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Dedication

This senior thesis is dedicated to my parents:

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Contents

Introduction

Fractional Brownian motion was introduced by *Kolmogorov* in connection to his work related to turbulence (1940). *Kolmogorov* gave a spectral representation of fBm using an orthogonally scattered Gaussian measure.
In 1968 *Mandelbroit* and *van Ness* gave another representation for Kolmogorov process, and they also renamed the process to fractional Brownian motion.
In 1969 *Molchan* and *Golosov* proved a Girsanov theorem for Kolmogorov process. They also gave a representation theorem for Kolmogorov process in terms of standard Brownian motion.

In the case of Brownian motion, the famous Lévy characterization theorem states that a continuous stochastic process $(B_t, t \geq 0)$ adapted to a right-continuous filtration $(\mathcal{F}_t, t \geq 0)$ is an $\mathcal{F}_t$-Brownian motion if and only if B is a local martingale and $\langle B \rangle_t = t$. A natural problem is the extension of Lévy characterization theorem to the fractional Brownian motion.

The purpose of this manuscript is to introduce and study the notion of a fractional martingale, and apply it to the above problem. The notion of fractional martingales has been introduced in [11] where the authors proved an extension of Lévy's characterization theorem to the fractional Brownian motion.
Fix $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. If $\{M_t, t \geq 0\}$ is a continuous local martingale, we denote by $M^{(\alpha)} = (M_t^{(\alpha)}, t \geq 0)$ the stochastic process defined by

$$M_t^{(\alpha)} = \int_0^t (t-s)^\alpha dM_s$$

provided this stochastic integral exists for all $t \geq 0$. The process $M^{(\alpha)}$ is called the Riemann-Liouville process of M. Notice that $M^{(\alpha)}$ is no longer a martingale and we will say that it is a fractional martingale.

We are interested here in the variation properties of fractional martingales. The process $M^{(\alpha)}$ has Hölder continuous trajectories of order γ on any finite interval, for any $\gamma < \frac{1}{2}+\alpha$, provided M has Hölder continuous trajectories of order $\frac{1}{2} - \epsilon$, on any finite interval, for any $\epsilon > 0$. Then, it is natural to expect that $M^{(\alpha)}$ has a finite and nonzero variation of order $\beta = (\frac{1}{2} + \alpha)^{-1}$. We show that (see Theorem 2.1.1) if $d\langle M\rangle_t = \xi_t^2 dt$, then $M^{(\alpha)}$ has a finite β-variation $c_\alpha \int_0^t |\xi_s|^\beta ds$ under some integrability conditions on ξ, where c_α is a constant depending only on α. The proof of this result is based on the variation properties of the fractional Brownian motion.

The fractional Brownian motion B_H is not a martingale unless $H = \frac{1}{2}$. But the process

$$M_t = \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} dB_s^H$$

is a martingale with respect to the filtration generated by the fBm, verifying $\langle M\rangle_t = d_H t^{2H}$ for some constant d_H. We show that if $B = (B_t, t \geq 0)$ is a continuous square integrable centered process with $B_0 = 0$, then B is a fractional Brownian motion with Hurst parameter H if and only if the process B has the following properties:

i) The sample paths of the process B are Hölder continuous of order γ for any $\gamma \in (0, H)$.

ii) The process M defined in above, where B^H is replaced by B, is a martingale with respect to the filtration generated by B. If $H > \frac{1}{2}$, we also assume that the quadratic variation of M is absolutely continuous with respect to the Lebesgue measure.

iii) For any $t > 0$, the process B has $\frac{1}{H}$-variation which equals to $c_H t$ on the interval $[0, t]$.

In order to prove that the conditions (i), (ii) and (iii) imply that B is a fractional Brownian motion, it suffices to show that the martingale M satisfies $\langle M\rangle_t = d_H t^{2H}$ for some constant d_H , and this will be a consequence of the condition (iii) and the general result on the β-variation of a fractional martingale.

In a recent work [11], *Mishura* and *Valkeila* have proved another extension of the Lévy characterization theorem, where condition (iii) is replaced by an assumption on the renormalized quadratic variation, and no restriction on the quadratic variation of M is required.

This senior thesis is orgnized as follows. chapter 1 divided on two section, In the first we give the definition of fractional Brownian motion (one parameter cases), and some properties, In the second section we give the necessary notion of Two-parameter fractional Brownian motion and its properties, Chapter 2 is devoted to study the β-variation of fractional martingales, and contains the proof of the Lévy characterization theorem for the fBm. Some technical lemmas are included in the Appendix.

Chapter 1

The elements of Fractional Brownian Motion

1.1 Fractional Brownian Motion

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space.

Definition 1.1.1 *The fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$ on $(\Omega, \mathcal{A}, \mathbb{P})$, having the properties:*

1. $B_0^H = 0$,
2. $\mathbb{E}[B_t^H] = 0$; $t \in \mathbb{R}$,
3. $\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big)$; $s, t \in \mathbb{R}$.

Remark 1.1.1 *Since $\mathbb{E}[B_t^H - B_s^H]^2 = |t-s|^{2H}$ and B_H is a Gaussian process, it has a continuous modification, according to the Kolmogorov criterion.*

Remark 1.1.2 *For $H = 1$, we set $B_t^H = B_t^1 = t\xi$, where ξ is a standard normal Random variable.*
For $H = \frac{1}{2}$, the characteristic function has the form

$$\phi_\lambda(t) = \mathbb{E}\Big[exp(i\sum_{k=1}^{n} \lambda_k B_{t_k}^H)\Big] = exp\Big(-\frac{1}{2}(C_t\lambda, \lambda)\Big),$$

where $C_t = (\mathbb{E}[B_{t_K}^H B_{t_i}^H])_{1 \le i,k \le n}$and $(.,.)$ is the inner product on $\mathbb{R}^n$.

1.1.1 Stochastic Integral Representation

Here we discuss some of the integral representations for the fBm. In [10] it is proved that the process

$$\begin{aligned} Z(t) &= \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}((t-s)_+^{H-\frac{1}{2}}-(-s)_+^{H-\frac{1}{2}})dB(s) \\ &= \frac{1}{\Gamma(H+\frac{1}{2})}\Big(\int_{-\infty}^{0}((t-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}})dB(s) \\ &+ \int_0^t (t-s)^{H-\frac{1}{2}}dB(s)\Big). \end{aligned}$$

Where $B(t)$ is a standard Brownian motion and Γ represents the gamma function, is a fBm with Hurst index $H \in (0,1)$. First we notice that $Z(t)$ is a continuous centered Gaussian process. Hence, we need only to compute the covariance functions. In the following computations we drop the constant $\frac{1}{\Gamma(H+\frac{1}{2})}$ for the sake of simplicity. We obtain

$$\begin{aligned} \mathbb{E}[Z^2(t)] &= \int_{\mathbb{R}}\Big[(t-s)_+^{H-\frac{1}{2}}-(-s)_+^{H-\frac{1}{2}}\Big]^2 ds \\ &= t^{2H}\int_{\mathbb{R}}\Big[(1-u)_+^{H-\frac{1}{2}}-(-u)_+^{H-\frac{1}{2}}\Big]^2 du \\ &= C(H)t^{2H}, \end{aligned}$$

where we have used the change of variable $s = tu$. Analogously, we have that

$$\begin{aligned} \mathbb{E}[|Z(t)-Z(s)|^2] &= \int_{\mathbb{R}}\Big[(t-u)_+^{H-\frac{1}{2}}-(s-u)_+^{H-\frac{1}{2}}\Big]^2 ds \\ &= t^{2H}\int_{\mathbb{R}}\Big[(t-s-u)_+^{H-\frac{1}{2}}-(-u)_+^{H-\frac{1}{2}}\Big]^2 du \\ &= C(H)|t-s|^{2H}. \end{aligned}$$

Now

$$\begin{aligned} \mathbb{E}[Z(t)\times Z(s)] &= -\frac{1}{2}\Big\{\mathbb{E}[|Z(t)-Z(s)|^2]-\mathbb{E}[Z(t)^2]-\mathbb{E}[Z(s)^2]\Big\} \\ &= \frac{1}{2}\Big(t^{2H}+s^{2H}-|t-s|^{2H}\Big). \end{aligned}$$

Hence we can conclude that $Z(t)$ is a fBm of Hurst index H.

We can also represent the fBm over a finite interval, i.e.

$$B_t^{(H)} = \int_0^t K_H(t,s)dB_s, \quad t \geq 0,$$

where

1. For $H > \frac{1}{2}$,

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

   where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right]^{\frac{1}{2}}$ and $t > s$,

2. For $H < \frac{1}{2}$,

$$K_H(t,s) = c_H \Big[\Big(\frac{t}{s}\Big)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \Big],$$

   with $c_H = \left[\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}\right]^{\frac{1}{2}}$ and $t > s$.

1.1.2 Correlation between two increments

For $H = \dfrac{1}{2}$, $B^{(H)}$ is a standard Brownian motion; hence, in this case the increments of the process are independent. On the contrary, for $H \neq \dfrac{1}{2}$ the increments are not independent. More precisely, by Definition (1.1.1) we know that the covariance between $B^H(t+h) - B^H(t)$ and $B^H(s+h) - B^H(s)$ with $s + h \leq t$ and $t - s = nh$ is

$$\rho^H(n) = \frac{1}{2} h^{2H} \Big[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \Big].$$

In particular, we obtain that two increments of the form $B^H(t+h) - B^H(t)$ and $B^H(t+2h) - B^H(t+h)$ are positively correlated for $H > \dfrac{1}{2}$, while they are negatively correlated for $H < \dfrac{1}{2}$. In the first case the process presents an aggregation behavior and this property can be used in order to describe (cluster) phenomena (systems with memory and persistence). In the second case it can be used to model sequences with intermittency and antipersistence.

1.1.3 Self-similarity

Definition 1.1.2 *We say that an $\mathbb{R}^d$-valued random process $X = (X_t)_{t\geq 0}$ is self-similar or satisfies the property of self-similarity if for every $a > 0$ there exist $b > 0$ such that:*

$$law\,(X_{at}, t \geq 0) = law\,(bX_t, t \geq 0) \tag{1.1}$$

Note that (1.1) means that the two process X_{at} and bX_t have the same finite-dimensional distribution functions, i.e., for every choice $t_1, ..., t_n \in \mathbb{R}$,

$$\mathbb{P}\,(X_{at_0} \leq x_0, ..., X_{at_n} \leq x_n) = \mathbb{P}(bX_{t_0} \leq x_0, ..., bX_{t_n} \leq x_n)$$

For every $x_0, ..., x_n \in \mathbb{R}$.

Remark 1.1.3 *If $b = a^{-H}$ in (1.1), then we say that $X = (X_t)_{t_0}$ is a self-similar process with Hurst index H or that it satisfies the property of (statistical) self-similarity with Hurst index H. The quantity $D = \frac{1}{H}$ is called the statistical fractal dimension of X. Since the covariance function of the fBm is homogeneous of order $2H$, we obtain that B^H is a self-similar process with Hurst index H, i.e., for any constant $a > 0$ the processes $B^H(at)$ and $a^{-H}B^H(t)$ have the same distribution law.*

1.1.4 Hölder continuity

We recall that according to the Kolmogorov criterion [16], a process $X = (X_t)_{t\in\mathbb{R}}$ admits a continuous modification if there exist constants $\alpha \geq 1$, $\beta > 0$, and $k > 0$ such that

$$\mathbb{E}[|X(t) - X(s)|^\alpha] \leq k|t - s|^{1+\beta}$$

for all $s, t \in \mathbb{R}$.

Theorem 1.1.1 *Let $H \in (0, 1)$. The fractional Brownian motion B^H admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*

Proof. We recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Hölder continuous of order α, $0 < \alpha \leq 1$ and write $f \in \mathcal{C}^\alpha(\mathbb{R})$, if there exists $M > 0$ such that

$$|f(t) - f(s)| \leq M|t - s|^\alpha,$$

for every $s, t \in \mathbb{R}$. For any $\alpha > 0$ we have

$$\mathbb{E}[|B^H(t) - B^H(s)|^\alpha] = \mathbb{E}[|B^H(1)|^\alpha]|t-s|^{\alpha H};$$

hence, by the Kolmogorov criterion we get that the sample paths of B^H are almost every where Hölder continuous of order strictly less than H. Moreover, by [1] we have

$$\limsup_{t \longrightarrow 0^+} \frac{|B^H(t)|}{t^H \sqrt{\log \log t^{-1}}} = c_H$$

with probability one, where c_H is a suitable constant. Hence B^H can not have sample paths with Hölder continuity's order greater than H. □

1.1.5 Path differentiability

By [9] we also obtain that the process B^H is not mean square differentiable and it does not have differentiable sample paths.

Proposition 1.1.1 *Let $H \in (0,1)$. The fractional Brownian motion sample path $B^H(.)$ is not differentiable. In fact, for every $t_0 \in [0, \infty)$*

$$\mathit{lim}\sup_{t \to t_0} \left| \frac{B^H_t - B^H_{t_0}}{t - t_0} \right| = \infty$$

With probability one.

1.1.6 The fBm is not a Semimartingale for $H \neq \frac{1}{2}$

The fact that the fBm is not a semimartingale for $H \neq \dfrac{1}{2}$ has been proved by several authors. In order to verify that B^H is not a semimartingale for $H \neq \dfrac{1}{2}$, it is sufficient to compute the p-variation of B^H.

Definition 1.1.3 *Let $(X(t))_{t \in [0,T]}$ be a stochastic process and consider a partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$. Put*

$$\mathcal{S}_p(x, \pi) := \sum_{i=1}^{n} |X(t_i) - X(t_{i-1})|^p$$

The p-variation of X over the interval $[0,T]$ is defined as

$$\mathcal{V}_p(X,[0,T]) := \sup_{\pi} \mathcal{S}_p(X,\pi),$$

where π is a finite partition of $[0,T]$. The index of p-variation of a process is defined as

$$I(X,[0,T]) := \inf\{p>0; \mathcal{V}_p(X,[0,T])<\infty\}.$$

We claim that

$$I(B^H,[0,T]) = \frac{1}{H}.$$

In fact, consider for $p>0$,

$$Y_{n,p} = n^{pH-1}\sum_{i=1}^{n}\left|B^H\Big(\frac{i}{n}\Big) - B^H\Big(\frac{i-1}{n}\Big)\right|^p.$$

Since B^H has the self-similarity property, the sequence $Y_{n,p}, n \in N$ has the same distribution as

$$\tilde{Y}_{n,p} = n^{-1}\sum_{i=1}^{n}\left|B^H(i) - B^H(i-1)\right|^p.$$

and By the Ergodic theorem the sequence $\tilde{y}_{n,p}$ converges almost surely and in L^1 to $\mathbb{E}[|B^H(1)|^p]$ as n tends to infinity. It follows that

$$V_{n,p} = \sum_{i=1}^{n}\left|B^H\Big(\frac{i}{n}\Big) - B^H\Big(\frac{i-1}{n}\Big)\right|^p$$

converges in probability respectively to 0 if $pH>1$ and to infinity if $pH<1$ as n tends to infinity. Thus we can conclude that $I(B^H,[0,T]) = \frac{1}{H}$. Since for every semimartingale X, the index $I(X,[0,T])$ must belong to $[0,1]\cup\{2\}$, the fBm B^H cannot be a semimartingale unless $H = \dfrac{1}{2}$.

1.1.7 Invariance principle

Here we present an invariance principle for fBms due to [2].

Assume that $\{X_n, n = 1,2,...\}$ is a stationary Gaussian sequence with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = 1$. Define

$$Z_n(t) = \frac{1}{n^H} \sum_{k=1}^{[nt]} X_k, \quad 0 \leq t \leq 1,$$

where [.] stands for the integer part. We will show that if the covariance of $\sum_{0}^{n} X_k$ is proportional to Cn^{2H} for large n, $Z_n(t), t \geq 0$ converges weakly to $\sqrt{C} B_t^{(H)}$ in a suitable metric space. Let as first introduce the real-valued function $\omega_\beta^\alpha(.)$ defined by

$$\omega_\beta^\alpha(t) = t^\alpha \left(1 + \log \frac{1}{t}\right)^\beta, \quad t > 0,$$

and we let

$$||f||_p^{\omega_\beta^\alpha} = ||f||_{L^p(I)} \sup_{0<t\leq 1} \frac{\omega_p(f,t)}{\omega_\beta^\alpha(t)}.$$

The Besov space $Lip_p(\alpha, \beta)$ is the class of functions f in $L^p(I)$ such that $||f||_p^{\omega_\beta^\alpha} < \infty$. $Lip_p(\alpha, \beta)$ endowed with the norm $||.||_p^{\omega_\beta^\alpha}$ is a nonseparable Banach space. Let $B_p^{\alpha,\beta}$ denote the separable subspace of $Lip_p(\alpha, \beta)$ formed by functions $f \in Lip_p(\alpha, \beta)$ satisfying $\omega_p(f,t) = \circ(\omega_\beta^\alpha(t))$ as $t \longrightarrow 0$. For a continuous function f, denote by $\{C_n(f), n \geq 0\}$ the coefficients of the decomposition of f in the Schauder basis given by

$$C_0(f) = f(0), \quad C_1(f) = f(1) - f(0),$$

and for $n = 2^j + k,\ j \geq 0$, and $k = 0, \ldots, 2^j - 1$,

$$C_n(f) = 2.2^{\frac{j}{2}} \left\{ f\left(\frac{2k+1}{2^{j+1}}\right) - \frac{1}{2}\left[f\left(\frac{2k}{2^{j+1}}\right) + f\left(\frac{2k-2}{2^{j+1}}\right)\right]\right\}.$$

Lemma 1.1.1 *Let $\alpha > \frac{1}{p}$ and $0 < \beta < \beta'$. The space $Lip_p(\alpha, \beta)$ is compactly embedded in $B_p^{\alpha,\beta'}$.*

We refer the reader to [10].

Lemma 1.1.2 *Let $(X_n^t, t \in I)_{n\geq 1}$ be a sequence of stochastic processes satisfying*

1. $X_0^n = 0$, *for all* $n \geq 1$.

2. *There exists a positive constant* C *and* $\alpha \in]0,1[$ *such that for* $p \geq 1$,

$$\mathbb{E}[|X_t^n - X_s^n|^p] \leq C|t-s|^{p\alpha};$$

for all $s, t \in I$. *Then* $(X^n(t), t \in I)_{n\geq 1}$ *is tight in* $B_p^{\alpha,\beta}, \beta > 0$ *for* $p > \max(\frac{1}{\alpha}, \frac{1}{\beta})$.

Proof. By the assumptions, we have $C_0(X^n) = 0$ and $C_1(X^n) = X_1^n$. To prove the lemma, by [10] it is enough to show that there exists a constant $C_p > 0$ such that, for $\lambda > 0$ and $\frac{1}{p} < \beta' < \beta$, we have

$$\mathbb{P}(||X^n||_p^{\omega_{\beta'}^{\alpha}} > \lambda) \leq C_p \lambda^{-p}$$

for all $n \geq 1$.Thus, it suffices to show that

$$\mathbb{P}\left(M(X^n) > \lambda\right) \leq C_p \lambda^{-p},$$

where $M(X^n)$ is the maximum of the set

$$\left\{ |C_0(X^n)|, |C_1(X^n)|, \sup_{j\geq 0} \frac{2^{-j(\frac{1}{2}-\alpha+\frac{1}{p})}}{(1+j)^{\beta'}} \left[\sum_{m=2^j+1}^{2^{j+1}} |C_m(x^n)|^p \right]^{\frac{1}{p}} \right\}.$$

Now, by the Chebyshev inequality, we have

$$\begin{aligned} I &= \mathbb{P}\left(\sup_{j\geq 0} \frac{2^{-j(\frac{1}{2}-\alpha+\frac{1}{p})}}{(1+j)^{\beta'}} \left[\sum_{m=2^j+1}^{2^{j+1}} |C_m(X^n)|^p \right]^{\frac{1}{p}} > \lambda \right) \\ &\leq \sum_{j\geq 0} \frac{2^{-jp(\frac{1}{2}-\alpha+\frac{1}{p})}}{(1+j)^{p\beta'}} \sum_{m=2^j+1}^{2^{j+1}} \mathbb{E}[|C_m(X^n)|^p]\lambda^{-p}. \end{aligned}$$

Recall that for $m = 2^j + k$,

$$C_m(X^n) = 2.2^{\frac{j}{2}} \left[X^n_{(2k-1)/2^{j+1}} - \frac{1}{2}\left(X^n_{(2k)/2^{j+1}} + X_{(2k-2)/2^{j+1}} \right) \right].$$

Thus,

$$\begin{aligned} I &\leq C_p\lambda^{-p}\sum_{j\geq 0}\frac{2^{-jp(\frac{1}{2}-\alpha+\frac{1}{p})}}{(1+j)^{p\beta'}}.\sum_{k=1}^{2^j}\Big(\mathbb{E}[|X^n_{(2k-1)/2^{j+1}}-X_{(2k)/2^{j+1}}|^p] \\ &+ \mathbb{E}[|X^n_{(2k-1)/2^{j+1}}-X^n_{(2k-2)/2^{j+1}}|^p]\Big) \\ &\leq \lambda^{-p}\left[C_p\sum_{j\geq 0}\frac{1}{(1+j)^{p\beta'}}\right]\leq C_p\lambda^{-p}. \end{aligned}$$

which completes the proof. □

Corollaire 1.1.1 *Let $H\in(0,1)$, $\beta>0$, and $p>\max(\frac{1}{H},\frac{1}{\beta})$.*
Assume that $\{X_n, n=1,2,\ldots\}$ is a stationary Gaussian sequence with spectral representation

$$X_n=\int_{-\pi}^{\pi}\exp(in\lambda)|\lambda|^{\frac{1}{2-H}}B(d\lambda), n=1,2\ldots,$$

where $B(d\lambda)$ is a Gaussian random measure with $\mathbb{E}[|B(d\lambda)|^2]=d\lambda$. Then there exists a positive constant C such that $(Z_n(t), t\in[0,1])$ converges weakly to $(CB^H_t), t\in[0,1])$ in the space $B^{H,\beta}_p$.

1.2 Two-parameter Fractional Brownian Motion

1.2.1 The Main Definition

For technical simplicity we consider two-parameter fbm (fbm field) $\{B^H_t, t\in\mathbb{R}^2_+\}$, where $t=(t_1,t_2)$. We suppose that $s\leq t$ if $s=(s_1,s_2), t=(t_1,t_2)$ and $s_i\leq t_i, i=1,2$.

Definition 1.2.1 *The two-parameter process $\{B^H_t, t\in\mathbb{R}^2_+\}$ is called a (normalized) two-parameter fBm with Hurst index $H=(H_1,H_2)\in(0,1)^2$, if it satisfies the assumptions:*

(a) B^H *is a Gaussian field,* $B_t=0$ *for* $t\in\partial\mathbb{R}^2_+$*;*

(b) $\mathbb{E}B^H_t=0,\quad \mathbb{E}B^H_tB^H_s=\frac{1}{4}\prod_{i=1,2}\left(t_i^{2H_i}+s_i^{2H_i}-\mid t_i-s_i\mid^{2H_i}\right)$

Evidently, such a process has the modification with continuous trajectoires, and we will always consider such a modification. Moreover, consider "two-parameter" increments: $\Delta_s B_t^H := B_t^H - B_{s_1 t_2}^H - B_{t_1 s_2}^H + B_s^H$ for $s \leq t$. Then they are stationary. Note, that for any fixed $t_i > 0$ the process $B_{(t_i,.)}^H$ will be the fbm with Hurst index $H_j, i = 1, 2, j = 3 - i$, evidently, nonnormalized.

1.2.2 Fractional Integrals and Fractional Derivatives of Two-parameter Functions

For $\overline{\alpha} = (\alpha_1, \alpha_2)$ denote $\overline{\Gamma}(\overline{\alpha}) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$

Definition 1.2.2 *[15] Let $f \in \mathcal{P} := [a, b] := \prod_{i=1,2} [a_i, b_i], a = (a_1, a_2), b = (b_1, b_2)$. Forward and backward Reimann-Liouville fractional integrals of orders $0 < \alpha_i < 1$ are defined as*

$$(I_{a^+}^{\alpha_1 \alpha_2} f)(x) := \overline{\Gamma}(\overline{\alpha}) \int_{[a,x]} \frac{f(u)}{\varphi(x, u, 1-\alpha)} du,$$

and

$$(I_{b^-}^{\alpha_1 \alpha_2} f)(x) := \overline{\Gamma}(\overline{\alpha}) \int_{[x,b]} \frac{f(u)}{\varphi(x, u, 1-\alpha)} du,$$

correspondingly, where $[a, x] = \prod_{i=1,2} [a_i, x_i], [x, b] = \prod_{i=1,2} [x_i, b_i], du = du_1 du_2$,

$\varphi(u, x, \alpha) =| u_1 - x_1 |^{\alpha_1} | u_2 - x_2 |^{\alpha_2}, \quad u, x \in [a, b]$.

Definition 1.2.3 *Forward and backward fractional Liouville derivatives of orders $0 < \alpha_i < 1$ are defined as*

$$(D_{a^+}^{\alpha_1 \alpha_2} f)(x) := \overline{\Gamma}(\overline{1-\alpha}) \frac{\partial^2}{\partial x_1 \partial x_2} \int_{[a,x]} \frac{f(u)}{\varphi(x, u, \alpha)} du,$$

and

$$(D_{b^-}^{\alpha_1 \alpha_2} f)(x) := \overline{\Gamma}(\overline{1-\alpha}) \frac{\partial^2}{\partial x_1 \partial x_2} \int_{[x,b]} \frac{f(u)}{\varphi(x, u, \alpha)} du, \quad x \in [a, b]$$

Definition 1.2.4 *Forward fractional Marchaud derivatives of orders $0 < \alpha_i < 1$ are defined as*

$$\begin{aligned}(\widetilde{D}_{a^+}^{\alpha_1\alpha_2} f)(x) &:= \overline{\Gamma}(\overline{1-\alpha})\Big(\frac{f(x)}{\varphi(x,u,\alpha)} + \alpha_1\alpha_2 \int_{[a,x]} \frac{\Delta_u f(x) du}{\varphi(x,u,1+\alpha)} \\ &+ \sum_{i=1,2,j=3-i} \frac{\alpha_i}{x_j - a_j} \alpha_j \int_{a_i}^{x_i} \frac{f(x) - f(u_i, x_j)}{(x_i - u_i)^{1+\alpha_i}} du_i\Big)\end{aligned}$$

and the backward derivatives can be defined in a similar way.

Let $1 \le p \le \infty$, the classes $I_+^{\alpha_1\alpha_2}(L_p(\mathcal{P})) := \{f | f = I_{a^+}^{\alpha_1\alpha_2}\varphi, \varphi \in L_p(\mathcal{P})\}$, $I_-^{\alpha_1\alpha_2}(L_p(\mathcal{P})) := \{f | f = I_{b^-}^{\alpha_1\alpha_2}\varphi, \varphi \in L_p(\mathcal{P})\}$

Further we denote $D_{a^+}^{\alpha_1\alpha_2} := I_{a^+}^{-(\alpha_1\alpha_2)}$. Of course, we can introduce the notion of fractional integrals and fractional derivatives on $\mathbb{R}_+^2$. For exemple, the Riemann-Liouville fractional integrals and derivatives on $\mathbb{R}_+^2$ are defined by the formulas

$$(I_+^{\alpha_1\alpha_2} f)(x) := \overline{\Gamma}(\overline{\alpha}) \int_{(-\infty,x]} \frac{f(t)}{\varphi(x,u,\alpha)} dt,$$

$$(I_-^{\alpha_1\alpha_2} f)(x) := \overline{\Gamma}(\overline{\alpha}) \int_{[x,\infty)} \frac{f(t)}{\varphi(x,u,\alpha)} dt,$$

$$(I_+^{-(\alpha_1\alpha_2)} f)(x) = (D_+^{\alpha_1\alpha_2} f)(x) := \overline{\Gamma}(\overline{1-\alpha}) \frac{\partial^2}{\partial x_1 \partial x_2} \int_{(-\infty,x]} \frac{f(t)}{\varphi(x,t,\alpha)} dt,$$

and

$$(I_-^{-(\alpha_1\alpha_2)} f)(x) = (D_-^{\alpha_1\alpha_2} f)(x) := \overline{\Gamma}(\overline{1-\alpha}) \frac{\partial^2}{\partial x_1 \partial x_2} \int_{[x,\infty)} \frac{f(t)}{\varphi(x,t,\alpha)} dt,$$

$0 < \alpha_i < 1$. Evidently, all these operators can be expanded into the product of the form $I_+^{\alpha_1\alpha_2} = I_+^{\alpha_1} \otimes I_+^{\alpha_2}$, and so on. In what follows we shall consider only the case $H_i \in (1/2, 1)$. Define the operator

$$M_\pm^{H_1H_2} f := \prod_{i=1,2} C_{H_i}^{(3)} I_\pm^{\alpha_1\alpha_2} f.$$

Definition 1.2.5 *A random field* $\{X_t, t \in \mathbb{R}^2_+\}$ *is a field with independent increments if its increments* $\{\Delta_{s_i} X_{t_i}, i = \overline{1,n}\}$ *for any family of disjoint rectangles* $\{(s_i, t_i], i = \overline{1,n}\}$ *are independent.*

Definition 1.2.6 *The random field* $\{W_t, t \in \mathbb{R}^2_+\}$ *is called the Wiener field if* $W = 0$ *on* $\partial\mathbb{R}^2_+$. *W is the field with the independent increments and*

$$\mathbb{E}(\Delta_s W_t)^2 = area((s,t]) = \prod_{i=1,2} (t_i - s_i).$$

Let we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with two-parameter filtration $\{\mathcal{F}_t, t \in \mathbb{R}^2_+\}$ on it. It means that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $s < t$. Denote $\mathcal{F}^*_s := \sigma\{\mathcal{F}_u, s \leq u\}$.

Definition 1.2.7 *An adapted random field* $\{X_t, \mathcal{F}_t, t \in \mathbb{R}^2_+\}$ *is a strong martingale if X vanishes on* $\partial\mathbb{R}^2_+$, $\mathbb{E}|X_t| < \infty$ *for all* $t \in \mathbb{R}^2_+$ *and for any* $s < t$ $\mathbb{E}(\Delta_s X_t | \mathcal{F}^*_s) = 0$.

Evidently, any random field with constant expectation and independent increments is a strong martingale, in particular, the Wiener field is a strong martingale.

Definition 1.2.8 *Let*

$$f \in L_2^{H_1 H_2} := \left\{ f : \mathbb{R}^2 \longrightarrow \mathbb{R} : \int_{\mathbb{R}^2} ((M_-^{H_1 H_2} f)(t))^2 dt < \infty \right\}$$

Then we denote $\int_{\mathbb{R}^2} f(t) dB_t^{H_1 H_2}$ as $\int_{\mathbb{R}^2} (M_-^{H_1 H_2} f)(t) dW_t$ for the underlying Wiener process W.

1.2.3 Hölder Properties of Two-parameter fbm

We fix $\alpha = (\alpha_1, \alpha_2), \alpha_i \in (0,1]$ and let $T = [a_1, b_1] \times [a_2, b_2]$. Let f the Riemann-Liouville fractional integral of order α i.e

$$(I^\alpha_{a^+} f)(x_1, x_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{a_1}^{x_1} \int_{a_2}^{x_2} \frac{f(t_1, t_2)}{(x_1 - t_1)^{1-\alpha_1} (x_2 - t_2)^{1-\alpha_2}} dt_1 dt_2, \quad (x_1, x_2) \in T$$

The space $\Lambda_{\alpha,p} = (I^\alpha_{a^+})(L_p(T))$ is called the Liouville space (or Besov space) and it becomes separable Banach space with respect to the norm

$$\|I^\alpha_{a^+} f\|_{\alpha,p} = \|f\|_p$$

Proposition 1.2.1 *[7] For every α, β*

$$I^{\alpha}_{a^+} I^{\beta}_{a^+} = I^{\alpha+\beta}_{a^+},$$

If $f \in \mathcal{C}^2_b(T)$ and $f = 0$ on $\partial_1 T = ([a_1, b_1] \times \{b_1\}) \cup (\{a_1\} \times [a_2, b_2])$then the function

$$D^{\alpha}_{a^+} f(x_1, x_2) = \frac{1}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} \int_{a_1}^{x_1} \int_{a_2}^{x_2} \frac{\partial^2 f(t_1, t_2)}{\partial t_1 \partial t_2} \frac{dt_1 dt_2}{(x_1 - t_1)^{\alpha_1}(x_2 - t_2)^{\alpha_2}} \tag{1.2}$$

is the unique function from $L_\infty(T)$ such that

$$I^{\alpha}_{a^+} D^{\alpha}_{a^+} f = f.$$

For a rectangle $D = [s_1, t_1] \times [s_2, t_2] \subset T$ we define the increment on D of the function $f : T \to \mathbb{R}$ by

$$f(D) = f(t_1, t_2) - f(t_1, s_2) - f(s_1, t_2) + f(s_1, s_2).$$

We denote by $\mathcal{C}^{\alpha_i}([a_i, b_i])$ the space of all α_i-Hölder functions on $[a_i, b_i]$ and

$$\|f\|_{[a_i,b_i],\alpha_i} = \sup_{u \neq v, a_i \leq u, v \leq b_i} \frac{|f(u) - f(v)|}{(u - v)^{\alpha_i}}.$$

Also, we denote by $\mathcal{C}^{\alpha_1,\alpha_2}(T)$ the space of all $(\alpha_1, \alpha_2)-$Hölder functions on T, i.e., $f \in \mathcal{C}^{\alpha_1,\alpha_2}(T)$ if f is continuous,

$$\|f(a_1, .)\|_{[a_2,b_2],\alpha_2} < \infty, \qquad \|f(., a_2)\|_{[a_1,b_1],\alpha_1} < \infty$$

and

$$\|f\|_{T,\alpha_1,\alpha_2} = \sup_{u_i \neq v_i} \frac{|f([u_1, v_1] \times [u_2, v_2])|}{|u_1 - v_1|^{\alpha_1} |u_2 - v_2|^{\alpha_2}} < \infty.$$

Proposition 1.2.2 *[4] Let $0 < \beta_1 < \alpha_1, 0 < \beta_2 < \alpha_2$ and $p \geq 1$. Then we have the continuous inclusions $\Lambda_{\alpha,p} \subset \Lambda_{\beta,p}$,*

$$\Lambda_{\alpha,p} \subset \mathcal{C}^{\alpha_1 - p^{-1}, \alpha_2 - p^{-1}}, \mathcal{C}^{\beta_1,\beta_2} \subset \Lambda_{\gamma,p} \qquad if \quad \alpha_i p > 1, \beta_i > \gamma_i > 0$$

ici il faut tout d'abord dfinir la fonction gnralis

Proposition 1.2.3 *[4] Assume that* f, g *are* $\mathcal{C}^1([a,b])-$*function with* $f(a)=0$. *Let* $\alpha, \beta \in (0,1]$ *be such that* $\alpha+\beta>1$ *and let* $\delta := \{a=t_0<...<t_n=b\}$ *be a partition of* $[a,b]$ *with the norm* $\|\delta\| = \max_j |t_{j+1}-t_j|$. *Then for every* $0<\varepsilon<\alpha+\beta-1$ *the following estimates hold:*

$$\left|\int_a^b f(t)dg(t)\right| \leq C(\alpha,\beta)\|f\|_{[a,b],\alpha}\|g\|_{[a,b],\beta}(b-a)^{1+\varepsilon}, \tag{1.3}$$

$$\left|\int_a^b f(t)dg(t) - \sum_i f(t_i)\left[g(t_{i+1})-g(t_i)\right]\right| \leq C(\alpha,\beta)\|f\|_{[a,b],\alpha}\|g\|_{[a,b],\beta}(b-a)^{\varepsilon}. \tag{1.4}$$

Chapter 2

Fractional martingales and characterization of the fractional Brownian motion

2.1 β-variation of α-martingales

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a right-continuous filtration $(\mathcal{F}_t, t \geq 0)$ such that $\mathcal{F}_0$ contains the $\mathbb{P}$-null sets. Fix a parameter $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. We introduce the following notion.

Definition 2.1.1 *A continuous $\mathcal{F}_t$-adapted process $(M_t^{(\alpha)}, t \geq 0)$ is called a fractional martingale of order α if there is a continuous local martingale $(M_t, t \geq 0)$ such that for all $t \geq 0$,*

$$\int_0^t (t-s)^{2\alpha} d\langle M \rangle_s < \infty \tag{2.1}$$

almost surely, and

$$M_t^{(\alpha)} = \int_0^t (t-s)^{\alpha} dM_s \tag{2.2}$$

Notice that by Fubini's theorem condition (2.1) holds true for almost all $t \geq 0$. if $\alpha \in (0, \frac{1}{2})$, then (2.1) is always fulfilled. Moreover, an integration by parts implies that the integral appearing in (2.2) exists as a Riemann-Stieltjes integral and $M_t^{(\alpha)} = \Gamma(\alpha+1) I_{0+}^{\alpha}(M_t)$, where I_{0+}^{α} is the left-sided fractional integral of order α.

For any $\alpha \in (-\frac{1}{2}, 0)$ we introduce the following hypothesis:
(H). The trajectories of M are α'-Hölder continuous on finite intervals for some $\alpha' > -\alpha$ Then we have the following result.

Lemma 2.1.1 *Fix $\alpha \in (-\frac{1}{2}, 0)$ and let M be a continuous local martingale satisfying condition (H). Then (2.1) holds, $M_t^{(\alpha)}$ exists as a Riemann-Stieltjes integral and it coincides with $M_t^{(\alpha)} = \Gamma(\alpha+1)D_0^{-\alpha}(M)_t$ where $D_{0+}^{-\alpha}$ is the left-sided fractional derivative of order $-\alpha$.*

Proof. Set

$$Z_t = |M_t| + \langle M \rangle_t + \sup_{0 \le s \le u \le t} \frac{|M_s - M_u|}{|s-u|^{-\alpha'}}$$

For any integer $n \geq 1$ we define

$$T_N = \inf\{t \geq 0 : Z_t > N\}.$$

Then, T_N is an nondecreasing sequence of stopping times such that $T_N \uparrow \infty$. For any $s < t$ we can write

$$\mathbb{E}(|\langle M \rangle_{t \wedge T_N} - \langle M \rangle_{s \wedge T_N}|^p) \leq C_p \mathbb{E}(|M_{t \wedge T_N} - M_{s \wedge T_N}|^{2p}) \leq C_p N^{2p} |t-s|^{2p\alpha'}.$$

By Kolmogorovs continuity criterion the sample paths of $\langle M \rangle$ are Hölder continuous of order γ for any $\gamma < 2\alpha'$, on any finite interval. This implies (2.1), and it is easy to check that the stochastic integral is a Riemann-Stieltjes integral and coincides with $\Gamma(\alpha+1)D_{0+}^{-\alpha}(M)_t$. □

From fractional calculus, assuming condition (H) if $\alpha < 0$,we have $M_t = \frac{1}{\Gamma(\alpha+1)} I_{0+}^{-\alpha}(M^{(\alpha)})_t$ where $I^{-\alpha} = D^{\alpha}$ if $\alpha > 0$. Using the definition of the leftsided fractional integral and derivative, we have

$$M_t = \begin{cases} \frac{1}{\Gamma(\alpha+1)\Gamma(-\alpha)} \int_0^t (t-s)^{-1-\alpha} dM_s^{(\alpha)}, & if \alpha < 0 \\ \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} dM_s^{(\alpha)}, & if \alpha > 0. \end{cases} \tag{2.3}$$

In order to define the β-variation, let us first introduce some notation. Fix a time interval $[a, b]$, and consider the uniform partition

$$\Pi^n = \{a = t_0^n < t_1^n < \ldots < t_n^n = b\},$$

where $t_i^n = a + \frac{i}{n}(b-a)$ for $i = 0, \dots, n$. let $\beta \geq 1$ and let $X = (X_t, t \geq 0)$ be a continuous stochastic process.

Definition 2.1.2 *We define the β-variation of X on the interval $[a, b]$, denoted by $\langle X \rangle_{\beta,[a,b]}$, as the limit in probability of*

$$S_{\beta,n}^{[a,b]} := \sum_{i=1}^{n} |\Delta_i^n X|^\beta, \tag{2.4}$$

if the limit exists, where $\Delta_i^n X = X_{t_i^n} - X_{t_{i-1}^n}$. We say that the β-variation of X on $[a, b]$ exist in L^1 if the above limit exists in L^1.

We also denote $\langle X \rangle_{\beta,[0,t]}$ by $\langle X \rangle_{\beta,t}$. For instance, a continuous local martingale as a finite 2-variation, denoted by $\langle M \rangle_t$ and the fractional Brownian motion B_t^H of Hurst parameter $H \in (0, 1)$ has $\frac{1}{H}$-variation which is equal to $c_H t$, where $c_H = (\mathbb{E}|B_1^H|)^{\frac{1}{H}}$.
A direct consequence of the above definition is that if $\langle X \rangle_{\beta,[a,b]}$ exists, then for any $a < b < c$, both $\langle X \rangle_{\beta,[a,b]}$ and $\langle X \rangle_{\beta,[b,c]}$ exist and

$$\langle X \rangle_{\beta,[a,c]} = \langle X \rangle_{\beta,[a,b]} + \langle X \rangle_{\beta,[b,c]}. \tag{2.5}$$

It is also easy to see that the following triangular inequality holds:

$$S_{\beta,n}^{[a,b]}(X+Y)^{\frac{1}{\beta}} \leq S_{\beta,n}^{[a,b]}(X)^{\frac{1}{\beta}} + S_{\beta,n}^{[a,b]}(Y)^{\frac{1}{\beta}}. \tag{2.6}$$

This inequality implies that if X and Y are two continuous stochastic processes such that $\langle X \rangle_{\beta,[a,b]}$ exists and $\langle Y \rangle_{\beta,[a,b]} = 0$

$$\langle X + Y \rangle_{\beta,[a,b]} = \langle X \rangle_{\beta,[a,b]}. \tag{2.7}$$

Let $W = (W_t, t \geq 0)$ be an $\mathcal{F}_t$-Brownian motion. We want to compute the β-variation of $M^{(\alpha)}$, where M is a martingale of the form $M_t = \int_0^t \xi_s dW_s$.
We will denote by C a generic constant that may depend on α. Consider first the case where the martingale is just a standard Wiener process. We recall that

$$\beta = \frac{2}{1 + 2\alpha}.$$

Lemma 2.1.2 *Let $W = (W_t, t \geq 0)$ be a Wiener process, and set*

$$X_t = W_t^{(\alpha)} = \int_0^t (t-s)^{(\alpha)} dW_s$$

Then the β-variation of X exists in L^1 and $\langle X \rangle_{\beta,[a,b]} = c_\alpha (b-a)$, where $c_\alpha = c_H k_H^{-\frac{1}{H}}$, $H = \frac{1}{2} + \alpha$, $c_H = (\mathbb{E}|B_1^H|)^{\frac{1}{H}}$, and

$$k_H = \left(\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2-2H)} \right)^{\frac{1}{2}}. \tag{2.8}$$

Proof. Because of (2.5), it is sufficient to show that $\langle X \rangle_{\beta,t} = c_\alpha t$ .We can extend the underlying probability space in such a way that $(W_{-t}, t \geq 0)$ is a Brownian motion independent of W. Then, the process B^H defined by

$$B_t^H = k_H \Big(\int_0^t (t-s)^\alpha dW_s + \int_{-\infty}^0 ((t-s)^\alpha - (-s)^\alpha) dW_s \Big);$$

is a fractional Brownian motion with Hurst parameter H (see Mandelbrot and Van Ness [8]). Hence,

$$X_t = k_H^{-1} B_t^H - Z_t,$$

where $Z_t = \int_{-\infty}^0 ((t-s)^\alpha - (-s)^\alpha) dW_s$. From the $\frac{1}{H}$-variation property of fractional Brownian motion we know $\langle B^H \rangle_{\beta,t} = c_H t$, in L^1, because $\beta = \frac{1}{H}$. Then, by (2.7) it suffices to show that $\underset{n\to\infty}{lim} \mathbb{E}(|S_{\beta,n}^{[0,t]}(Z)|) = 0$ for all $t \geq 0$. We have

$$\begin{aligned}
\sum_{i=1}^n \mathbb{E}(|Z_{t_i^n} - Z_{t_{i-1}^n}|^\beta) &= C \sum_{i=1}^n \Big(\int_{-\infty}^0 ((t_i^n - s)^\alpha - (t_{i-1}^n - s)^\alpha)^2 ds \Big)^{\frac{\beta}{2}} \\
&= C \sum_{i=1}^n \Big(\int_0^\infty \Big(\Big(t_{i-1}^n + \frac{t}{n} + s \Big)^\alpha - (t_{i-1}^n + s)^\alpha ds \Big) \Big)^{\frac{\beta}{2}} \\
&\leq C \Big(\int_0^\infty \Big(\Big(\tfrac{t}{n} + s \Big)^\alpha - s^\alpha \Big)^2 ds \Big)^{\frac{\beta}{2}} + \tfrac{C}{n^\beta} \sum_{i=2}^n \Big(\int_0^\infty (t_{i-1}^n + s)^{2\alpha - 2} ds \Big)^{\frac{\beta}{2}} \\
&= I_1 + I_2
\end{aligned}$$

It is easy to see by the dominated convergence theorem that $I_1 \to 0$ as $n \to \infty$. On the other hand,

$$I_2 \le Ctn^{-1}\sum_{i=2}^{n}(i-1)^{(2\alpha-1)\frac{\beta}{2}} \le Ctn^{\frac{(2\alpha-1)}{(2\alpha+1)}} \to 0$$

since $\alpha < \frac{1}{2}$. This proves the lemma. □

We will make use of the following lemma.

Lemma 2.1.3 *Fix $a > 0$. For $t \ge a$ let $X_t = \int_0^a (t-s)^\alpha dW_s$ where $W = (W_t, t \ge 0)$ is a Wiener process. Then, for all $t \ge a$*

$$\lim_{n\to\infty} \mathbb{E}(|S^{[a,t]}_{\beta,n}(X)|) = 0 \tag{2.9}$$

Proof. Take $\beta = \frac{2}{(1+2\alpha)}$. First we have

$$\sum_{i=1}^{n}\mathbb{E}\Big|\int_0^a [(t_i^n - s)^\alpha - (t_{i-1}^n - s)^\alpha]dW_s\Big|^\beta \le C\sum_{i=1}^{n}\left\{\int_0^a [(t_i^n - s)^\alpha - (t_{i-1}^n - s)^\alpha]^2 ds\right\}^{\frac{\beta}{2}},$$

where $t \ge a$ and $\{t_i^n\}$ is a uniform partition on $[a,t]$. Then we apply a similar argument as in the proof of Lemma 2.1.2 □

The following theorem is the main result of this section.

Theorem 2.1.1 *Set $\beta = \frac{2}{(1+2\alpha)}$. Consider a continuous local martingale of the form $M_t = \int_0^t \xi_s dW_s$, where $\xi = (\xi_t, t \ge 0)$ is a progressively measurable process such that, for all $t \ge 0$,*

$$\begin{cases} \int_0^t (\mathbb{E}(|\ \xi_s\ |^\beta))^{\frac{\beta}{\beta'}} ds < \infty, & for \quad some \quad \beta' > \beta, \quad if \quad \alpha < 0, \\ \int_0^t (\mathbb{E}(\xi_s^2))^{\frac{\beta}{2}} ds < \infty, & if \quad \alpha > 0. \end{cases} \tag{2.10}$$

Then, the β-variation of $M^{(\alpha)}$ on any interval $[0,t]$ exists in L^1 and $\langle M^{(\alpha)}\rangle_{\beta,t} = c_\alpha \int_0^t (|\ \xi_s\ |)^\beta ds$, where $c_\alpha = c_H k_H^{\frac{-1}{H}}, H = \frac{1}{2} + \alpha$, and k_H is defined in (2.8)

Proof. We can represent the martingale M as a stochastic integral $M_t = \int_0^t \xi_s dW_s$, where $W = (W_t, t \geq 0)$ is a Brownian motion defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ of our original probability space $(\Omega, \mathcal{F}, P)$. The space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ is the product of $(\Omega, \mathcal{F}, P)$, and another space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ supporting a Brownian motion independent of M. Clearly, if the conclusion of the theorem holds in the extended space, it also holds in the original space. Notice that if $\alpha < 0$, by Hölder's inequality condition (2.10) implies that

$$\int_0^t (t-s)^{-2\alpha} \mathbb{E}(\xi_s^2) ds < \infty,$$

and (2.1) holds.
Suppose first that the process ξ has the form $\xi_t = Y \mathbb{I}_{(t_1,t_2]}(t)$, where $0 \leq t_1 < t_2$ and Y is a bounded $\mathcal{F}_{t_1}$-measurable random variable. In this case the process $M^{(\alpha)}$, denoted by X, is given by

$$X_t = Y \mathbb{I}_{[t_1,\infty)} \int_{t_1}^{t \wedge t_2} (t-s)^\alpha dW_s.$$

For $t \in [0, t_1]$, we clearly have $\langle X \rangle_{\beta,t} = 0$. For $t \in [t_1, t_2]$,

$$X_t = Y \int_0^t (t-s)^\alpha dW_s - Y \int_0^{t_1} (t-s)^\alpha dW_s,$$

and by Lemmas (2.1.2)and (2.1.3), for any interval $[a, b] \in [t_1, t_2]$, the β-variation of X exists in L^1, and

$$\langle X \rangle_{\beta,[a,b]} = c_\alpha \mid Y \mid^\beta (b-a).$$

Finally, by Lemma (2.1.3), for any interval $[a, b] \subset [t_2, \infty)$, $\langle X \rangle_{\beta,[a,b]} = 0$, in L^1.
Hence, we have proved that

$$\langle X \rangle_{\beta,t} = c_\alpha \mid Y \mid^\beta (t \wedge t_2 - t_1)_+ = c_\alpha \int_0^t \mid \xi_s \mid^\beta ds.$$

Let us denote by $\mathcal{S}$ the space of step functions of the form

$$\xi_t = \sum_{i=1}^{n} Y_i \mathbb{I}_{(t_{i-1},t_i]}(t),$$

where Y_i is $\mathcal{F}_{t_{i-1}}-$ measurable and bounded, and $0 = t_0 < \ldots < t_n$. For $\xi \in \mathcal{S}$, we have $X_t = \sum_{i=1}^{n} X_t^i$, where $X_t^i = \int_0^t \xi_t^i (t-s)^{\alpha} dW_s$ and $\xi_t^i = Y_i \mathbb{I}_{(t_{i-1},t_i]}(t)$. From (2.1.3) we have

$$\langle X \rangle_{\beta,t} = \sum_{i=1}^{n} \langle X \rangle_{\beta,[t_{i-1},t_i]\cap[0,t]}.$$

From the first part of the proof we see that

$$\langle X^j \rangle_{\beta,[t_{i-1},t_i]\cap[0,t]} = \begin{cases} c_\alpha |Y_i|^\beta (t_i \wedge t - t_{i-1})_+, & if \quad j = i, \\ 0 & if \quad j \neq i. \end{cases}$$

and applying the triangular inequality (2.6), we see then that

$$\langle X \rangle_{\beta,[t_{i-1},t_i]\cap[0,t]} = \langle X^i \rangle_{\beta,[t_{i-1},t_i]\cap[0,t]}.$$

Hence,

$$\langle X \rangle_{\beta,[0,t]} = c_\alpha \sum_{i=1}^{n} \mid Y_i \mid^\beta (t_i \wedge t - t_{i-1})_+ = c_\alpha \int_0^t \mid \xi_s \mid^\beta ds, \tag{2.11}$$

and this proves the result for step functions.
To complete the proof, we use a density argument. Fix a time interval $[0,T]$. We can find a sequence of step functions $(\xi^k, k \geq 1)$ in $\mathcal{S}$ such that if $\alpha > 0$, then

$$\lim_{k\to\infty} \int_0^T (\mathbb{E}(\mid \xi_s - \xi_s^k \mid^2))^{\frac{\beta}{2}} ds = 0,$$

and if $\alpha < 0$, then

$$\lim_{k\to\infty} \int_0^T (\mathbb{E}(\mid \xi_s - \xi_s^k \mid^\beta))^{\frac{\beta'}{\beta}} ds = 0.$$

Define $X_t^k = \int_0^t (t-s)^\alpha \xi_s^k dB_s$ for $t \in [0,T]$. From the triangular inequality (2.6) and the Burkholder-Davis-Gundy inequality (see, for instance,[6]), we have, for $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}(|S_{\beta,n}^{[0,t]}(X)^{\frac{1}{\beta}} - S_{\beta,n}^{[0,t]}(X^k)^{\frac{1}{\beta}}|) &\leq \mathbb{E}((S_{\beta,n}^{[0,t]}(X-X^k))^{\frac{1}{\beta}}) \\
&\leq C\left(\mathbb{E}\left(\sum_{i=1}^n \Big| \int_0^{t_i^n} ((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)(\xi_s - \xi_s^k) dW_s \Big|^\beta\right)\right)^{\frac{1}{\beta}} \\
&\leq C\left(\mathbb{E}\left(\sum_{i=1}^n \Big| \int_0^{t_i^n} ((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)^2(\xi_s - \xi_s^k)^2 ds \Big|^{\frac{\beta}{2}}\right)\right)^{\frac{1}{\beta}}.
\end{aligned} \tag{2.12}$$

Now we will consider two cases depending on the sign of α.

1. If $\alpha > 0$, namely, $\beta < 2$, then by the concavity of $x^{\frac{\beta}{2}}$ and Lemma in [6]), we have

$$\begin{aligned}
\mathbb{E}(|S_{\beta,n}^{[0,t]}(X)^{\frac{1}{\beta}} - S_{\beta,n}^{[0,t]}(X^k)^{\frac{1}{\beta}}|) &\leq C\left(\sum_{i=1}^n \Big| \int_0^{t_i^n} ((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)^2 \mathbb{E}(|\xi_s - \xi_s^k|^2) ds \Big|^{\frac{\beta}{2}}\right)^{\frac{1}{\beta}} \\
&\leq C\Big(\int_0^t (\mathbb{E}(|\xi_s - \xi_s^k|^2)^{\frac{\beta}{2}} ds)\Big)^{\frac{1}{\beta}}.
\end{aligned} \tag{2.13}$$

Then

$$\begin{aligned}
\mathbb{E}\left(\left|S_{\beta,n}^{[0,t]}(X)^{\frac{1}{\beta}} - \left(c_\alpha \int_0^t \Big|\xi_s|^\beta ds\right)^{\frac{1}{\beta}}\right|\right) &\leq \mathbb{E}(|S_{\beta,n}^{[0,t]}(X)^{\frac{1}{\beta}} - S_{\beta,n}^{[0,t]}(X^k)^{\frac{1}{\beta}}|) \\
&+ \mathbb{E}\left(\left|S_{\beta,n}^{[0,t]}(X)^{\frac{1}{\beta}} - \left(c_\alpha \int_0^t |\xi_s^k|^\beta ds\right)^{\frac{1}{\beta}}\right|\right) \\
&+ c_\alpha^{\frac{1}{\beta}} \mathbb{E}\left(\left|\left(\int_0^t |\xi_s^k|^\beta ds\right)^{\frac{1}{\beta}} - \left(\int_0^t |\xi_s|^\beta ds\right)^{\frac{1}{\beta}}\right|\right).
\end{aligned}$$

From (2.13) and (2.11) we obtain

$$\limsup_{n\to\infty} \mathbb{E}\left(\left|S^{[0,t]}_{\beta,n}(X)^{\frac{1}{\beta}} - \left(c_\alpha \int_0^t \left|\xi_s\right|^\beta ds\right)^{\frac{1}{\beta}}\right|\right)$$

$$\begin{aligned} &\leq C\left(\int_0^t (\mathbb{E}(|\ \xi_s - \xi_s^k\ |^2))^{\frac{\beta}{2}} ds\right)^{\frac{1}{\beta}} \\ &+ c_\alpha^{\frac{1}{\beta}} \mathbb{E}\Big(\Big|\left(\int_0^t |\ \xi_s^k\ |^\beta\ ds\right)^{\frac{1}{\beta}} - \left(\int_0^t |\xi_s\ |^\beta\ ds\right)^{\frac{1}{\beta}}\Big|\Big) \end{aligned}$$

and letting k tend to zero, we prove the desired result.

2. If $\alpha < 0$, namely, $\beta > 2$, then applying the Minkovski inequality in (2.12) and using Lemma 3.1.3, we have
$\mathbb{E}(|\ S^{[0,t]}_{\beta,n}(X)^{\frac{1}{\beta}} - |\ S^{[0,t]}_{\beta,n}(X^k)^{\frac{1}{\beta}}\ |)$

$$\begin{aligned} &\leq C\Big(\sum_{i=1}^n \Big|\int_0^{t_i^n} ((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)^2 (\mathbb{E}(|\ \xi_s - \xi_s^k\ |^\beta))^{\frac{2}{\beta}} ds\Big|^{\frac{\beta}{2}}\Big)^{\frac{1}{\beta}} \\ &\leq C\left(\int_0^t (\mathbb{E}(|\ \xi_s - \xi_s^k\ |^\beta))^{\frac{\beta'}{\beta}}\right)^{\frac{1}{\beta'}}. \end{aligned}$$

Now in the same way as for the case $\alpha > 0$, we can show

$$\lim_{n\to\infty} \mathbb{E}\left(\left|S^{[0,t]}_{\beta,n}(X)^{\frac{1}{\beta}} - \left(c_\alpha \int_0^t |\xi_s\ |^\beta\ ds\right)^{\frac{1}{\beta}}\right|\right) = 0.$$

This proves the theorem. □

Remark 2.1.1 *If $\alpha > 0$ and $\int_0^t \mathbb{E}(\xi_s^2) ds < \infty$, then $\int_0^t (\mathbb{E}(\xi_s^2))^{\frac{\beta}{2}} ds < \infty$, and the β - variation of the fractional martingale $M^{(\alpha)}$ exists in L^1, and $\langle M^{(\alpha)}\rangle_{\beta,t} = c_\alpha \int_0^t |\xi_s|^\beta ds$. Using a localization argument, we can prove that this result remains true with the convergence in probability, for any continuous local martingale such that $\langle M\rangle_t = \int_0^t \xi_s^2 ds$ for all $t \geq 0$.*
On the other hand, if $\alpha < 0$ and $\langle M\rangle_t = \int_0^t \mathbb{E}(|\xi_s|^{\beta'}) ds < \infty$ for all $t \geq 0$, and for some $\beta' > \beta$, then the β-variation of the fractional martingale $M^{(\alpha)}$ exists in L^1 and

$\langle M^{(\alpha)}\rangle_{\beta,t} = c_\alpha \int_0^t |\xi_s|^\beta ds$. *As a consequence, again by a localization argument, the result remains true with the convergence in probability, for any continuous local martingale such that* $\langle M\rangle_t = \int_0^t \xi_s^2 ds$, *assuming that* $\int_0^t |\xi_s|^{\beta'} ds < \infty$ *almost surely, for all* $t \geq 0$,*and for some* $\beta' > \beta$.

Corollaire 2.1.1 *Consider a continuous local martingale* $M = (M_t, t \geq 0)$ *with* $M_0 = 0$ *and* $\langle M\rangle_t = \int_0^t \xi_s^2 ds$, *where* $\xi = (\xi_t, t \geq 0)$ *is a progressively measurable process. Suppose that* M *satisfies (2.1) for some* $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. *Then there exists* $C > 0$, *such that*

$$\liminf_{n\to\infty} \mathbb{E}(S^{[a,b]}_{\beta,n}(M^{(\alpha)})) \geq C \int_a^b \mathbb{E}(|\xi_s|^\beta) ds$$

.

Proof. For each integer $N \geq 1$ let $\psi N(x) = x$ if $|x| \leq N$ and $\psi N(x) = \frac{N}{x}$ if $|x| > N$ .Denote $M_t^{(\alpha),N} = \int_0^t (t-s)^\alpha \psi N(\xi_s) dM_s$. An application of Burkholders inequality yields

$$\begin{aligned}
\mathbb{E}(S^{[a,b]}_{\beta,n}(M^{(\alpha)})) &= \mathbb{E}\Big(\sum_{i=1}^n \Big| \int_0^{t_i^n} ((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha) dM_s\Big|^\beta\Big) \\
&\geq C\mathbb{E}\Big(\sum_{i=1}^n \Big| \int_0^{t_i^n} ((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)^2 |\xi_s|^2 \Big|^{\frac{\beta}{2}}\Big) \\
&\geq C\mathbb{E}\Big(\sum_{i=1}^n \Big| \int_0^{t_i^n} ((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)^2 (|\xi_s| \wedge N)^2 ds \Big|^{\frac{\beta}{2}}\Big) \\
&\geq C\mathbb{E}(S^{[a,b]}_{\beta,n}(M^{(\alpha),N})).
\end{aligned}$$

By Theorem (2.1.1) $S^{[a,b]}_{\beta,n}(M^{(\alpha),N})$ converges to $\int_a^b (|\xi_s| \wedge N)^\beta ds$ in L^1 as n tends to infinity. So,

$$\lim_{n\to\infty} \mathbb{E}(S^{[a,b]}_{\beta,n}(M^{(\alpha),N})) = \int_a^b (|\xi_s| \wedge N)^\beta ds$$

and, consequently,

$$\lim_{n\to\infty} \inf \mathbb{E}(S^{[a,b]}_{\beta,n}(M^{(\alpha)})) \geq C \int_a^b \mathbb{E}(|\xi_s|^\beta) ds.$$

□

So far we have considered continuous local martingales such that $\langle M_t\rangle$ is absolutely continuous with respect to the Lebesgue measure. The next result says that in the case $\alpha < 0$

if the quadratic variation of the martingale is not absolutely continuous with respect to the Lebesgue measure with positive probability, then the β-variation is infinite.

Proposition 2.1.1 *Fix $-\frac{1}{2} < \alpha < 0$. Suppose that $M = (M_t, t \geq 0)$ is a continuous local martingale, satisfying (2.1). Consider the Lebesgue decomposition of its quadratic variation given by $\langle M \rangle_t = \mu_t + \nu_t$ where μ_t and ν_t are continuous nondecreasing adapted processes such that $d\mu_t$ is absolutely continuous with respect to the Lebesgue measure, and $d\nu_t$ is singular. If $\mathbb{P}(\nu_t \neq 0) > 0$, then we have $\lim_{n\to\infty} \mathbb{E}(S^{[a,b]}_{\beta,n}(M^{(\alpha)})) = \infty$, for all $t \geq 0$*

Proof By Burkholders inequality, we have

$$\begin{aligned}
\mathbb{E}\Big(\sum_{i=1}^{n} |M^{(\alpha)}_{t_i^n} - M^{(\alpha)}_{t_{i-1}^n}|^\beta\Big) &\geq C\sum_{i=1}^{n}\mathbb{E}\Big(\int_0^{t_i^n}((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)^2 d\langle M\rangle_s\Big)^{\frac{\beta}{2}} \\
&\geq C\sum_{i=1}^{n}\mathbb{E}\Big(\int_0^{t_i^n}((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)^2 d\mu_s\Big)^{\frac{\beta}{2}} \\
&+ C\sum_{i=1}^{n}\mathbb{E}\Big(\int_0^{t_i^n}((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)^2 d\nu_s\Big)^{\frac{\beta}{2}}.
\end{aligned}$$

Then the result follows from the above inequality and Lemma 3.1.3. □

On the other hand, the next result says that in the case $\alpha \in (0, \frac{1}{4})$, the β-variation is zero if the quadratic variation of the martingale is singular.

Proposition 2.1.2 *Suppose that $M = (M_t, t \geq 0)$ is a continuous local martingale, such that almost surely the measure $d\langle M\rangle_t$ is singular with respect to the Lebesgue measure. Then, if $\alpha \in (0, \frac{1}{4})$, we have*

$$\lim_{n\to\infty} \mathbb{E}(S^{[a,b]}_{\beta,n}(M^{(\alpha)})) = 0,$$

for all $t \geq 0$.

Proof. The result is an immediate consequence of Lemma 3.1.3. □

2.2 Characterization of fractional Brownian motion

Suppose that B^H is a fractional Brownian motion with Hurst parameter $H \in (0,1)$. The process B^H admits the following representation (see [6]):

$$B_t^H = \int_0^t Z_H(t,s) dW_s \tag{2.14}$$

where

$$Z_H(t,s) = k_H \Big[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \Big] \tag{2.15}$$

with k_H defined in (2.8).

The next theorem is the main result of this chapter and provides an extension of Lévy characterization to the fractional Brownian motion.

Theorem 2.2.1 *Fix $H \in (0,1), H \neq \frac{1}{2}$. Suppose that $B = (B_t, t \geq 0)$ is a zero mean continuous stochastic process. The following two conditions are equivalent:*

1. *B is a fractional Brownian motion with Hurst parameter H.*
2. *The process B satisfies the following conditions:*

   i) *The trajectories of B are Hölder continuous of order $H - \epsilon$ for any $H - \epsilon \in (0, H)$.*

   ii) *Let*

   $$M_t = \int_0^t s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} dB_s. \tag{2.16}$$

   Then M is a local martingale. Furthermore, if $H > \frac{1}{2}$, the quadratic variation of the martingale M is absolutely continuous with respect to the Lebesgue measure almost surely.

   iii) *For any $t > 0$, the $\frac{1}{H}$-variation of B in the interval $[0,t]$ exists in L^1, and $\langle B \rangle_{\frac{1}{H},t} = c_H t$, where $c_H = \mathbb{E}(|\xi|^{\frac{1}{H}})$ and ξ is a standard normal random variable.*

Remark 2.2.1 *Notice that condition (i) is always true if $H < \frac{1}{2}$, and the Riemann-Stieltjes integral in (2.16) exists*

Proof of theorem2.2.1 From the properties of the fractional Brownian motion we know that (1) implies (2). Suppose that (2) holds. Fix $H-\epsilon \in (0,H)$, and $T>0$. We are going to show that B is a fractional Brownian motion with Hurst parameter H in the time interval $[0,T]$. Denote by $\|B\|_{H-\epsilon}$ the Hölder norm of order $H-\epsilon$ on $[0,T]$. The proof is divided into several steps.

Step 1. From (2.16), we can solve the integral equation to express B as a functional of M. This can be done as in the proof of Theorem 5.2 of [12]. In this way we obtain

$$B_t = d_H \left[t^{H-\frac{1}{2}} R_t - \left(H - \frac{1}{2} \right) Y_t \right],$$

where $d_H = B(\frac{3}{2}-H, H+\frac{1}{2})^{-1}$,

$$R_t = \int_0^t (t-s)^{H-\frac{1}{2}} dM_s,$$

and

$$Y_t = \int_0^t \left(\int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right) dM_s$$

Comparing with the representation formula (2.14) for the fractional Brownian motion, it suffices to prove that

$$d\langle M \rangle_s = (k_H d_H^{-1} s^{\frac{1}{2}-H})^2 ds, \tag{2.17}$$

because this implies that M is a Gaussian martingale, and B has the covariance of the fractional Brownian motion with Hurst parameter H. In order to show (2.17), we are going to compute the $\frac{1}{H}$-variation of R, from the decomposition

$$R_t = d_H^{-1} t^{\frac{1}{2}-H} B_t + \left(H - \frac{1}{2} \right) t^{\frac{1}{2}-H} Y_t. \tag{2.18}$$

Step 2. Fix $0<\epsilon<H \wedge \frac{1}{2} \wedge (1-H)$ and suppose that $E(\|B\|_{H-\varepsilon}^{\frac{1}{H}}) < \infty$. We will first show that the $\frac{1}{H}$-variation of the process $Z_t = t^{\frac{1}{2}-H} B_t$ exists in L^1 in any interval $[0,t] \subset [0,T]$, and

$$\langle Z \rangle_{\frac{1}{H},t} = 2H c_H t^{\frac{1}{(2H)}}. \tag{2.19}$$

An application of the triangular inequality yields

$$S^{[0,t]}_{\frac{1}{H},n}(Z) \leq \Bigg| \left(\sum_{i=1}^{n} (t_i^n)^{\frac{1}{(2H)-1}} |B_{t_i^n} - B_{t_{i-1}^n}|^{\frac{1}{H}} \right)^H + \left(\sum_{i=1}^{n} |(t_i^n)^{\frac{1}{2}-H} - (t_{i-1}^n)^{\frac{1}{2}-H}|^{\frac{1}{H}} |B_{t_{i-1}^n}|^{\frac{1}{H}} \right)^H \Bigg|^{\frac{1}{H}}, \tag{2.20}$$

and

$$S^{[0,t]}_{\frac{1}{H},n}(Z) \geq \Bigg| \left(\sum_{i=1}^{n} (t_i^n)^{\frac{1}{(2H)-1}} |B_{t_i^n} - B_{t_{i-1}^n}|^{\frac{1}{H}} \right)^H - \left(\sum_{i=1}^{n} |(t_i^n)^{\frac{1}{2}-H} - (t_{i-1}^n)^{\frac{1}{2}-H}|^{\frac{1}{H}} |B_{t_{i-1}^n}|^{\frac{1}{H}} \right)^H \Bigg|^{\frac{1}{H}}. \tag{2.21}$$

We have

$$\begin{aligned} \sum_{i=1}^{n} |(t_i^n)^{\frac{1}{2}-H} - (t_{i-1}^n)^{\frac{1}{2}-H}|^{\frac{1}{H}} |B_{t_{i-1}^n}|^{\frac{1}{H}} &\leq C \|B\|^{\frac{1}{H}}_{H-\epsilon} \left(\tfrac{t}{n}\right)^{\frac{1}{2H}-\frac{\epsilon}{H}} \textstyle\sum_{i=2}^{n} (i-1)^{\frac{-1}{(2H)}-\frac{\epsilon}{H}} \\ &\leq C \|B\|^{\frac{1}{H}}_{H-\epsilon} t^{\frac{1}{2H}-\frac{\epsilon}{H}} n^{1-\frac{1}{H}}, \end{aligned} \tag{2.22}$$

which converges in L^1 to 0 as n tends to infinity. From (2.20) to (2.22) we obtain

$$\lim_{n \longrightarrow +\infty} S^{[0,t]}_{\frac{1}{H},n}(Z) = \lim_{n \longrightarrow +\infty} \sum_{i=1}^{n} (t_i^n)^{\frac{1}{2H}-1} |B_{t_i^n} - B_{t_{i-1}^n}|^{\frac{1}{H}} \tag{2.23}$$

in L^1, provided that the limit on the right-hand side of (2.23) exists. Denote $I_j^n = [t_{j-1}^n, t_j^n]$ for $j = 1, 2, \ldots, n$. We divide every subinterval I_j^n into m parts, and we get a finer partition $0 = t_0^{nm} < \ldots < t_{nm}^{nm} = t$. Then, we have

$$\left| \sum_{i=1}^{nm} (t_i^{nm})^{\frac{1}{(2H)}-1} |B_{t_i^{nm}} - B_{t_{i-1}^{nm}}|^{\frac{1}{H}} - \sum_{j=1}^{n} c_H (t_j^n)^{\frac{1}{(2H)}-1} (t_j^n - t_{j-1}^n) \right|$$

$$
\begin{aligned}
&= \Big|\sum_{j=1}^{n}\Big(\sum_{i=(j-1)m+1}^{jm}((t_i^{nm})^{\frac{1}{2H}-1}-(t_j^n)^{\frac{1}{2H}-1})|B_{t_i^{nm}}-B_{t_{i-1}^{nm}}|^{\frac{1}{H}} \\
&\quad +(t_j^n)^{\frac{1}{2H}-1}\Big(\sum_{i=(j-1)m+1}^{jm}|B_{t_i^{nm}}-B_{t_{i-1}^{nm}}|^{\frac{1}{H}}-c_H(t_j^n-t_{j-1}^n)\Big)\Big)\Big| \\
&\le \sum_{j=1}^{n}|(t_j^n)^{\frac{1}{2H}-1}-(t_{j-1}^n)^{\frac{1}{2H}-1}|\sum_{i=(j-1)m+1}^{jm}|B_{t_i^{nm}}-B_{t_{i-1}^{nm}}|^{\frac{1}{H}} \\
&\quad +(t_j^n)^{\frac{1}{2H}-1}\Big|\sum_{i=(j-1)m+1}^{jn}|B_{t_i^{nm}}-B_{t_{i-1}^{nm}}|^{\frac{1}{H}}-c_H(t_j^n)^{\frac{1}{2H}-1})\Big|.
\end{aligned}
$$

Letting m tend to infinity and using assumption (ii), we obtain

$$
\lim_{n\longrightarrow+\infty}\sum_{i=1}^{n}(t_i^n)^{\frac{1}{(2H)}-1}|B_{t_i^n}-B_{t_{i-1}^n}|^{\frac{1}{H}}=2Hc_H t^{\frac{1}{(2H)}},
$$

in L^1, which shows (2.19).

Step 3. We claim that the $\frac{1}{H}$-variation of the process $V_t = t^{\frac{1}{2}-H}Y_t$ in L^1 is zero. The increment $|Y_t - Y_s|$ can be estimated by Lemma 3.1.3 in the Appendix with $\alpha = \frac{1}{2} - H$, f being a trajectory of the process B and $\beta = H - \epsilon$. Notice that $\alpha + \beta = \frac{1}{2} - \epsilon$, and $2\alpha + \beta = 1 - H - \epsilon$. Hence, for any $s, t \in [0, T]$, we have

$$
|Y_t - Y_s| \le C\|B\|_{H-\epsilon}(t\beta - s^{\beta}).
$$

Therefore, as in (2.20), we have

$$
\begin{aligned}
\mathbb{E}(S^{[0,t]}_{\frac{1}{H},n}(V)) &\le C\sum_{i=1}^{n}(t_i^n)^{\frac{1}{(2H)}-1}\mathbb{E}(|Y_{t_i^n}-Y_{t_{i-1}^n}|^{1/H}) \\
&\quad +C\sum_{i=1}^{n}((t_i^n)^{\frac{1}{2}-H}-(t_{i-1}^n)^{\frac{1}{2}-H})^{\frac{1}{H}}\mathbb{E}(|Y_{t_{i-1}^n}|^{\frac{1}{H}}) \\
&= A_n + B_n.
\end{aligned}
$$

For the term A_n we have

$$
\begin{aligned}
A_n &\leq C\|B\|_{H-\epsilon}^{\frac{1}{H}} \sum_{i=1}^{n} (t_i^n)^{\frac{1}{(2H)}-1}((t_i^n)^{H-\epsilon} - (t_{i-1}^n)^{H-\epsilon})^{\frac{1}{H}} \\
&= C\|B\|_{H-\epsilon}^{\frac{1}{H}} \left(\frac{t}{n}\right)^{\frac{1}{(2H)}-\frac{\epsilon}{H}} \sum_{i=1}^{n} i^{\frac{1}{(2H)}-1}(i-1)^{1-\frac{\epsilon}{H}-\frac{1}{H}} \\
&\leq C\|B\|_{H-\epsilon}^{\frac{1}{H}} \left(\frac{t}{n}\right)^{\frac{1}{(2H)}-\frac{\epsilon}{H}} n^{-\frac{1}{(2H)}-\frac{\epsilon}{H}+1}.
\end{aligned}
$$

By Lemma 3.2.3, $\underset{n\to\infty}{lim}\ \mathbb{E}(A_n) = 0$.

For the term B_n, using that

$$\mathbb{E}(|Y_{t_{i-1}^n}|_{\frac{1}{H}}) \leq CE(\|B\|_{H-\varepsilon}^{\frac{1}{H}})|t_{i-1}^n|^{1-\frac{\varepsilon}{H}}$$

, we obtain

$$
\begin{aligned}
\mathbb{E}(B_n) &\leq C\mathbb{E}(\|B\|_{H-\varepsilon}^{\frac{1}{H}}) \sum_{i=1}^{n} (t_{i-1}^n)^{-\frac{1}{2H}} - \frac{\varepsilon}{H}\left(\frac{t}{n}\right)^{\frac{1}{H}} \\
&\leq C\mathbb{E}(\|B\|_{H-\varepsilon}^{\frac{1}{H}}) \left(\tfrac{t}{n}\right)^{-1+\frac{1}{H}-\frac{\varepsilon}{H}} \to 0.
\end{aligned}
$$

Hence, $\langle Y\rangle_{\frac{1}{H},t} = 0$, in L^1 , for $t \in [0,T]$.

Step 4. From (2.18), (2.19), Step 3 and (2.7), we get that the $\frac{1}{H}$ -variation of the process R in any interval $[0,t] \subset [0,T]$ exists in L^1, and

$$\langle R\rangle_{\frac{1}{H}.t} = c_H d_H^{-\frac{1}{H}} 2Ht^{\frac{1}{(2H)}}. \tag{2.24}$$

On the other hand, since R_t is an $H - \frac{1}{2}$ martingale, Theorem (2.1.1) and Proposition (2.1.1) imply that if $H < \frac{1}{2}$, the quadratic variation $d\langle M\rangle_s$ must be absolutely continuous with respect to the Lebesgue measure, almost surely. In the case $H > \frac{1}{2}$ this is true by the assumption (ii). This implies that $\langle M\rangle_t = \int_0^t \xi_s^2 ds$. where $\xi = (\xi_t, t \geq 0)$ is a progressively measurable process.

By Corollary (2.1.1), there is a positive constant C such that, for any $t_1, t_2 \in [0,T]$,

$C\int_{t_1}^{t_2} s^{\frac{1}{(2H)}-1}ds \geq \int_{t_1}^{t_2}\mathbb{E}(|\xi_s|^{\frac{1}{H}})ds$ Then $\mathbb{E}(|\xi_s|^{\frac{1}{H}}) \leq Cs^{\frac{1}{(2H)}-1}$. Thus, we can apply Theorem (2.1.1) to obtain $\langle R\rangle_{\frac{1}{H}.t} = c_H k_H^{-\frac{1}{H}} \int_0^t |\xi_s|^{\frac{1}{H}} ds.$

Comparing this with (2.24), we obtain

$$|\xi_s| = k_H d_H^{-1} s^{\frac{1}{2}-H}, \quad 0 \leq s \leq t,$$

and (2.17) holds. This proves that B is a fractional Brownian motion with Hurst parameter H under the condition $\mathbb{E}(\|B\|_{H-\varepsilon}^{\frac{1}{H}} < \infty)$.

Step 5. If $\mathbb{E}(\|B\|_{H-\varepsilon}^{\frac{1}{H}})$ is not necessarily finite, we can use a localization argument. Denote

$$T_k = inf\{t \geq 0 : \|B\|_{t,H-\varepsilon} \geq K\} \wedge T.$$

and $B_t^K = B_{t\wedge T_k}$. Since $\sum_{i=1}^{n} |B_{t_i^n}^K - B_{t_{i-1}^n}^K|^{\frac{1}{H}} \leq \sum_{i=1}^{n} |B_{t_i^n}^K - B_{t_{i-1}^n}^K|^{\frac{1}{H}} + (K\frac{t}{n})^{\frac{1}{H}}$, by the dominated convergence theorem, we can also get

$$\lim_n \mathbb{E}\left(\left|\sum_{i=1}^{n} |B_{t_i^n}^K - B_{t_{i-1}^n}^K|^{\frac{1}{H}} - c_H(t \wedge T_k)\right|\right) = 0.$$

By modifying the proof in Steps 1-4 slightly, we get

$$|\xi_s| = k_H d_H^{-1} s^{\frac{1}{2}-H}, \quad 0 \leq s \leq t \wedge T_K.$$

Clearly, $\lim_{K\to\infty} T_K = T$, and then

$$|\xi_s| = k_H d_H^{-1} s^{\frac{1}{2}-H}, \quad 0 \leq s \leq T.$$

□

Remark 2.2.2 *Notice that in the case $H > \frac{1}{2}$ we have imposed the additional assumption that the martingale (2.16) has an absolutely continuous quadratic variation. This is true, for instance, if the filtration generated by the process B is included in the filtration generated by a Brownian motion.*

The next proposition shows that this condition is necessary at least in the case $H \in (\frac{1}{2}, \frac{3}{4})$.

Proposition 2.2.1 *Suppose that $H \in (\frac{1}{2}, \frac{3}{4})$. There exists a process B , satisfying conditions (i) and (iii) of Theorem 2.2.1, such that the process M defined in (2.16) is a local martingale, and B is not a fractional Brownian motion.*

Proof. Let B^H be a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, \frac{3}{4})$. Define

$$M_t = \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} dB_s^H.$$

Let $N_t = W_{\phi(t)}$, where W is a Brownian motion independent of B^H, and ϕ is a strictly increasing, Holder continuous function of exponent γ for any $\gamma < 1$, null at zero, such that the measure $d\phi(t)$ is singular with respect to the Lebesgue measure (for the existence of such function, see Lemma 3.3.1 in the Appendix). Set

$$\widetilde{M_t} = M_t + N_t \quad \textit{and} \quad \widetilde{B}_t^H = B_t^H + Y_t.$$

where

$$Y_t = d_H \left(t^{H-\frac{1}{2}} \int_0^t (t-s)^{H-\frac{1}{2}} dN_s \quad - \left(H - \frac{1}{2}\right) \int_0^t \left(\int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right) dN_s \right).$$

The process $\widetilde{B}^H$ clearly satisfies (i) and it is not a fractional Brownian motion. Finally, $\langle \widetilde{B}^H \rangle_{\frac{1}{H},t} = c_H t$ in L^1, because the $\frac{1}{H}$-variation of $\int_0^t (t-s)^{H-\frac{1}{2}} dN_s$ is zero by Proposition 2.1.2, and, by the same arguments as in the proof of Theorem 2.2.1, we can show that the $\frac{1}{H}$-variation of Y vanishes. □

Chapter 3

Appendice

3.1 Some technical lemmas.

Lemma 3.1.1 *Let $\alpha \in (0, \frac{1}{2})$. Fix an interval $[0, t]$. For any natural number m, we define $t_i^m = \frac{i}{m}t, 0 \le i \le m$. Let g be a measurable function on $[0, \infty)$ such that, for all $t \ge 0, \int_0^t |g(s)|ds < \infty$. Then there exists a function $C(t) > 0$ satisfying*

$$\limsup_{m\to\infty} \sum_{i=1}^m \Big(\int_0^{t_i^m} ((t_i^m - s)^\alpha - (t_{i-1}^m - s)_+^\alpha)^2 |g(s)| ds \Big)^{\frac{\beta}{2}} \le C(t) \int_0^t |g(s)|^{\frac{\beta}{2}} ds.$$

Lemma 3.1.2 *Let $\alpha \in (-\frac{1}{2}, 0)$. Fix an interval $[0, t]$. For any natural number m, we define $t_i^m = \frac{i}{m}t, 0 \le i \le m$. Let g be a measurable function on $[0, \infty)$ such that, for all $t \ge 0, \int_0^t |g(s)|^{\frac{\beta'}{2}} ds < \infty$. for some $\beta' > \beta$. Then there exists a constant C depending on t such that*

$$\sum_{i=1}^m \Big(\int_0^{t_i^m} ((t_i^m - s)^\alpha - (t_{i-1}^m - s)_+^\alpha)^2 |g(s)| ds \Big)^{\frac{\beta}{2}} \le C \Big(\int_0^t |g(s)|^{\frac{\beta'}{2}} ds \Big)^{\frac{\beta'}{\beta}}.$$

Lemma 3.1.3 *Suppose that v is a measure on an interval $[0, t]$, which is singular with respect to the Lebesgue measure. We have the following conditions:*

(i) If $\alpha \in (-\frac{1}{2}, 0)$, then

$$\lim_{n\to\infty} \sum_{i=1}^n \Big(\int_0^{t_i^n} ((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)^2 dv_s \Big)^{\frac{\beta}{2}} = \infty.$$

(ii) If $\alpha \in (0, \frac{1}{4})$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Big(\int_0^{t_i^n} ((t_i^n - s)^\alpha - (t_{i-1}^n - s)_+^\alpha)^2 dv_s \Big)^{\frac{\beta}{2}} = 0.$$

3.2 Transformations of Hölder continuous functions.

Let $\beta \in (0,1]$. We denote by $C^\beta \in ([0,T])$ the set of Hölder continuous functions on $[0,T]$. For any function f in $C^\beta \in ([0,T])$ and any $0 < a < b \leq T$, we will write

$$\|f\|_\beta, a, b = \sup_{a \leq s < t \leq b} \frac{|f(t) - f(s)|}{|t-s|^\beta}. \tag{3.1}$$

We also set $\|f\|_\beta = \|f\|_{\beta,0,T}$.

Lemma 3.2.1 *Suppose that $f \in C^\beta \in ([0,T])$, and assume that $0 \leq a < b < v \leq T$. Let, $\gamma \geq 0$ and $\alpha + \beta \neq 0$. Then*

$$\Big| \int_a^b s^\gamma (v-s)^\alpha df(s) \Big| \leq \|f\|_\beta \Big(2 + \Big| \frac{\alpha}{\alpha+\beta} \Big| \Big) b^\gamma ((v-b)^{\alpha+\beta} + (v-a)^{\alpha+\beta}).$$

Lemma 3.2.2 *Suppose that $f \in C^\beta \in ([0,T])$, and suppose $\alpha < 0, \alpha + \beta > 0$. Let $g(t) = \int_0^t s^\alpha df(s)$. Then, $g \in C^{\alpha+\beta} \in ([0,T])$, and*

$$\|g\|_{\alpha+\beta} \leq \frac{\beta}{\alpha+\beta} \|f\|_\beta.$$

Proposition 3.2.1 *Fix $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ and $\beta \in (0,1]$ such that $0 < \alpha + \beta \leq 1$. Suppose that $f \in C^\beta \in ([0,T])$, and let $g(t) = \int_0^t s^\alpha (t-s)^\alpha df(s)$. Then:*

1. *If $\alpha > 0, g \in C^{\alpha+\beta} \in ([0,T])$ and for any $0 \leq a < b \leq T$, we have*

$$|g(b) - g(a)| \leq C \|f\|_\beta b^\alpha (b-a)^{\alpha+\beta}. \tag{3.2}$$

2. *If $\alpha < 0$ and $0 < 2\alpha + \beta \leq 1$, then $g \in C^{2\alpha+\beta} \in ([0,T])$ and*

$$|g(b) - g(a)| \leq C \|f\|_\beta (b-a)^{2\alpha+\beta}. \tag{3.3}$$

Lemma 3.2.3 *fix $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ and $\beta \in (0,1]$ such that $0 < \alpha + \beta \leq 1$. Suppose that $f \in C^\beta \in ([0,T])$, and let $g(t) = \int_0^t s^\alpha (t-s)^\alpha df_s$. Set:*

$$h(t) = \int_0^t u^{-\alpha-1}\Big(\int_0^u (u-s)^{-\alpha} dg_s\Big) du.$$

Then for any $0 \leq a < b \leq T$, we have

$$|h(b) - h(a)| \leq C\|f\|_\beta (b^\beta - a^\beta).$$

3.3 Existence of singular Hölder continuous distribution functions.

Let $0 < H < 1$ and $\rho > 1$. Suppose that $X = (X_t, t \geq 0)$ is a zero mean Gaussian process with stationary increments and a variance $\sigma^2(t) = \mathbb{E}(X_t^2)$ given by

$$\sigma^2(t) = \int_0^\infty (1 - \cos(xt)) g(x) dx, \tag{3.4}$$

where $g(x) = x^{-2H-1}\mathbb{I}_{[0,2)}(x) + (|\log x|^\rho x)^{-1}\mathbb{I}_{[2,\infty)}(x)$. If we replace $g(x)$ by $gH(x) = x^{-2H-1}$ in equation (3.4) then the process X is a fractional Brownian motion with Hurst parameter H. Taking into account that $g(x) \geq CgH(x)$ for some constant $C > 0$, it follows that the process X satisfies the local nondeterminism property in some interval $(0, d)$ (see Theorem 4.1 in [3]). The following lemma implies the existence of finite measures on the real line which are singular with respect to the Lebesgue measure, and whose distribution function is Hölder continuous of order γ , for any $\gamma < 1$ on any finite interval.

Lemma 3.3.1 *Let X be the Gaussian process introduced above. Then, there exists a version of its local time $L(t,x)$, jointly continuous in t and x, with the following properties:*

(i) For each $x \in \mathbb{R}$ and $\gamma < 1, L(t,x)$ is Hölder continuous of order γ with respect to t , On any finite interval.

(ii) $L(t,x)$ is a nondecreasing function of t .

(iii) For each $x \in \mathbb{R}$ the support of the measure $L(dt,x)$ is the set $S, X_S = x$, which has a Lebesgue measure 0.

Conclusion

This manuscript has investigated the fractional martingales and characterization of the fractional Brownian motion. Our future goal is to study an possible extension of lévy characterization to the Two parameter fractional Brownian motion defined in the first chapter. We think that it's possible to employ the obtained result in the study of some class of stochastic differential equations

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