People's Democratic Republic of Algeria
Ministry of High Education and Scientific Research

Number assigned by the library

Academic Year: 2022/2023

University of Saida Dr. Moulay Tahar.
Faculty of Sciences, Department of Mathematics
on the collaboration with
Btu Senftenburg University, Cottbus.

Thesis presented for the award of the diploma of
Academic Masters
Sector: Mathematics
Specialty: Stochastic analysis, statistic of processes and applications (SASPA)

Presented by
Bouchra MEBARKI [1]

Supervised by
Pr. Abdeljebbar Kandouci and **Dr. Omar Kebiri**

Theme:

Existence and uniqueness and properties of solution for nonlinear stochastic differential equations in infinite dimension

Board of Examiners

Dr. Mme R. Hazeb	University of Saida Dr. Moulay Tahar	President
Pr. A. Kandouci	University of Saida Dr. Moulay Tahar	Supervisor
Dr. O. Kebiri	University of Cottbus Btu senftenburg	Co-Supervisor
Dr. Mme F. Benziadi	University of Saida Dr. Moulay Tahar	Examiner
Pr. C. Hartmann	University of Cottbus Btu senftenburg	Invited

[1] e-mail: bouchramebarki727@yahoo.fr

Dedication

*All praise to **Allah**, today we fold the day's tiredness and the errand summing up between the cover of this humble work.*

I dedicate my work to:

*My great teacher and messenger, **Mohammed-peace and grace from Allah be upon him***, who taught us the purpose of life.

***My parents** , especially **my mother** who have been our source of inspiration and gave us strength when we thought of giving up, who continually provide their moral, spiritual, emotional, and financial. God save them.*

*Last but not least I am dedicating this to my sisters **Douab** and **Sara** and my friends **Hayat, Roumaissa** and **Manel**.*

Acknowledgments

In the Name of Allah, the Most Merciful, the Most Compassionate, all praise be to Allah, and prayers and peace be upon Prophet Muhammad, His servant and messenger.

I am deeply grateful and profoundly appreciative to my supervisors, Pr. A. Kandouci and Dr. O. Kebiri, for their continuous guidance and expertise. Their astute insights and constructive comments have greatly influenced the direction of my work. I would also like to express my gratitude for their generous allocation of time, immense patience, and the wealth of knowledge they shared with me as I prepared for this master's thesis.

I want to express my gratitude to the committee members, Dr. R. Hazeb and Dr. F. Benziadi, for examining my work and honoring their presence at our defense.

I would like to extend my thanks to the faculty members and staff at University of Saida Dr Moulay Tahar and BTU Cottbus-Senftenberg for their dedication to providing a conducive learning and research environment. Their commitment to education and scholarly pursuits has been instrumental in shaping my academic journey.

To all those mentioned above and to anyone who has contributed in any way, I offer my sincerest thanks. Your support, guidance, and encouragement have played a pivotal role in the successful completion of this thesis. May Allah bless you all abundantly.

Contents

Notations

$(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$	**Separable Hilbert space.**
$(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$	**Separable Hilbert space**
$(\Omega, \mathcal{F}, \mathbb{P})$	**Probability space.**
$\mathcal{C}(\mathbb{H}) = \mathcal{C}^0(\mathbb{H})$	**Collection of real-valued continuous functions on** $\mathbb{H}$
$\mathscr{M}_T^2(\mathbb{H})$	**Hilbert space of** $\mathbb{H}$**-valued continuous square integrable martingales.**
$\mathscr{B}(\mathbb{H})$	**Borel** σ**-field on** $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$.
$\mathscr{E}(\mathscr{L}(\mathbb{K}, \mathbb{H}))$	$\mathscr{L}(\mathbb{K}, \mathbb{H})$**-valued elementary processes**
$\mathscr{L}(\mathbb{K}, \mathbb{H})$	**Bounded linear operators from** $\mathbb{X}$ **to** $\mathbb{Y}$
$\mathscr{L}_1(\mathbb{H})$	**Trace-class operators on** $\mathbb{H}$
$\mathscr{L}^2(\mathbb{K}, \mathbb{H})$	**Hilbert-Schmidt operators from** $\mathbb{K}$ **to** $\mathbb{H}$
$\mathscr{D}(A)$	**Domain of an operator** A
A_λ	**Yosida approximation of an operator** A
$tr(A)$	**Trace of an operator** A
$S \star \phi(t)$	**Stochastic convolution**
Q_t	**Covariance of the stochastic convolution** $\int_0^t S(t-s)dW_s$
W	**Q-Wiener process**
$\tilde{W}$	**Cylindrical Wiener process**
$\langle M \rangle$	**Increasing process of a martingale** M_t
$\langle\langle M \rangle\rangle$	**Quadratic variation process of a martingale** M_t
$\mathbb{H}^k(0,1)$	**Sobolev space of square-integrable functions on** $[0,1]$ **with existing weak derivatives up to order** $k \in \mathbb{N}$.
$\mathbb{H}_0^1(0,1)$	**Sobolev space of square-integrable functions on** $[0,1]$ **with existing first order weak derivative . and zero boundary conditions.**
$\dot{\mathbb{H}}^s$	$\dot{\mathbb{H}}^s = dom(A^{\frac{s}{2}})$**, domain of** $A^{\frac{s}{2}}$
$\mathbb{1}_A$	**Indicator function for the measurable set** A
I_h	**Interpolation operator**
T^*	**Adjoint of a linear operator** T

Introduction

A non-linear stochastic differential equation (SDE) in infinite dimensions is a mathematical model used to describe the evolution of complex systems that exhibit non-linear behavior and involve an infinite number of variables. Non-linearity captures the intricate interactions and dependencies among variables that are often present in real-world phenomena, providing a more accurate representation of their dynamics. On the other hand, infinite dimensions are employed when dealing with systems that possess an uncountably infinite number of degrees of freedom, such as continuous media or fields, enabling a comprehensive description of the system's behavior across all possible variations and interactions. Together, the combination of non-linearity and infinite dimensions in non-linear SDEs allows for a powerful mathematical framework to analyze and understand the complex dynamics of diverse systems in various fields of study for examples in quantum field theory, SDEs in infinite dimensions are used to describe the dynamics of quantum fields, these equations help model particle interactions and phenomena at the smallest scales, contributing to our understanding of fundamental physics see [Liu+21], also in finance to model complex systems such as option pricing, portfolio optimization, and risk management where nonlinear SDEs in infinite dimensions can capture more intricate market dynamics and provide more accurate pricing models see [MT06], also they are utilized in statistical physics to describe the collective behavior of large systems, they help analyze phase transitions, critical phenomena, and the dynamics of complex systems such as spin glasses and disordered materials see [Bré21].

Backward stochastic differential equations (BSDEs) in infinite dimensions are powerful mathematical tools that extend the theory of BSDEs to the realm of infinite-dimensional spaces. Unlike their finite-dimensional counterparts, BSDEs in infinite dimensions deal with stochastic processes taking values in infinite-dimensional Hilbert spaces. These equations were first studied in [HP91] by Hu and Peng and after in [ØZ01] Oksendal and Zhang, they give an existence and uniqueness result with an operator infinitesimal generator of a strongly continuous semigroup and the coefficient Lipschitz in both variables. Pardoux and Rascanu [PR99] replaced the operator with the subdifferential of a convex function and assume that the coefficient is dissipative, everywhere defined and continuous with respect to the first variable,Lipschitz with respect to the second variable and with linear growth in both variables. BSDEs in infinite dimensions were also studied in [Con06], [Con07], [Fuh02], [GT05] and [PR99] in the more general case when the driver can be random. Also BSDEs was studied in [Fuh03], [FT04], [FT02].BSDEs play a crucial role in various areas of mathematical, for example, in stochastic optimal control see e.g. [Pen93] and references therein. See also the work of Oksendal in [Øks01] in this respect. BSDEs are applicable to some financial problems, as seen from the work of El Karoui et al. [EPQ97] and the work of Duffie and Epstein in [DE92] and references therein. In relation to PDEs, BSDEs have been proven to be important and useful, particularly, in giving a representation to the solution of certain PDEs. This representation generalizes the so-called Feynman-Kac formula, see [Pen91], [BL97], [PP05], [PR98] and references therein for other treatments of several cases in this subject.

This master memory falls into three chapters.

In chapter 1, we delve into the fascinating realm of stochastic calculus by exploring the concept of stochastic integration with respect to the Wiener process.First we give the definition of the Wiener process where they serve as a fundamental building block for modeling random phenomena with continuous paths. We will disccuse scheme of the construction of the stochastic starting by considering a certain class of elementary process and after that we extend the stochastic integral to the linear space by localization procedure. We also examine the Itô formula, a powerful tool that allows us to perform calculus on stochastic processes. We investigate its applicability in both the standard Wiener process and the cylindrical Wiener process, which plays a crucial role in the study of stochastic differential equations and mathematical finance. By the end of this chapter, readers will have gained a comprehensive understanding of the construction of the stochastic integral in general Hilbert spaces.

In chapter 2, we will focuses on the analysis of nonlinear stochastic differential equations (SDEs) driven by Wiener processes. We will start by giving some preliminary about element of semi-group and stochastic convolution we will define the different notions for a solution to the semi-linear SDE, strong, weak, and mild solutions. First, we will study existence and uniqueness of equations with Lipschitz nonlinearities, second we will review also the existence and uniqueness of nonlinear equations on Banach spaces in both cases additive and multiplicative noise.

In chapter 3, we will examine the fundamental issues of whether unique solutions to BSDEs in infinite dimensions exist and if they do. We will examine the complex interplay between the infinite dimensionality, stochasticity, and nonlinear dynamics of these equations. We will present a summary of the notable result in this field. We will go over how each result differs from the others, as well as the assumptions provided and the proof's technique. After that we will solve this equation first by project them on sup-space on finite dimension and then solve the corresponding finite dimension BSDE using least-squares Monte Carlo and plot the solution.

Chapter 1

Generalities on stochastic integral via Q-Wiener processes

Let $\mathbb{H}$ and $\mathbb{K}$ be two separable Hilbert spaces. This chapter is devoted to the construction of the stochastic Itô integral

$$\int_0^t \Phi(s)dW(s), \quad t \in [0,T],$$

where $W(\cdot)$ is a Wiener process on a Hilbert space $\mathbb{K}$ and Φ is a process with values that are linear but not necessarily bounded operators from $\mathbb{K}$ into a Hilbert space $\mathbb{H}$.

We begin by collecting basic facts about Hilbert space valued Wiener processes, including cylindrical Wiener processes. The stochastic integral is then defined in stages, starting from elementary processes and ending up with the most general.We also establish basic properties of the stochastic integral, including the Itô formula.

Definition 1.1 (Gaussian Measure). *A probability measure μ on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ is called Gaussian if for arbitrary $h \in \mathbb{H}$ there exist $m \in \mathbb{R}$, $q \geq 0$ such that,*

$$\mu(\{x \in \mathbb{H}; \langle\, h, x\rangle \in \mathrm{A}\}) = \mathcal{N}(m,q)(\mathrm{A}), \quad \forall \mathrm{A} \in \mathscr{B}(\mathbb{R}).$$

In particular, if μ is Gaussian, the following functional:

$$\mathbb{H} \to \mathbb{R}, \quad h \to \int_{\mathbb{H}} \langle\, h, x\rangle \mu(dx)$$

$$\mathbb{H} \times \mathbb{H} \to \mathbb{R}, \quad (h_1, h_2) \to \int_{\mathbb{H}} \langle\, h_1, x\rangle \langle\, h_2, x\rangle \mu(dx)$$

are well defined and continuous.

Definition 1.2 (Bochner integral)**.** *Let (X, Σ, μ) be a measure space, and $\mathbb{B}$ a Banach space. The Bochner integral of a function $f : X \to \mathbb{B}$ is defined in the same way as the Lebesgue integral. First, define a simple function to be any finite sum of the form:*

$$s(x) = \sum_{i=1}^{n} \mathbb{1}_{E_i}(x) b_i,$$

where the E_i are disjoint members of the σ-algebra Σ, the b_i are distinct elements of $\mathbb{B}$, and $\mathbb{1}_E$ is the characteristic function of E.

If $\mu(E_i)$ is finite whenever $b_i \neq 0$, then the simple function is integrable, and the integral is then defined by:

$$\int_X \left[\sum_{i=1}^{n} \mathbb{1}_{E_i}(x) b_i \right] d\mu = \sum_{i=1}^{n} \mu(E_i) b_i$$

exactly as it is for the ordinary Lebesgue integral.

1.1 Hilbert-Space-Valued and Cylindrical Wiener Processes

We first establish cylindrical Gaussian random variables and Hilbert-space-valued Gaussian random variables before naturally defining cylindrical Wiener process and Hilbert-space-valued Wiener process. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and $\mathbb{K}$ be a real separable Hilbert space with the norm and scalar product denoted by $\|.\|_{\mathbb{K}}$ and $\langle .,. \rangle_{\mathbb{K}}$ respectively. We always assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is complete.

Definition 1.3. *We say that $\tilde{X}$ is a cylindrical standard Gaussian random variable on $\mathbb{K}$ if $\tilde{X} : \mathbb{K} \to L^2(\Omega, \mathscr{F}, \mathbb{P})$ satisfies the following conditions:*

i) The mapping $\tilde{X}$ is linear.

ii) For an arbitrary $k \in \mathbb{K}$, $\tilde{X}(k)$ is a Gaussian random variable with mean zero and variance $\|k\|^2_{\mathbb{K}}$.

iii) If $k, k' \in \mathbb{K}$ are orthogonal, i.e., $\langle k, k' \rangle_{\mathbb{K}} = 0$, then the random variables $\tilde{X}(k)$ and $\tilde{X}(k')$ are independent.

Note that if $\{f_j\}_{j=1}^{\infty}$ is an orthonormal basis (ONB) in $\mathbb{K}$, then $\left\{\tilde{X}(f_j)\right\}_{j=1}^{\infty}$ is a sequence of independent Gaussian random variables with mean zero and variance one. By linearity of the mapping $\tilde{X} : \mathbb{K} \to L^2(\Omega, \mathscr{F}, \mathbb{P})$. We can represent $\tilde{X}$ as

$$\tilde{X}(k) = \sum_{j=1}^{\infty} \langle k, f_j \rangle_{\mathbb{K}} \tilde{X}(f_j),$$

for all k the series convergent $\mathbb{P}$-a.s. by Kolmogorov's three-series theorem.

However, it is not true that there exists a $\mathbb{K}$-valued random variable X such that

$$\tilde{X}(k)(\omega) = \langle X(\omega), k \rangle_{\mathbb{K}} .$$

This can be easily seen since we can express

$$\|X(\omega)\|^2_{\mathbb{K}} = \sum_{j=1}^{\infty} \langle X(\omega), f_j \rangle^2_{\mathbb{K}} ,$$

with the series being $\mathbb{P}$-a.s. divergent by the strong law of large numbers.

In order to produce a $\mathbb{K}$*-valued Gaussian random variable, we proceed as follows. Denote by* $\mathscr{L}_1(\mathbb{K})$ *the space of trace-class operators on* $\mathbb{K}$*.*

$$\mathscr{L}_1(\mathbb{K}) = \left\{ L \in \mathscr{L}(\mathbb{K}) : \tau(L) := \operatorname{tr}\left((LL^*)^{1/2} \right) < \infty \right\}, \tag{1.1}$$

where the trace of the operator $[L] = (LL^*)^{1/2}$ *is defined by*

$$\operatorname{tr}([L]) = \sum_{j=1}^{\infty} \langle [L] f_j, f_j \rangle_{\mathbb{K}},$$

for an ONB $\{f_j\}_{j=1}^{\infty} \subset \mathbb{K}$*. It is well known that* $\operatorname{tr}([L])$ *is independent of the choice of the ONB and that* $\mathscr{L}_1(K)$ *equipped with the trace norm* τ *is a Banach space. Let* $Q : \mathbb{K} \to \mathbb{K}$ *be a symmetric nonnegative definite trace-class operator.*

Assume that $X : \mathbb{K} \to L^2(\Omega, \mathscr{F}, \mathbb{P})$ *satisfies the following conditions:*

- *i) The mapping* X *is linear.*
- *ii) For an arbitrary* $k \in K$*,* $X(k)$ *is a Gaussian random variable with mean zero.*
- *iii) For arbitrary* $k, k' \in \mathbb{K}, \mathbb{E}\left(X(k) X\left(k'\right)\right) = \langle Qk, k' \rangle_{\mathbb{K}}$*.*

Let $\{f_j\}_{j=1}^{\infty}$ *be an ONB in* $\mathbb{K}$ *diagonalizing* Q*, and let the eigenvalues corresponding to the eigenvectors* f_j *be denoted by* λ_j*, so that* $Qf_j = \lambda_j f_j$*. We define*

$$X(\omega) = \sum_{j=1}^{\infty} X\left(f_j\right)(\omega) f_j.$$

Since $\sum_{j=1}^{\infty} \lambda_j < \infty$*, the series converge in* $L^2((\Omega, \mathscr{F}, \mathbb{P}), \mathbb{H})$ *and hence* $\mathbb{P}$*-a.s. In this case,* $\mathbb{P}$*-a.s.,*

$$\langle X(w), k\rangle_{\mathbb{K}} = X(k)(\omega),$$

so that $X : \Omega \to \mathbb{K}$ is $(\mathscr{F}, \mathscr{B}((K)))$-measurable, where $\mathscr{B}(\mathbb{K})$ denotes the Borel σ field on $\mathbb{K}$.

Definition 1.4. *The mapping $X : \Omega \to \mathbb{K}$ defined above is called a $\mathbb{K}$-valued Gaussian random variable with covariance Q.*

Definition 1.5. *Let $\mathbb{K}$ be a separable Hilbert space. The measure $\mathbb{P} \circ X^{-1}$ induced by a $\mathbb{K}$-valued Gaussian random variable X with covariance Q on the measurable Hilbert space $(\mathbb{K}, \mathscr{B}(\mathbb{K})$ is called a Gaussian measure with covariance Q on $\mathbb{K}$.*

1.1.1 Cylindrical and Q-Wiener Processes

Let $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$be a filtered probability space, and, as above, $\mathbb{K}$ a real separable Hilbert space. We always assume that the filtration $\mathscr{F}_t$ satisfies the usual conditions

i) $\mathscr{F}_0$ contains all the negligible parts.

ii) $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$.

Definition 1.6. *A $\mathbb{K}$-valued stochastic process $\{X_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called Gaussian if for any positive integer n and $t_1, \dots, t_n \geq 0$, $(X_{t_1}, \dots, X_{t_n})$ is a $\mathbb{K}^n$-valued Gaussian random vector.*

A standard cylindrical Wiener process can now be introduced using the concept of a cylindrical random variable.

Definition 1.7. *We call a family $\{\tilde{W}_t, t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ a cylindrical Wiener process in a Hilbert space $\mathbb{K}$ if:*

i) For an arbitrary $t \geq 0$, the mapping $\tilde{W}_t$: $\mathbb{K} \longrightarrow \mathscr{L}^2(\Omega, \mathscr{F}, \mathbb{P})$ is linear.

ii) For an arbitrary $k \in \mathbb{K}$, $\{\tilde{W}_t(k), t \geq 0\}$ is an $\mathbb{F}$-Brownian motion.

iii) For an arbitrary $k, k' \in \mathbb{K}$ *and* $t \geq 0, \quad \mathbb{E}(\tilde{W}_t(k)\tilde{W}_t(k')) = t\langle k, k'\rangle_{\mathbb{K}}$

Remark 1.1. *For every* $t > 0$, $\tilde{W}_t/\sqrt{t}$ *is a standard cylindrical Gaussian random variable, so that for any* $k \in \mathbb{K}$, $\tilde{W}_t(k)$ *can be represented as a* $\mathbb{P}$*-a.s. convergent series*

$$\tilde{W}_t(k) = \sum_{j=1}^{\infty} \langle k, k'\rangle_{\mathbb{K}} \tilde{W}_t(f_j), \tag{1.2}$$

where $\{f_j\}_{j=1}^{\infty}$ *is an ONB in* $\mathbb{K}$.
For the same reason a cylindrical Gaussian random variable cannot be realized as a $\mathbb{K}$*-valued random variable, there is no* $\mathbb{K}$*-valued process* (W_t) *such that*

$$\tilde{W}_i(k)(\omega) = \langle W_t(\omega), k\rangle_{\mathbb{K}}.$$

However, if Q *is a non-negative definite symmetric trace class operator on* $\mathbb{K}$*, then a* $\mathbb{K}$*-valued* Q*-Wiener process can be defined.*

Definition 1.8. *Let* Q *be a non-negative definite symmetric trace-class operator on a separable Hilbert space* $\mathbb{K}$,$\{f_j\}_{j=1}^{\infty}$ *be an ONB in* $\mathbb{K}$ *of the self-adjoint trace- class (hence compact) operator* Q*, and let the corresponding eigenvalues be* $\{\lambda_j\}_{j=1}^{\infty}$. *Let* $\{w_j(t)\}_{t\geq 0}, j = 1, 2\ldots,$ *a sequence of independent Brownian motions defined on* $(\Omega, \mathcal{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$*. In this case, the process admit the regular representation*

$$W_t = \sum_{j=1}^{\infty} \lambda_j^{1/2} w_j(t) f_j \tag{1.3}$$

is called a Q*-Wiener process in* $\mathbb{K}$.

We can assume that the Brownian motions $w_j(t)$ *are continuous. Then, the series (1.3) converge in* $L^2(\Omega, \mathcal{C}([0, T], \mathbb{K}))$ *for every interval* $[0, T]$.

Therefore, the $\mathbb{K}$*-valued* Q*-Wiener process can be assumed to be continuous.*

We denote

$$W_t(k) = \sum_{j=1}^{\infty} \lambda_j^{1/2} w_j(t) \langle f_j, k \rangle_{\mathbb{K}},$$

for any $k \in \mathbb{K}$*, with the series converging in* $L^2(\Omega, \mathcal{C}([0,T], \mathbb{R}))$ *on every interval* $[0,T]$.

Remark 1.2. *A stronger convergence result can be obtained for these series since*

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le T}\left\|\sum_{j=m}^{n} \lambda_j^{1/2} w_j(t) f_j\right\|_{\mathbb{K}} > \epsilon\right) &\le \frac{1}{\epsilon^2}\mathbb{E}\left\|\sum_{j=m}^{n} \lambda_j^{1/2} w_j(T) f_j\right\|_{\mathbb{K}}^2 \\ &= \frac{T}{\epsilon^2}\sum_{j=m}^{n} \lambda_j \to 0\end{aligned}$$

with $m \le n, m, n \to \infty$*, the series converge uniformly on* $[0,T]$ *in probability* $\mathbb{P}$*, and by the Levy-lio-Nisio theorem, it also converges* $\mathbb{P}$*-a.s. uniformly on* $[0,T]$.

The basic properties of a Q-Wiener process are summarized in the next theorem.

Theorem 1.1. *A* $\mathbb{K}$*-valued Q-Wiener process* $\{W_t\}_{t\ge 0}$ *has the following properties:*

1. $W_0 = 0$
2. *W has continuous trajectories in* $\mathbb{K}$.
3. *W has independent increments.*
4. *W is a Gaussian process with the covariance operator Q, i.e., for any* $k, k' \in \mathbb{K}$ *and* $s, t \ge 0$,

$$\mathbb{E}\left(W_t(k) W_s(k')\right) = (t \wedge s)\langle Qk, k' \rangle_{\mathbb{K}}$$

5. *For an arbitrary $k \in \mathbb{K}$,we have the law*

$$\mathcal{L}\left(\left(W_t - W_s\right)(k)\right) \sim \mathcal{N}\left(0, (t-s)\left\langle Qk, k\right\rangle_{\mathbb{K}}\right).$$

Remark 1.3. *A Q-Wiener process is a Cylindrical Wiener process if the operator Q is the identity operator.*

1.2 Stochastic Integral with Respect to a Wiener Process

In this section, we introduce the concept of Itô's stochastic integral with respect to a Q-Wiener process and with respect to a cylindrical Wiener process.

Let $\mathbb{K}$ and $\mathbb{H}$ be a separable Hilbert spaces, $\mathscr{L}(\mathbb{H}_1, \mathbb{H}_2)$ denotes the space of bounded linear operators from $\mathbb{H}_1$ to $\mathbb{H}_2$ and Q be either a symmetric non-negative definite trace class operator on $\mathbb{K}$ or $Q = \mathbf{I}_{\mathbb{K}}$, the identity operator on $\mathbb{K}$. If Q of trace class, we will always assume that its all eigenvalues are positives, i.e. $\lambda_j > 0$, $j = 1, 2, \ldots$; otherwise we can start with the Hilbert space $ker(Q)^{\perp}$ instead of $\mathbb{K}$. The associated eigenvectors forming an ONB in $\mathbb{K}$ will be denoted by f_k. Then the space $\mathbb{K}_Q = Q^{1/2}\mathbb{K}$ equipped with the scalar product

$$\langle u, v\rangle_{\mathbb{K}_Q} = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \langle u, f_j\rangle_{\mathbb{K}} \langle v, f_j\rangle_{\mathbb{K}}$$

is a separable Hilbert space with an ONB $\left\{\lambda_j^{1/2} f_j\right\}_{j=1}^{\infty}$. If $\mathbb{H}_1$, $\mathbb{H}_2$ are two real separable Hilbert spaces with $\{e_i\}_{i=1}^{\infty}$ an ONB in $\mathbb{H}_1$, then the space of Hilbert-Schmidt operators $\mathbb{H}_1$ to $\mathbb{H}_2$ is defined as :

$$\mathscr{L}^2(\mathbb{H}_1, \mathbb{H}_2) = \left\{ L \in \mathscr{L}(\mathbb{H}_1, \mathbb{H}_2) : \sum_{i=1}^{\infty} \|Le_i\|_{\mathbb{H}_2}^2 < \infty \right\}. \tag{1.4}$$

It is well known that $\mathscr{L}^2(\mathbb{H}_1, \mathbb{H}_2)$ *equipped with the norm*

$$\|L\|_{\mathscr{L}^2(\mathbb{H}_1,\mathbb{H}_2)} = \left(\sum_{i=1}^{\infty} \|Le_i\|_{\mathbb{H}_2}^2\right)^{1/2}$$

is a Hilbert space. Since the Hilbert spaces $\mathbb{H}_1$ *and* $\mathbb{H}_2$ *are separable, the space* $\mathscr{L}^2(\mathbb{H}_1, \mathbb{H}_2)$ *is also separable, as Hilbert-Schmidt operators are limites of sequences of finite-dimensional linear operators.*

Consider $\mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})$*,if* $\{e_j\}_{j=1}^{\infty}$ *is an ONB in* $\mathbb{H}$ *then the Hilbert-Schmidt norm of an operator* $L \in \mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})$ *is given by :*

$$\begin{aligned}\|L\|^2_{\mathscr{L}^2,(\mathbb{K}_Q,\mathbb{H})} &= \sum_{j,i=1}^{\infty} \left\langle L\left(\lambda_j^{1/2} f_j\right), e_i\right\rangle_{\mathbb{H}}^2 = \sum_{j,i=1}^{\infty} \left\langle LQ^{1/2} f_j, e_i\right\rangle_{\mathbb{H}}^2 \\ &= \left\|LQ^{1/2}\right\|^2_{\mathscr{L}^2(\mathbb{K},\mathbb{H})} = \mathrm{Tr}\left(\left(LQ^{1/2}\right)\left(LQ^{1/2}\right)^*\right).\end{aligned} \tag{1.5}$$

The scalar product between two operators $L, M \in \mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})$ *is defined by:*

$$\langle L, M\rangle_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})} = \mathrm{Tr}\left(\left(LQ^{1/2}\right)\left(MQ^{1/2}\right)^*\right). \tag{1.6}$$

Since the Hilbert spaces $\mathbb{K}_Q$ *and* $\mathbb{H}$ *are separables, the space* $\mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})$ *is also separable.*

Let $L \in \mathscr{L}(\mathbb{K}, \mathbb{H})$*. If* $k \in \mathbb{K}_Q$*, then*

$$k = \sum_{j=1}^{\infty} \left\langle k, \lambda_j^{1/2} f_j\right\rangle_{\mathbb{K}_Q} \lambda_j^{1/2} f_j$$

and L*, considered as an operator from* $\mathbb{K}_Q$ *to* $\mathbb{H}$*, defined as*

$$Lk = \sum_{j=1}^{\infty} \left\langle k, \lambda_j^{1/2} f_j\right\rangle_{\mathbb{K}_Q} \lambda_j^{1/2} L f_j$$

has a finite Hilbert-Schmidt norm, since

$$\|L\|^2_{\mathscr{L}^2\left(\mathbb{K}_Q,\mathbb{H}\right)} = \sum_{j=1}^{\infty} \left\| L\left(\lambda_j^{1/2} f_j\right)\right\|^2_{\mathbb{H}} = \sum_{j=1}^{\infty} \lambda_j \|Lf_j\|^2_{\mathbb{H}} \leq \|L\|^2_{\mathscr{L}(\mathbb{K},\mathbb{H})}\, tr\,(Q)\,.$$

As previously stated, in order to define the stochastic integral, we begin with elementary processes.

1.2.1 Stochastic Itô Integral for Elementary Processes

Let $\mathscr{E}(\mathscr{L}(\mathbb{K},\mathbb{H}))$ *denote the class of* $\mathscr{L}(\mathbb{K},\mathbb{H})$*-valued elementary processes adapted to the filtration* $\{\mathscr{F}_t\}_{t\leq T}$ *that are of the form*

$$\Phi(t,\omega) = \phi(\omega)1_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j(\omega)1_{(f_j,t_{j+1}]}(t). \tag{1.7}$$

where $0 = t_0 \leq t_1 \leq \cdots t_n = T$*, and* $\phi, \phi_j, j = 0, 1, \ldots, n-1$*, are respectively* $\mathscr{F}_0$*-measurable and* $\mathscr{F}_{t_j}$*-measurable* $\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})$*-valued random variables such that* $\phi(\omega), \phi_j(\omega) \in \mathscr{L}(\mathbb{K},\mathbb{H}), j = 0, 1, \ldots, n-1$ *(recall that* $\mathscr{L}(\mathbb{K},\mathbb{H}) \subset \mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})$*).*

Note that if $Q = I_K$*, then the random variables* ϕ_j *are, in fact,* $\mathscr{L}^2(\mathbb{K},\mathbb{H})$*-valued.*

We say that an elementary process $\phi \in \mathscr{E}(\mathscr{L}(\mathbb{K},\mathbb{H}))$ *is bounded if it is bounded in* $\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})$*.*

Remark 1.4. *We have defined elementary processes to be left-continuous as opposed to being right-continuous. There is no difference if the Itô stochastic integral is constructed with respect to a Wiener process. Our choice, however, is consistent with the construction of a stochastic integral with respect to square-integrable martingales.*

Definition 1.9. *For an elementary process* $\phi \in \mathscr{E}(\mathscr{L}(\mathbb{K},\mathbb{H}))$*, we define the Itô stochastic integral with respect to a* Q*-Wiener process* W *by*

$$\int_0^t \Phi(s)dW_= \sum_{j=0}^{\infty} \phi_j \left(W_{t_{j+1}\wedge t} - W_{t_j\wedge t}\right)$$

Note that in reality the above is a finite sum, so there is no danger in how we take the limit. Indeed it can alternatively be expressed as

$$\sum_{j=0}^{k-1} \phi_j \left(W_{t_{j+1}} - W_{t_j}\right) + \phi_k \left(W_t - W_{t_k}\right)$$

where k is such that $t \in (t_k, t_{k+1})$.

We define the Itô cylindrical stochastic integral of an elementary process $\phi \in \mathcal{E}(\mathscr{L}(\mathbb{K},\mathbb{H}))$ with respect to a cylindrical Wiener process $\tilde{W}$ by

$$\left(\int_0^t \phi(s)d\tilde{W}_s\right)(h) = \sum_{j=0}^{n-1} \left(\tilde{W}_{t_{j+1}\wedge t}\left(\phi_j^*(h)\right) - \tilde{W}_{t_j\wedge t}\left(\phi_j^*(h)\right)\right) \tag{1.8}$$

for $t \in [0,T]$ and $h \in \mathbb{H}$.

Now we define the integral for a more general class of integrands, using approximations by simple processes.

1.2.2 Stochastic Itô Integral with Respect to a Q-Wiener Process

We are ready to extend the definition of the Itô stochastic integral with respect to a Q-Wiener process to adapted stochastic processes $\Phi(s)$ satisfying the condition:

$$\mathbb{E}\left(\int_0^T \|\Phi(s)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})}\, ds\right) < \infty$$

In other words, $\Psi \in L^2(\Omega \times [0,T]; \mathcal{H})$ where the domain space $\Omega \times [0,T]$ is a *measure space equipped with the product measure* $\mathbb{P} \times \lambda$.

Unlike what is typically found in the literature, we have made the definition for progressively measurable processes rather than previsible processes. Progressive measurability is a weaker condition than previsibility, but thankfully most reasonably behaved processes (adapted and left continuous for example) are both progressively measurable and previsible. Here, we define the more broad class of integrands in situations when the integrator is continuous. Other author's may search for previsible processes as these become necessary in retaining nice properties (e.g. martingality) when defining the stochastic integral with respect to discontinuous integrators, or even in making the definition itself. which will be further relaxed to the condition

$$\mathbb{P}\left(\int_0^T \|\Phi(s)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})}\, ds < \infty\right) = 1$$

Let denote elementary processes satisfying the above condition are elements of Λ_2*.This set represents our class of integrands.*

The stochastic integral of a process $\phi \in \Lambda_2(\mathbb{K}_Q, \mathbb{H})$ *with respect to a* $\mathbb{K}$*-valued* Q*-Wiener process* $\{W(t)\}$ *is the unique isometric linear extension of the mapping*

$$\Phi(\cdot) \to \int_0^T \Phi(s) dW_s$$

from the class of bounded elementary processes to $\mathscr{L}^2(\Omega, \mathbb{H})$*, to a mapping from* $\Lambda_2(\mathbb{K}_Q, \mathbb{H})$ *to* $\mathscr{L}^2(\Omega, \mathbb{H})$.

Such that the image of

$$\phi(t) = \phi 1_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j 1_{(t_j, t_{j+1}]}(t) \text{ is } \sum_{j=0}^{n-1} \phi_j \left(W_{t_{j+1}} - W_{t_j}\right).$$

Definition 1.10. *We define the stochastic integral process* $\int_0^t \Phi(s) dW_s, \quad 0 \leq$

$t \leq T$, *for* $\Phi \in \Lambda_2(\mathbb{K}_Q, \mathbb{H})$ *by:*

$$\int_0^t \Phi(s) dW_s = \int_0^T \Phi(s) \mathbb{1}_{[0,t]} dW_s$$

We provide some background knowledge on martingales in order to describe the relationship between $\Lambda_2(\mathbb{K}_Q, \mathbb{H})$ *and the space of continuous square-integrable martingales* $\mathscr{M}_{\mathrm{T}}^2(\mathbb{H})$

Definition 1.11. *A martingale* $\{M_t\}_{0 \leq t \leq T}$ *is called square integrable if*

$$\mathbb{E} \|M_T\|_{\mathbb{H}}^2 < \infty$$

The class of continuous square-integrable martingales will be denoted by $\mathscr{M}_T^2(\mathbb{H})$.

Since $M_t \in \mathscr{M}_T^2(H)$ *is determined by the relation* $M_t = E(M_T \mid \mathscr{F}_t)$, *the space* $\mathscr{M}_T^2(\mathbb{H})$ *is a Hilbert space with scalar product*

$$\langle M, N \rangle_{\mathscr{M}_{\mathbb{H}}^2} = E(\langle M_T, N_T \rangle_{\mathbb{H}}).$$

In the case of real-valued martingales $M_t, N_t \in \mathscr{M}_T^2(\mathbb{R})$. *there exist unique quadratic variation and cross quadratic variation processes, denoted by* $\langle M \rangle_t$, *and* $\langle M, N \rangle_t$, *respectively, such that* $M_t^2 - \langle M \rangle_t$ *and* $M_t N_t - \langle M, N \rangle_t$ *are continuous martingales. For Hilbert-space-valued martingales, we have the following definition.*

Definition 1.12. *Let* $M_t \in \mathscr{M}_T^2(\mathbb{H})$. *We denote by* $\langle M \rangle_t$ *the unique adapted continuous increasing process starting from 0 such that* $\|M_t\|_{\mathbb{H}}^2 - \langle M_t \rangle_t$ *is a continuous martingale. We define a quadratic variation process* $\langle\langle M \rangle\rangle_t$ *of* M_t *as an adapted continuous process starting from 0 , with values in the space of nonnegative definite trace-class operators on* $\mathbb{H}$, *such that for all* $h, g \in \mathbb{H}$.

$$\langle M_{\mathrm{t}}, h \rangle_{\mathbb{H}} \langle M_{\mathrm{t}}, g \rangle_{\mathbb{H}} - \langle \langle\langle M \rangle\rangle_t (h), g \rangle_{\mathbb{H}}$$

is a martingale.

Theorem 1.2. *The stochastic integral* $\Phi \to \int_0^t \Phi(s) dW_s$ *with respect to a* $\mathbb{K}$*-valued* Q*-Wiener process* W *is an isometry between* $\Lambda_2(\mathbb{K}_Q, \mathbb{H})$ *and the space of continuous square-integrable martingales* $\mathscr{M}_T^2(\mathbb{H})$*,*

$$E \left\| \int_0^t \Phi(s) dW_s \right\|_{\mathbb{H}}^2 = E \int_0^t \|\Phi(s)\|_{\mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})}^2 ds < \infty \tag{1.9}$$

for $t \in [0, T]$.

The quadratic variation process of the stochastic integral process $\int_0^t \Phi(s) dW_s$*, and the increasing process related to* $\left\| \int_0^t \Phi(s) dW_s \right\|_{\mathbb{H}}^2$ *are given by:*

$$\ll \int_0^{\cdot} \Phi(s) dW_s \gg_t = \int_0^t \left(\Phi(s) Q^{1/2}\right) \left(\Phi(s) Q^{1/2}\right)^* ds$$

and

$$\left\langle \int_0^{\cdot} \Phi(s) dW_s \right\rangle_t = \int_t^0 tr \left(\left(\Phi(s) Q^{1/2}\right) \left(\Phi(s) Q^{1/2}\right)^* \right) ds = \int_0^t \|\Phi(s)\|_{\mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})}^2 ds$$

Proof. We note that the stochastic integral process for a bounded elementary process in $\mathscr{E}(\mathscr{L}(\mathbb{K}, \mathbb{H}))$ is a continuous square-integrable martingale. Let the sequence of bounded elementary processes $\{\phi_n\}_{n=1}^{\infty} \subset \mathscr{E}(\mathscr{L}(\mathbb{K}, \mathbb{H}))$ approximate $\Phi \in \Lambda_2(\mathbb{K}_Q, \mathbb{H})$. We can assume that $\phi_1 = 0$ and

$$\|\phi_{n+1} - \phi_n\|_{\Lambda_2(\mathbb{K}_Q, \mathbb{H})} < \frac{1}{2^n}. \tag{1.10}$$

□

Theorem 1.3 (Doob's Maximal Inequalities)**.** *If* $M_t \in L^p(\Omega, \mathscr{F}, \mathbb{P})$ *is an* H*-valued martingale, then*

(1) $P\left(\sup_{0\leq t\leq T}\|M_t\|_{\mathbb{H}} > \lambda\right) \leq \dfrac{1}{\lambda^p} E\|M_T\|_{\mathbb{H}}^p, p \geq 1, \lambda > 0;$

(2) $E\left(\sup_{0\leq t\leq T}\|M_t\|_{\mathbb{H}}^p\right) \leq \left(\dfrac{p}{p-1}\right)^p E\|M_T\|_{\mathbb{H}}^p, p > 1.$

Let us now finish the proof, by Doob's inequality, Theorem 1.3, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\sup_{t\leq T}\left\|\int_0^t \phi_{n+1}(s)dW_s - \int_0^t \phi_n(s)dW_s\right\|_{\mathbb{H}} > \frac{1}{n^2}\right)\\
&\leq \sum_{n=1}^{\infty} n^4 E\left\|\int_0^T (\phi_{n+1}(s) - \phi_n(s))\, dW_s\right\|_{\mathbb{H}}^2\\
&= \sum_{n=1}^{\infty} n^4 E\int_0^T \|\phi_{n+1}(s) - \phi_n(s)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})}\, ds \leq \sum_{n=1}^{\infty}\frac{n^4}{2^n}.
\end{aligned}$$

By the Borel-Cantelli lemma,

$$\sup_{t\leq T}\left\|\int_0^t \phi_{n+1}(s)dW_s - \int_0^t \phi_n(s)dW_s\right\|_{\mathbb{H}} \leq \frac{1}{n^2}, \quad n > N(\omega),$$

for some $N(\omega)$, $\mathbb{P}$-a.s. Consequently, the series

$$\sum_{n=1}^{\infty}\left(\int_0^t \phi_{n+1}(s)dW_s - \int_0^t \phi_n(s)dW_s\right)$$

converges to $\int_0^t \Phi(s)dW_s$ in $L^2(\Omega,\mathbb{H})$ for every $t \leq T$ and converges $\mathbb{P}$-a.s. as a series of $\mathbb{H}$-valued continuous functions to a continuous version of $\int_0^t \Phi(s)dW_s$.

Thus, the mapping $\Phi \to \int_0^{\cdot} \Phi(s)dW_s$ is an isometry from the subset of bounded elements in $\mathscr{E}(\mathscr{L}(\mathbb{K},\mathbb{H}))$ into the space of continuous square-integrable martingales $\mathscr{M}_T^2(\mathbb{H})$, and it extends to $\Lambda_2(\mathbb{K}_Q,\mathbb{H})$ with images in $\mathscr{M}_T^2(\mathbb{H})$ by the completeness argument. We only need to verify the formula for the quadratic variation process. We use the representation 1.3, we have for $h \in \mathbb{H}$,

$$\left\langle \int_0^t \Phi(s) dW_s, h \right\rangle_{\mathbb{H}} = \sum_{j=1}^{\infty} \int_0^t \left\langle \lambda_j^{1/2} \Phi(s) f_j, h \right\rangle_{\mathbb{H}} dw_j(t),$$

with the series convergent in $L^2(\Omega, \mathbb{R})$.

If $h, g \in \mathbb{H}$, *then*

$$\begin{aligned}\left\langle \left(\Phi(s)Q^{1/2}\right)\left(\Phi(s)Q^{1/2}\right)^* h, g \right\rangle_{\mathbb{H}} &= \sum_{j=1}^{\infty} \left\langle h, \Phi(s)Q^{1/2} f_j \right\rangle_{\mathbb{H}} \left\langle g, \Phi(s)Q^{1/2} f_j \right\rangle_{\mathbb{H}} \\ &= \sum_{j=1}^{\infty} \lambda_j \left\langle h, \Phi(s) f_j \right\rangle_{\mathbb{H}} \left\langle g, \Phi(s) f_j \right\rangle_{\mathbb{H}}.\end{aligned}$$

Now, for $0 \leq u \leq t$,

$$\begin{aligned}&E\Bigg(\left\langle \int_0^t \Phi(s) dW_s, h \right\rangle_{\mathbb{H}} \left\langle \int_0^t \Phi(s) dW_s, g \right\rangle_{\mathbb{H}} \\ &- \left\langle \left(\int_0^t \left(\Phi(s)Q^{1/2}\right)\left(\Phi(s)Q^{1/2}\right)^* ds \right)(h), g \right\rangle_H \mid \mathscr{F}_u \Bigg) \\ &= E\Bigg(\sum_{i=1}^{\infty} \int_0^t \lambda_i^{1/2} \langle \Phi(s) f_i, h \rangle_{\mathbb{H}} dw_i(s) \sum_{j=1}^{\infty} \int_0^t \lambda_j^{1/2} \langle \Phi(s) f_j, g \rangle_{\mathbb{H}} dw_j(s) \\ &- \sum_{j=1}^{\infty} \int_0^t \lambda_j \langle h, \Phi(s) f_j \rangle_{\mathbb{H}} \langle g, \Phi(s) f_j \rangle_{\mathbb{H}} ds \mid \mathscr{F}_u \Bigg) \\ &= \left\langle \int_0^u \Phi(s) dW_s, h \right\rangle_{\mathbb{H}} \left\langle \int_0^u \Phi(s) dW_s, h \right\rangle_{\mathbb{H}} \\ &- \left\langle \left(\int_0^u \left(\Phi(s)Q^{1/2}\right)\left(\Phi(s)Q^{1/2}\right)^* ds \right)(h), g \right\rangle_{\mathbb{H}}\end{aligned}$$

The formula for the increasing process follows from this Lemma:

Lemma 1.1. *The quadratic variation process of a martingale* $M_t \in \mathscr{M}_T^2(\mathbb{H})$ *exists and is unique. Moreover,*

$$\langle M \rangle_t = tr \, \langle\langle M \rangle\rangle_t$$

Proof. The lemma follows by applying the classical one-dimensional results. We can assume without loss of generality that $M_0 = 0$. Denote

$$M_t^i = \langle M_t, e_i \rangle_{\mathbb{H}}$$

where $\{e_i\}_{i=1}^{\infty}$ is an ONB in $\mathbb{H}$. Note that the quadratic variation process has to satisfy

$$\langle \langle\langle M \rangle\rangle_t (e_i), e_j \rangle_H = \langle M_i, M_j \rangle_t$$

Consequently, we define the quadratic variation process by

$$\langle \langle\langle M \rangle\rangle_t (h), g \rangle_{\mathbb{H}} = \sum_{i,j=1}^{\infty} \langle M_i, M_j \rangle_t \langle e_i, h \rangle_{\mathbb{H}} \langle e_j, g \rangle_{\mathbb{H}} \tag{1.11}$$

The sum in (1.11) converges $\mathbb{P}$-a.s. and defines a nonnegative definite trace-class operator on $\mathbb{H}$, since

$$\mathbb{E} \, \mathrm{tr} \, \langle\langle M \rangle\rangle_t = \mathbb{E} \sum_{i=1}^{\infty} \langle M_t, e_i \rangle_{\mathbb{H}}^2 = \mathbb{E} \, \|M_t\|_{\mathbb{H}}^2 < \infty \tag{1.12}$$

□

Now equality (1.11) follows from (1.12).

The following corollary follows from the proof of Theorem 1.2.

Corollary 1.1. *For the sequence of bounded elementary processes approximating $\Phi \in \mathscr{E}(\mathscr{L}(\mathbb{K}, \mathbb{H}))$ and satisfying condition (1.10), the corresponding* *stochastic integrals converge uniformly with probability one,*

$$\mathbb{P}\left(\sup_{t\leq T}\left\|\int_0^t \Phi(s)_n dW_s - \int_0^t \Phi(s) dW_s\right\|_{\mathbb{H}} \to 0\right) = 1.$$

Remark 1.5. *For $\Phi \in \Lambda_2(\mathbb{K}_Q, \mathbb{H})$ such that $\Phi(s) \in \mathscr{L}(\mathbb{K}, \mathbb{H})$, the quadratic variation process of the stochastic integral process $\int_0^t \Phi(s) dW_s$ and the increasing process related to $\left\|\int_0^t \Phi(s) dW_s\right\|_{\mathbb{H}}^2$ simplify to*

$$\ll \int_0^{\cdot} \Phi(s) dW_s \gg_t = \int_0^t \Phi(s) Q \Phi(s)^* ds$$

and

$$\left\langle \int_0^{\cdot} \Phi(s) dW_s \right\rangle = \int_0^t tr(\Phi(s) Q \Phi(s)^*) ds.$$

The final step in constructing the Itô stochastic integral is to extend it to the class of integrands satisfying a less restrictive assumption on their second moments.

This extension is necessary if one wants to study Itô formula even for functions as simple as $x \to x^2$. In this chapter, we will only need the concept of a real-valued progressively measurable process, but in Chap. 2 we will have to refer to $\mathbb{H}$-valued progressively measurable processes. Therefore we include a more general definition here.

Definition 1.13. *An H-valued stochastic process $X(t), t \geq 0$, defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is called progressively measurable if for every $t \geq 0$, the mapping*

$$X(.,.) : ([0,t], \mathscr{B}([0,t])) \times (\Omega, \mathscr{F}_t) \to (\mathbb{H}, \mathscr{B}(\mathbb{H}))$$

is measurable with respect to the indicated σ-fields.

It is well known that an adapted rightcontinuous (or left-continuous) process is progressively measurable.

Let $\mathscr{P}(\mathbb{K}_Q, \mathbb{H})$ *denote the class of* $\mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})$*-valued stochastic processes adapted to the filtration* $\{\mathscr{F}_t\}_{t\leq T}$ *, measurable as mappings from* $([0,T] \times \Omega, \mathscr{B}([0,T]) \otimes \mathscr{F}_T)$ *to* $(\mathscr{L}^2(\mathbb{K}_Q, \mathbb{H}), \mathscr{B}(\mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})))$*, and satisfying the condition*

$$\mathbb{P}\left\{\int_0^T \|\Phi(t)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})}\, dt < \infty\right\} = 1 \tag{1.13}$$

Obviously, $\Lambda_2(\mathbb{K}_Q, \mathbb{H}) \subset \mathscr{P}(\mathbb{K}_Q, \mathbb{H})$*. We will show that processes from* $\mathscr{P}(\mathbb{K}_Q, \mathbb{H})$ *can be approximated in a suitable way by processes from* $\Lambda_2(\mathbb{K}_Q, \mathbb{H}))$ *and, in fact, by bounded elementary processes from* $\mathscr{E}(\mathscr{L}(\mathbb{K}, \mathbb{H}))$*. This procedure will allow us to derive basic properties of the extended stochastic integral.*

Lemma 1.2. *Let* $\Phi \in \mathscr{P}(\mathbb{K}_Q, \mathbb{H})$*. Then there exists a sequence of bounded processes* $\phi_n \in \mathscr{E}(\mathscr{L}(\mathbb{K}, \mathbb{H})) \subset \Lambda_2(\mathbb{K}_Q, \mathbb{H})$ *such that*

$$\int_0^T \|\Phi(t,\omega) - \phi_n(t,\omega)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})}\, dt \to 0 \quad \text{as} \quad n \to \infty \tag{1.14}$$

in probability and $\mathbb{P}$ *-a.s.*

1.3 The Itô Formula

One of the fundamental results in stochastic calculus is discussed in this section. The Itô formula is used to compute the derivative of a random process with unpredictability, also known as "noise." It enables us to describe the derivative of both a random process and another process function.

1.3.1 The case of a Q-Wiener process

If $\phi \in \mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})$ *and* $\psi \in \mathbb{H}$*, then* $\phi^*\psi \in \mathscr{L}^2(\mathbb{K}_Q, \mathbb{R})$*, since*

$$\begin{aligned}\|\phi^*\psi\|^2_{\mathscr{L}^2\left(\mathbb{K}_Q,\mathbb{R}\right)} &= \sum_{j=1}^{\infty} \left((\phi^*\psi)\left(\lambda^{1/2} f_j\right)\right)^2 = \sum_{j=1}^{\infty} \left\langle \psi, \phi\left(\lambda^{1/2} f_j\right)\right\rangle^2_{\mathbb{H}} \\ &\leq \|\psi\|^2_{\mathbb{H}} \|\phi\|^2_{\mathscr{L}^2\left(\mathbb{K}_Q,\mathbb{H}\right)}\end{aligned}$$

Hence, if $\Phi(s) \in \mathscr{P}(\mathbb{K}_Q, \mathbb{H})$ *and* $\Psi(s) \in \mathbb{H}$ *are* $\mathscr{F}_t$*-adapted processes, then the process* $\Phi^*(s)\Psi(s)$ *defined by*

$$(\Phi^*(s)\Psi(s))(k) = \langle \Psi(s), \Phi(s)(k)\rangle_{\mathbb{H}}$$

has values in $\mathscr{L}^2(\mathbb{K}_Q, \mathbb{R})$*. If, in addition,* $\mathbb{P}$*-a.s.,* $\Psi(s)$ *is bounded as a function of s, then*

$$\mathbb{P}\left(\int_0^T \|\Phi^*(s)\Psi(s)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{R})} ds < \infty\right) = 1,$$

so that $\Phi^*(s)\Psi(s) \in \mathscr{P}(\mathbb{K}_Q, \mathbb{R})$*, and we can define*

$$\int_0^T \langle \Psi(s), \Phi(s) dW_s\rangle_{\mathbb{H}} = \int_0^T \Phi^*(s)\Psi(s) dW_s.$$

Theorem 1.4. *Let Q be a symmetric non-negative trace class operator on a separable Hilbert space* $\mathbb{K}$*, and let* $\{W_t\}_{0\le t\le T}$ *be a Q-Wiener process on a filtered probability space* $(\Omega, \mathscr{F}, \mathbb{F} = \{\mathscr{F}_t\}, \mathbb{P})$*. Assume that a stochastic process* $X(t), 0 \le t \le T$*, is given by:*

$$X(t) = X(0) + \int_0^t \Psi(s)ds + \int_0^t \Phi(s)dW_s \tag{1.15}$$

Let $\mathscr{P}(\mathbb{K}_Q, \mathbb{H})$ *denote the class of* $\mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})$*-valued stochastic processes adapted to the filtration* $\{\mathscr{F}_t\}_{t\le T}$*, measurable as mappings from* $([0,T] \times \Omega,\ \mathscr{B}([0,T]) \otimes \mathscr{F}_T)$ *to* $\left(\mathscr{L}^2(\mathbb{K}_Q, \mathbb{H}), \mathscr{B}\left(\mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})\right)\right)$ *, and satisfying the condition:*

$$\mathbb{P}\left\{\int_0^T \|\Phi(t)\|^2_{\mathscr{L}^2(K_Q,H)} dt < \infty\right\} = 1.$$

Obviously, $\Lambda_2(\mathbb{K}_Q, \mathbb{H}) \subset \mathscr{P}(\mathbb{K}_Q, \mathbb{H})$*. where* $X(0)$ *is an* $\mathscr{F}_0$*-measurable* $\mathbb{H}$*-valued random variable,* $\Psi(s)$ *is an* $\mathbb{H}$*-valued* $\mathscr{F}_s$*-measurable* $\mathbb{P}$*-a.s. Bochner-integrable process on* $[0,T]$*,* $\int_0^T \|\Psi(s)\|_{\mathbb{H}} ds < \infty$ $\mathbb{P}$*-a.s and* $\Phi \in \mathscr{P}(\mathbb{K}_Q, \mathbb{H})$*.*

Assume that a function $F : [0,T] \times \mathbb{H} \to \mathbb{R}$ *is such that* F *is continuous and its Frechet partial derivatives* F_t, F_x, F_{xx} *are continuous and bounded an bounded subsets of* $[0,T] \times \mathbb{H}$. *Then the following Itô's formula holds:*

$$\begin{aligned} F(t, X(t)) =& F(0, X(0)) + \int_0^t \langle F_x(s, X(s)), \Phi(s) dW_s \rangle_{\mathbb{H}} \\ &+ \int_0^t \{F_t(s, X(s)) + \langle F_x(s, X(s)), \Psi(s) \rangle_{\mathbb{H}} \\ &+ \frac{1}{2} \left[F_{xx}(s, X(s))(\Phi(s)Q^{1/2})(\Phi(s)Q^{1/2})^* \right] ds\} \end{aligned} \tag{1.16}$$

$\mathbb{P}$*-a.s for all* $t \in [0,T]$.

For the proof see Section 2.3.1 in [GM10].

1.3.2 The Case of a Cylindrical Wiener Process

As in the case of a Q-Wiener process, for $\phi(s) \in \mathscr{P}(\mathbb{K}, \mathbb{H})$ *and a* $\mathbb{P}$*-a.s. bounded* H*-valued* F_t*-adapted process* $\psi(s), \phi^*(s)\psi(s) \in \mathscr{P}(\mathbb{K}, \mathbb{R})$. *In addition, since*

$$\sum_{j=1}^{\infty} \left(\left(\phi^*(s)\psi(s) \right) (f_j) \right)^2 = \sum_{j=1}^{\infty} \langle \psi(s), \phi(s)(f_j) \rangle_{\mathbb{H}}^2 \leq \|\psi(s)\|_{\mathbb{H}}^2 \|\Phi(s)\|_{\mathscr{L}^2\left(\mathbb{K}_Q, \mathbb{H}\right)}^2:$$

the process $\phi^*(s)\psi(s)$ *can be considered as being* $\mathbb{K}$*-or* $\mathbb{K}^*$*-valued, and we can define*

$$\int_0^T \left\langle \psi(s), \Phi(s) d\tilde{W}_s \right\rangle_{\mathbb{H}} = \int_0^T \left\langle \phi^*(s)\psi(s), d\tilde{W}_s \right\rangle_{\mathbb{K}} = \int_0^T \phi^*(s)\psi(s) d\tilde{W}_s$$

Theorem 1.5. *Let* $\mathbb{K}$ *and* $\mathbb{H}$ *be real separable Hilbert spaces, and* $\left\{\tilde{W}_t\right\}_{0 \leq t \leq T}$ *be a* K*-valued cylindrical Wiener process on a filtered probability space* $\left(\Omega, \mathscr{F}, \mathbb{F} = \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P}\right)$.

Assume that a stochastic process $X(t), 0 \leq t \leq T$, is given by

$$X(t) = X(0) + \int_0^t \Psi(s)ds + \int_0^t \phi(s)d\tilde{W}_s \tag{1.17}$$

where $X(0)$ is an $\mathscr{F}_0$-measurable $\mathbb{H}$ valued random variable, $\psi(s)$ is an $\mathbb{H}$-valued $\mathscr{F}_1$-measurable $\mathbb{P}$-a.s. Bochner-integroble process on $[0,T]$.

$$\int_0^T \|\psi(s)\|_{\mathbb{H}} ds < \infty, \quad P - a.s.$$

and $\mathscr{P} \in \rho(\mathbb{K}, \mathbb{H})$.

Assume that a function $F : [0,T] \times \mathbb{H} \to \mathbb{R}$ is such that F is continuous and its Frechet partial derivatives $F_{\mathrm{t}}, F_x, F_{xx}$ are continuous and bounded on bounded subsets of $[0,T]\times$ $\mathbb{H}$. Then the following Ito's formula holds:

$$\begin{aligned} F(t, X(t)) =& F(0, X(0)) + \int_0^t \left\langle F_x(s, X(s)), \phi(s)\tilde{W}_s \right\rangle_{\mathbb{H}} \\ &+ \int_0^t \{F_t(s, X(s)) \left\langle F_x(s, X(s)), \psi(s) \right\rangle_{\mathbb{H}} \\ &+ \frac{1}{2} \operatorname{tr}\left[F_{xx}(s, X(s))\phi(s)(\phi(s))^*\right]\} ds \end{aligned} \tag{1.18}$$

$\mathbb{P}$ -a.s. for all $t \in [0,T]$.

For the proof see Section 2.3.2 in [GM10].

Chapter 2

Analysis of non-Linear SDEs driven by Wiener processes

Before we begin, we review the fundamentals theory of linear operator semi-group, as well as basic facts about the sub-differential of the norm in a Banach space $\mathbb{E}$ and general properties of dissipative mappings in $\mathbb{E}$. We will also discuss stochastic convolution in order to investigate the existence and uniqueness of solutions.

2.1 Elements of semi-group theory

Semi-group theory is a branch of abstract algebra that deals with the study of collections of transformations or "operators" that satisfy certain algebraic properties. In the context of SDEs, the relevant semi-group is the Markov semi-group, which describes the evolution of the probability distribution of a stochastic process over time.

By using semi-group theory, it is possible to prove various existence and uniqueness results for solutions to SDEs in infinite dimensions. In particular, the Hille-Yosida theorem is a powerful tool that is often used to prove the existence of a unique strong solution to an SDE in an infinite-dimensional space.

Another important concept in semi-group theory that is useful for studying SDEs is the notion of a generator. The generator of a semi-group is an operator that describes the infinitesimal behavior of the semi-group, and can be used to study the long-term behavior of solutions to SDEs.

This section is similar with Sec 1.2 in [GM10] with a small modification .

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ *and* $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ *be Banach spaces. Denote by* $\mathscr{L}(\mathbb{X}, \mathbb{Y})$ *the family of bounded linear operators from* $\mathbb{X}$ *to* $\mathbb{Y}$*.* $\mathscr{L}(\mathbb{X}, \mathbb{Y})$ *becomes a Banach space when equipped with the norm*

$$\|T\|_{\mathscr{L}(\mathbb{X},\mathbb{Y})} = \sup_{x\in\mathbb{X}, \|x\|_{\mathbb{X}}=1} \|Tx\|_{\mathbb{Y}}, \quad T \in \mathscr{L}(\mathbb{X}, \mathbb{Y}).$$

For brevity, $\mathscr{L}(\mathbb{X})$ *will denote the Banach space of bounded linear operators on* $\mathbb{X}$*. The identity operator on* $\mathbb{X}$ *is denoted by* I*.*

Let $\mathbb{X}^*$ *denote the dual space of all bounded linear functionals* x^* *on* $\mathbb{X}, \mathbb{X}^*$ *is again a Banach space under the supremum norm*

$$\|x^*\|_{\mathbb{X}^*} = \sup_{x\in\mathbb{X}, \|x\|_{\mathbb{X}}=1} |\langle x, x^*\rangle|,$$

Where $\langle\cdot,\cdot\rangle$ *denotes the duality on* $\mathbb{X} \times \mathbb{X}^*$*.*

For $T \in \mathscr{L}(\mathbb{X}, \mathbb{Y})$*, the adjoint operator* $T^* \in \mathscr{L}(\mathbb{Y}^*, \mathbb{X}^*)$ *is defined by*

$$\langle x, T^*y^*\rangle = \langle Tx, y^*\rangle, \quad x \in \mathbb{X}, y^* \in \mathbb{Y}.$$

Let $\mathbb{H}$ *be a real Hilbert space. A linear operator* $T \in \mathscr{L}(\mathbb{H})$ *is called symmetric if for all* $h, g \in \mathbb{H}$*.*

$$\langle Th, g\rangle_{\mathbb{H}} = \langle h, Tg\rangle_{\mathbb{H}}.$$

A symmetric operator T is called nonnegative definite if for every $h \in \mathbb{H}$,

$$\langle Th, h \rangle_{\mathbb{H}} \geq 0.$$

Definition 2.1. *A family $S(t) \in \mathscr{L}(\mathbb{H}), t \geq 0$, of bounded linear operators on a Banach space $\mathbb{X}$ is called a strongly continuous semi-group(or a*

Let $S(t)$ be a

$$\|S(t)\|_{\mathscr{L}(\mathbb{H})} \leq Me^{\alpha t}, \quad t \geq 0.$$

If $M = 1$, then $S(t)$ is called a pseudo-contraction semi-group. If $\alpha = 0$, then $S(t)$ called uniformly bounded, and if $\alpha = 0$ and $M = 1$ (i.e., $\|S(t)\|_{\mathscr{L}(\mathbb{H})} \leq 1$, then $S(t)$ is called a semi-group of contractions. If for every $x \in \mathbb{X}$, the mapping $t \to S(t)x$ is differentiable for $T > 0$, then $S(t)$ is called a differentiable semi-group. A semi-group of linear operators $\{S(t), t \geq 0\}$ is called compact if the operators $S(t), t > 0$, are compact.

For any

$$t \in \mathbb{R}_+ \to S(t)x \in \mathbb{X}$$

is continuous.

Definition 2.2. *Let $S(t)$ be a*

$$\mathbb{D}(A) = \left\{ x \in \mathbb{X} : \lim_{t \to 0^+} \frac{S(t)x - x}{t} \textit{exists} \right\} \tag{2.1}$$

defined by

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t} \tag{2.2}$$

is called the infinitesimal generator of the semi-group $S(t)$.

A semi-group $S(t)$ is called uniformly continuous if

$$\lim_{t\to 0^+} \|S(t)-I\|_{\mathscr{L}(\mathbb{X})} = 0$$

Theorem 2.1. *A linear operator Ais the infinitesimal generator of a uniformly continuous semi-group $S(t)$ on a Banach space $\mathbb{X}$ if and only if $A \in \mathscr{L}(\mathbb{X})$. We have*

$$S(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

the series converge in norm for every $t \geq 0$.

We will interested in the case where $A \in \mathscr{L}(\mathbb{X})$.
The following theorem provides useful facts about semi-groups.

Theorem 2.2. *Let A be an infinitesimal generator of a*

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} S(t)xds = S(t)x. \tag{2.3}$$

For $x \in \mathbb{X}, S(t)t \in \mathscr{D}(A)$ and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax. \tag{2.4}$$

For $x \in \mathbb{X}$, $\displaystyle\int_0^t S(s)xds \in \mathscr{D}(A)$. and

$$A\left(\int_0^t S(s)xds\right) = S(t)x - x. \tag{2.5}$$

If $S(t)$ is differentiable then for $n = 1, 2, \ldots S(t) : \mathbb{X} \to \mathscr{D}(A^n)$ we have

$$S^{(n)}(t) = A^n S(t) \in \mathscr{L}(\mathbb{X}).$$

If $S(t)$ is compact then $S(t)$ is continuous in the operator topology for $t > 0$, i.e.,

$$\lim_{s\to t, s,t>0} \|S(s) - S(t)\|_{\mathscr{L}(\mathbb{H})} = 0. \tag{2.6}$$

For $x \in \mathscr{D}(A)$,

$$S(t)x - S(s)x = \int_s^t S(u)Axdu = \int_s^t AS(u)xdu. \tag{2.7}$$

$\mathscr{D}(A)$ is dense in $\mathbb{X}$, and A is a closed linear operator.

The intersection $\bigcap_{n=1}^{\infty} \mathscr{D}(A^n)$ is dense in $\mathbb{X}$.

Let $\mathbb{X}$ be a reflexive Banach space. Then the adjoint semi-group $S(t)^*$ of $S(t)$ is a

If $\mathbb{X} = \mathbb{H}$, a real separable Hilbert space, then for $h \in \mathbb{H}$, define the graph norm

$$\|h\|_{\mathscr{D}(A)} = \left(\|h\|_{\mathbb{H}}^2 + \|Ah\|_{\mathbb{H}}^2\right)^{1/2}. \tag{2.8}$$

Then $(\mathscr{D}(A), \|\cdot\|_{\mathscr{D}(A)})$ is a real separable Hilbert space.

Let $\mathscr{B}(\mathbb{H})$ denote the Borel σ-field on $\mathbb{H}$. Then $\mathscr{D}(A) \in \mathscr{B}(\mathbb{H})$, and

$$A : \left(\mathscr{D}(A), \mathscr{B}(\mathbb{H})|_{\mathscr{D}(A)}\right) \to (\mathbb{H}, \mathscr{B}(\mathbb{H})).$$

Consequently, the restricted Borel σ -field $\mathscr{B}(\mathbb{H})|_{\mathscr{D}(A)}$ coincides with the Borel σ -field on the Hilbert space $(\mathscr{D}(A), \|\cdot\|_{\mathscr{D}(A)})$, and measurability of $\mathscr{D}(A)$-valued functions can be understood with respect to either Borel σ -field.

Theorem 2.3. *Let $f : [0,T] \to \mathscr{D}(A)$ be measurable, and let $\int_0^t \|f(s)\|_{\mathscr{D}(A)} < \infty$. Then*

$$\int_0^t f(s)ds \in \mathscr{D}(A) \text{ and } \int_0^t Af(s)ds = A\int_0^t f(s)ds. \tag{2.9}$$

Definition 2.3. *The resolvent set $\rho(A)$ of a closed linear operator A on a Banach space $\mathbb{X}$ is the set of all complex numbers λ for which $\lambda I - A$ has a bounded inverse, i.e., the operator $(\lambda I - A)^{-1} \in \mathscr{L}(\mathbb{X})$.The family of bounded linear operators*

$$R(\lambda, A) = (\lambda I - A)^{-1}, \quad \lambda \in \rho(A), \tag{2.10}$$

is called the resolvent of A.

We note that $R(\lambda, A)$ is a one-to-one transformation of $\mathbb{X}$ onto $\mathscr{D}(A)$,i.e.,

$$\begin{aligned} (\lambda I - A)R(\lambda, A)x = x, \quad x \in \mathbb{X}, \\ R(\lambda, A)(\lambda I - A)x = x, \quad x \in \mathscr{D}(A). \end{aligned} \tag{2.11}$$

In particular,

$$AR(\lambda, A)x = R(\lambda, A)Ax, \quad x \in \mathscr{D}(A). \tag{2.12}$$

In addition, we have the following commutativity property:

$$R(\lambda_1, A)\, R(\lambda_2, A) = R(\lambda_2, A)\, R(\lambda_1, A), \quad \lambda_1, \lambda_2 \in \rho(A). \tag{2.13}$$

The following statement is true in greater generality; however, we will use it only in the real domain.

Proposition 2.1. *[GM10]Let $S(t)$ be a*

Furthermore, for each $x \in \mathbb{X}$,

$$\lim_{\lambda \to \infty} \|\lambda R(\lambda, A)x - x\|_{\mathbb{X}} = 0. \tag{2.14}$$

Theorem 2.4 (Hille-Yosida)**.** *[GM10]Let $A : \mathscr{D}(A) \subset \mathbb{X} \to \mathbb{X}$ be a linear operator on a Banach space $\mathbb{X}$. Necessary and sufficient conditions for A to generate a*

1) A is closed and $\overline{\mathscr{D}(A)} = \mathbb{X}$.

2) There exist real numbers M *and* α *such that for every* $\lambda > \alpha, \lambda \in \rho(A)$ *(the resolvent set) and*

$$\|(R(\lambda, A))^r\|_{\mathscr{L}(\mathbb{X})} \leq M\,(\lambda - \alpha)^{-r}, \quad r = 1, 2, ... \tag{2.15}$$

In this case,we have

$$\|S(t)\|_{\mathscr{L}(\mathbb{X})} \leq Me^{\alpha t}, \quad t \geq 0.$$

We provide now an essential approximation of an operator A *and the*

For $\lambda \in \rho(A)$, *consider the family of operators*

$$R_\lambda = \lambda R(\lambda, A). \tag{2.16}$$

Since the range $\mathscr{R}\,(R(\lambda, A)) \subset \mathscr{D}(A)$, *we can define the Yosida approximation of* A *by*

$$A_\lambda x = AR_\lambda x, \quad x \in \mathbb{X}. \tag{2.17}$$

Note that by (2.12)

$$A_\lambda x = R_\lambda Ax, \quad x \in \mathscr{D}(A).$$

Since $\lambda(\lambda I - A)R(\lambda, A) = \lambda I$, *we have* $\lambda^2 R(\lambda, A) - \lambda I = \lambda AR(\lambda, A)$, *so that*

$$A_\lambda x = \lambda^2 R(\lambda, A) - \lambda I,$$

proving that $A_\lambda \in \mathscr{L}(\mathbb{X})$. *Denote by* $S_\lambda(t)$ *the (uniformly continuous) semi-group generated by* A_λ,

$$S_\lambda(t)x = e^{tA_\lambda}x, \quad x \in \mathbb{X}. \tag{2.18}$$

Using the commutativity of the resolvent (2.13), we have

$$A_{\lambda_1} A_{\lambda_2} = A_{\lambda_2} A_{\lambda_1} \tag{2.19}$$

and, by the definition of $S_\lambda(t)$ *(2.18) ,*

$$A_\lambda S_\lambda(t) = S_\lambda(t) A_\lambda. \tag{2.20}$$

Proposition 2.2 (Yosida Approximation). *Let* A *be an infinitesimal generator of a*

and

$$\lim_{\lambda \to \infty} S_\lambda(t)x = S(t)x, \quad x \in \mathbb{X}. \tag{2.21}$$

The convergence in (2.21) is uniform on compact subsets of $\mathbb{R}_+$*. The following estimate holds:*

$$\|S_\lambda(t)\|_{\mathscr{L}(\mathbb{X})} \leq M \exp\{t\lambda\alpha/(\lambda - \alpha)\} \tag{2.22}$$

with the constants M,α *determined by the Hille-Yosida theorem.*

2.2 Stochastic convolution

Stochastic convolution allows us to represent a stochastic process as a convolution of a deterministic function with a stochastic process, and is often used to solve non-linear SDEs.

In the context of infinite-dimensional SDEs, stochastic convolution is particularly useful because it allows us to represent a stochastic process as a linear combination of functions in an infinite-dimensional space. This makes it possible to apply techniques from functional analysis and operator theory to study the behavior of the process.

Significant application of stochastic convolution is in the construction of so-called "mild" solutions to SDEs. A mild solution is a weak solution to a SDE *that can be expressed in terms of a stochastic convolution.This representation allows us to prove various existence and uniqueness results for solutions to SDEs in infinite dimensions.*

We study first solutions to a SDE corresponding to the deterministic abstract inhomogeneous Cauchy problem.

Let A be a linear operator on a real separable Hilbert space $\mathbb{H}$, and let us consider the abstract Cauchy problem given by

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t), \quad 0 < t < T, \\ u(0) = x, \quad x \in \mathbb{H}. \end{cases} \tag{2.23}$$

Definition 2.4. *A function $u : [0, T[\to \mathbb{H}$ is a (classical) solution of the problem (2.23) on $[0, T[$ if u is continuous on $[0, T[$, continuously differentiable and $u(t) \in \mathscr{D}(A)$ for $t \in]0, T[$, and (2.23) is satisfied on $[0, T[$.*

Remark 2.1. *If A is an infinitesimal generator of a*

In this case, $u^x(t) = S(t)x$ is not a solution in the usual sense, but it can be viewed as a generalized solution, which will be called a mild solution. In fact, the concept of mild solution can be introduced to study the following nonhomogeneous initial-value problem:

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + f(t), \quad 0 < t < T, \\ u(0) = x, \quad x \in \mathbb{H}, \end{cases} \tag{2.24}$$

where $f : [0, T[\to \mathbb{H}$.

Remark 2.2. *We assume that A is an infinitesimal generator of a*

Definition 2.5. *Let A be an infinitesimal generator of a*

Note that for $x \in \mathbb{H}$ and $f \equiv 0$, the mild solution is $S(t)x$, which is not in general a classical solution.
When $x \in \mathscr{D}(A)$, the continuity of f is insufficient to assure the existence of a classical solution. To see this, consider $f(t) = S(t)x$ for $x \in \mathbb{H}$ such

that $S(t)x \in \mathscr{D}(A)$. *Then (2.24) may not have a classical solution even if* $u(0) = 0 \in \mathscr{D}(A)$, *as the mild solution given by*

$$u(t) = \int_0^t S(t-s)S(s)xds = tS(t)x$$

is not, in general, differentiable.

One has the following theorem

Theorem 2.5. *[GM10]*

Let A *be an infinitesimal generator of a*

The role of the deterministic convolution $\int_0^t S(t-s)f(s)ds$ *will now be played by the stochastic process*

$$S \star \Phi(t) = \int_0^t S(t-s)\Phi(s)dW_s, \quad \Phi \in \mathscr{P}(\mathbb{K}_Q, \mathbb{H}) \tag{2.25}$$

which will be called stochastic convolution. Let $\|.\|_{\mathscr{D}(A)}$ *be the graph norm on* $\mathscr{D}(A)$,

$$\|h\|_{\mathscr{D}(A)} = (\|h\|_{\mathbb{H}}^2 + \|Ah\|_{\mathbb{H}}^2)^{1/2}$$

The space $(\mathscr{D}(A), \|.\|_{\mathscr{D}(A)})$ *is a separable Hilbert space . If* $f : [0,T] \to \mathscr{D}(A)$ *is a measurable function and* $\int_0^T \|f(s)\|_{\mathscr{D}(A)} < \infty$, *then for any* $t \in [0,T]$,

$$\int_0^t f(s)ds \in \mathscr{D}(A) \quad \text{and} \quad \int_0^t Af(s)ds = A\int_0^t f(s)ds.$$

The following proposition is the stochastic counterpart to the previous theorem.

Proposition 2.3 (Lemma 2.2, Chap. 2,[GM10])**.** *Assume that* A *is the infinitesimal generator of a*

$$\mathbb{P}\left(\int_0^T \|\Phi(t)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})} dt < \infty\right) = 1,$$

$$\mathbb{P}\left(\int_0^T \|A\Phi(t)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})}\, dt < \infty\right) = 1,$$

then $\mathbb{P}\left(\int_0^T \Phi(t)dW_t \in \mathscr{D}(A)\right) = 1$ *and*

$$A\int_0^T \Phi(t)dW_t = \int_0^T A\Phi(t)dW_t \quad \mathbb{P}\text{-}a.s. \tag{2.26}$$

Proof. Equality (2.26) is true for bounded elementary processes in $\mathscr{E}(\mathscr{L}(\mathbb{K},\mathscr{D}(A))$. Let $\phi_n \in \mathscr{E}(\mathscr{L}(\mathbb{K},\mathscr{D}(A))$ be bounded elementary processes approximating Φ as in Lemma 1.2,

$$\int_0^T \|\Phi(t,\omega) - \phi_n(t,\omega)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathscr{D}(A)}\, dt \to 0 \quad \text{as} \quad n\to\infty$$

$\mathbb{P}$ -a.s. and hence,

$$\int_0^t \phi_n(s)dW_s \to \int_0^t \Phi(s)dW_s.$$

$$A\int_0^t \phi_n(s)dW_s = \int_0^t A\phi_n(s)dW_s \to \int_0^t A\Phi(s)dW_s$$

in probability as $n\to\infty$. Now (2.26) follows since the infinitesimal generator A is a closed operator. □

Theorem 2.6. *Assume that A is an infinitesimal generator of a*

Proof. (a) Assume that (**??**) holds and let

$$u(s,x) = \langle x, S^*(t-s)h\rangle_{\mathbb{H}},$$

where $h \in \mathscr{D}(A^*)$ is arbitrary but fixed, $x\in\mathbb{H}$, and $0\le s\le t\le T$. The problem is to determine the differential of $u(s,X(s))$.
Since the adjoint semi-group $S^*(t)$ is a

$$\begin{aligned} u(s,X(s))-u(0,X(0)) &= \sum_{j=1}^{n-1} u_s\left(s_j, X\left(s_{j+1}\right)\right)\Delta s_j + \sum_{j=1}^{n-1}\left\langle u_x\left(s_j, X\left(s_j\right)\right), \Delta X_j\right\rangle_{\mathbb{H}} \\ &+ \sum_{j=1}^{n-1}\left[u_s\left(\tilde{s_j}, X\left(s_{j+1}\right)\right) - u_s\left(s_j, W_{s_{j+1}}\right)\right]\Delta s_j. \end{aligned} \tag{2.27}$$

Due to the continuity of $u_s(s,X(s))$,

$$\sum_{j=1}^{n-1} u_s\left(s_j, X\left(s_{j+1}\right)\right)\Delta s_j \to \int_0^s u_s\left(r, X(r)\right)dr = \int_0^s \left\langle X(r), -A^*S^*(t-r)h\right\rangle_{\mathbb{H}} dr.$$

We consider the second sum,

$$\begin{aligned} &\sum_{j=1}^{n-1}\left\langle u_x\left(s_j, X\left(s_j\right)\right), \Delta X_j\right\rangle_{\mathbb{H}} = \sum_{j=1}^{n-1}\left\langle S^*\left(t-s_j\right)h, X\left(s_{j+1}\right) - X\left(s_j\right)\right\rangle_{\mathbb{H}} \\ &= \sum_{j=1}^{n-1}\left(\left\langle \int_{s_j}^{s_{j+1}} X(r)dr, A^*S^*\left(t-s_j\right)h\right\rangle_{\mathbb{H}} + \left\langle \int_{s_j}^{s_{j+1}} \Phi(r)dW_r, S^*\left(t-s_j\right)h\right\rangle_{\mathbb{H}}\right). \end{aligned}$$

Due to the continuity of $A^*S^*(t-s)h = S^*(t-s)A^*h$, the first sum converges to

$$\int_0^s \left\langle X(r), A^*S^*(t-r)h\right\rangle_{\mathbb{H}} dr.$$

Denote $M_s = \displaystyle\int_0^s \Phi r dW_r \in \mathscr{M}_T^2(\mathbb{H})$. Then, $\displaystyle\int_{s_j}^{s_{j+1}} \phi(r)dW_r = M_{s_{j+1}} - M_{s_j}$. The second sum converges in $\mathscr{L}^2(\Omega,\mathbb{R})$ to

$$\left\langle \int_0^s S(t-r)dM_r, h \right\rangle_{\mathbb{H}} = \left(\int_0^s S(t-r)\Phi(r)dW_r, h \right)_{\mathbb{H}}.$$

The last sum in (2.27) converges to zero, since it is bounded by

$$t \sup_{0\le j\le n-1} \left| \langle W(s_{j+1}), A^*S^*(t-\tilde{s}_j)h\rangle_{\mathbb{H}} - \langle W(s_{j+1}), A^*S^*(t-s_j)h\rangle_{\mathbb{H}} \right| \to 0$$

due to the uniform continuity of $S(s)h$ on finite intervals and commutativity of A^* and S^* on the domain of A^*. We have proved that

$$\begin{aligned} u(s, X(s)) - u(0, X(0)) &= \langle X(s), S^*(t-s)h\rangle_{\mathbb{H}} \\ &= \left\langle \int_0^s S(t-r)\Phi(r)dW_r, h \right\rangle_{\mathbb{H}}. \end{aligned} \tag{2.28}$$

For $s = t$, we have

$$\langle X(t), h\rangle_{\mathbb{H}} = \left\langle \int_0^t S(t-r)\Phi(r)dW_r, h \right)_{\mathbb{H}}$$

Since $\mathscr{D}(A^*)$ is dense in $\mathbb{H}$, (a) follows.
(b) For $h \in \mathscr{D}(A^*)$ and $k \in \mathbb{K}$, consider the process defined by

$$\begin{aligned} \Psi(s,\omega,t)(k) &= \left(\mathbb{1}_{\{(0,t]\}}(s) \left(S(t-s)\Phi(s) \right)^* A^*h \right)(k) \\ &= \left\langle \mathbb{1}_{\{(0,t]\}}(s) S(t-s)\Phi(s)(k), A^*h \right\rangle_{\mathbb{H}}. \end{aligned}$$

Then the mapping Ψ defined on $[0,T]\times\Omega\times[0,T] \to \mathscr{L}^2(\mathbb{K}_Q,\mathbb{R})$ verifies for every $0 \le t \le T$, $\Psi(\cdot,\cdot,t)$ is $\{\mathscr{F}_s\}_{0\le s\le T}$-adapted, and

$$\begin{aligned} & \left\| \Psi \right\| = \int_0^T \|\Psi(\cdot,\cdot,t)\|_{\Lambda_2(\mathbb{K}_Q,\mathbb{R})} dt \\ & = \int_0^T \left(\mathbb{E} \left\| \mathbb{1}_{\{(0,t]\}}(s) \left(S(t-s)\Phi(s) \right)^* A^*h \right\|^2_{\Lambda_2(\mathbb{K}_Q,\mathbb{R})} ds \right)^{1/2} dt \end{aligned}$$

$$\leq T \left\| A^* h \right\|_{\mathbb{H}} M e^{\alpha t} \left(\mathbb{E} \int_0^T \left\| \Phi(s) \right\|^2_{\Lambda_2(\mathbb{K}_Q, \mathbb{H})} ds \right)^{1/2}$$
$$= C \|\Phi\|_{\Lambda_2(\mathbb{K}_Q, \mathbb{H})} < \infty,$$

so that the assumptions of the stochastic Fubini theorem (Theorem 2.8 in [GM10]), are satisfied. We obtain

$$\left\langle \int_0^1 S \star \Phi(s) ds, A^* h \right\rangle_{\mathbb{H}} = \int_0^t \int_0^s \left\langle S(s-u)\Phi(u) dW_u, A^* h \right\rangle_{\mathbb{H}} ds$$
$$= \int_0^t \left(\int_0^s \Psi(u, \omega, s) dW_u \right) ds$$
$$= \int_0^t \left(\int_0^t \Psi(u, \omega, s) ds \right) dW_u$$
$$= \int_0^t \left(\int_0^t \mathbb{1}_{\{(0,t]\}}(u) \left\langle S(s-u)(\cdot), A^* h \right\rangle_{\mathbb{H}} ds \right) dW_u$$
$$= \int_0^t \left\langle \left(A \int_u^t S(s-u)\Phi(u)(\cdot) ds \right), h \right\rangle_{\mathbb{H}} dW_u$$
$$= \int_0^t \left\langle \left(S(t-u)\Phi(u) - \Phi(u) \right)(\cdot), h \right\rangle_{\mathbb{H}} dW_u$$
$$= \int_0^t \left\langle \left(S(t-u)\Phi(u) - \Phi(u) \right) dW_u, h \right\rangle_{\mathbb{H}},$$

where we have used the fact that for $x \in \mathbb{H}$, the integral $\int_0^t S(r) x dr \in \mathscr{D}(A)$, and

$$A \left(\int_0^t S(r) x dr \right) = S(t)x - x$$

(Theorem 2.2). Thus we conclude that

$$\left\langle \int_0^t S \star \Phi(s) ds, A^* h \right\rangle_{\mathbb{H}} = \left\langle S \star \Phi(t), h \right\rangle_{\mathbb{H}} - \left\langle \int_0^t \Phi(s) dW_s, h \right\rangle_{\mathbb{H}}$$

this completes the proof of (b).

(c) Recall from (2.17), the Yosida approximation $A_n = AR_n$ of A_r and let

$S_n(s) = e^{sA_n}$ *be the corresponding semi-groups.Then part (b) implies that*

$$S_n \star \Phi(t) = \int_0^t A_n S_n \star \Phi(s) ds + \int_0^t \Phi(s) dW_s. \tag{2.29}$$

To complete the proof, we need the following lemma

Lemma 2.1. *(Lemma 2.2, Chap. 2 [GM10])*

i) Let $T, T_n \in \mathscr{L}(\mathbb{H})$ be such that for every $h \in \mathbb{H}, T_n h \to Th$, and let $L \in \mathscr{L}^2(\mathbb{K}_Q, \mathbb{H})$. Then

$$\|T_n L - TL\|_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})} \to 0. \tag{2.30}$$

ii) Let Abe the generator of a

Part (ii) of Lemma 2.1, implies that

$$\sup_{0 \le t \le T} \mathbb{E} \|S_n \star \Phi(t) - S \star \Phi(t)\|^2_{\mathbb{H}} \to 0 \tag{2.31}$$

Recall the commutativity property (2.12) that for $x \in \mathscr{D}, AR_n x = R_n Ax$. In addition, $AS_n(t)x = S_n(t)Ax$ for $x \in \mathscr{D}(A)$. Using Proposition (2.3, we obtain

$$\begin{aligned} A_n S_n \star \Phi(t) &= AR_n \int_0^t S_n(t-s)\Phi(s) dW_s \\ &= R_n \int_0^t S_n(t-s) A\Phi(s) dW_s \\ &= R_n S_n \star A\Phi(t). \end{aligned}$$

Hence,

$$
\begin{aligned}
&\sup_{0\le t\le T} \mathbb{E}\left\| \int_0^t \left(A_nS_n\star\Phi(s) - AS\star\Phi(s)\right) ds\right\|_{\mathbb{H}}^2\\
&\le T^2 \sup_{0\le t\le T} \mathbb{E}\int_0^t \left\|A_nS_n\star\Phi(s) - AS\star\Phi(s)\right\|_{\mathbb{H}}^2 ds\\
&\le T^2 E\int_0^T \left\|R_nS_n\star A\Phi(s) - S\star A\Phi(s)\right\|_{\mathbb{H}}^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&\le T^2\mathbb{E}\int_0^T \left\|R_n\left(S_n\star A\Phi(s) - S\star A\Phi(s)\right)\right\|_{\mathbb{H}}^2 ds\\
&+ T^2\mathbb{E}\int_0^T \left\|\left(R_n - I\right)S\star A\Phi(s)\right\|_{\mathbb{H}}^2 ds\\
&\le C\Bigg(\mathbb{E}\int_0^T \left\|S_n\star A\Phi(s) - S\star A\Phi(s)\right\|_{\mathbb{H}}^2 ds\\
&+ \mathbb{E}\int_0^T \left\|\left(R_n - I\right)S\star A\Phi(s)\right\|_{\mathbb{H}}^2 ds.
\end{aligned}
$$

The first summed converges to zero by (iii) of Lemma 2.1,

Since $R_nx \to x$ for $x \in \mathbb{H}$. we have

$$\left\|\left(R_n - I\right)S\star\Phi(s)\right\|_{\mathbb{H}} \to 0$$

and

$$\left\|\left(R_n - I\right)S\star A\Phi(s)\right\|_{\mathbb{H}}^2 \le C_1\left\|S\star A\Phi(s)\right\|_{\mathbb{H}}^2$$

with

$$
\begin{aligned}
&\mathbb{E}\int_0^T \left\|S\star A\Phi(s)\right\|_{\mathbb{H}}^2 ds = \int_0^T \mathbb{E}\int_0^s \left\|S(s-u)A\Phi(u)\right\|_{\mathbb{L}(\mathbb{K}_Q,\mathbb{H})}^2 duds\\
&\le C_2\left\|A\Phi\right\|_{\Lambda_2(\mathbb{K}_Q,\mathbb{H})}^2 < \infty,
\end{aligned}
$$

and the second summand converges to zero by the Lebesgue dominated convergence theorem DCT (theorem 4.5.1 in [Bur04]).

Summarizing,

$$\sup_{0 \le t \le T} \mathbb{E} \int_0^1 \|(A_n S_n \star \Phi(s) - AS \star \Phi(s))\, ds\|_{\mathbb{H}}^2 \to 0. \tag{2.32}$$

Combining (2.31) and (2.32),we obtain that both terms in (2.29) converge uniformly in mean square to the desired limits, so that (??) is satisfied by $S \star \Phi(t)$. *This concludes the proof.* □

2.3 Stochastic Differential Equations and their solutions

After briefly discussing the fundamental concepts of semi-group theory, we turn our attention to the general issue of the existence and uniqueness of solutions to the semi-linear and nonlinear SDE.

Let $\mathbb{K}$ *and* $\mathbb{H}$ *be real separable Hilbert spaces, and* W_t *be a* $\mathbb{K}$*-valued* Q*-Wiener process on a complete filtered probability space* $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t \ge 0}, \mathbb{P})$ *with the filtration* $\mathscr{F}_t$ *satisfying the usual conditions. We consider semi-linear SDEs (SSDEs for short) on* $[0,T]$ *in* $\mathbb{H}$*. The general form of such SSDE is*

$$\begin{cases} dX = (AX(t) + F(t, X(t)))dt + B(t, X(t))dW(t), \\ X(0) = \xi \end{cases} \tag{2.33}$$

Here, $A : \mathscr{D}(A) \in \mathbb{H} \to \mathbb{H}$ *is the generator of a*

$\|S(t)\|_{\mathscr{L}(\mathbb{H})} \le M \exp(\alpha t)$ *and if* $M = 1$*, then* $S(t)$ *is called a pseudo-contraction semi-group.*
The coefficients F *and* B *are, in general, nonlinear mappings,*

$$F : \Omega \times [0,T] \times \mathcal{C}([0,T], \mathbb{H}) \to \mathbb{H}$$

$$B : \Omega \times [0,T] \times \mathcal{C}([0,T], \mathbb{H}) \to \mathscr{L}^2(\mathbb{K}_Q, \mathbb{H}).$$

The initial condition ξ *is* $\mathscr{F}_0$*-measurable* $\mathbb{H}$*-valued random variable.*

We study the existence and uniqueness problem under various regularity assumptions on the coefficients of (2.33) that include:

A) *F and B are jointly measurable, and for every $0 \leq t \leq T$, they are measurable with respect to the product σ-field $\mathscr{F}_t \otimes \mathscr{C}_t$ on $(\Omega \times \mathcal{C}([0,T],\mathbb{H}))$, where $\mathscr{C}_t$ is a σ-field generated by cylinders with bases over $[0,t]$.*

B) *F and B are jointly continuous.*

C) *There exists a constant k such that for all $x \in \mathcal{C}([0,T],\mathbb{H})$,*

$$\|F(\omega,t,x)\|_{\mathbb{H}} + \|B(\omega,t,x)\|_{\mathbb{L}_2(\mathbb{K}_Q,\mathbb{H})} \leq k(1 + \sup_{0\leq s\leq T} \|x(s)\|_{\mathbb{H}})$$

for $\omega \in \Omega$ and $0 \leq t \leq T$

D) *For all $x,y \in \mathcal{C}([0,T],\mathbb{H}), \omega \in \Omega, 0 \leq t \leq T$,there exists $C > 0$ such that*

$$\|F(t,\omega,x) - F(t,\omega,y\|_{\mathbb{H}} + \|B(t,\omega,x) - B(t,\omega,y)\|_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})} \leq C \sup_{0\leq s\leq T} \|x(s) - y(s)\|_{\mathbb{H}}.$$

To simplify the notation, we will not indicate the dependence of F and B on ω whenever this does not lead to confusion.

There exist different notions of a solution to the semi-linear SDE (2.33). Now we define strong, weak, and mild solutions.

Definition 2.6. *A stochastic process $X(t)$ defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ and adapted to the filtration $\{\mathscr{F}_t\}_{t\leq T}$*

A) *is a strong solution of (2.33) if*

1. *$X(.) \in \mathcal{C}([0,T],\mathbb{H})$;*
2. *$X(t,\omega) \in \mathbb{D}(A) dt \otimes d\mathbb{P}$-almost everywhere;*

3. the following conditions hold:

$$\mathbb{P}(\int_0^T \|AX(t)\|_{\mathbb{H}}\, dt < \infty) = 1$$

$$\mathbb{P}(\int_0^T (\|F(t,X)\|_{\mathbb{H}} + \|B(t,X)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})})dt < \infty) = 1;$$

4. for every $t \leq T$, $\mathbb{P}$ *-a.s.,*

$$X(t) = \xi + \int_0^t (AX(s) + F(s,X))ds + \int_0^t B(s,X)dW_s; \quad (2.34)$$

B) is a weak solution of (2.33) (in the sense of duality) if

1. the following conditions hold:

$$\mathbb{P}(\int_0^T \|X(t)\|_{\mathbb{H}}\, dt < \infty) = 1, \quad (2.35)$$

$$\mathbb{P}(\int_0^T (\|F(t,X)\|_{\mathbb{H}} + \|B(t,X)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})})dt < \infty) = 1; \quad (2.36)$$

2. for every $h \in \mathscr{D}(A^*)$, $\mathbb{P}$ *-a.s.,*

$$\begin{aligned}\langle X(t), h\rangle_{\mathbb{H}} = \langle \xi, h\rangle_{\mathbb{H}} &+ \int_0^t (\langle X(s), A^*h\rangle_{\mathbb{H}} + \langle F(s,X), h\rangle_{\mathbb{H}})ds \\ &+ \int_0^t \langle h, B(s,X)dW_s\rangle_{\mathbb{H}}\,;\end{aligned} \quad (2.37)$$

C) is a mild solution of (2.33) if

1. conditions (2.35) and (2.36) hold;

2. *for all* $t \leq T$, $\mathbb{P}$ *-a.s.,*

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s,X)ds + \int_0^t S(t-s)B(s,X)dW_s. \tag{2.38}$$

We say that a process X *is a martingale solution of the equation*

$$\begin{cases} dX(t) = (AX(t) + F(t,X))dt + B(t,X)dW_t, \\ X(0) = x \in \mathbb{H} \text{ (deterministic)}, \end{cases} \tag{2.39}$$

if there exists a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ *and, on this probability space, a* Q*-Wiener process* W_t *, relative to the filtration*$\{\mathscr{F}_t\}_{t\leq T}$ *, such that* X_t *is a mild solution of (2.39).*

Unlike the strong solution, where the filtered probability space and the Wiener process are given, a martingale solution is a system $((\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}), W, X)$ *where the filtered probability space and the Wiener process are part of the solution.*

If $A = 0, S(t) = \mathbb{1}_{\mathbb{H}}$ *(identity on* $\mathbb{H}$*), we obtain the SDE*

$$\begin{cases} dX(t) = F(t,X(t))dt + B(t,X(t))dW_t, \\ X(0) = x \in \mathbb{H} \text{ (deterministic)}, \end{cases} \tag{2.40}$$

and a martingale solution of (2.40) is called a weak solution (in the stochastic sense,).

Remark 2.3. *In the presence or absence of the operator* A*, there should be no confusion between a weak solution of (2.33) in the sense of duality and a weak solution of (2.40) in the stochastic context.*

Obviously, a strong solution is a weak solution (either meaning) and a mild solution is a martingale solution.

Remembering the preliminary discussion of stochastic convolution, we move on to a more general problem of the relationship between various types of solutions to the semi-linear SDE (2.33).

Theorem 2.7. *A weak solution to (2.33) is a mild solution. Conversely, if X is a mild solution of (2.33) and*

$$\mathbb{E}\int_0^T \|B(t,X)\|^2_{\mathscr{L}^2(\mathbb{K}_Q,\mathbb{H})}\,dt < \infty,$$

then $X(t)$ is a weak solution of (2.33). If, in addition $X(t) \in \mathscr{D}(A)$,$d\mathbb{P}\otimes dt$ almost everywhere, then $X(t)$ is a strong solution of (2.33).

Proof. The techniques for proving parts (a) and (b) of Theorem 2.6 are applicable to a more general case. Consider the process $X(t)$ satisfying the equation

$$\begin{aligned}\langle X(t),h\rangle_{\mathbb{H}} = \langle \xi,h\rangle_{\mathbb{H}} + \int_0^t \left(\langle X(t),A^*h\rangle_{\mathbb{H}} + \langle f(s),h\rangle_{\mathbb{H}}\right)ds \\ + \int_0^t \langle h,\Phi(s)dW_s\rangle_{\mathbb{H}}\end{aligned} \tag{2.41}$$

with an adapted process $f(\cdot)\in L^1(\Omega,\mathbb{H}), \Phi\in\mathscr{P}(\mathbb{K}_Q,\mathbb{H})$, and $h\in\mathscr{D}(A^*)$. As in (a) of Theorem 2.6, we let

$$u(s,x) = \langle x, S^*(t-s)h\rangle_{\mathbb{H}},$$

where $h\in\mathscr{D}(A^*)$ is arbitrary but fixed, $x\in\mathbb{H}$, and $0\le s\le t\le T$. Then, formula 2.27 takes the form

$$\begin{aligned} u(s,X(s)) - u(0,X(0)) &= \langle X(s),S^*(t-s)h\rangle_{\mathbb{H}} - \langle X(0),S^*(t)h\rangle_{\mathbb{H}} \\ &= \left\langle \int_0^s S(t-r)\Phi(r)dW_r, h\right\rangle_{\mathbb{H}} + \lim_{n\to\infty}\sum_{j=1}^{n-1}\left\langle S^*(t-s_j)h, \int_{s_j}^{s_{j+1}} f(r)dr\right\rangle_{\mathbb{H}} \\ &= \left\langle \int_0^s S(t-r)\Phi(r)dW_r, h\right\rangle_{\mathbb{H}} + \left\langle \int_0^s S(t-r)f(r)dr, h\right\rangle_{\mathbb{H}}. \end{aligned}$$

For $s = t$, we have

Now it follows that $X(t)$ is a mild solution if we substitute $f(t) = F(t, X)$ and $\Phi(t) = B(t, X)$ and use the fact that $\mathscr{D}(A^*)$is dense in $\mathbb{H}$.

To prove the converse statement, consider the process

$$X(t) = S(t)\xi + \int_0^t S(t-s)f(s)ds + S \star \Phi(t),$$

where $f(t)$ is as in the first part, and $\Phi \in \Lambda_{2(\mathbb{K}_Q,\mathbb{H})}$. We need to show that

$$\begin{aligned}\langle X(t), h\rangle_{\mathbb{H}} &= (\xi, h)_{\mathbb{H}} \\ &\quad + \int_0^t \left\langle S(s)\xi + \int_0^s S(s-u)f(u)du + S \star \Phi(s), A^*h \right\rangle_{\mathbb{H}} ds \\ &\quad\quad + \int_0^t \langle f(s), h\rangle_{\mathbb{H}} ds + \left\langle \int_0^t \Phi(s)dW_s, h \right\rangle_{\mathbb{H}}.\end{aligned}$$

Using the result in (b) of Theorem 2.6, we have that

$$\langle S \star \Phi(t), h\rangle_{\mathbb{H}} = \left\langle \int_0^t S \star \Phi(s)ds, A^*h \right\rangle_{\mathbb{H}} + \left\langle \int_0^t \Phi(s)dW_s, h \right\rangle_{\mathbb{H}}.$$

Since (see Theorem 2.2) for any $\xi \in \mathbb{H}$, $\int_0^t S(s)\xi ds \in \mathscr{D}(A)$ and

$$A\int_0^t S(s)\xi ds = S(t)\xi - \xi,$$

We get

$$\langle S(t)\xi, h\rangle_{\mathbb{H}} = \langle \xi, h\rangle_{\mathbb{H}} + \int_0^t \langle S(s)\xi, A^*h\rangle_{\mathbb{H}} ds.$$

Finally, using (deterministic) Fubini's theorem (see [Sol07]),

$$\begin{aligned}\left\langle \int_0^t \int_0^s S(s-u)f(u)duds, A^*h \right\rangle_{\mathbb{H}} &= \left\langle \int_0^t \int_u^t S(s-u)f(u)dsdu, A^*h \right\rangle_{\mathbb{H}} \\ &= \left\langle \int_0^t A \int_u^t S(s-u)f(u)dsdu, h \right\rangle_{\mathbb{H}} \\ &= \left\langle \int_0^t A \int_0^{t-u} S(v)f(u)dvdu, h \right\rangle_{\mathbb{H}} \\ &= \left\langle \int_0^t \left(S(t-u)f(u) - f(u)\right) du, h \right\rangle_{\mathbb{H}},\end{aligned}$$

completing the calculations.

The last statement of the theorem is now obvious. □

The following existence and uniqueness result for linear SDEs is a direct application of Theorem 2.7.

Corollary 2.1. *Let $\{W_t, 0 \leq t \leq T\}$ be a Q-Wiener process defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, and A be the infinitesimal generator of a*

$$\begin{cases} dX(t) = (AX(t) + f(t))dt + BdW(t), \\ X(0) = \xi \end{cases} \tag{2.42}$$

has a unique weak solution given by

$$X(t) = S(t)\xi + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)BdW_s, \quad 0 \leq t \leq T.$$

2.4 Existence and uniqueness of solution for nonlinear equations

The existence and uniqueness of solutions for nonlinear SDEs in infinite dimensions refer to the question of whether solutions to these equations exist and, if they do, whether they are unique. In other words, can we find a process that satisfies the given SDE, and is this process the only solution?

Answering this question is nontrivial due to the interplay between the nonlinearities, stochasticity, and infinite-dimensional structure. It requires the use of advanced mathematical tools, such as stochastic analysis, functional analysis, and probability theory. Researchers in this field have developed various approaches, including the theory of mild solutions, stochastic convolution, and fixed point methods, to establish existence and uniqueness results.

We distinguish two cases according to which the Wiener process has nuclear or identity covariance.

Then we relax the Lipschitz condition in two important situations: when the diffusion term is additive or multiplicative. In the former case the nonlinear drift term is defined on an embedded Banach subspace and it is either locally Lipschitz or dissipative.

Existence is determined in the multiplicative case using a dissipativity assumption including the drift and diffusion terms. Finally, the topic of powerful solutions is discussed.

2.4.1 Equations with Lipschitz nonlinearities

We proceed to study nonlinear equations

$$\begin{cases} dX(t) = (AX(t) + F(t, X(t)))dt + B(t, X(t))dW(t), \\ X(0) = \xi \end{cases} \tag{2.43}$$

starting from the case when $F(\cdot)$ and $B(\cdot)$ satisfy properly formulated Lipschitz and linear growth conditions.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ *a probability space with a normal filtration* $\mathscr{F}_t, t \geq 0$*, are given.* $\mathscr{P}$ *and* $\mathscr{P}_T$ *will denote predictable* σ*-fields on* $\Omega_\infty = [0, +\infty) \times \Omega$ *and on* $\Omega_T = [0, T] \times \Omega$ *respectively. For any* $T > 0$ *we define* $\mathbb{P}_T$ *to be the product of the Lebesgue measure in* $[0, T]$ *and the measure* $\mathbb{P}$*.*

We assume also that $\mathbb{U}$ *and* $\mathbb{H}$ *are separable Hilbert spaces and that* W *is a* Q*-Wiener process on* $\mathbb{U}_1 \supset \mathbb{U}$ *and* $\mathbb{U}_0 = Q^{\frac{1}{2}}\mathbb{U}$*. The spaces* $\mathbb{U}, \mathbb{H}$ *and* $\mathscr{L}_0^2 = \mathscr{L}^2(\mathbb{U}_0, \mathbb{H})$ *are equipped with Borel* σ*-fields* $\mathscr{B}(\mathbb{U}), \mathscr{B}(\mathbb{H})$ *and* $\mathscr{B}(\mathscr{L}_0^2)$*. Moreover* ξ *is an* $\mathbb{H}$ *- valued random variable* $\mathscr{F}_0$ *- measurable.*

We fix $T > 0$ *and impose first the following conditions on coefficients* A*,* F *and* B *of the equation.*

Hypotheses 2.1.

i) A *is the generator of a*

Assume Hypothesis 2.1 (iv) and that for arbitrary $x, h \in \mathbb{H}, u \in \mathbb{U}$ *the processes*$\langle F(.,.,x), h\rangle, \langle B(.,.,x)Q^{1/2}u, h\rangle$ *are predictables. Then Hypotheses 2.1 (ii)(iii) are fulfilled.*
A predictable $\mathbb{H}$*-valued process* $X(t), t \in [0, T]$ *is said to be a mild solution of (2.43)if*

$$P\left(\int_0^T |X(s)|^2 ds < +\infty\right) = 1 \tag{2.44}$$

and, for arbitrary $t \in [0, T]$*, we have*

$$\begin{aligned} X(t) =& S(t)\xi + \int_0^t S(t-s)F(s, X(s))ds \\ &+ \int_0^t S(t-s)B(s, X(s))dW(s), \mathbb{P}\text{-}as \end{aligned} \tag{2.45}$$

The condition(2.44) implies that the integrals in (2.45)are well defined.

The main result of the present section is the following.

Theorem 2.8. *Assume that* ξ *is an* $\mathscr{F}_0$*-measurable* $\mathbb{H}$*-valued random variable and Hypothesis 2.1 is satisfied* *i) There exists a mild solution X to (2.43) unique, up to equivalence, among the processes satisfying*

$$\mathbb{P}\left(\int_0^T |X(s)|^2 ds < +\infty\right) = 1.$$

Moreover X possesses a continuous modification.

ii) For any $p \geq 2$ there exists a constant $C_{p,T} > 0$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\,|X(t)|^p \leq C_{p,T}\left(1 + \mathbb{E}\,|\xi|^p\right). \tag{2.46}$$

iii) For any $p > 2$ there exists a constant $\hat{C}_{p,T} > 0$ such that

$$\mathbb{E} \sup_{t\in[0,T]} |X(t)|^p \leq \hat{C}_{p,T}\left(1 + \mathbb{E}\,|\xi|^p\right). \tag{2.47}$$

Proof. We first prove uniqueness. We show that if $X_1()$ and $X_2()$ are two processes satisfying (2.44) and (2.45) then, for arbitrary $t \in [0,T]$, $P(X_1(t) = X_2(t)) = 1$. For a fixed number $R > 0$ we define

$$\tau_t = \inf\left\{t \leq T \int_0^t |F(s, X_T(s))|\, ds \geq R \quad \text{or} \quad \int_0^t \|B(s, X_t(s))\|^2_{\mathscr{L}_0^2}\, ds \geq R\right\}, i = 1, 2$$

and $\tau = \tau_1 \wedge \tau_2$.

Let $\hat{X}_t(t) = \mathbb{1}_{[0,\tau]}(t) X_t(t), t \in [0,T], i = 1, 2$, then for arbitrary $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\begin{aligned}\hat{X}_t(t) = \mathbb{1}_{[0,\tau]}(t) S(t)\xi + \mathbb{1}_{[0,\tau]}(t) \int_0^t \mathbb{1}_{[0,\tau]}(s) S(t-s) F(s, \hat{X}_t(s)) ds \\ + \mathbb{1}_{[0,\tau]}(t) \int_0^t \mathbb{1}_{[0,\tau]}(s) S(t-s) B(s, \hat{X}_t(s)) dW(s).\end{aligned}$$

Consequently, for arbitrary $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\begin{aligned}\mathbb{E}\left|\hat{X}_1(t)-\hat{X}_2(t)\right|^2 \leq & 2\mathbb{E}\left\{\int_0^t \mid F\left(s,\hat{X}_1(s)\right)-F\left(s,\hat{X}_2(s)\right)ds\right\}^2 \\ & + 2\mathbb{E}\left\{\int_0^t \|B\left(s,\hat{X}_1(s)\right)-B\left(s,\hat{X}_2(s)\|^2_{\mathscr{L}^2}ds\right\}\end{aligned} \tag{2.48}$$

By the definition of the stopping times τ, $\tau_{1]}$, τ_2 we find that the both sides of (2.48) is a bounded function on $t \in [0,T]$. Again by (**??**)-(**??**) we have

$$\mathbb{E}\left|\hat{X}_1(t)-\hat{X}_2(t)\right|^2 \leq 2C^2(T+1)\int_0^t \mathbb{E}\left|\hat{X}_1(s)-\hat{X}_2(s)\right|^2 ds.$$

The boundedness of $\mathbb{E}\left|\hat{X}_1(t)-\hat{X}_2(t)\right|^2, t \in [0,T]$, and the Gronwall lemma imply $\mathbb{E}\left|\hat{X}_1(t)-\hat{X}_2(t)\right|^2 = 0$. Therefore, for all $t \in [0,T]$, one has $\mathbb{P}\left(\hat{X}_1(t)=\hat{X}_2(t)\right) = 1$. So the predictable processes $\hat{X}_1(\cdot), \hat{X}_2(\cdot)$ are $\mathbb{P}_T$-a.s. identical. Since this is true for arbitrary $R > 0$ therefore $X_1(\cdot)$ and $X_2(\cdot)$ are $\mathbb{P}_T$-a.s. identical. Taking into account that X_1 and X_2 are solutions of the equation 2.45 one easily deduces that for arbitrary $t \in [0,T], X_1(t) = X_2(t)$, $\mathbb{P}$ -a.s.

The proof of existence is based on the classical fixed point theorem for contractions.

Let $\mathscr{H}_p, p \geq 2$, the Banach space of all the $\mathbb{H}$-valued predictable processes Y defined on the time interval $[0,T]$ such that

$$\|Y\|_p = \left(\sup_{t\in[0,T]} \mathbb{E}\left|Y(t)^p\right|\right)^{1/p} < \infty.$$

If one identifies processes which are identical $\mathbb{P}_T$-a.s then $\mathscr{H}_p$, with the norm $\|\cdot\|_p$. becomes a Banach space. Let $\mathscr{K}$ be the following transformation:

$$\begin{aligned}\mathscr{K}(Y)(t) &= S(t)\xi + \int_0^t S(t-s)F(s,Y(s))ds + \int_0^t S(t-s)B(s,Y(s))dW(s)\\ &= S(t)\xi + \mathscr{K}_1(Y)(t) + \mathscr{K}_2(Y)(t), \quad t\in[0,T], Y\in\mathscr{H}_p.\end{aligned}$$

We assume that $\mathbb{E}(|\xi|^p) < +\infty$ and show that $\mathscr{K}$ maps $\mathscr{H}_p$ into $\mathscr{H}_p$. As the composition of measurable mappings is measurable therefore, taking into account Hypothesis 2.1, one obtains that the transformations $\mathscr{K}_1$ and $\mathscr{K}_2$ are well defined. Moreover:

$$\begin{aligned}&\|\mathcal{A}_1(Y)\|_p^p \le M^p\mathbb{E}\left(\int_0^T |F(s,Y(s))|\,ds\right)^p\\ &\le T^{p-1}M^p\int_0^T |F(s,Y(s))|^p\,ds\\ &\le 2^{p/2-1}T^{p-1}M^pC^p\mathbb{E}\int_0^T (1+|Y(s)|^p)\,ds\\ &\le 2^{p/2-1}(TMC)^p\,(1+\|Y\|^p),\end{aligned}$$

where $M = \sup_{t\in[0,T]}\|S(t)\|$. Consequently,$\mathscr{K}_1$ maps $\mathscr{H}_p$ into $\mathscr{H}_p$. To show the same property for $\mathscr{K}_2$ we remark that,

$$\begin{aligned}&\|\mathscr{K}_2(Y)\|_p^p \le \sup_{1<0,T]}\mathbb{E}\left(\left|\int_0^t S(t-s)B(s,Y(s))dW(s)\right|^p\right)\\ &\le M^pC_{p/2}CT^{p/2-1}2^{p/2-1}\mathbb{E}\int_0^T \left(1+|Y(s)|^P\right)ds\\ &\le M^pC_{p/2}C(2T)^{p/2-1}\left(T+\|Y\|_p\right)^p.\end{aligned}$$

Now let Y_1 and Y_2 be arbitrary processes from $\mathscr{H}_p$ then

$$\|\mathscr{K}(Y_1)-\mathscr{K}(Y_2)\|_p \le \|\mathscr{K}_1(Y_1)-\mathscr{K}_1(Y_2)\|_p + \|\mathscr{K}_2(Y_1)-\mathscr{K}_2(Y_2)\|_p \;= \mathbb{1}_1+\mathbb{1}_2.$$

and

$$\begin{aligned}
\mathbb{1}_1^p &\leq \sup_{t\in[0,T]} \mathbb{E}\left\{\left|\int_0^t [S(t-s)(F(s,Y_1(s))-F(s,Y_2(s)))]\,ds\right|^p\right\}\\
&\leq M^p \sup_{t\in[0,T]} \mathbb{E}\left\{\int_0^t [|\ F\,(s,Y_1(s))-F\,(s,Y_2(s))]\,ds\right\}^p\\
&\leq (MC)^p T^p \left\|Y_1-Y_2\right\|_p^p
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\mathbb{1}_2^p &\leq C_{p/2}M^p\mathbb{E}\left\{\int_0^T \left\|B\,(s,Y_1(s))-B\,(s,Y_2(s))\right\|_{L_2^0}^2 ds\right\}^{p/2}\\
&\leq C_{P/2}(MC)^p T^{p/2-1}\mathbb{E}\left\{\left\|B\,(s,Y_1(s))-B\,(s,Y_2(s))\right\|_{\mathscr{L}_0^2}^p\right\} ds\\
&\leq C_{P/2}(MC)^p T^{p/2}\left\|Y_1-Y_2\right\|_p^p.
\end{aligned}$$

Summing up the obtained estimates we have:

$$\left\|\mathscr{K}(Y_1)-\mathscr{K}(Y_2)\right\|_p \leq CM(T^p+C_{p/2}T^{p/2})^{1/p}\left\|Y_1-Y_2\right\|_p^p, \tag{2.49}$$

for all $Y_1, Y_2 \in \mathscr{K}$. Consequently if

$$MCT\left(1+c_{p/2}T^{1/2}\right)^{1/p} < 1, \tag{2.50}$$

then the transformation $\mathscr{K}$ has unique fixed point X in $\mathscr{H}_p$ which, it is evident to see that it is a solution of the equation of the equation (2.43). The extra condition (2.50) on T can be easily removed by considering the equation on intervals $[0,\tilde{T}],[\tilde{T},2\tilde{T}]\ldots$ with $\tilde{T}$ satisfying (2.50). Thus we have proved assertion (ii) of the theorem since (2.8) follows easily by using Gronwall's lemma.

To construct a solution when $\mathbb{E}|\xi|^P = +\infty$, we show first that if ξ and η are two initial conditions satisfying $\mathbb{E}|\xi|^P < +\infty, \mathbb{E}|\eta|^P < +\infty$, and if $X, Y \in \mathscr{H}_p$ are the corresponding solutions of equation (2.43), then

$$\mathbb{1}_\Gamma X(\cdot) = \mathbb{1}_\Gamma Y(\cdot), \quad \mathbb{P}\text{ -a.s.}, \tag{2.51}$$

where

$$\Gamma = \{\omega \in \Omega : \xi(\omega) = \eta(\omega)\}.$$

To see this define

$$X^0 = S(\cdot)\xi, \quad X^{k+1} = \mathscr{K}\left(X^k\right), \quad t \in [0,T], k \in \mathbb{N}.$$

Thus for $t \in [0,T]$, $\mathbb{P}$ -a.s.

$$X^{k+1}(t) = S(t)\xi + \int_0^t S(t-s)F(s,X^k(s))ds + \int_0^t S(t-s)B(s,X^k(s))dW(s).$$

Since $\mathbb{1}_\Gamma$ is an $\mathscr{F}_0$-measurable random variable, therefore $\mathbb{1}_\Gamma B\left(\cdot, X^k(\cdot)\right)$ is an $\mathscr{L}^2$-predictable process and for $t \in [0,T]$,

$$\int_0^t S(t-s)\mathbb{1}_\Gamma B\left(s, X^k(s)\right) dW(s) = \mathbb{1}_\Gamma \int_0^t S(t-s)B\left(s, X^k(s)\right) dW(s).$$

Thus, for $t \in [0,T]$,

$$\begin{aligned} \mathbb{1}_\Gamma X^{k+1}(t) =& S(t)\mathbb{1}_\Gamma \xi + \int_0^t S(t-s)\mathbb{1}_\Gamma F(s,X^k(s))ds \\ &+ \int_0^t S(t-s)\mathbb{1}_\Gamma B(s,\mathbb{1}_\Gamma X^k(s))dW(s). \end{aligned} \tag{2.52}$$

If for a defined sequence

$$Y^0(t) = S(t)\eta, \quad Y^{k+1}(t) = \mathscr{K}(Y^k), \quad k \in \mathbb{N}, t \in [0, T],$$

for some k, we have

$$\mathbb{1}_\Gamma X^k(\cdot) = \mathbb{1}_\Gamma Y^k(\cdot) \quad \mathbb{P}_T - a.s.$$

then also

$$\mathbb{1}_\Gamma F\left(\cdot, X^k(\cdot)\right) = \mathbb{1}_\Gamma F\left(\cdot, Y^k(\cdot)\right), \mathbb{1}_\Gamma B\left(\cdot, X^k(\cdot)\right) = B\left(\cdot, Y^k(\cdot)\right), \quad \mathbb{P}_T\text{-}a.s.$$

Consequently

$$\mathbb{1}_\Gamma X^{k+1}(\cdot) = \mathbb{1}_\Gamma Y^{t+1}(\cdot) \quad \mathbb{P}_T\text{-}a.s.$$

Since the processes X and Y are limits in the $\|\cdot\|_p$ norm of the sequences $\left\{X^k(\cdot)\right\}$ and $\left\{Y^k(\cdot)\right\}$ respectively, therefore (2.51) must be true. Moreover the process $\mathbb{1}_\Gamma X(\cdot)$ satisfies the equation (2.43) with the initial condition $\mathbb{1}_\Gamma \xi = \mathbb{1}_\Gamma \eta$.

We now prove existence. Let us define, for $n \in \mathbb{N}$

$$\xi_n = \begin{cases} \xi & \text{if } |\xi| \leq n \\ 0 & \text{if } |\xi| > n, \end{cases}$$

and denote by $X_n(\cdot)$ the corresponding solution of (2.50). By the previous argument we have

$$X_n(t) = X_{n+1}(t) \text{ on } \{\omega \in \Omega : |\xi| \leq n\}.$$

It is now easy to see that the process

$$X(t) = \lim_{n\to\infty} X_n(t), t \in [0,T],$$

is $\mathbb{P}$-a.s. well defined and satisfies the equation (2.43).

For proof of existence of continuous modification of the mild solution we assume first that $\mathbb{E}\,|\xi|^{2r} < +\infty$ for some $r > 1$. From the first part of the theorem one knows that

$$\sup_{t\in[0,T]} \mathbb{E}\,\|X(t)\|^{2r} < +\infty. \tag{2.53}$$

Notons

$$\Phi(t) = B(t, X(t)), \quad t \in [0,T].$$

and

$$I = \mathbb{E}\int_0^T \|\Phi(t)\|_{L_2^0}^{2r}\,dt = \mathbb{E}\int_0^T \|B(t,X(t))\|_{L_2^0}^{2r}\,dt.$$

*By (**??**)-(**??**) we have*

$$I \le C^{2r}\mathbb{E}\left(\int_0^T \left(1+|X(t)|^2\right)^r dt\right) < +\infty.$$

Consequently Proposition 2.4 implies that the processes

$$\int_0^t S(t-s)B(s,X(s))dW(s), \quad t \in [0,T],$$

and $X(t), t \in [0,T]$, have a continuous modifications.
The case of initial conditions satisfying $\mathbb{E}\,|\xi|^{2r} = +\infty$ can be reduced to the case just considered by regarding initial conditions ξ_n

$$\xi_n = \begin{cases} \xi & \textit{if } |\xi| \leq n, \\ 0 & \textit{if } |\xi| > n, \end{cases}$$

as in the proof of existence. Finally (2.47) follows again from Gronwall's lemma, The proof is complete.

□

Now, we conside the approximating problem

$$\begin{cases} dX_n = (A_n X + F(t, X_n)\, dt + B(t, X_t)\, dW(t), \\ X_n(0) = \xi, \end{cases} \tag{2.54}$$

where A_n are the Yosida approximations of A. Clearly problem (2.54) has a unique solution X_n for any random variable $\xi, \mathscr{F}_0$-measurable.

We will need the following result

Proposition 2.4. *Let $p > 2, T > 0$ and let Φ be an $\mathscr{L}_0^2$-valued predictable process such that $\mathbb{E}\left(\int_0^T \|\Phi(s)\|_{L_2^0}^p\, ds\right) < +\infty$. There exists a constant $C_T > 0$ such that*

$$\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t S(t-s)\Phi(s)dW(s) \right|^p \leq C_T \mathbb{E}\left(\int_0^T \|\Phi(s)\|_{L_2^0}^p\, ds \right). \tag{2.55}$$

Moreover

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T]} \left| W_A^\Phi(t) - W_{A,n}(t) \right|^p = 0, \tag{2.56}$$

where $W_{A,n}^\Phi$ is defined as

$$W_{A,n}^\Phi(t) = \int_0^t e^{(t-s)A_n}\Phi(s)dW(s), \quad t \in [0,T],$$

and A_n Yosida approximations of A.

Finally $W_A^\Phi = \int_0^t S(t-s)\Phi(s)dW(s)$ *has a continuous modification.*

Proposition 2.5 (Proposition 7.3 in [DZ14]). *Under the hypotheses of Theorem 2.8, assume that* $\xi \in L^P(\Omega, \mathscr{F}, \mathbb{P})$*, with* $p \geq 2$*, and let* X *and* X_n *be the solutions of problems (2.43) and (2.54) respectively. Then we have*

$$\lim_{n\to\infty} \sup_{t=[0,T]} \mathbb{E}\left(|X(t) - X_n(t)|^p\right) = 0. \tag{2.57}$$

Moreover,if $p > 2$

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T]} |X(t) - X_n(t)|^p = 0. \tag{2.58}$$

Proof. The result follows from a straightforward application of the contraction principle depending on the parameter n, Proposition 2.4. □

2.4.2 Nonlinear equations on Banach spaces: additive noise

The nonlinear operators F *and* B *are only defined on a portion of the Hilbert space* $\mathbb{H}$ *in many interesting cases. A typical example is provided by the so called polynomial nonlinearities. One way of treating such cases is to consider equation (2.43) on a smaller state space* $\mathbb{E}$ *on which the nonlinear operators* F *and* B *are well defined and sufficiently regular, say locally Lipschitz continuous or simply continuous. This method requires that the initial condition takes values in the smaller space* $\mathbb{E}$*. We also demonstrate how the idea of mild solution can be expanded to include all initial conditions in* $\mathbb{H}$ *in some significant cases.*

In this section, we restrict our attention to equations with additive noise. Thus we consider the problem

$$\begin{cases} dX(t) = (AX(t) + F(X(t))dt + dW(t) \\ X(0) = \xi \in \mathbb{E} \end{cases} \tag{2.59}$$

where $\mathbb{E}$ is a Banach space continuously, densely and as Borel subset ,embedded in $\mathbb{H}, \mathbb{U} = \mathbb{E}$ and A generates a $\mathcal{C}_0$-semi-group in $\mathbb{H}$. We denote by $A_{\mathbb{E}}$ the part of A in $\mathbb{E}$. We will need the following assumptions.

Hypotheses 2.2. *either*

i) $A_{\mathbb{E}}$ generates a $\mathcal{C}_0$-semi-group $S(\cdot)$ on $\mathbb{E}$ or

ii) A generates an analytic semi-group $S_{\mathbb{E}}(\cdot)$ on $\mathbb{E}$.

Moreover the stochastic convolution W_A, has an $\mathbb{E}$-continuous version.

We recall that if the semi-group $S_{\mathbb{E}}$ is analytic, then it is strongly continuous at 0 if and only if $D(A_{\mathbb{E}})$ is dense in $\mathbb{E}$.

Locally Lipschitz nonlinearities

We impose the following conditions on F.

Hypotheses 2.3.

i) $\mathscr{D}(F) \supset \mathbb{E}$, F maps $\mathbb{E}$ into $\mathbb{E}$ and the restriction $F_{\mathbb{E}}$ of F to $\mathbb{E}$ is locally Lipschitz continuous and bounded on bounded subsets of $\mathbb{E}$.

ii) There exists an increasing function $a : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\langle A_{\mathbb{E},n}x + F(x+y), x^* \rangle \leq a(\|y\|)(1 + \|x\|), \forall x, y \in \mathbb{E}, x^* \in \partial \|x\|, n \in \mathbb{N}.$$

$A_{\mathbb{E},n}$ the Yosida approximations of $A_{\mathbb{E}}$. Moreover $\langle \cdot, \cdot \rangle$ is the duality form on $\mathbb{E} \times \mathbb{E}^*$ ($\mathbb{E}^*$ is the topological dual of $\mathbb{E}$) and $\partial \|x\|$ the subdifferential of the $\mathbb{E}$-norm $\|\cdot\|$ at the point $x \in \mathbb{E}$.

Theorem 2.9. *Assume that A generates a $\mathcal{C}_0$-semi-group and that Hypotheses 2.2,2.3 hold. So,*

i) If the condition 2.2(i) holds then equation (2.59)has a unique mild solution in $\mathcal{C}([0, +\infty); \mathbb{E})$.

ii) If the condition 2.2(ii) holds then equation (2.59)has a unique mild solution in $\mathcal{C}([0, +\infty); \mathbb{E}) \cap L^{\infty}_{loc}([0, +\infty); \mathbb{E})$.

Proof. Define $v(t) = X(t) - z(t), t \in [0,T]$, where $z(\cdot) = W_A(\cdot)$, and note that for $\mathbb{E}$-valued processes X the mild version of (2.59) can be written as

$$v(t) = S_{\mathbb{E}}(t)\xi + \int_0^t S_{\mathbb{E}}(t-s)F_{\mathbb{E}}(v(s)+z(s))ds, \quad t \in [0,T]. \tag{2.60}$$

We'll attempt a pathwise solution to (2.60), assuming that $W_A(\cdot)$ is $\mathbb{E}$-continuous.
We first prove the theorem when $D(A_{\mathbb{E}})$ is dense in $\mathbb{E}$. In this case, for any $T>0$ we set $Z_T = \mathcal{C}([0,T];\mathbb{E})$ and moreover

$$(\gamma(v))(t) = S_{\mathbb{E}}(t)\xi + \int_0^t S_{\mathbb{E}}(t-s)F_{\mathbb{E}}(v(s)+z(s))ds. \tag{2.61}$$

Clearly γ maps Z_T into Z_T and, by using the local inversion theorem, it is easy to show that if T is small enough, then there exists a unique mild solution on $[0,T]$. Since $F_{\mathbb{E}}$ is bounded on bounded sets of $\mathbb{E}$, to obtain global existence it is sufficient to deduce a priori estimate for $\|v(\cdot)\|$. Let $v(\cdot)$ be a mild solution of (2.60) on a (stochastic) interval $[0,T_0]$ and let $\{v_n\}$ be the sequence in $C^1([0,T_0];\mathbb{E}) \cap C([0,T_0];D(A_{\mathbb{E}}))$ given by

$$v_n(t) = nR(n,A_{\mathbb{E}})S_{\mathbb{E}}(t)x + \int_0^t nR(n,A_{\mathbb{E}})F_{\mathbb{E}}(v(s)+W_A(s))ds,$$

then it is easy to check that ,

$$v_n \to v \quad \frac{dv_n}{dt} - Av_n - F_{\mathbb{E}}(v_n+z) = \delta_n \to 0,$$

uniformly on $[0,T_0]$ as $n \to \infty$. Now, for $t \geq 0$ and $x^*_{t,n} \in \partial \|\mathrm{v}_n(t)\|$

$$\begin{aligned}\frac{d^-}{dt}\|v_n(t)\| &\leq \left\langle Av_n(t) + F_{\mathbb{E}}(v_n(t)+z(t)), x^*_{t,n}\right\rangle + \left\langle \delta_n(t), x^*_{t,n}\right\rangle \\ &\leq a(\|z(t)\|)(1+\|v_n(t)\|) + \|\delta_n(t)\|.\end{aligned}$$

Consequently

$$\|v_n(t)\| \leq e^{\int_0^{T_0} a(\|z(s)\|)ds} \|v_n(s)\| \int_0^{T_0} e^{\int_s^t a(\|z(u)\|)du} \left[a(\|z(s)\|) + \|\delta_n(s)\|\right] ds.$$

Therefore, letting n tend to infinity, and taking into account continuous dependence of the solutions on n we get

$$\|v_n(t)\| \leq e^{\int_0^{T_0} a(\|z(s)\|)ds} \|v(0)\| \int_0^{T_0} e^{\int_s^t a(\|z(u)\|)du} a(\|z(s)\|)ds.$$

This finishes the proof for case (i). The proof of case (ii) is similar, one has only to replace the Banach space Z_T with $\tilde{Z}_T = \mathcal{C}([0,T];\mathbb{E}) \cap L^\infty([0,T];\mathbb{E})$, endowed with the sup norm. □

Dissipative nonlinearities

Before we begin, let's define dissipative mapping.

Definition 2.7. *A mapping $f : \mathscr{D}(f) \subset \mathbb{E} \to \mathbb{E}$ is said to be dissipative if*

$$\|x - y\| \leq \|x - y - \alpha\left(f(x) - f(y)\right)\|, \quad \forall x, y \in \mathscr{D}(f).$$

for more information about dissipative mappings see Appendix D in [DZ14].

We first point out that the continuity alone cannot substitute the local Lipschitz requirement imposed on the mapping F. Godunov's theorem [God75] says that on an arbitrary infinite dimensional Banach space $\mathbb{E}$ one can define a mapping F continuous and bounded such that the deterministic equation

$$X(t) = x + \int_0^t F(X(s))ds, \quad t \geq 0, \tag{2.62}$$

does not have a local solution. However, continuity and dissipativity of F implies existence of a global solution of (2.62),see [Mar70], and we show that this is true also in the stochastic case. We shall assume the following.

Hypotheses 2.4. *The mapping $F_{\mathbb{E}}$ is dissipative and uniformly continuous on bounded sets of $\mathbb{E}$.*

We now give the proof of the main result of this subsection done by Da Prato and al. in [DZ14] following [Pra92]. Earlier results for the case of A equal to the second derivative, with Dirichlet boundary conditions in a bounded interval of $\mathbb{R}$, can be found in [FJ82], [Man86], [Man88].

Theorem 2.10. *Assume that A generates a $\mathcal{C}_0$-semi-group on $\mathbb{H}$, that Hypotheses 2.2 and 2.4 hold and that $\|S_{\mathbb{E}}(t)\| \leq e^{\omega t}$, for some $\omega \in \mathbb{R}$ and all $t \geq 0$. Then*

i) If the condition 2.2(i) holds then equation (2.59)has a unique mild solution in $\mathcal{C}([0,+\infty);\mathbb{E})$.

ii) If the condition 2.2(ii) holds then equation (2.59)has a unique mild solution in $\mathcal{C}([0,+\infty);\mathbb{E}) \cap L^{\infty}_{loc}([0,+\infty);\mathbb{E})$.

Proof. We restrict our considerations to the case (i) and set $F = F_{\mathbb{E}}$ for simplicity. As in the proof of Theorem 2.9, define

$$v(t) = X(t) - z(t),$$

Where $z(\cdot) = W_A(\cdot)$, and consider equation (2.60). Let us introduce, for any $\alpha > 0$, the approximating equation

$$v^{\alpha}(t) = S_{\mathbb{E}}(t)\xi + \int_0^t S_{\mathbb{E}}(t-s)F_{\alpha}(v_{\alpha}(s) + z(s))ds, \tag{2.63}$$

where F_{α} are the Yosida approximations of F. Let us check that assumption (2.3), with F replaced by F_{α} holds true. 2.3(i) is fulfilled since F_{α} is Lipschitz continuous, by (Proposition D.10 in [DZ14]).

Moreover, for any $x, y \in \mathbb{E}$ and some $x^* \in \partial\|x\|$. we have

$$\begin{aligned}\langle F_a(x+y), x^*\rangle &= \langle F_{\alpha}(x+y) - F(y), x^*\rangle + \langle F_{\alpha}(y), x^*\rangle \\ &\leq \langle F_{\alpha}(y), x^*\rangle \|F(y)\|,\end{aligned}$$

and, recalling that $\|S_{\mathbb{E}}(t)\| \le e^{\omega t}, t \ge 0$, we have $\langle A_{\mathbb{E},n}x, x^*\rangle \le \omega\|x\|$. Consequently

$$\langle A_{\mathbb{E},n}x + F_\alpha(x+y), x^*\rangle \le \omega\|x\| + \|F(y)\|,$$

and 2.3(ii) is also fulfilled.

Now by Theorem 2.9 , equation (2.63) has a unique global solution v^α. Fix $T>0$ and let $\{v_n^\alpha\} \subset C^1([0,T],\mathbb{E}) \cap C\left([0,T]; D\left(A_{\mathbb{E}}\right)\right)$ and $\{\delta_n^\alpha\} \subset \mathcal{C}([0,T];\mathbb{E})$ be sequences such that, uniformly on $[0,T]$,

$$v_{\mathrm{n}}^\alpha \to v_\alpha, \frac{dv_n^\alpha}{dt} - Av_n^\alpha - F\left(v_n^\alpha + z\right) = \delta_n^\alpha \to 0,$$

see the proof of Theorem 2.9. Now, for some $x_{t,n,\alpha}^* \in \partial\left\|v_n^\alpha(t)\right\|$ we get the estimate

$$\begin{aligned}\frac{d^-}{dt}\left\|v_n^\alpha(t)\right\| &\left\langle Av_n^\alpha(t) + F_\alpha(v_n^\alpha(t) + z(t)), x_{t,n,\alpha}^*\right\rangle + \left\langle \delta_n^\alpha(t), x_{t,n,\alpha}^*\right\rangle \\ &\le \omega\left\|v_n^\alpha(t)\right\| + \left\|F_\alpha(z(t))\right\| + \left\|\delta_{n,\alpha}(t)\right\|.\end{aligned}$$

Since $\|F_\alpha(z)\| \le \|F(z)\|$ for all $z \in \mathbb{E}$, therefore, letting $n \to \infty$, we have

$$\|v^\alpha(t)\| \le e^{\omega t}\|x\| + \int_0^t e^{\omega(t-s)}\|F(z(s))\|ds. \tag{2.64}$$

This shows that the sequence $\{v^\alpha(\cdot)\}$ is bounded uniformly on bounded sets. To show convergence of the sequence, we set for any $\alpha, \beta > 0$,

$$g^{\alpha,\beta} = v^\alpha - v^\beta, \quad u^\alpha = v^\alpha + z, \quad u^\beta = v^\beta + z$$

Then $g^{\alpha,\beta}$ is a classical solution to the problem

$$\begin{cases} \dfrac{d}{dt}g^{\alpha,\beta}(t) = Ag^{\alpha,\beta}(t) + F_\alpha(u^\alpha(t)) - F_\beta\left(u^\beta(t)\right), \\ g^{\alpha,\beta}(0) = 0. \end{cases}$$

Let $y^*_{\alpha,\beta,t} \in \partial \left\| g^{\alpha,\beta}(t) \right\|$, then we have

$$\frac{d^- \left\| g^{\alpha,\beta}(t) \right\|}{dt} \leq \omega \left\| g^{\alpha,\beta}(t) \right\| + \left\langle F_\alpha\left(u^\alpha(t)\right) - F_\beta\left(u^\beta(t)\right), y^*_{\alpha,\beta,t)} \right\rangle$$
$$\leq \omega \left\| g^{\alpha,\beta}(t) \right\| + \left\| F\left(J_\alpha\left(u^\alpha(t)\right)\right) - F\left(u^\alpha(t)\right) \right\|$$
$$+ \left\| F\left(J_\beta\left(u^\beta(t)\right)\right) - F\left(u^\beta(t)\right) \right\|.$$

Now by (2.64) and recalling that F is bounded on bounded subsets of $\mathbb{E}$, for a fixed $T > 0$ there exists $R > 0$ such that

$$\|u^\alpha(t)\| \leq R, \quad \text{and} \ \|F\left(u^\alpha(u)\right)\| \leq R, \quad \forall t \in [0,T], \forall \alpha \in (0,1].$$

Moreover

$$\|u^\alpha(t) - J^\alpha(u^\alpha(t))\,\| \leq \alpha \|F\left(u^\alpha(t)\right)\|\alpha R,$$

and so

$$\|F\left(J_\alpha\left(u^\alpha(t)\right)\right) - F\left(u^\alpha(t)\right)\| + \left\| F\left(J_\beta\left(u^\beta(t)\right)\right) - F\left(u^\beta(t)\right) \right\| \leq \rho_F(\alpha R) + \rho_F(\beta R),$$

which implies

$$\|g_{\alpha,\beta}(t)\| \leq [\rho_F(\alpha R) + \rho_F(\beta R)] \int_0^t e^{\omega s} ds,$$

where ρ_F is the modulus of continuity of F restricted to $B(0,R)$(i.e any function $\rho_F : [0,+\infty] \to [0,+\infty)$ such that $\lim_{r\to 0} \rho_F(r = 0)$ and $|F(x) - F(y)| \leq \rho_F(\|x-y\|), \forall x,y \in B(0,R)$, is called a continuity modulus of F restricted to $B(0,R)$) . This yields the convergence of the sequence $\{v^\alpha\}$ in $\mathcal{C}([0,T];\mathbb{E})$ to a function v. It is easily seen that v solves (2.61). This finishes the proof of existence of a solution to (2.60) on arbitrary time interval.

To show uniqueness, let $\widehat{X}$ be another solution to (2.59). Then we have

$$\frac{d^- \|X(t) - \widehat{X(t)}\|}{dt} \leq \left\langle A(X(t) - \widehat{X}(t)) + E(X(t)) - F(\widehat{X}(t)), \widehat{x}^*_t \right\rangle$$

for $t \in [0,T]$ and for an appropriate $\widehat{x}_t^* \in \partial \left\| X(t) - \widehat{X}(t) \right\|$. Consequently

$$\frac{d^- \|X(t) - \widehat{X}(t)\|}{dt} \leq \rho \|X(t) - \widehat{X}(t)\|, \quad t \in [0,T].$$

Since $X(0) = \widehat{X}(0)$ therefore $X(t) = \widehat{X}(t), t \in [0, I]$. The proof is complete. □

2.4.3 Nonlinear equations on Banach spaces: multiplicative noise

The results of this section were essentially obtained by variational methods. We follow here the semi-group approach .We set $\mathbb{H} = L^2(\mathscr{O})$ and $E = \mathcal{C}(\bar{\mathscr{O}})$, where $\mathscr{O}$ is an open bounded subset of $\mathbb{R}^N$. Let us consider the following problem

$$\begin{cases} dX(t) = (AX(t) + F(X(t))dt + B(X(t))dW(t) \\ X(0) = x \in \mathbb{H} \end{cases} \tag{2.65}$$

where A and F are as in the previous section (Section 2.4.2) and B is a linear closed operator from its domain $\mathscr{D}(B)(\mathscr{D}(A) \subset \mathscr{D}(B) \subset \mathbb{H})$ into L_2^0 . In the formulation of the next theorem, A_n are the Yosida approximations of A and $B_n(x) = B(J_n x)$, $x \in \mathbb{H}$. By $\Xi := \mathscr{N}_W^2(0,T;\mathscr{D}(B))$ we denote the space of all predictable $\mathscr{D}(B)$-valued processes $Y(t), t \in [0,T]$ such that

$$\begin{aligned} \|Y\|_\Xi^2 &= \mathbb{E}\int_0^T |Y(s)|_{\mathbb{H}}^2 \, ds + \mathbb{E}\left|\int_0^T B(Y(s))dW(s)\right|^2 \\ &= \|Y\|^2_{\mathscr{N}_W^2(0,T;\mathscr{D}(B))} + \|B(Y)\|^2_{\mathscr{N}_W^2(0,T;L_2^0(B))}. \end{aligned} \tag{2.66}$$

Theorem 2.11. *Assume that $F : E \to E$ is a dissipative mapping uniformly continuous on bounded subsets of E and that there exists $\eta \in \mathbb{R}$ and $\delta > 1$*

such that

$$2\langle A_n(x-y), x-y\rangle + \delta \|B_n(x-y)\|^2_{L_2^0} \leq \eta |x-y|^2, \quad \forall x, y \in E, n \in \mathbb{N}. \tag{2.67}$$

Then there exists a unique predictable process X such that

1. $X \in \mathscr{N}_W^2(0,T;\mathscr{D}(B))$,
2. $X \in L^p([0,T]\,\mathscr{O})$,
3. *X fulfills the integral equation*

$$x(t) = S(t)x + \int_0^t \tilde{S}(t-s)\tilde{F}(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s). \tag{2.68}$$

Proof. Fix $T>0$ and, for any $Y \in \mathscr{N}_W^2(0,T;\mathscr{D}(B))$ denote by $Z = \Gamma(Y)$ the solution to

$$\begin{cases} dZ(t) = (AZ(t) + \tilde{F}(Z(t))dt + B(Y(t))dW(t), \\ Z(0) = x \in \mathbb{H} \end{cases} \tag{2.69}$$

provided by the following Lemma.

Lemma 2.2 (Proposition 7.18 in [DZ14])**.** *Assume that the hypotheses of Theorem 2.11 are satisfied and that for some $k>0, \mathscr{D}(A^k)$ is continuously embedded in E. Define*

$$\phi_n = J_n^{k+1}\Phi, \quad x_n = J_n^{k+1}x,$$

where $J_n = nR(n,A), n \in \mathbb{N}$. Then this equation

$$\begin{cases} dX_n(t) = (AX_n(t) + F(X_n(t))dt + \phi_n dW(t), \\ X_n(0) = x_n \in \mathbb{H} \end{cases}$$

has a unique E-valued solution $X_n, n \in \mathbb{N}$, and there exists an $\mathbb{H}$-valued, *predictable mean square continuous process X, such that*

$$\lim_{n\to\infty} \sup \mathbb{E}(|X(t) - X_n(t)|^2) = 0, \quad T > 0.$$

In order to prove the theorem we are going to show that Γ is a contraction in the space Ξ. Let $Y_t \in \mathscr{N}_W^2(0, T : \mathscr{D}(B)), Z_t = \Gamma(Y_t)$ and $Z_{t,n}$ be solution of 2.69 corresponding to $Y = Y_t, i = 1, 2$. We denote by $Z_{t,n,m}$ solutions to the following equation

$$\begin{cases} dZ_{t,n,m}(t) = (A_n Z_{t,n,m}(t) + \tilde{F}(Z_{t,n,m}(t))dt + J_n^m B_t(Y_t(t))dW(t), \\ Z_{t,n,m}(0) = x \in \mathbb{H} \end{cases} \tag{2.70}$$

we have:

$$\begin{aligned} &\mathbb{E}|Z_{1,n,m}(t) - Z_{2,n,m}(t)|^2 + \delta\mathbb{E}\int_0^t \|B(Z_{1,n,m}(s)) - B(Z_{2,n,m}(s))\|^2_{L_2^0} ds \\ &+ \mathbb{E}\int_0^t |Z_{1,n,m}(s) - Z_{2,n,m}(s)|_p^p ds \\ &\le \mathbb{E}\int_0^t \|J_n^m B(Y_1(s)) - J_n^m B(Y_2(s))\|^2_{L_2^0} ds + \eta\mathbb{E}\int_0^t |Z_{1,m}(s) - Z_{2,m}(s)|^2 ds. \end{aligned} \tag{2.71}$$

In particular,

$$\begin{aligned} \mathbb{E}|Z_{1,n,m}(t) - Z_{2,n,m}(t)|^2 &\le \eta\mathbb{E}\int_0^t |Z_{1,n,m}(t) - Z_{2,n,m}(t)|^2 ds \\ &+ \mathbb{E}\int_0^t \|J_n^m B(Y_1(s)) - J_n^m B(Y_2(s))\|^2_{L_2^0} ds. \end{aligned}$$

Using Gronwall's lemma and letting m and n tend to infinity, we find the following two estimates

$$\mathbb{E}|Z_1(t) - Z_2(t)|^2 \le \int_0^t e^{(t-s)\eta}\mathbb{E}\|B(Y_1(s)) - B(Y_2(s))\|^2_{L_2^0} ds,$$

and

$$\delta\mathbb{E}\int_0^t \|B(Z_1(s)) - B(Z_2(s))\|^2_{L_2^0} ds \leq \mathbb{E}\int_0^t \|B(Y_1(s)) - B(Y_2(s))\|^2_{L_2^0} ds$$
$$+ \eta T e^{\eta T}\mathbb{E}\int_0^t \|B(Y_1(s)) - B(Y_2(s))\|^2_{L_2^0} ds.$$

It follows from the above inequalities that

$$\|Z_1 - Z_2\|^2_{\mathcal{N}_W^2(0,T;\mathbb{H})} \leq Te^{T\eta}\|B(Y_1) - B(Y_2)\|^2_{\mathcal{N}_W^2(0,T;L_2^0)},$$

$$\delta\|BZ_1 - BZ_2\|^2_{\mathcal{N}_W^2(0,T;\mathbb{H})} \leq (1 + \eta Te^{T\eta})\|B(Y_1) - B(Y_2)\|^2_{\mathcal{N}_W^2(0,T;L_2^0)}.$$

Therefore

$$\|\Gamma(Y_1) - \Gamma(Y_2)\|^2_{\Xi} \leq [Te^{T\eta} + \frac{1 + \eta Te^{T\eta}}{\delta}]\|Y_1 - Y_2\|^2_{\Xi}.$$

Since $\delta > 1$, the transformation Γ is a contraction on Ξ, provided T is sufficiently small, and so it has a unique a fixed point Y in Ξ.

Moreover, by using (2.71) it follows that Y is the unique solution with the required integrability properties.

□

Remark 2.4. *If the operator A is self-adjoint negative definite, then the condition (2.67) can be replaced by the following, simpler to handle, condition. There exists $\eta \in \mathbb{R}$ and $\delta > 1$ such that*

$$2\langle Ax, x\rangle + \delta\|B(x)\|^2_{L_2^0} \leq \eta|x|^2, \quad \forall x, y \in E. \tag{2.72}$$

To check that this is the case, it is enough to use the following consequence of spectral decomposition for self-adjoint operators:

$$\langle AJ_n x, J_n x\rangle \geq \langle AJ_n x, x\rangle, \quad \forall x \in \mathscr{D}(A).$$

2.4.4 Strong solutions

In this section we will choose as Hilbert space $\mathbb{H} = L^2(\mathbb{R}^d)$*, where* d *is a positive integer. We want to find strong solutions to the equation*

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + \sum_{k=1}^{N} B_k X(t) d\beta_k, \\ X(0) = x \in \mathbb{H}, \end{cases} \tag{2.73}$$

which it is in a Banach space of continuous functions. We assume that A*:* $\mathscr{D}(A) \subset \mathbb{H} \to \mathbb{H}$*, and* $B_k : \mathscr{D}(B_k) \subset \mathbb{H} \to \mathbb{H}, k \in \mathbb{N}$*, are generators of semi-groups* $S(t) = e^{tA}$ *and* $S_k(t) = e^{tB_k}$ *respectively, such that this following Hypothesis are satisfied .*

Hypotheses 2.5.

i) Operators $B_1, \ldots, B_N$ *generate mutually commuting* C_0*-groups* $e^{tB_1}, \ldots, e^{tB_N}$ *respectively.*

ii) For $k = 1, \ldots, N, \mathscr{D}(B_k^2) \supset \mathscr{D}(A)$ *and* $\bigcap_{k=1}^{N} \mathscr{D}((B_k^*)^2)$ *is dense in* $\mathbb{H}$*.*

iii) The operator

$$C = A - \frac{1}{2} \sum_{k=1}^{N} B_k^2, \quad \mathscr{D}(C) = \mathscr{D}(A),$$

is the infinitesimal generator of a $\mathcal{C}_0$*-semi-group* $S_0(t) = e^{tC}, t \geq 0$*.*

Moreoverwe introduce the Banach space E *of all functions* u *on* $\mathbb{R}^d$ *which are uniformly continuous and bounded on bounded subsets and such that*

$$\|u\| = \sup_{\xi \in \mathbb{R}^d} \frac{|f(\xi)|}{(1 + |\xi|^2)^{d/2}}$$

Clearly $E \subset \mathbb{H} = L^2(\mathbb{R}^d)$*. We denote by* $A_1, B_{E,1}, \ldots, B_{E,N}$ *the parts of* $A, B_1, \ldots, B_N$ *in* E *and we assume the following.*

Hypotheses 2.6.

i) *The operators* $B_{E,1}, \ldots, B_{E,N}$ *generate mutually commuting* C_0*-groups* $S_{E,k}(t), t \in \mathbb{R}$*, in* E*.*

ii) *The operator*

$$C_E = A_E - \frac{1}{2}\sum_{k=1}^{N} B_{E,k}^2, \qquad \text{with the domain } \mathscr{D}(A_E)$$

is the infinitesimal generator of a $\mathcal{C}_0$*-semi-group* $S_{E,0}(t) = e^{tC_E}$ *. We set*

$$U_E(t) = \prod_{k=1}^{N} S_{E,k}(\beta_k(t)).$$

Concerning the nonlinear mapping $F : \mathscr{D}(F) \subset \mathbb{H} \to \mathbb{H}$ *we assume that*

Hypotheses 2.7. $E \subset \mathscr{D}(F)$ *and the restriction* F_E *of* F *to* E *is uniformly continuous and bounded on bounded sets of* E*.*

we set $v(t) = U_E^1(t)X(t)$ *and reduce problem (2.73) to the deterministic equation*

$$\begin{cases} v'(t) = U_E^{-1}(t)C_E U_E(t)v(t) + U_E^{-1}F_E(U_E(t)v(t)), \\ v(0) = x \in E, \end{cases} \tag{2.74}$$

We have the following result.

Theorem 2.12 (Theorem 7.23 in [DZ14])**.** *Assume Hypotheses 2.6-2.7. If* X *is a strong solution to (2.73) then the process* v *satisfies (2.74). Conversely, if* v *is a predictable process such that*

i) *trajectories of* v *are of class* C^1 *and satisfy (2.74),*

ii) *the process* $X(\cdot) = U_E(\cdot)v(\cdot)$ *takes values in* $\mathscr{D}(C), \mathbb{P}_T$ *-a.s.,*

then the process X *is a strong solution of (2.73).*

Chapter 3

Backward Stochastic Differential Equations

In the field of stochastic calculus, Backward Stochastic Differential Equations (BSDEs) have gained a significant amount of attention due to their versatile nature and wide range of applications in finance, insurance, and other fields. BSDEs were first introduced by Pardoux and Peng in the early 1990s [PP90] as a natural extension of the theory of stochastic differential equations.

Unlike SDEs, which describe the evolution of a state variable over time, BSDEs describe the evolution of a function of the state variable over time. This makes BSDEs a powerful tool for modeling and analyzing complex systems with non-linear dependencies.

In this chapter, we will delve into the mathematical foundations of BSDEs. We will explore the basic theory of BSDEs and their properties, such as existence, uniqueness, and stability of solutions. We will also discuss the numerical methods used for solving BSDEs, such as the Euler scheme and the backward Monte Carlo method.

BSDEs have a lot of application in stochastic optimal control and solving PDEs by the connection using the non-linear Feynman-Kac formula.

The framework for BSDEs in finite dimensions is well established and has been extensively studied. This includes the study of existence and uniqueness of solutions, as well as numerical methods for their approximation.

On the other hand, the framework for BSDEs in infinite dimensions is more challenging due to the complications that arise from the infinite-dimensional nature of the underlying space. Nonetheless, BSDEs in infinite dimensions have numerous applications in areas such as mathematical physics, control theory, and stochastic analysis.

3.1 Existence and Uniqueness of solution for BSDEs

BSDEs in the infinite dimensional were first studied in [HP91]. In this article Y. Hu and S. Peng looked for an adapted pair process $\{(x(t), y(t)) : t \in [0, T]\}$ *with values in* $\mathbb{H}$ *and* $\mathscr{L}^2(\mathbb{K}, \mathbb{H})$ *respectively (where* $\mathbb{K}$ *and* $\mathbb{H}$ *are two separable Hilbert spaces and* $\mathscr{L}^2(\mathbb{K}, \mathbb{H})$ *is the set of Hilbert-Schmidt operators from* $\mathbb{K}$ *into* $\mathbb{H}$*) which solves a semi-linear stochastic evolution equation of the backward form :*

$$\begin{cases} dx(t) + Ax(t)dt = f(t, x(t), y(t))dt + (g(t, x(t)) + y(t))dW(t), \\ x(T) = X. \end{cases} \tag{3.1}$$

where $\{W(t), t \in [0, T]\}$ *is a cylindrical Wiener process with values in* $\mathbb{K}$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathscr{F}_t\}$ *denote its natural filtration and* $X \in \mathrm{L}^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{H})$, A *is the infinitesimal generator of a* $\mathcal{C}_0$*-semi-group* $\{e^{At}\}$ *on* $\mathbb{H}$ *. The precise meaning of the equation is*

$$\begin{aligned} x(t) &+ \int_t^T e^{A(s-t)} f(s, x(s), y(s))ds + \int_t^T e^{A(s-t)}(g(s, x(s)) + y(s))dW(s) \\ &= e^{A(T-t)}X, \end{aligned} \tag{3.2}$$

where f *maps* $(\Omega \times [0,T], \mathbb{H}, \mathscr{L}^2(\mathbb{K},\mathbb{H}))$ *into* $\mathbb{H}$, f *is assumed to be* $(\mathcal{P} \otimes \beta(\mathbb{H}) \otimes \beta(\mathscr{L}^2(\mathbb{K},\mathbb{H}))/\beta(\mathbb{H}))$ *measurable, the function* g $(\Omega \times [0,T], \mathbb{H})$ *into* $\mathscr{L}^2(\mathbb{K},\mathbb{H})$, g *is assumed to be* $(\mathcal{P} \otimes \beta(\mathbb{H})/\beta(\mathscr{L}^2(\mathbb{K},\mathbb{H})))$ *measurable, with the conditions that*

$$f(\cdot,0,0) \in \mathrm{L}^2_{\mathcal{F}}(0,T;\mathbb{H}), \text{ and } g(\cdot,0) \in \mathrm{L}^2_{\mathcal{F}}(0,T;\mathscr{L}^2(\mathbb{K},\mathbb{H})) \tag{3.3}$$

for any Hilbert space $\mathbb{H}_1$, *we denote by* $\mathrm{L}^2_{\mathcal{F}}(0,T;\mathbb{H}_1)$ *the set of all* $(\mathscr{F}_t)$*-progressively measurable processes with values in* $\mathbb{H}_1$ *such that*

$$\|x(\cdot)\| = \left(\mathbb{E}\int_0^T |x(t)|^{\frac{1}{2}}\,dt\right) < +\infty$$

And there exists $c > 0$ *such that*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq c\left(|x_1 - x_2| + |y_1 - y_2|\right), \tag{3.4}$$

and

$$|g(t,x_1) - g(t,x_2)| \leq c\left(|x_1 - x_2|\right) \tag{3.5}$$

$\forall x_1, x_2 \in \mathbb{H}, y_1, y_2 \in \mathscr{L}^2(\mathbb{K},\mathbb{H}), (w,t)$ *almost everywhere. Note also that 3.3), (3.4) and (3.5) imply that* $f(\cdot,x(\cdot),y(\cdot)) \in \mathrm{L}^2_{\mathcal{F}}(0,T;\mathbb{H})$ *and* $g(\cdot,x(\cdot)) \in \mathrm{L}^2_{\mathcal{F}}(0,T;\mathscr{L}^2(\mathbb{K},\mathbb{H}))$ *whenever* $x \in \mathrm{L}^2_{\mathcal{F}}(0,T;\mathbb{H})$ *and* $y \in \mathrm{L}^2_{\mathcal{F}}(0,T;\mathscr{L}^2(\mathbb{K},\mathbb{H}))$.

Theorem 3.1. *Given* $X \in \mathrm{L}(\Omega,\mathscr{F}_T,\mathbb{P};\mathbb{H})$, f *and* g *given as above and satisfying in particular (3.3)-(3.5),then there exists a unique pair* $(x,y) \in \mathrm{L}^2_{\mathcal{F}}(0,T;\mathbb{H}) \times \mathrm{L}^2_{\mathcal{F}}(0,T;\mathscr{L}^2(\mathbb{K},\mathbb{H}))$ *which solves equation (3.2)*

However, in his paper [Al-04] A. Al-hussein establish some results regarding the existence and uniqueness of solutions to BSDEs of type (3.6) given below, under some weaker condition by imposing first a global Lipschitz condition on the coefficients f *and* g. *Secondly, he set a monotonicity condition on the drift* f, *with respect to the first variable* y.

We shall explain below that this latter condition is weaker than f *being Lip-*

schitzian in y. These parameters appear in the following BSDE as follows.

$$\begin{cases} -dY(t) = f(t, Y(t), Z(t))dt - (g(t, Y(t), Z(t))dW(t), & 0 \leq t \leq T, \\ Y(T) = \xi. \end{cases} \tag{3.6}$$

where W is a Q-Wiener which taking values in $\mathbb{H}$. This equation is usually understood from its integral form:

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T g(s, Y(s), Z(s))dW(s), \quad 0 \leq t \leq T. \tag{3.7}$$

Consider the natural filtration $\{\mathscr{F}_t, 0 \leq t \leq T\}$ of W. Assume that f is a mapping from $([0,T] \times \Omega \times \mathbb{K} \times \mathscr{L}^2(\mathbb{H},\mathbb{K}))$ to $\mathbb{K}$ that is $\mathcal{P} \otimes \mathscr{B}(\mathbb{K}) \otimes \mathscr{B}(\mathscr{L}^2(\mathbb{H},\mathbb{K}))/\mathscr{B}(\mathbb{K})$-measurable,where $\mathcal{P}$ is the σ-algebra of $\mathscr{F}_T$ progressively measurable subsets of $[0,T] \times \Omega$. Also g is a mapping from $([0,T] \times \Omega \times \mathbb{K} \times \mathscr{L}^2(\mathbb{H},\mathbb{K}))$ to $\mathbb{K}$,and is assumed to be $\mathcal{P} \otimes \mathscr{B}(\mathbb{K}) \otimes \mathscr{B}(\mathscr{L}^2(\mathbb{H},\mathbb{K}))/\mathscr{B}(\mathbb{K})$ measurable.

Let put some conditions on these mappings f, g and ξ.

- *(A1) The two mappings $f(\cdot, 0, 0)$ and $g(\cdot, 0, 0)$ in $\mathrm{L}^2_{\mathcal{F}}(0,T;\mathbb{K})$ and $\mathrm{L}^2_{\mathcal{F}}(0,T;\mathscr{L}^2(\mathbb{H},\mathbb{K}))$ respectively, and $\xi \in \mathrm{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{K})$.*
- *(A2) $\exists k > 0$ such that $\forall y, y^* \in \mathbb{K}$ and $\forall z, z^* \in \mathscr{L}^2(\mathbb{H},\mathbb{K})$, we have*

  $$|f(t,y,z) - f(t,y^*,z^*)|^2_{\mathbb{K}} \leq \left(|y - y^*|^2_{\mathbb{K}} + |z - z^*|^2_{\mathscr{L}^2(\mathbb{H},\mathbb{K})}\right),$$

  and

  $$|g(t,y,z) - g(t,y^*,z^*)|^2_{\mathscr{L}^2(\mathbb{H},\mathbb{K})} \leq \left(|y - y^*|^2_{\mathbb{K}} + |z - z^*|^2_{\mathscr{L}^2(\mathbb{H},\mathbb{K})}\right),$$

  (t, w) $dt \otimes \mathbb{P}$*-a.e.*
- *(A3) $\exists \alpha > 0$ such that*

  $$|g(t,y,z) - g(t,y^*,z^*)|^2_{\mathscr{L}^2(\mathbb{H},\mathbb{K})} \geq \alpha\, |z - z^*|^2_{\mathscr{L}^2(\mathbb{H},\mathbb{K})},$$

$\forall z, z^* \in \mathscr{L}^2(\mathbb{H}, \mathbb{K})$ *and* $\forall y \in \mathbb{K}, (t, w)$ *almost everywhere.*

- *(A4) For each* (t, w, y) *the mapping* $z \mapsto g(t, y, z)$ *is surjective on* $\mathscr{L}^2(\mathbb{H}, \mathbb{K})$.

A strong solution of (3.6) is a pair (Y, Z) *in* $(\mathrm{L}^2_{\mathcal{F}}(0, T; \mathbb{K}) \times \mathrm{L}^2_{\mathcal{F}}(0, T; \mathscr{L}^2(\mathbb{H}, \mathbb{K})))$, *such that (3.7) holds. The following theorem is an infinite dimensional version of Theorem 4.1 in [PP90].*

Theorem 3.2. *Under (A1)-(A4), the BSDE (3.6) has a unique strong solution* $Y(\cdot), Z(\cdot)$*in* $\mathrm{L}^2_{\mathcal{F}}(0, T; \mathbb{K}), \mathrm{L}^2_{\mathcal{F}}(0, T; \mathscr{L}^2(\mathbb{H}, \mathbb{K}))$, *respectively.*

Proof. Uniqueness: Suppose that (Y, Z) and (Y', Z') are two solutions of 3.23. From Itô formula it follows that:

$$
\begin{gathered}
|Y(t) - Y'(t)|^2_{\mathbb{K}} = \\
2\int_t^T \langle Y(s) - Y'(s), f(s, Y(s), Z(s)) - f\left(s, Y'(s), Z'(s)\right)\rangle_{\mathbb{K}}\, ds \\
-2\int_t^T \langle Y(s) - Y'(s), \left(g(s, Y(s), Z(s)) - g\left(s, Y'(s), Z'(s)\right)\right) dW(s)\rangle_{\mathbb{K}} \\
-\int_t^T |g(s, Y(s), Z(s)) - g\left(s, Y'(s), Z'(s)\right)|_{\mathscr{L}^2(\mathbb{H},\mathbb{K}) ds}.
\end{gathered}
\tag{3.8}
$$

We shall suppress writing subscripts under the norms in the rest of this proof.

Note that from (A2) and (A3) we find that

$$
\begin{aligned}
&-\left|g(s,Y(s),Z(s))-g\left(s,Y'(s),Z'(s)\right)\right|^2\leq\\
&-\left|g(s,Y(s),Z(s))-g\left(s,Y(s),Z'(s)\right)\right|^2\\
&+2\left|g(s,Y(s),Z(s))-g\left(s,Y(s),Z'(s)\right)\right|\times\\
&\quad\left|g\left(s,Y(s),Z'(s)\right)-g\left(s,Y'(s),Z'(s)\right)\right|\\
&-\left|g\left(s,Y(s),Z'(s)\right)-g\left(s,Y'(s),Z'(s)\right)\right|^2\\
&\leq-\alpha\left|Z(s)-Z'(s)\right|^2+2k\left|Z(s)-Z'(s)\right|\left|Y(s)-Y'(s)\right|\\
&-\left|g\left(s,Y(s),Z'(s)\right)-g\left(s,Y'(s),Z'(s)\right)\right|^2\\
&\leq(-\alpha+\epsilon)\left|Z(s)-Z'(s)\right|^2+\frac{k^2}{\epsilon}\left|Y(s)-Y'(s)\right|^2,
\end{aligned}
$$

for any $\epsilon > 0$. Also, by using (A2), we get

$$
\begin{aligned}
&\left\langle Y(s)-Y'(s),f(s,Y(s),Z(s))-f\left(s,Y'(s),Z'(s)\right)\right\rangle\\
&\leq\left(\frac{1}{\epsilon}+\epsilon k\right)\left|Y(s)-Y'(s)\right|^2+\epsilon k\left|Z(s)-Z'(s)\right|^2,\forall\epsilon>0.
\end{aligned}
$$

Hence, by installing these two latter inequalities in 3.1, we conclude that for all $t \in [0, T]$

$$
\begin{aligned}
\mathbb{E}\left|Y(t)-Y'(t)\right|^2\leq&\left(\frac{1}{\epsilon}+\epsilon k+\frac{k^2}{\epsilon}\right)\mathbb{E}\int_t^T\left|Y(s)-Y'(s)\right|^2ds\\
&+(-\alpha+\epsilon+\epsilon k)\mathbb{E}\int_t^T\left|Z(s)-Z'(s)\right|^2ds,
\end{aligned}
\tag{3.9}
$$

$\forall\epsilon > 0$. Therefore, by choosing $\epsilon = \dfrac{\alpha}{2(1+k)}$, we conclude from Gronwall's inequality that $\mathbb{E}\left|Y(t)-Y'(t)\right|^2 = 0$, for all $t \in [0, T]$ and so 3.9 implies that $\mathbb{E}\displaystyle\int_0^T\left|Z(s)-Z'(s)\right|^2ds = 0$

Existence: We shall divide the proof into two steps.

Step 1: We study the following simplified version of 3.7,

$$
Y(t)=\xi+\int_t^T f(s)ds-\int_t^T g(s,Z(s))dW(s),
$$

where f does not depend on Y and Z, while g depends on Z but not on Y. In this case define, for each t,

$$Y(t) := \mathbb{E}\left[\xi + \int_t^T f(s)ds \mid \mathscr{F}_t\right]$$

There exists a unique $\tilde{Z} \in L^2_{\mathcal{F}}\left(0, T; \mathscr{L}^2(\mathbb{H}, \mathbb{K})\right)$, such that

$$\mathbb{E}\left[\xi + \int_0^T f(s)ds \mid \mathscr{F}_t\right] = \mathbb{E}\left[\xi + \int_0^T f(s)ds\right] + \int_0^t \tilde{Z}(s)dW(s),$$

which implies that

$$Y(t) = \xi + \int_t^T f(s)ds - \int_t^T \tilde{Z}(s)dW(s).$$

It remains to show that given $\tilde{Z} \in L^2_{\mathcal{F}}\left(0, T; \mathscr{L}^2(\mathbb{H}, \mathbb{K})\right)$, there exists $Z \in L^2_{\mathcal{F}}\left(0, T; \mathscr{L}^2(\mathbb{H}, \mathbb{K})\right)$ such that $g(t, Z(t)) = \tilde{Z}(t)$. Since g is a bijection in the Z-variable, for any $(t, \omega, z) \in [0, T] \times \Omega \times \mathscr{L}^2(\mathbb{H}, \mathbb{K})$, there exists a unique $\phi(t, \omega, z) \in \mathscr{L}^2(\mathbb{H}, \mathbb{K})$, such that $g(t, \phi(t, \omega, z)) = z$.

Thus we have only to show that ϕ is $\mathcal{P} \otimes \mathscr{B}\left(\mathscr{L}^2(\mathbb{H}, \mathbb{K})\right) / \mathscr{B}\left(\mathscr{L}^2(\mathbb{H}, \mathbb{K})\right)$ measurable. We may assume without loss of generality that $\Omega \equiv C_0([0, T]; H)$, $\mathcal{F}_T \equiv \mathscr{B}(\Omega)$ and $W(t)(\omega) = \omega(t), \forall t \in [0, T]$.

From the properties of g we see that the mapping $(t, \omega, z) \mapsto G(t, \omega, z) := (t, \omega, g(t, \omega, z))$, defined from $E \equiv [0, T] \times \Omega \times \mathscr{L}^2(\mathbb{H}, \mathbb{K})$ into itself is Borel measurable and a bijection.

Since E is a complete separable metric space, it follows from Kuratowski's theorem (cf. e.g. [Par05], Theorem 3.9 and Corollary 3.3) that G^{-1}, given by $G^{-1}(t, \omega, z) = (t, \omega, \phi(t, \omega, z))$, is $\mathscr{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathscr{B}\left(\mathscr{L}^2(\mathbb{H}, \mathbb{K})\right)$ measurable.

This implies that the restriction of G^{-1} to each sub-interval $[0, t]$ is $\mathscr{B}([0, t]) \otimes \mathcal{F}_t \otimes \mathscr{B}\left(\mathscr{L}^2(\mathbb{H}, \mathbb{K})\right)$ measurable for any $t \in [0, T]$.

Thus ϕ is $\mathcal{P} \otimes \mathscr{B}\left(\mathscr{L}^2(\mathbb{H}, \mathbb{K})\right) / \mathscr{B}\left(\mathscr{L}^2(\mathbb{H}, \mathbb{K})\right)$ measurable.

Step 2: We let f and g satisfy (A1)-(A4). By making use of Step 1, this case follows directly in the same way the standard result of Pardoux and Peng [PP90], in the finite dimensional case, was proved.

□

I used [Con06] as my foundation to find the outcome below. We are looking for an adapted pair of processes $\{(Y_t, Z_t) : t \in [0,T]\}$ *with values in* $\mathbb{H}$ *and* $\mathscr{L}^2(\mathbb{H},\mathbb{K}))$ *respectively,which solves the stochastic equation of the backward form:*

$$dY_t = -AY_t dt - f(t, Y_t, Z_t)dt + Z_t dW_t, \quad Y_T = \xi, \tag{3.10}$$

where A *is a dissipative operator which generates a* $\mathcal{C}_0$*-semi-group* $\left\{e^{tA}\right\}$ *on* $\mathbb{H}$ *and* ξ *is a* $\mathscr{F}_T$ *-measurable random variable in* $\mathscr{L}^2(\mathbb{H},\mathbb{K}))$. *More precisely, the above equation should be written as:*

$$Y_t - \int_t^T e^{(s-t)A} f(s, Y_s, Z_s)dt + \int_t^T e^{(s-t)A} Z_s dW_s = e^{(T-t)A}\xi. \tag{3.11}$$

We assume the coeffcient f *to be a continuous and dissipative function with respect to* y*, uniformly Lipschitz with respect to* z*. More generally, we consider* f *such that* $f - \mu I$ *is dissipative. Moreover,* f *is a function with sublinear growth with respect to* y *and* z*.The equation was previously considered in [HP91], and more recently in [ØZ01], but in these papers the drift* f *was required to be Lipschitz also with respect to* y*. In [PR99] the authors have solved BSDEs involving the sub-differential of a convex function in the coefficients. Instead we consider occurrence of a dissipative term together with an unbounded linear drift. The author followed the approach of [PR99] by providing a direct proof in the infinite-dimensional case based on the classical Yosida approximation of dissipative mappings. In [PR99] no proof is given since the authors state that the proof is the same as in the finite-dimensional case, which it had early shown in [PR98].*

The proof is based on the construction of a sequence of BSDEs

$$dY_t^\varepsilon = -AY_t^\varepsilon dt - f_\varepsilon(t, Y_t^\varepsilon, Z_t^\varepsilon)dt + Z_t^\varepsilon dW_t, \quad Y_T^\varepsilon = \xi,$$

where $f_\varepsilon(t,y,z)$ are the Yosida approximations of the dissipative mapping f. We show that these solutions $(Y_t^\varepsilon, Z_t^\varepsilon)$ converge to a pair of processes which solves our equation.This technique allows us to state the existence and uniqueness theorem for the infinite dimensional case, and also to simplify the proof of [PR98] in finite dimension. Moreover,we are able to deal with equations where A is not necessarily a sub-differential.

Hypotheses 3.1.

(i) *The final condition ξ is a $\mathscr{F}_T$ -measurable random variable in* $\mathrm{L}^2(\Omega;\mathbb{H})$.

(ii) *The operator A is the infinitesimal generator of a $\mathcal{C}_0$-semi-group $\{e^{tA}\}$ of linear bounded operators on $\mathbb{H}$. We denote by $\mathscr{D}(A)$ the domain of A and we assume that A is dissipative (i.e. $\langle Ay, y\rangle \leq 0, \forall y \in \mathscr{D}(A)$).*

(iii) *The coefficient $f : \Omega \times [0,T] \times \mathbb{H} \times \mathscr{L}^(\mathbb{K},\mathbb{H}) \to \mathbb{H}$ is such that for some $\mathbb{R}_+$-valued progressively measurable process $\left\{\bar{f}_t : 0 \leq t \leq T\right\}$ and some numbers $\mu \in \mathbb{R}$ and $K > 0$*

(1) *$f(\cdot,y,z)$ is progressively measurable, for all $y \in \mathbb{H}, z \in \mathscr{L}^2(\mathbb{K},\mathbb{H})$;*

(2) $\mathbb{P}$*-a.s.*

$$|f(t,y,z)| \leq \bar{f}_t + K\left(|y| + \|z\|\right), \quad t \in [0,T], \quad y \in \mathbb{H}, \quad z \in \mathscr{L}^2(\mathbb{K},\mathbb{H}); \tag{3.12}$$

(3) $\mathbb{E}\displaystyle\int_0^T \left|\bar{f}_t\right|^2 dt < \infty$;

(4) $\mathbb{P}$*-a.s.*

$$|f(t,y,z) - f(t,y^*,z^*)| \leq K\|z - z^*\|, \quad y \in \mathbb{H}, \quad z, z^* \in \mathscr{L}^2(\mathbb{K},\mathbb{H})$$

(5) $\mathbb{P}$*-a.s.*

$$\langle y - y^*, f(t,y,z) - f(t,y^*,z)\rangle \leq \mu\,|y - y^*|^2,$$

where $t \in [0,T], \quad y, y^* \in \mathbb{H}, \quad z \in \mathscr{L}^2(\mathbb{K},\mathbb{H})$.

(6) $\mathbb{P}$*-a.s. $y \mapsto f(t,y,z)$ is continuous, for all $t \in [0,T], z \in \mathscr{L}^2(\mathbb{K},\mathbb{H})$.*

We are looking for solutions of the (3.10), in the so-called mild sense (already introduced in the literature by Hu and Peng [HP91]). Before stating the main theorem of this section, we define a typical tool which allows to treat dissipative functions.

Given f satisfying (e) in hypothesis 3.1 , we note that $f-\mu I$ is a continuous and dissipative function with respect to y. We introduce its Yosida approximations defined, for $\varepsilon>0$, by

$$f_\varepsilon(t,y,z)=f(t,J_\varepsilon(t,y,z),z)-\mu J_\varepsilon(t,y,z), \tag{3.13}$$

for $y\in\mathbb{H}, t\in[0,T], z\in\mathscr{L}^2(\mathbb{K},\mathbb{H})$, where

$$J_\varepsilon(t,y,z)=(I-\varepsilon(f-\mu I))^{-1}(t,y,z),\quad y\in\mathbb{H}, z\in\mathscr{L}^2(\mathbb{K},\mathbb{H}). \tag{3.14}$$

The following properties are well known when f does not depend on ω,t,z and can be proved as in the general case (see Chap. 5 of [DZ96]).

(i) *For any $\varepsilon>0$ we have*

$$|J_\varepsilon(t,x,z)-J_\varepsilon(t,y,z)|\le|x-y|,\quad \forall x,y\in\mathbb{H}, t\in[0,T], z\in\mathscr{L}^2(\mathbb{K},\mathbb{H}). \tag{3.15}$$

(ii) *For any $\varepsilon>0, f_\varepsilon$ is dissipative and Lipschitz continuous with respect to y:*

$$|f_\varepsilon(t,x,z)-f_\varepsilon(t,y,z)|\le\frac{2}{\varepsilon}|x-y|,\quad \forall x,y\in\mathbb{H}, t\in[0,T], z\in\mathscr{L}^2(\mathbb{K},\mathbb{H}).$$

and

$$|f_\varepsilon(t,y,z)|\le|(f-\mu I)(t,y,z)|,\quad \forall y\in\mathscr{D}(f), t\in[0,T], z\in\mathscr{L}^2(\mathbb{K},\mathbb{H}).$$

(iii) *We have*

$$\lim_{\varepsilon\to0}J_\varepsilon(t,y,z)=y,\quad \forall\in\overline{\mathscr{D}(f)}, t\in[0,T], z\in\mathscr{L}^2(\mathbb{K},\mathbb{H}).$$

(iv) For any $\varepsilon, \sigma > 0, t \in [0,T], z \in \mathscr{L}^2(\mathbb{K}, \mathbb{H})$ *the following inequality holds:*

$$\langle f_\varepsilon(t,x,z) - f_\sigma(t,y,z), x-y\rangle \leq (\varepsilon+\sigma)(|f_\varepsilon(t,x,z)|+|f_\sigma(t,y,z)|)^2, \quad \forall x,y \in \mathbb{H}. \tag{3.16}$$

Theorem 3.3. *Under Hypothesis 3.1, the BSDE (3.10) has a unique mild solution* (Y,Z)*.*

For the proof see [Con06].

F.Confortola in his article [Con07] extended the previous result in a special direction.This backward stochastic differential equation is one that we are interested in solving.

$$dY_t = -AY_t dt - f(t,Y_t,Z_t)dt + Z_t dW_t, \quad 0 \leq t \leq T, \quad Y_T = \xi \tag{3.17}$$

and

$$f(t,Y_t,Z_t) = f_0(t,Y_t) + f_1(t,Y_t,Z_t)$$

where ξ *is a random variable with values in* $\mathbb{H}$ *and* f_0, f_1 *are given functions.The coefficient* $f_1(t,y,z)$ *is assumed to be bounded and Lipschitz with respect to* y *and* z*. The non-linearity of* $f_0(t,y)$ *is dissipative, sectorial and defined for* y *only taking values in a suitable subspace* $\mathbb{H}_\alpha$ *of* $\mathbb{H}$ *and it satisfies the following growth condition for some* $1 < \gamma < 1/\alpha, S \geq 0, \mathbb{P}$*-a.s.*

$$|f_0(t,y)|_{\mathbb{H}} \leq S\left(1 + \|y\|^\gamma_{\mathbb{H}_\alpha}\right) \quad \forall t \in [0,T], \forall y \in \mathbb{H}_\alpha.$$

We can write Eq.3.17 in the following integral form

$$Y_t - \int_t^T e^{(s-t)A}\left[f_0(s,Y_s) + f_1(s,Y_s,Z_s)\right] ds + \int_t^T e^{(s-t)A} Z_s dW_s = e^{(T-t)A}\xi, \tag{3.18}$$

in particular, that Y *takes values in* $\mathbb{H}_\alpha$*. This requires generally that the final condition also takes values in the smaller space* $\mathbb{H}_\alpha$*. We take* $\mathbb{H}_\alpha$ *as a real interpolation space which belongs to the class* J_α *between* $\mathbb{H}$ *and the domain of* *an operator* A . *Moreover* $f_0(t,\cdot)$ *is assumed to be locally Lipschitz from* $\mathbb{H}_\alpha$ *into* $\mathbb{H}$ *and dissipative in* $\mathbb{H}$. *Based on Fulvia's paper, we can demonstrated that if* ξ *takes its values in the closure of* $\mathscr{D}(A)$ *in* $\mathbb{H}_\alpha$ *and is such that* $\|\xi\|_{\mathbb{H}_\alpha}$ *is essentially bounded, we determine that Eq.3.18 has a unique mild solution.*

Hypotheses 3.2.

1. $A : \mathscr{D}(A) \subset \mathbb{H} \to \mathbb{H}$ *is a sectorial operator. We also assume that* A *is dissipative, i.e. it satisfies* $\langle Ay, y\rangle \le 0, \forall y \in \mathscr{D}(A)$;
2. *For some* $0 < \alpha < 1$ *there exists a Banach space* $\mathbb{H}_\alpha$ *continuously embedded in* $\mathbb{H}$ *and such that*
   - *(i)* $\mathscr{D}_A(\alpha, 1) \subset \mathbb{H}_\alpha \subset \mathscr{D}_A(\alpha, \infty)$;
   - *(ii) the part of* A *in* $\mathbb{H}_\alpha$ *is sectorial in* $\mathbb{H}_\alpha$.
3. ξ *is an* $\mathscr{F}_T$ *-measurable random variable defined on* Ω *with values in the closure of* $\mathscr{D}(A)$ *with respect to* $\mathbb{H}_\alpha$*-norm. We denote this set* $\overline{\mathscr{D}(A)}^{\mathbb{H}_\alpha}$. *Moreover* ξ *belongs to* $\mathrm{L}^\infty(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{H}_\alpha)$;
4. $f_0 : \Omega \times [0,T] \times \mathbb{H}_\alpha \to \mathbb{H}$ *satisfies:*
   - *(i)* $\{f_0(t,y)\}_{t\in[0,T]}$ *is progressively measurable* $\forall y \in \mathbb{H}_\alpha$;
   - *(ii) There exist constants* $S > 0, 1 < \gamma < 1/\alpha$ *such that* $\mathbb{P}$*-a.s.*

     $$|f_0(t,y)|_{\mathbb{H}} \le S(1 + \|y\|^\gamma_{\mathbb{H}_\alpha}) \quad t \in [0,T], y \in \mathbb{H}_\alpha;$$

   - *(iii) For every* $R > 0$ *there is* $L_R > 0$ *such that* $\mathbb{P}$*-a.s.*

     $$|f_0(t,y_1) - f_0(t,y_2)|_{\mathbb{H}} \le L_R \|y_1 - y_2\|_{\mathbb{H}_\alpha}$$

     for $t \in [0,T], y_i \in \mathbb{H}_\alpha$ *with* $\|y_i\|_{\mathbb{H}_\alpha} \ge R$;
   - *(iv) There exists a number* $\mu \in \mathbb{R}$ *such that* $\mathbb{P}$*-a.s.,* $\forall t \in [0,T], y_1, y_2 \in \mathbb{H}_\alpha$,

     $$\langle f_0(t,y_1) - f_0(t,y_2), y_1 - y_2\rangle_{\mathbb{H}} \le \mu \|y_1 - y_2\|^2_{\mathbb{H}_\alpha}; \qquad (3.19)$$

5. $f_1 : \Omega \times [0,T] \to \mathbb{H}$ *is progressively measurable and for some constant* $C > 0$ *it satisfies* $\mathbb{P}$*-a.s.* $|f_1(t)|_{\mathbb{H}} \leq C$, *for* $t \in [0,T]$.

6. *There exists* K *positive such that* $\mathbb{P}$*-a.s.*

$$|f_1(t,y,z) - f_1(t,y^*,z^*)|_{\mathbb{H}} \leq K\,|y-y^*|_{\mathbb{H}} + K\,|z-z^*|_{\mathscr{L}^2(\mathbb{K},\mathbb{H})}\,,$$

for every $t \in [0,T], y, y^* \in \mathbb{H}, z, z^* \in \mathscr{L}^2(\mathbb{K},\mathbb{H})$.

7. *There exists* $C \geq 0$ *such that* $\mathbb{P}$*-a.s.* $|f_1(t,y,z)|_{\mathbb{H}} \leq C$, *for every* $t \in [0,T], y \in \mathbb{H}, z \in \mathscr{L}^2(\mathbb{K},\mathbb{H})$.

Theorem 3.4. *If Hypotheses 3.2 hold, then the equation 3.18 has a unique solution in* $\mathrm{L}^2(\Omega;\mathcal{C}([0,T];\mathbb{H}_\alpha)) \times \mathrm{L}^2(\Omega \times [0,T];\mathscr{L}^2(\mathbb{K},\mathbb{H}))$.

In [FH07] M.Fuhrman and Y. Hu prove the existence of a solution to the following BSDE assuming that ψ *is only continuous with respect to* (y,z). *We consider the following BSDE:*

$$dY_t = -BY_t dt - \psi(t,X_t,Y_t,Z_t)dt + Z_t dW_t, \quad Y_T = \varphi(X_T). \tag{3.20}$$

where W *is a cylindrical Wiener process in a Hilbert space* Ξ, B *is the infinitesimal generator of a strongly continuous dissipative compact semi-group* (e^{tB}) *in a Hilbert space* $\mathbb{K}$, X *is a Markov process with respect to the filtration generated by* W, *and* ψ *and* φ *are deterministic functions with values in* $\mathbb{K}$. *The solution* (Y,Z) *takes values in* $\mathbb{K} \times \mathscr{L}^2(\Xi,\mathbb{K})$, *where* $\mathscr{L}^2(\Xi,\mathbb{K})$ *denotes the space of Hilbert-Schmidt operators from* Ξ *to* $\mathbb{K}$.

Let H, Ξ *be a Hilbert spaces.We are given two linear operators* $A : \mathscr{D}(A) \subset \mathbb{H} \to \mathbb{H}$ *and* $G \in \mathrm{L}(\Xi,\mathbb{H})$ *such that*

Hypotheses 3.3.

(i) *The operator* $A : \mathscr{D}(A) \subset \mathbb{H} \to \mathbb{H}$ *is the infinitesimal generator of a strongly continuous semi-group* $\left\{e^{tA}, t \geq 0\right\}$ *of bounded linear operators in* $\mathbb{H}$.

(ii) $G : \Xi \to \mathbb{H}$ *is a bounded linear operator.*

(iii) The operators

$$Q_t x = \int_0^t e^{sA} G G^* e^{sA^*} x ds, \quad x \in \mathbb{H},$$

are of trace class for all $t \geq 0$.

(iv) $e^{tA}(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H})$, *for all* $t > 0$.

We define the Ornstein Uhlenbeck process as the solution of the following stochastic equation:

$$dX_t = AX_t dt + G dW_t, \quad X_0 = x, \tag{3.21}$$

where $x \in \mathbb{H}$ *is given and* W *is a cylindrical Wiener process in* Ξ. *The equation (3.21) It is considered in the so-called mild sense: the solution by definition is the following process*

$$X_t = e^{tA} x + \int_0^t e^{(t-s)A} G dW_s, \quad t \geq 0. \tag{3.22}$$

We consider a BSDE of the form

$$dY_t = -BY_t dt - \psi(t, X_t, Y_t, Z_t) dt + Z_t dW_t, \quad Y_T = \varphi(X_t), \tag{3.23}$$

for t *varying on a bounded time interval* $[0, T]$. W *is a cylindrical Wiener process in a Hilbert space* Ξ *and we denote by* $(\mathscr{F}_t)$ *its Brownian filtration. The unknown processes* Y *and* Z *take values in a Hilbert space* $\mathbb{K}$ *and in the Hilbert space* $\mathscr{L}^2(\Xi, \mathbb{K})$, *respectively.* X *is a given* $(\mathscr{F}_t)$*-predictable process in another Hilbert space* $\mathbb{H}$. *We assume the following.*

Hypotheses 3.4.

(i) The operator $B : \mathscr{D}(B) \subset \mathbb{K} \to \mathbb{K}$ *is the infinitesimal generator of a strongly continuous dissipative semi-group* $\left\{ e^{tB}, t \geq 0 \right\}$ *of linear bounded operators on* $\mathbb{K}$.

(ii) $\varphi : \mathbb{H} \to \mathbb{K}$ *and* $\psi : [0, T] \times \mathbb{H} \times \mathbb{K} \times \mathscr{L}^2(\Xi, \mathbb{K}) \to \mathbb{K}$ *are Borel measurable*

functions, and there exist two constants $C > 0$ and $p \geq 1$ such that

$$|\varphi(x)| \leq C\left(1 + |x|^p\right), \quad x \in \mathbb{H}$$

$$|\psi(t,x,y,z)| \leq \mathcal{C}(1 + |x|^p + |y| + |z|),$$
$$t \in [0,T], \quad x \in \mathbb{H}, \quad y \in \mathbb{K}, \quad z \in \mathscr{L}^2(\Xi,\mathbb{K})$$

.

(iii) For every $t \in [0,T]$ and $x \in \mathbb{H}$, the function $\psi(t,x,\cdot,\cdot) : \mathbb{K}\times\mathscr{L}^2(\Xi,\mathbb{K}) \to \mathbb{K}$, is continuous.

Theorem 3.5. *Assume that Hypotheses 3.3 and 3.4 hold and suppose that the operators e^{tB} are compact for $t > 0$. Let X be the Ornstein Uhlenbeck process defined by (3.21). Then there exists a mild solution (Y,Z) to (3.23). Moreover there exist a Borel measurable functions $u : [0,T] \times \mathbb{H} \to \mathbb{K}$, and $v : [0,T] \times \mathbb{H} \to \mathscr{L}^2(\Xi,\mathbb{K})$ such that, $\mathbb{P}$-a.s.,*

$$Y_t = u(t,X_t), \quad \textit{for all} \quad t \in [0,T];$$
$$Z_t = v(t,X_t), \quad \textit{for almost all} \quad t \in [0,T];$$

Based on [BC08], we consider this BSDE:

$$Y_T = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s \tag{3.24}$$

where W is a cylindrical Wiener process in some infinite dimensional Hilbert space Ξ and the generator f is Lipschitz but with random Lipschitz constants where $f : [0,T] \times \Omega \times \mathbb{R} \times \Xi^* \to \mathbb{R}$.

We want to consider the situation where the Lipschitz constant, say K, depends also on (t,ω). The main goal is to study the previous BSDE when this stochastic process K is such that its stochastic integral is a bounded mean oscillation martingale (BMO martingale in the sequel). BSDEs under stochastic Lipschitz condition have already been studied in [EH97] and more *recently in [CRW06].*

However, the results in these section do not fit BMO framework which arises naturally in the study of the regularity of solutions to quadratic BSDEs with bounded terminal condition.

Such quadratic BSDEs have been intensively studied by Kobylanski [Kob00] and then by Lepeltier and San Martin in [LM98]. He refer the reader to [BH06] for the case of an unbounded terminal condition.

We work with the following assumption.

Hypotheses 3.5.

(i) There exist a real process K and a constant $\alpha \in (0,1)$ such that $\mathbb{P}$-a.s.:

- *for each $t \in [0,T], (y,z) \to f(t,y,z)$ is continuous;*
- *for each $(t,z) \in [0,T] \times \Xi^*$,*

$$\forall y,p \in \mathbb{R}, \quad (y-p)(f(t,y,z)-f(t,p,z)) \leq K_t^{2\alpha} |y-p|^2 ;$$

- *for each $(t,y) \in [0,T] \times \mathbb{R}$,*

$$\forall (z,q) \in \Xi^* \times \Xi^*, \quad |f(t,y,z)-f(t,y,q)| \leq K_t |y-q|_{\Xi^*} ;$$

(ii) $\{K_s\}_{s\in[0,T]}$ is a predictable real process bounded from below by 1 such that there is a constant C such that, for any stopping time $\tau \leq T$,

$$\mathbb{E}\left(\int_\tau^T |K_s|^2 \, ds|\mathscr{F}_\tau\right) \leq C^2.$$

N denotes the smallest constant C for which the previous statement is true.

In order to explain the meaning of this assumption we have to introduce the space of Bounded Mean Oscillation martingales (BMO martingales for short).

We refer the reader to [Kaz06] for the theory of BMO martingales and we just recall the properties that we will use in the sequel. Let M be a continuous local $(\mathbb{P},\mathcal{F})$-martingale satisfying $M_0 = 0$. Let $1 \leq p < \infty$. Then M is in the normed linear space BMO_p if

$$\|M\|_{BMO_p} = \sup_{\tau} \left\| \mathbb{E}\left(|M_T - M_\tau|^p \,|\mathscr{F}_\tau\right)^{1/p} \right\|_\infty < \infty$$

where the supremum is taken over all stopping times $\tau \leq T$. By Corollary 2.1 in [Kaz06], M is a BMO_p martingale if and only if it is a BMO_q martingale for every $q \geq 1$. Therefore, it is simply called a BMO martingale. In particular, M is a BMO martingale if and only if

$$\|M\|_{BMO_2} = \sup_{\tau} \left\| \mathbb{E}\left[\langle M\rangle_T - \langle M\rangle_\tau \,|\mathscr{F}_\tau\right]^{1/2} \right\|_\infty < \infty$$

where the supremum is taken over all stopping times $\tau \leq T$; $\langle M\rangle$ denotes the quadratic variation of M. This means local martingales of the form $M_t = \int_0^t \xi_s dW_s$ are BMO martingales if and only if

$$\|M\|_{BMO_2} = \sup_{\tau} \left\| \mathbb{E}\left[\int_0^t \|\xi_s\|^2 \, ds |\mathcal{F}_\tau\right]^{1/2} \right\|_\infty < \infty$$

Hence this assumption says that, for any $u \in \Xi^*$ such that $\|u\|_\Xi^* = 1$, the martingale $M_t = \int_0^t K_s u dW_s, 0 \leq t \leq T$ is a BMO martingale with $\|M\|_{BMO_2} = N$. It follows from the inequality ([Kaz06], p. 26)

$$\forall n \in \mathbb{N}^*, \quad \mathbb{E}\left[\langle M\rangle_T^n\right] = \mathbb{E}\left[\left(\int_0^T |K_s|^2 \, ds\right)^n\right] \leq n! N^{2n}$$

that M belongs to $\mathbb{H}^P$ for all $p \geq 1$ and moreover

$$\forall \alpha \in (0,1), \quad \eta(p)^p = \mathbb{E}\left[\exp\left(p \int_0^T |K_s|^{2\alpha} \, ds\right)\right] < \infty.$$

The very important feature of BMO martingales is the following: the *exponential martingale*

$$\varepsilon(M)_t = \varepsilon_t = \exp\left(\int_0^t K_s u.dW_s - \frac{1}{2}\int_0^t |K_s|^{2\alpha}\, ds\right)$$

is a uniformly integrable martingale. More precisely, $\{\varepsilon_t\}_{0\le t\le T}$ *satisfies a reverse Hölder inequality. Let* Φ *be the function defined on* $(1, +\infty)$ *by*

$$\Phi(p) = \left(1 + \frac{1}{p^2}\log\frac{2p-1}{2(p-1)}\right)^{1/2} - 1;$$

Φ *is nonincreasing with* $\lim_{p\to 1}\Phi(p) = \infty$, $\lim_{p\to\infty}\Phi(p) = 0$,*. Let* q_* *be such that* $\Phi(q_*) = N$*. Then, for each* $1 < q < q_*$ *and for all stopping times* $\tau \le T$,

$$\mathbb{E}\left(\varepsilon(M)_T^q | \mathscr{F}_\tau\right) \le K(q, N)\varepsilon(M)_\tau^q \tag{3.25}$$

where the constant $K(q, N)$ *can be chosen depending only on* q *and* $N = \|M\|_{BMO_2}$ *, e.g.*

$$K(q, N) = \frac{2}{1 - 2(q-1)(2q-1)^{-1}\exp(q^2(N^2 + 2N))}.$$

Remark 3.1. *If we denote by* $\mathbb{P}^*$ *the probability measure on* $(\Omega, \mathscr{F}_T)$ *whose density with respect to* $\mathbb{P}$ *is given by* ε_T *then* $\mathbb{P}$ *and* $\mathbb{P}^*$ *are equivalent.*

Moreover, it follows from (3.25) and Hölder inequality that if X *belongs to* $L^p(\mathbb{P})$ *then* X *belongs to* $L^s(\mathbb{P}^*)$ *for all* $s < p/p_*$ *where* p_* *is the conjugate exponent of* q_**.*

We assume also some integrability conditions. For this, let p_* *be the conjugate exponent of* q_**.*

(iii) There exists $p^* > p_*$ *such that*

$$\mathbb{E}\left[|\xi|^{p^*} + \left(\int_0^T |f(s, 0, 0)|\, ds\right)^{p^*}\right] < \infty.$$

(iv) There exists a nonnegative predictable process f such that

$$\mathbb{E}\left[\left(\int_0^T f(s)ds\right)^{p^*}\right] < \infty.$$

and $\mathbb{P}$-a.s.

$$\forall (t,y,z) \in [0,T]\times\mathbb{R}\times\Xi^*, \quad |f(t,y,z)| \le f(t) + K_t^{2\alpha}|y| + K_t|z|.$$

Theorem 3.6. *Let the Assumptions 3.5 hold. Then BSDE (3.24) has a unique solution (Y,Z) which belongs to $\mathcal{S}^p\times\mathcal{M}^p$ for all $p<p^*$, where for any $p>0, \mathcal{S}^p(\mathbb{K})$, or $\mathcal{S}^p$ denotes the set of $\mathbb{K}$-valued, adapted and càdlàg processes $\{Y_t\}_{t\in[0,T]}$ such that*

$$\|Y\|_{\mathcal{S}^p} := \mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^p\right]^{1/p} < \infty.$$

If $p\ge 1, |\cdot|_{\mathcal{S}^p}$ is a norm on $\mathcal{S}^p$ and if $p\in(0,1),(X,\grave{X})\mapsto \left\|X-\grave{X}\right\|_{\mathcal{S}^p}$ defines a distance on $\mathcal{S}^p$. Under this metric, $\mathcal{S}^p$ is complete.

$\mathcal{M}^p(\mathscr{L}^2(\Xi,\mathbb{K}))$ denotes the set of (equivalent classes of) predictable processes $\{Z_t\}_{t\in[0,T]}$ with values in $\mathscr{L}^2(\Xi,\mathbb{K})$ such that

$$\|Z\|_{\mathcal{M}^p} := \mathbb{E}\left[\left(\int_0^T |Z_s|^2\,ds\right)^{p/2}\right]^{1\wedge 1/p}.$$

For $p\ge 1$, $\mathcal{M}^p$ is a Banach space endowed with this norm and for $p\in(0,1),\mathcal{M}^p$ is a complete metric space with the resulting distance.

Proof. We need this corollary in the proof

Corollary 3.1. *If (Y,Z) is a solution to 3.24 such that, for some $r>p_*$, $Y\in\mathcal{S}^r$, then, for each $p\in(p_*,p^*)$,$(Y,Z)\in\mathcal{S}^p\times\mathcal{M}^p$ and*

$$\|Y\|_{\mathcal{S}^p}+\|Z\|_{\mathcal{M}^p} \le C\left\||\xi|+\int_0^T|f(S,0,0)|\,ds\right\|_{p^*}\left(1+\left\|\left(\int_0^T\left(K_s^{2\alpha}+K_s^2\right)ds\right)^{1/2}\right\|_d\right)$$

where $d = \dfrac{p(p^*+p)}{(p^*-p)}$ *and* C *depends on* p, p^*, p_* *and* N.

Let us prove first uniqueness. Let (Y^1, Z^1) and (Y^2, Z^2) be solutions to (3.24) such that Y^1 and Y^2 belongs to $\mathcal{S}^p$ for $p > p_*$. Then by Corollary 3.1, (Y^1, Z^1) and (Y^2, Z^2) belong to $\mathcal{S}^p \times \mathcal{M}^p$ for all $p < p_*$. Moreover, $U = Y^1 Y^2$ and $V = Z^1 Z^2$ solve the BSDE

$$U_t = \int_t^T F(s, U_s, V_s)ds - \int_t^T V_s.dW_s,$$

where $F(s,u,v) = f(t, Y_t^2 + u, Z_t^2 + v) - f(t, T_t^2, Z_t^2)$.We have $F(t,0,0) = 0$ and F satisfies Assumption (i) in 3.5 with the same process K. It follows from Corollary 3.1 $(U,V) \equiv (0,0)$. Let us turn to existence. For each integer $n \geq 1$, let τ_n be the following stopping time:

$$\tau_n = \inf \left\{ t \in [0,T] : \int_0^t (f(s) + K_s^2)ds \geq n \right\} \wedge T.$$

Let $\xi^n = \xi \mathbb{1}_{|\xi| \leq n}$ and (Y^n, Z^n) be the solution to the BSDE

$$Y_t^n = \xi^n + \int_t^T \mathbb{1}_{s \leq \tau_n} f(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dW_s.$$

The existence of the solution (Y^n, Z^n) to the previous equation comes from [Mor09]. Indeed, we have, setting $f^n(t,y,z) = \mathbb{1}_{t \leq \tau_n} f(t,y,z)$,

$$|f^n(t,y,z)| \leq \mathbb{1}_{t \leq \tau_n}(f(t) + K_t^{2\alpha} + K_t^2/2)(1 + |y|) + |z|^2/2,$$

and, $\mathbb{P}$-a.s.,

$$\int_0^T \mathbb{1}_{t \leq \tau_n}(f(t) + K_t^{2\alpha} + K_t^2/2)dt \leq 5n/2.$$

Since ξ^n is bounded by n, the previous BSDE has a unique solution (Y^n, Z^n) such that Y^n is a bounded process and $Z^n \in \mathcal{M}^2$. Since

$$\int_0^T |f^n(t,0,0)|\, dt \leq n,$$

we know, from Corollary 3.1 , that $(Y^n, Z^n) \in \mathcal{S}^p \times \mathcal{M}^p$ for all p. Moreover, still by Corollary 3.1 [BC08], the sequence $((Y^n, Z^n))_{n\geq 1}$ is bounded in $\mathcal{K}^p := \mathcal{S}^p \times \mathcal{M}^p$ for all $p < p^*$. Let us show that $((Y^n, Z^n))_{n\geq 1}$ is a Cauchy sequence in $\mathcal{K}^p := \mathcal{S}^p \times \mathcal{M}^p$ for all $p < p^*$. Let $m > n1$ and let us set as before $U = Y^m Y^n, V = Z^m Z^n$. Then (U, V) solves the BSDE

$$U_t = \xi^m - \xi^n + \int_t^T F(s, U_s, V_s)ds - \int_t^T V_s dW_s$$

where

$$F(t, u, v) = \mathbb{1}_{t\leq\tau_m}(f(t, u+Y_t^n, v+Z_t^n) - f(t, Y_t^n, Z_t^n)) - \mathbb{1}_{\tau_n<t\leq\tau_m} f(t, Y_t^n, Z_t^n).$$

□

3.2 Numerical Analysis

We consider the separable Hilbert space $\mathscr{L}^2([0,1], \mathscr{B}([0,1]), dx; \mathbb{R})$ *and let* $-A$ *the Laplace operator with zero boundary conditions. We can define fractional powers of* A *(see appendix B.2 in [Kru14]). For any* $r \geq 0$ *let the operator* $A^{\frac{r}{2}} : \mathscr{D}(A^{\frac{r}{2}}) \subset \mathbb{H} \to \mathbb{H}$ *be given by*

$$A^{\frac{r}{2}} x = \sum_{n=1}^{\infty} \lambda_n^{\frac{r}{2}} (x, e_n) e_n$$

for all

$$x \in \mathscr{D}(A^{\frac{r}{2}}) = \left\{ x \in \mathbb{H} : \|x\|_r^2 := \sum_{n=1}^{\infty} \lambda_n^r (x, e_n)^2 < \infty \right\}$$

where $(e_n)_{n\geq 1}$ *an orthonormal basis of eigenvectors obtain by applying the spectral theorem for linear compact and self-adjoint operators to* A^{-1}.

By setting $\dot{\mathbb{H}}^r := \mathscr{D}(A^{\frac{r}{2}})$ *and* $\dot{\mathbb{H}}^r := (A^{\frac{r}{2}} \cdot, A^{\frac{r}{2}} \cdot)$ *obtain a separable Hilbert space* $(\dot{\mathbb{H}}^r, \dot{\mathbb{H}}^r, \|\cdot\|_r)$ *for every* $r > 0$.

We have here $\dot{\mathbb{H}}^1 = \mathbb{H}_0^1(0,1)$ *and* $\dot{\mathbb{H}}^2 = \mathbb{H}_0^1(0,1) \cap \mathbb{H}^2(0,1)$ *where* $\mathbb{H}_0^1(0,1)$ *is the Sobolev space of square-integrable functions on* $[0,1]$ *with existing first order* *weak derivative and zero boundary conditions and* $\mathbb{H}^2(0,1)$ *is the Sobolev space of square-integrable functions on* $[0,1]$ *with existing weak derivatives up to order 2.*

Further, the eigenvalues and eigenfunctions of A *are given by*

$$\lambda_j = j^2\pi^2 \quad \text{and } e_j(y) = \sqrt{2}\sin(j\pi y) \quad \forall j \in \mathbb{N}, j \geq 1, y \in [0,1].$$

We are concerned with a spatial semidiscretization in terms of the standard finite element method from (see section 6.1 in [Kru14]).

For this, let $N_h \in \mathbb{N}$ *and* $h := \dfrac{1}{N_h + 1}$*. We consider an equidistant partition of the spatial domain* $[0,1]$ *with* $x_j := jh, j := 0, \cdots, N_h + 1$*.Then, for every* h *the spaces* $\mathbb{S}_h$ *are given as the set of all continuous functions on* $[0,1]$*, which are piecewise (affine) linear on all intervals* $[x_{j-1}, x_j], j = 1, \cdots, N_h + 1$*, and satisfy the zero boundary condition. we recall that* $\mathbb{S}_h \subset \dot{\mathbb{H}}^1$ *and* $\dim(\mathbb{S}_h) = N_h$*.Further, the set* $(\Phi_j)_{j=1}^{N_h}$ *of so called hat functions or pyramid functions forms a basis of* $\mathbb{S}_h$ *, where* $\Phi_j, j \in \{1, \cdots, N_h\}$ *, is defined by*

$$\phi_j(x_i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

As a result, every $u_h \in \mathbb{S}_h$ *has the representation*

$$u_h(x) = \sum_{j=1}^{\infty} u_h(x)\phi_j(x), \quad \forall x \in [0,1]. \tag{3.26}$$

Besides the orthogonal projectors $P_h : \dot{\mathbb{H}}^{-1} \to \mathbb{S}_h$ *and* $R_h : \dot{\mathbb{H}}^{-1} \to \mathbb{S}_h$*, which are presented in (chapter 3 in [Kru14]), a given element* $u \in \dot{\mathbb{H}}^{-1}$ *may also be approximated by its interpolant* $I_h u$*. Here, the interpolation operator* I_h *is given by*

$$I_h u = \sum_{j=1}^{\infty} u(x_j)\phi_j. \tag{3.27}$$

Note, that by the Sobolev embedding theorem in [Led03], all elements in $\dot{\mathbb{H}}^{-1} = \mathbb{H}_0^1(0,1)$ *have a continuous function in their equivalence class. Thus, the evaluation of* $u \in \dot{\mathbb{H}}^{-1}$ *at grid points is a well-defined operation if we say that we always evaluate the continuous representative of* u.

From Th. 3.1.5 in [Cia02] it is well-known that I_h *satisfies the following error estimates*

$$\|I_h u - u\| \leq Ch^s \|u\|_s \quad \forall u \in \dot{\mathbb{H}}^s, s \in 1, 2, N_h \in \mathbb{N} \tag{3.28}$$

Note that these estimates are comparable to Assumption in [Kru14]. Next, we recall from section 3.2 in [Kru14] about Galerkin finite element methods the discrete Laplacian is uniquely determined by the relationship

$$(u_h, v_h)_1 = (A_h u_h, v_h), \quad \forall v_h \in \mathbb{S}_h. \tag{3.29}$$

In particular, by inserting 3.26 and $v_h = \phi_i$*for* $i \in \{1, \cdots, N_h\}$ *we obtain*

$$(A_h u_h, \phi_i) = (u_h, \phi_i)_1 = \sum_{j=1}^{N_h} (\phi_i, \phi_j)_1 \, u_h(x_j).$$

Therefore, in terms of the basis $(\phi_j)_{j=1}^{N_h}$ *the operator Ah is represented by the so called stiffness matrix with entries* $a_{ij} = (\phi_i, \phi_j)_1,$*for* $i, j = 1, \cdots, N_h$.

$$a_{ij} = \int_0^1 \phi_i'(x)\phi_j'(x)\mathrm{d}x = \begin{cases} \frac{2}{h}, & i = j \\ -\frac{1}{h}, & |i-j| = 1 \\ 0, & |i-j| > 1 \end{cases}$$

That is, we obtain a well-known tridiagonal matrix of the form

$$(a_{ij})_{i,j=1}^{N_h} = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & -1 \\ 0 & & & -1 & 2 \end{pmatrix} \tag{3.30}$$

The typical method for conducting numerical analysis for BSDEs is to project our equation on sup-space on finite dimension and then solve the corresponding finite dimensional BSDE using the least squares Monte Carlo given in the following.

3.2.1 Least-squares Monte Carlo

This section covers the numerical discretization of uncoupled FBSDE and explains how the two relate to the solution of the BSD equation. Here, we suggest a semi-parametric method based on radial basis functions. The main difference between this study and the results in the article [KNH19] is that in this paper, the dimension of Y is equal to 1, whereas in our work, we assume that Y is a multidimensional process.

3.2.2 Time stepping scheme

The FBSDE can be discretized using an explicit time-stepping technique because it is decoupled. Here, we use a modification of the least-squares Monte Carlo approach that was first introduced in [KNH19]. The least-squares Monte Carlo scheme is based on the Euler discretisation of specifically,

$$\begin{aligned} \hat{X}_{n+1} &= \hat{X}_n + \Delta t\, b(\hat{X}_n, t_n) + \sqrt{\Delta t}\,\sigma(\hat{X}_n)\xi_{n+1} \\ \hat{Y}_{n+1} &= \hat{Y}_n - \Delta t\, h(\hat{X}_n, \hat{Y}_n, \hat{Z}_n) + \sqrt{\Delta t}\,\hat{Z}_n \cdot \xi_{n+1}\,, \end{aligned} \tag{3.31}$$

where $(\hat{X}_n, \hat{Y}_n, \hat{Z}_n)$ denotes the numerical discretisation of the joint process (X_s, Y_s, Z_s), where we set $X_s \equiv X_T$, and $(\xi_i)_{i\geqslant 1}$ are an i.i.d. sequence of *normalised Gaussian random variables. Now let*

$$\mathcal{F}_n = \sigma\big(\big\{\hat{B}_k : 0 \leqslant k \leqslant n\big\}\big)$$

be the σ*-algebra generated by the discrete Brownian motion* $\hat{B}_n := \sqrt{\Delta t}\sum_{i\leqslant n} \xi_i$*. By definition, the continuous-time process* (X_s, Y_s, Z_s) *is adapted to the filtration generated by* $(B_r)_{0\leqslant r\leqslant s}$*. For the discretised process, this implies*

$$\hat{Y}_n = \mathbb{E}\big[\hat{Y}_n|\mathcal{F}_n\big] = \mathbb{E}\big[\hat{Y}_{n+1} + \Delta t\, h(\hat{X}_n, \hat{Y}_n, \hat{Z}_n)|\mathcal{F}_n\big]\,, \tag{3.32}$$

using that $\hat{Z}_n$ *is independent of* ξ_{n+1}*. In order to compute* $\hat{Y}_n$ *from* $\hat{Y}_{n+1}$*, it is convenient to replace* $(\hat{Y}_n, \hat{Z}_n)$ *on the right hand side by* $(\hat{Y}_{n+1}, \hat{Z}_{n+1})$*, so that we end up with the fully explicit time stepping scheme*

$$\hat{Y}_n = \mathbb{E}\big[\hat{Y}_{n+1} + \Delta t\, h(\hat{X}_n, \hat{Y}_{n+1}, \hat{Z}_{n+1})|\mathcal{F}_n\big]\,. \tag{3.33}$$

We replace $\hat{Z}_{n+1}$ *in the last equation by*

$$\hat{Z}_{n+1} = \sigma(\hat{X}_{n+1})^T \nabla V_K(\hat{X}_{n+1}, t_{n+1})\,, \tag{3.34}$$

where $V_K(x,t) = \sum_{k=1}^{K} \alpha_k(t)\phi_k(x)$.

3.2.3 Conditional expectation

The next section deals with how to calculate conditional expectations with respect to $\mathcal{F}_n$*. In light of this, it is important to remember that the conditional expectation can be thought of as a best approximation in* L^2*:*

$$\mathbb{E}\big[S|\mathcal{F}_n\big] = \underset{Y\in L^2,\, \mathcal{F}_n\text{-measurable}}{argmin} \mathbb{E}[|Y - S|^2]\,.$$

(Hence the name least-squares Monte Carlo*.) Here measurability* $\mathcal{F}_n$ *means that* $(\hat{Y}_n, \hat{Z}_n)$ *can be expressed as functions of* $\hat{X}_n$*. In view of the parametric*

V_K and equation (3.33), this suggests the approximation scheme

$$\hat{Y}_n \approx \underset{Y=Y(\hat{X}_n)}{argmin} \frac{1}{M}\sum_{m=1}^{M}\left|Y - \hat{Y}_{n+1}^{(m)} - \Delta t\, h\big(\hat{X}_n^{(m)}, \hat{Y}_{n+1}^{(m)}, \hat{Z}_{n+1}^{(m)}\big)\right|^2 , \tag{3.35}$$

where the data at time t_{n+1} is given in form of M independent realisations of the forward process, $\hat{X}_n^{(m)}$, $m = 1, \ldots, M$, the resulting values for $\hat{Y}_{n+1}$,

$$\hat{Y}_{n+1}^{(m)} = \sum_{k=1}^{K} \alpha_k(t_{n+1})\phi_k\big(\hat{X}_{n+1}^{(m)}\big) , \tag{3.36}$$

and

$$\hat{Z}_{n+1}^{(m)} = \sigma\big(\hat{X}_{n+1}^{(m)}\big)^T \sum_{k=1}^{K} \alpha_k(t_{n+1})\nabla\phi_k\big(\hat{X}_{n+1}^{(m)}\big) . \tag{3.37}$$

At time $T := N\Delta t$, the data are determined by the terminal cost:

$$\hat{Y}_N^{(m)} = g\big(X_N^{(m)}\big) , \quad \hat{Z}_N^{(m)} = \sigma\big(\hat{X}_N^{(m)}\big)^T \nabla g\big(X_N^{(m)}\big) \tag{3.38}$$

The unknowns that have to be computed in every iteration step are the coefficients α_k, which makes them functions of time, i.e. $\alpha_k = \alpha_k(t_{n+1})$. We call $\hat{\alpha} = (\alpha_1, \ldots, \alpha_K)$ the vector of the unknowns, so that the least-squares problem that has to be solved in the n-th step of the backward iteration is of the form

$$\hat{\alpha}(t_n) = \underset{\alpha\in\mathbb{R}^K}{argmin} \left\|A_n\alpha - c_n\right\|^2 , \tag{3.39}$$

with coefficients

$$A_n = \Big(\phi_k\big(\hat{X}_n^{(m)}\big)\Big)_{m=1,\ldots,M;k=1,\ldots,K} \tag{3.40}$$

and data

$$c_n = \Big(\hat{Y}_{n+1}^{(m)} + \Delta t\, h\big(\hat{X}_n^{(m)}, \hat{Y}_{n+1}^{(m)}, \hat{Z}_{n+1}^{(m)}\big)\Big)_{m=1,\ldots,M} . \tag{3.41}$$

Assuming that the coefficient matrix $A_n \in \mathbb{R}^{M\times K}$, $K \leqslant M$ defined by (3.40) has maximum rank K, then the solution to (3.39) is given by

$$\hat{\alpha}(t_n) = \big(A_n^T A_n\big)^{-1} A_n^T b_n . \tag{3.42}$$

Algorithm 1 An algorithm with caption

Define K, M, N and $\Delta t = T/N$.
Set initial condition $x \in \mathbb{R}^d$.
Choose radial basis functions $\{\phi_k \in C^1(\mathbb{R}^d, \mathbb{R})\colon k = 1, \ldots, K\}$.
Generate M independent realisations $\hat{X}^{(1)}, \ldots, \hat{X}^{(M)}$ of length N from

$$\hat{X}_{n+1} = \hat{X}_n + \Delta t\, b(\hat{X}_n, t_n) + \sqrt{\Delta t}\, \sigma(\hat{X}_n)\xi_{n+1}\,, \ \hat{X}_0 = x\,.$$

Initialise BSDE by

$$\hat{Y}_N^{(m)} = g\big(\hat{X}_N^{(m)}\big)\,, \quad \hat{Z}_N^{(m)} = \sigma\big(\hat{X}_N^{(m)}\big)^T \nabla g\big(\hat{X}_N^{(m)}\big)\,.$$

for $n = N-1\colon 1$ **do**
  Assemble linear system $A_n \hat{\alpha}(t_n) = c_n$ according to (3.39)–(3.41).
  Evaluate $\hat{Y}_n^{(m)}$ and $\hat{Z}_n^{(m)}$ according to

$$\hat{Y}_n^{(m)} = \sum_{k=1}^{K} \alpha_k(t_n)\phi_k\big(\hat{X}_n^{(m)}\big)\,, \quad \hat{Z}_n^{(m)} = \sigma\big(\hat{X}_n^{(m)}\big)^T \sum_{k=1}^{K} \alpha_k(t_n)\nabla\phi_k\big(\hat{X}_n^{(k)}\big)\,.$$

  If necessary, adapt basis functions ϕ_k.
end for

3.2.4 Examples

Example 1: *We consider this FBSDE :*

$$\begin{cases} dX_t = AX_tdt + 0.2VdW_t, & X_0 = x \\ dY_t = (AY_t + V)dt + Z_tdW_t, & Y_T = X_T. \end{cases} \tag{3.43}$$

where $-A$ is the Laplace operator with zero boundary conditions and W is the Wiener process and x is the initial condition and V is an infinite vector.

Following the preceding steps in section 3.2 discretize 3.43 in space and time, project it into a finite space with dimension equal to four, and then you get this FBSDE in finite dimensions **??** *. From there, we may use the Monte Carlo approach (section 3.2.1).*
We obtain:

$$\begin{cases} d\tilde{X}_t = \tilde{A}_n\tilde{X}_tdt + 0.2\tilde{V}_nd\tilde{W}_t, & \tilde{X}_0 = \tilde{x}_n \\ d\tilde{Y}_t = (\tilde{A}_n\tilde{Y}_t + \tilde{V}_n)dt + \tilde{Z}_td\tilde{W}_t, & \tilde{Y}_T = \tilde{X}_T, \end{cases} \tag{3.44}$$

where $\tilde{x}_n = (2,3,4,5)^t$ and $\tilde{V}_n = (1,1,1,1)^t$ and $\tilde{A}'_n$ is the matrix 3.30.

We obtain the following result by supposing that $K = 3, M = 100, dt = 0.01$:

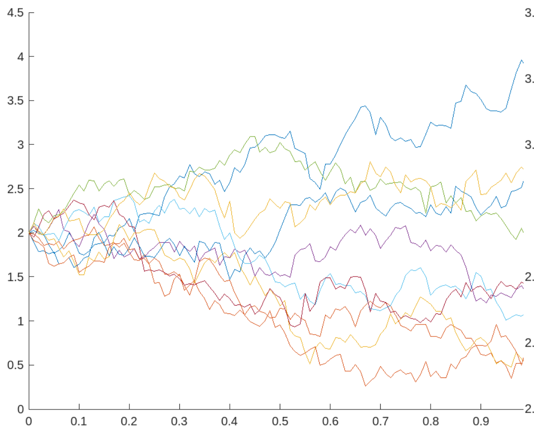

Figure 3.1 Figure 3.2
Simulation of the process X. We plot the process's first component, X, for 10 realizations in figure 3.1. Additionally, we display the realizations of the second component of X for 10 realizations in figure 3.2.

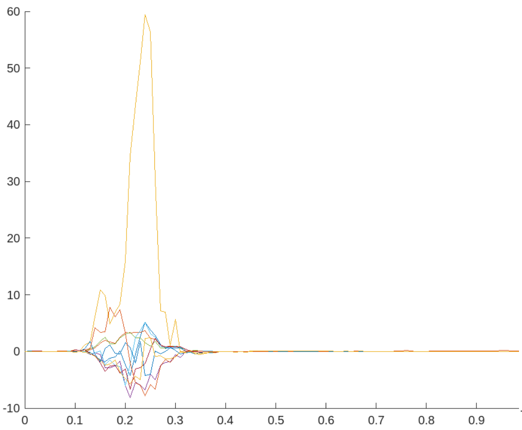

Figure 3.3

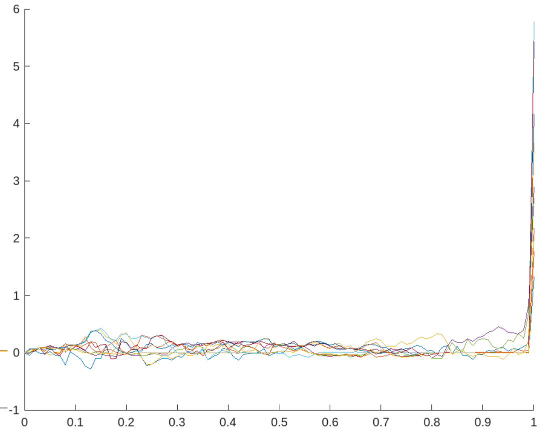

Figure 3.4

Simulation of the process Y. Figure 3.3 shows the initial component of the process, Y, for 10 realizations. In figure 3.4, we also show the realizations of the second component of Y for a total of 10 realizations.

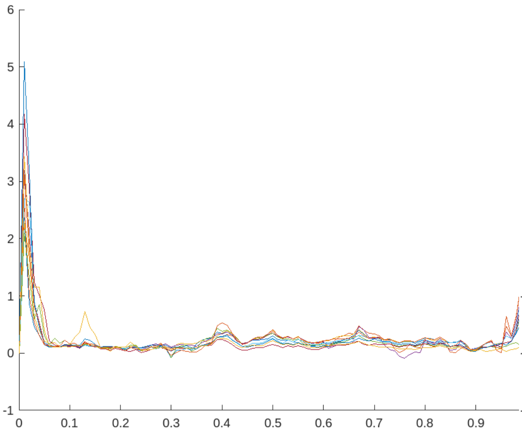

Figure 3.5

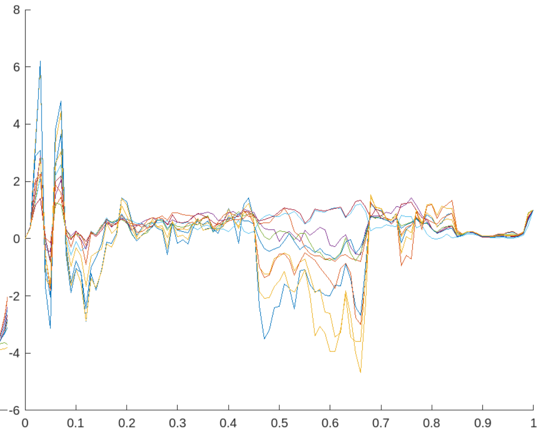

Figure 3.6

Simulation of the process Z, Figure 3.5 displays the initial phase of the process, Z, with 10 different realization. Additionally, in Figure 3.6, we present the realizations of the second aspect of Z using a set of 10 realization.

We give an other example

Example 2: *We consider this FBSDE in infinite dimesion:*

$$\begin{cases} dX_t = AX_t dt + 0.2V dW_t, \quad X_0 = x \\ dY_t = (AY_t + AX_t + V)dt + Z_t dW_t, \quad Y_T = 10^{-2} X_T. \end{cases} \tag{3.45}$$

We follow the same previous procedure we obtain:

$$\begin{cases} d\tilde{X}_t = \tilde{A}_n \tilde{X}_t dt + 0.2\tilde{V}_n d\tilde{W}_t, & \tilde{X}_0 = \tilde{x}_n \\ d\tilde{Y}_t = (\tilde{A}_n \tilde{X}_t + \tilde{A}_n \tilde{Y}_t + \tilde{V}_n)dt + \tilde{Z}_t d\tilde{W}_t, & \tilde{Y}_T = \tilde{X}_T, \end{cases} \tag{3.46}$$

where $\tilde{x}_n = (2, 3, 4, 5)^t$ *and* $\tilde{V}_n = (1, 1, 1, 1)^t$ *and* $\tilde{A}'_n$ *is the matrix 3.30.*

We obtain the following result by supposing that $K = 3, M = 100, dt = 0.01$*:*

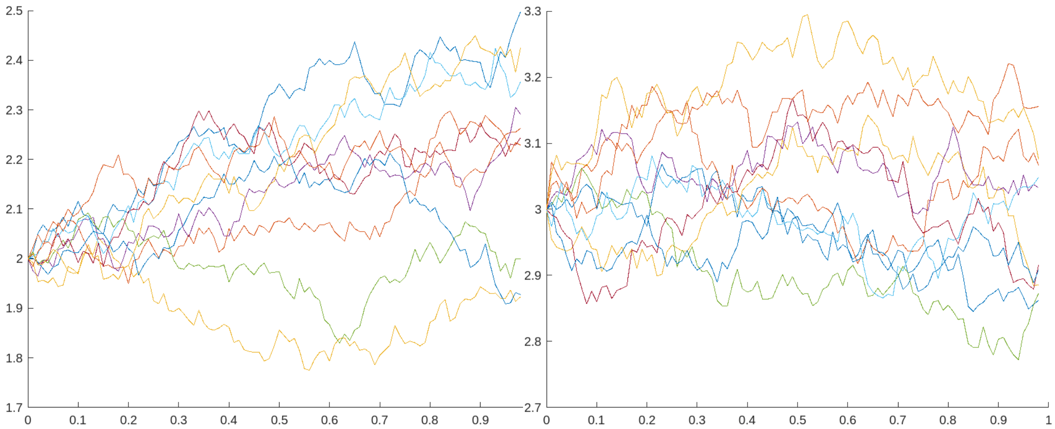

Figure 3.7 Figure 3.8

Simulating the process X. The initial part of the process, X, for ten realizations was described in Figure 3.7. Additionally, for a total of 10 realizations, we display the realizations of the second component of X in figure 3.8.

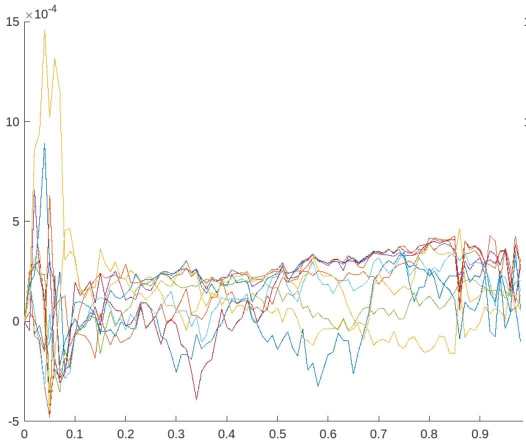

Figure 3.9

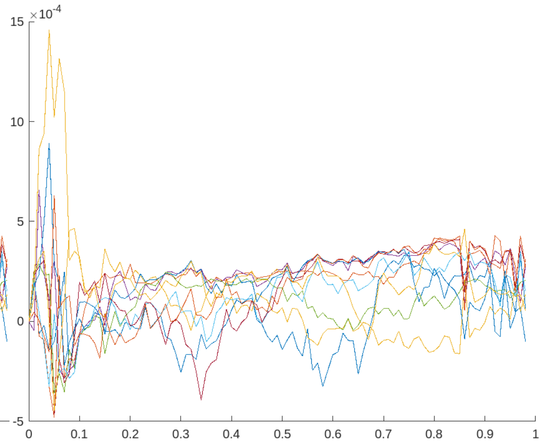

Figure 3.10

Process Y is being simulated. Figure 3.9 detailed the Y initial component of the process for ten realizations. Additionally, we show the realizations of the second component of Y in figure 3.10 for a total of 10 realizations.

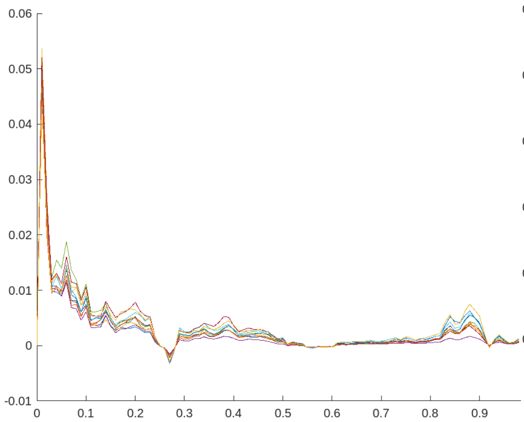

Figure 3.11

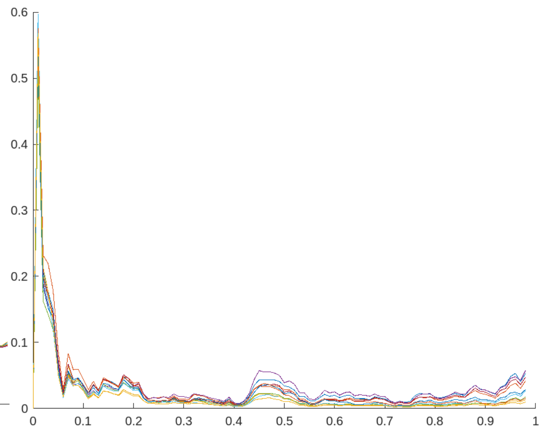

Figure 3.12

Simulation of the process Z. Figure 3.11 described the Z initial component of the process for ten realizations. Furthermore, we provide the realizations of the second component of Z for a total of 10 realizations in figure 3.11.

Conclusion

In my master's thesis, I embarked on an in-depth exploration of stochastic nonlinear equations in infinite dimensions. After that, I concentrated on the fascinating world of infinite-dimensional backward stochastic differential equations (BSDEs). My research primarily revolved around establishing the existence and uniqueness of solutions under various assumptions, while also critically examining the notable advancements achieved in this field. Leveraging the powerful least-squares Monte Carlo method, I successfully solved a range of BSDE examples, generating insightful plots that vividly illustrated the behavior of the solutions.

After reviewing the literature on infinite (BSDEs), I have come to a significant conclusion. To comprehensively explore and elucidate complex phenomena, it is imperative to extend existing findings by incorporating alternative operator assumptions and investigating the non-linear situation. This strategic approach stems from the synthesis of extensive researches on BSDEs in infinite dimensions, which has demonstrated their remarkable potential in explaining intricate phenomena. By embracing these advancements and venturing into uncharted territories, we can achieve a more comprehensive and nuanced understanding of the underlying dynamics, thereby paving the way for groundbreaking insights and transformative breakthroughs.

My thesis provides a thorough and in-depth analysis of BSDEs in infinite dimensions, making substantial contributions to the dynamic field of study that is BSDEs. Theoretical analysis, a thorough literature assessment, and numerical simulation are all smoothly integrated to achieve this.

Bibliography

[Al-04] ABDULRAHMAN Al-Hussein. "Backward stochastic differential equations in infinite dimensions and applications". In: *Arab J. Math. Sc* 10.2 (2004), pp. 1–42.

[BC08] Philippe Briand and Fulvia Confortola. "BSDEs with stochastic Lipschitz condition and quadratic PDEs in Hilbert spaces". In: *Stochastic Processes and their Applications* 118.5 (2008), pp. 818–838.

[BH06] Philippe Briand and Ying Hu. "BSDE with quadratic growth and unbounded terminal value". In: *Probability Theory and Related Fields* 136.4 (2006), pp. 604–618.

[BL97] G Barles and E Lesigne. "Sde, bsde and pde". In: *Pitman Research Notes in Mathematics Series* (1997), pp. 47–82.

[Bré21] Charles-Edouard Bréhier. "Contributions to stochastic numerics: simulation of infinite dimensional, multiscale and metastable processes". PhD thesis. Université Claude Bernard Lyon 1, 2021.

[Bur04] John Charles Burkill. *The lebesgue integral.* 40. Cambridge University Press, 2004.

[Cia02] Philippe G Ciarlet. *The finite element method for elliptic problems.* SIAM, 2002.

[Con06] Fulvia Confortola. "Dissipative backward stochastic differential equations in infinite dimensions". In: *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 9.01 (2006), pp. 155–168.

[Con07] Fulvia Confortola. "Dissipative backward stochastic differential equations with locally Lipschitz nonlinearity". In: *Stochastic processes and their applications* 117.5 (2007), pp. 613–628.

[CRW06] Q Chen, Q Ran, and J Wang. "Lp solutions of BSDEs with stochastic Lipschitz condition". In: *Preprint* (2006).

[DE92] Darrell Duffie and Larry G Epstein. "Stochastic differential utility". In: *Econometrica: Journal of the Econometric Society* (1992), pp. 353–394.

[DZ14] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*. Cambridge university press, 2014.

[DZ96] Giuseppe Da Prato and Jerzy Zabczyk. *Ergodicity for infinite dimensional systems*. Vol. 229. Cambridge University Press, 1996.

[EH97] Nicole El Karoui and SJ Huang. "A general result of existence and uniqueness of backward stochastic differential equations". In: *Pitman research notes in mathematics series* (1997), pp. 27–38.

[EPQ97] Nicole El Karoui, Shige Peng, and Marie Claire Quenez. "Backward stochastic differential equations in finance". In: *Mathematical finance* 7.1 (1997), pp. 1–71.

[FH07] Marco Fuhrman and Ying Hu. "Backward stochastic differential equations in infinite dimensions with continuous driver and applications". In: *Applied Mathematics and Optimization* 56.2 (2007), pp. 265–302.

[FJ82] William G Faris and Giovanni Jona-Lasinio. "Large fluctuations for a nonlinear heat equation with noise". In: *Journal of Physics A: Mathematical and General* 15.10 (1982), p. 3025.

[FT02] Marco Fuhrman and Gianmario Tessitore. "The Bismut-Elworthy formula for backward SDE's and applications to nonlinear Kolmogorov equations and control in infinite dimensional spaces". In: *Stochastics: An International Journal of Probability and Stochastic Processes* 74.1-2 (2002), pp. 429–464.

[FT04] Marco Fuhrman and Gianmario Tessitore. "Infinite horizon backward stochastic differential equations and elliptic equations in Hilbert spaces". In: *The Annals of Probability* 32.1B (2004), pp. 607–660.

[Fuh02] Marco Fuhrman. "Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control". In: *The Annals of Probability* 30.3 (2002), pp. 1397–1465.

[Fuh03] Marco Fuhrman. "A class of stochastic optimal control problems in Hilbert spaces: BSDEs and optimal control laws, state constraints, conditioned processes". In: *Stochastic processes and their applications* 108.2 (2003), pp. 263–298.

[GM10] Leszek Gawarecki and Vidyadhar Mandrekar. *Stochastic differential equations in infinite dimensions: with applications to stochastic partial differential equations*. Springer Science & Business Media, 2010.

[God75] Alexandr Nikolaevich Godunov. "Peano's theorem in Banach spaces". In: *Functional Analysis and Its Applications* 9.1 (1975), pp. 53–55.

[GT05] Giuseppina Guatteri and Gianmario Tessitore. "On the backward stochastic Riccati equation in infinite dimensions". In: *SIAM journal on control and optimization* 44.1 (2005), pp. 159–194.

[HP91] Ying Hu and Shige Peng. "Adapted solution of a backward semilinear stochastic evolution equation". In: *Stochastic Analysis and Applications* 9.4 (1991), pp. 445–459.

[Kaz06] Norihiko Kazamaki. *Continuous exponential martingales and BMO*. Springer, 2006.

[KNH19] Omar Kebiri, Lara Neureither, and Carsten Hartmann. "Adaptive importance sampling with forward-backward stochastic differential equations". In: *Stochastic Dynamics Out of Equilibrium: Institut Henri Poincaré, Paris, France, 2017*. Springer. 2019, pp. 265–281.

[Kob00] Magdalena Kobylanski. "Backward stochastic differential equations and partial differential equations with quadratic growth". In: *the Annals of Probability* 28.2 (2000), pp. 558–602.

[Kru14] Raphael Kruse. *Strong and weak approximation of semilinear stochastic evolution equations*. Springer, 2014.

[Led03] M Ledoux. "On improved Sobolev embedding theorems". In: *Mathematical Research Letters* 10.5 (2003), pp. 659–669.

[Liu+21] Jin-Peng Liu et al. "Efficient quantum algorithm for dissipative nonlinear differential equations". In: *Proceedings of the National Academy of Sciences* 118.35 (2021), e2026805118.

[LM98] J-P Lepeltier and J San Martín. "Existence for BSDE with superlinear–quadratic coefficient". In: *Stochastics: An International Journal of Probability and Stochastic Processes* 63.3-4 (1998), pp. 227–240.

[Man86] Ralf Manthey. "Existence and Uniqueness of a Solution of a Reaction-Diffusion Equation with Polynomial Nonlinearity and White Noise Disturbance". In: *Mathematische Nachrichten* 125.1 (1986), pp. 121–133.

[Man88] Ralf Manthey. "On the Cauchy Problem for Reaction-Diffusion Equations with White Noise". In: *Mathematische Nachrichten* 136.1 (1988), pp. 209–228.

[Mar70] RH Martin. "A global existence theorem for autonomous differential equations in a Banach space". In: *Proceedings of the American Mathematical Society* 26.2 (1970), pp. 307–314.

[Mor09] Marie-Amélie Morlais. "Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem". In: *Finance and Stochastics* 13 (2009), pp. 121–150.

[MT06] Paul Malliavin and Anton Thalmaier. *Stochastic calculus of variations in mathematical finance*. Springer, 2006.

[Øks01] Bernt Øksendal. "Optimal Control of Stochastic Partial Differential Equations and Application to Partial Observation Control". In: (2001).

[ØZ01] Bernt Øksendal and Tusheng Zhang. "On Backward Stochastic Partial Differential Equations." In: *Preprint series. Pure mathematics http://urn. nb. no/URN: NBN: no-8076* (2001).

[Par05] Kalyanapuram Rangachari Parthasarathy. *Probability measures on metric spaces*. Vol. 352. American Mathematical Soc., 2005.

[Pen91] Shige Peng. "Probabilistic interpretation for systems of quasilinear parabolic partial differential equations". In: *Stochastics and stochastics reports (Print)* 37.1-2 (1991), pp. 61–74.

[Pen93] Shige Peng. "Backward stochastic differential equations and applications to optimal control". In: *Applied Mathematics and Optimization* 27.2 (1993), pp. 125–144.

[PP05] Etienne Pardoux and Shige Peng. "Backward stochastic differential equations and quasilinear parabolic partial differential equations". In: *Stochastic Partial Differential Equations and Their Applications: Proceedings of IFIP WG 7/1 International Conference University of North Carolina at Charlotte, NC June 6–8, 1991*. Springer. 2005, pp. 200–217.

[PP90] Etienne Pardoux and Shige Peng. "Adapted solution of a backward stochastic differential equation". In: *Systems & control letters* 14.1 (1990), pp. 55–61.

[PR98] Etienne Pardoux and Aurel Rășcanu. "Backward stochastic differential equations with subdifferential operator and related variational inequalities". In: *Stochastic Processes and their applications* 76.2 (1998), pp. 191–215.

[PR99] Étienne Pardoux and Aurel Rășcanu. "Backward stochastic variational inequalities". In: *Stochastics: An International Journal of Probability and Stochastic Processes* 67.3-4 (1999), pp. 159–167.

[Pra92] Da Prato. "Stochastic equations in infinite dimensions". In: *(No Title)* 44 (1992).

[Sol07] Sławomir Solecki. "A Fubini theorem". In: *Topology and its Applications* 154.12 (2007), pp. 2462–2464.