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Mémoire de MASTER

The Stochastic Integral in General Hilbert Spaces (w.r.t. Brownian Motion)

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To my dear parents who have supported me throughout my studies,
To my brothers and sisters,
To my all family,
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Contents

Introduction

Stochastic calculus is a branch of mathematics that operates on stochastic processes. It allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to stochastic processes, the subject of stochastic integral in infinite dimensional space and stochastic differential equations has grown in importance since the publication of K. Itô's CBMS monograph, J. Walsh's article in the École d'Étè de probabilités de Saint Flow, and two recent books: by B. L. Rorovskii in 1990 and by G. Da Prato and J. Zabczyk in 1992 [8]. First results on infinite dimensional Ito's equations started to appear in the mid 1960s and were motivated by development of analysis and theory of stochastic processes on one side, and by a need to describe random phenomena studied in natural sciences like physics, chemistry, biology, and in control theory, on the other side.

In this working we are interested essentially to the stochastic integral for Infinite dimensional Wiener processes, the best-known stochastic process to which stochastic calculus is applied is the Wiener process (named in honor of Norbert Wiener), which is used for modeling Brownian motion as described by Louis Bachelier in 1900 and by Albert Einstein in 1905 and other physical diffusion processes in space of particles subject to random forces. Since the 1970s, the Wiener process has been widely applied in financial mathematics and economics to model the evolution in time of stock prices and bond interest rates.

This senior thesis is orgnized as follows. Firstly, we will give definitions and properties of the Wiener process in infinite dimensions, mainly Hilbert and Banach spaces. To follow it of chapter we will give the construction and properties of the stochastic integral w.r.t Q-Wiener processes. This concept is important for the stochastic evolution equations on infinite dimensional spaces, that will given shortly in the next chapter.

In the next chapter, we introduce first various concepts of solution so certain type of stochastic evolution problems and discuss their elementary properties. Then we prove existence and uniqueness of weak solutions. At the end we give sufficient conditions for existence of strong solutions.

Symbols

$L(U,H)$	Space of all bounded and linear operators from U to H
$L(U)$	$L(U,U)$
$L_1(U,H)$	Space of all nuclear operators from U to H
tr Q	Trace of Q,
$L_2(U,H)$	Space of all Hilbert-Schmidt operators from U to H
$\\| \\ \\|_{L_2}$	Hilbert-Schmidt norm
$L^p(\Omega,\mathcal{F},\mu;X)$	Set of all with respect to μ p-integrable mappings from Ω to X
$L^p(\Omega,\mathcal{F},\mu)$	$L^p(\Omega,\mathcal{F},\mu;\mathbb{R})$
$\\| \\ \\|_T$	L^2-norm on $L^2(\Omega_T,\mathcal{P}_T,\mathbb{P}_T;L_2^0)$
L_0^p	$L^p(\Omega,\mathcal{F}_0,\mathbb{P};H)$
$\mathcal{N}(m,Q)$	Gaussian measure with mean m and covariance Q
$W(t), t\in[0,T]$	Standard Wiener process
$\mathcal{M}_T^2(U)$	Space of all continuous U-valued, square integrable martingales
ξ	Class of all $L(U,H)$-valued elementary processes
Ω_T	$[0,T]\times\Omega$
dx	Lebesgue measure
$\mathbb{P}_T$	$\mathrm{d}x_{\|[0,T]}\otimes\mathbb{P}$
$\mathcal{P}_T$	Predictable σ-field on Ω_T
$\int\Phi(s)dW(s)$	Stochastic integral w.r.t. W
$\mathcal{N}_W^2(0,T;H)$	$L^2(\Omega_T,\mathcal{P}_T,\mathbb{P}_T;L_2^0)$
$\mathcal{N}_W^2(0,T)$	$\mathcal{N}_W^2(0,T;H)$
$\mathcal{N}_W^2$	$\mathcal{N}_W^2(0,T;H)$
$\mathcal{N}_W(0,T;H)$	Space of all stochastically integrable processes
$\mathcal{N}_W(0,T)$	$\mathcal{N}_W^2(0,T;H)$
A^*	Adjoint operator of $A\in L(U,H)$
$Q^{\frac{1}{2}}$	Square root of $Q\in L(U)$
T^{-1}	(Pseudo) inverse of $T\in L(U,H)$
U_0	$Q^{\frac{1}{2}}(U)$,
L_2^0	$L_2(U_0,H)$

$\langle u, v\rangle_0$	$\langle Q^{\frac{1}{2}}u, Q^{\frac{1}{2}}v\rangle_U$
$L(U,H)_0$	$\{T_{U_0} \mid T \in L(U,H)\}$
$D(A)$	The infinitesimal generator A of $S(.)$
$S(.)$	Semigroup in H

Chapter 1

The stochastic integral for Infinite-dimensional Wiener processes

We fix two separable Hilbert spaces $(U, \langle\ .\ \rangle_U)$ and $(H, \langle\ .\ \rangle_H)$. The chapter is dedicated to the construction of the stochastic integral

$$\int_0^t \Phi(s)\mathrm{d}W(s), t \in [0,T],$$

where $W(t), t \in [0,T]$, is a Wiener process on U and Φ is a process with values that are linear but not necessarily bounded operators from U to H.

For that we first will have to introduce the notion of the standard Wiener process in infinite dimensions. This concept is important for the construction of the stochastic integral that will be explained in to follow it of chapter.

1.1 Infinite-dimensional Wiener processes

1.1.1 Linear Operators

Let $(U, \|\ \|)$ be a Banach space, $\mathcal{B}(U)$ the Borel σ-field of U and $(\Omega, \mathcal{F}, \mu)$ a measure space with finite measure μ.

Proposition 1.1.1.1. *Let $f \in L_1(\Omega, \mathcal{F}, \mu; U)$. Then*

$$\int L \circ f \, d\mu = L(\int f \, d\mu)$$

holds for all $L \in L(U,H)$, where H is another Banach space.

Proof. see [5], Proposition E.11, p. 356. □

Proposition 1.1.1.2. *Let $(\Omega, \mathcal{F})$ be a measurable space and let U be a Banach space. Then:*

1. *The set of $\mathcal{F}/\mathcal{B}(U)$-measurable functions from Ω to U is closed under the formation of pointwise limits, and*

2. *The set of strongly measurable functions from Ω to U is closed under the formation of pointwise limits.*

Proof. see [5], Proposition E.1, p. 350. □

Proposition 1.1.1.3. *Let E be a metric space with metric d and let $f : \Omega \longrightarrow E$ be strongly measurable. Then there exists a sequence $f_n, n \in \mathbb{N}$, of simple Evalued functions (i.e. f_n is $\mathcal{F}/\mathcal{B}(U)$-measurable and takes only a finite number of values) such that for arbitrary $\omega \in \Omega$ the sequence $d(f_n(\omega), f(\omega)), n \in \mathbb{N}$, is monotonely decreasing to zero.*

Proof. see [8], Lemma 1.1, p. 16. □

1.1.2 Gaussian random variable

Definition 1.1.2.1. *A probability measure μ on $(U, \mathcal{B}(U))$ is called Gaussian if for all bounded linear mapping*

$$\begin{aligned} \upsilon' \ : \ & U \longrightarrow \mathbb{R} \\ & \mu \longmapsto \langle u, \upsilon \rangle_U, \quad u \in U \end{aligned}$$

have a Gaussian law, i.e. for all $\upsilon \in U$ there exist $m = m(\upsilon) \in \mathbb{R}$ and $\sigma = \sigma(\upsilon) > 0$ such that

$$(\mu \circ (\upsilon')^{-1})(A) = \mu \circ (\upsilon' \in A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{(x-m)^2}{2\sigma^2}} dx \ for \ all \ A \in \mathcal{B}(\mathbb{R}),$$

or

$$\mu = \delta_u \ for \ one \ u \in U \ where \ \delta_u \ is \ the \ Dirac \ measur \ in \ u.$$

Theorem 1.1.2.1. *A measure μ on $(U, \mathcal{B}(U))$ is Gaussian if and only if*

$$\hat{\mu}(u) := \int_U e^{i\langle u,v\rangle_U} \mu(dv) = e^{i\langle m,u\rangle_U - \frac{1}{2}\langle Qu,u\rangle_U}, \ u \in U.$$

where $m \in U$ and $Q \in L(U)$ is nonnegative, symmetric, with finite trace. In this case μ will be denoted by $\mathcal{N}(m, Q)$ where m is called mean and Q is called covariance (operator). The measure μ is uniquely determined by m and Q.

Furthermore, for all $h, g \in U$

$$\int \langle x, h\rangle_U \mu(dx) = \langle m, h\rangle_U,$$

$$\int (\langle x, h\rangle_U - \langle m, h\rangle_U)(\langle x, g\rangle_U - \langle m, g\rangle_U)\mu(dx) = \langle Qh, g\rangle_U,$$

$$\int \|x - m\|_U^2 \mu(dx) = trQ.$$

Proof. (cf. [8]) Obviously, a probability measure with this Fourier transform is Gaussian. Now let us conversely assume that μ is Gaussian. We need the following:

Lemma 1.1.2.1. *Let ν be a probability measure on $(U, \mathcal{B}(U))$. Let $k \in \mathbb{N}$ be such that*

$$\int_U |\langle z, x\rangle_U|^k \nu(dx) < \infty \quad \forall z \in U.$$

Then there exists a constant $C = C(k, \nu) > 0$ such that for al $h_1, \ldots, h_k \in U$

$$\int_U |\langle h_1, x\rangle_U \ldots \langle h_k, x\rangle_U| \nu(dx) \leq C\|h_1\|_U \ldots \|h_K\|_U.$$

In particular, the symmetric k-linear form

$$U^k \ni (h_1, \ldots, h_k) \longmapsto \int \langle h_1, x\rangle_U \ldots \langle h_k, x\rangle_U \nu(dx) \in \mathbb{R}$$

is continuous.

Proof. For $n \in \mathbb{N}$ define

$$U_n = \left\{ z \in U \,\middle|\, \int_U |\langle z, x\rangle|^k \nu(\mathrm{d}x) \leq n \right\}.$$

By assumption

$$U = \bigcup_{n=1}^{\infty} U_n.$$

Since U is a complete metric space, by the Baire category theorem, there exists $n_0 \in \mathbb{N}$ such that U_{n_0} has non-empty interior, so there exists a ball (with centre z_0 and radius r_0) $B(z_0, r_0) \subset U_{n_0}$. Hence

$$\int_U |\langle z_0 + y, x\rangle_U|^k \nu(dx) \leq n_0 \quad \forall y \in B(0, r_0),$$

therefore for all $y \in B(0, r_0)$

$$\begin{aligned}
&\int_U |\langle y, x\rangle_U|^k \nu(\mathrm{d}x) = \int_U |\langle z_0 + y, x\rangle_U - \langle z_0, x\rangle_U|^k \nu(\mathrm{d}x) \\
&\leq 2^{k-1} \int_U |\langle z_0 + y, x\rangle_U|^k \nu(\mathrm{d}x) + 2^{k-1} \int_U |\langle z_0, x\rangle_U|^k \nu(\mathrm{d}x) \\
&\leq 2^k n_0.
\end{aligned}$$

Applying this for $y := r_0 z$, $z \in U$ with $|z|_U = 1$, we obtain

$$\int_U |\langle z, x\rangle_U|^k \nu(\mathrm{d}x) \leq 2^k n_0 r_0^{-k}.$$

Hence, if $h_1 \ldots h_k \in U\backslash\{0\}$, then by the generalized Hölder inequality

$$\begin{aligned}
&\int_U \left| \left\langle \frac{h_1}{|h_1|_U}, x \right\rangle_U \cdots \left\langle \frac{h_k}{|h_k|_U}, x \right\rangle_U \right|^k \nu(\mathrm{d}x) \\
&\leq \left(\int_U \left| \left\langle \frac{h_1}{|h_1|_U}, x \right\rangle_U \right|^k \nu(\mathrm{d}x) \right)^{\frac{1}{k}} \cdots \left(\int_U \left| \left\langle \frac{h_k}{|h_k|_U}, x \right\rangle_U \right|^k \nu(\mathrm{d}x) \right)^{\frac{1}{k}} \\
&\leq 2^k n_0 r_0^{-k}.
\end{aligned}$$

and the assertion follows. $\square$

Applying Lemma 1.1.2.1 for $k = 1$ and $\nu := \mu$ we obtain that

$$U \ni h \longmapsto \int \langle h, x\rangle_U \mu(\mathrm{d}x) \in \mathbb{R}$$

is a continuous linear map, hence there exists $m \in U$ such that

$$\int_U \langle x, h\rangle_U \mu(\mathrm{d}x) = \langle m, h\rangle \quad \forall h \in H.$$

Applying Lemma 1.1.2.1 for $k = 2$ and $\nu := \mu$ we obtain that

$$U^2 \ni (h_1, h_2) \longmapsto \int \langle x, h_1\rangle_U \langle x, h_2\rangle_U \mu(\mathrm{d}x) - \langle m, h_1\rangle_U \langle m, h_2\rangle_U$$

is a continuous symmetric bilinear map, hence there exists a symmetric $Q \in L(U)$ such that this map is equal to

$$U^2 \ni (h_1, h_2) \longmapsto \langle Qh_1, h_2\rangle_U.$$

Since for all $h \in U$

$$\langle Qh, h\rangle_U = \int \langle x, h\rangle_U^2 \mu(\mathrm{d}x) - \left(\int \langle x, h\rangle_U \mu(\mathrm{d}x)\right)^2 \geq 0,$$

Q is positive definite. It remains to prove that Q is trace class (i.e.

$$\mathrm{tr}Q := \sum_{i=1}^{\infty} \langle Qe_i, e_i\rangle_U < \infty$$

for one (hence every) orthonormal basis $\{e_i | i \in \mathbb{N}\}$ of U).

We may assume without loss of generality that μ has mean zero, i.e. $m = 0$, since the image measure of μ under the translation $U \ni x \longmapsto x - m$ is again Gaussian with mean zero and the same covariance Q. Then we have for all $h \in U$ and all $c \in (0, \infty)$

$$\begin{aligned} 1 - e^{-\frac{1}{2}\langle Qh,h\rangle_U} &= \int_U (1 - \cos\langle h, x\rangle_U)\mu(\mathrm{d}x) \\ &\leq \int_{\{|x|_U \leq c\}} (1 - \cos\langle h, x\rangle_U)\mu(\mathrm{d}x) + 2\mu(\{x \in U|\ |x|_U > c\}) \\ &\leq \frac{1}{2}\int_{\{|x|_U \leq c\}} |\langle h, x\rangle_U|^2 \mu(\mathrm{d}x) + 2\mu(\{x \in U|\ |x|_U > c\}) \qquad (1.1) \end{aligned}$$

(since $1 - \cos x \leq \frac{1}{2}x^2$). Defining the positive definite symmetric linear operator Q_c on U by

$$\langle Q_c h_1, h_2\rangle_U := \int_{\{|x|_U \leq c\}} \langle h_1, x\rangle_U . \langle h_2, x\rangle_U \mu(\mathrm{d}x), \quad h_1, h_2 \in U,$$

we even have that Q_c is trace class because for every orthonormal basis $\{e_k \mid k \in \mathbb{N}\}$ of U we have (by monotone convergence)

$$\sum_{k=1}^{\infty} \langle Q_c e_k, e_k\rangle_U = \int_{\{|x|_U \leq c\}} \sum_{k=1}^{\infty} \langle e_k, x\rangle_U^2 \mu(\mathrm{d}x) = \int_{\{|x|_U \leq c\}} |x|_U^2 \mu(\mathrm{d}x)$$

$$\leq c^2 < \infty.$$

Claim: There exists $c_0 \in (0, \infty)$ (large enough) so that $Q \leq 2\log 4 Q_{c_0}$ (meaning that $\langle Qh, h\rangle_U \leq 2\log 4 \langle Q_{c_0}h, h\rangle_U$ for all $h \in U$).

To prove the claim let c_0 be so big that $\mu(\{x \in U \mid |x|_U > c_0\}) \leq \frac{1}{8}$. Let $h \in U$ such that $\langle Q_{c_0}h, h\rangle_U \leq 1$. Then (1.1) implies

$$1 - e^{-\frac{1}{2}\langle Qh,h\rangle_U} \leq \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$

hence $4 \geq e^{\frac{1}{2}\langle Qh,h\rangle_U}$, so $\langle Qh, h\rangle_U \leq 2\log 4$. If $h \in U$ is arbitrary, but $\langle Q_{c_0}h, h\rangle_U \neq 0$, then we apply what we have just proved to $h/\langle Q_{c_0}h, h\rangle_U^{\frac{1}{2}}$ and the claim follows for such h. If, however, $\langle Q_{c_0}h, h\rangle_U = 0$, then for all $n \in \mathbb{N}$, $\langle Q_{c_0}nh, nh\rangle_U = 0 \leq 1$, hence by the above $\langle Qh, h\rangle_U \leq n^{-2} 2\log 4$. Therefore, $\langle Q_{c_0}h, h\rangle_U = 0$ and the claim is proved, also for such h.

Since Q_{c_0} has finite trace, so has Q by the claim and the theorem is proved, since the uniqueness part follows from the fact that the Fourier transform is one-to-one. □

The following result is then obvious.

Proposition 1.1.2.1. *Let X be a U-valued Gaussian random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. there exist $m \in U$ and $Q \in L(U)$ nonnegative, symmetric, with finite trace such that $\mathbb{P} \circ X^{-1} = \mathcal{N}(m, Q)$.*

Then $\langle X, u\rangle_U$ is normally distributed for all $u \in U$ and the following statements hold:

- $\mathbb{E}(\langle X, u\rangle_U) = \langle m, u\rangle_U$ *for all* $u \in U$,
- $\mathbb{E}(\langle X - m, u\rangle_U . \langle X - m, v\rangle_U) = \langle Qu, v\rangle_U$ *for all* $v, u \in U$,
- $\mathbb{E}(\|X - m\|_U^2) = trQ$.

The following proposition will lead to a representation of a U-valued Gaussian random variable in terms of real-valued Gaussian random variables.

Proposition 1.1.2.2. *If $Q \in L(U)$ is nonnegative, symmetric, with finite trace then there exists an orthonormal basis $e_k, k \in \mathbb{N}$, of U such that*

$$Qe_k = \lambda_k e_k, \quad \lambda_k \geq 0, \;\; k \in \mathbb{N}.$$

and 0 is the only accumulation point of the sequence $(\lambda_k)_{k\in\mathbb{N}}$.

Proof. See [13], Theorem VI.21; Theorem VI.16 (Hilbert-Schmidt theorem). □

Proposition 1.1.2.3. ***(Representation of a Gaussian random variable)*** *Let $m \in U$ and $Q \in L(U)$ be nonnegative, symmetric, with $trQ < \infty$. In addition, we assume that $e_k, k \in \mathbb{N}$, is an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $\lambda_k, k \in \mathbb{N}$, as in Proposition 1.1.2.2, numbered in decreasing order.*

Then a U-valued random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is Gaussian with $\mathbb{P} \circ X^{-1} = \mathcal{N}(m, Q)$ if and only if

$$X = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k e_k + m \quad (\textit{as objects in } L^2(\Omega, \mathcal{F}, \mathbb{P}; U)).$$

where $\beta_k, k \in \mathbb{N}$, are independent real-valued random variables $\mathbb{P} \circ \beta_k^{-1} = \mathcal{N}(0,1)$ for all $k \in \mathbb{N}$ with $\lambda_k > 0$. The series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$.

Proof.

1. Let X be a Gaussian random variable with mean m and covariance Q. Below we set $\langle\ ,\ \rangle := \langle\ ,\ \rangle_U$.
   Then $X = \sum_{k \in \mathbb{N}} \langle X, e_k \rangle_{e_k}$ in U where $\langle X, e_k \rangle$ is normally distributed with mean $\langle m, e_k \rangle$ and variance $\lambda_k, k \in \mathbb{N}$, by Proposition 1.1.2.1. If we now define

$$\beta_k : \begin{cases} = & \frac{\langle X, e_k \rangle - \langle m, e_k \rangle}{\sqrt{\lambda_k}} & if & k \in \mathbb{N} \textit{ with } \lambda_k > 0, \\ \equiv & 0 \in \mathbb{R} & else, & \end{cases}$$

then we get that $\mathbb{P} \circ \beta_k^{-1} = \mathcal{N}(0,1)$ and $X = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k e_k + m$. To prove the independence of $\beta_k, k \in \mathbb{N}$, we take an arbitrary $n \in \mathbb{N}$ and $\alpha_K \in \mathbb{R}$, $1 \le k \le n$, and obtain that for $c := -\sum_{k=1, \lambda_k \neq 0}^{n} \frac{\alpha_k}{\sqrt{\lambda_k}} \langle m, e_k \rangle$

$$\sum_{k=1}^{n} \alpha_k \beta_k = \sum_{k=1, \lambda_k \neq 0}^{n} \frac{\alpha_k}{\sqrt{\lambda_k}} \langle X, e_k \rangle + c = \left\langle X, \sum_{k=1, \lambda_k \neq 0}^{n} \frac{\alpha_k}{\sqrt{\lambda_k}} e_k \right\rangle + c$$

which is normally distributed since X is a Gaussian random variable. Therefore we have that $\beta_k, k \in \mathbb{N}$, form a Gaussian family. Hence, to get the independence, we only have to check that the covariance of β_i and β_j, $i, j \in \mathbb{N}, i \neq j$, with $\lambda_i \neq 0 \neq \lambda_j$, is equal to zero. But this is clear since

$$\begin{aligned}\mathbb{E}(\beta_i\beta_j) &= \tfrac{1}{\sqrt{\lambda_i\lambda_j}}\mathbb{E}(\langle X-m, e_i\rangle\langle X-m, e_j\rangle) = \tfrac{1}{\sqrt{\lambda_i\lambda_j}}\langle Qe_i, e_j\rangle \\ &= \tfrac{\lambda_i}{\sqrt{\lambda_i\lambda_j}}\langle e_i, e_j\rangle = 0\end{aligned}$$

for $i \neq j$.

Besides, the series $\sum_{k=1}^{n}\sqrt{\lambda_k}\beta_k e_k, n \in \mathbb{N}$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$ since the space is complete and

$$\mathbb{E}\left(\left\|\sum_{k=m}^{n}\sqrt{\lambda_k}\beta_k e_k\right\|^2\right) = \sum_{k=m}^{n}\lambda_k\mathbb{E}(|\beta_k|^2) = \sum_{k=m}^{n}\lambda_k.$$

Since $\sum_{n\in\mathbb{N}}\lambda_k = \mathrm{tr}Q < \infty$ this expression becomes arbitrarily small for m and n large enough.

2. Let $e_k, k \in \mathbb{N}$, be an orthonormal basis of U such that $Qe_k = \lambda_k e_k, k \in \mathbb{N}$ and let $\beta_k, k \in \mathbb{N}$ be a family of independent real-valued Gaussian random variables with mean 0 and variance 1. Then it is clear that the series $\sum_{k=1}^{n}\sqrt{\lambda_k}\beta_k e_k + m, n \in \mathbb{N}$, converges to $X := \sum_{k\in\mathbb{N}}\sqrt{\lambda_k}\beta_k e_k + m.$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$. Now we fix $u \in U$ and get that

$$\left\langle \sum_{k=1}^{n}\sqrt{\lambda_k}\beta_k e_k + m, u\right\rangle = \sum_{k=1}^{n}\sqrt{\lambda_k}\beta_k\langle e_k, u\rangle + \langle m, u\rangle$$

is normally distributed for all $n \in N$ and the sequence converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. This implies that the limit $\langle X, u\rangle$ is also normally distributed where

$$\begin{aligned}\mathbb{E}(\langle X, u\rangle) &= \mathbb{E}\left(\sum_{k\in\mathbb{N}}\sqrt{\lambda_k}\beta_k\langle e_k, u\rangle + \langle m, u\rangle\right) \\ &= \lim_{n\to\infty}\mathbb{E}\left(\sum_{k=1}^{n}\sqrt{\lambda_k}\beta_k\langle e_k, u\rangle\right) + \langle m, u\rangle = \langle m, u\rangle\end{aligned}$$

and concerning the covariance we obtain that

$$\begin{aligned}
&\mathbb{E}\Big((\langle X,u\rangle-\langle m,u\rangle)(\langle X,v\rangle-\langle m,v\rangle)\Big)\\
&\quad= \lim_{n\to\infty}\mathbb{E}\left(\sum_{k=1}^{n}\sqrt{\lambda_k}\beta_k\langle e_k,u\rangle\sum_{k=1}^{n}\sqrt{\lambda_k}\beta_k\langle e_k,v\rangle\right)\\
&\quad= \sum_{k\in\mathbb{N}}\lambda_k\langle e_k,u\rangle\langle e_k,v\rangle \;=\; \sum_{k\in\mathbb{N}}\langle Qe_k,u\rangle\langle e_k,v\rangle\\
&\quad= \sum_{k\in\mathbb{N}}\langle e_k,Qu\rangle\langle e_k,v\rangle \;=\; \langle Qu,v\rangle
\end{aligned}$$

for all $u,v\in U$. □

1.1.3 The definition of the standard Q-Wiener process

After these preparations we will give the definition of the standard Q-Wiener process. To this end we fix an element $Q\in L(U)$, nonnegative, symmetric and with finite trace and a positive real number T.

Definition 1.1.3.1. *A U-valued stochastic process $W(t), t\in[0,T]$, on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ is called normal (standard) Q-Wiener process if:*

- $W(0)=0$
- *W has $\mathbb{P}$-a.s. continuous trajectories,*
- *the increments of W are independent, i.e. the random variables*

  $$W(t_1), W(t_2)-W(t_1),\cdots,W(t_n)-W(t_{n-1})$$

  are independent for all $0\le t_1<\ldots<t_n\le T, n\in\mathbb{N}$,
- *the increments have the following Gaussian laws:*

  $$\mathbb{P}\circ(W(t)-W(s))^{-1}=\mathcal{N}(0,(t-s)Q) \text{ for all } 0\le s\le t\le T.$$

Proposition 1.1.3.1. *For arbitrary trace class symmetric nonnegative operator Q on a separable Hilbert space U there exists a Q-Wiener process $W(t), t\ge 0$.*

Proof. See [8], Proposition 4.2, p. 88. □

Proposition 1.1.3.2. *Let $T > 0$ and $W(t), t \in [0,T]$, be a U-valued Q-Wiener process with respect to a normal filtration $\mathcal{F}_t, t \in [0,T]$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then $W(t), t \in [0,T]$, is a continuous square integrable $\mathcal{F}_t$-martingale, i.e. $W \in \mathcal{M}_T^2(U)$.*

Proof. The continuity is clear by definition and for each $t \in [0,T]$ we have that $\mathbb{E}(\|W(t)\|_U^2) = t \ \mathrm{tr}Q < \infty$ (see Proposition 1.1.2.1). Hence let $0 \leq s \leq t \leq T$ and $A \in \mathcal{F}_s$. Then we get by Proposition 1.1.1.1 that

$$\left\langle \int_A (W(t) - W(s))\mathrm{d}\mathbb{P}, u \right\rangle_U = \int_A \langle W(t) - W(s), u\rangle_U \mathrm{d}\mathbb{P}$$

$$= \mathbb{P}(A) \int \langle W(t) - W(s), u\rangle_U \mathrm{d}\mathbb{P} = 0$$

for all $u \in U$ as $\mathcal{F}_s$ is independent of $W(t)-W(s)$ and $\mathbb{E}(\langle W(t)-W(s), u\rangle_U) = 0$ for all $u \in U$. Therefore,

$$\begin{aligned} \int_A W(t)\mathrm{d}\mathbb{P} &= \int_A W(s) + (W(t) - W(s))\mathrm{d}\mathbb{P} \\ &= \int_A W(s)\mathrm{d}\mathbb{P} + \int_A W(t) - W(s)\mathrm{d}\mathbb{P} \\ &= \int_A W(s)\mathrm{d}\mathbb{P}, \ \text{for all } A \in \mathcal{F}_s. \end{aligned}$$

□

1.1.4 Representation of the Q-Wiener process

Proposition 1.1.4.1. *Let $e_k, \ k \in \mathbb{N}$, be an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $\lambda_k, k \in \mathbb{N}$. Then a U-valued stochastic process $W(t)$, $t \in [0,T]$, is a Q-Wiener process if and only if*

$$W(t) = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k}\beta_k(t)e_k, \quad t \in [0,T], \tag{1.2}$$

where $\beta_k, k \in \{n \in \mathbb{N} \mid \lambda_n > 0\}$, are independent real-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The series even converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0,T], U))$, and thus always has a $\mathbb{P}$-a.s. continuous modification. (Here the space $C([0,T], U)$ is equipped with the sup norm).

Proof.

1. Let $W(t),\ t \in [0,T]$, be a Q-Wiener process in U.
   Since $\mathbb{P} \circ W(t)^{-1} = \mathcal{N}(0, tQ)$, we see as in the proof of Proposition 1.1.2.3 that
   $$W(t) = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k(t) e_k, \quad t \in [0,T],$$
   with
   $$\beta_k(t) : \begin{cases} = & \frac{\langle W(t), e_k \rangle}{\sqrt{\lambda_k}} & if & k \in \mathbb{N}\ with\ \lambda_k > 0, \\ \equiv & 0 & else, & \end{cases}$$
   for all $t \in [0,T]$. Furthermore, $\mathbb{P} \circ \beta_k^{-1}(t) = \mathcal{N}(0,t), k \in \mathbb{N}$ and $\beta_k(t), k \in \mathbb{N}$ are independent for each $t \in [0,T]$.
   Now we fix $k \in \mathbb{N}$. First we show that $\beta_k(t), t \in [0,T]$, is a Brownian motion:
   If we take an arbitrary partition $0 = t_0 \leq t_1 < \ldots < t_n \leq T, n \in \mathbb{N}$, of $[0,T]$ we get that
   $$\beta_k(t_1), \beta_k(t_2) - \beta_k(t_1), \ldots, \beta_k(t_n) - \beta_k(t_{n-1})$$
   are independent for each $k \in \mathbb{N}$ since for $1 \leq j \leq n$
   $$\beta_k(t_j) - \beta_k(t_{j-1}) = \begin{cases} \frac{1}{\sqrt{\lambda_k}} & \langle W(t_j) - W(t_{j-1}), e_k \rangle & if & \lambda_k > 0, \\ 0 & & else, & \end{cases}$$
   Moreover, we obtain that for the same reason $\mathbb{P} \circ (\beta_k(t) - \beta_k(s))^{-1} = \mathcal{N}(0, t-s)$ for all $0 \leq s \leq t \leq T$.
   In addition,
   $$t \longmapsto \frac{1}{\sqrt{\lambda_k}} \langle W(t), e_k \rangle = \beta_k(t)$$
   is $\mathbb{P}$-a.s. continuous for all $k \in \mathbb{N}$.
   Secondly, it remains to prove that $\beta_k, k \in \mathbb{N}$, are independent.
   We take $k_1, \ldots, k_n \in \mathbb{N}, n \in \mathbb{N}, k_i \neq k_j$ if $i \neq j$ and an arbitrary partition $0 = t_0 \leq t_1 < \ldots < t_m \leq T, m \in \mathbb{N}$.
   Then we have to show that
   $$\sigma(\beta_{k_1}(t_1), \ldots, \beta_{k_1}(t_m)), \ldots, \sigma(\beta_{k_n}(t_1), \ldots, \beta_{k_n}(t_m))$$
   are independent.
   We will prove this by induction with respect to m:
   If $m = 1$ it is clear that $\beta_{k_1}(t_1), \ldots, \beta_{k_n}(t_1)$ are independent as observed above. Thus, we now take a partition $0 = t_0 \leq t_1 \leq \ldots \leq t_{m+1} \leq T$ and assume that
   $$\sigma(\beta_{k_1}(t_1), \ldots, \beta_{k_1}(t_m)), \ldots, \sigma(\beta_{k_n}(t_1), \ldots, \beta_{k_n}(t_m))$$

are independent. We note that

$$\sigma(\beta_{k_i}(t_1),\dots,\beta_{k_i}(t_m),\beta_{k_i}(t_{m+1}))$$
$$= \sigma(\beta_{k_i}(t_1),\dots,\beta_{k_i}(t_m),\beta_{k_i}(t_{m+1})-\beta_{k_i}(t_m)) \quad 1 \le i \le n,$$

and that

$$\beta_{k_i}(t_{m+1})-\beta_{k_i}(t_m) = \begin{cases} \frac{1}{\sqrt{\lambda_{k_i}}} & \langle W(t_{m+1})-W(t_m), e_{k_i}\rangle_U \quad if\ \lambda_k > 0, \\ 0 & else, \end{cases}$$

$1 \le i \le n$, are independent since they are pairwise orthogonal in $L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R})$ and since $W(t_{m+1})-W(t_m)$ is a Gaussian random variable. If we take $A_{i,j}\in\mathcal{B}(\mathbb{R}), 1\le i\le n, 1\le j\le m+1$, then because of the independence of $\sigma(W(s)|s\le t_m)$ and $\sigma(W(t_{m+1})-W(t_m))$ we get that

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{i=1}^n \{\beta_{k_i}(t_1)\in A_{i,1},\dots,\beta_{k_i}(t_m)\in A_{i,m},\right.\\
&\quad\left.\beta_{k_i}(t_{m+1})-\beta_{k_i}(t_m)\in A_{i,m+1}\}\right)\\
=\ &\mathbb{P}\left(\underbrace{\bigcap_{i=1}^n\bigcap_{j=1}^m\{\beta_{k_i}(t_j)\in A_{i,j}\}}_{\in\sigma(W(s)|s\le t_m)} \cap \underbrace{\bigcap_{i=1}^n\{\beta_{k_i}(t_{m+1})-\beta_{k_i}(t_m)\in A_{i,m+1}\}}_{\in\sigma(W(t_{m+1})-W(t_m))}\right)\\
=\ &\mathbb{P}\left(\bigcap_{i=1}^n\bigcap_{j=1}^m\{\beta_{k_i}(t_j)\in A_{i,j}\}\right).\mathbb{P}\left(\bigcap_{i=1}^n\{\beta_{k_i}(t_{m+1})-\beta_{k_i}(t_m)\in A_{i,m+1}\}\right)\\
=\ &\left(\prod_{i=1}^n\mathbb{P}(\bigcap_{j=1}^m\{\beta_{k_i}(t_j)\in A_{i,j}\})\right)\\
&\quad.\left(\prod_{i=1}^n\mathbb{P}\{\beta_{k_i}(t_{m+1})-\beta_{k_i}(t_m)\in A_{i,m+1}\}\right)\\
=\ &\prod_{i=1}^n\mathbb{P}\left(\bigcap_{j=1}^m\{\beta_{k_i}(t_j)\in A_{i,j}\}\cap\{\beta_{k_i}(t_{m+1})-\beta_{k_i}(t_m)\in A_{i,m+1}\}\right).
\end{aligned}$$

and therefore the assertion follows.

2. If we define

$$W(t) := \sum_{k\in\mathbb{N}}\sqrt{\lambda_k}\beta_k(t)e_k, \quad t\in[0,T],$$

where $\beta_k, k \in \mathbb{N}$, are independent real-valued continuous Brownian motions then it is clear that $W(t), t \in [0,T]$, is well-defined in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$. Besides, it is obvious that the process $W(t), t \in [0,T]$, starts at zero and that

$$\mathbb{P} \circ (W(t) - W(s))^{-1} = \mathcal{N}(0, (t-s)Q), \quad 0 \le s < t \le T,$$

by Proposition 1.1.2.3 It is also clear that the increments are independent. Thus it remains to show that the above series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0,T], U))$. To this end we set

$$W^N(t, w) := \sum_{k=1}^{N} \sqrt{\lambda_k} \beta_k(t, w) e_k$$

for all $(t, w) \in \Omega_T := [0,T] \times \Omega$ and $N \in \mathbb{N}$. Then $W^N, N \in \mathbb{N}$, is $\mathbb{P}$-a.s. continuous and we have that for $M < N$

$$\mathbb{E}\left(\sup_{t \in [0,T]} \|W^N(t) - W^M(t)\|_U^2 \right) = \mathbb{E}\left(\sup_{t \in [0,T]} \sum_{k=M+1}^{N} \lambda_k \beta_k^2(t) \right)$$

$$\le \sum_{k=M+1}^{N} \lambda_k \mathbb{E}(\sup_{t \in [0,T]} \beta_k^2(t)) \le c \sum_{k=M+1}^{N} \lambda_k$$

where $c = \mathbb{E}(\sup_{t \in [0,T]} \beta_k^2(t)) < \infty$ because of Doob's maximal inequality for real-valued submartingales. As $\sum_{k \in \mathbb{N}} \lambda_k = \mathrm{tr} Q < \infty$, the assertion follows. □

Definition 1.1.4.1. ***(Normal filtration).*** *A filtration $\mathcal{F}_t, t \in [0,T]$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called normal if:*

- *$\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ and*
- *$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0,T]$.*

Definition 1.1.4.2. *(Q-**Wiener process with respect to a filtration**). A Q-Wiener process $W(t), t \in [0,T]$, is called a Q-Wiener process with respect to a filtration $\mathcal{F}_t, t \in [0,T]$, if:*

- *$W(t), t \in [0,T]$, is adapted to $\mathcal{F}_t, t \in [0,T]$ and*
- *$W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \le s \le t \le T$.*

In fact it is possible to show that any U-valued Q-Wiener process $W(t), t \in [0,T]$, is a Q-Wiener process with respect to a normal filtration.

1.2 The definition of the stochastic integral

For the whole section we fix a positive real number T and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we define $\Omega_T := [0,T] \times \Omega$ and $\mathbb{P}_T := dx \otimes \mathbb{P}$ where dx is the Lebesgue measure.

Moreover, let $Q \in L(U)$ be symmetric, nonnegative and with finite trace and we consider a Q-Wiener process $W(t)$, $t \in [0,T]$, with respect to a normal filtration $\mathcal{F}_t$, $t \in [0,T]$.

1.2.1 Hilbert-Schmidt operator

Definition 1.2.1.1. *(**Hilbert-Schmidt operator**). Let $e_k, k \in \mathbb{N}$, be an orthonormal basis of U. An operator $A \in L(U,H)$ is called Hilbert Schmidt if*

$$\sum_{k \in \mathbb{N}} \langle Ae_k, Ae_k \rangle < \infty.$$

Remark 1.2.1.1.

(i) The definition of Hilbert-Schmidt operator and the number

$$\|A\|_{L_2(U,H)} := \left(\sum_{k \in \mathbb{N}} \|Ae_k\|^2 \right)^{\frac{1}{2}}$$

does not depend on the choice of the orthonormal basis $e_k, k \in \mathbb{N}$, and we have that $\|A\|_{L_2(U,H)} = \|A^*\|_{L_2(H,U)}$. For simplicity we also write $\|A\|_{L_2(U,H)}$ instead of $\|A\|_{L_2}$.

(ii) $\|A\|_{L(U,H)} \leq \|A\|_{L_2(U,H)}$.

(iii) Let G be another Hilbert space and $B_1 \in L(H,G), B_2 \in L(G,U), A \in L_2(U,H)$. Then $B_1 A \in L_2(U,G)$ and $AB_2 \in L_2(G,H)$ and

$$\|B_1 A\|_{L(U,G)} \leq \|B_1\|_{L(H,G)} \|A\|_{L_2(U,H)},$$

$$\|AB_2\|_{L(G,H)} \leq \|A\|_{L_2(U,H)} \|B_2\|_{L(G,U)}.$$

Proposition 1.2.1.1. *Let $B, A \in L_2(U,H)$ and let $e_k, k \in \mathbb{N}$, be an orthonormal basis of U. If we define*

$$\langle A, B \rangle_{L_2} := \sum_{k \in \mathbb{N}} \langle Ae_k, Be_k \rangle$$

we obtain that $(L_2(U,H), \langle\ ,\ \rangle_{L_2})$ *is a separable Hilbert space.*

If $f_k, k \in \mathbb{N}$, *is an orthonormal basis of* H *we get that* $f_j \otimes e_k := f_j \langle e_k, . \rangle_U, j, k \in \mathbb{N}$, *is an orthonormal basis of* $L_2(U,H)$.

Proof. We have to prove the completeness and the separability.

1. $L_2(U,H)$ is complete:
   Let $A_n, n \in \mathbb{N}$, be a Cauchy sequence in $L_2(U,H)$. Then it is clear that it is also a Cauchy sequence in $L(U,H)$. Because of the completeness of $L(U,H)$ there exists an element $A \in L(U,H)$ such that $\|A_n - A\|_{L(U,G)} \to 0$ as $n \mapsto \infty$. But by the lemma of Fatou we also have for any orthonormal basis $e_k, k \in \mathbb{N}$, of U that

$$\begin{aligned}
\|A_n - A\|_{L_2}^2 &= \sum_{k\in\mathbb{N}} \langle (A_n - A)e_k, (A_n - A)e_k \rangle \\
&= \sum_{k\in\mathbb{N}} \liminf_{m\to\infty} \|(A_n - A_m)e_k\|^2 \\
&\leq \liminf_{m\to\infty} \sum_{k\in\mathbb{N}} \|(A_n - A_m)e_k\|^2 \\
&= \liminf_{m\to\infty} \|A_n - A_m\|_{L_2}^2 < \varepsilon
\end{aligned}$$

   for all $n \in \mathbb{N}$ big enough. Therefore the assertion follows.

2. $L_2(U,H)$ is separable:
   If we define $f_j \otimes e_k := f_j \langle e_k, .\ \rangle_U, j, k \in \mathbb{N}$, then it is clear that $f_j \otimes e_k \in L_2(U,H)$ for all $j, k \in \mathbb{N}$ and for arbitrary $A \in L_2(U,H)$ we get that

$$\langle f_j \otimes e_k, A \rangle_{L_2} = \sum_{k\in\mathbb{N}} \langle e_k, e_n \rangle_U . \langle f_j, Ae_n \rangle = \langle f_j, Ae_k \rangle$$

   Therefore it is obvious that $f_j \otimes e_k, j, k \in \mathbb{N}$, is an orthonormal system. In addition, $A = 0$ if $\langle f_j \otimes e_k, A \rangle_{L_2} = 0$ for all $j, k \in \mathbb{N}$, and therefore $span(f_j \otimes e_k | j, k \in \mathbb{N})$ is a dense subspace of $L_2(U,H)$. □

Besides we recall the following fact.

Proposition 1.2.1.2. *If* $Q \in L(U)$ *is nonnegative and symmetric then there exists exactly one element* $Q^{\frac{1}{2}} \in L(U)$ *nonnegative and symmetric such that* $Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}} = Q$.

If, in addition, $trQ < \infty$ we have that $Q^{\frac{1}{2}} \in L_2(U)$ where $\|Q^{\frac{1}{2}}\|_{L_2}^2 = trQ$ and of course $L \circ Q^{\frac{1}{2}} \in L_2(U, H)$ for all $L \in L(U, H)$.

Proof. [13], Theorem VI.9, p. 196. □

Proposition 1.2.1.3. *Let $T \in L(U)$ and T^{-1} the pseudo inverse of T.*

1. *If we define an inner product on $T(U)$ by*

   $$\langle x, y\rangle_{T(U)} := \langle T^{-1}x, T^{-1}y\rangle_U \text{ for all } x, y \in T(U),$$

   then $(T(U), \langle\ ,\ \rangle_{T(U)})$ is a Hilbert space.

2. *Let $e_k, k \in \mathbb{N}$, be an orthonormal basis of $(KerT)^{\perp}$. Then $Te_k, k \in \mathbb{N}$, is an orthonormal basis of $(T(U), \langle\ ,\ \rangle_{T(U)})$.*

Proof. $T : (KerT)^{\perp} \longrightarrow T(U)$ is bijective and an isometry if $(KerT)^{\perp}$ is equipped with $\langle\ ,\ \rangle_U$ and $T(U)$ with $\langle\ ,\ \rangle_{T(U)}$. □

1.2.2 Scheme of the construction of the stochastic integral

Step 1: First we consider a certain class ξ of elementary $L(U, H)$-valued processes and define the mapping

$$\begin{array}{rccl} Int: & \xi & \longrightarrow & \mathcal{M}_T^2(H) := \mathcal{M}_T^2 \\ & \Phi & \longmapsto & \int_0^t \Phi(s)\mathrm{d}W(s), \quad t \in [0, T]. \end{array}$$

Step 2: We prove that there is a certain norm on ξ such that

$$Int : \xi \longrightarrow \mathcal{M}_T^2$$

is an isometry. Since $\mathcal{M}_T^2$ is a Banach space this implies that Int can be extended to the abstract completion $\overline{\xi}$ of ξ. This extension remains isometric and it is unique.

Step 3: We give an explicit representation of $\overline{\xi}$.

Step 4: We show how the definition of the stochastic integral can be extended by localization.

1.2.3 The construction of the stochastic integral in detail

Step 1: First we define the class ξ of all elementary processes as follows.

Definition 1.2.3.1. *(**Elementary process**). An $L = L(U, H)$-valued process $\Phi(t)$, $t \in [0, T]$, on $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $\mathcal{F}_t$, $t \in [0, T]$, is said to be* elementary *if there exist $0 = t_o < \cdots < t_k = T$, $k \in \mathbb{N}$*

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi_m \mathbb{1}_{]t_m, t_{m+1}]}(t), \quad t \in [0, T],$$

where:

- $\Phi_m : \Omega \longrightarrow L(U, H)$ is $\mathcal{F}_{t_m}$-measurable, w.r.t. strong Borel σ-algebra on $L(U, H)$, $\quad 0 \leq m \leq k - 1$,
- Φ_m takes only a finite number of values in $L(U, H)$, $\quad 1 \leq m \leq k - 1$.

If we define now

$$Int(\Phi)(t) := \int_0^t \Phi(s)\mathrm{d}W(s) := \sum_{m=0}^{k-1} \Phi_m(W(t_{m+1} \wedge t) - W(t_m \wedge t)), \quad t \in [0, T],$$

(this is obviously independent of the representation) for all $\Phi \in \xi$, we have the following important result.

Proposition 1.2.3.1. *Let $\Phi \in \xi$. Then the stochastic integral $\displaystyle\int_0^t \Phi(s)dW(s), t \in [0, T]$, defined in the previous way, is a continuous square integrable martingale with respect to $\mathcal{F}_t, t \in [0, T]$ i.e.*

$$Int : \xi \longrightarrow \mathcal{M}_T^2$$

Proof. Let $\Phi \in \xi$ be given by

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi(m) \mathbb{1}_{]t_m, t_{m+1}]}(t), \quad t \in [0, T],$$

as in Definition1.2.3.1. Then it is clear that

$$t \longmapsto \int_0^t \Phi(s)\mathrm{d}W(s) = \sum_{m=0}^{k-1} \Phi_m(W(t_{m+1} \wedge t) - W(t_m \wedge t))$$

is $\mathbb{P}$-a.s. continuous because of the continuity of the Wiener process and the continuity of $\Phi_m(w) : U \longmapsto H, 0 \le m \le k-1, w \in \Omega$. In addition, we get for each summand that

$$\|\Phi_m(W(t_{m+1} \wedge t) - W(t_m \wedge t))\| \le \|\Phi_m\|_{L(U,H)} \|W(t_{m+1} \wedge t) - W(t_m \wedge t)\|_U.$$

Since $W(t), t \in [0,T]$, is square integrable this implies that $\int_0^t \Phi(s)\mathrm{d}W(s)$ is square integrable for each $t \in [0,T]$.

To prove the martingale property we take $0 \le s \le t \le T$ and a set A from $\mathcal{F}_s$. If $\{\Phi_m(w) | w \in \Omega\} := \{L_1^m, \dots, L_{k_m}^m\}$ we obtain by Proposition 1.1.1.1 and the martingale property of the Wiener process (more precisely using optional stopping) that

$$\begin{aligned}
&\int_{\mathrm{A}} \sum_{m=0}^{k-1} \Phi_m(W(t_{m+1} \wedge t) - W(t_m \wedge t))d\mathbb{P} \\
= &\sum_{\substack{0 \le m \le k-1, \\ t_{m+1} < s}} \int_{\mathrm{A}} \Phi_m(W(t_{m+1} \wedge s) - W(t_m \wedge s))\mathrm{d}\mathbb{P} \\
+ &\sum_{\substack{0 \le m \le k-1, \\ s \le t_{m+1}}} \sum_{j=1}^{k_m} \int_{\mathrm{A} \cap \{\Phi_m = L_j^m\}} L_j^m (W(t_{m+1} \wedge t) - W(t_m \wedge t))\mathrm{d}\mathbb{P} \\
= &\sum_{\substack{0 \le m \le k-1, \\ t_{m+1} < s}} \int_{\mathrm{A}} \Phi_m(W(t_{m+1} \wedge s) - W(t_m \wedge s))\mathrm{d}\mathbb{P} \\
+ &\sum_{\substack{0 \le m \le k-1, \\ s \le t_{m+1}}} \sum_{j=1}^{k_m} L_j^m \int_{\underbrace{\mathrm{A} \cap \{\Phi_m = L_j^m\}}_{\mathcal{F}_{s \vee t_m}}} (W(t_{m+1} \wedge t) - W(t_m \wedge t))\mathrm{d}\mathbb{P} \\
= &\sum_{\substack{0 \le m \le k-1, \\ t_{m+1} < s}} \int_{\mathrm{A}} \Phi_m(W(t_{m+1} \wedge s) - W(t_m \wedge s))\mathrm{d}\mathbb{P} \\
+ &\sum_{\substack{0 \le m \le k-1, \\ t_m \le s \le t_{m+1}}} \sum_{j=1}^{k_m} L_j^m \int_{\mathrm{A} \cap \{\Phi_m = L_j^m\}} (W(t_{m+1} \wedge s) - W(t_m \wedge s))\mathrm{d}\mathbb{P}
\end{aligned}$$

$$= \int_A \sum_{m=0}^{k-1} \Phi_m (W(t_{m+1} \wedge s) - W(t_m \wedge s)) \mathrm{d}\mathbb{P} \qquad \square$$

Step 2: To verify the assertion that there is a norm on ξ such that $Int : \xi \longrightarrow \mathcal{M}_T^2$ is an isometry.
We calculate the $\mathcal{M}_T^2$-norm of

$$\int_0^t \Phi(s) \mathrm{d}W(s), t \in [0,T],$$

and get the following result.

Proposition 1.2.3.2. *If* $\Phi = \sum_{m=0}^{k-1} \Phi_m \mathbb{1}_{]t_m, t_{m+1}]}$ *is an elementary* $L(U,H)$-*valued process then*

$$\left\| \int_0^\cdot \Phi(s) dW(s) \right\|_{\mathcal{M}_T^2}^2 = \mathbb{E}\left(\int_0^T \left\| \Phi(s) \circ Q^{\frac{1}{2}} \right\|_{L_2}^2 ds \right) =: \|\Phi\|_T^2 \quad (\text{"Itô-isometry"}).$$

Proof. If we set $\triangle_m := W(t_{m+1}) - W(t_m)$ then we get that

$$\begin{aligned} \left\| \int_0^\cdot \Phi(s) dW(s) \right\|_{\mathcal{M}_T^2}^2 &= \mathbb{E}\left(\left\| \int_0^T \Phi(s) dW(s) \right\|_H^2 \right) = \mathbb{E}\left(\left\| \sum_{m=0}^{K-1} \Phi_m \triangle_m \right\|_H^2 \right) \\ &= \mathbb{E}\left(\sum_{m=0}^{K-1} \left\| \Phi_m \triangle_m \right\|_H^2 \right) + 2\mathbb{E}\left(\sum_{0 \le m < n \le k-1} \langle \Phi_m \triangle_m, \Phi_n \triangle_n \rangle_H \right). \end{aligned}$$

Claim 1:

$$\begin{aligned} \mathbb{E}\left(\sum_{m=0}^{K-1} \left\| \Phi_m \triangle_m \right\|_H^2 \right) &= \sum_{m=0}^{K-1} (t_{m+1} - t_m) \mathbb{E}\left(\left\| \Phi_m \circ Q^{\frac{1}{2}} \right\|_{L_2}^2 \right) \\ &= \int_0^T \mathbb{E}\left(\left\| \Phi(s) \circ Q^{\frac{1}{2}} \right\|_{L_2}^2 \right) \mathrm{d}s. \end{aligned}$$

To prove this we take an orthonormal basis f_k, $k \in \mathbb{N}$ of H and get by the Parseval identity and Levi's monotone convergence theorem that

$$\mathbb{E}(\|\Phi_m \triangle_m\|_H^2) = \sum_{l \in \mathbb{N}} \mathbb{E}(\langle \Phi_m \triangle_m, f_l \rangle_H^2) = \sum_{l \in \mathbb{N}} \mathbb{E}(\mathbb{E}(\langle \triangle_m, \Phi_m^* f_l \rangle_U^2 \mid \mathcal{F}_{t_m})).$$

Taking an orthonormal basis e_k, $k \in \mathbb{N}$ of U we obtain that

$$\Phi_m^* f_l = \sum_{l \in \mathbb{N}} \langle f_l, \Phi_m e_k \rangle_H e_k.$$

Since $\langle f_l, \Phi_m e_k \rangle_H$ is $\mathcal{F}_{t_m}$-measurable, this implies that $\Phi_m^* f_l$ is $\mathcal{F}_{t_m}$-measurable by Proposition 1.1.1.2. Using the fact that $\sigma(\Delta_m)$ is independent of $\mathcal{F}_{t_m}$ we have, for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\begin{aligned}
\mathbb{E}(\langle \triangle_m, \Phi_m^* f_l \rangle_U^2 \mid \mathcal{F}_{t_m})(\omega) &= \mathbb{E}(\langle \triangle_m, \Phi_m^*(\omega) f_l \rangle_U^2) \\
&= (t_{m+1} - t_m) \langle Q(\Phi_m^*(\omega) f_l), \Phi_m^*(\omega) f_l \rangle_U,
\end{aligned}$$

Since $\mathbb{E}(\langle \triangle_m, u \rangle_U^2 = (t_{m+1} - t_m) \langle Qu, u \rangle_U$ for all $u \in U$. Thus, the symmetry of $Q^{\frac{1}{2}}$ finally implies that

$$\begin{aligned}
\mathbb{E}(\|\Phi_m \triangle_m\|_H^2) &= \sum_{l \in \mathbb{N}} \mathbb{E}\Big(\mathbb{E}(\langle \triangle_m, \Phi_m^* f_l \rangle_U^2 \mid \mathcal{F}_{t_m}) \Big) \\
&= (t_{m+1} - t_m) \sum_{l \in \mathbb{N}} \mathbb{E}(\langle Q \Phi_m^* f_l, \Phi_m^* f_l \rangle_U) \\
&= (t_{m+1} - t_m) \sum_{l \in \mathbb{N}} \mathbb{E}\Big(\|Q^{\frac{1}{2}} \Phi_m^* f_l\|_U^2 \Big) \\
&= (t_{m+1} - t_m) \mathbb{E}\Big(\|(\Phi_m \circ Q^{\frac{1}{2}})^*\|_{L_2(H,U)}^2 \Big) \\
&= (t_{m+1} - t_m) \mathbb{E}\Big(\|\Phi_m \circ Q^{\frac{1}{2}}\|_{L_2(U,H)}^2 \Big).
\end{aligned}$$

Hence the first assertion is proved and it only remains to verify the following claim.

Claim 2:

$$\mathbb{E}(\langle \Phi_m \triangle_m, \Phi_n \triangle_n \rangle_H) = 0, \quad 0 \leq m < n \leq k - 1.$$

But this can be proved in a similar way to Claim 1:

$$\begin{aligned}
\mathbb{E}(\langle \Phi_m \triangle_m, \Phi_n \triangle_n \rangle_H) &= \mathbb{E}\Big(\mathbb{E}(\langle \Phi_n^* \Phi_m \triangle_m, \triangle_n \rangle_U \mid \mathcal{F}_{t_m}) \Big) \\
&= \int \mathbb{E}\Big(\langle \Phi_n^*(\omega) \Phi_m(\omega) \triangle_m(\omega), \triangle_n \rangle_U \Big) \mathbb{P}(\mathrm{d}\omega) = 0,
\end{aligned}$$

since $\mathbb{E}(\langle u, \triangle_n\rangle_U) = 0$ for all $u \in U$. Hence the assertion follows. □

Hence the right norm on ξ has been identified. But strictly speaking $\| \, \|_T$ is only a seminorm on ξ. Therefore, we have to consider equivalence classes of elementary processes with respect to $\| \, \|_T$ to get a norm on ξ. For simplicity we will not change the notation but stress the following fact.

Remark 1.2.3.1. *If two elementary processes Φ and $\tilde{\Phi}$ belong to one equivalence class with respect to $\| \, \|_T$ it does not follow that they are equal $\mathbb{P}_T$-a.e. because their values only have to correspond on $Q^{\frac{1}{2}}(U)$ $\mathbb{P}_T$-a.e.*

Thus we finally have shown that

$$Int : (\xi, \| \, \|_T) \longrightarrow (\mathcal{M}_T^2, \| \, \|_{\mathcal{M}_T^2})$$

is an isometric transformation. Since ξ is dense in the abstract completion $\bar{\xi}$ of ξ with respect to $\| \, \|_T$ it is clear that there is a unique isometric extension of Int to $\bar{\xi}$.

Step 3: To give an explicit representation of $\bar{\xi}$ it is useful, at this moment, to introduce the subspace $U_0 = Q^{\frac{1}{2}}(U)$ with the inner product given by

$$\langle u_0, v_0\rangle_0 = \langle Q^{\frac{1}{2}} u_0, Q^{\frac{1}{2}} v_0\rangle_U.$$

$u_0, v_0 \in U_0$, where $Q^{-\frac{1}{2}}$ is the pseudo inverse of $Q^{\frac{1}{2}}$ in the case that Q is not one-to-one. Then we get by Proposition 1.2.1.3(1) that $(U_0, \langle \, , \, \rangle_0)$ is again a separable Hilbert space.

The separable Hilbert space $L_2(U_0, H)$ is called L_2^0.
By Proposition 1.2.1.3(2) we know that $Q^{\frac{1}{2}} g_k, k \in \mathbb{N}$, is an orthonormal basis of $(U_0, \langle \, , \, \rangle_0)$ if $g_k, k \in \mathbb{N}$, is an orthonormal basis of $(Ker Q^{\frac{1}{2}})^{\perp}$. This basis can be supplemented to a basis of U by elements of $Ker Q^{\frac{1}{2}}$. Thus we obtain that

$$\|L\|_{L_2^0} = \|L \circ Q^{\frac{1}{2}}\|_{L_2} \text{ for each } L \in L_2^0.$$

Define $L(U,H)_0 := \{T|_{U_0} \;\; | \;\; T \in L(U,H)\}$. Since $Q^{\frac{1}{2}} \in L_2(U)$ it is clear that $L(U,H)_0 \subset L_2^0$ and that the $\| \, \|_T$-norm of $\Phi \in \xi$ can be written in the following way:

$$\|\Phi\|_T = \left(\mathbb{E}\left(\int_0^T \|\Phi(s)\|_{L_2^0}^2 ds\right)\right)^{\frac{1}{2}}$$

Besides we need the following σ-field:

$$\begin{aligned}\mathcal{P}_T &:= \sigma\Big(\{]s,t] \times F_s | 0 \le s < t \le T, F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 | F_0 \in \mathcal{F}_0\}\Big) \\ &= \sigma\Big(Y : \Omega_T \to \mathbb{R} \mid Y \text{ is left-continuous and adapted to } \mathcal{F}_t, t \in [0,T]\Big).\end{aligned}$$

Let $\tilde{H}$ be an arbitrary separable Hilbert space. If $Y : \Omega_T \to \tilde{H}$ is $\mathcal{P}_T/\mathcal{B}(\tilde{H})$-measurable it is called $(\tilde{H})$-predictable.

If, for example, the process Y itself is continuous and adapted to $\mathcal{F}_t, t \in [0,T]$, then it is predictable.

So, we are now able to characterize $\bar{\xi}$.

Claim: There is an explicit representation of $\bar{\xi}$ and it is given by

$$\begin{aligned}\mathcal{N}_W^2(0,T;H) &= \{\Phi : [0,T] \times \Omega \longrightarrow L_2^0 | \ \Phi \text{ is predictable and } \|\Phi\|_T < \infty\} \\ &= L^2\Big([0,T] \times \Omega, \mathcal{P}_T, dt \otimes \mathbb{P}; L_2^0\Big).\end{aligned}$$

For simplicity we also write $\mathcal{N}_W^2(0,T)$ or $\mathcal{N}_W^2$ instead of $\mathcal{N}_W^2(0,T;H)$.

To prove this claim we first notice the following facts:

1. Since $L(U,H)_0 \subset L_2^0$ and since any $\Phi \in \xi$ is L_2^0-predictable by construction we have that $\xi \subset \mathcal{N}_W^2$.

2. Because of the completeness of L_2^0 we get by Appendix A in [9] that

$$\mathcal{N}_W^2 = L^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; L_2^0)$$

   is also complete.

Therefore $\mathcal{N}_W^2$ is at least a candidate for a representation of $\bar{\xi}$. Thus there only remains to show that ξ is a dense subset of $\mathcal{N}_W^2$. But this is formulated in Proposition 1.2.3.3 below, which can be proved with the help of the following lemma.

Lemma 1.2.3.1. *There is an orthonormal basis of L_2^0 consisting of elements of $L(U,H)_0$. This implies especially that $L(U,H)_0$ is a dense subset of L_2^0.*

Proof. Since Q is symmetric, nonnegative and $trQ < \infty$ we know by Proposition 1.1.2.2 that there exists an orthonormal basis $e_k, k \in \mathbb{N}$,of U

such that $Qe_k = \lambda_k e_k,\ \lambda_k \geq 0, k \in \mathbb{N}$. In this case $Q^{\frac{1}{2}} e_k = \sqrt{\lambda_k} e_k,\ k \in \mathbb{N}$. with $\lambda_k > 0$, is an orthonormal basis of U_0.

If $f_k, k \in \mathbb{N}$, is an orthonormal basis of H then by Proposition1.2.1.1 we know that

$$f_j \otimes \sqrt{\lambda_k} e_k = f_j \langle \sqrt{\lambda_k} e_k,\ .\ \rangle_{U_0} = \frac{1}{\lambda_k} f_j \langle e_k,\ .\ \rangle_U,\quad j,k \in \mathbb{N}, \lambda_k > 0,$$

form an orthonormal basis of L_0^2 consisting of operators in $L(U,H)$. But, of course,

$$\overline{span\left(\frac{1}{\sqrt{\lambda_k}} f_j \otimes e_k \middle| j,k \in \mathbb{N} \text{ with } \lambda_k > 0 \right)} = L_2^0.$$

$\square$

Proposition 1.2.3.3. *If Φ is a L_2^0-predictable process such that $\|\Phi\|_T < \infty$ then there exists a sequence $\Phi_n, n \in \mathbb{N}$, of $L(U,H)_0$-valued elementary processes such that*

$$\|\Phi - \Phi_n\|_T \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Proof.

Step 1: If $\Phi \in \mathcal{N}_W^2$ there exists a sequence of simple random variables $\Phi_n = \sum_{k=1}^{M_n} L_k^n \mathbb{1}_{A_k^n},\ A_k^n \in \mathcal{P}_T$ and $L_k^n \in L_2^0,\ n \in \mathbb{N}$ such that

$$\|\Phi - \Phi_n\|_T \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

As L_0^2 is a Hilbert space this is a simple consequence of Lemma 1.1.1.3 and Lebesgue's dominated convergence theorem.

Thus the assertion is reduced to the case that $\Phi = L\mathbb{1}_A$ where $L \in L_2^0$ and $A \in \mathcal{P}_T$.

Step 2: Let $A \in \mathcal{P}_T$ and $L \in L_2^0$. Then there exists a sequence $L_n,\ n \in \mathbb{N}$, in $L(U,H)_0$ such that

$$\|L\mathbb{1}_A - L_n\mathbb{1}_A\|_T \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

This result is obvious by Lemma 1.2.3.1 and thus now we only have to consider the case that $\Phi = L\mathbb{1}_A, L \in L(U,H)_0$ and $A \in \mathcal{P}_T$.

Step 3: If $\Phi = L\mathbb{1}_A, L \in L(U,H)_0, A \in \mathcal{P}_T$ then there is a sequence $\Phi_n, n \in \mathbb{N}$, of elementary $L(U,H)_0$-valued processes in the sense of Definition 1.2.3.1 such that

$$\|L\mathbb{1}_A - \Phi_n\|_T \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

To show this it is sufficient to prove that for any $\varepsilon > 0$ there is a finite union $\Lambda = \bigcup_{n=1}^{N} A_n$ of pairwise disjoint predictable rectangles

$$A_n \in \{]s,t] \times F_s \mid 0 \leq s < t \leq T, F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 \mid F_0 \in \mathcal{F}_0\} =: \mathcal{A}$$

such that

$$\mathbb{P}_T((A \backslash \Lambda) \cup (\Lambda \backslash A)) < \varepsilon.$$

For then we get that $\sum_{n=1}^{N} L 1\!\!1_{A_n}$ differs from an elementary process by a function of type $1\!\!1_{\{0\}\times F_0}$ with $F_0 \in \mathcal{F}_0$, which has $\| \, . \, \|_T$-norm zero and

$$\Big\| L 1\!\!1_A - \sum_{n=1}^{N} L 1\!\!1_{A_n} \Big\|_T^2 = \mathbb{E}\bigg(\int_0^T \Big\| L(1\!\!1_A - \sum_{n=1}^{N} 1\!\!1_{A_n}) \Big\|_{L_2^0}^2 ds \bigg) \leq \varepsilon \|L\|_{L_2^0}^2.$$

Hence we define

$$\mathcal{K} = \bigg\{ \bigcup_{i \in I} A_i \;\bigg|\; I \text{ is finite and } A_i \in \mathcal{A}, i \in I \bigg\}.$$

Then $\mathcal{K}$ is an algebra and any element in $\mathcal{K}$ can be written as a finite disjoint union of elements in $\mathcal{A}$. Now let $\mathcal{G}$ be the family of all $A \in \mathcal{P}_T$ which can be approximated by elements of $\mathcal{K}$ in the above sense. Then $\mathcal{G}$ is a Dynkin system and therefore $\mathcal{P}_T = \sigma(\mathcal{K}) = \mathcal{D}(\mathcal{K}) \subset \mathcal{G}$ as $\mathcal{K} \subset \mathcal{G}$. □

Step 4: Finally the so-called localization procedure provides the possibility to extend the definition of the stochastic integral even to the linear space

$$\begin{aligned} \mathcal{N}_W(0,T;H) \;:=\; & \bigg\{ \Phi : \Omega \longrightarrow L_2^0 \;\bigg|\; \Phi \text{ is predictable with} \\ & \mathbb{P}\bigg(\int_0^T \|\Phi(s)\|_{L_2^0}^2 ds < \infty \bigg) = 1 \bigg\}. \end{aligned}$$

For simplicity we also write $\mathcal{N}_W(0,T)$ or $\mathcal{N}_W$ instead of $\mathcal{N}_W(0,T;H)$ and $\mathcal{N}_W$ is called the class of *stochastically integrable* processes on $[0,T]$.

The extension is done in the following way:

For $\Phi \in \mathcal{N}_W$ we define

$$\tau_n := \inf \bigg\{ t \in [0,T] \;\bigg|\; \int_0^t \|\Phi(s)\|_{L_2^0}^2 \mathrm{d}s > n \bigg\} \wedge T. \tag{1.3}$$

Then by the right-continuity of the filtration $\mathcal{F}_t, t \in [0,T]$, we get that

$$\begin{aligned}
\{\tau_n \le t\} &= \bigcap_{m\in\mathbb{N}} \left\{\tau_n < t + \frac{1}{m}\right\} \\
&= \underbrace{\bigcap_{m\in\mathbb{N}} \bigcup_{q\in[0,t+\frac{1}{m}[\cap\mathbb{Q}} \underbrace{\left\{\int_0^q \|\Phi(s)\|^2_{L_2^0}\mathrm{d}s > n\right\}}_{\in\mathcal{F}_q \text{ by the real Fubini theorem}}}_{\in\mathcal{F}_{t+\frac{1}{m}} \text{ and decreasing in } m} \quad \in \mathcal{F}_t.
\end{aligned}$$

Therefore $\tau_n, n \in \mathbb{N}$ is an increasing sequence of stopping times with respect to $\mathcal{F}_t, t \in [0,T]$, such that

$$\mathbb{E}\bigg(\int_0^T \|1\!\!1_{]0,\tau_n]}(s)\Phi(s)\|^2_{L_2^0} ds\bigg) \le n < \infty.$$

In addition, the processes $1\!\!1_{]0,\tau_n]}\Phi, n \in \mathbb{N}$, are still L_2^0-predictable since $1\!\!1_{]0,\tau_n]}$ is left-continuous and $(\mathcal{F}_t)$-adapted or since

$$\begin{aligned}
]0,\tau_n] &:= \{(s,w) \in \Omega_T \mid 0 < s \le \tau_n(w)\} \\
&= \Big(\{(s,w) \in \Omega_T \mid \tau_n(w) < s \le T\} \cup \{0\} \times \Omega\Big)^c \\
&= \Big(\underbrace{\bigcup_{q\in\mathbb{Q}} (]q,T] \times \underbrace{\{\tau_n \le q\}}_{\in\mathcal{F}_q})}_{\in\mathcal{P}_T} \cup \{0\} \times \Omega\Big)^c \in \mathcal{P}_T.
\end{aligned}$$

Thus we get that the stochastic integrals

$$\int_0^t 1\!\!1_{]0,\tau_n]}(s)\Phi(s)\mathrm{d}W(s),\ t \in [0,T],$$

are well-defined for all $n \in \mathbb{N}$. For arbitrary $t \in [0,T]$ we set

$$\int_0^t \Phi(s)\mathrm{d}W(s) := \int_0^t 1\!\!1_{]0,\tau_n]}(s)\Phi(s)\mathrm{d}W(s), \tag{1.4}$$

where n is an arbitrary natural number such that $\tau_n \ge t$. (Note that the sequence $\tau_n, n \in \mathbb{N}$, even reaches T $\mathbb{P}$-a.s., in the sense that for $\mathbb{P}$-a.e. $\omega \in \Omega$ there exists $n(\omega) \in \mathbb{N}$ such that $\tau_n(\omega) = T$ for all $n \ge n(\omega)$.)

To show that this definition is consistent we have to prove that for arbitrary natural numbers $m < n$ and $t \in [0,T]$

$$\int_0^t 1\!\!1_{]0,\tau_m]}(s)\Phi(s)\mathrm{d}W(s) = \int_0^t 1\!\!1_{]0,\tau_n]}(s)\Phi(s)\mathrm{d}W(s)\ \mathbb{P}\text{-a.s.}$$

on $\{\tau_m \geq t\} \subset \{\tau_n \geq t\}$. This result follows from the following lemma, which implies that the process in (1.4) is a continuous H-valued local martingale.

Lemma 1.2.3.2. *Assume that $\Phi \in \mathcal{N}_W$ and that τ is an $\mathcal{F}_t$-stopping time such that $\mathbb{P}(\tau \leq T) = 1$. Then there exists a $\mathbb{P}$-null set $N \in \mathcal{F}$ independent of $t \in [0,T]$ such that*

$$\int_0^t 1\!\!1_{]0,\tau]}(s)\Phi(s)dW(s) = Int(1\!\!1_{]0,\tau]}\Phi)(t) = Int(\Phi)(\tau \wedge t)$$

$$= \int_0^{\tau\wedge t} \Phi(s)dW(s) \text{ on } N^c \text{ for all } t \in [0,T].$$

Proof. Since both integrals which appear in the equation are $\mathbb{P}$-a.s. continuous we only have to prove that they are equal $\mathbb{P}$-a.s. at any fixed time $t \in [0,T]$.

Step 1: We first consider the case that $\Phi \in \xi$ and that τ is a simple stopping time which means that it takes only a finite number of values.

Let $0 = t_0 < t_1 < \ldots < t_k \leq T,\ k \in \mathbb{N}$, and

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi_m 1\!\!1_{]t_m,t_{m+1}]}$$

where $\Phi_m : \Omega \longrightarrow L(U,H)$ is $\mathcal{F}_{t_m}$-measurable and only takes a finite number of values for all $0 \leq m \leq k-1$.

If τ is a simple stopping time there exists $n \in \mathbb{N}$ such that $\tau(\Omega) = \{a_0, \ldots, a_n\}$ and

$$\tau = \sum_{j=0}^{n} a_j 1\!\!1_{A_j}$$

where $0 \leq a_j < a_{j+1} \leq T$ and $A_j = \{\tau = a_j\} \in \mathcal{F}_{a_j}$. In this way we get that $1\!\!1_{]\tau,T]}(s)\Phi$ is an elementary process since

$$\begin{aligned}
1\!\!1_{]\tau,T]}(s)\Phi(s) &= \sum_{m=0}^{k-1} \Phi_m 1\!\!1_{]t_m,t_{m+1}]\cap]\tau,T]}(s) \\
&= \sum_{m=0}^{k-1}\sum_{j=0}^{n} 1\!\!1_{A_j}\Phi_m 1\!\!1_{]t_m,t_{m+1}]\cap]a_j,T]}(s) \\
&= \sum_{m=0}^{k-1}\sum_{j=0}^{n} \underbrace{1\!\!1_{A_j}\Phi_m}_{\mathcal{F}_{t_m\vee a_j}-measurable} 1\!\!1_{]t_m\vee a_j,t_{m+1}\vee a_j]}(s)
\end{aligned}$$

and concerning the integral we are interested in, we obtain that

$$\int_0^t \mathbb{1}_{]0,\tau]}(s)\Phi(s)\mathrm{d}W(s) = \int_0^t \Phi(s)dW(s) - \int_0^t \mathbb{1}_{]\tau,T]}(s)\Phi(s)\mathrm{d}W(s)$$

$$= \sum_{m=0}^{k-1} \Phi_m(W(t_{m+1}\wedge t) - W(t_m \wedge t))$$

$$- \sum_{m=0}^{k-1}\sum_{j=0}^{n} \mathbb{1}_{A_j}\Phi_m\Big(W((t_{m+1}\vee a_j)\wedge t) - W((t_m \vee a_j)\wedge t)\Big)$$

$$= \sum_{m=0}^{k-1} \Phi_m(W(t_{m+1}\wedge t) - W(t_m \wedge t))$$

$$- \sum_{m=0}^{k-1}\sum_{j=0}^{n} \mathbb{1}_{A_j}\Phi_m\Big(W((t_{m+1}\vee \tau)\wedge t) - W((t_m \vee \tau)\wedge t)\Big)$$

$$= \sum_{m=0}^{k-1} \Phi_m(W(t_{m+1}\wedge t) - W(t_m \wedge t))$$

$$- \sum_{m=0}^{k-1} \Phi_m\Big(W((t_{m+1}\vee \tau)\wedge t) - W((t_m \vee \tau)\wedge t)\Big)$$

$$= \sum_{m=0}^{k-1} \Phi_m\Big(W(t_{m+1}\wedge t) - W(t_m \wedge t)$$

$$-W((t_{m+1}\vee \tau)\wedge t) - W((t_m \vee \tau)\wedge t)\Big)$$

$$= \sum_{m=0}^{k-1} \Phi_m\Big(W(t_{m+1}\wedge t \wedge \tau) - W(t_m \wedge t\wedge \tau)\Big) = \int_0^{t\wedge\tau} \Phi(s)\mathrm{d}W(s).$$

Step 2: Now we consider the case that Φ is still an elementary process while τ is an arbitrary stopping time with $\mathbb{P}(\tau \le T) = 1$.

Then there exists a sequence

$$\tau_n = \sum_{k=0}^{2^n-1} T(k+1)2^{-n}\mathbb{1}_{]Tk2^{-n},T(k+1)2^{-n}]} \circ \tau,\ n \in \mathbb{N},$$

of simple stopping times such that $\tau_n \downarrow \tau$ as $n \to \infty$ and because of the continuity of the stochastic integral we get that

$$\int_0^{\tau_n \wedge t} \Phi(s) \mathrm{d}W(s) \overset{n\to\infty}{\longrightarrow} \int_0^{\tau \wedge t} \Phi(s) \mathrm{d}W(s) \text{ } \mathbb{P}\text{-a.s.}$$

Besides, we obtain (even for non-elementary processes Φ) that

$$\left\| 1\!\!1_{]0,\tau_n]}\Phi - 1\!\!1_{]0,\tau]}\Phi \right\|_T^2 = \mathbb{E}\Big(\int_0^T 1\!\!1_{]\tau,\tau_n]}(s) \|\Phi(s)\|_{L_2^0}^2 \mathrm{d}s \Big) \overset{n\to\infty}{\longrightarrow} 0,$$

which by the definition of the integral implies that

$$\mathbb{E}\Big(\Big\| \int_0^t 1\!\!1_{]0,\tau_n]}(s)\Phi(s) \mathrm{d}W(s) - \int_0^t 1\!\!1_{]0,\tau]}(s)\Phi(s) \mathrm{d}W(s) \Big\|^2 \Big) \overset{n\to\infty}{\longrightarrow} 0,$$

for all $t \in [0,T]$. As by Step 1

$$\int_0^t 1\!\!1_{]0,\tau_n]}(s)\Phi(s) \mathrm{d}W(s) = \int_0^{\tau_n \wedge t} \Phi(s) \mathrm{d}W(s), \ n \in \mathbb{N}, \ t \in [0,T],$$

the assertion follows.

Step 3: Finally we generalize the statement to arbitrary $\Phi \in \mathcal{N}_W^2(0,T)$:

If $\Phi \in \mathcal{N}_W^2(0,T)$ then there exists a sequence of elementary processes $\Phi_n, n \in \mathbb{N}$, such that

$$\|\Phi_n - \Phi\|_T \overset{n\to\infty}{\longrightarrow} 0.$$

By the definition of the stochastic integral this means that

$$\int_0^\cdot \Phi_n(s) \mathrm{d}W(s) \overset{n\to\infty}{\longrightarrow} \int_0^\cdot \Phi(s) \mathrm{d}W(s) \text{ in } \mathcal{M}_T^2.$$

Hence it follows that there is a subsequence $n_k, k \in \mathbb{N}$, and a $\mathbb{P}$-null set $N \in \mathcal{F}$ independent of $t \in [0,T]$ such that

$$\int_0^t \Phi_{n_k}(s) \mathrm{d}W(s) \overset{k\to\infty}{\longrightarrow} \int_0^t \Phi(s) \mathrm{d}W(s) \text{ on } \mathrm{N}^c.$$

for all $t \in [0,T]$ and therefore we get for all $t \in [0,T]$ that

$$\int_0^{\tau \wedge t} \Phi_{n_k}(s) \mathrm{d}W(s) \overset{k\to\infty}{\longrightarrow} \int_0^{\tau \wedge t} \Phi(s) \mathrm{d}W(s) \ \ \mathbb{P}\text{-a.s.}$$

In addition, it is clear that

$$\| 1\!\!1_{]0,\tau]}\Phi_n - 1\!\!1_{]0,\tau]}\Phi \|_T \overset{n\to\infty}{\longrightarrow} 0$$

which implies that for all $t \in [0,T]$

$$\mathbb{E}\left(\left\| \int_0^t 1\!\!1_{]0,\tau]}(s)\Phi_n(s)\mathrm{d}W(s) - \int_0^t 1\!\!1_{]0,\tau]}(s)\Phi(s)\mathrm{d}W(s)\right\|^2\right) \overset{n\to\infty}{\longrightarrow} 0.$$

As by Step 2

$$\int_0^t 1\!\!1_{]0,\tau]}(s)\Phi_{n_k}(s)\mathrm{d}W(s) = \int_0^{\tau\wedge t} \Phi_{n_k}(s)\mathrm{d}W(s) \quad \mathbb{P}\text{-a.s.}$$

for all $k \in \mathbb{N}$ the assertion follows. □

Therefore, for $m < n$ on $\{\tau_m \geq t\} \subset \{\tau_n \geq t\}$

$$\int_0^t 1\!\!1_{]0,\tau_n]}(s)\Phi(s)\mathrm{d}W(s) = \int_0^{\tau_m\wedge t} 1\!\!1_{]0,\tau_n]}(s)\Phi(s)\mathrm{d}W(s)$$

$$= \int_0^t 1\!\!1_{]0,\tau_m]}(s)1\!\!1_{]0,\tau_n]}(s)\Phi(s)\mathrm{d}W(s) = \int_0^t 1\!\!1_{]0,\tau_m]}(s)\Phi(s)\mathrm{d}W(s) \ \mathbb{P}\text{-a.s.},$$

where we used Lemma 1.2.3.2 for the second equality. Hence the definition is consistent.

Remark 1.2.3.2. *In fact it is easy to see that the definition of the stochastic integral does not depend on the choice of $\tau_n, n \in \mathbb{N}$. If $\sigma_n, n \in \mathbb{N}$ is another sequence of stopping times such that $\sigma_n \uparrow T$ as $n \to \infty$ and $1\!\!1_{]0,\sigma_n]}(s)\Phi \in \mathcal{N}_W^2$ for all $n \in \mathbb{N}$ we also get that*

$$\int_0^t \Phi(s)dW(s) = \lim_{n\to\infty}\int_0^t 1\!\!1_{]0,\sigma_n]}(s)\Phi(s)dW(s) \ \mathbb{P}\text{-}a.s. \text{ for all } t \in [0,T].$$

Proof. Let $t \in [0,T]$. Then we get that on the set $\{\tau_m \geq t\}$

$$\begin{aligned}
\int_0^t \Phi(s)dW(s) &= \int_0^t 1\!\!1_{]0,\tau_m]}(s)\Phi(s)\mathrm{d}W(s) \\
&= \lim_{n\to\infty}\int_0^{t\wedge\sigma_n} 1\!\!1_{]0,\tau_m]}(s)\Phi(s)\mathrm{d}W(s) \\
&= \lim_{n\to\infty}\int_0^{t\wedge\tau_m} 1\!\!1_{]0,\sigma_n]}(s)\Phi(s)\mathrm{d}W(s) \\
&= \lim_{n\to\infty}\int_0^{t} 1\!\!1_{]0,\sigma_n]}(s)\Phi(s)\mathrm{d}W(s) \quad \mathbb{P}\text{-a.s.}.
\end{aligned}$$

□

1.3 Properties of the stochastic integral

Let T be a positive real number and $W(t), t \in [0,T]$, a Q-Wiener process as described at the beginning of the previous section.

Lemma 1.3.1. *Let Φ be a L_2^0-valued stochastically integrable process, $(\tilde{H}, \| \, \|_{\tilde{H}})$ a further separable Hilbert space and $L \in L(H, \tilde{H})$.*

Then the process $L(\Phi(t)), t \in [0,T]$, is an element of $\mathcal{N}_W(0,T;\tilde{H})$ and

$$L\Big(\int_0^T \Phi(t)dW(t)\Big) = \int_0^T L(\Phi(t))dW(t) \quad \mathbb{P}\text{-}a.s.$$

Proof. Since Φ is a stochastically integrable process and

$$\Big\| L(\Phi(t)) \Big\|_{L_2(U_0,\tilde{H})} \leq \|L\|_{L(H,\tilde{H})} \|\Phi(t)\|_{L_2^0},$$

it is obvious that $L(\Phi(t)), t \in [0,T]$, is $L_2(U_0, \tilde{H})$-predictable and

$$\mathbb{P}\Big(\int_0^T \Big\|L(\Phi(t))\Big\|^2_{L_2(U_0,\tilde{H})} dt < \infty\Big) = 1.$$

Step 1: As the first step we consider the case that Φ is an elementary process, i.e.

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi_m \mathbb{1}_{]t_m,t_{m+1}]}(t), \quad t \in [0,T],$$

where $0 = t_o < \cdots < t_k = T, \Phi_m : \ \Omega \longrightarrow L(U,H)$ is $\mathcal{F}_{t_m}$-measurable with $|\Phi_m(\Omega)| < \infty$ for $0 \leq m \leq k$. Then

$$L\Big(\int_0^T \Phi(t)\mathrm{d}W(t)\Big) = L\Big(\sum_{m=0}^{k-1} \Phi_m(W(t_{m+1}) - W(t_m))\Big)$$

$$= \sum_{m=0}^{k-1} L\Big(\Phi_m(W(t_{m+1}) - W(t_m))\Big) = \int_0^T L(\Phi(t))\mathrm{d}W(t).$$

Step 2: Now let $\Phi \in \mathcal{N}_W^2(0,T)$. Then there exists a sequence $\Phi_n, n \in \mathbb{N}$, of elementary processes with values in $L(U,H)_0$ such that

$$\|\Phi_n - \Phi\|_T = \Big(\mathbb{E}\Big(\int_0^T \|\Phi_n(t) - \Phi(t)\|^2_{L_2^0} dt\Big)\Big)^{\frac{1}{2}} \overset{n\to\infty}{\longrightarrow} 0.$$

Then $L(\Phi_n), n \in \mathbb{N}$ is a sequence of elementary processes with values in $L(U, \tilde{H})_0$ and

$$\|L(\Phi_n) - L(\Phi)\|_T \leq \| L\|_{L(H,\tilde{H})} \|\Phi_n - \Phi\|_T \stackrel{n\to\infty}{\longrightarrow} 0.$$

By the definition of the stochastic integral, Step 1 and the continuity of L we get that there is a subsequence $n_k, k \in \mathbb{N}$, such that

$$\begin{aligned}\int_0^T L(\Phi(t))\mathrm{d}W(t) &= \lim_{k\to\infty} \int_0^T L(\Phi_{n_k}(t))\mathrm{d}W(t) \\ &= \lim_{k\to\infty} L\Big(\int_0^T \Phi_{n_k}(t)\mathrm{d}W(t)\Big) \\ &= L\Big(\lim_{k\to\infty} \int_0^T \Phi_{n_k}(t)\mathrm{d}W(t)\Big) \\ &= L\Big(\int_0^T \Phi(t)\mathrm{d}W(t)\Big) \ \mathbb{P}\text{-a.s.}\end{aligned}$$

Step 3: Finally let $\Phi \in \mathcal{N}_W(0,T)$.

Let $\tau_n, n \in \mathbb{N}$ be a sequence of stopping times such that $\tau_n \uparrow T$ as $n \to \infty$ and $1\!\!1_{]0,\tau_n]}\Phi \in \mathcal{N}_W^2(0,T,H)$. Then $1\!\!1_{]0,\tau_n]}L(\Phi) \in \mathcal{N}_W^2(0,T,H)$ for all $n \in \mathbb{N}$ and we obtain by Remark 1.2.3.2 and Step 2 (selecting a subsequence if necessary)

$$\begin{aligned}\int_0^T L(\Phi(t))dW(t) &= \lim_{n\to\infty} \int_0^T 1\!\!1_{]0,\tau_n]}(t)L(\Phi(t))dW(t) \\ &= \lim_{n\to\infty} L\Big(\int_0^T 1\!\!1_{]0,\tau_n]}(t)\Phi(t)dW(t)\Big) \\ &= L\Big(\lim_{n\to\infty} \int_0^T 1\!\!1_{]0,\tau_n]}(t)\Phi(t)dW(t)\Big) \\ &= L\Big(\int_0^T \Phi(t)dW(t)\Big) \ \mathbb{P}\text{-a.s.}\end{aligned}$$

□

Lemma 1.3.2. *Let $\Phi \in \mathcal{N}_W(0,T)$ and f an $(\mathcal{F}_t)$-adapted continuous H-valued process. Set*

$$\int_0^T \langle f(t), \Phi(t)dW(t)\rangle := \int_0^T \tilde{\Phi}_f(t)dW(t) \tag{1.5}$$

with

$$\tilde{\Phi}_f(t)(u) := \langle f(t), \Phi(t)u\rangle,\ \ u \in U_0.$$

Then the stochastic integral in (1.5) is well-defined as a continuous $\mathbb{R}$-valued stochastic process. More precisely, $\tilde{\Phi}_f$ is a $\mathcal{P}_T/\mathcal{B}(L_2(U_0,\mathbb{R}))$-measurable map from $[0,T]\times\Omega$ to $L_2(U_0,\mathbb{R})$,

$$\|\tilde{\Phi}_f(t,\omega)\|_{L_2(U_0,\mathbb{R})} = \|\Phi^*(t,\omega)f(t,\omega)\|_{U_0}$$

for all $(t,\omega)\in[0,T]\times\Omega$ and

$$\int_0^T \|\tilde{\Phi}_f(t)\|^2_{L_2(U_0,\mathbb{R})}dt \le \sup_{t\in[0,T]}\|f(t)\|\int_0^T \|\Phi(t)\|^2_{L_2^0}dt < \infty \ \mathbb{P}\text{-}a.e..$$

Proof. Since f is continuous, $\tilde{\Phi}_f$ is clearly predictable. Let $e_k, k\in\mathbb{N}$, be an orthonormal basis of U_0. Then for all $(t,\omega)\in[0,T]\times\Omega$

$$\begin{aligned}
\|\tilde{\Phi}_f(t,\omega)\|^2_{L_2(U_0,\mathbb{R})} &= \sum_{k=1}^{\infty}\langle f(t,\omega),\Phi(t,\omega)_{e_k}\rangle^2 \\
&= \sum_{k=1}^{\infty}\langle \Phi^*(t,\omega)f(t,\omega),e_k\rangle^2_{U_0} \\
&= \|\Phi^*(t,\omega)f(t,\omega)\|^2_{U_0} \\
&\le \|\Phi^*(t,\omega)\|^2_{L(H,U_0)}\|f(t,\omega)\|^2_H \\
&\le \|\Phi^*(t,\omega)\|^2_{L_2(H,U_0)}\|f(t,\omega)\|^2_H \\
&= \|\Phi(t,\omega)\|^2_{L_2^0}\|f(t,\omega)\|^2_H,
\end{aligned}$$

where we used Remark 1.2.1.1 in the last step. Now all assertions follow. □

Lemma 1.3.3. *let $\Phi\in\mathcal{N}_W(0,T)$ and $M(t) := \int_0^t \Phi(s)dW(s), \quad t\in[0,T]$. Define*

$$\langle M\rangle_t := \int_0^t \|\Phi(s)\|^2_{L_2^0}ds, \quad t\in[0,T].$$

Then $\langle M\rangle$ is the unique continuous increasing $(\mathcal{F}_t)$-adapted process starting at zero such that $\|M(t)\|^2 - \langle M\rangle_t \quad t\in[0,T]$, is a local martingale. If $\Phi\in\mathcal{N}_W^2(0,T)$, then for any sequence

$$I_l := \{0 = t_0^l < t_1^l < \cdots < t_{k_l}^l = T\}, \quad l\in\mathbb{N},$$

of partitions with

$$\max_i(t_i^l - t_{i-1}^l)\to 0 \text{ as } l\to\infty$$

$$\lim_{l\to\infty} \mathbb{E}\Bigg(\Bigg|\sum_{t^l_{j+1}\le t} \|M(t^l_{j+1}) - M(t^l_j)\|^2 - \langle M\rangle_t\Bigg|\Bigg) = 0.$$

Proof. For $n \in \mathbb{N}$ let τ_n be as in (1.3) and τ an $\mathcal{F}_t$-stopping time with $\mathbb{P}[\tau \le T] = 1$. Then by Lemma 1.2.3.2 for $\sigma := \tau \wedge \tau_n,\ t \in [0,T]$

$$\begin{aligned}
\mathbb{E}\Bigg(\Bigg\|\int_0^{t\wedge\sigma} \Phi(s)dW(s)\Bigg\|^2\Bigg) &= \mathbb{E}\Bigg(\Bigg\|\int_0^t \mathbb{1}_{]0,\sigma]}\Phi(s)\mathrm{d}W(s)\Bigg\|^2\Bigg)\\
&= \mathbb{E}\Bigg(\int_0^t \|\mathbb{1}_{]0,\sigma]}\Phi(s)\|^2_{L_2^0}\mathrm{d}s\Bigg)\\
&= \mathbb{E}\Bigg(\int_0^{t\wedge\sigma} \|\Phi(s)\|^2_{L_2^0}\mathrm{d}s\Bigg),
\end{aligned}$$

and the first assertion follows, because the uniqueness is obvious, since any real-valued local martingale of bounded variation is constant.
To prove the second assertion we fix an orthonormal basis $\{e_i | i \in \mathbb{N}\}$ of H and note that by the theory of real-valued martingales we have for each $i \in \mathbb{N}$

$$\lim_{l\to\infty} \mathbb{E}\Bigg(\Bigg|\sum_{t^l_{j+1}\le t} \langle e_i, M(t^l_{j+1}) - M(t^l_j)\rangle^2_H - \int_0^t \|\Phi(s)^* e_i\|^2_{U_0}\mathrm{d}s\Bigg|\Bigg) = 0, \tag{1.6}$$

since by the first part of the assertion and Lemmas 1.5 and 1.6

$$\Bigg\langle \int_0^t \langle e_i, \Phi(s)\mathrm{d}W(s)\rangle_H \Bigg\rangle_t = \int_0^t \|\Phi(s)^* e_i\|^2_{U_0}\mathrm{d}s \quad t \in [0,T],$$

Furthermore, for all $i \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E}\Bigg(\Bigg|\sum_{t^l_{j+1}\le t} \langle e_i, M(t^l_{j+1}) - M(t^l_j)\rangle^2_H - \int_0^t \|\Phi(s)^* e_i\|^2_{U_0}\mathrm{d}s\Bigg|\Bigg)\\
&\le \sum_{t^l_{j+1}\le t} \mathbb{E}\Bigg[\Bigg(\int_{t^l_j}^{t^l_{j+1}} \langle e_i, \Phi(s)\mathrm{d}W(s)\rangle_H\Bigg)^2\Bigg] + \mathbb{E}\Bigg(\int_0^t \|\Phi(s)^* e_i\|^2_{U_0}\mathrm{d}s\Bigg)\\
&= \sum_{t^l_{j+1}\le t} \mathbb{E}\Bigg(\int_{t^l_j}^{t^l_{j+1}} \|\Phi(s)^* e_i\|^2_{U_0}\mathrm{d}s\Bigg) + \mathbb{E}\Bigg(\int_0^t \|\Phi(s)^* e_i\|^2_{U_0}\mathrm{d}s\Bigg)
\end{aligned}$$

$$\le 2\mathbb{E}\Bigg(\int_0^t \|\Phi(s)^* e_i\|^2_{U_0}\mathrm{d}s\Bigg) \tag{1.7}$$

which is summable over $i \in \mathbb{N}$. Here we used the isometry property of Int in the second to last step. But

$$\mathbb{E}\left(\left|\sum_{t^l_{j+1}\le t}\|M(t^l_{j+1})-M(t^l_j)\|2-\int_0^t\|\Phi(s)\|^2_{U_0}\mathrm{d}s\right|\right)$$

$$= \mathbb{E}\left(\left|\sum_{i=1}^{\infty}\left(\sum_{t^l_{j+1}\le t}\langle e_i, M(t^l_{j+1})-M(t^l_j)\rangle^2_H-\int_0^t\|\Phi(s)^* e_i\|^2_{U_0}ds\right)\right|\right)$$
$$\le \sum_{i=1}^{\infty}\mathbb{E}\left(\left|\sum_{t^l_{j+1}\le t}\langle e_i, M(t^l_{j+1})-M(t^l_j)\rangle^2_H-\int_0^t\|\Phi(s)^* e_i\|^2_{U^0}\mathrm{d}s\right|\right)$$

where we used Remark 1.2.1.1 in the second step. Hence the second assertion follows by Lebesgue dominated convergence theorem from (1.6) and (1.7). □

Chapter 2

Stochastic Differential Equations on Hilbert spaces

In the present chapter we introduce solution concepts to certain type of stochastic evolution problems and prove existence and uniqueness of their solutions. The mathematical framework is based on the theory of strongly continuous operator semigroups.

2.1 Basic concepts

2.1.1 Concept of solutions

We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a normal filtration $\mathcal{F}_t, t \geq 0$. We consider two Hilbert spaces H and U, and a Q-Wiener process W on $(\Omega, \mathcal{F}, \mathbb{P})$, with the covariance operator $Q \in L(U)$.
We will consider the following linear equation

$$\left\{ \begin{array}{lcl} \mathrm{d}X_t & = & [AX_t + f(t)]\mathrm{d}t + B\mathrm{d}W(t) \\ X_0 & = & \alpha \end{array} \right. \tag{2.1}$$

where we make the following assumptions.

$A : D(A) \subset H \longrightarrow H$ and $B : U \longrightarrow H$ are linear operators, f is an H-valued stochastic process.
with

- A generates a strongly continuous semigroup $S(.)$ in H,
- $B \in L(U, H)$, (2.2)

It is natural to require that

- f is a predictable process with Bochner integrable trajectories on arbitrary finite interval $[0, T]$, (2.3)
- α is $\mathcal{F}_0$-measurable.

Remark 2.1.1.1. *If W is a Q-Wiener process in U, then $W_t = BW$ is a BQB^*-Wiener process in H. So we could assume, without any restriction, that $U = H$. However in sorne applications, for example wave or delay equations, it is convenient to have B different from identity.*

- **Strong solution**: An H-valued predictable process $X(t), t \in [0, T]$, is said to be a strong solution to (2.1) if X takes values in $D(A)$, $\mathbb{P}_T$-a.s., $\displaystyle\int_0^T |AX(s)|\mathrm{d}s < +\infty$, $\mathbb{P}$-a.s., and for $t \in [0, T]$

$$X_t = \alpha + \int_0^t [AX_s + f(s)]\mathrm{d}s + BW(t), \ \mathbb{P}\text{-a.s.}, \tag{2.4}$$

  This definition is meaningful only if BW is a U-valued process and therefore requires that $BQB^* < +\infty$. Note that a strong solution should necessarily have continuous modification.

- **Weak solution**: An H-valued predictable process $X(t), t \in [0, T]$, is said to be a weak solution of (2.1) if the trajectories of $X(.)$ are $\mathbb{P}$-a.s. Bochner integrable and if for all $\varphi \in D(A^*)$ and all $t \in [0, T]$ we have

$$\begin{aligned} \langle X(t), \varphi \rangle &= \langle \alpha, \varphi \rangle_H + \int_0^t [\langle X(s), A^*\varphi \rangle + \langle f(s), \varphi \rangle]\mathrm{d}s \\ &+ \int_0^t \langle BW(t), \varphi \rangle, \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.5}$$

We will need the following lemma about interchanging the stochastic integral with closed operators.

Lemma 2.1.1.1. *Let E be a separable Hilbert space. Let $\Phi \in \mathcal{N}_W^2$ $A : D(A) \in H \longrightarrow E$ be a closed, linear operator with $D(A)$ being a Borel subset of H. If $\Phi(t)u \in D(A)$ $\mathbb{P}$-almost surely for all $t \in [0; T]$ and $u \in U$ and $A\Phi \in \mathcal{N}_W^2$, then*

$$\mathbb{P}\left(\int_0^T \Phi(s)\ \mathrm{d}W(s) \in D(A)\right) = 1$$

and

$$A\left(\int_0^T \Phi(s)\ \mathrm{d}W(s)\right) = \int_0^T A\Phi(s)\ \mathrm{d}W(s) \ \ \mathbb{P}\text{-a.s.} \tag{2.6}$$

Proof. The lemma is a special case of [8], Proposition 4.15. □

2.1.2 Stochastic convolution

It is of great importance in our study of linear and nonlinear equations to establish first the basic properties of the process

$$W_A(t) = \int_0^t S(t-s)B\mathrm{d}W(s),$$

which is called a stochastic convolution. They are collected in the following theorem.

Theorem 2.1.2.1. *Assume 2.2, and*

$$\int_0^T \|S(r)B\|_{L_2^0}^2 \,\mathrm{d}r = \int_0^t Tr[S(r)BQB^*S^*(r)] \,\mathrm{d}r < +\infty. \tag{2.7}$$

Then

1. *the process $W_A(.)$ is Gaussian, continuous in mean square and has a predictable version.*
2. *we have*

$$CovW_A(t) = \int_0^t S(r)BQB^*S^*(r) \,\mathrm{d}r, \quad t \in [0,T]. \tag{2.8}$$

Proof. see [8], Theorem 5.2, p. 120. □

2.2 Existence and uniqueness of weak solutions

The main result of this section is the following

Theorem 2.2.1. *Assume (2.2), (2.3) and (2.7). Then equation (2.1) has exactly one weak solution which is given by the formula*

$$\begin{aligned} X(t) &= S(t)\varphi + \int_0^t S(t-s)f(s)\,\mathrm{d}s \\ &+ \int_0^t S(t-s)f(s)B\,\mathrm{d}W(s), \quad t \in [0,T]. \end{aligned} \tag{2.9}$$

Proof. It easily follows from Proposition A.4 in [8], that the process X is a weak solution to (2.1) if and only if the process $\widetilde{X}$ given by the formula

$$\widetilde{X}(t) = X(t) - \left(S(t)\varphi + \int_0^t S(t-s)f(s)\mathrm{d}s\right), \quad t \in [0,T],$$

is a weak solution to

$$\mathrm{d}\widetilde{X} = A\widetilde{X}\mathrm{d}t + B\mathrm{d}W, \quad \widetilde{X}(0) = 0.$$

So, we can assume, without any Joss of generality, that $\varphi = 0$ and $f \equiv O$. To prove existence we show that equation (2.1) with $\varphi = 0$ and $f \equiv 0$ is satisfied by the process $W_A(.)$. Fix $t \in [0,T]$ and let $\varphi \in D(A^*)$. Note that

$$\int_0^t \langle A^*\varphi, W_A(s)\rangle \mathrm{d}s = \int_0^t \left\langle A^*\varphi, \int_0^t 1\!\!1_{[0,s]}(r)S(s-r)B\mathrm{d}W(r)\right\rangle \mathrm{d}s$$

and consequently,

$$\begin{aligned}
\int_0^t \langle A^*\varphi, W_A(s)\rangle \mathrm{d}s &= \int_0^t \left\langle \int_0^t 1\!\!1_{[0,s]}(r)B^*S^*(s-r)A^*\varphi \mathrm{d}s, \mathrm{d}W(r)\right\rangle \\
&= \int_0^t \left\langle \int_r^t B^*S^*(s-r)A^*\varphi \mathrm{d}s, \mathrm{d}W(r)\right\rangle \\
&= \int_0^t \left\langle \int_r^t (\frac{\mathrm{d}}{\mathrm{d}s}B^*S^*(s-r)\varphi)S^*(s-r)A^*\mathrm{d}s, \mathrm{d}W(r)\right\rangle \\
&= \int_0^t \langle B^*S^*(t-r)\varphi, \mathrm{d}W(r)\rangle \\
&= \langle \varphi, W_A(s)\rangle - \langle \varphi, BW(t)\rangle.
\end{aligned}$$

Therefore $W_A(.)$ is a weak solution. □

To prove uniqueness we need the following lemma.

Lemma 2.2.1. *Let X be a weak solution of problem (2.1) with $\varphi = 0$, $f \equiv O$. Then,for arbitrary function $\varphi(.) \in C^1([0,T]; D(A^*))$ and $t \in [0,T]$, we have*

$$\begin{aligned}
\langle X(t), \varphi(t)\rangle &= \int_0^t [\langle X(s), \varphi'(s) + A^*\varphi(s)\rangle] ds \\
&+ \int_0^t \langle \varphi(s), B dW(s)\rangle.
\end{aligned}$$

Proof. Consider first functions of the form $\varphi = \varphi_{0\phi(s)}$, $s \in [0,T]$, where $\phi \in C^1([0,T])$ and $\varphi_0 \in D(A^*)$. Let

$$F_{\varphi_0}(t) = \int_0^t \langle X(s), A^*\varphi_0\rangle \mathrm{d}s + \langle BW(t), \varphi_0\rangle.$$

Applying lto's formula to the process $F_{\varphi_0}(s)\phi(s)$ we get

$$\mathrm{d}[F_{\varphi_0}(s)\phi(s)] = \phi(s)\mathrm{d}F_{\varphi_0}(s) + \phi'(s)F_{\varphi_0}(s)\mathrm{d}s.$$

In particular

$$\begin{aligned} F_{\varphi_0}(t)\phi(t) &= \int_0^t \langle \varphi(s), B\mathrm{d}W(s)\rangle \\ &+ \int_0^t [\phi(s)\langle X(s), A^*\varphi_0\rangle + \phi'(s)\langle X(s), \varphi_0\rangle]\mathrm{d}s. \end{aligned}$$

Since $F_{\varphi_0}(.) = \langle X(.), \varphi_0\rangle$ the lemma is proved for the special function $\varphi(t) = \varphi_{0\phi(t)}$. Since these functions are linearly dense in $C^1([0,T]; D(A^*))$, the lemma is true in general. □

Let X be a weak solution and let $\varphi_0 \in D(A^*)$. Applying Lemma 2.2.1 to the function $\varphi(s) = S^*(t-s)\varphi_0,\ s \in [0,t]$ we have

$$\langle X(t), \varphi_0\rangle = \langle \int_0^t S(t-s)B\ dW(s), \varphi_0\rangle$$

and, since $D(A^*)$ is dense in H, we find that $X = W_A$. □

Example: *Delay equations.* We are concerned with the problem

$$\left.\begin{aligned} \mathrm{d}Z(t) &= \left[\int_{-r}^{\theta} a(\mathrm{d}\theta)Z(t+\theta)\right]\mathrm{d}t + f(t)\mathrm{d}t + \mathrm{d}W(t),\ t \geq 0, \\ Z(0) &= h_0, \\ Z(\theta) &= h_1(\theta),\ \theta \in [-r, 0], \text{a.s.} \end{aligned}\right\} \qquad (2.10)$$

where $a(.)$ is an $n \times n$ matrix valued finite measure on $[-r, 0]$, $f : [0, +\infty[\longrightarrow \mathbb{R}^N$ is a locally integrable function, $h_0 \in \mathbb{R}^N$, $h_1 \in L^2([-r, 0]; \mathbb{R}^N)$ and r is a positive number representing the maximal delay. In a similar way to that in the deterministic case we can associate with the equation (2.10) a stochastic linear equation

$$\mathrm{d}X_t = AX_t\mathrm{d}t + f(t)\mathrm{d}t + B\mathrm{d}W(t), \qquad X(0) = \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} \tag{2.11}$$

on the space $H = \mathbb{R}^N \oplus L^2([-r, O]; \mathbb{R}^N)$, where the generator A is given by:

$$\left.\begin{array}{rcl} D(A) & = & \left\{ \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} \in H; \;\; h_0 \in \mathbb{R}^N, \;\; \begin{array}{l} h_1 \in W^{1,2}(-r, 0, \mathbb{R}^N), \\ h_1(0) = h_o \end{array} \right\}, \\ \\ Ah & = & \begin{pmatrix} \displaystyle\int_{-r}^{\theta} a(\mathrm{d}\theta)h_1\theta \\ \mathrm{d}h_1/\mathrm{d}\theta \end{pmatrix}, \end{array} \right\} \tag{2.12}$$

and $B = (I, 0)$. In the present situation U is equal to $\mathbb{R}^N$. Obviously hypothesis (2.7) is fulfilled in this case and therefore the equation (2.11) has a unique weak solution. It bas been shown by several authors ([2], [6]) under different sets of assumptions that $X(t) = (Z(t), Z_t),\ t > 0$, where Z and Z_t are the solutions of the equation (2.10) and its segment respectively. This fact is important in the application of delay systems to control problems and stability.

Let us consider the special case when

$$a(.) = a_0\delta_0(.) + a_1\delta_{-r}(.)$$

where a_0, a_1 are $N \times N$ matrices. The equation (2.11) can now be solved by successive steps. In particular, for $t \in [0, r]$,

$$Z(t) = e^{ta_0}h_0 + \int_0^t e^{(t-r)a_1}h_1(s-r)\mathrm{d}r + \int_0^t e^{(t-r)a_0}\mathrm{d}W(r). \tag{2.13}$$

Taking into account that trajectories of the stochastic convolution part in (2.13) are never absolutely continuous, we can see that in this case weak solutions to (2.11) are never strong. □

2.3 Existence of strong solutions

We give in this section general sufficient condition for existence of strong solutions to (2.1). In addition to the hypothesis Tr$Q < +\infty$ we shall assume that $U = H$ and $B = I$.

Theorem 2.3.1. *Assume that*

1. *Tr* $Q < +\infty$, $U = H$, $B = I$ *and*[1] $A \in L_2(H)$,
2. $f \in C^1([0,T];H) \cap C([O,T] : D(A))$, *P-a.s.*,
3. $\alpha \in D(A)$, *P-a.s.*

Then problem (2.1) has a strong solution.

Proof. As remarked before we can assume that $f \equiv 0$, $\alpha = O$. Note that

$$\begin{aligned}\int_0^T \|AS(r)\|^2_{L_2(H)}\mathrm{d}r &= \int_0^T \|AS(r)Q^{\frac{1}{2}}\|^2_{L_2(H)}\mathrm{d}r \\ &\leq \|A\|^2_{L_2(H)}\int_0^T \|S(r)\|^2\mathrm{d}r < +\infty.\end{aligned}$$

Consequently, by Proposition 2.1.1.1 we have $W_A(t) \in D(A)$, $\mathbb{P}$-a.s and

$$AW_A(t) = \int_0^t AS(t-r)\mathrm{d}W(r), \quad t \in [0,T].$$

Since $\mathbb{E}\int_0^T \|AW_A(t)\|^2\mathrm{d}t < +\infty$, the process $AW_A(.)$ has square integrable trajectories. Since $W_A(.)$ is a weak solution of (2.1), for $t \in [0,T]$ and $\varphi \in D(A^*)$, $\mathbb{P}$-a.s.

$$\begin{aligned}\langle W_A(t),\varphi\rangle &= \int_0^t \langle W_A(s), A^*\varphi\rangle\mathrm{d}s + \langle W(t),\varphi\rangle \\ &= \int_0^t \langle AW_A(s),\varphi\rangle\mathrm{d}s + \langle W(t),\varphi\rangle \\ &= \langle A\int_0^t W_A(s)\mathrm{d}s,\varphi\rangle + \langle W(t),\varphi\rangle\end{aligned}$$

Consequently

$$W_A(t) = \int_0^t AW_A(s)\mathrm{d}s + W(t), \quad \mathbb{P}\text{-a.s.}$$

This finishes the proof. □

[1]This is equivalent to require that $ImQ^{\frac{1}{2}} \subset D(A)$ and $\mathrm{Tr}[AQ^{\frac{1}{2}}] < +\infty$

Theorem 2.3.2. *Assume that*

1. $(-A)^{\beta} \in L_2(H)$ *for sorne* $\beta \in]\frac{1}{2}, 1[$,
2. $f \in C^{\kappa}([0,T]; H) + C([0,T]; D_A(\kappa, \infty))$ *for sorne* $\kappa \in]0,1[$,
3. $\alpha \in D(A)$, $\mathbb{P}$*-a.s.*

Then problem (2.1) has a strong solution.

Proof. By Propositions A.23 in [8], we have that

$$\widetilde{X} = S(.)\varphi + \int_0^{\cdot} S(t-s) f(s) \mathrm{d}s \in C^1([0,T]; H) \cap C([0,T]; D(A)),$$

and moreover

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{X}(t) = A\widetilde{X}(t) + f(t), \quad t \in [0,T], \quad \widetilde{X}(0) = \alpha,$$

so, without any loss of generality, one can assume that $f = 0$ and $\alpha = O$. It follows from (2) that the process

$$W^{\beta}(t) = (-A)^{\beta} W(t), \quad t \in [0,T],$$

is a well-defined H -valued Wiener process with a nuclear covariance.
The process

$$Z(t) = (-A)^{1-\beta} \int_0^t S(t-s) \mathrm{d}W^{\beta}(s), \quad t \in [0,T],$$

is κ-Hölder continuous for $\kappa \in]0, \beta - \frac{1}{2}[$. But the distributions of the processes $Z(.)$ and $AW_A(.)$ are identical; so the process AW_A has a continuous modification. In a similar way to that in the proof of the previous theorem, we easily show that $W_A(.)$ is the strong solution. $\square$

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