Dedication

There are a number of people without whom this thesis might not have been written, and to whom I am greatly indebted.

To my mother, who continues to learn, grow and develop and who has been a source of encouragement and inspiration to me throughout my life, a very special thank you for providing a "writing space" and for nurturing me through the months of writing. And also for the myriad of ways in which, throughout my life, you have actively supported me in my determination to find and realise my potential, and to make this contribution to our world.

To my father, who taught me that the best kind of knowledge to have is that which is learned for its own sake. My sister Nedjma, has never left my side and is very special. My brothers Mohammed, Lakhdar and Nourdinne who have supported me throughout the process.

Loving thanks to my friends / learning partners, Soumia Idrissi, Soumia Dani, and Sara, who played such important roles along the journey, as we mutually engaged in making sense of the various challenges we faced and in providing encouragement to each other at those times when it seemed impossible to continue.

Acknowledgments

First of all, I thank Allah, the almighty, for giving me the strength to carry on this thesis and for blessing me with many great people who have been my greatest support in both my personal and professional life.

*T*here are a number of people without whom this thesis might not have been written, and to whom I am greatly indebted:

I would like to express my great appreciation and gratitude to my supervisor, Doctor *Toufik Geundouzi.* Thank you for your patience, advices and careful guidance, encouragement, answers to my questions.

I wish to thank president *Ouakkas Seddik* and members of the jury *Kandouci Abdeldjebbar*, *Gheriballah Abdelkader*, *Laksaci Ali*, *Attouch Mohammed*, for having accepted judging this noble work.

*M*any thanks to *Soumia Idrissi* and all my colleagues, Institute of Mathematics, for their support, encouragement. Specially, *Dani Soumia*(for their friendship), *Samiha Tigrine.*Thus, acknowledging them will be the least of what I can do to show my appreciation.

*T*he most important persons are saved for last: My parents deserve special mention for their inseparable support and prayers.

Contents

Introduction

During the past two decades, fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, and engineering [24, 43]. There has been a great deal of interest in the solutions of fractional differential equations in analytical and numerical senses. One can see the monographs of Kilbas et al. [28], Miller and Ross [33], Podlubny [38], and Lakshmikantham et al. [30] and the survey of Agarwal et al. [3, 5].

To study the theory of abstract differential equations with fractional derivatives in infinite dimensional spaces, the first step is how to introduce new concepts of mild solutions. A pioneering work has been reported by El-Borai [11]. Very recently, Hernández et al.[22] showed that some recent papers of fractional differential equations in Banach spaces were incorrect and used another approach to treat abstract equations with fractional derivatives based on the well developed theory of resolvent operators for integral equation. Moreover, Wang and Zhou [45], Zhou and Jiao [48] also introduced a suitable definition of mild solutions based on Laplace transform and probability density functions.

On the other hand, the theory of impulsive differential equations has become an active area of invetigation due to its applications in fields such as mechanics, electrical engineering, medicine, biology, and ecology. One can refer to [47] and the references therein. Recently, the problems of existence of solutions of impulsive differential equations have been extensively studied [13]. Benedetti in [6] proved an existence result for impulsive functional differential in Banach spaces. Obukhovskii and Yao [37] considered local and global existence results for semilinear functional differential inclusions with infinite delay and impulse characteristics in a Banach space. Some existence results were obtained for certain classes of functional differential equations in Banach spaces under assumption that the linear part generates an compact semigroup (see, e.g, [1, 2]). The problem of existence of solution in general and the existence of almost periodic solution in particular of impulsive differential equations have been generalized to stochastic differential equations with impulsive conditions [8, 4, 22].

We would like to mention that the impulsive effects also widely exist in fractional stochastic differential systems [21, 41], and it is important and necessary to discuss the qualitative properties for stochastic fractional equations with impulsive perturbations and delay.

At present, there are few works in the existence problem of impulsive fractional stochastic differentian equation with delay and the aim of this thesis is to fill this gap.

This thesis is structured as follows: The thesis begins in chapter 1 with a brief summary of the theory of stochastic and fractional calculus. In this chapter we will give definitions and properties of the needed theory. We briefly recall some basic notions of the Brownian motion, then we skim through the fractional Brownian motion we review rapidly the basic concepts, then we discuss integration with respect to Brownian motion and fractional Brownian motion.

Next, In chapter 2 , we briefly present some basic notations and preliminaries, and discuss the existence of solutions for a class of impulsive fractional stochastic differential equations with infinite delay by using some appropriate fixed point theorems and evolution system theory. This chapter is concluded with an example to illustrate the obtained results.

Finally, in chapter 3, we introduce a class of impulsive stochastic differential equations with delays, and the relating notations, definitions and lemmas which would be used later, in Section 2, a new sufficient condition is proposed to ensure the existence and uniqueness of mean square almost periodic solutions. In Section 3, an example is constructed to show the effectiveness of our results.

All chapters of this thesis are the subject of communications and publications.

Chapter 1

Preliminary Background

In this chapter the basic concepts and results concerning stochastic calculus of continuous stochastic processes in Euclidean spaces are established. We omit some introductory facts from probability theory. For more detail we refer the reader to [10, 16, 23, 27, 18]. We first start with stochastic process, Wiener process and fractional Brownian motion.

1.1 Notations and definitions

A stochastic process X is an umbrella term for any collection of random variables $\{X(t,w)\}$ depending on time t, which is defined on the same probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. Time can be discrete, for example, $t = 0, 1, 2, \dots$, or continuous, $t \geq 0$.

For fixed time t, the observation is described by a random variable which we denote by X_t or $X(t)$.
For fixed $\omega \in \Omega$, $X(t)$ is a single realization (single path) of this process. Any single path is a function of time t, $x_t = x(t)$, $t \geq 0$.

At a fixed time t, properties of the random variable $X(t)$ are described by a probability distribution of $X(t)$, $P(X(t) \leq x)$. A stochastic process is determined by all its finite dimensional distributions, that is, probabilities of the form

$$P(X(t_1) \leq x_1, X(t_2) \leq x_2, \dots, X(t_n) \leq x_n),$$

for any choice of time points $0 \leq t_1 < t_2 < \dots < t_n$, any $n \geq 1$ with $x_1, \dots, x_n \in \mathbb{R}$.

Definition 1.1.1 [*18*] *If all finite dimensional distributions of a stochastic process is Gaussian (multi normal), then the process is called a Gaussian process. Because, a multivariate normal distribution is determined by its mean and covariance matrix, a Gaussian process is determined by it mean function $m(t) = \mathbb{E}X(t)$ and covariance function $\gamma(t,s) = Cov\{X(t), X(s)\}$.*

1.2 Brownian Motion

We start by recalling the definition of Brownian motion, which is a fundamental example of a stochastic process. The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of Brownian motion can be constructed on the space $\Omega = \mathcal{C}_0(\mathbb{R}_+)$ of continuous real-valued functions on $\mathbb{R}_+$ started at 0.

Definition 1.2.1 [*35*] *The standard Brownian motion is a stochastic process* $(W(t))_{t\in\mathbb{R}_+}$ *such that*
(i) $W(0) = 0$ *almost surely,*
(ii) The sample trajectories $t \mapsto W(t)$ *are continuous, with probability* 1.
(iii) For any finite sequence of times $t_0 < t_1 < \ldots < t_n$, *the increments*

$$W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$$

are independent.
(iv) For any given times $0 \leq s < t, W(t) - W(s)$ *has the Gaussian distribution* $\mathcal{N}(0; t-s)$ *with mean zero and variance* $t - s$.

We refer the reader theorem $10, 28$ of [17] and to Chapter 1 of [35] for the proof about the existence of Brownian motion as a stochastic process $(W(t))_{t\in\mathbb{R}_+}$ satisfying the above properties (i)-(iv).

Definition 1.2.2 *Brownian motion is a continuous adapted real-valued process* $W(t)$, $t \geq 0$ *such that*

- $W(0) = 0$.
- $W(t) - W(s)$ *is independent of* $\mathcal{F}_s$ *for all* $0 \leq s < t$.
- $W(t) - W(s)$ *is* $\mathcal{N}(0; t-s)$*-distributed for all* $0 \leq s \leq t$.

1.2.1 Simple Properties of Brownian Motion

Let $W(t)$ be a fixed Brownian motion. We give below some simple properties that follow directly from the definition of Brownian motion.

Proposition 1.2.1.1 [*18*]. *Brownian motion is a Gaussian process because the increments* $W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$ *are independent and normal distributed, as their linear transform, the random variables* $W(t_1), W(t_2), \ldots, W(t_m)$ *are jointly* *normally distributed, that is, the finite dimensional of Brownian motion is multivariate normal. So Brownian motion is a Gaussian process with mean* 0 *and covariance function*

$$\gamma(t,s) = Cov\{X(t), X(s)\} = \mathbb{E}W(t)W(s).$$

If $t < s$*, then* $W(s) = W(t) + W(s) - W(t)$*, and*

$$\mathbb{E}W(t)W(s) = \mathbb{E}W^2(t) + \mathbb{E}W(t)(W(s) - W(t)) = \mathbb{E}W^2(t) = t.$$

Similarly if $t > s$, $\mathbb{E}W(t)W(s) = s$*. Therefore*

$$\gamma(t,s) = min(t,s)$$

Proposition 1.2.1.2 [*18*] *(Translation invariance.) For fixed* $t_0 \geq 0$*, the stochastic process* $\widetilde{W}(t) = W(t + t_0) - W(t_0)$ *is also a Brownian motion.*

Proof. The stochastic process $\widetilde{W}(t)$ obviously satisfies the usual conditions of a Brownian motion. For any $s < t$,

$$\widetilde{W}(t) - \widetilde{W}(s) = W(t + t_0) - W(s + t_0). \tag{1.1}$$

We see that $\widetilde{W}(t)-\widetilde{W}(s)$ is normally distributed with mean 0 and variance $(t+t_0)-(s+t_0) = t-s$. Thus $\widetilde{W}(t)$ satisfies condition to independent increments. To check independent increments for $\widetilde{W}(t)$, we may assume that $t_0 > 0$. Then for any $0 \leq t_1 < t_2 < \ldots < t_n$, we have $0 < t_0 \leq t_1+t_0 < \ldots < t_n + t_0$. Hence by condition independent increments of $W(t)$, $W(t_k + t_0) - B(t_{k-1} + t_0)$, $k = 1, 2, \ldots, n$, are independent random variables. Thus by Equation (1.1), the random variables $\widetilde{W}(t_k)-\widetilde{W}(t_{k-1})$, $k = 1, 2, \ldots, n$, are independent and so $\widetilde{W}(t)$ has independent increments. □

1.2.2 Quadratic variation and Brownian motion

Proposition 1.2.2.1 *Let* $W(t)_{t\in\mathbb{R}+}$ *be a Brownian motion. For* $t \geq 0$*, for all sequence of subdivisions* $\Delta_n[0,t]$*, such that* $\lim\limits_{n\to\infty} |\Delta_n[0,t]| = 0$ *we have*

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} \left(W_{\frac{it}{2^n}} - W_{\frac{(i-1)t}{2^n}} \right)^2 = t, \qquad p.s.$$

Proof. The proof can be found in ([23], p 46).

1.2.3 Brownian paths

Proposition 1.2.3.1 [27]. *A Brownian motion has its paths a.s, locally γ-Hölder continuous for $\gamma \in [0, 1/2)$.*

Proof. Let $T > 0$; $n \in \mathbb{N}$ and $0 \leq s \leq t$. Then we have,

$$\mathbb{E}((W_t - W_s)^{2n}) = \frac{2n!}{2^n n!}(t-s)^n.$$

Hence, by using the Kolomogorov continuity theorem , there exists a continuous modification $(\widetilde{W_t})_{0\leq t\leq T}$ of $(W_t)_{0\leq t\leq T}$, whose the paths are locally γ-Hölder continuous $\forall \gamma \in [0, \frac{n-1}{2n}]$. Moreover, we have

$$\mathbb{P}(\forall t \in [0, T], W_t = \widetilde{W_t}) = 1$$

because the two processes are continuous. It implies that also almost all the paths of $(W_t)_{0\leq t\leq T}$ are locally γ-Hölder continuous.

$\square$

Proposition 1.2.3.2 [27]. *The Brownian motion's sample paths are a.s., nowhere differentiable.*

1.2.4 Brownian motion and martingales

As a stochastic process, we could ask, knowing all well properties of martingales, if the Brownian motion is one.

Proposition 1.2.4.1 [16]. *Let $(W_t)_{t\in\mathbb{R}_+}$ be a Brownian motion. Then the following processes are $(\mathcal{F}_t^W)$-martingales:*

1. $(W(t))_{t\in\mathbb{R}_+}$;

2. $(W^2(t) - t)_{t\in\mathbb{R}_+}$;

3. $(e^{uW(t)-\frac{u^2}{2}t})_{t\in\mathbb{R}_+}$ For any $u \in \mathbb{R}$.

Proof. The proof can be found in ([16], p.40).

1.2.5 Cylindrical stochastic processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Similarly to the correspondence between measures and random variables there is an analogue random object associated to cylindrical measures.

Definition 1.2.5.1 *A cylindrical random variable X in U is a linear map*

$$X : U^* \to L^0(\Omega).$$

A cylindrical process X in U is a family $(X(t) : t \geq 0)$ of cylindrical random variables in U, where U be a separable Banach space with dual U^* The dual pairing is denoted by $\langle u, u^* \rangle$, for $u \in U$ and $u^* \in U^*$.
The characteristic function of a cylindrical random variable X is defined by

$$\varphi X : U^* \to \mathbb{C}, \qquad \varphi X(u^*) = \mathbb{E}[exp(iXu^*)].$$

The concepts of cylindrical measures and cylindrical random variables match perfectly. Because the characteristic function of a cylindrical random variable is positive-definite and continuous on finite subspaces there exists a cylindrical measure μ with the same characteristic function. We call μ the cylindrical distribution of X. Vice versa, for every cylindrical measure μ on $\mathcal{C}(U)$ there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a cylindrical random variable $X : U^* \to L^0(\Omega)$ such that μ is the cylindrical distribution of X, see ([32], VI.3.2).

Remark 1.2.5.1 Our definition of cylindrical processes is based on the definitions in [7]. In [32] cylindrical random variables are considered which have values in $L^p(\Omega)$ for $p > 0$. They assume in addition that a cylindrical random variable is continuous. The continuity of a cylindrical variable is reflected by continuity properties of its characteristic function, see [[32], Prop.IV. 3.4]. The notion of weakly independent increments origins from [7].

Definition 1.2.5.2 *An adapted cylindrical process $W = (W(t) : t > 0)$ in U is a weakly cylindrical Wiener process, if for all $u_1^* \ldots, u_n^*$ and $n \in N$ the $\mathbb{R}^n$-valued stochastic process $((W(t)u_1^* \ldots, W(t)u_n^*) : t \geq 0)$ is a Wiener process.*

Our definition of a weakly cylindrical Wiener process is an obvious extension of the definition of a finite-dimensional Wiener processes and is exactly in the spirit of cylindrical processes.

Lemma 1.2.5.1 *For an adapted cylindrical process $W = (W(t) : t \geq 0)$ the following are equivalent:*
(a) W is a weakly cylindrical Wiener process;
(b) W satisfies

(i) W has weakly independent increments;

(ii) $(W(t)u^* : t \geq 0)$ is a Wiener process for all $u^* \in U^*$.

Proof. We have only to show that (b) implies (a). By linearity we have

$$\beta_1(W(t)-W(s))u^* + \ldots + \beta_n(W(t)-W(s))u_n^* = (W(t)-W(s))\left(\sum_{i=1}^{n}\beta_i u_n^*\right),$$

for all $\beta_i \in \mathbb{R}$ and $u^* \in U^*$. Which shows that the increments of $((W(t)u_1^*, \ldots, W(t)u_n^*)) : t \geq 0)$ are normally distributed and stationary. The independence of the increments follows by (i). □

Let $(\Omega, \mathcal{F}, \{F_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space equipped with some filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (i.e, it is right continuous and $\mathcal{F}_0$ contains all P-null sets) and $\mathcal{K}$, $\mathcal{H}$ be two separable Hilbert spaces with the inner product $\langle\cdot,\cdot\rangle$, and we will use the notation $||\cdot||$ to denote the norms in $\mathcal{H}$,$\mathcal{K}$. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of $\mathcal{K}$. Suppose that $\{W(t) : t \geq 0\}$ is a cylindrical $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{i=1}^{\infty}\lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $W(t) = \sum_{i=1}^{\infty}\sqrt{\lambda_i}W_i(t)e_i$, where $\{W_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F} = \{\sigma W(s) : 0 \leq s \leq t$ is the σ-algebra generated by W and $\mathcal{F}_b = \mathcal{F}$.

Now,we introduce the following notions which can be used in the next chapters. Let $L(\mathcal{K}, \mathcal{H})$ denote the space of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ equipped with the usual operator norm $||\cdot||_{L(\mathcal{K},\mathcal{H})}$. For $\psi(t) \in L(\mathcal{K}, \mathcal{H})$ we define

$$||\psi||_Q^2 = Tr(\psi Q\psi*) = \sum_{n=1}^{\infty}||\sqrt{\lambda_n}\psi e_n||^2.$$

If $||\psi||_Q^2 < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(\mathcal{K}, \mathcal{H})$ denote the space of all Q-Hilbert-Schmidt operators. The completion $L_Q(\mathcal{K}, \mathcal{H})$ of $L(\mathcal{K}, \mathcal{H})$ with respect to the topology induced by the norm $||Q||$ where $||\psi||_Q^2 = \langle\psi, \psi\rangle$ is a Hilbert space with the above norm topology.

1.3 Fractional Brownian Motion

The fractional Brownian motion (fBm) is a generalization of the more simple and more studied stochastic process of standard Brownian motion. More precisely, the fractional Brownian motion is a centered continuous Gaussian process with stationary increments and H-self similar properties. The Hurst parameter H, due to the British hydrologist H. E. Hurst, is between 0 and 1. The case $H = \frac{1}{2}$ corresponds to standard Brownian motion.

Let us start with some basic facts about the fractional Brownian motion (fBm) and the stochastic calculus that can be developed with respect to this process.

Definition 1.3.1 *The fractional brownian motion (fbm) with Hurst index* ($H \in (0,1)$) *is a Gaussian process* $B^H = \{B_t^H, t \in \mathbb{R}\}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$*, having the properties:*

1. $B_0^H = 0$,
2. $\mathbb{E}[B_t^H] = 0$; $t \in \mathbb{R}$,
3. $\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big)$; $s, t \in \mathbb{R}$.

Remark 1.3.1 Since $\mathbb{E}[B_t^H - B_s^H]^2 = |t-s|^{2H}$ and B^H is a Gaussian process, it has a continuous modification, according to the Kolmogorov theorem.

Remark 1.3.2 For $H = 1$, we set $B_t^H = B_t^1 = t\xi$, where ξ is a standard normal random variable. For $H = \frac{1}{2}$, the characteristic function has the form

$$\phi_\lambda(t) = \mathbb{E}\Big[exp(i\sum_{k=1}^{n} \lambda_k B_{t_k}^H)\Big] = exp\Big(-\frac{1}{2}(C_t\lambda, \lambda)\Big),$$

where $C_t = (\mathbb{E}[B_{t_K}^H B_{t_i}^H])_{1 \le i,k \le n}$ and $\langle ., . \rangle$ is the inner product on $\mathbb{R}^n$.

1.3.1 Stochastic Integral Representation

Here we discuss some of the integral representations for the fBm. In [10] it is proved that the process

$$\begin{aligned} Z(t) &= \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}})dB(s) \\ &= \frac{1}{\Gamma(H+\frac{1}{2})}\Big(\int_{-\infty}^{0}((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}})dB(s) \\ &\quad + \int_0^t (t-s)^{H-\frac{1}{2}} dB(s)\Big), \end{aligned}$$

where $B(t)$ is a standard Brownian motion and Γ represents the gamma function, is a fBm with Hurst index $H \in (0,1)$. First we notice that $Z(t)$ is a continuous centered Gaussian process.

Hence, we need only to compute the covariance functions. In the following computations we drop the constant $\frac{1}{\Gamma(H+\frac{1}{2})}$ for the sake of simplicity. We obtain

$$\begin{aligned}\mathbb{E}[Z^2(t)] &= \int_{\mathbb{R}} \Big[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\Big]^2 ds \\ &= t^{2H} \int_{\mathbb{R}} \Big[(1-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\Big]^2 du \\ &= C(H) t^{2H},\end{aligned}$$

where we have used the change of variable $s = tu$.

Analogously, we have that

$$\begin{aligned}\mathbb{E}[|Z(t)-Z(s)|^2] &= \int_{\mathbb{R}} \Big[(t-u)_+^{H-\frac{1}{2}} - (s-u)_+^{H-\frac{1}{2}}\Big]^2 ds \\ &= t^{2H} \int_{\mathbb{R}} \Big[(t-s-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\Big]^2 du \\ &= C(H)|t-s|^{2H}.\end{aligned}$$

Now,

$$\begin{aligned}\mathbb{E}[Z(t)-Z(s)] &= -\frac{1}{2}\Big\{\mathbb{E}[|Z(t)-Z(s)|^2] - \mathbb{E}[Z(t)^2] - \mathbb{E}[Z(s)^2]\Big\} \\ &= \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big).\end{aligned}$$

Hence we can conclude that $Z(t)$ is a fBm of Hurst index H.
We can also represent the fBm over a finite interval, i.e.

$$B_t^{(H)} = \int_0^t K_H(t,s)dB_s, \quad t \geq 0,$$

where

1. For $H > \frac{1}{2}$,

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

   with $c_H = [\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}]^{\frac{1}{2}}$ and $t > s$,

2. For $H < \frac{1}{2}$,

$$K_H(t,s) = c_H[(\frac{t}{s})^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du],$$

   with $c_H = [\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}]^{\frac{1}{2}}$ and $t > s$.

1.3.2 Correlation between two increments

For $H = \dfrac{1}{2}$, $B^{(H)}$ is a standard Brownian motion; hence, in this case the increments of the process are independent. On the contrary, for $H \neq \dfrac{1}{2}$ the increments are not independent. More precisely, by Definition 1.3.1 we know that the covariance between $B^{(H)}(t+h) - B^{(H)}(t)$ and $B^{(H)}(s+h) - B^{H}(s)$ with $s + h \leq t$ and $t - s = nh$ is

$$\rho^H(n) = \frac{1}{2} h^{2H} \Big[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \Big].$$

In particular, we obtain that two increments of the form $B^H(t+h) - B^H(t)$ and $B^H(t+2h) - B^H(t+h)$ are positively correlated for $H > \dfrac{1}{2}$, while they are negatively correlated for $H < \dfrac{1}{2}$. In the first case the process presents an aggregation behavior and this property can be used in order to describe (cluster) phenomena (systems with memory and persistence). In the second case it can be used to model sequences with intermittency and antipersistence.

1.3.3 Self-similarity and long-range dependence

We will first define the self-similarity and long-range dependence in the framework of general stationary stochastic processes.

Definition 1.3.3.1 *We say that an $\mathbb{R}^d$-valued random process $X = (X_t)_{t \geq 0}$ is self-similar or satisfies the property of self-similarity if for every $a > 0$ there exists $b > 0$ such that*

$$law(X_{at}, t \geq 0) = law(bX_t, t \geq 0). \tag{1.2}$$

Note that (1.2) means that the two processes X_{at} and bX_t have the same finite-dimensional distribution functions, i.e., for every choice $t_1, ..., t_n$ in $\mathbb{R}$,

$$\mathbb{P}(X_{at_0} \leq x_0, ..., X_{at_n} \leq x_n) = \mathbb{P}(bX_{t_0} \leq x_0, ..., bX_{t_n} \leq x_n)$$

For every $x_0, ..., x_n$ in $\mathbb{R}$.

Definition 1.3.3.2 *If $b = a^{-H}$ in definition (1.2), then we say that $X = (X_t)_{t_0}$ is a self-similar process with Hurst index H or that it satisfies the property of (statistical) self-similarity with Hurst index H. The quantity $D = \frac{1}{H}$ is called the statistical fractal dimension of X. Since the covariance function of the fBm is homogeneous of order $2H$, we obtain that B^H is a self-similar* *process with Hurst index* H*, i.e., for any constant* $a > 0$ *the processes* $B^H(at)$ *and* $a^{-H}B^H(t)$ *have the same distribution law.*

Theorem 1.3.3.1 *The fractional Brownian motion* $\{B^H(t), t \geq 0\}$ *is a* H*-self-similar process (H-ss, for short).*

Definition 1.3.3.3 *A stationary sequence* $(X_n)_{n\in\mathbb{N}}$ *exhibits long-range dependence if the auto-covariance functions* $\rho(n) := cov(X_k, X_{k+n})$ *satisfy*

$$\lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1,$$

for some constant c *and* $\alpha \in (0,1)$*. In this case, the dependence between* X_k *and* X_{k+n} *decays slowly as* n *tends to infinity and*

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

Hence, we obtain immediately that the increments $X_k := B^H(k) - B^H(k-1)$ of B^H and $X_{k+n} := B^H(k+n) - B^H(k+n-1)$ of B^H have the long-range dependence property for $H > \frac{1}{2}$ since

$$\rho_H(n) := \frac{1}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}] \sim H(2H-1)n^{2H-2},$$

as n goes to infinity. In particular,

$$\lim_{n\to\infty} \frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1.$$

- If $H \in (0, \frac{1}{2})$, then $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$.

- If $H \in (\frac{1}{2}, 1)$, then $\sum_{n=1}^{\infty} \rho_H(n) = \infty$, in this case we say that fBm B^H has the property of long-range dependence.

In general, self-similarity and long-range dependence are not equivalent. As an example, the increments of a standard Brownian motion are self-similar with Hurst parameter $H = 1/2$, but clearly not long-range dependent (the increments are even independent).

1.3.4 Hölder continuity

We recall that according to the Kolmogorov criterion [44], a process $X = (X_t)_{t\in\mathbb{R}}$ admits a continuous modification if there exist constants $\alpha \geq 1$, $\beta > 0$, and $k > 0$ such that

$$\mathbb{E}[|X(t) - X(s)|^{\alpha}] \leq k|t-s|^{1+\beta}$$

for all $s, t \in \mathbb{R}$.

Theorem 1.3.4.1 *Let $H \in (0,1)$. The fbm $B^{(H)}$ admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*

Proof. We recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Hölder continuous of order α, $0 < \alpha \leq 1$ and write $f \in \mathcal{C}^{\alpha}(\mathbb{R})$, if there exists $M > 0$ such that

$$|f(t) - f(s)| \leq M|t-s|^{\alpha},$$

for every $s, t \in \mathbb{R}$. For any $\alpha > 0$ we have

$$\mathbb{E}[|B^H(t) - B^H(s)|^{\alpha}] = \mathbb{E}[|B^H(1)|^{\alpha}]|t-s|^{\alpha H};$$

hence, by the Kolmogorov criterion we get that the sample paths of B^H are almost every where Hölder continuous of order strictly less than H. Moreover, by [12] we have

$$\limsup_{t\longrightarrow 0^+} \frac{|B^{(H)}|(t)}{t^H\sqrt{\log\log t^{-1}}} = c_H$$

with probability one, where c_H is a suitable constant. Hence B^H can not have sample paths with Hölder continuity's order greater than H.

□

1.3.5 Path differentiability

By [31] we also obtain that the process B^H is not mean square differentiable and it does not have differentiable sample paths.

Proposition 1.3.5.1 *Let $H \in (0,1)$. The fBm sample path $B^H(.)$ is not differentiable. In fact, for every $t_0 \in [0,\infty)$*

$$\mathop{lim\,\sup}_{t\to t_0} \left|\frac{B^H(t) - B^H(t_0)}{t - t_0}\right| = \infty$$

with probability one.

Proof. Here we recall the proof of [31]. Note that we assume $B^H(0) = 0$. The result is proved by exploiting the self-similarity of B^H. Consider the random variable

$$\mathcal{R}_{t,t_0} := \frac{B^H(t) - B^H(t_0)}{t - t_0}$$

that represents the incremental ratio of B^H. Since B^H is self-similar (*see*[10]), we have that the law of $\mathcal{R}_{t,t_0}$ is the same of $(t-t)_0^{H-1}B^H(1)$. If one considers the event

$$A(t,w) := \left\{ \sup_{0\leq s\leq t} \left| \frac{B^H(s)}{s} \right| > d \right\},$$

then for any sequence $(t_n)_{n\in\mathbb{N}}$ decreasing to 0, we have

$$A(t_n, w) \supseteq A(t_{n+1}, w),$$

and

$$A\left(t_n, w\right) \supseteq (|\frac{B^H(t_n)}{t_n}| > d) = \left(|B^H(1)| > t_n^{1-H} d\right).$$

The thesis follows since the probability of the last term tends to 1 as $n \longrightarrow \infty$.

1.3.6 The fBm is not a Semimartingale for $H \neq \frac{1}{2}$

The fact that the fBm is not a semimartingale for $H \neq \dfrac{1}{2}$ has been proved by several authors. In order to verify that B^H is not a semimartingale for $H \neq \dfrac{1}{2}$, it is sufficient to compute the p-variation of B^H.

Definition 1.3.2 *Let $(X(t))_{t\in[0,T]}$ be a stochastic process and consider a partition $\pi = \{0 = t_0 < t_1 < ...... < t_n = T\}$. Put*

$$\mathcal{S}_p(x,\pi) := \sum_{i=1}^{n} |X(t_i) - X(t_{i-1})|^p.$$

The p-variation of X over the interval $[0,T]$ is defined as

$$\mathcal{V}_p(X,[0,T]) := \sup_{\pi} \mathcal{S}_p(X,\pi),$$

where π *is a finite partition of* $[0,T]$. *The index of* p*-variation of a process is defined as*

$$I(X,[0,T]) := \inf\{p>0; \mathcal{V}_p(X,[0,T])<\infty\}.$$

We claim that

$$I(B^H,[0,T]) = \frac{1}{H}.$$

In fact, consider for $p>0$,

$$Y_{n,p} = n^{pH-1}\sum_{i=1}^{n}\left|B^H(\frac{i}{n}) - B^H(\frac{i-1}{n})\right|^p.$$

Since B^H *has the self-similarity property, the sequence* $Y_{n,p}, n\in N$ *has the same distribution as*

$$\widetilde{y}_{n,p} = n^{-1}\sum_{i=1}^{n}\left|B^H(i) - B^H(i-1)\right|^p,$$

and by the Ergodic theorem (see, for example, [39]) the sequence $\widetilde{y}_{n,p}$ *converges almost surely and in* L^1 *to* $\mathbb{E}[|B^H(1)|^p]$ *as* n *tends to infinity. It follows that*

$$V_{n,p} = \sum_{i=1}^{n}\left|B^H(\frac{i}{n}) - B^H(\frac{i-1}{n})\right|^p$$

converges in probability respectively to 0 *if* $pH>1$ *and to infinity if* $pH<1$ *as* n *tends to infinity. Thus we can conclude that* $I(B^H,[0,T]) = \frac{1}{H}$. *Since for every semimartingale* X, *the index* $I(X,[0,T])$ *must belong to* $[0,1]\cup\{2\}$, *the fBm* B^H *cannot be a semimartingale unless* $H=\frac{1}{2}$.

1.3.7 Invariance principle

Here we present an invariance principle for fBms due to [12].

Assume that $\{X_n, n=1,2,...\}$ is a stationary Gaussian sequence with $\mathbb{E}[X_i]=0$ and $\mathbb{E}[X_i^2]=1$. Define

$$Z_n(t) = \tfrac{1}{n^H}\sum_{k=1}^{[nt]-1} X_k, \quad 0\le t\le 1,$$

where $[\cdot]$ stands for the integer part. We will show that if the covariance of $\sum_{0}^{n} X_k$ is proportional to C_n^{2H} for large n, $Z_n(t), t \geq 0$ converges weakly to $\sqrt{C}B_t^{(H)}$ in a suitable metric space. Let as first introduce the real-valued function $\omega_\beta^\alpha(.)$ defined by

$$\omega_\beta^\alpha(t) = t^\alpha \left(1 + \log\frac{1}{t}\right)^\beta, \quad t > 0,$$

and we let

$$||f||_p^{\omega_\beta^\alpha} = ||f||_{L^p(I)} \sup_{0<t\leq 1} \frac{\omega_p(f,t)}{\omega_\beta^\alpha(t)}.$$

The Besov space $Lip_p(\alpha,\beta)$ is the class of functions f in $L^p(I)$ such that $||f||_p^{\omega_\beta^\alpha} < \infty$. $Lip_p(\alpha,\beta)$ endowed with the norm $||.||_p^{\omega_\beta^\alpha}$ is a nonseparable Banach space. Let $B_p^{\alpha,\beta}$ denote the separable subspace of $Lip_p(\alpha,\beta)$ formed by functions $f \in Lip_p(\alpha,\beta)$ satisfying $\omega_p(f,t) = \circ(\omega_\beta^\alpha(t))$ as $t \longrightarrow 0$. For a continuous function f, denote by $\{C_n(f), n \geq 0\}$ the coefficients of the decomposition of f in the Schauder basis given by

$$C_0(f) = f(0), C_1(f) = f(1) - f(0),$$

and for $n = 2^j + k, \quad j \geq 0$, and $k = 0, \ldots, 2^j - 1$,

$$C_n(f) = 2.2^{\frac{j}{2}} \left\{ f\left(\frac{2k+1}{2^{j+1}}\right) - \frac{1}{2}\left[f\left(\frac{2k}{2^{j+1}}\right) + f\left(\frac{2k-2}{2^{j+1}}\right)\right]\right\}.$$

Lemma 1.3.7.1 *Let $\alpha > \frac{1}{p}$ and $0 < \beta < \beta'$. The space $Lip_p(\alpha,\beta)$ is compactly embedded in $B_p^{\alpha,\beta'}$.*

We refer the reader to [12].

Lemma 1.3.7.2 *Let $(X_n^t, t \in I)_{n\geq 1}$ be a sequence of stochastic processes satisfying*

1. *$X_0^n = 0$, for all $n \geq 1$.*

2. *There exists a positive constant C and $\alpha \in]0,1[$ such that for $p \geq 1$,*

$$\mathbb{E}[|X_t^n - X_s^n|^p \leq C|t-s|^{p\alpha};$$

for all $s,t \in I$. Then $(X^n(t), t \in I)_{n\geq 1}$ is tight in $B_p^{\alpha,\beta}, \beta > 0$ for $p > \max(\frac{1}{\alpha}, \frac{1}{\beta})$.

Proof. By the assumptions, we have $C_0(X^n) = 0$ and $C_1(X^n) = X_1^n$. To prove the lemma, by lemma (1.9.3, see [36]) it is enough to show that there exists a constant $C_p > 0$ such that , for $\lambda > 0$ and $\frac{1}{p} < \beta' < \beta$, we have

$$\mathbb{P}(||X^n||_p^{\omega_{\beta'}^{\alpha}} > \lambda) \leq C_p \lambda^{-p}$$

for all $n \geq 1$.Thus, it suffices to show that

$$\mathbb{P}\left(M(X^n) > \lambda\right) \leq C_p \lambda^{-p},$$

where $M(X^n)$ is the maximum of the set

$$\left\{ |C_0(X^n)|, |C_1(X^n)|, \sup_{j\geq 0} \frac{2^{-j(\frac{1}{2}-\alpha+\frac{1}{p})}}{(1+j)^{\beta'}} \left[\sum_{m=2^j+1}^{2^{j+1}} |C_m(x^n)|^p\right]^{\frac{1}{p}} \right\}.$$

Now, by the Chebyshev inequality, we have

$$\begin{aligned} I &= \mathbb{P}\left(\sup_{j\geq 0} \frac{2^{-j(\frac{1}{2}-\alpha+\frac{1}{p})}}{(1+j)^{\beta'}} \left[\sum_{m=2^j+1}^{2^{j+1}} |C_m(X^n)|^p\right]^{\frac{1}{p}} > \lambda\right) \\ &\leq \sum_{j\geq 0} \frac{2^{-jp(\frac{1}{2}-\alpha+\frac{1}{p})}}{(1+j)^{p\beta'}} \sum_{m=2^j+1}^{2^{j+1}} \mathbb{E}[|C_m(X^n)|^p] \lambda^{-p}. \end{aligned}$$

Recall that for $m = 2^j + k$,

$$C_m(X^n) = 2.2^{\frac{j}{2}} \left[X^n_{(2k-1)/2^{j+1}} - \frac{1}{2}\left(X^n_{(2k)/2^{j+1}} + X_{(2k-2)/2^{j+1}}\right)\right].$$

Thus,

$$\begin{aligned} I &\leq C_p \lambda^{-p} \sum_{j\geq 0} \frac{2^{-jp(\frac{1}{2}-\alpha+\frac{1}{p})}}{(1+j)^{p\beta'}} \sum_{k=1}^{2^j} (\mathbb{E}[|X^n_{(2k-1)/2^{j+1}} - X_{(2k)/2^{j+1}}|^p] \\ &+ \mathbb{E}\left[|X^n_{(2k-1)/2^{j+1}} - X^n_{(2k-2)/2^{j+1}}|^p\right]) \\ &\leq \lambda^{-p} \left[C_p \sum_{j\geq 0} \frac{1}{(1+j)^{p\beta'}}\right] \leq C_p \lambda^{-p}. \end{aligned}$$

which completes the proof.

□

Corollary 1.3.7.1 [12]. *Let $H \in (0,1)$, $\beta > 0$, and $p > \max(\frac{1}{H}, \frac{1}{\beta})$. Assume that $\{X_n, n = 1, 2, \ldots\}$ is a stationary Gaussian sequence with spectral representation*

$$X_n = \int_{-\pi}^{\pi} \exp(in\lambda)|\lambda|^{\frac{1}{2-H}} B(d\lambda), n = 1, 2 \ldots,$$

where $B(d\lambda)$ is a Gaussian random measure with $\mathbb{E}[|B(d\lambda)|^2] = d\lambda$. Then there exists a positive constant C such that $(Z_n(t), t \in [0,1])$ converges weakly to $(CB_t^H), t \in [0,1])$ in the space $B_p^{H,\beta}$.

1.4 Stochastic Integration w.r.t SBM

This section is devoted to the study of an integration where the integrator is Brownian motion. In fact, we would like to define

$$\int_T f_s dW_s, \tag{1.3}$$

where f_s is a certain stochastic process. Note that there are various notation for the stochastic integral. We use (1.3) or $I_W(f)$ as well as $f \cdot W$.
Besides, we would like that this integral satisfies the common property of the usual Riemann (Lebesgue) integral. For example, if the integrands is $f_s = 1$, then $\int_0^T dW_t = W_T - W_0$, we want the integral satisfies the (splitting) property, i.e. the integration over $[0, T]$ is equal to the sum of the integration over $[0, a)$ and $[a, T]$. Also, we ask for the linearity.

1.4.1 Wiener Integral

Now let us consider the following integral:

$$\int_a^b f(t) dW(t, \omega),$$

where f is a deterministic function (i.e, it does not depend on ω) and $W(t, \omega)$ is a Brownian motion. Suppose for each $\omega \in \Omega$ we want to use the integration by parts formula to define this integral in the Riemann-Stieltjes sense by

$$\int_a^b f(t) dW(t, \omega) = f(t) dW(t, \omega)]_a^b - \int_a^b W(t, \omega) df(t).$$

Then the class of functions $f(t)$ for which the integral $\int_a^b f(t)dW(t,\omega)$ is defined for each $\omega \in \Omega$ is rather limited, i.e, $f(t)$ needs to be a continuous function of bounded variation.

We need a different idea in order to define the integral $\int_a^b f(t)dW(t,\omega)$ for a wider class of functions $f(t)$. This new integral, called the Wiener integral of f, is defined for all functions $f \in L^2[a,b]$. Here $L^2[a,b]$ denotes the Hilbert space of all real-valued square integrable functions on $[a,b]$.

Now we define the Wiener integral in two steps:

Step 1. Suppose f is a step function given by $f = \sum_{i=1}^{n} a_i \mathbf{1}_{[t_{i-1},t_i)}$, where $t_0 = a$ and $t_n = b$. In this case, define

$$I(f) = \sum_{i=1}^{n} a_i(W(t_i), W(t_{i-1})). \tag{1.4}$$

Obviously, $I(af + bg) = aI(f) + bI(g)$ for any $a, b \in \mathbb{R}$ and step functions f and g. Moreover, we have the following lemma.

Lemma 1.4.1.1 *For a step function f, the random variable $I(f)$ is Gaussian with mean* 0 *and variance*

$$\mathbb{E}(I(f)^2) = \int_a^b f(t)^2 dt. \tag{1.5}$$

Proof. It is well known that a linear combination of independent Gaussian random variables is also a Gaussian random variable. Hence by definition 1.2.2, the random variable $I(f)$ defined by Equation (1.4) is Gaussian with mean 0. To check Equation (1.5), note that

$$\mathbb{E}(I(f)^2) = \mathbb{E} \sum_{i,j=1}^{n} a_i a_i (W(t_i), W(t_{i-1}))(W(t_j), W(t_{j-1})).$$

By Definition 1.2.2, we have

$$\mathbb{E}(W(t_i), W(t_{i-1})) = t_i - t_{i-1},$$

and for $i \neq j$,

$$\mathbb{E}(W(t_i), W(t_{i-1}))(W(t_j), W(t_{j-1})) = 0.$$

Therefore,

$$\mathbb{E}(I(f)^2) = \sum_{i=1}^{n} a_i^2 (t_i - t_{i-1}) = \int_a^b f(t)^2 dt.$$

$\square$

Step 2. We will use $L^2(\Omega)$ to denote the Hilbert space of square integrable real-valued random variables on Ω with inner product $\langle X, Y\rangle = E(XY)$. Let $f \in L^2[a,b]$. Choose a sequence $\{f_n\}_{n=1}^{\infty}$ of step functions such that $f_n \to f$ in $L^2[a,b]$. By Lemma 1.4.1.1 the sequence $\{I(f_n)\}$ is Cauchy in $L^2(\Omega)$. Hence it converges in $L^2(\Omega)$. Define

$$I(f) = \lim_{n\to\infty} I(f_n), \qquad in \quad L^2(\Omega). \tag{1.6}$$

Question. Is $I(f)$ well-defined?

In order for $I(f)$ to be well-defined, we need to show that the limit in Equation (1.6) is independent of the choice of the sequence $\{f_n\}$. Suppose $\{g_m\}$ is another such sequence, i.e, the g_m are step functions and $g_m \to f$ in $L^2[a,b]$. Then by the linearity of the mapping I and equation (1.6),

$$\mathbb{E}(|I(f_n) - I(g_m))|^2) = \mathbb{E}(|I(f_n - g_m)|^2) = \int_a^b (f_n(t) - g_m(t))^2 dt.$$

Write $f_n(t) - g_m(t) = [f_n(t) - f(t)] - [g_m(t) - f(t)]$ and then use the inequality $(x-y)^2 \le 2(x^2+y^2)$ to get

$$\begin{aligned} \int_a^b (f_n(t) - g_m(t))^2 dt &\le 2\int_a^b \left([f_n(t) - f(t)]^2 + [g_m(t) - f(t)]^2\right) dt \\ &\to 0, \qquad as \quad n, m \to \infty. \end{aligned}$$

It follows that $\lim_{n\to\infty} I(f_n) = \lim_{m\to\infty} I(g_m)$ in $L^2(\Omega)$. This shows that $I(f)$ is well-defined.

Definition 1.4.1.1 *Let $f \in L^2[a,b]$. The limit $I(f)$ defined in Equation (1.6) is called the Wiener integral of f.*

The Wiener integral $I(f)$ of f will be denoted by

$$I(f)(\omega) = \left(\int_a^b f(t)dW(t)\right)(\omega), \qquad \omega \in \Omega, \quad almost \quad surely.$$

For simplicity, it will be denoted by $\int_a^b f(t)dW(t)$ or $\int_a^b f(t)dW(t,\omega)$. Note that the mapping I is linear on $L^2[a,b]$.

Theorem 1.4.1.1 *For each $f \in L^2[a,b]$, the Wiener integral $\int_a^b f(t)dW(t,\omega)$ is a Gaussian random variable with mean 0 and variance $||f||^2 = \int_a^b f(t)^2 dt$.*

Proof. By Lemma 1.4.1.1, the assertion is true when f is a step function. For a general $f \in L^2[a,b]$, the assertion follows from the following well-known fact: If X_n is Gaussian with mean μ_n and variance σ_n^2 and X_n converges to X in $L^2(\Omega)$, then X is Gaussian with mean $\mu = \lim\limits_{m\to\infty} \mu_n$ and variance $\sigma = \lim\limits_{m\to\infty} \sigma_n$.

$\square$

Example 1.4.1 *The Wiener integral* $\int_0^1 s^2 dW_s$ *is a gaussian r.v. with mean zero and variance* $\int_0^1 (s^2)^2 ds = 1$.

Definition 1.4.1.2 [*35*]. *The stochastic integral with respect to Brownian motion* $(W_t)_{t\in\mathbb{R}_+}$ *of the simple step functions* f *is defined by*

$$\int_0^\infty f(t)dW_t := \sum_{i=1}^n (W_{t_i} - W_{t_{i-1}}).$$

1.4.2 Itô integral

In this section, we will study the simplest stochastic integral, where the integrand and the integrator are random variable. The first who defined this integral was K.Itô in 1944. Therefore we named this integral after him. In fact, the integrand will be an adapted stochastic process w.r.t the natural filtration of the Brownian motion. Besides, to be well defined, we will need another hypothesis on the integrand.

There are many reason to developp the stochastic integration. For example, we showed in theorem 8 in [26] that the following stochastic process given by the random variables

$$M_t = \int_a^t f(s)dW_s, \qquad a \le t \le b, \tag{1.7}$$

is an $\mathcal{F}_t^W$-martingale, where $f \in L^2([a,b])$.

A natural question is whether the following process (yet not defined)

$$M_t = \int_a^t f_s(\omega)dW_s(\omega), \qquad a \le t \le b,$$

is a martingale, where $(f_t)_{t\in\mathbb{R}_+}$ is now a stochastic process. (We emphazise the randomness by adding the ω.)

Riemann-type Approach

As soon as we want to define an integral, often we would like to have a Riemann approach, i.e. we want the integral to be the a.s. limit of the so-called Riemann sums $\sum_i H_{u_i}(W_{t_i}, W_{t_{i+1}})$, where (H_t) is the integrand process and $u_i \in [t_i, t_{i+1}]$. Moreover, since the Brownian motion is a martingale, if we consider discrete-time processes, we have seen definition 39 in [26]. that the so-called martingale transform

$$\sum_i H_{u_i}(W_{t_{i+1}\wedge t}, W_{t_i \wedge t})$$

is a martingale. Thus, we could take the limit and define as Riemann did an integral with continuous-time processes.

However, we cannot have yet a such approach. Indeed, as we proved in Corollary 4 in [26] the Brownian motion is not of bounded variation. Besides, the Riemann-Stieltjes integration theory says that we can have a such approach only if the paths are locally of bounded variation. Thus, the Riemann sums does not converge pathwise almost surely. However, it can be shown that it converges in probability.

Therefore, we will use a generalization of the Wiener type integral.

Wiener-type Approach

We have seen in (1.7) how to define an integral, where the integrand was a non-stochastic function and the integrator was a Brownian motion. Then, we will adopt the same way as we did in (1.7).

First, we will consider in the sequel a Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ defined on the filtered probability space $(\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ satisfying the usual conditions.

Definition 1.4.2.1 *[35] We denote by $L^2_{ad}(\Omega \times T, (\mathcal{F}_t)_{t\in T})$ the set of càglàd $(\mathcal{F}_t)$-adapted processes $(H_t)_{t\in T}$ satisfying*

$$\mathbb{E}\left(\int_T H_s^2 ds\right) < \infty \tag{1.8}$$

Remark 1.4.2.1 These are These are càglàd (continu à gauche, limité à droite) processes. Note that, for the integration with respect to a Brownian, we can also take right-continuous functions. But, the point is that when we change the integrator, as in the next section, when we deal with martingales, we can take only the left-continuous functions.

Remark 1.4.2.2 Recall that the càglàd $(\mathcal{F}_t)$-adapted processes are equivalent to the progressively measurable processes.

Lemma 1.4.2.1 $\left(L^2_{ad}(\Omega \times T, (\mathcal{F}_t)_{t\in T}, ||\cdot||_{L^2_{ad}(\Omega\times T)}\right)$ *is a Hilbert space with the following norm*

$$||H||_{L^2_{ad}(\Omega\times T)} = \mathbb{E}\left(\int_T H_s^2 ds\right).$$

This Hilbert space will be the space of our integrands. let us start with the simplest case of random integrands.

Integrands as stochastic step processes

Let us denote by ξ the set of simples $(\mathcal{F}_t)$-predictables processes $(H_t)_{t\in\mathbb{R}_+}$, i.e.

$$H_t(\omega) = \sum_{i=1}^{n} h_i(\omega)\mathbf{1}_{[t_{i-1},t_i]}(t), \qquad t \in T. \tag{1.9}$$

with $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$ tn and h_i a $\mathcal{F}_{t_{i-1}}$-measurable random variable which belongs to $L^2(\Omega)$.

Then we can define the integral for $H \in \xi$ w.r.t a brownian motion by.

$$I(H) = (H\cdot W)_s = \int_T H_s dW_s = \begin{cases} \displaystyle\sum_{i=1}^{n} h_i(W_{t_i}, W_{t_{i-1}}) \qquad if \quad T = \mathbb{R}_+, \\ \displaystyle\sum_{i=1}^{n} h_i(W_{t_i\wedge T}, W_{t_{i-1}\wedge T}) \qquad if \quad T = [0,T] \end{cases} \tag{1.10}$$

Proposition 1.4.2.1 *For $I(H)$ defined by (1.10), we have*

$$\mathbb{E}\left(I(H)\right) = 0 \quad and \quad \mathbb{E}\left(I(H)^2\right) = \mathbb{E}\left(\int_T H_t^2 dt\right).$$

Proof. The proof can be found in ([26], p.47).

Remark 1.4.2.3 Note that if $H \notin \xi \subset L^2(\Omega)$, we would not have a finite variance.

Integrands as square integrable stochastic processes

The idea is to extend, by density of ξ in $L^2(\Omega)$, the definition of $I(H)$ in (1.10) to larger processes, i.e. processes in $L^2(\Omega)$ as the limit of processes in ξ, like we did for the Wiener integral. Indeed, by density, we have for each $(H_t)_{t\in\mathbb{R}_+} \in L^2(\Omega)$ there exists a sequence $(H_t)_{t\in\mathbb{R}_+} \in L^2(\Omega))_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} (|H_t - H_{t,n}|)dt = 0. \tag{1.11}$$

However, our integrands, as the L^2 limit of processes in ξ, must satisfy certain constraints to be well-defined. Therefore, we will take as the space of integrands $L^2_{ad}(\Omega \times T, (\mathcal{F}_t)_{t\in T})$.

Obviously we have $\xi \subset L^2_{ad}(\Omega\times T, (\mathcal{F}_t)_{t\in T})$ and $\tilde{\xi} = L^2_{ad}(\Omega\times T, (\mathcal{F}_t)_{t\in T})$. In this way, we have the following theorem which defines the so-called Itô integral.

Theorem 1.4.2.1 *[35] There exists a unique linear application*

$$\mathcal{I} : L^2_{ad}(\Omega \times T, (\mathcal{F}_t)_{t\in T}) \to L^2(\Omega, \mathcal{F}_t, \mathbb{P})$$

such that:

1. *For* $H_t(\omega) = \sum_{i=1}^n h_i(\omega)\mathbf{1}_{(t_{i-1},t_1]}(t) \in \xi$,

$$I(H) = \begin{cases} \displaystyle\sum_{i=1}^n h_i(W_{t_i}, W_{t_{i-1}}) & if \quad T = \mathbb{R}_+, \\ \displaystyle\sum_{i=1}^n h_i(W_{t_i\wedge T}, W_{t_{i-1}\wedge T}) & if \quad T = [0,T] \end{cases} \tag{1.12}$$

2. *For* $\widetilde{H} \in L^2_{ad}(\Omega \times T, (\mathcal{F}_t)_{t\in T})$

$$\mathbb{E}(I(\widetilde{H})^2) = \mathbb{E}(\int_T \widetilde{H}_s^2 ds). \tag{1.13}$$

Definition 1.4.2.2 *The application defined in Theorem 1.4.2.1 is called the Itô integral w.r.t the brownian motion.*

Proposition 1.4.2.2 *The definition of the stochastic integral* $\displaystyle\int_0^\infty f(t)dW_t$ *can be extended to any measurable function* $f \in L^2(\mathbb{R}_+)$, *i.e, to* f *such that*

$$\int_0^\infty |f(t)|^2 dt < \infty.$$

In this case, $\int_0^\infty f(t)dW_t$ *has a centered Gaussian distribution*

$$\int_0^\infty f(t)dW_t \simeq \mathcal{N}(0, \int_0^\infty |f(t)|^2 dt)$$

with variance $\int_0^\infty |f(t)|^2 dt$ *and we have the Itô isometry*

$$\mathbb{E}\left[(\int_0^\infty f(t)dW_t)^2 \right] = \int_0^\infty |f(t)|^2 dt.$$

proof. We refer the reader to proposition 4.1 in [35]

1.5 Stochastic Integration w.r.t FBM

Fractional Brownian motion is not a semimartingale, and hence the stochastic integral with respect to fractional Brownian motion B^H becomes more challenging. It turns out that fractional calculus creates a path to defining a kind of integral with respect to paths of fractional Brownian motion. For a complete treatment of deterministic fractional calculus, see the book by Samko [28].

1.5.1 Fractional calculus on a finite interval

Let $a < b$ be two real numbers and $f : [a, b] \to R$ be a function. Then by a straightforward induction argument, a multiple integral of f can be expressed as

$$\int_a^{t_n} \cdots \int_a^{t_2} \int_a^{t_1} f(u)dt_1 dt_{n-1} = \frac{1}{(n-1)!} \int_a^{t_n} f(u)(t_n - u)^{n-1} du, \tag{1.14}$$

where $t_n \in [a, b]$ and $n \geq 1$. (By convention, $(0)! = 1$ and $a^0 = 1$.) We know that $(n-1)! = \Gamma(n)$. So replacing n by a real number $\alpha > 0$ in (1.14), we are motivated to define the so-called fractional integrals as follows.

Definition 1.5.1.1 *Let* $f \in L^1[a, b]$ *and* $\alpha > 0$. *The integrals*

$$(I_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \tag{1.15}$$

for $t \in (a,b)$*, and*

$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1}dt, \tag{1.16}$$

where $t \in (a,b)$*, are called fractional integrals of order* α*.*

The fractional integral I^{α}_{a+} is called left-sided since the integration in (1.16) is over the left hand side of the interval $[a,t]$ of the interval $[a,b]$. Similarly, the fractional integral I^{α}_{b-} is called right-sided. Both integral I^{α}_{a+} and I^{α}_{b-} are also called Riemann-Liouville fractional integrals

Remark 1.5.1.1 [34]. The fractional integrals I^{α}_{a+} and I^{α}_{b-} are well-defined for functions $f \in L^1[a,b]$, and so also for functions $f \in L^p[a,b]$, for $p > 1$ as well, i.e. the integrals in (1.15) and (1.16) converge for almost all $t \in (a,b)$ with respect to Lebesgue measure.

Remark 1.5.1.2 The left (right)-sided fractional integrals can be defined on the whole real line in a similar way.

Proposition 1.5.1.1 [*34*]. *For* $\alpha > 0$*, the fractional integrals* I^{α}_{a+} *and* I^{α}_{b-}*. have the following properties:*
(i) Semigroup property: for $f \in L^1[a,b]$ *and* $\alpha, \beta > 0$

$$I^{\alpha}_{a+}I^{\beta}_{a+}f = I^{\alpha+\beta}_{a+}f \quad \textit{and} \quad I^{\alpha}_{b-}I^{\beta}_{b-}f = I^{\alpha+\beta}_{b-} \tag{1.17}$$

(ii) Fractional integration by parts formula: let $f \in L^p[a,b]$ *and* $g \in L^q[a,b]$ *either with* $p, q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$*, or with* $p, q > 1$ *and* $p, q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$*.*
Then we have

$$\int_a^b f(t)(I^{\alpha}_{a+}g)(t)dt = \int_a^b g(t)(I^{\alpha}_{b-}f)(t)dt. \tag{1.18}$$

(iii) If $I^{\alpha}_{a+}f = 0$ *or* $I^{\alpha}_{b-}f = 0$ *then* $f(u) = 0$ *almost everywhere.*

For $0 < \alpha < 1$, we define the operator $I^{-\alpha}_{a+}(I^{-\alpha}_{b-})$ as the inverse of the fractional integral operator in the following way.

Definition 1.5.1.2 *Let* $0 < \alpha < 1$*. The integrals*

$$D^{\alpha}_{a+}f(t) = (I^{-\alpha}_{a+})(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(s)(t-s)^{-\alpha}ds, \tag{1.19}$$

$$D_{b-}^{\alpha} f(t) = (I_{b-}^{-\alpha})(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b f(s)(t-s)^{-\alpha} ds, \tag{1.20}$$

for $t \in (a,b)$, are called fractional derivatives of order α. Both (1.19) and (1.20) are also called the Riemann-Liouville fractional derivatives.

Remark 1.5.1.3 The fractional derivatives $D_{a+}^{\alpha} f$ and $D_{b-}^{\alpha} f$ are well defined if, for example, function f can be expressed as $f = D_{a+}^{\alpha}\phi$ or $f = D_{b-}^{\alpha}\phi$, for some $\phi \in L^p[a,b]$ and $p \geq 1$.

1.5.2 The Lebesgue-Stieltjes integral

Let us starting by recall the Riemann-Liouville fractional derivatives

$$(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x f(t)(x-t)^{-\alpha} dt,$$

and

$$(D_{b-}^{\alpha} f)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b f(t)(x-t)^{-\alpha} dt,$$

for $\alpha > 0$.
We refer the reader to [34] for futher information in this topics.

Consider two nonrandom functions f and g defined on some interval $[a,b] \subset \mathbb{R}$ and suppose that the limits $f(u+) := \lim_{\delta \downarrow 0} f(u+\delta)$ and $g(u-) := \lim_{\delta \downarrow 0} g(u-\delta), a \leq u \leq b$, exist. Put $f_{a+}(x) := (f(x) - f(a+))\mathbf{1}_{(a,b)}(x)$, $g_{b-}(x) := (g(b-) - g(x))\mathbf{1}_{(a,b)}(x)$.
Suppose also that $f_{a+} \in I_{a+}^{\alpha}(L_p[a,b])$, $g_{b-} \in I_{b-}^{1-\alpha}(L_p[a,b])$ for some $p \geq 1, q \geq 1, 1/p + 1/q \leq 1, 0 \leq \alpha \leq 1$. Then, evidently, $D_{a+}^{\alpha} f_{a+} \in L_p[a,b]$, $D_{b-}^{1-\alpha} g_{b-} \in L_q[a,b]$.

Let $\alpha > 0$ (and in most cases below $\alpha < 1$ though this is not obligatory). Define the Riemann-Liouville left-sided and right-sided fractional integrals on (a,b) of order α by

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt,$$

and

$$(I_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt,$$

respectively.

We say that the function $f \in \mathcal{D}(I_{a+(b-)})$ (the symbol $\mathcal{D}(.)$ denotes the domain of the corresponding operator),

Definition 1.5.2.1 [*34*]. *The generalized (fractional) Lebesgue-Stieltjes integral* $\int_a^b f(x)dg(x)$ *is defined as*

$$\int_a^b f(x)dg(x) := \int_a^b (D^{\alpha}_{a_+} f_{a_+})(x)(D^{1-\alpha}_{b_-} g_{b_-})(x)dx + f(a_+)(g(b_-) - g(a_+)).$$

Lemma 1.5.2.1 *Definition* 1.5.2.1 *does not depend on the possible choice of* α*.*

Proof. Let $f_{a+} \in (I^{\alpha}_{a+} \cap I^{\alpha+\beta}_{a+}(L_p[a,b]),\ g_{b-} \in (I^{1-\alpha}_{b-} \cap I^{1-\alpha-\beta}_{b-}(L_p[a,b])$ for some α, β such that $0 \leq \alpha \leq 1$, $0 \leq \alpha + \beta \leq 1$, $1/p + 1/q \leq 1$. Then, according to (1.1.5) (composition formula for fractional derivatives) and (1.1.6) (integration-by-parts formula),[34].

$$\begin{aligned}
\int_a^b (D^{\alpha+\beta}_{a_+} f_{a_+})(x)(D^{1-\alpha-\beta}_{b_-} g_{b_-})(x)dx &= \int_a^b (D^{\beta}_{a_+} D^{\alpha}_{a_+} f_{a_+})(x)(D^{1-\alpha-\beta}_{b_-} g_{b_-})(x)dx \\
&= \int_a^b (D^{\alpha}_{a_+} f_{a_+})(x)(D^{\beta}_{b_-} D^{1-\alpha-\beta}_{b_-} g_{b_-})(x)dx \\
&= \int_a^b (D^{\alpha}_{a_+} f_{a_+})(x)(D^{1-\alpha}_{b_-} g_{b_-})(x)dx.
\end{aligned}$$

□

Chapter 2

Stochastic Evolution Equations With Infinite Delay

In this chapter[1], we are interested in studying the existence of mild solutions of the following impulsive fractional stochastic differential equations with infinite delay in the form

$$\left\{\begin{array}{rcl} {}^cD_t^\alpha[x(t)-g(t,x_t)] & = & A[x(t)-g(t,x_t)]+f(t,x_t,B_1x(t))+\sigma(t,x_t,B_2x(t))\frac{dw(t)}{dt}, \\ & & t\in J:=[0,T], \quad T>0, \quad t\neq t_k, \\ \Delta x(t_k) & = & I_kx(t_k^-), \quad k=1,2,\ldots,m, \\ x(t) & = & \phi(t), \qquad \phi(t)\in\mathcal{B}_h, \end{array}\right. \tag{2.1}$$

Where ${}^cD_t$ is the Caputo fractional derivative of order $\alpha, 0<\alpha<1$; $x(.)$ takes the value in the separable Hilbert space $\mathcal{H}; A:\mathcal{D}(A)\subset\mathcal{H}\to\mathcal{H}$ is the infinitesimal generator of an α-resolvent family $S_\alpha(t)_{t\geq 0}$. The history $x_t:(-\infty,0]\to\mathcal{H}, x_t(\theta)=x(t+\theta),\theta\leq 0$, belongs to an abstract phase space $\mathcal{B}_h$, $g:J\times\mathcal{B}_h\to\mathcal{H}, f:J\times\mathcal{B}_h\times\mathcal{H}\to\mathcal{H}$ and $\sigma:J\times\mathcal{B}_h\times\mathcal{H}\to L_0^2$ are appropriate functions to be specified later; $I_k:\mathcal{B}_h\to\mathcal{H}, k=1,2,...,m$, are appropriate functions. The terms $B_1x(t)$ and $B_2x(t)$ are given by

$$B_1x(t)=\int_0^t K(t,s)x(s)ds$$

and

$$B_2x(t)=\int_0^t P(t,s)x(s)ds$$

respectively, where $K,P\in\mathcal{C}(\mathcal{D},\mathbb{R}^+)$ are the set of all positive continuous functions on $\mathcal{D}=\{(t,s)\in\mathbb{R}^2:0\leq s\leq t\leq T\}$. Here $0=t_0<t_1<...<t_m<t_{m+1}=T$, $\Delta x(t_k)=$

[1]The chapter is based on the paper [19].

$x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The initial data $\phi = \{\phi(t), t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued random variable independent of with finite second moments.

2.1 Preliminaries And Basic Properties

Let $\mathcal{H}, \mathcal{K}$ be two separable Hilbert spaces and $L(\mathcal{K}, \mathcal{H})$ be the space of bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$. For convenience, we will use the same notation $||.||$ to denote the norms in $\mathcal{H}, \mathcal{K}$ and $L(\mathcal{K}, \mathcal{H})$, and use $\langle ., . \rangle$ to denote the inner product of $\mathcal{H}$ and $\mathcal{K}$ without any confusion. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. $W = (W_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathcal{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, ...$ and a sequence $\{\beta_k\}_{k\geq 1}$ of independent Brownian motions such that

$$(w(t), e)_{\mathcal{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_{\mathcal{K}} \beta_k(t), \quad e \in \mathcal{K}, t \in [0, b].$$

Let $L_0^2 = L^2(Q^{1/2}\mathcal{H}, \mathcal{H})$ be the space of all Hilbert Schmidt operators from $Q^{1/2}\mathcal{K}$ into $\mathcal{H}$ with the inner product $\langle \psi, \pi \rangle_{L_0^2} = tr[\psi Q \pi^\star]$.

Assume that $h : (-\infty, 0] \to (0, \infty)$ with $l = \int_{-\infty}^{0} h(t)dt < \infty$ a continuous function. We define the abstract phase space $\mathcal{B}_h$ by

$$\mathcal{B}_h = \Big\{ \phi : (-\infty, 0] \to \mathcal{H}, for \quad any \quad a > 0, (\mathbb{E}|\phi(\theta)|^2)^{1/2} \quad is \quad bounded \quad and \quad measurable$$

$$function\, on\, [-a, 0] \quad with \quad \phi(0) = 0 \quad and \quad \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}|\phi(\theta))|^2)^{1/2} ds < \infty \Big\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}|\phi(\theta))|^2)^{1/2} ds, \quad \phi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space [14].

We consider the space

$$\mathcal{B}_b = \Big\{ x : (-\infty, 0] \to \mathcal{H}, such \quad that \quad x|_{J_k} \in \mathcal{C}(J_k, \mathcal{H}) \quad and \quad there \quad exist$$

$$x(t_k^+) \quad and \quad x(t_k^-) \quad with \quad x(t_k) = x(t_k^-), \quad x = \phi \in \mathcal{B}_h, \quad k = 1, 2, ..., m \Big\},$$

where $x|_{J_k}$ is the restriction of $x \quad to \quad J_k = (t_k, t_{k+1}], \quad k = 1, 2, ..., m.$ the function $||.||_{\mathcal{B}_h}$ to be a seminorm in $\mathcal{B}_b$, it is defined by

$$\|x\|_{\mathcal{B}_b} = \|\phi\|_{\mathcal{B}_h} + \sup_{0 \leq s \leq T} (\mathbb{E}|\phi(\theta)|^2)^{1/2}, \quad x \in \mathcal{B}_b$$

Lemma 2.1.1 *Assume that $x \in \mathcal{B}_h$; then for $t \in J, \quad x_t \in \mathcal{B}_h$. Moreover,*

$$l(\mathbb{E}||x(t)||^2)^{1/2} \leq l \sup_{0 \leq s \leq T} (\mathbb{E}||x(s)||^2)^{1/2} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \displaystyle\int_{-\infty}^{0} h(s)ds < \infty.$

Let us recall the following known definitions. For more details see [28].

Definition 2.1.1 *The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \quad \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.1.2 *Riemann-Liouville derivative of order α with lower limit 0 for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \qquad t > 0, n-1 \leq \alpha \leq n. \tag{2.2}$$

Definition 2.1.3 *The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^c D^\alpha f(t) =^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0) \right), \qquad t > 0, \quad n-1 < \alpha < n. \tag{2.3}$$

If $f(t) \in C^n[0,\infty)$, then

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s)ds = I^{n-\alpha} f^n(s), \quad t > 0, n-1 < \alpha < n,$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{^cD^\alpha f(t); s\} = s^\alpha \hat{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); \quad n-1 \leq \alpha < n.$$

Definition 2.1.4 *[28] A two parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \qquad \alpha, \beta \in C, \mathcal{R}(\alpha) > 0,$$

where C is a contour which starts and ends at $-\infty$ end encircles the disc $|\mu| \leq |z|^{1/2}$ counter clockwise.

For short, $E_\alpha(z) = E_{\alpha,1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_1(z) = e^z$ and the cosine function: $E_2(z^2) = cosh(z), E_2(-z^2) = cos(z)$, and plays a vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad Re\lambda > \omega^{1/\alpha}, \omega > 0,$$

and for more details see [28].

Definition 2.1.5 *[41]. A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi], M > 0$, such that the following two conditions are satisfied:*

- $\rho(A) \subset \Sigma_{\theta,\omega} = \{\lambda \in C, \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}$,
- $||R(\lambda, \omega)|| = ||(\lambda I - A)^{-1}|| \leq \frac{M}{|\lambda - \omega|}, \lambda \in \Sigma_{\theta,\omega}$.

Definition 2.1.6 [15]. *Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space H. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(H)$, where $L(H)$ is a Banach space of all bounded linear operators from H into H and the corresponding norm is denoted by $||.||$, such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)xdt, \quad Re\lambda > \omega, x \in H, \tag{2.4}$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.1.7 *Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space H and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(H)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)xdt, \qquad Re\lambda > \omega, x \in H, \tag{2.5}$$

where $S_\alpha(t)$ is called the solution operator generated by A.

The concept of the solution operator is closely related to the concept of a resolvent family. For more details on α-resolvent family and solution operators, we refer the reader to [15].

Lemma 2.1.2 [15]. *If f satisfies the uniform Hölder condition with the exponent $\beta \in (0, 1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem*

$$\begin{array}{rcl} {}^cD_t^\alpha &=& Ax(t) + f(t, x_t, Bx(t)), \quad t > t_0, t_0 \geq 0, 0 < \alpha < 1, \\ x(t) &=& \phi(t), \quad t \leq t_0. \end{array} \tag{2.6}$$

is given by

$$x(t) = T_\alpha(t - t_0)(x(t_0^+)) + \int_{t_0}^t S_\alpha(t - s)f(s, x_s, Bx(s))ds, \tag{2.7}$$

where

$$T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda, \tag{2.8}$$

$$S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i}\int_{\hat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda, \tag{2.9}$$

here $\widehat{B}_r$ denotes the Bromwich path; $S_\alpha(t)$ is called the α-resolvent family and $T_\alpha(t)$ is the solution operator generated by A.

Proof. Let $t - t_0 = u$, then we get

$$D_\alpha^u x(u + t_0) = Ax(u + t_0) + f(u + t_0, x_{u+t_0}, Bx(u + t_0)), \quad u > 0. \tag{2.10}$$

Taking the Laplace transform of (2.10), we have

$$\lambda^\alpha L\{x(u + t_0)\} - \lambda^{\alpha-1} x(t_0^+) = AL\{x(u + t_0)\} + L\{f(u + t_0, x_{u+t_0}, Bx(u + t_0))\}. \tag{2.11}$$

Since $(\lambda^\alpha I - A)^{-1}$ exists, that is, $\lambda^\alpha \in \rho(A)$, from (2.11), we obtain

$$L\{x(u + t_0)\} = \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} x(t_0^+) + (\lambda^\alpha I - A)^{-1} L\{f(u + t_0, x_{u+t_0}, Bx(u + t_0))\}. \tag{2.12}$$

By the inverse Laplace transform of (2.12), we get

$$x(u+t_0) = E_{\alpha,1}(Au^\alpha)x(t_0^+) + \int_0^u (u-s)^{\alpha-1} E_{\alpha,\alpha}(A(u-s)^\alpha) f(s+t_0, x_{s+t_0}, Bx(s+t_0))ds. \tag{2.13}$$

Set $u + t_0 = t$, in (2.13), we have

$$\begin{aligned} x(t) &= E_{\alpha,1}(A(t - t_0)^\alpha)x(t_0^+) \\ &\quad + \int_0^{t-t_0} (t - t_0 - s)^{\alpha-1} E_{\alpha,\alpha}(A(t - t_0 - s)^\alpha) f(s + t_0, x_{s+t_0}, Bx(s + t_0))ds. \end{aligned} \tag{2.14}$$

On simplification, we obtain

$$\begin{aligned} x(t) &= E_{\alpha,1}(A(t - t_0)^\alpha)x(t_0^+) \\ &\quad + \int_0^{t-t_0} (t - \theta)^{\alpha-1} E_{\alpha,\alpha}(A(t - \theta)^\alpha) f(\theta, x_\theta, Bx(\theta))d\theta. \end{aligned} \tag{2.15}$$

Set $T_\alpha(t) = E_{\alpha,1}(At^\alpha)$ and $S_\alpha(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)$, in (2.15). We have

$$x(t) = T_\alpha(t - t_0)x(t_0^+) + \int_{t_0}^t S_\alpha(t + \theta) f(\theta, x_\theta, Bx(\theta))d\theta.$$

This completes the proof of the lemma.

□

Theorem 2.1.1 *[15]. Let B be a nonempty closed convex of a Banach space $(\mathcal{H}, ||\cdot||)$. Suppose that P and Q map B into $\mathcal{H}$ such that*

- *$Px + Qy \in B$ whenever $x, y \in B$;*

- *P is compact and continuous;*
- *Q is a contraction mapping.*
  Then there exists $z \in B$ such that $z = Pz + Qz$.

Theorem 2.1.2 *[42]. If $\alpha \in (0,1)$ and $A \in \mathcal{A}^{\alpha}(\theta_0, \omega_0)$ is a sectorial operator, then for any $x \in \mathcal{H}$ and $t > 0$, we have*

$$||S_\alpha(t)|| \leq Ce^{\omega t}(1 + t^{\alpha-1}), \quad t > 0, \omega > \omega_0,$$

where C is a constant depending only on θ and ω.

At the end of this section, we recall the fixed point theorem of Sadovskii [40] which is used to establish the existence of the mild solution to the impulsive fractional system (2.1).

Theorem 2.1.3 *[40]. Let Φ be a condensing operator on a Banach space $\mathcal{H}$, that is, Φ is continuous and takes bounded sets into bounded sets, and $\mu(\Phi(B)) \leq \mu(B)$ for every bounded set B of $\mathcal{H}$ with $\mu(B) > 0$. If $\Phi(N) \subset N$ for a convex, closed and bounded set N of $\mathcal{H}$, then Φ has a fixed point in $\mathcal{H}$ (where $\mu(.)$ denotes Kuratowski's measure of noncompactness).*

2.2 The mild solution and existence

In this section, we consider the fractional impulsive system (2.1). We first present the definition of mild solutions for the system based on the paper [15].

Definition 2.2.1 *An $\mathcal{H}$-valued stochastic process $\{x(t), t \in (-\infty, T]\}$ is said to be a mild solution of the system* (2.1) *if $x_0 = \phi \in \mathcal{B}_h$ satisfying $x_0 \in L^2_\alpha(\Omega, \mathcal{H})$ and the following conditions hold.*

i. $x(t)$ is $\mathcal{F}_t$ adapted and measurable, $t \geq 0$;

ii. x_t is $\mathcal{B}_h$-valued and the restriction of $x(.)$ to the interval $(t_k, t_{k+1}], k = 1, 2, ..., m$ is continuous;

iii. for each $t \in J$, $x(t)$ satisfies the following integral equation

$$
x(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ T_\alpha(t)[\phi(0) + g(0,\phi)] - g(t, x_t) + \int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds \\ + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) d\omega(s), \quad t \in [0, t_1], \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + T_\alpha(t-t_1) I_1((t_1^-)) - g(t, x_t) \\ + T_\alpha(t-t_1)[g(t_1, x_{t_1} + I_1(t_1^-)) - g(t_1, x_{t_1})] \\ + \int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) d\omega(s), \quad t \in (t_1, t_2], \\ \vdots \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + \sum_{k=1}^m T_\alpha(t-t_k) I_k(x(t_k^-)) - g(t, x_t) \\ + \sum_{k=1}^m T_\alpha(t-t_k)[g(t_{t_K}, x_{t_k}, I_k(x_{t_k^-})) - g(t_k, x_{t_k})] \\ + \int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) d\omega(s),\ t \in (t_m, T]. \end{cases} \tag{2.16}
$$

iv. $\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, ..., m$ *the restriction of* $x(.)$ *to the interval* $[0, T)\backslash\{t_1, \dots, t_m\}$ *is continuous.*

In order to explain our theorem, we need the following assumptions.

(**H1**): If $\alpha \in (0,1)$ and $A \in \mathcal{A}_\alpha(\theta_0, \omega_0)$, then for $x \in \mathcal{H}$ and $t > 0$ we have $||S_\alpha(t)|| \leq Ce^{\omega t}(1 + t^{\alpha-1})$ and $||T_\alpha(t)|| \leq Me^{\omega t}$,$\omega > \omega_0$. Thus we have

$$||T_\alpha(t)|| \leq \widetilde{M_T} \quad \textit{and} \quad ||S_\alpha(t)|| \leq t^{\alpha-1}\widetilde{M_S},$$

where $\widetilde{M_T} = \sup_{0\leq t\leq T} ||T_\alpha(t)||, \widetilde{M_S} = \sup_{0\leq t\leq T} Ce^{\omega t}(1 + t^{1-\alpha})$ (fore more details, see [42]).

(**H2**): The function $g : J \times \mathcal{B}_h \to \mathcal{H}$ is continuous and there exists some constant $M_g > 0$ such that

$$\mathbb{E}||g(t,\psi_1) - g(t,\psi_2)||^2_{\mathcal{H}} \leq M_g||\psi_1 - \psi_2||^2_{\mathcal{B}_h}, \qquad (t, \psi_i) \in J \times \mathcal{B}_h, \quad i = 1, 2,$$

$$\mathbb{E}||g(t,\psi)||^2_{\mathcal{H}} \leq M_g \left(||\psi||^2_{\mathcal{B}_h} + 1\right).$$

(**H3**): The function $f : J \times \mathcal{B}_h \times \mathcal{H} \to \mathcal{H}$ satisfies the following properties:

i. $f(t, ., .) : \mathcal{B}_h \to \mathcal{H}$ is continuous for each $t \in J$ and for each $(\psi, x) \in \mathcal{B}_h \times \mathcal{H}, f(., \psi, x) : J \to \mathcal{H}$ is strongly measurable;

ii. there exist two positive integrable functions $\mu_1, \mu_2 \in L^1([0,T])$ and a continuous nondecreasing function $\Xi_f : [0,\infty) \to (0,\infty)$ such that for every $(t,\psi,x) \in J \times \mathcal{B}_h \times \mathcal{H}$, we have

$$\mathbb{E}||f(t,\psi,x)||^2_{\mathcal{H}} \leq \mu_1(t)\Xi_f(||\psi||^2_{\mathcal{B}_h}) + \mu_1(t)\mathbb{E}||x||^2_{\mathcal{H}}, \quad \liminf_{q\to\infty} \frac{\Xi_f(q)}{q} = \Lambda < \infty.$$

iii. there exist two positive integrable functions $\mu_1, \mu_2 \in L^1([0,T])$ such that

$$\mathbb{E}||f(t,\psi,x) - f(t,\varphi,y)||^2_{\mathcal{H}} \leq \mu_1(t)||\psi - \varphi||^2_{\mathcal{B}_h} + \mu_2(t)\mathbb{E}||x-y||^2_{\mathcal{H}},$$

for every (t,ψ,x) and $(t,\varphi,y) \in J \times \mathcal{B}_h \times \mathcal{H}$.

(**H4**): The function $\sigma : J \times \mathcal{B}_h \times \mathcal{H} \to L^2_0$ satisfies the following properties:

i. $\sigma(t,.,.) : \mathcal{B}_h \times \mathcal{H} \to L^2_0$ is continuous for each $t \in J$ and for each $(\psi,x) \in \mathcal{B}_h \times \mathcal{H}, \sigma(.,\psi,x) : J \to L^2_0$ is strongly measurable;

ii. there exist two positive integrable functions $\nu_1, \nu_2 \in L^1([0,T])$ and a continuous nondecreasing function $\Xi_\sigma : [0,\infty) \to (0,\infty)$ such that for every $(t,\varphi,x) \in J \times \mathcal{B}_h \times \mathcal{H}$, we have

$$\mathbb{E}||\sigma(t,\psi,x)||^2_{L^2_0} \leq \nu_1(t)\Xi_\sigma(||\psi||^2_{\mathcal{B}_h}) + \nu_1(t)\mathbb{E}||x||^2_{\mathcal{H}}, \quad \liminf_{q\to\infty} \frac{\Xi_\sigma(q)}{q} = \Upsilon < \infty.$$

iii. there exist two positive integrable functions $\nu_1, \nu_2 \in L^1([0,T])$ such that

$$\mathbb{E}||\sigma(t,\psi,x) - \sigma(t,\varphi,y)||_{L^2_0} \leq \nu_1(t)||\psi - \varphi||^2_{\mathcal{B}_h} + \nu_2(t)\mathbb{E}||x-y||^2_{\mathcal{H}},$$

for every (t,ψ,x) and $(t,\varphi,y) \in J \times \mathcal{B}_h \times \mathcal{H}$.

(**H5**): The function $I_k : \mathcal{H} \to \mathcal{H}$ is continuous and there exists $\Theta > 0$ such that

$$\Theta = \max_{1\leq k\leq m, \quad x\in B_q} \left\{\mathbb{E}||I_k(x)||^2_{\mathcal{H}}\right\},$$

where $B_q = \{y \in B^0_b, ||y||^2_{B^0_b} \leq q, \quad q > 0\}$.

The set B_q is clearly a bounded closed convex set in B^0_b for each q and for each $y \in B_q$. From

Lemma 2.1.1, we have

$$\begin{aligned} \|y_t - \bar{z}_t\|^2_{\mathcal{B}_h} &\le 2(\|y_t\|^2_{\mathcal{B}_h} + \|\bar{z}_t\|^2_{\mathcal{B}_h}) \\ &\le 4\left(l^2 \sup_{0\le t\le T} \mathbb{E}\|y(t)\|^2_{\mathcal{H}} + \|y_0\|^2_{\mathcal{B}_h}\right) + 4\left(l^2 \sup_{0\le t\le T} \mathbb{E}\|y(t)\|^2_{\mathcal{H}}) + \|\bar{z}_0\|^2_{\mathcal{B}_h}\right) \\ &\le 4(\|\phi\|^2_{\mathcal{B}_h}) + l^2 q). \end{aligned} \tag{2.17}$$

The main object of this chapter is to explain and prove the following theorem.

Theorem 2.2.1 *Assume that the assumptions (**H1**)-(**H5**) hold. Then the impulsive stochastic fractional system* (2.1) *has a mild solution on* $(-\infty, T]$ *provided that*

$$\widetilde{C} + 16M_g l^2 + 7\widetilde{M}_s^2 T^{2\alpha}\left[\frac{\eta_1}{\alpha^2} + \frac{\eta_2}{T(2\alpha-1)}\right] < 1. \tag{2.18}$$

and

$$M_g l^2 + 7\widetilde{M}_s^2 T^{2\alpha}\left[\frac{\upsilon_1}{\alpha^2} + \frac{\upsilon_2}{T(2\alpha-1)}\right] < 1, \tag{2.19}$$

$\widetilde{C}$ *is a positive constant depending only on* $\widetilde{M_T}$ *,*M_g *and* l.

Proof. Consider the operator $\mathcal{P} : \mathcal{B}_b \to \mathcal{B}_b$ defined by

$$\mathcal{P}(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ T_\alpha(t)[\phi(0) + g(0,\phi)] - g(t, x_t) + \int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) d\omega(s), \quad t \in [0, t_1], \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + T_\alpha(t-t_1) I_1(x(t_1^-)) - g(t, x_t) \\ \quad + T_\alpha(t-t_1)[g(t_1, x_{t_1} + I_1(t_1^-)) - g(t_1, x_{t_1})] \\ \quad + \int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) d\omega(s), t \in (t_1, t_2], \\ \vdots \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + \sum_{k=1}^m T_\alpha(t-t_k) I_k((t_k^-)) - g(t, x_k) \\ \quad + \sum_{k=1}^m T_\alpha(t-t_k)[g(t_{t_k}, x_{t_k}, I_k(x_{t_k^-})) - g(t_k, x_{t_k})] \\ \quad + \int_0^t S_\alpha(t-s) f(s, x_s, B_1 x(s)) ds + \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s)) d\omega(s), \quad t \in (t_m, T]. \end{cases} \tag{2.20}$$

We shall show that $\mathcal{P}$ has a fixed point, which is then a mild solution for the impulsive system (2.1).
For $\phi \in \mathcal{B}_h$, define

$$\bar{z}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ 0, & t \in J. \end{cases}$$

Then $\bar{z}(t) \in \mathcal{B}_b$. Let $x(t) = y(t) + \bar{z}(t), t \in (-\infty, T]$. It is easy to check that x satisfies (2.1) if and only if $y_0 = 0$ and

$$y(t) = \begin{cases} T_\alpha(t)[\phi(0) + g(0,\phi)] - g(t, y_t + \bar{z}_t) + \int_0^t S_\alpha(t-s) f(s, y_s + \bar{z}_s, B_1(y(t) + \bar{z}(t)))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, B_2(y(t) + \bar{z}(t)))d\omega(s), \quad t \in [0, t_1], \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + T_\alpha(t-t_1) I_1(y(t_1^-)) - g(t, y_t + \bar{z}_t) \\ \quad + T_\alpha(t-t_1)[g(t_1, y_{t_1} + \bar{z}_{t_1} + I_1(y_{t_1^-} + \bar{z}_{t_1^-})) - g(t_1, y_{t_1} + \bar{z}_{t_1})] \\ \quad + \int_0^t S_\alpha(t-s) f(s, y_s + \bar{z}_s, B_1(y(t) + \bar{z}(t))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, B_2(y_s + \bar{z}_s)d\omega(s), \quad t \in (t_1, t_2], \\ \vdots \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + \sum_{k=1}^m T_\alpha(t-t_k) I_k(y(t_k^-)) - g(t, y_t + \bar{z}_t) \\ \quad + \sum_{k=1}^m T_\alpha(t-t_k)[g(t_{t_k}, y_{t_k} + \bar{z}_{t_k}, I_k(x_{t_k^-})) - g(t_k, x_{t_k})] \\ \quad + \int_0^t S_\alpha(t-s) f(s, y_s + \bar{z}_s, B_1(y(t) + \bar{z}(t)))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, B_2(y(t) + \bar{z}(t))d\omega(s), \quad t \in (t_m, T]. \end{cases}$$

Set

$$\mathcal{B}_b^0 = \{y \in \mathcal{B}_b, y_0 = 0 \in \mathcal{B}_h\}$$

Thus, for any $y \in \mathcal{B}_b^0$ we have

$$||y||_b = ||y_0||_{\mathcal{B}_h} + \sup_{0 \le s \le T} \left(\mathbb{E}||y(s)||^2\right)^{\frac{1}{2}} = \sup_{0 \le s \le T} \left(\mathbb{E}||y(s)||^2\right)^{\frac{1}{2}}.$$

Therefore, $(\mathcal{B}_b^0, ||.||_b)$ is a Banach space.

Consider the map Π on $\mathcal{B}_b^0$ defined by

$$(\Pi y)(t) = \begin{cases} T_\alpha(t)[\phi(0) + g(0,\phi)] - g(t, y_t + \bar{z}_t) + \int_0^t S_\alpha(t-s) f(s, y_s + \bar{z}_s, B_1(y(s) + \bar{z}(s)))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, B_2(y(s) + \bar{z}(s))d\omega(s), \quad t \in [0, t_1], \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + T_\alpha(t-t_1)I_1(y(t_1^-)) - g(t, y_t + \bar{z}_t) \\ \quad + T_\alpha(t-t_1)[g(t_1, y_{t_1} + \bar{z}_{t_1} + I_1(y_{t_1^-} + \bar{z}_{t_1^-})) - g(t_1, y_{t_1} + \bar{z}_{t_1})] \\ \quad + \int_0^t S_\alpha(t-s) f(s, y_s + \bar{z}_s, B_1(y(s) + \bar{z}(s)))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, B_2(y(s) + \bar{z}(s))d\omega(s), \quad t \in (t_1, t_2], \\ \vdots \\ T_\alpha(t)[\phi(0) + g(0,\phi)] + \sum_{k=1}^m T_\alpha(t-t_k)I_k(y(t_k^-)) - g(t, y_t + \bar{z}_t) \\ \quad + \sum_{k=1}^m T_\alpha(t-t_k)[g(t_{t_k}, y_{t_k} + \bar{z}_{t_k}, I_k(x_{t_K^-})) - g(t_k, x_{t_k})] \\ \quad + \int_0^t S_\alpha(t-s) f(s, y_s + \bar{z}_s, B_1(y(s) + \bar{z}(s))ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, B_2(y(s) + \bar{z}(s))d\omega(s), \quad t \in (t_m, T]. \end{cases}$$

It is clear that the operator $\mathcal{P}$ has a fixed point if and only if Π has a fixed point. So let us prove that Π has a fixed point. Now, we decompose Π as $\Pi = \Pi_1 + \Pi_2$, where

$$(\Pi_1 y)(t) = \begin{cases} 0, \qquad t \in [0, t_1], \\ T_\alpha(t-t_1)I_1(y(t_1^-)) \\ + T_\alpha(t-t_1)[g(t_1, y_{t_1} + \bar{z}_{t_1} + I_1(y_{t_1^-} + \bar{z}_{t_1^-})) - g(t_1, y_{t_1} + \bar{z}_{t_1})], \quad t \in [t_1, t_2], \\ \vdots \\ \sum_{k=1}^m T_\alpha(t-t_k)I_k(y(t_k^-)) \\ + \sum_{k=1}^m T_\alpha(t-t_k)[g(t_1, y_{t_k} + \bar{z}_{t_k} + I_k(y_{t_k^-} + \bar{z}_{t_k^-})) - g(t_1, y_{t_k} + \bar{z}_{t_k})], \quad t \in [t_m, T], \end{cases}$$

$$\begin{aligned} (\Pi_2 y)(t) &= T_\alpha(t)g(0,\phi) - g(t, y_t + \bar{z}_t) + \int_0^t S_\alpha(t-s) f(s, y_s + \bar{z}_s, B_1(y(t) + \bar{z}(t)))ds \\ &\quad + \int_0^t S_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, B_2(y(s) + \bar{z}(s))d\omega(s)), \quad t \in J. \end{aligned}$$

In order to use Theorem 2.1.3 we will verify that Π_1 is compact and continuous while Π_2 is a contraction operator. For the sake of convenience, we divide the proof into several steps.

Step1. We show that there exists a positive number q such that $\Pi(B_q) \subset B_q$. If this is not true, then for each $q > 0$, there exists a function $y^q(.) \in B_q$, but $\Pi(y^q) \in B_q$, that is $\mathbb{E}||(\Pi y^q)(t)||^2_{\mathcal{H}} > q$. An elementary inequality can show that, for $t \in [0, t_1]$.

$$
\begin{aligned}
q &\leq \mathbb{E}\,\|(\Pi(y^q)(t)\|^2_{\mathcal{H}} \\
&\leq 4\mathbb{E}||T_\alpha(t)g(0,\phi)||^2_{\mathcal{H}} + 4\mathbb{E}\|g(t, y^q_t + \bar{z}_t)\|^2_{\mathcal{H}} + 4\mathbb{E}\left\|\int_0^t S_\alpha(t-s)f(s, y^q_s + \bar{z}_s, B_1(y^q(s) + \bar{z}(s)))ds\right\|^2_{\mathcal{H}} \\
&\quad +4\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, B_2(y(s) + \bar{z}(s))d\omega(s)\right\|^2_{\mathcal{H}} \\
&= 4\textstyle\sum_{i=1}^4 I_i.
\end{aligned}
\tag{2.21}
$$

Let us now estimate each term above $I_i, i = 1, \ldots, 4$. By Lemma 2.1.1 and assumptions (**H1**) − (**H2**), we have

$$
I_1 \leq \widetilde{M^2_T}\mathbb{E}||g(0,\phi)||^2_{\mathcal{H}} \leq \widetilde{M^2_T}M_g(||\phi||^2_{\mathcal{B}_h} + 1), \tag{2.22}
$$

$$
I_2 \leq M_g(||y^q_t + \bar{z}_t||^2_{\mathcal{B}_h} + 1) \leq M_g\left[4(||\phi||^2_{\mathcal{B}_h} + l^2 q) + 1\right]. \tag{2.23}
$$

Together with assumption (**H3**) and (2.17), we have

$$
\begin{aligned}
I_3 &\leq \int_0^t ||S_\alpha(t-s)||ds + \int_0^t ||S_\alpha(t-s)||\mathbb{E}||f(s, y^q_s + \bar{z}_s, B_1(y^q(s) + \bar{z}(s)))||^2_{\mathcal{H}}ds \\
&\leq \widetilde{M}^2_s \int_0^t (t-s)^{\alpha-1}ds + \int_0^t (t-s)^{\alpha-1}[\mu_1(s)\Xi_f(||y^q_t + \bar{z}_t||^2_{\mathcal{B}_h}) + \mu_2\mathbb{E}||B_1(y^q(s) + \bar{z}(s))||^2_{\mathcal{H}}]ds \\
&\leq \widetilde{M}^2_S \tfrac{T^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1}\left[\Xi_f(4(||\phi||^2_{\mathcal{B}_h} + l^2 q))\mu^*_1 + B^*_1\mu^*_2 \sup_{0\leq s\leq T}\mathbb{E}||y^q_s + \bar{z}_s||^2_{\mathcal{H}}ds\right] \\
&\leq \widetilde{M}^2_S \tfrac{T^{2\alpha}}{\alpha^2} \int_0^t (t-s)^{\alpha-1}\left[\Xi_f 4(||\phi||^2_{\mathcal{B}_h} + l^2 q)\mu^*_1 + B^*_1\mu^*_2 q\right],
\end{aligned}
\tag{2.24}
$$

where $B_1^* = \sup\limits_{t\in[0,T]} \int_0^t K(t,s)ds < \infty$, $\mu_1^* = \sup\limits_{s\in[0,t]} \mu_1(s)$, $\mu_2^* = \sup\limits_{s\in[0,t]} \mu_2(s)$.

A similar argument involves assumption (**H4**), we obtain

$$\begin{aligned} I_4 &\leq \int_0^t ||S_\alpha(t-s)||^2 \mathbb{E}||\sigma(s, y_s^q + \bar{z}_s, B_1(y^q(s)+\bar{z}(s)))||_{L_0^2}^2 ds \\ &\leq \widetilde{M}_S^2 \int_0^t (t-s)^{\alpha-1}\left[\Xi_\sigma 4(||\phi||_{\mathcal{B}_h}^2 + l^2 q)\nu_1^* + B_1^*\nu_2^* \sup_{0\leq s\leq T}\mathbb{E}||y_s^q + \bar{z}_s||_{\mathcal{H}}^2 ds\right] \\ &\leq \widetilde{M}_S^2 \tfrac{T^{2\alpha-1}}{2\alpha-1}\int_0^t (t-s)^{\alpha-1}\left[\Xi_\sigma 4(||\phi||_{\mathcal{B}_h}^2 + l^2 q)\nu_1^* + B_1^*\nu_2^* q\right], \end{aligned} \tag{2.25}$$

where $B_2^* = \sup\limits_{t\in[0,T]} \int_0^t K(t,s)ds < \infty$, $\nu_1^* = \sup\limits_{s\in[0,t]} \nu_1(s), \nu_2^* = \sup\limits_{s\in[0,t]} \nu_2(s)$.

Combining these estimates (2.21)-(2.25) yields

$$\begin{aligned} q &\leq \mathbb{E}||(\Pi y^q)(t)||_{\mathcal{H}}^2 \\ &\leq L_0 + 16M_g l^2 q + 4\widetilde{M}_S^2\tfrac{T^{2\alpha}}{\alpha^2}\int_0^t \left[\Xi_\sigma 4(||\phi||_{\mathcal{B}_h}^2 + l^2 q)\mu_1^* + B_1^*\mu_2^* q\right] \\ &\quad + 4\widetilde{M}_S^2\tfrac{T^{2\alpha-1}}{2\alpha-1}\int_0^t (t-s)^{\alpha-1}\left[\Xi_\sigma 4(||\phi||_{\mathcal{B}_h}^2 + l^2 q)\nu_1^* + B_1^*\nu_2^* q\right], \end{aligned} \tag{2.26}$$

where

$$L_0 = 4\widetilde{M_T^2} M_g(||\phi||_{\mathcal{B}_h}^2 + 1) + 4M_g(1 + 4||\phi||_{\mathcal{B}_h}^2).$$

Dividing both sides of (2.26) by q and taking $q \to \infty$, we obtain

$$\begin{aligned} &16M_g l^2 + 4\widetilde{M}_S^2\tfrac{T^{2\alpha}}{\alpha^2}[4\Lambda\mu_1^* + B_1^*\mu_2^*] + 4\widetilde{M}_S^2\frac{T^{2\alpha-1}}{2\alpha-1}[4\Upsilon\nu_1^* + B_2^*\nu_2^*] \\ &= 16M_g l^2 + 4\widetilde{M}_S^2 T^{2\alpha}\left[\tfrac{\eta_1}{\alpha^2} + \tfrac{\eta_2}{T(2\alpha-1)}\right] \geq 1. \end{aligned}$$

which is a contradiction to our assumption in (2.18).

For $t \in (t_1, t_2]$, we have

$$\begin{aligned} q &\leq \mathbb{E}||(\Pi y^q)(t)||_{\mathcal{H}}^2 \\ &\leq 7||T_\alpha(t-t_1)||^2\mathbb{E}||I_1(y^q(t_1^-))||_{\mathcal{H}}^2 + 7||T_\alpha(t-t_1)||^2\mathbb{E}||g(t_1, y_{t_1}^q + \bar{z}_{t_1} + I_1(y_{t_1^-}^q + \bar{z}_{t_1^-}))||_{\mathcal{H}}^2 \\ &\quad + 7||T_\alpha(t-t_1)||^2||g(t_1, y_{t_1}^q + \bar{z}_{t_1}||_{\mathcal{H}}^2 + 7\mathbb{E}||T_\alpha(t)g(0,\phi)||_{\mathcal{H}}^2 + 7\mathbb{E}||g(t, y_t^q + \bar{z}_t||_{\mathcal{H}}^2 \\ &\quad + 4\mathbb{E}\left\|\int_0^t S_\alpha(t-s)f(s, y_s^q + \bar{z}_s, B_1(y^q(s)+\bar{z}(s)))ds\right\|_{\mathcal{H}}^2 \\ &\quad + 7\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\sigma(s, y_s^q + \bar{z}_s, B_2(y^q(s)+\bar{z}(s)))d\omega(s)\right\|_{\mathcal{H}}^2. \end{aligned} \tag{2.27}$$

Using assumptions (**H1**)-(**H5**) we obtain

$$\begin{aligned}
& \mathbb{E}||(\Pi y^q)(t)||^2_{\mathcal{H}} \\
\leq\ & L_1 + 70\widetilde{M_T^2} M_g l^2 q + 28 M_g l^2 q + 7\widetilde{M}_S^2 \tfrac{T^{2\alpha}}{\alpha^2} \int_0^t \left[\Xi_f(4(||\phi||^2_{\mathcal{B}_h} + l^2 q))\mu_1^* + B_1^*\mu_2^* q\right] \\
& + 7\widetilde{M}_S^2 \tfrac{T^{2\alpha-1}}{2\alpha-1} \int_0^t \left[\Xi_\sigma(4(||\phi||^2_{\mathcal{B}_h} + l^2 q))\nu_1^* + B_1^*\nu_2^* q\right],
\end{aligned}$$

where

$$L_1 = 7\widetilde{M_T^2}\left(\Theta + M_g\left[1 + 6\left(||\phi||^2_{\mathcal{B}_h} + l^2\Theta\right)\right]\right) + 7\widetilde{M_T^2} M_g (1 + ||\phi||^2_{\mathcal{B}_h}) + 7M_g\left(1 + 4||\phi||^2_{\mathcal{B}_h}\right).$$

A Similar argument gives

$$\begin{aligned}
& 70\widetilde{M_T^2} M_g l^2 + 28 M_g l^2 + 7\widetilde{M}_S^2 \tfrac{T^{2\alpha}}{\alpha^2}\left[4\Lambda\mu_1^* + B_1^*\mu_2^*\right] + 7\widetilde{M}_S^2 \tfrac{T^{2\alpha-1}}{2\alpha-1}\left[4\Upsilon\nu_1^* + B_2^*\nu_2^*\right] \\
& = 70\widetilde{M_T^2} M_g l^2 + 28 M_g l^2 + 7\widetilde{M}_S^2 T^{2\alpha}\left[\tfrac{\eta_1}{\alpha^2} + \tfrac{\eta_2}{T(2\alpha-1)}\right] \geq 1,
\end{aligned}$$

which is a contradiction to our assumption in (2.18). Similarly for $t \in (t_i, t_{i+1}], i = 1, 2, \ldots, m$, we obtain

$$\begin{aligned}
& \widetilde{C} + 16 M_g l^2 + 7\widetilde{M}_S^2 \tfrac{T^{2\alpha}}{\alpha^2}\left[4\Lambda\mu_1^* + B_1^*\mu_2^*\right] + 7\widetilde{M}_S^2 \tfrac{T^{2\alpha-1}}{2\alpha-1}\left[4\Upsilon\nu_1^* + B_2^*\nu_2^*\right] \\
& = \widetilde{C} + 16 M_g l^2 + 7\widetilde{M}_S^2 T^{2\alpha}\left[\tfrac{\eta_1}{\alpha^2} + \tfrac{\eta_2}{T(2\alpha-1)}\right] \geq 1,
\end{aligned}$$

with $\eta_1 = 4\Lambda\mu_1^* + B_1^*\mu_2^*$,$\eta_2 = 4\Upsilon\nu_1^* + B_2^*\nu_2^*$ and $\widetilde{C}$ is a positive constant depending only on $\widetilde{M}_t$,M_g and l. This is a contradiction to our assumption in (2.18).
Thus, for some positive number q, $\Pi(B_q) \subset B_q$.
Step 2. The map Π_1 is continuous on B_q.
Let $\{y^n\}_{n=1}^{\infty}$ be a sequence in B_q with $lim\, y^n \to y \in B_q$. Then for $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned}
& \mathbb{E}||(\Pi_1 y^n)(t) - (\Pi_1 y)(t)|| \\
\leq\ & 3\textstyle\sum_{k=1}^{i} ||T_\alpha(t - t_k)||^2 \Big[\mathbb{E}||I_k(y^n(t_k^-)) - I_k(y(t_1^-))||^2_{\mathcal{H}} \\
& + \mathbb{E}\left\|g(t_k, y^n_{t_k} + \bar{z}_{t_k} + I_k(y^n_{t_k^-} + \bar{z}_{t_k^-}) - g(t_k, y_{t_k} + \bar{z}_{t_k} + I_k(y_{t_k^-} + \bar{z}_{t_k^-}))\right\|^2_{\mathcal{H}} \\
& + \mathbb{E}\left\|g(t_k, y^n_{t_k} + \bar{z}_{t_k}) - g(t_k, y_{t_k} + \bar{z}_{t_k}))\right\|^2_{\mathcal{H}}\Big].
\end{aligned}$$

Since the functions g, $I_i, i = 1, 2, \ldots, m$ are continuous, hence $\lim\limits_{n\to\infty} \mathbb{E}\|\Pi_1 y^n - \Pi_1 y\|^2 = 0$ which implies that the mapping Π_1 is continuous on B_q.

Step 3. Π_1 maps bounded sets into bounded sets in B_q.

Let us prove that for $q > 0$ there exists a $\delta > 0$ such that for each $y \in B_q$, we have $\mathbb{E}||(\Pi_1 y)(t)||^2_{\mathcal{H}} \leq \delta$ for $t \in (t_i, t_{i+1}], i = 0, 1, \ldots, m.$ We have

$$\begin{aligned}\mathbb{E}||(\Pi_1 y)(t)||^2_{\mathcal{H}} &\leq 3\sum_{k=1}^{i}||T_\alpha(t-t_k)||^2\Big[\mathbb{E}||I_k(y(t_k^-))||^2_{\mathcal{H}} + \mathbb{E}||g(t_k, y_{t_k} + \bar{z}_{t_k})||^2_{\mathcal{H}} \\ &\quad + \mathbb{E}||g(t_k, y_{t_k} + \bar{z}_{t_k} + I_k(y_{t_k^-} + \bar{z}_{t_k^-}))||^2_{\mathcal{H}}\Big] \\ &\leq 3\widetilde{M_T^2}\left[\Theta(1+6M_g l^2) + 2M_g + 10M_g(||\phi||^2_{\mathcal{B}_h} + l^2 q)\right] \\ &:= \delta,\end{aligned}$$

which proves the desired result.

Step 4. The set $\{\Pi_1 y, y \in B_q\}$ is an equicontinuous family of functions on J.

Let $u, v \in (t_i, t_{i+1}], t_i \leq u < v \leq t_{i+1}, \quad i = 0, 1, \ldots, m, y \in B_q$. We have

$$\begin{aligned}&\mathbb{E}||(\Pi_1 y)(v) - (\Pi_1 y)(u)||^2_{\mathcal{H}} \\ \leq\ & 3\sum_{k=1}^{i}||T_\alpha(v-t_k) - T_\alpha(u-t_k)||^2\Big[\mathbb{E}||I_k(y(t_k^-))||^2_{\mathcal{H}} + \mathbb{E}||g(t_k, y_{t_k} + \bar{z}_{t_k})||^2_{\mathcal{H}} \\ & + \mathbb{E}||g(t_k, y_{t_k} + \bar{z}_{t_k} + I_k(y_{t_k^-} + \bar{z}_{t_k^-}))||^2_{\mathcal{H}}\Big] \\ \leq\ & 3\Big[\Theta(1+6M_g l^2) + 2M_g + 10M_g(||\phi||^2_{\mathcal{B}_h} + l^2 q)\Big]\sum_{k=1}^{i}||T_\alpha(v-t_k) - T_\alpha(u-t_k)||^2.\end{aligned}$$

Since T_α is strongly continuous and it allows us to conclude that $\lim_{u\to v} ||T_\alpha(v-t_k) - T_\alpha(u-t_k)||^2 = 0$ for all $k = 1, 2, \ldots, m$, which implies that the set $\{\Pi_1 y, y \in B_q\}$ is equicontinuous.

Finally, combining Step 1 to Step 4 together with Ascoli's theorem, we conclude that the operator Π_1 is compact.

Step5. Π_2 is contractive. Let $y, y^* \in B_q$ and $t \in (t_i, t_{i+1}], \quad i = 0, 1, \ldots, m.$ Then

$$\begin{aligned}&\mathbb{E}||(\Pi_2 y)(t) - (\Pi_2 y^*)(t)||^2_{\mathcal{H}} \\ \leq\ & 3\mathbb{E}||g(t, y_t + \bar{z}_t) - g(t, y_t^* + \bar{z}_t)||^2_{\mathcal{H}} \\ & + 3\mathbb{E}\left\|\int_0^t S_\alpha(t-s)\Big[f(s, y_s + \bar{z}_s, B_1(y(s) + \bar{z}(s)) - f(s, y_s^* + \bar{z}_s, B_1(y^*(s) + \bar{z}(s)))\Big]ds\right\|^2_{\mathcal{H}} \\ & + 3\mathbb{E}\left\|\int_0^t S_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s, B_2(y(s) + \bar{z}(s))) - \sigma(s, y_s^* + \bar{z}_s, B_2(y^*(s) + \bar{z}(s)))d\omega(s)\right\|^2_{\mathcal{H}}\end{aligned}$$

$$
\begin{aligned}
&\leq\ 3\mathbb{E}||g(t,y_t+\bar{z}_t)-g(t,y^*_t+\bar{z}_t)||^2_{\mathcal{H}}+3\int_0^t||S_\alpha(t-s)||ds\int_0^t||S_\alpha(t-s)||\\
&\quad\times\mathbb{E}||f(s,y_s+\bar{z}_s,B_1(y(t)+\bar{z}(t))-f(s,y^*_s+\bar{z}_s,B_1(y^*(s)+\bar{z}(s))||^2_{\mathcal{H}}ds\\
&\quad+3\int_0^t||S_\alpha(t-s)||^2\mathbb{E}||\sigma(s,y_s+\bar{z}_s,B_2(y(s)+\bar{z}(s))-\sigma(s,y^*_s+\bar{z}_s,B_2(y^*(s)+\bar{z}(s))||^2_{L^2_0}ds\\
&\leq\ 3M_g||y_t-y^*_t||^2_{\mathcal{B}_h}+3\widetilde{M}^2_S\int_0^t(t-s)^{\alpha-1}ds\\
&\quad\times\left[\mu_1(s)||y_s-y^*_s||^2_{\mathcal{B}_h}+\mu_2(s)\mathbb{E}||B_1(y(t)+\bar{z}(t))-B_1(y^*(s)+\bar{z}(s))||^2_{\mathcal{H}}\right]ds\\
&\quad+3\widetilde{M}^2_S\int_0^t(t-s)^{2(\alpha-1)}\left[\nu_1(s)||y_s-y^*(s)||^2_{\mathcal{B}_h}+\nu_2\mathbb{E}||B_2(y(s)+\bar{z}(s))-B_2(y^*(s)+\bar{z}(s))||^2_{\mathcal{H}}\right]ds\\
&\leq\ 3M_g||y_t-y^*_t||^2_{\mathcal{B}_h}+3\widetilde{M}^2_S\frac{T^\alpha}{\alpha}\\
&\quad\times\int_0^t(t-s)^{\alpha-1}\left[\mu^*_1l^2\sup\mathbb{E}||y(s)-y^*(s)||^2_{\mathcal{H}}+\mu^*_2B^*_1\sup\mathbb{E}||y(s)-y^*(s)||_{\mathcal{H}^2}\right]ds\\
&\quad+3\widetilde{M}^2_S\int_0^t(t-s)^{2(\alpha-1)}\left[\nu^*_1l^2\sup||y_t-y^*(t)||^2_{\mathcal{B}_h}+\nu^*_2B^*_1\sup||y_t-y^*(t)||^2_{\mathcal{B}_h}\right]ds\\
&\leq\ 3\left(l^2M_g+\widetilde{M}^2_S[\frac{1}{\alpha^2}(\mu^*_1l^2+\mu^*_2B^*_1)+\frac{1}{T(2\alpha-1)}(\nu^*_1l^2+\nu^*_2B^*_2)]\right)||y-y^*||^2_{\mathcal{B}^0_b}\\
&=\ 3\left(l^2M_g+\widetilde{M}^2_S\left[\frac{v_1}{\alpha^2}+\frac{v_2}{T(2\alpha-1)}\right]\right)||y-y^*||^2_{\mathcal{B}^0_b}.
\end{aligned}
$$

So Π_2 is a contraction by our assumption in (2.19). Hence, by Sadovskii's fixed point theorem we can conclude that the problem (2.1) has at least one solution on $(-\infty,T]$. This completes the proof of the theorem.

$\square$

2.3 An example.

In this section, we consider an example to illustrate our main theorem. We examine the existence of solutions for the following fractional stochastic partial differential equation of the form

$$\begin{aligned}
&D_t^q\left[u(t,x)+\int_{-\infty}^{t}a(t,x,s-t)Q_1(u(s,x))ds\right]=\tfrac{\partial^2}{\partial x^2}[u(t,x)+\int_{-\infty}^{t}a(t,x,s-t)Q_1(u(s,x))ds]\\
&\quad+\int_{-\infty}^{t}H(t,x,s-t)Q_2(u(s,x))ds+\int_0^t k(s,t)e^{-u(s,x)}ds\\
&\quad+\left[\int_{-\infty}^{t}V(t,x,s-t)Q_3(u(s,x))ds+\int_0^t p(s,t)e^{-u(s,x)}ds\right]\tfrac{d\beta(t)}{dt},\\
&x\in[0,\pi],\quad t\in[0,b],t\neq t_k\\
&u(t,0)=0=u(t,\pi),\quad t\geq 0\\
&u(t,x)=\phi(t,x),\quad t\in(-\infty,0],\quad x\in[0,\pi],\\
&\triangle u(t_i)(x)=\int_{-\infty}^{t}q_i(t_i-s)u(s,x)ds,\quad x\in[0,\pi],
\end{aligned}\tag{2.28}$$

where $\beta(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega,\{\mathcal{F}_t\},\mathcal{F},\mathbb{P})$; D_t^q is the Caputo fractional derivative of order $0<q<1; 0<t_1<t_2<\ldots<t_n=T$ are prefixed numbers; a,Q_1,H,Q_2,V,Q_3 are continuous; $\phi\in\mathcal{B}_h$.

Let $\mathcal{H}=L_2([0,\pi])$ with the norm $||\cdot||$. Define $A:\mathcal{H}\to\mathcal{H}$ by $Ay=y''$ with the domain

$$\mathcal{D}(A)=\left\{y\in\mathcal{H}; y,y' \textit{ are absolutely continuous}, y''\in\mathcal{H} \textit{ and } y(0)=y(\pi)=0\right\}.$$

Then, $Ay=\sum_{n=1}^{\infty}n^2(y,y_n)y_n,\quad y\in\mathcal{D}(A)$, where $y_n(x)=\sqrt{\frac{2}{\pi}}sin(nx), n=1,2,\ldots,$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in $\mathcal{H}$ is given by

$$T(t)y=\sum_{n=1}^{\infty}\exp^{-n^2t}(y,y_n)y_n,\quad for\quad all\quad y\in\mathcal{H},t>0.$$

It follows from the above expressions that $(T(t))_{t\geq 0}$ is a uniformly bounded compact semigroup, so that, $R(\lambda,A)=(\lambda I-A)^{-1}$ is a compact operator for all $\lambda\in\rho(A)$.

Let $h(s)=e^{2s}, s<0$, then $l=\int_{-\infty}^{t}h(s)ds=\dfrac{1}{2}$ and define

$$||\phi||_{\mathcal{B}_h}=\int_{-\infty}^{t}h(s)\sup_{s\leq\theta\leq 0}(\mathbb{E}|\phi(\theta)|^2)^{\frac{1}{2}}ds.$$

Hence for $(t,\phi) \in [0,T] \times \mathcal{B}_h$, where $\phi(\theta)(y) = \phi(\theta,y), (\theta,y) \in (-\infty,0] \times [0,\pi]$. Set $u(t)(x) = u(t,x)$,

$$\begin{aligned} g(t,\phi)(x) &= \int_{-\infty}^{0} a(t,x,\theta)Q_1(\phi(\theta)(x))d\theta, \\ f(t,\phi,B_1u(t))(x) &= \int_{-\infty}^{0} H(t,x,\theta)Q_1(\phi(\theta)(x))d\theta + B_1u(t)(x), \\ \sigma(t,\phi,B_2u(t))(x) &= \int_{-\infty}^{0} V(t,x,\theta)Q_3(\phi(\theta)(x))d\theta + B_2u(t)(x), \\ I_i(\phi)(x) &= \int_{-\infty}^{0} q_i(-\theta)\phi(\theta)(x)d\theta, \end{aligned}$$

where $B_1u(t) = \displaystyle\int_0^t k(s,t)e^{-u(s,x)}ds$ and $B_2u(t) = \displaystyle\int_0^t p(s,t)e^{-u(s,x)}ds$. Then with these settings the equations in (2.28) can be written in the abstract form of Eq. (2.1). All conditions of Theorem 2.2.1 are now fulfilled, so we deduce that the system (2.28) has a mild solution on $(-\infty,T]$.

Chapter 3

Almost periodic mild solutions for stochastic delay functional differential equations driven by a FBM

In the present chapter[1], we investigate the existence and stability of quadratic-mean almost periodic mild solutions for stochastic delay functional differential equations

$$\left\{\begin{array}{lll} dx(t) & = & (Ax(t)+b(t,x(t),x_t))dt+\sigma_H(t)dB_Q^H(t), \quad t\in[0,T], \\ x(t) & = & \varphi(t), \qquad -r\leq t\leq 0, \qquad r\geq 0, \end{array}\right. \tag{3.1}$$

where $B_Q^H=\{B_Q^H(t),\quad t\in[0,T]\}$ is a fBm with Hurst index $H\in(\frac{1}{2},1)$. For more detail we refer the reader to [8, 46].

3.1 Preliminaries

In this section we introduce some notations, definitions, a technical lemmas and preliminary fact which are used in what follows.

Let $(\Omega,\mathcal{F},(\mathcal{F}_t,t\in[0,T]),\mathbb{P})$ be a complete probability space with a filtration satisfying the standard conditions. Let $T>0$ and denote by Υ the linear space of $\mathbb{R}$-valued step functions on $[0,T]$, that is, $\phi\in\Upsilon$ if

$$\phi(t)=\sum_{i=1}^{n-1} z_i\chi_{[t_i,t_{i+1})}(t),$$

[1]The chapter is based on the paper [20]

where $t \in [0,T], z_i \in \mathbb{R}$ and $0 = t_1 < t_2 < . < t_n = T$. For $\phi \in \Upsilon$ its Wiener integral with respect to B^H is

$$\int_0^T \phi(s)dB^H(s) = \sum_{i=1}^{n-1} z_i(B^H(t_{i+1}) - B^H(t_i)).$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of Υ with respect to the scalar product $\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s)$. Then the mapping

$$\phi = \sum_{i=1}^{n-1} z_i \chi_{[t_i,t_{i+1})}(t) \mapsto \int_0^T \phi(s)dB^H(s)$$

is an isometry between Υ and the linear space $span\{B^H(t), t \in [0,T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\overline{span}^{L^2(\Omega)}\{B^H(t), t \in [0,T]\}$, ($see$[34]). The image of an element $\phi \in \mathcal{H}$ by this isometry is called the Wiener integral of ϕ with respect to B^H.

Let us now consider the Kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du$$

Where $c_H = (\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})})^{\frac{1}{2}}$, where β denoting the Beta function, and $t > s$. It is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t,s) = H(\frac{t}{s})^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}.$$

Let $\mathcal{K}_H : \Upsilon \mapsto L^2([0,T])$ be the linear operator given by

$$\mathcal{K}_H \phi(s)(s) = \int_s^t \phi(t) \frac{\partial K_H}{\partial t}(t,s)dt.$$

Then $(\mathcal{K}_H \chi_{[0,t]})(s) = K_H(t,s)\chi_{[0,t]}(s)$ is an isometry between Υ and $L^2([0,T])$ that can be extended to $\mathcal{H}$. Denoting $L^2_{\mathcal{H}}([0,T]) = \{\phi \in \mathcal{H}, \mathcal{K}_H \phi \in L^2([0,T])\}$. since $H > 1/2$, we have

$$L^{1/H}([0,T]) \subset L^2_{\mathcal{H}}([0,T]). \tag{3.2}$$

Moreover the following result hold:

Lemma 3.1.1 [34]. *For $\phi \in L^{1/H}([0,T])$,*

$$H(2H-1)\int_0^T\int_0^T |\phi(r)||\phi(u)||r-u|^{2H-2}drdu \leq c_H||\phi||^2_{L^{1/H}([0,T])}.$$

Let us now consider two separable Hilbert spaces $(U, |\cdot|_U, \langle\cdot,\cdot\rangle_U)$ and $(V, |\cdot|_V, \langle\cdot,\cdot\rangle_V)$, Let $L(V,U)$ denote the space of all bounded linear operator from V to U and $Q \in L(V,V)$ be a non-negative self adjoint operator. Denote by $L^0_Q(V,U)$ the space of all $\xi \in L(V,U)$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. the norm is given by

$$|\xi|^2_{L^0_Q(V,U)} = |\xi Q^{\frac{1}{2}}|^2_{HS} = tr(\xi Q \xi^*).$$

Then ξ is called a Q-Hilbert-Schmidt operator from V to U.

Let $\{B^H_n(t)\}_{n\in\mathbb{N}}$ be a sequence of two-side one-dimensional fBm mutually independent on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal basis in V.
Define the V-valued stochastic process $B^H_Q(t)$ by

$$B^H_Q(t) = \sum_{n=1}^{\infty} B^H_n(t) Q^{\frac{1}{2}} e_n, t \geq 0.$$

If Q is a non-negative self-adjoint trace class operator, then this series converges in the space V, that is, it holds that $B^H_Q(t) \in L^2(\Omega, V)$. Then, we say that $B^H_Q(t)$ is a V-valued Q-cylindrical fBm with covariance operator Q. Let $\psi : [0,T] \to L^0_Q(V,U)$ such that

$$\sum_{n=1}^{\infty} \|\mathcal{K}_H(\psi Q^{\frac{1}{2}})e_n\|_{L^2([0,T],U)} < \infty. \tag{3.3}$$

Definition 3.1.1 *Let $\psi : [0,T] \to L^0_Q(V,U)$ satisfy (3.3). Then, its stochastic integral with respect to the fBm B^H_Q is defined for $t \geq 0$ as*

$$\int_0^t \psi(s) dB^H_Q(s) := \sum_{n=1}^{\infty} \int_0^t \psi(s) Q^{1/2} e_n dB^H_n(s) = \sum_{n=1}^{\infty} \int_0^t (\mathcal{K}_H(\psi Q^{1/2} e_n))(s) dW(s),$$

where W is a Wiener process.

Notice that if

$$\sum_{n=1}^{\infty} \|\psi Q^{\frac{1}{2}} e_n\|_{L^{1/H}([0,T],U)} < \infty, \tag{3.4}$$

then in particular (3.4) holds, which follows immediately from (3.3).

The following lemma is proved in [34] and obtained as a simple application of Lemma 3.1.1.

Lemma 3.1.2 *([34]). For any $\psi : [0,T] \to L_Q^0(V,U)$ such that (3.4) holds, and for any $\alpha, \beta \in [0,T]$ with $\alpha > \beta$,*

$$\mathbb{E}\left|\int_\beta^\alpha \psi(s) dB_Q^H(s)\right|_U^2 \leq cH(2H-1)(\alpha-\beta)^{(2H-1)} \sum_{n=1}^{\infty} \left|\int_\beta^\alpha \psi Q^{\frac{1}{2}} e_n\right|_U^2 ds,$$

where $c = c(H)$. If in addition

$$\sum_{n=1}^{\infty} \left|\psi Q^{\frac{1}{2}} e_n\right|_U \quad is \quad uniformly \quad convergent \quad for \quad t \in [0,T], \tag{3.5}$$

then

$$\mathbb{E}\left|\int_\beta^\alpha \psi(s) dB_Q^H(s)\right|_U^2 \leq cH(2H-1)(\alpha-\beta)^{(2H-1)} \int_\beta^\alpha t|\psi(s)|^2_{L_Q^0(V,U)} ds. \tag{3.6}$$

For more detail (see [8]). Now, we recall the following

Definition 3.1.2 *1. A stochastic process $X : [0,T] \to L^2(\Omega, U)$ is said to be continuous, provided that, for any $s \in [0,T]$, $\lim_{t\to s} \mathbb{E}|X(t) - X(s)|_U^2 = 0$.*

2. A stochastic process $X : [0,T] \to L^2(\Omega, U)$ is said to be stochastically bounded, whenever $\lim_{N\to\infty} \mathbb{P}[|X(t)|_U > N] = 0$.

Let us consider the Banach space $C([0,T]; L^2(\Omega, U)) = C([0,T]; L^2(\Omega, \mathcal{F}, \mathbb{P}, U))$ of all continuous and uniformly bounded processes from $[0,T]$ in to $L^2(\Omega, U)$ equipped with the sup norm.

Definition 3.1.3 *A continuous stochastic process $X : [0,T] \to L^2(\Omega, U)$ is said to be quadratic-mean almost periodic, provided that, for each $\epsilon > 0$, the set*

$$J(X,\epsilon) := \left\{ k : \sup_{t\in[0,T]} \mathbb{E}|X(t+k) - X(t)|_U^2 < \epsilon \right\}$$

is relatively dense in $\mathbb{R}$*, i.e., there exists a constant* $c = c(\epsilon) > 0$ *such that* $J(X, \epsilon) \cap [t, t+c] \neq \varnothing$*, for any* $t \in [0, T]$.

Denote the set of all quadratic-mean almost periodic stochastic processes by $\widehat{C}([0,T], L^2(\Omega, U))$. Notice that this set is a closed subspace of $C([0,T]; L^2(\Omega, U))$. therefore, $\widehat{C}([0,T], L^2(\Omega, U))$ equipped with the sup norm is a Banach space.

Definition 3.1.4 *A function* $b(t, Y) : [0,T] \times L^2(\Omega, U) \to L^2(\Omega, V)$*, which is jointly continuous, is said to be quadratic-mean almost periodic in* $t \in [0,T]$*, uniformly for* $Y \in \mathbb{K}$*, where* $\mathbb{K} \subset L^2(\Omega, U)$ *is compact; if for any* $\epsilon > 0$*, there exists a constant* $c(\epsilon, \mathbb{K}) > 0$ *such that any interval of length* $c(\epsilon, \mathbb{K})$ *contains at least a number* k *satisfying*

$$\sup_{t \in [0,T]} \left(\mathbb{E} |b(t+k, Y) - b(t, Y)|_V^2 \right) < \epsilon,$$

for each stochastic process $Y : [0,T] \to \mathbb{K}$.
The collection of such functions will be denoted by $\widehat{C}([0,T] \times L^2(\Omega, U)), L^2(\Omega, V))$.

The following lemma is also proved in [8].

Lemma 3.1.3 *Let* $\widetilde{C}([-r, 0]; L^2(\Omega, U))$ *be the space of all continuous functions from* $[-r, 0]$ *into* $L^2(\Omega, U)$ *with the sup norm*

$$||Z||_{\widetilde{C}([-r,T]; L^2(\Omega,U))} = sup\{|Z(s)|_U; Z \in \widetilde{C}, -r \leq s \leq 0\},$$

$\mathbb{K} \subset L^2(\Omega, U) \times \widetilde{C}([-r, 0]; L^2(\Omega, U))$ *be a compact set. Assume that the function* $b(t, x, y) : [0,T] \times L^2(\Omega, U) \times \widetilde{C}([-r, 0]; L^2(\Omega, U)) \to L^2(\Omega, V))$ *be quadratic-mean almost periodic in* $t \in [0,T]$*, uniformly for* $(x, y) \in \mathbb{K}$*; furthermore, there exists a constant* $c_1 > 0$ *such that*

$$|b(t, x, y) - b(t, \tilde{x}, \tilde{y})|_V^2 \leq c_1 \left(|x - \tilde{x}|_U^2 + ||y - \tilde{y}||_{\widetilde{C}^2([-r,0]; L^2(\Omega,U))} \right),$$

for $t \in [0,T]$ *and* $(x, y), (\tilde{x}, \tilde{y}) \in L^2(\Omega, U) \times \widetilde{C}([-r, 0]; L^2(\Omega, U))$*, then for any quadratic-mean almost periodic stochastic process* $\psi : [0,T] \to L^2(\Omega, U)$*, the stochastic process* $t \to b(t, \psi(t), \psi_t)$ *is quadratic-mean almost periodic.*

3.2 Almost Periodic Mild Solutions

In this section we study the existence of quadratic-mean almost periodic mild solutions for stochastic delay functional differential equations

$$\begin{aligned} dx(t) &= (Ax(t) + b(t, x(t), x_t))dt + \sigma_H(t)dB_Q^H(t), \quad t \in [0, T], \\ x(t) &= \varphi(t), \quad -r \leq t \leq 0, \quad r \geq 0, \end{aligned} \tag{3.7}$$

where $B_Q^H(t)$ is the fractional Brownian motion which was introduced in the previous section, the initial data $\varphi \in \widetilde{C}([-r, 0]; L^2(\Omega, U))$ is a function defined by $\varphi_t(s) = \varphi(t + s), s \in [-r, 0]$, and $A : Dom(A) \subset U \to U$ is the infinitesimal generator of a strongly continuous semigroup $S(.)$ on U, that is, for $t \geq 0$, it holds $|S(t)|_U \leq Me^{\rho t}, M \geq 1, \rho \in \mathbb{R}$. The coefficients $b : [0, T] \times U \times \widetilde{C}([-r, 0]; U) \to U$ and $\sigma_H : [0, T] \to L_Q^0(U, V)$ are appropriate functions.

Definition 3.2.1 *A U-valued process $x(t)$ is called a mild solution of* (3.7) *if* $x \in \widetilde{C}([-r, T]; L^2(\Omega, U))$, $x(t) = \varphi(t)$ *for* $t \in [-r, 0]$, *and, for* $t \in [0, T]$, *satisfies*

$$x(t) = S(t)\varphi(0) + \int_0^t S(t - s)b(s, x(s), x_s)ds + \int_0^t S(t - s)\sigma_H(s)dB_Q^H(s) \quad \mathbb{P} - a \cdot s. \tag{3.8}$$

Now, we state our first main result. We will make use of the following assumptions on the coefficients.

($\mathbf{H}b$) The function $b \in \widehat{C}([0, T] \times U \times \widetilde{C}, U)$, and there exists a constant $c_b > 0$ such that

$$|b(t, x, y) - b(t, \widetilde{x}, \tilde{y})|_U^2 \leq c_b \left(|x - \tilde{x}|_U^2 + ||y - \widetilde{y}||_{\widetilde{C}}^2, \right),$$

where the space $\widetilde{C}$ is defined in Section 1, $(x, y), (\widetilde{x}, \widetilde{y}) \in U \times \widetilde{C}, t \in [0, T]$.

($\mathbf{H}\sigma_H$) The function $\sigma_H : [0, T] \to L_Q^0(U, V)$ satisfies the following conditions: for the complete orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ in V , we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \|\sigma_H Q^{\frac{1}{2}} e_n\|_{L^2([0,T],U)} < \infty. \\ &\sum_{n=1}^{\infty} |\sigma_H(t, x(t)) Q^{\frac{1}{2}} e_n|_U \quad is \quad uniformly \quad convergent \quad for \quad t \in [0, T]. \end{aligned}$$

Note that assumption ($\mathbf{H}\sigma_H$) immediately imply that, for every $t \in [0, T]$, $\int_0^t |\sigma_H(s)|_{L_Q^0(U,V)}^2 < \infty$.

Theorem 3.2.1 *Under the assumptions on A, the conditions* (***Hb***) *and* ($\boldsymbol{H}\sigma_H$) *, for every* $\varphi \in \widetilde{C}([-r,T];L^2(\Omega,U))$*, Eq.* (3.7) *has a unique quadratic-mean almost periodic mild solution* x *whenever*

$$\gamma = 2Me^{\rho T}\sqrt{Tc_b} < 1,$$

where c_b *is a positive constant.*

Proof. We can assume that $\rho > 0$, otherwise we can take $\rho_0 > 0$ such that, for $t \geq 0$, $|S(t)| \leq Me^{\rho_0 t}$. Define the operator $\mathcal{L}$ on $\widehat{C}([0,T],U)$ by

$$\begin{aligned}(\mathcal{L}x)(t) &:= S(t)\varphi(0) + \int_0^t S(t-s)b(s,x(s),x_s)ds + \int_0^t S(t-s)\sigma_H(s)dB_Q^H(s) \\ &:= S(t)\varphi(0) + \Phi x(t) + \Psi(t). \qquad\qquad \mathbb{P}-a\cdot s.\end{aligned} \tag{3.9}$$

Firstly, it suffices to show that $\Phi x(.)$ is quadratic-mean almost periodic whenever x is quadratic-mean almost periodic.

Indeed, assuming that x is quadratic-mean almost periodic, using condition ($\mathbf{H}b$) and Lemma 3.1.3, one can see that $s \mapsto b(s,x(s),x_s)$ is quadratic-mean almost periodic. Therefore, for each $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that any interval of length $c(\epsilon)$ contains at least κ satisfying

$$\sup_{0\leq t\leq T} \mathbb{E}|b(t+\kappa,x(t+\kappa),x_{t+\kappa}) - b(t,x(t),x_t)|_U^2 \leq \frac{\epsilon}{(TMe^{\rho T})^2}, \tag{3.10}$$

for $T > 0$ fixed. Furthermore

$$\begin{aligned} &\mathbb{E}|\Phi x(t+\kappa) - \Phi x(t)|_U^2 \\ =\ & \mathbb{E}\left|\int_0^t S(t-s)b(s+\kappa,x(s+\kappa),x_{s+\kappa})ds - \int_0^t S(t-s)b(s,x(s),x_s)ds\right|_U^2 \\ \leq\ & t\mathbb{E}\int_0^t |S(t-s)\left(b(s+\kappa,x(s+\kappa),x_{s+\kappa}) - b(s,x(s),x_s)\right)|_U^2\, ds \\ \leq\ & tM^2e^{2\rho T}\mathbb{E}\int_0^t |S(t-s)(b(s+\kappa,x(s+\kappa),x_{s+\kappa}) - b(s,x(s),x_s))|_U^2 ds \\ \leq\ & TM^2e^{2\rho T}\int_0^t \sup_{0\leq\tau\leq s} \mathbb{E}|b(\tau+\kappa,x(\tau+\kappa),x_{\tau+\kappa}) - b(\tau,x(\tau),x_\tau))|_U^2 ds \\ <\ & \epsilon. \end{aligned}$$

Secondly, for the chosen $\upsilon > 0$ small enough, we have

$$\begin{aligned}
& \mathbb{E}|\Psi(t+\upsilon)-\Psi(t)|^2 \\
= \;& \mathbb{E}\left|\int_0^{t+\upsilon} S(t+\upsilon-s)\sigma_H(s)dB_Q^H(s) - \int_0^t S(t-s)\sigma_H(s)dB_Q^H(s)\right|^2 \\
\leq \;& 2\mathbb{E}\left|\int_0^t [S(t+\upsilon-s)-S(t-s)]\sigma_H(s)dB_Q^H(s)\right|^2 + 2\mathbb{E}\left|\int_t^{t+\upsilon} S(t-s)\sigma_H(s)dB_Q^H(s)\right|^2 \\
:= \;& I_1 + I_2.
\end{aligned}$$

Applying inequality (3.5) to I_1 we get

$$\begin{aligned}
I_1 &\leq 2cH(2H-1)t^{2H-1}\int_0^t |S(t-s)(S(\upsilon)-Id)\sigma_H(s)|^2_{L_Q^0(U,V)}\,ds \\
&\leq 2cH(2H-1)t^{2H-1}M^2e^{2\rho T}\int_0^t |(S(\upsilon)-Id)\sigma_H(s)|^2_{L_Q^0(V,U)}\,ds \\
&\leq 2cH(2H-1)t^{2H-1}M^4e^{2\rho T}(1+e^{2\rho\upsilon})\int_0^t |\sigma_H(s)|^2_{L_Q^0(V,U)}\,ds.
\end{aligned}$$

Applying now inequality (3.5) to I_2 we obtain

$$I_2 \leq 2cH(2H-1)\upsilon^{2H-1}M^2e^{2\rho\upsilon}\int_0^{t+\upsilon} |\sigma_H(s)|^2_{L_Q^0(V,U)}\,ds.$$

We observe that the condition $(\mathbf{H}\sigma_H)$ ensures the existence of a positive constants c_1 and c_2 such that

$$2cH(2H-1)t^{2H-1}M^4e^{2\rho T}(1+e^{2\rho\upsilon})\int_0^t |\sigma_H(s)|^2_{L_Q^0(V,U)}\,ds \leq c_1,$$

and

$$2cH(2H-1)\upsilon^{2H-1}M^2e^{2\rho\upsilon}\int_0^{t+\upsilon} |\sigma_H(s)|^2_{L_Q^0(V,U)}\,ds \leq c_2.$$

Therefore, for the chosen $\upsilon > 0$ and all $t \geq 0$ we have

$$\mathbb{E}|\Psi(t+\upsilon)-\Psi(t)|^2 \leq c_1 + c_2 = c_3.$$

From the above discussion, it is clear that the operator $\mathcal{L}$ maps $\widehat{C}([0,T],U)$ into itself.

Finally we claim that $\mathcal{L}$ is a contraction mapping on $\widehat{C}([0,T],U)$. We have

$$\begin{aligned}
\mathbb{E}|(\mathcal{L}x)(t)-(\mathcal{L}y)(t)|^2 &= \mathbb{E}\left|\int_0^t S(t-s)[b(s,x(s),x_s)-b(s,y(s),y_s)]ds\right|^2 \\
&\leq 2M^2e^{2\rho t}\mathbb{E}\int_0^t |b(s,x(s),x_s)-b(s,y(s),y_s)|_U^2 ds \\
&\leq 2M^2e^{2\rho T}\mathbb{E}\int_0^t \sup_{0\leq\tau\leq s} |b(\tau,x(\tau),x_\tau)-b(\tau,y(\tau),y_\tau)|_U^2 ds \\
&\leq 2TM^2e^{2\rho T}c_b \sup_{0\leq\tau\leq s}\left(|x-y|_U^2+||x-y||_{\widetilde{C}}^2\right) \\
&\leq 4TM^2e^{2\rho T}\sup_{0\leq\tau\leq s}||x-y||_\infty^2.
\end{aligned}$$

Hence,

$$||(\mathcal{L}x)(t)-(\mathcal{L}y)(t)||_\infty \leq 2Me^{\rho T}\sqrt{c_b}||x-y||_\infty = \gamma||x-y||_\infty. \tag{3.11}$$

As $\gamma < 1$,by (3.11), we know that $\mathcal{L}$ is a contraction mapping. Hence, by the contraction mapping principle, $\mathcal{L}$ has a unique fixed point x, which obviously is the unique quadratic-mean almost periodic mild solution to Eq. (3.7).

□

Now, we give another main result. We first need to state the following conditions:
($\mathbf{H}'$) The semigroup $\{S(t)\}_{t\geq 0}$ is bounded, i.e., there exists a constant $M_1 > 0$ such that $|S(t)|_U \leq M_1$;
($\mathbf{H}'b$) The function $b \in \widehat{C}([0,T\times U\times \widetilde{C},U)$, and for each natural number n, there exists a function $\eta_n : \mathbb{R}\to\mathbb{R}^+$ such that

$$\sup_{|x|\leq n}\mathbb{E}|b(t,x(t),x_t)|_U^2 \leq \eta_n(t), \quad for \quad (x,x_t)\in U\times\widetilde{C}, t\in[0,T];$$

($\mathbf{H}'\sigma_H$) The function $\sigma_H : [0;T]\to L_Q^0(U,V)$, and there exists a function $\vartheta : \mathbb{R}\to\mathbb{R}^+$ such that

$$|\sigma_H(t)|_{L_Q^0(U,V)}^2 \leq \vartheta(t), \quad for \quad t\in[0,T];$$

($\mathbf{H}''$) $\liminf_{n\to\infty}\frac{1}{n}\left(\int_0^T \eta_n(s)ds + tr(Q)cH(2H-1)T^{2H-1}\int_0^T\vartheta(s)ds\right) = \Omega < \infty.$

Theorem 3.2.2 *Let the conditions ,* ($\boldsymbol{H}'b$)*,* ($\boldsymbol{H}'\sigma_H$) *and* ($\boldsymbol{H}''$) *be satisfied. Then Eq.* (3.7) *has a quadratic-mean almost periodic mild solution whenever* $\Omega M_1^2 < \frac{1}{3}$.

Proof. Let $\mathcal{L}$ be the operator defined by (3.9). First, we use the Schauder fixed point theorem to prove that $\mathcal{L}$ has a fixed point. The proof will be given in several steps.

Step 1. Let $\{x_n\}$ be a sequence such that $x_n \to x$. Using the continuity of $b(t, x(t), x_t)$ with respect to $x(t)$ and x_t, we get $b(t, x_n(t), (x_n)_t) \to b(t, x(t), x_t)$ as $n \to \infty$. For each $0 \le t \le T$ we have

$$\begin{aligned}
\mathbb{E}|(\mathcal{L}x_n)(t) - (\mathcal{L}x)(t)|^2 &= \mathbb{E}\left|\int_0^t S(t-s)[b(s, x_n(s), (x_n)_s) - b(s, x(s), x_s)]ds\right|^2 \\
&\le 2M_1^2 \mathbb{E}\int_0^t |b(s, x_n(s), (x_n)_s) - b(s, x(s), x_s)|_U^2 ds \\
&\le 2M_1^2 \mathbb{E}\int_0^t \sup_{0\le\tau\le s} |b(\tau, x_n(\tau), (x_n)_\tau) - b(\tau, x(\tau), x_\tau)|_U^2 ds,
\end{aligned}$$

which implies that $\mathcal{L}$ is continuous.

Step 2. Let $D_n = \{x \in \widehat{C}([0, T], U); |x| \le n\}$, for each natural number n. We want to show that the operator $\mathcal{L}$ maps bounded sets into bounded sets, i.e. there exists a natural number n^* such that $\mathcal{L}D_{n^*} \subset D_{n^*}$. If it is not true, then for each n, there exist $x_n \in D_n$ and $t_n \in [0, T]$ such that $\mathcal{L}x_n(t_n) > n$. This, together with $(\mathbf{H}')$, $(\mathbf{H}'b)$, $(\mathbf{H}'\sigma_H)$ and $(\mathbf{H}'')$ yields

$$\begin{aligned}
n &< |\mathcal{L}x_n(t_n)|_U^2 \\
&= \mathbb{E}\left|S(t)\varphi(0) + \int_0^{t_n} S(t_n - s)b(s, x_n(s), (x_n)_s)ds + \int_0^{t_n} S(t_n - s)\sigma_H(s)dB_Q^H(s)\right|^2 \\
&\le 3\mathbb{E}\left|S(t)\varphi(0)\right|^2 + 3\mathbb{E}\left|\int_0^{t_n} S(t_n - s)b(s, x_n(s), (x_n)_s)ds\right|^2 \\
&\quad +3\mathbb{E}\left|\int_0^{t_n} S(t_n - s)\sigma_H(s)dB_Q^H(s)\right|^2 \\
&\le 3M_1^2\mathbb{E}\left|\varphi(0)\right|^2 + 3\int_0^T \mathbb{E}\left|S(t-s)b(s, x(s), x_s)\right|^2 ds \\
&\quad +3M_1^2 tr(Q)cH(2H-1)T^{2H-1}\int_0^T |\sigma_H(s)|^2_{L_Q^0(V,U)}ds \\
&\le 3M_1^2\mathbb{E}\left|\varphi(0)\right|^2 + 3M_1^2\int_0^T \eta_n(s)ds + 3M_1^2 tr(Q)cH(2H-1)T^{2H-1}\int_0^T \vartheta(s)ds.
\end{aligned} \tag{3.12}$$

Dividing both sides of (3.12) by n and taking the lower limit as $n \to \infty$, one obtains

$$1 < \liminf_{n\to\infty} \frac{3M_1^2}{n}\int_0^T \eta_n(s)ds + \frac{3M_1^2 tr(Q)cH(2H-1)T^{2H-1}}{n}\int_0^T \vartheta(s)ds.$$

This is a contradiction to the assumption $\Omega M_1^2 < \frac{1}{3}$. Then $\mathcal{L}D_{n^*} \subset D_{n^*}$.

Step 3. Let D_n^* be a bounded set as in Step 2, and $x \in D_{n^*}$. Then for $t_1 < t_2$ we have

$$
\begin{aligned}
& \mathbb{E}|(\mathcal{L}x)(t_2)-(\mathcal{L}x)(t_1)|_U^2 \\
\leq\ & 3\mathbb{E}\left|[S(t_2)-S(t_1)]\varphi(0)\right|^2+3\mathbb{E}\left|\int_0^{t_2} S(t_2-s)b(s,x(s),x_s)ds-\int_0^{t_1} S(t_1-s)b(s,x(s),x_s)ds\right|^2 \\
& +3\mathbb{E}\left|\int_0^{t_2} S(t_2-s)\sigma_H(s)dB_Q^H(s)-\int_0^{t_1} S(t_1-s)\sigma_H(s)dB_Q^H(s)\right|^2 \\
\leq\ & 3\mathbb{E}\left|[S(t_2)-S(t_1)]\varphi(0)\right|^2 \\
& +3\mathbb{E}\left|\int_0^{t_2} S(s)b(t_2-s,x(t_2-s),x_{t_2-s})ds-\int_0^{t_1} S(s)b(t_1-s,x(t_1-s),x_{t_1-s})ds\right|^2 \\
& +3\mathbb{E}\left|\int_0^{t_2} S(s)\sigma_H(t_2-s)dB_Q^H(s)-\int_0^{t_1} S(s)\sigma_H(t_1-s)dB_Q^H(s)\right|^2 \\
\leq\ & \mathbb{E}\left|[S(t_2)-(t_1)]\varphi(0)\right|^2+6\mathbb{E}\left|\int_{t_1}^{t_2} S(t)b(t_2-s,x(t_2-s),x_{t_2-s})ds\right|^2 \\
& +6\mathbb{E}\left|\int_0^{t_1} S(s)[b(t_2-s,x(t_2-s),x_{t_2-s})-b(t_2-s,x(t_2-s),x_{t_2-s})]ds\right|^2 \\
& +6\mathbb{E}\left|\int_0^{t_1} S(s)[\sigma_H(t_2-s)-\sigma_H(t_1-s)]dB_Q^H(s)\right|^2+6\mathbb{E}\left|\int_{t_1}^{t_2} S(s)\sigma_H(t_2-s)dB_Q^H(s)\right|^2.
\end{aligned}
$$

Applying (3.6) of Lemma 3.1.2, the assumptions $(\mathbf{H}'b)$ and $(\mathbf{H}'\sigma_H)$, we obtain

$$
\begin{aligned}
& \mathbb{E}|(\mathcal{L}x)(t_2)-(\mathcal{L}x)(t_1)|_U^2 \\
\leq\ & 3\mathbb{E}\left|[S(t_2)-S(t_1)]\varphi(0)\right|^2+6M_1^2\int_{t_1}^{t_2}\mathbb{E}\left|b(t_2-s,x(t_2-s),x_{t_2-s})\right|^2 ds \\
& +6M_1^2\int_0^{t_1}\mathbb{E}\left|b(t_2-s,x(t_2-s),x_{t_2-s})-b(t_1-s,x(t_1-s),x_{t_1-s})\right|^2 ds \\
& +6M_1^2 tr(Q)cH(2H-1)T^{2H-1}\int_0^{t_1}|\sigma_H(t_2-s)-\sigma_H(t_1-s)|^2_{L_Q^0(U,V)}ds \\
& +6M_1^2 tr(Q)cH(2H-1)T^{2H-1}\int_{t_1}^{t_2}|\sigma_H(t_2-s)|^2_{L_Q^0(U,V)}ds \\
\leq\ & 3\mathbb{E}\left|[S(t_2)-S(t_1)]\varphi(0)\right|^2+6M_1^2\int_{t_1}^{t_2}\eta_{t_2-n}(s)ds \\
& +6M_1^2 tr(Q)cH(2H-1)T^{2H-1}\int_{t_1}^{t_2}\vartheta(t_2-s)ds \\
& +6M_1^2\int_0^{t_1}\mathbb{E}\left|b(t_2-s,x(t_2-s),x_{t_2-s})-b(t_1-s,x(t_1-s),x_{t_1-s})\right|^2 ds \\
& +6M_1^2 tr(Q)cH(2H-1)T^{2H-1}\int_0^{t_1}|\sigma_H(t_2-s)-\sigma_H(t_1-s)|^2_{L_Q^0(U,V)}ds.
\end{aligned}
$$

Thus $\mathcal{L}$ is equicontinuous.

It remains to prove that $\Theta(t)=\{\mathcal{L}x(t); x\in D_{n^*}\}$ is relatively compact in U. $S(t)$ is compact in U, since it is generated by the dense operator A. Then $\Theta(0)=S(0)x_0$ is relatively compact in U.

Now, for t fixed and for each $\epsilon \in (0,t), x \in D_{n^*}$ we define $L_\epsilon x(t)$ as follow

$$L_\epsilon x(t) = S(t)\varphi(0) + \int_0^{t-\epsilon} S(t-s)b(s,x(s),x_s)ds + \int_0^{t-\epsilon} S(t-s)\sigma_H(s)dB_Q^H(s). \quad (3.13)$$

Then the sets $\Theta_\epsilon(t) = \{L_\epsilon x(t); x \in D_{n^*}\}$ are relatively compact in U . Moreover, for each $x \in D_{n^*}$, one has

$$|\mathcal{L}x(t) - L_\epsilon x(t)|_U^2 \leq 2M_1^2 \left(\int_{t-\epsilon}^t \eta_n(s)ds + tr(Q)cH(2H-1)T^{2H-1} \int_{t-\epsilon}^t \vartheta(s)ds \right), \quad (3.14)$$

from which, by combining the condition $(\mathbf{H}'')$, follows that there are relatively compact sets arbitrarily close to $\Theta(t)$ and hence $\Theta(t)$ is also relatively compact in U. Thus, the Arzela-Ascoli theorem implies that LD_n^* is relatively compact, and $\mathcal{L}$ is completely continuous on D_{n^*}.
As a consequence of Steps1-3 together with the Schauder fixed point theorem, we deduce that $\mathcal{L}$ has a fixed point in D_{n^*} which is a quadratic-mean almost periodic mild solution to Eq. (3.7).

□

Now, we give the third main result. In this sequence, we require the following assumptions.
$(\mathbf{H}''b)$ The function $b \in \widehat{C}([0,T] \times U \times \widetilde{C}, U)$, and there exists a function $\eta : \mathbb{R} \to \mathbb{R}^+$ such that

$$\sup \mathbb{E}|b(t,x(t),x_t)|_U^2 \leq \eta(t), \quad for \quad (x,x_t) \in U \times \widetilde{C}, t \in [0,T];$$

$(\mathbf{H}''')$ The integral $\int_0^t \eta(t-s)ds + tr(Q)cH(2H-1)T^{2H-1} \int_0^t \vartheta(t-s)ds$ exists for all $t \in [0,T]$.

Theorem 3.2.3 *Let the conditions* $(\boldsymbol{H}''b)$, $(\boldsymbol{H}'\sigma_H)$ *and* $(\boldsymbol{H}''')$ *be satisfied. Then Eq.* (3.7) *has a quadratic-mean almost periodic mild solution.*

Proof. We shall also apply the Schauder fixed point theorem to prove this theorem. The proof of Step 1 in this theorem is the same as the proof of Step 1 in Theorem 3.2.2 and so is omitted. Now, we start our proof from Step 2.

Step 2. Let $D = \{x \in \widehat{C}([0,T],U); |x| \leq k\}$, where $k = 3M_2^1(\mathbb{E}|\varphi(0)|^2 + M_2)$ and M_2 is the integral defined in $(\mathbf{H}''')$. We have

$$\begin{aligned}
|(\mathcal{L}x)(t)|_U^2 &= \mathbb{E}\left|S(t)\varphi(0) + \int_0^t S(s)b(t-s,x(t-s),x_{t-s})ds + \int_0^t S(s)\sigma_H(t-s)dB_Q^H(s)\right|^2 \\
&\leq 3M_1^2\mathbb{E}|\varphi(0)|^2 + 3\int_0^t \mathbb{E}|S(s)b(t-s,x(t-s),x_{t-s})|^2 ds \\
&\quad +3M_1^2 tr(Q)cH(2H-1)T^{2H-1}\int_0^t |\sigma_H(t-s)|^2_{L_Q^0(U,V)} ds \\
&\leq 3M_1^2\left(\mathbb{E}|\varphi(0)|^2 + \int_0^t \eta(t-s)ds + tr(Q)cH(2H-1)T^{2H-1}\int_0^t \vartheta(t-s)ds\right) = k.
\end{aligned}$$

Therefore, $\mathcal{L} : D \to D$.

Step 3. Let D be a bounded set as in Step 2, $t_1 < t_2$ and $x \in D$. We have

$$\begin{aligned}
& \mathbb{E}|(\mathcal{L}x)(t_2) - (\mathcal{L}x)(t_1)|^2 \\
\leq\ & 3\mathbb{E}\,|S(t_2) - S(t_1)\varphi(0)|^2 + 6M_1^2 \int_{t_1}^{t_2} \eta(t_2 - s)ds + 6M_1^2 tr(Q)cH(2H-1)T^{2H-1} \\
& \times \int_{t_1}^{t_2} \vartheta(t-s)ds + 6M_1^2 \int_0^{t_1} \mathbb{E}\,|b(t_2 - s, x(t_2 - s), x_{t_2 - s}) - b(t_1 - s, x(t_1 - s), x_{t_1 - s})|^2\, ds \\
& + 6M_1^2 cH(2H-1)T^{2H-1} tr(Q) \int_0^{t_1} |\sigma_H(t_2 - s) - \sigma_H(t_1 - s)|^2_{L_Q^0(U,V)} ds.
\end{aligned}$$

Thus, $\mathcal{L}$ is equicontinuous.

Set $\Theta(t) = \{\mathcal{L}x(t) : x \in D$. Fix t, for each $\epsilon \in (0,t)$ and $x \in D$. Let $\mathcal{L}_\epsilon$ be the operator defined by (3.13); then the sets $\Theta_\epsilon(t) = \{\mathcal{L}_\epsilon x(t) : x \in D\}$ are relatively compact in U. Meanwhile, (3.15) implies that $\mathcal{L}_\epsilon$ arbitrarily close to $\Theta(t)$ and $\Theta(t)$ is also relatively compact in U. Thus, the ArzelaAscoli theorem implies that $\mathcal{L}D$ is relatively compact, $\mathcal{L}$ is completely continuous on D.

Finally, we can conclude from Step 1-2 that $\mathcal{L}D \to D$ is continuous and completely continuous. Thus,$\mathcal{L}$ has a fixed point in D by using the Schauder fixed point theorem. So, it follow that Eq. (3.7) has at least a quadratic-mean almost periodic mild solution.

□

3.3 Stability

As in this section we first assume that the operator A is a closed linear operator generating a strongly continuous exponentially stable semigroup $S(.)$ on U, that is, for $t \geq 0$, it holds $|S(t)|_U \leq Me^{-\lambda t}, M, \lambda > 0$. We also assume in addition to assumption $(\mathbf{H}\sigma_H)$ that $\int_0^\infty e^{\lambda s}|\sigma_H(s)|^2_{L_Q^0(U,V)} ds < \infty$. Our first result on the stability of the quadratic-mean almost periodic mild solution is the following theorem.

Theorem 3.3.1 *Under the assumptions on A, the conditions ($\boldsymbol{H}b$) and ($\boldsymbol{H}\sigma_H$), the quadratic-mean almost periodic mild solution $x(t)$ to Eq.* (3.7) *is globally exponentially stable.*

Proof. Using the assumptions, one can choose a positive constant η such that $0 < \eta < \lambda$. One has

$$\begin{aligned} e^{\eta t}\mathbb{E}|x(t)|^2 &\leq 3e^{\eta t}\mathbb{E}|S(t)\varphi(0)|^2 + 3e^{\eta t}\mathbb{E}\left|\int_0^t S(t-s)b(s,x(s),x_s)ds\right|^2 \\ &\quad + e^{\eta t}\mathbb{E}\left|\int_{-\infty}^t S(t-s)\sigma_H(s)dB_Q^H(s)\right|^2 \\ &= 3e^{\eta t}\mathbb{E}|S(t)\varphi(0)|^2 + I_1 + I_2. \end{aligned} \tag{3.15}$$

Estimating the terms on the right-hand side of (3.15) yields

$$3e^{\eta t}\mathbb{E}|S(t)\varphi(0)|^2 \leq 3e^{(\eta-\rho)t}M^2\mathbb{E}|\varphi(0)|^2 \to 0 \quad as \quad t \to \infty, \tag{3.16}$$

and

$$I_1 \leq 3e^{\eta t}M^2 c_b \int_0^t e^{-\lambda(t-s)}ds \int_0^t e^{-\lambda(t-s)}\left(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\right)ds.$$

For the chosen parameter θ, and any $x(t) \in U$ we have

$$\begin{aligned} I_1 &\leq \tfrac{3}{\lambda}M^2 c_b e^{\eta t}\int_0^t e^{-\lambda(t-s)}\left(|x(s)|_U^2 + \|x(s)\|_{\widetilde{C}}^2\right)ds \\ &= \tfrac{3}{\lambda}M^2 c_b e^{-\theta t}\int_0^t e^{\theta s}e^{\eta s}\left(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\right)ds. \end{aligned}$$

Now, for any $\epsilon > 0$, there exists a constant $\upsilon > 0$ such that $e^{\eta s}(|x(s-r)|_U^2 < \epsilon$, for $s \geq \upsilon$, and we have

$$\begin{aligned} I_1 &\leq \tfrac{3}{\lambda}M^2 c_b e^{-\theta t}\int_\upsilon^t e^{\theta s}e^{\eta s}\left(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\right)ds \\ &\quad + \tfrac{3}{\lambda}c_b e^{-\theta t}\int_0^\upsilon e^{\theta s}e^{\eta s}\left(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\right)ds \\ &\leq \tfrac{6M^2 c_b\epsilon}{\lambda\theta} + \tfrac{3}{\lambda}M^2 c_b e^{-\theta t}\textstyle\int_0^\upsilon e^{\theta s}e^{\eta s}(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2)ds. \end{aligned} \tag{3.17}$$

Using the fact that $e^{-\theta t} \to 0$ as $t \to \infty$, it follows that there exists a constant $u \geq \upsilon$ such that for any $t \geq u$,

$$\frac{3}{\lambda}M^2 c_b e^{-\theta t}\int_{-\infty}^{\vartheta} e^{\theta s}e^{\eta s}\left(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\right)ds < \epsilon - \frac{6M^2 c_b\epsilon}{\lambda\theta}. \tag{3.18}$$

Thus, from (3.17) and (3.18), we get for any $t \geq u$,

$$I_1 = 4e^{\eta t}\mathbb{E}\left|\int_0^t S(t-s)b(s,x(s),x_s)ds\right|^2 < \epsilon,$$

which implies

$$I_1 = 4e^{\eta t}\mathbb{E}\left|\int_0^t S(t-s)b(s,x(s),x_s)ds\right|^2 \to 0 \quad as \quad t\to\infty. \tag{3.19}$$

Estimating I_2, for any $x(t)\in U$, $t\geq -r$, we have

$$\begin{aligned} I_2 &\leq 3cH(2H-1)M^2t^{2H-1}e^{\eta t}\int_0^t e^{-2\lambda(t-s)}|\sigma_H(s)|^2_{L_Q^0(U,V)}ds \\ &\leq 3cH(2H-1)M^2t^{2H}\int_0^t e^{\lambda s}|\sigma_H(s)|^2_{L_Q^0(U,V)}ds, \end{aligned}$$

and the additional assumption to $(\mathbf{H}\sigma_H)$ ensures the existence of a positive constant ϵ such that

$$3cH(2H-1)M^2T^{2H}\int_0^t e^{\lambda s}|\sigma_H(s)|^2_{L_Q^0(U,V)}ds \leq \epsilon \quad for \quad all \quad t\geq -r. \tag{3.20}$$

Thus, from (3.15), (3.19) and (3.20), we obtain $e^{\eta t}|x(s)|^2_{L_Q^0(U,V)} \to 0$ as $t\to\infty$. The quadratic-mean almost periodic mild solution of (3.7) is exponentially stable.

□

Now we study the uniform stability of the quadratic-mean almost periodic mild solution to Eq. (3.7). We first require the following assumption:
$(\mathbf{H}'''b)$ The function $b\in\widehat{C}([0,T]\times U\times\widetilde{C},U)$, and there exists a function $c_b:\mathbb{R}\to\mathbb{R}^+$ such that

$$|b(t,x,y)|^2_U \leq c_b(t)\left(|x|^2_U + ||y||^2_{\widetilde{C}}\right),$$

where $(x,y)\in U\times\widetilde{C}, t\in[0,T]$.

Theorem 3.3.2 *Under the assumptions* $(\boldsymbol{H}')$, $(\boldsymbol{H}\sigma_H)$ *and* $(\boldsymbol{H}'''b)$, *the quadratic-mean almost periodic mild solution to Eq.* (3.7) *is uniformly stable whenever* $M_1^2 I\leq\frac{1}{6}$ *where* $I=\int_0^t c_b(s)ds$.

Proof. Let $x(t)$ be a solution of

$$x(t) = S(t)\varphi(0) + \int_0^t S(t-s)b(s,x(s),x_s))ds + \int_0^t S(t-s)\sigma_H(s)dB_Q^H(s), \tag{3.21}$$

such that $x(0) = x_0$, where $x_0 \in U$. Then

$$\begin{aligned} |x(t)|_U^2 &\leq 3\mathbb{E}\,|S(t)\varphi(0)|^2 + 3\mathbb{E}\left|\int_0^t S(t-s)b(s,x(s),x_s)ds\right|^2 \\ &\quad +3\mathbb{E}\left|\int_0^t S(t-s)\sigma_H(s)dB_Q^H(s)\right|^2 \\ &\leq 3M_1^2\mathbb{E}\,|\varphi(0)|^2 + 3M_1^2\int_0^t c_b(s)\left(|x(s)|_U^2 + ||x_s||_{\widetilde{C}}^2\right)ds \\ &\quad +3M_1^2cH(2H-1)T^{2H-1}tr(Q)\int_0^t |\sigma_H(s)|^2_{L_Q^0(U,V)}ds. \end{aligned}$$

Using the assumption $(\mathbf{H}\sigma_H)$ we obtain

$$\begin{aligned} |x(t)|_U^2 &\leq 3M_1^2||\varphi(0)||_\infty^2 + 6M_1^2\left(\int_0^t c_b(s)ds\right)||x||_\infty^2 + c_3 \\ &= 3M_1^2||\varphi(0)||_\infty^2 + 6M_1^2I||x||_\infty^2 + c_3, \end{aligned}$$

c_3 is a positive positive constant.
Thus

$$||x(t)||_\infty^2 \leq 3M_1^2||\varphi(0)||_\infty^2 + 6M_1^2I||x||_\infty^2 + c_3,$$

$6M_1^2I \leq 1$ yields

$$||x(t)||_\infty^2 \leq \frac{1}{1-6M_1^2I}\left(c_3 + 3M_1^2||\varphi(0)||_\infty^2\right).$$

Therefore, if $||\varphi(0)||_\infty^2 < \lambda(\epsilon)$, then $||x||_\infty^2 < \epsilon$, which implies that the quadratic-mean almost periodic mild solution to Eq. (3.7) is uniformly stable.

□

3.4 Example

Consider the following stochastic evolution equation:

$$\begin{cases} d\xi(t,x) &= \left[\frac{\partial^2}{\partial x^2}\xi(t,x) + \delta[\xi(t,x)(\sin(t)+\sin(\sqrt{2}t))]\right]dt + \sigma_H(t)dB_Q^H(t), \quad t\in[0,t], x\in[0,\pi] \\ \xi(t,0) &= \xi(t,\pi) = 0, \\ \xi(t,x) &= \varphi(t,x) = 0, \quad t\in[-r,0], \end{cases} \tag{3.22}$$

where $r \in (0,1), \varphi(\cdot,x) \in \widetilde{C}([-r,0],\mathbb{R})$ and $B_Q^H(t)$ is a Q-cylindrical fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$ satisfying $tr(Q) = 1$. Denote $U = L^2(\mathbb{P}; L^2[0,\pi])$, and define

$A : D(A) \subset U \to U$ given by $A = \frac{\partial^2}{\partial x^2}$ with $D(A) = \{\xi(.) \in U : \xi'' \in U, \xi' \in U$ is absolutely continuous on $[0, \pi], \xi(0) = \xi(\pi) = 0\}$.

It is well known that a strongly continuous semigroup S, generated by the operator A, satisfies $|S(t)| \leq e^{-t}$, for $t \geq 0$. Taking $b(t, \varphi, \varphi_t)(\theta) = \delta[\varphi(\theta)(\sin(t) + \sin(\sqrt{2}t))]$, and σ_H satisfies assumption $(\mathbf{H}\sigma_H)$. Thus one has

$$|b(t, x, x_t) - b(t, y, y_t)|_U^2 \leq 4\delta^2 |x - y|_U^2.$$

Therefore, Eq. 3.20 has a quadratic-mean almost periodic mild solution, provided that, $\delta \leq \frac{\sqrt{3}}{6}$ according to Theorem 3.2.2.

Let $\eta_n(t) = \delta_n(t) = \delta^2(\sin(t) + \sin(\sqrt{2}tt))^2$ for $n \in \mathbb{N}$, Eq. 3.20 has a quadratic-mean almos periodic mild solution according to Theorem 3.2.2.

Let $\eta(t) = \delta_n(t) = \delta^2(\sin(t) + \sin(\sqrt{2}tt))^2$, Eq. 3.22 has a quadratic-mean almost periodic mild solution according to Theorem 3.2.3.
The quadratic-mean almost periodic mild solution to Eq. 3.22 is exponentially stable according to Theorem 3.3.1.
The quadratic-mean almost periodic mild solution to Eq. 3.22 is uniformly stable, provided that, $\delta < \frac{\sqrt{3}}{6}$according to Theorem 3.3.2.

Conclusion

In this thesis, We have studied the existence of mild solutions for a class of impulsive fractional stochastic differential equations in Hilbert spaces, which is new and allow us to develop the existence of various fractional differential equations and stochastic fractional differential equations. An example is provided to illustrate the applicability of the new result. The results presented in this chapter extend and improve the corresponding ones announced by Dabas et al [15], Dabas and Chauhan [15], Shu et al [40], Sakthivel et al [42] and others.

At the same time, We conquer the difficulty of existence of impulsive, delay and stochastic factors in a dynamic system, and give a result for the existence and uniqueness of mean almost periodic solutions. Moreover, our results have important applications in almost periodic stochastic delayed networks with impulsive stability.

Bibliography

[1] N. Abada, R. P. Agarwal, M. Benchohra, and H. Hammouche, Existence results for non-densely defined impulsive semilinear functional differential equations with state-dependent delay, *Asian-European Journal of Mathematics*, vol. 1, no. 4, pp. 449-468, 2008.

[2] N. Abada, M. Benchohra, and H. Hammouche, Nonlinear impulsive partial functional differential inclusions with statedependent delay and multivalued jumps, *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 4, pp. 791-803, 2010

[3] R. P. Agarwal, M. Belmekki, and M. Benchohra, A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative, *Advances in Difference Equations*, vol. 2009, Article ID 918728, 47 pages, 2009.

[4] A. Anguraj, A. Vinodkumar, Existence, uniqueness and stability results of impulsive stochastic semilinear neutral functional differential equations with infinite delays, *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 67, pp. 1-13, 2009.

[5] M. Benchohra, J. Henderson, S.K. Ntouyas, Existence results for impulsive semilinear neutral functional differential equations in Banach spaces. *Memoirs on Diff. Equ. Math. Phys*, 25 (2002), 105-120.

[6] I. Benedetti, An existence result for impulsive functional differential inclusions in Banach spaces, *Discussiones Mathematicae Differential Inclusions, Control and Optimization*, vol. 24, pp. 13-30, 2004.

[7] N. Bermanand, W. L. Root. *A weak stochastic integral in Banach space with application to a linear stochastic differential equation.* Appl. Math. Optimization, 10:97-125, 1983.

[8] P. Bezandry, T. Diagana, Existence of almost periodic solutions to some stochastic differential equations, *Applicable Analysis*, vol. 86, no. 7, pp. 819-827, 2007

[9] P. Bezandry, Existence of almost periodic solutions to some functional integro-differential stochastic evolution equations, *Statistics and Probability Letters*, vol. 78, no. 17, pp. 2844-2849, 2008.

[10] F. Biagini, Y. Hu, B.Ôksendal, T. Zhang, *Stocastic calculus for fractional brownian motion and application*, Springer-Verlag London Limited 2008.

[11] M. M. Borai, Some probability densities and fundamental solutions of fractional evolution equations, *Chaos, Solitons and Fractals*, vol. 14, no. 3, pp. 43-440, 2002.

[12] B. Boufoussi, Y. Ouknine, On a SDE driven by a fractional Brownian motion and with monotone drift, *Electron. Comm. Proba*, 8 (2003), 122-134.

[13] Y. K. Chang, W.T. Li, Existence results for second order impulsive functional differential inclusions, *Journal of Mathmatical Analysis and Applications*, vol. 301, no. 2, pp. 477-490, 2005.

[14] J. Cui, L. Yan, Existence result for fractional neutral stochastic integro-differential equations with infinite delay. *J. Phys. A: Math. Theor.* 44 (2011) 335201 (16)

[15] J. Dabas, A. Chauhan, M. Kumar, Existence of the mild solutions for impulsive fractional equations with infinite delay. *Int. J. Differ. Equ*, 20 (2011) .

[16] R. Durrett, *Probability: Theory and Examples*, Cambridge University Press,V3, 2010.

[17] J. Eriksson, J. Persson, Pricing turbo warrants. *Preprint*, 2006. 31-178

[18] C. Fima, *Introduction to Stochastic Calculus with Applications*, Imperial College Press, 1998.

[19] T. Guendouzi, K. Mehdi, Existence of mild solutions for impulsive fractional stochastic equations with infinite delay, *Malaya Journal of Matematik.* 4 (1)(2013), 30-43.

[20] T. Guendouzi, K. Mehdi, Almost periodic mild solutions for stochastic delay functional differential equations driven by a fractional Brownian motion, *Romanian Journal of Mathimatics and computer science*, 4 (2014), 12-26.

[21] T. Guendouzi, Existence and controllability of fractional-order impulsive stochastic system with infinite delay, *Discussiones Mathematicae Differential Inclusions, Control and Optimization*, vol. 33, no. 1, 2013

[22] E. Hernández. M. Pierri, G. Goncalves, Existence results for an impulsive abstract partial differential equation with state-dependent delay, *Computers and Mathematics with Applications*, 52 (3-4)(2006), 411-420.

[23] T. Hida, M. Hitsuda, *Gaussian Processes*, American Mathem. Society: Translations of Math. Monographs 1976.

[24] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, River Edge, NJ, USA, 2000

[25] J. Hull, A. White. Pricing interest rate derivative securities. *The Review of Financial Studies*, 3:537-592, 1990.

[26] Y. Joachim, Nahmani, Introduction to Stochastic Integration with respect to Fractional Brownian Motion, *Institute of Mathematics A - Chair of Stochastic Processes*, 2009.

[27] I. Karatzas, Steven E.Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag New York, 1991.

[28] A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.

[29] H. Kuo, *Introduction to Stochastic Integration*, Springer Science Business Media, 2006.

[30] V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Academic, Cambridge, UK, 2009.

[31] B. Maslowski, D. Nualart, Evolution equations driven by a fractional Brownian motion. *J. Funct. Anal*, 202 (2003), 277-305.

[32] M. Metivier, J. Pellaumail. *Stochastic integration.* New York: Academic Press, 1980.

[33] K. S. Miller, and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, New York, NY, USA, 1993.

[34] Y. Mishura, *stochastic calculus for fractional brownian motion and related processes*, Springer-Verlag Berhim Heidelberg 2008.

[35] P. Nicolas, Stochastic Calculus in Finance, *School of Physical and Mathematical Sciences*, (2013),67-92

[36] D. Nualart, Y. Ouknine, Besov regularity of stochastic integrals with respect to the fractional Brownian motion with parameter $H > 1/2$. *J. Theor. Prob*, 16 (2003), 451-470.

[37] V. Obukhovskii, J.C. Yao, On impulsive functional differential inclusions with Hille-Yosida operators in Banach spaces, *Nonlinear Analysis: Theory, Methods and Applications*, vol. 73, no. 6, pp. 1715-1728, 2010.

[38] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999

[39] G. Prato, J. Jabczyk, *Ergodicity for infinite dimensional systems.* Cambridge University Press, 1996.

[40] B.N. Sadovskii, On a fixed point principle, *Funct. Anal. Appl*, 1 (1967), 71-74.

[41] R. Sakthivel, Y. Ren, N.I. Mahmudov, On the approximate controllability of semilinear fractional differential systems, *Computers and Mathematics with Applications*, 62 (2011), 1451-1459.

[42] X.B. Shu, Y. Lai, Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal. TMA*, 74 (2011), 2003-2011.

[43] V. E. Tarasov, Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particales, *Fields and Media, Springer, Heidelberg, Germany*, 2010.

[44] A. D. Venttsel, *A course in the theory of stochastic processes.* Mc. Graw Hill, New York, 1981.

[45] J. R. Wang, Y. Zhou, A class of fractional evolution equations and optimal controls, *Nonlinear Analysis: Real World Applications*, vol. 12, pp. 262-272, 2011.

[46] L. C. Young, An inequality of Hölder type connected with Stieltjes integration. *Acta Math*, 67 (1936), 251-282.

[47] H. Zhang, Yan, and Q. Chen, Stability and dissipative analysis for a class of stochastic system with time-delay, *Journal of the Franklin Institute*, 347(2010), 882-893.

[48] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Computers and Mathematics with Applications*, vol. 59, no. 3, pp. 1063-1077, 2010.