\alpha$ denotes the Caputo fractional derivative operator of order α; $x(.)$ takes its values in the Hilbert space X; $A : D(A) \subset X \to X$ is the infinitesimal generator of an α-resolvent family $\{S_\alpha(t), t \geq 0\}$; the control function $u(.)$ is given in $L^2_{\mathcal{F}}([0, T], U)$ of admissible control functions, U is a Hilbert space. B is a bounded linear operator from U into X; $f : J \times X \to X$ and $\sigma : J \times X \to L_2^0$ are appropriate functions to be specified later; x_0 is a suitable initial random function independent of $w(t)$ and $g \in C(X, X)$ is a given function.

Let us recall the following known definitions. For more details see [10].

Definition 2.1. *The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.2. *Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, n-1 < \alpha < n. \tag{2}$$

Definition 2.3. *The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^c D^\alpha f(t) = {}^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0) \right), \quad t > 0, n-1 < \alpha < n. \tag{3}$$

If $f(t) \in C^n[0, \infty)$, then

$$^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds = I^{n-\alpha} f^n(s), \quad t > 0, n-1 < \alpha < n$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$\mathcal{L}\{^c D^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); \quad n-1 \leq \alpha < n.$$

Definition 2.4. *A two parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \quad \alpha, \beta \in C, \mathcal{R}(\alpha) > 0,$$

where C is a contour which starts and ends at $-\infty$ end encircles the disc $|\mu| \leq |z|^{1/2}$ counter clockwise.

For short, $E_\alpha(z) = E_{\alpha,1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_1(z) = e^z$ and the cosine function: $E_2(z^2) = \cos h(z)$, $E_2(-z^2) = \cos(z)$, and plays a vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad Re\lambda > \omega^{\frac{1}{\alpha}}, \omega > 0,$$

and for more details see [10].

Definition 2.5 ([27]). *A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied:*
- $\rho(A) \subset \Sigma_{\theta,\omega} = \{\lambda \in C : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}$,
- $\|R(\lambda, A)\| \leq \frac{M}{|\lambda-\omega|}$, $\lambda \in \Sigma_{\theta,\omega}$.

Definition 2.6. *Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(X)$, where $L(X)$ is a Banach space of all bounded linear operators from X into X and the corresponding norm is denoted by $\|.\|$, such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{\lambda t} S_\alpha(t) x dt, \quad Re\lambda > \omega, x \in X, \tag{4}$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.7. *Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)xdt, \quad Re\lambda > \omega, x \in X, \tag{5}$$

where $S_\alpha(t)$ is called the solution operator generated by A.

The concept of the solution operator is closely related to the concept of a resolvent family. For more details on α-resolvent family and solution operators, we refer the reader to [10].

Now, we give the definition of the mild solution of (1) based on the paper [20].

Definition 2.8 ([20]). *A continuous stochastic process $x : J \to X$ is called a mild solution of* (1) *if the following conditions hold:*

(**i**) *$x(t)$ is measurable and $\mathcal{F}_t$-adapted.*
(**ii**) $x(0) + g(x) = x_0$.
(**iii**) *x satisfies the following equation*

$$x(t) = T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t-s)[Bu(s) + f(s, x(s))]ds + \int_0^t S_\alpha(t-s)\sigma(s, x(s))dw(s), \tag{6}$$

where $T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i}\int_{\hat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda$, $S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i}\int_{\hat{B}_r} e^{\lambda t}\frac{1}{\lambda^\alpha - A}d\lambda$, $\hat{B}_r$ denotes the Bromwich path, $S_\alpha(t)$ is the α-resolvent family and $T_\alpha(t)$ is the solution operator generated by $-A$.

Definition 2.9 ([26]). *Let $x_T(x_0; u)$ be the state value of* (1) *at the terminal time T corresponding to the control u and the initial value x_0. Introduce the set*

$$\mathbf{R}(T, x_0) = \{x(T) = x_T(x_0; u) : u(.) \in L^2_{\mathcal{F}}([0, T], U)\},$$

which is called the reachable set of (1) *at the terminal time T. Then the controlled system* (1) *is said to be relatively controllable at T if $\mathbf{R}(T, x_0) = L^2(\Omega, \mathcal{F}_T, X)$.*

Definition 2.10 ([26]). *The control system* (1) *is said to be relatively approximately controllable at T if the closure set $\overline{\mathbf{R}(T, x_0)} = L^2(\Omega, \mathcal{F}_T, X)$.*

To study the relative controllability of the fractional system (1), we will introduce the following equivalent conditions.

Lemma 2.11 ([16]). *The following conditions are equivalent:*
(**iv**) *The corresponding linear system with respect to* (1) *is relatively controllable on* $[0, T]$.
(**v**) *The corresponding linear system with respect to* (1) *is relatively approximately controllable on* $[0, T]$.
(**vi**) *The corresponding linear deterministic system with respect to* (1) *is relatively controllable on* $[0, T]$.

The following lemma is required to define the control function. The reader can refer to [17] for the proof.

Lemma 2.12. *For any $\tilde{x}_T \in L^2(\mathcal{F}_T, X)$, there exists $\tilde{g} \in L^2_{\mathcal{F}}(\Omega; L^2(0, T; L^0_2))$ such that $\tilde{x}_T = \mathbb{E}\tilde{x}_T + \int_0^T \tilde{g}(s)dw(s)$.*

Now, we define the control function in the following form

$$\begin{aligned} u(t,x) &= B^\star S^\star_\alpha(T-s)\Big((\psi_0^T)^{-1}[\mathbb{E}\tilde{x}_T - T_\alpha(T)(x_0 - g(x))] + \int_0^t (\psi_0^T)^{-1}\tilde{g}(s)dw(s)\Big) \\ &- B^\star S^\star_\alpha(T-t)\int_0^t (\psi_0^T)^{-1} S_\alpha(T-s)f(s,x(s))ds \\ &- B^\star S^\star_\alpha(T-t)\int_0^t (\psi_0^T)^{-1} S_\alpha(T-s)\sigma(s,x(s))dw(s), \end{aligned}$$

where $\psi_0^T = \int_0^T S_\alpha(T-s)BB^\star S^\star_\alpha(T-s)$ is the controllability Gramian, $B^\star$ denotes the adjoint of B and $S^\star_\alpha(t)$ the adjoint of $S_\alpha(t)$.

3. Controllability results

In this section it will be shown that the system (1) is relatively (approximately) controllable under appropriate conditions.

Let us assume the following conditions:

(**vii**) The corresponding linear system with respect to (1) is relatively controllable

(**viii**) If $\alpha \in (0,1)$ and $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$, then for $x \in X$ and $t > 0$ we have $\|T_\alpha(t)\| \leq Me^{\omega t}$ and $\|S_\alpha(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1})$, $\omega > \omega_0$. Thus we have

$$\|T_\alpha(t)\| \leq \widetilde{M}_T \quad \text{and} \quad \|S_\alpha(t)\| \leq t^{\alpha-1}\widetilde{M}_S,$$

where $\widetilde{M}_T = \sup_{0\leq t\leq T} \|T_\alpha(t)\|$, and $\widetilde{M}_S = \sup_{0\leq t\leq T} Ce^{\omega t}(1 + t^{1-\alpha})$ (fore more details, see [27]).

(**ix**) $f \in C(J \times X, X)$, $g \in C(X, X)$ and $\sigma \in C(J \times X, L_2^0)$. Moreover, there exists a constant $C_1 > 0$ such that for $x \in X$, $\mathbb{E}\|g(x)\|_X^2 \leq C_1$, and for $s \in J$, $x \in B_r$ there exist two continuous functions $\widetilde{L}_f, \widetilde{L}_\sigma : J \to (0,\infty)$ such that

$$\mathbb{E}\|f(t,x)\|_X^2 \leq \widetilde{L}_f(t)\phi(\mathbb{E}\|x\|_X^2), \qquad \mathbb{E}\|\sigma(t,x)\|_{L_2^0}^2 \leq \widetilde{L}_\sigma(t)\varphi(\mathbb{E}\|x\|_X^2),$$

where $\phi, \varphi : [0,\infty) \to (0,\infty)$ are a continuous nondecreasing functions with

$$\int_0^T \xi(s)ds \leq \int_c^\infty \frac{ds}{\phi(s) + \varphi(s)},$$

where $\xi(t) = \max\left\{\frac{5\widetilde{M}_S^2 T^\alpha}{\alpha} t^{\alpha-1}\eta\widetilde{L}_f(t), 5\widetilde{M}_S^2 t^{2(\alpha-1)}\eta\widetilde{L}_\sigma(t)\right\}$, $\quad c = 5\widetilde{M}_T^2(\mathbb{E}\|x_0\|_X^2 + C_1)$, and $\eta = \left[1 + 3\widetilde{M}_S^4 \frac{T^{2\alpha}}{2\alpha-1}\|B\|^4 T^{2\alpha-1} l^2 \frac{T^\alpha}{\alpha}\right]$.

Our result is based on the following Schaefer's fixed point theorem.

Theorem 3.1. *Let K be a closed convex subset of a Banach space H such that $0 \in K$. Let $\mathcal{P} : K \to K$ be a completely continuous map. Then the set $\{x \in K; x = \nu\mathcal{P}x; 0 \leq \nu \leq 1\}$ is unbounded or $\mathcal{P}$ has a fixed point.*

Theorem 3.2. *The fractional stochastic system* (1) *is relatively controllable if* (**vii**)-(**ix**) *are satisfied.*

Proof. First, it will be show that the fractional stochastic system (1) has at least one mild solution on J.

Let $\lambda : H_2 \to H_2$ be the operator defined by

$$(\lambda x)(t) = T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t-s)[Bu(s,x) + f(s,x(s))]ds + \int_0^t S_\alpha(t-s)\sigma(s,x(s))dw(s).$$

In order to use the Schaefer's fixed point theorem, it will be shown that λ is a completely continuous operator. We note that the operator λ is well defined in H_2.
For the sake of convenience, we divide the proof into several steps.

Step 1. We prove that λ is continuous.
Let $\{x^n\}_{n=0}^{\infty}$ be a sequence in H_2 such that $x^n \to x$ in H_2. Since the function f, g, u and σ are continuous, $\lim_{n\to\infty} \mathbb{E}\|\lambda x^n(t) - \lambda x(t)\|_X^2 = 0$ in H_2 for every $t \in J$. This implies that the mapping λ is continuous on H_2.

Step 2. Next we prove that λ maps bounded sets into bounded sets in H_2.
To prove that for any $r > 0$, there exists a $\gamma > 0$ such that for $x \in B_r = \{x \in H_2 : \mathbb{E}\|x\|_X^2 \le r\}$, we have $\mathbb{E}\|\lambda x\|_X^2 \le \gamma$. For any $x \in B_r$, $t \in J$, we have

$$\begin{aligned} & \mathbb{E}\|\lambda x(t)\|_X^2 \\ \le\ & 5\|T_\alpha(t)\|^2 \mathbb{E}\|x_0\|_X^2 + 5\|T_\alpha(t)\|^2 \mathbb{E}\|g(x)\|_X^2 + 5\int_0^t \|S_\alpha(t-s)\|ds \times \int_0^t \|S_\alpha(t-s)\|\mathbb{E}\|f(s,x(s))\|_X^2 ds \\ +\ & 5\int_0^t \|S_\alpha(t-s)\|ds \times \int_0^t \|S_\alpha(t-s)\|\mathbb{E}\|Bu(s,x)\|^2 ds + 5\int_0^t \|S_\alpha(t-s)\|\mathbb{E}\|\sigma(s,x(s))\|_{L_2^0}^2 ds. \end{aligned}$$

For simplicity, let $L_{\tilde{g}} = \max\{\|\tilde{g}(s)\|^2 : s \in [0,T]\}$. Note that if (**vii**) folds, the operator ψ_0^T is strictly positive definite and thus the inverse linear operator $(\psi_0^T)^{-1}$ is bounded, say, by l (see [14] for more details).
We have

$$\begin{aligned} & \mathbb{E}\|u(s,x)\|^2 \\ \le\ & 3\mathbb{E}\left\| B^\star S_\alpha^\star(T-t)\left((\psi_0^T)^{-1}[\mathbb{E}\tilde{x}_T - T_\alpha(T)(x_0 - g(x))] + \int_0^t (\psi_0^T)^{-1}\tilde{g}(s)dw(s)\right)\right\|^2 \\ +\ & 3\mathbb{E}\left\| B^\star S_\alpha^\star(T-t)\int_0^t (\psi_0^T)^{-1} S_\alpha(T-s) f(s,x(s))ds \right\|^2 \\ +\ & 3\mathbb{E}\left\| B^\star S_\alpha^\star(T-t)\int_0^t (\psi_0^T)^{-1} S_\alpha(T-s)\sigma(s,x(s))dw(s) \right\|^2 \\ \le\ & 3\|B\|^2 T^{2\alpha-2}\widetilde{M}_S^2 l^2\left[\mathbb{E}\|\tilde{x}_T\|^2 + \widetilde{M}_T^2 r + \widetilde{M}_T^2 C_1 + TL_{\tilde{g}}\right] + 3\|B\|^2 T^{2\alpha-2}\widetilde{M}_S^4 l^2 \frac{T^\alpha}{\alpha} \\ & \times \int_0^t (T-s)^{\alpha-1}\widetilde{L}_f(s)ds + 3\|B\|^2 T^{2\alpha-2}\widetilde{M}_S^4 l^2 \varphi(r)\int_0^t (T-s)^{2\alpha-2}\widetilde{L}_\sigma(s)ds. \end{aligned}$$

Thus

$$\begin{aligned} & \mathbb{E}\|\lambda x(t)\|_X^2 \\ \le\ & 5\widetilde{M}_T^2 r + 5\widetilde{M}_T^2 C_1 + 15\widetilde{M}_S^4 \frac{T^\alpha}{\alpha}\|B\|^4 \frac{t^{2\alpha-1}}{2\alpha-1} T^{2\alpha-2} l^2\left[\mathbb{E}\|\tilde{x}_T\|^2 + \widetilde{M}_T^2 r + \widetilde{M}_T^2 C_1 + TL_{\tilde{g}}\right] \\ +\ & 5\widetilde{M}_S^2 \frac{T^\alpha}{\alpha}\phi(r)\left[1 + 3\widetilde{M}_S^4 \frac{t^{2\alpha-1}}{2\alpha-1}\|B\|^4 T^{2\alpha-1} l^2 \frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{\alpha-1}\widetilde{L}_f(s)ds \\ +\ & 5\widetilde{M}_S^2 \varphi(r)\left[1 + 3\widetilde{M}_S^4 \frac{t^{2\alpha-1}}{2\alpha-1}\|B\|^4 T^{2\alpha-1} l^2 \frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{2\alpha-2}\widetilde{L}_\sigma(s)ds \\ =\ & \gamma, \quad t \in J. \end{aligned}$$

Step 3. We show that λ maps bounded sets into equicontinuous sets of B_r.

Let $0 < t_1 < t_2 \le T$, for each $x \in B_r$, we have

$$\begin{aligned}
&\mathbb{E}\|\lambda x(t_2) - \lambda x(t_1)\|_X^2 \\
\le\ & 8\|T_\alpha(t_2) - T_\alpha(t_1)\|^2 \mathbb{E}\|x_0\|_X^2 + 8\|T_\alpha(t_2) - T_\alpha(t_1)\|^2 \mathbb{E}\|g(x)\|_X^2 \\
+\ & 8\mathbb{E}\left\| \int_0^{t_1} [S_\alpha(t_2 - s) - S_\alpha(t_1 - s)] f(s, x(s)) ds \right\|_X^2 + 8\mathbb{E}\left\| \int_{t_1}^{t_2} S_\alpha(t_2 - s) f(s, x(s)) ds \right\|_X^2 \\
+\ & 8\mathbb{E}\left\| \int_0^{t_1} [S_\alpha(t_2 - s) - S_\alpha(t_1 - s)] \sigma(s, x(s)) dw(s) \right\|_X^2 + 8\mathbb{E}\left\| \int_{t_1}^{t_2} S_\alpha(t_2 - s) \sigma(s, x(s)) dw(s) \right\|_X^2 \\
+\ & 8\mathbb{E}\left\| \int_0^{t_1} [S_\alpha(t_2 - s) - S_\alpha(t_1 - s)] Bu(s, x) ds \right\|_X^2 + 8\mathbb{E}\left\| \int_{t_1}^{t_2} S_\alpha(t_2 - s) Bu(s, x) ds \right\|_X^2.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\mathbb{E}\|\lambda x(t_2) - \lambda x(t_1)\|_X^2 \\
\le\ & 8(r + C_1)\|T_\alpha(t_2) - T_\alpha(t_1)\|^2 + \int_0^{t_1} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| ds \\
\times\ & \int_0^{t_1} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| \mathbb{E}\|f(s, x(s))\|_X^2 ds \\
+\ & 8\int_{t_1}^{t_2} \|S_\alpha(t_2 - s)\| ds \int_{t_1}^{t_2} \|S_\alpha(t_2 - s)\| \mathbb{E}\|f(s, x(s))\|_X^2 ds \\
+\ & \int_0^{t_1} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| ds \int_0^{t_1} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| \|B\|^2 \mathbb{E}\|u(s, x)\|^2 ds \\
+\ & 8\int_{t_1}^{t_2} \|S_\alpha(t_2 - s)\| ds \int_{t_1}^{t_2} \|S_\alpha(t_2 - s)\| \|B\|^2 \mathbb{E}\|u(s, x)\|_X^2 ds \\
+\ & 8\int_0^{t_1} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\|^2 \mathbb{E}\|\sigma(s, x(s))\|_{L_2^0}^2 ds + 8\int_{t_1}^{t_2} \|S_\alpha(t_2 - s)\|^2 \mathbb{E}\|\sigma(s, x(s))\|_{L_2^0}^2 ds.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}\|\lambda x(t_2) - \lambda x(t_1)\|_X^2 \\
\le\ & 8(r + C_1)\|T_\alpha(t_2) - T_\alpha(t_1)\|^2 + 8\phi(r)\tilde{\eta} \int_0^{t_1} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| ds \\
\times\ & \int_0^{t_1} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| \widetilde{L}_f(s) ds \\
+\ & 8\widetilde{M}_S^2 \frac{(t_2 - t_1)^\alpha}{\alpha} \phi(r) \left[1 + 3\widetilde{M}_S^4 \frac{t^{2\alpha-1}}{2\alpha - 1} \|B\|^4 T^{2\alpha-1} l^2 \frac{T^\alpha}{\alpha} \right] \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \widetilde{L}_f(s) ds \\
+\ & 8\varphi(r)\tilde{\eta} \int_0^{t_1} \|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\|^2 \widetilde{L}_\sigma(s) ds \\
+\ & 8\widetilde{M}_S^2 \varphi(r) \left[1 + 3\widetilde{M}_S^4 \frac{t^{2\alpha-1}}{2\alpha - 1} \|B\|^4 T^{2\alpha-1} l^2 \frac{T^\alpha}{\alpha} \right] \int_{t_1}^{t_2} (t_2 - s)^{2\alpha-2} \widetilde{L}_\sigma(s) ds,
\end{aligned}$$

where $\tilde{\eta}$ is a positive constant depending only on α, l, B, T and $\widetilde{M}_S$. Since $T_\alpha(t)$ and $S_\alpha(t)$ are strongly continuous, $\|T_\alpha(t_2) - T_\alpha(t_1)\| \to 0$ and $\|S_\alpha(t_2 - s) - S_\alpha(t_1 - s)\| \to 0$ as $t_1 \to t_2$. Thus, from the above inequality we have $\lim_{t_1 \to t_2} \mathbb{E}\|\lambda x(t_2) - \lambda x(t_1)\|_X^2 = 0$. Thus, the set $\{\lambda x, x \in B_r\}$ is equicontinuous. Finally, combining Step 1 to 3 with Ascoli's theorem, we conclude that the operator λ is compact.

Step 4. Next, we show that the set

$$N = \{x \in H_2 \text{ such that } x = q\lambda x(t) \text{ for some } 0 < q < 1\}$$

is bounded. Let $x \in N$ then $x(t) = q\lambda x(t)$ for some $0 < q < 1$. Then for each $t \in J$, we have

$$x(t) = q\left(T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t - s)[Bu(s, x) + f(s, x(s))] ds + \int_0^t S_\alpha(t - s)\sigma(s, x(s)) dw(s) \right),$$

which implies that

$$\begin{aligned}
&\mathbb{E}\|x(t)\|_X^2\\
\le\ & 5\|T_\alpha\|^2\mathbb{E}\|x_0\|_X^2 + 5\|T_\alpha\|^2\mathbb{E}\|g(x)\|_X^2 + 5\int_0^t \|S_\alpha(t-s)\|ds\int_0^t \|S_\alpha(t-s)\|\mathbb{E}\|f(s,x(s))\|_X^2 ds\\
+\ & 5\int_0^t \|S_\alpha(t-s)\|ds\int_0^t \|S_\alpha(t-s)\|\mathbb{E}\|Bu(s,x)\|^2 ds + 5\int_0^t \|S_\alpha(t-s)\|^2\mathbb{E}\|\sigma(s,x(s))\|_{L_2^0}^2 ds\\
\le\ & 5\widetilde{M}_T^2\mathbb{E}\|x_0\|_X^2 + 5\widetilde{M}_T^2 C_1\\
+\ & 5\widetilde{M}_S^2\frac{T^\alpha}{\alpha}\left[1+3\widetilde{M}_S^4\frac{T^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{\alpha-1}\widetilde{L}_f(s)\phi(\mathbb{E}\|x(s)\|_X^2)ds\\
+\ & 5\widetilde{M}_S^2\left[1+3\widetilde{M}_S^4\frac{T^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{2\alpha-2}\widetilde{L}_\sigma(s)\varphi(\mathbb{E}\|x(s)\|_X^2)ds.
\end{aligned}$$

Consider the function $\mu(t)$ defined by
$\mu(t) = \sup\{\mathbb{E}\|x(s)\|_X^2; 0 \le s \le t\}, 0 \le t \le T.$

$$\begin{aligned}
\mu(t)\ \le\ & 5\widetilde{M}_T^2[\mathbb{E}\|x_0\|_X^2 + C_1]\\
+\ & 5\widetilde{M}_S^2\frac{T^\alpha}{\alpha}\left[1+3\widetilde{M}_S^4\frac{T^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{\alpha-1}\widetilde{L}_f(s)\phi(\mu(s))ds\\
+\ & 5\widetilde{M}_S^2\left[1+3\widetilde{M}_S^4\frac{T^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{2\alpha-2}\widetilde{L}_\sigma(s)\varphi(\mu(s))ds.
\end{aligned}$$

Denoting by $\nu(t)$ the right hand side of the last inequality, we have $\nu(0) = c = 5\widetilde{M}_T^2[\mathbb{E}\|x_0\|_X^2 + C_1]$, $\mu(t) \le \nu(t)$, $t \in J$.
Moreover,

$$\begin{aligned}
\nu'(t)\ =\ & 5\widetilde{M}_S^2\frac{T^\alpha}{\alpha}\left[1+3\widetilde{M}_S^4\frac{T^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]t^{\alpha-1}\widetilde{L}_f(t)\phi(\mu(t))\\
+\ & 5\widetilde{M}_S^2\left[1+3\widetilde{M}_S^4\frac{T^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]t^{2\alpha-2}\widetilde{L}_\sigma(t)\varphi(\mu(t))\\
\le\ & 5\widetilde{M}_S^2\frac{T^\alpha}{\alpha}\left[1+3\widetilde{M}_S^4\frac{T^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]t^{\alpha-1}\widetilde{L}_f(t)\phi(\nu(t))\\
+\ & 5\widetilde{M}_S^2\left[1+3\widetilde{M}_S^4\frac{T^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]t^{2\alpha-2}\widetilde{L}_\sigma(t)\varphi(\nu(t)),
\end{aligned}$$

or equivalently by (**ix**), we have

$$\int_{\nu(0)}^{\nu(t)}\frac{ds}{\phi(s)+\varphi(s)} \le \int_0^T \xi(s)ds < \int_c^\infty \frac{ds}{\phi(s)+\varphi(s)}, \quad 0 \le t \le T.$$

This inequality implies that there is a constant k such that $\nu(t) \le k$, $t \in J$, and hence, $\mu(t) \le k$. Furthermore, we get $\|x(t)\|^2 \le \mu(t) \le \nu(t) \le k$, $t \in J$. By the Schaefer's fixed point theorem, we deduce that λ has a fixed point $x(t)$ on J, with $x(T) = x_T$, which is a mild solution of (1). That means it is along this trajectory that the solution of (1) will be steered by u from x_0 to x_T. That completes the proof.

□

In order to study the approximate controllability for the fractional stochastic control system (1), we introduce the approximate controllability of its linear part

$$\begin{aligned}
{}^cD_t^\alpha x(t) &= Ax(t) + (Bu)(t), \quad t \in J = [0,T],\\
x(0) + g(x) &= x_0.
\end{aligned} \tag{7}$$

For this purpose, we need to introduce the relevant operator

$$\begin{aligned} \psi_0^T &= \int_0^T S_\alpha(T-s)BB^\star S_\alpha^\star(T-s) \\ R(q,\psi_0^T) &= (qI+\psi_0^T)^{-1}, \end{aligned}$$

where $q>0$ and ψ_0^T is a linear bounded operator.

We assume the following additional conditions

(**x**) $qR(q,\psi_0^T)\to 0$ as $q\to 0^+$ in the strong operator topology.

(**xi**) $f(t,x): J\times X\to X$ and $\sigma(t,x): J\times X\to L_2^0$ are bounded for $t\in J$ and $x\in X$.

Remark 3.3. *From [17] (Theorem 2), the condition* (**x**) *is equivalent to the fact that the linear fractional control system (7) is approximately controllable on* $J:=[0,T]$. *Hence, by Lemma 2.11,* (**vii**) *is equivalent to* $qR(q,\psi_0^T):=(qI+\psi_0^T)^{-1}\to 0$ *as* $q\to 0^+$ *in the strong operator. Moreover,* (**vii**) *can be replaced by the following more verifiable criterion:*
There exists some positive constant $\tilde{\gamma}$ *such that* $\langle\psi_s^T z,z\rangle\le\tilde{\gamma}\|z\|^2$ *for all* $0\le s<T$ *and all* $z\in X$.

Theorem 3.4. *Under the conditions* (**vii**) – (**xi**)*, and if* $S_\alpha(t)$ *is a compact, then system* (1) *is relatively approximately controllable on* $[0,T]$.

Proof. For all $q>0$ define the control function as

$$\begin{aligned} u^q(t,x) &= B^\star S_\alpha^\star(T-s)\bigg((qI+\psi_0^T)^{-1}[\mathbb{E}\tilde{x}_T - T_\alpha(T)(x_0-g(x))] + \int_0^t (qI+\psi_0^T)^{-1}\tilde{g}(s)dw(s)\bigg) \\ &- B^\star S_\alpha^\star(T-t)\int_0^t (qI+\psi_0^T)^{-1}S_\alpha(T-s)f(s,x(s))ds \\ &- B^\star S_\alpha^\star(T-t)\int_0^t (qI+\psi_0^T)^{-1}S_\alpha(T-s)\sigma(s,x(s))dw(s), \end{aligned} \tag{8}$$

and the operator $\lambda_q: H_2\to H_2$ as follows

$$(\lambda_q x)(t) = T_\alpha(t)(x_0-g(x)) + \int_0^t S_\alpha(t-s)[Bu^q(s,x)+f(s,x(s))]ds + \int_0^t S_\alpha(t-s)\sigma(s,x(s))dw(s). \tag{9}$$

Replacing l^2 with $\frac{1}{q^2}$ and using the same procedure as in the proof of Theorem 3.2, one can prove that λ_q has a unique fixed point x_q.

By using the stochastic Fubini theorem, it is easy to see that

$$\begin{aligned} x_q(T) &= \tilde{x}_T - q(qI+\psi)^{-1}[\mathbb{E}\tilde{x}_T - T_\alpha(T)(x_0-g(x))] + q\int_0^T (qI+\psi_s^T)^{-1}S_\alpha(T-s)f(s,x_q(s))ds \\ &+ q\int_0^T (qI+\psi_s^T)^{-1}[S_\alpha(T-s)\sigma(s,x_q(s)) - \tilde{g}(s)]dw(s). \end{aligned} \tag{10}$$

It follows from the properties of f and σ that $\|f(s,x_q(s))\|^2+\|\sigma(s,x_q(s))\|^2\le L_1$. Then there is a subsequence denoted by $\{f(s,x_q(s)),\sigma(s,x_q(s))\}$ weakly converging to say $\{f(s),\sigma(s)\}$. Thus from the above equation, we have

$$\begin{aligned}
\mathbb{E}\|x_q(T)-\tilde{x}_T\|^2 &\leq 6\|q(qI+\psi_0^T)^{-1}[\mathbb{E}\tilde{x}_T - T_\alpha(T)(x_0-g(x))]\|^2 + 6\mathbb{E}\bigg(\int_0^T \|q(qI+\psi_s^T)^{-1}\tilde{g}(s)\|^2_{L_2^0}ds\bigg)\\
&+ 6\mathbb{E}\bigg(\int_0^T \|q(qI+\psi_0^T)^{-1}\|\|S_\alpha(T-s)(f(s,x_q(s))-f(s))\|ds\bigg)^2\\
&+ 6\mathbb{E}\bigg(\int_0^T \|q(qI+\psi_s^T)^{-1}S_\alpha(T-s)f(s)\|ds\bigg)^2\\
&+ 6\mathbb{E}\bigg(\int_0^T \|q(qI+\psi_s^T)^{-1}\|\|S_\alpha(T-s)(\sigma(s,x_q(s))-\sigma(s))\|^2_{L_2^0}ds\bigg)\\
&+ 6\mathbb{E}\bigg(\int_0^T \|q(qI+\psi_s^T)^{-1}S_\alpha(T-s)\sigma(s)\|^2_{L_2^0}ds\bigg).
\end{aligned}$$

On the other hand, by assumption (**x**) for all $0 \leq s \leq T$, the operator $q(qI+\psi_s^T)^{-1} \to 0$ strongly as $q \to 0^+$, and moreover $\|q(qI+\psi_s^T)^{-1}\| \leq 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $S_\alpha(t)$, we obtain $\mathbb{E}\|x_q(T)-\tilde{x}_T\|^2 \to 0$ as $q \to 0^+$. This gives the approximate controllability of (1). Hence the proof is complete.

□

4. Example

Consider the fractional partial stochastic differential equation in the following form

$$\begin{aligned}
{}^cD_t^\alpha[z(t,x)] &= \frac{\partial^2}{\partial x^2}[z(t,x)] + h(t,z(t,x)) + \tilde{h}(t,z(t,x))\frac{d\beta}{dt}, \quad 0 \leq t \leq T, 0 \leq x \leq \pi,\\
z(t,0) &= z(t,\pi) = 0,\\
z(0,x) &+ \sum_{i=0}^{p}\int_0^\pi K(x,y)z(t,y)dy = z_0(x), \quad 0 \leq x \leq \pi,
\end{aligned} \tag{11}$$

where p is a positive integer, $T \leq \pi$, $0 < t_0 < t_1, \ldots < t_p < T$, $z_0(x) \in X = L^2([0,\pi])$, $K(x,y) \in L^2([0,\pi]\times[0,\pi])$ and ${}^cD_t^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$.

We take $X = L^2([0,\pi])$ and let A be the operator defined by $Ay = y''$ with domain $D(A) = \{y \in X : y, y'$ are absolutely continuous $, y'' \in X, y(0) = y(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ in X. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2$ and the corresponding normalized eigenfunctions are given by $x_n(z) = \sqrt{\frac{2}{\pi}}\sin(nz)$. In addition $\{x_n : n \in \mathbb{N}\}$ is an orthonormal basis for X, $T(t)y = \sum_{n=1}^{\infty} e^{-n^2t}(y,x_n)x_n$, for all $y \in X$, and every $t > 0$. From these expressions it follows that $\{T(t)\}_{t\geq 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda,A) = (\lambda - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$ i.e., $A \in \mathcal{A}^\alpha(\theta_0,\omega_0)$.

To represent the above fractional system (11) into the abstract form of (1), we introduce the functions $f : J\times X \to X$, $\sigma : J\times X \to L_2^0$ and $g : X \to X$ by $f(t,z)(x) = h(t,z(x))$, $\sigma(t,z)(x) = \tilde{h}(t,z(x))$ and $g(\omega) = \sum_{i=0}^{p} K\omega(t_i)$, where $K(z)(x) = \int_0^\pi K(x,y)z(y)dy$. Thus f, σ and g satisfy the assumption of Theorem 3.2. Hence by Theorem 3.2, if the corresponding linear system with respect to (11) is relatively controllable, then system (11) is relatively controllable on $[0,T]$. So one can deduce by Lemma 2.11 that the linear system corresponding to (11) is relatively approximately controllable, therefore (**x**) is fulfilled. In addition, if (**xi**) is satisfied, then system (11) is relatively approximately controllable on $[0,T]$ by Theorem 3.4.

5. Conclusion

This paper has investigated the relative controllability for a class of dynamical control systems described by semilinear fractional stochastic differential equations with nonlocal conditions in Hilbert space. A new set of sufficient conditions for the relative controllability of the considered system have been formulated and proved. As the differential inclusion system is considered as a generalization of the system described by differential equations, it should be pointed out that under some suitable conditions on f and σ, one can establish the relative controllability of fractional stochastic differential inclusions with nonlocal conditions by adapting the techniques and ideas established in this paper and suitably introducing the technique of single valued maps defined in [4]. This is one of our future goals.

References

[1] K. Balachandran, Yong Zhou, J. Kokila, Relative controllability of fractional dynamical systems with delays in control, Commun Nonlinear Sci Numer Simulat 17 (2012) 3508–3520
[2] K. Balachandran, J.P Dauer, Controllability of nonlinear systems via fixed point theorems, J Optim Theory Appl 53 (1987) 345–52.
[3] K. Balachandran, J.P Dauer, Controllability of nonlinear systems in Banach spaces: a survey, J Optim Theory Appl 115 (2002) 7–28.
[4] P. Balasubramaniam, S.K. Ntouyas, D. Vinayagam, Existence of solutions of semilinear stochastic delay evolution inclusions in a Hilbert space, J. Math. Anal. Appl. 305 (2005) 438–451.
[5] M. Benchohra, J. Henderson, S.K. Natouyas, A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, J. Math. Anal. Appl. 338 (2008) 1340–1350.
[6] Y.-K. Chang, Z.-H. Zhao, G.M. NGurkata, R. Ma, Stepanov-like almost automorphy for stochastic processes and applications to stochastic differential equations, Nonlinear Anal. RWA 12 (2011) 1130–1139.
[7] R. Hilfer, On fractional relaxation, Fractals 11 (2003) 251–257.
[8] R. Hilfer, Experimental evidence for fractional time evolution in glass forming materials, Chem. Phys. 284 (2002) 399–408.
[9] R. Hilfer, Application of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore, 2000.
[10] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V, Amsterdam, 2006.
[11] J. Klafter, S.C. Lim, R. Metzler, Fractional Dynamics, Recent Advances, World Scientific, Singapore, 2011.
[12] J. Klamka, Stochastic controllability of systems with multiple delays in control, International Journal of Applied Mathematics and Computer Science 19 (2009) 39–47.
[13] J. Klamka, Relative controllability of nonlinear systems with distributed delay in control, International Journal of Control 28 (1978) 307–312.
[14] J. Klamka, Stochastic controllability of systems with variable delay in control, Bull. Polon. A: Tech. 56 (2008) 279–284.
[15] J. Klamka, Stochastic controllability of linear systems with delay in control, Bulletin of the Polish Academy of Sciences: Technical Sciences 55 (2007) 23–29.
[16] J. Klamka, Stochastic controllability of linear systems with state delays, International Journal of Applied Mathematics and Computer Science 17 (2007) 5-13.
[17] N.I. Mahmudov, A. Denker, On controllability of linear stochastic systems, International Journal of Control 73 (2000) 144–151.
[18] X. Mao, Stochastic Differential Equations and Applications, Horwood, Chichester, 1997.
[19] Y. Ren, L. Hu, R. Sakthivel, Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay, Journal of Computational and Applied Mathematics , 235 (2011) 2603–2614.
[20] R. Sakthivel, P. Revathi, Yong Ren, Existence of solutions for nonlinear fractional stochastic differential equations, Nonlinear Anal. TMA 81 (2013) 70–86.
[21] R. Sakthivel, S. Suganya, S.M. Anthoni, Approximate controllability of fractional stochastic evolution equations, Computers & Mathematics with Applications 63 (2012) 660–668.
[22] R. Sakthivel, R. Ganesh, S. Suganya, Approximate controllability of fractional neutral stochastic system with infinite delay, R. Math. Physics. 70 (2012) 291–311.
[23] R. Sakthivel, N.I. Mahmudov, J.J. Nieto, Controllability for a class of fractional-order neutral evolution control systems, Applied Mathematics and Computation, 218 (2012) 10334–10340
[24] R. Sakthivel, Y. Ren, Complete controllability of stochastic evolution equations with jumps, Reports on Mathematical Physics, 68 (2011) 163–174.
[25] R. Sakthivel, Y. Ren, N.I. Mahmudov, On the approximate controllability of semilinear fractional differential systems, Computers and Mathematics with Applications 62 (2011) 1451–1459.
[26] L. Shen, J. Sun, Relative controllability of stochastic nonlinear systems with delay in control, Nonlinear Analysis: RWA 13 (2012) 2880–2887.
[27] X.B. Shu, Y. Lai, Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, Nonlinear Anal. TMA 74 (2011) 2003–2011.