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Solutions of fractional differential equations

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Dedication

I dedicate this modest work :

To my parents.

To my uncle and my aunt.

To my brothers and sisters.

For the whole family.

For all my friends.

To anyone who helped me to cross

a horizon in my life.

For everyone who contributed

to shaping my college career.

Bouchra HAMADENE

Acknowledgement

In the name of " ALLAH", the most gracious, the most merciful.

First of all, I would like to thank "ALLAH" almighty who provided me the power to complete this work.

I would like to thank my supervisor, "Dr. Hafida Abbas" who did not spare me with his advice and directions you have all the meanings of apprection and gratitude.

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I will never forget my teachers, from the primary to the university,to every one help me in my student. I thank them. Of course my thanks and gratitude to my family.

Bouchra HAMADENE

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Notations

- $\mathbb{R}$ The field of real numbers.
- $\mathbb{C}$ Set of complex numbers.
- $\mathbb{N}$ Set of natural numbers.
- $L^p[a,b]$ Space of measurable integrable power functions $p \in [0,+\infty[$.
- $C([a,b])$ Space of continuous functions on $[a,b]$.
- Ω a non-empty open set.
- $\overline{\Omega} = \Omega + \partial\Omega$ it's closing of Ω.
- $\Gamma(\cdot)$ Euler Gamma Functions.
- $\mathcal{B}(\cdot,\cdot)$ Beta Functions.
- $\|\cdot\|$ The norm.
- I_a^α Fractional integral in the sense of Riemann-Liouville of order $\alpha > 0$.
- D_a^α Fractional derivative in the sense of Riemann-Liouville of order $\alpha > 0$.
- $R - L$ Riemann-Liouville.

Introduction

It is to Leibniz that belongs the glory of having cleared a new path which is the theory fractional calculus, this theory which extended the derivation and integration of order integer to non-integer order is currently enjoying great popularity.

In fact, the history of fractional calculus began with a key question from Leibniz when he introduced the symbol $\frac{\mathrm{d}^n y}{x^n}$ to denote the n^{th} derivative of a function f where n is a positive integer. This symbolic representation then pushed the Hospital to wonder about the possibility of having n fractional and he sent a letter to Leibniz in 1695 wondering if $n = 1/2$?, Leibniz replied "that this is a paradox, from which useful consequences will one day be drawn".

Since then, many mathematicians have embarked on this question to overcome this difficulty and several contributions have been developed and many forms fractional differential operators have been introduced, we can cite the derivatives fractional type Riemann-Liouville, Caputo, Grunwald-Letnikow Weyl, Marchaud, Hadamard, Riesz (Hilfer 2000, Kilbas et al. 2006, Podlubny 1999, Samko et al.1993) and other more recent works by Klimek (2005), Kilbas and Saigo (2004), Cresson (2007), Katu- gampola (2011), where fractional operators of variable order introduced by Samko and Ross (1993). For more historical details, see [17].

However, the first real application of fractional calculus seems to be proposed by N. H. Abel in 1823 who showed that the generalized Tautochrone problem was written using a differential equation of non-integer order, the

solution of which he expressed using an integral equation.
During the last three decades, more interest has been lent to fractional calculus and many researchers have invested themselves in highlighting the results already established and the fields of application have diversified. As examples, we will cite some of these fields of application.

- Fractional derivatives have been widely used in the mathematical model of viscoelastic matter. The advantage of introducing fractional derivatives in theory of viscoelasticity is that it offers possibilities to obtain constitutive equations for the elastic complex modulus of viscoelastic materials with only some experimentally determined parameters. Fractional derivatives have also been used in the study of complex moduli and resistance see [2].

- Electromagnetic problems can be described using the equations Fractional integro-differentials.

- In biology, it turns out that the membranes of biological organism cells have the fractional order electrical conductance [14].

- In physicochemistry, the current is proportional to the fractional derivatives of the voltage when the fractal interface is placed between a metal and an ionic medium [10].

- In economics, some systems of nance can display a dynamic of order fractional, examples on this dynamic can be seen in the reference [18].

Other applications of fractional calculus have been described in several fields such as: Image processing [10], signal processing [21], automatic control matics and robotics, and analysis of dynamic systems with order models fractional.

In this work, we discuss existence of solutions for fractional differential equations, these results are determined, by applying Guo–Krasnoselskii fixed point theorem of cone expansion and compression of norm type. Our assumed problem will general than the problems considered [5] and [1].

This work is structured as follows.

The first chapter contains some basic concepts and theorems in addition to the notions of the functions play an important role in the fractional calculus.

The second chaper, we present the definitions of fractional derivative and integrals in the sense of Riemann-Liouville and its properties.

In the third chapter,we discuss existence of the solutions of a class of nonlinear boundary value problem of fractional differential equations with integral boundary conditions. To state our results, we study Green function and by applying Guo–Krasnoselskii fixed point theorem, end this chapter with an illustrative example.

In the final chapter, we discuss existence of positive solution for a Riemann fractional boundary value problem with integral boundary conditions and parameter dependence, these results are determined, by applying Guo–Krasnoselskii fixed point theorem of cone expansion and compression of norm type. Some examples are shown to point out the applicability of the obtained results.

Chapter 1

Basic tools

In this chapter, we provide some definitions and properties that we will use the remainder of this work (see [11], [12]).

1.1 Definitions

Definition 1.1.1. *(Banach space).*
A normed vector space $(E, \|\cdot\|)$ is said to be complete if any sequence de Cauchy of elements of E is convergent in $(E, \|\cdot\|)$. An e.v.n. complete is said Banach space.

The space $C\left(\mathbb{R}, |\cdot|\right)$ is a simple example of a Banach space.

Definition 1.1.2. *(L^P Space).*
Let Ω a finite or infinite interval of $\mathbb{R}$. We denote by $L^p(\Omega)$ with $1 \leq p < +\infty$ the Lebesgue space such that $f : \Omega \Rightarrow \mathbb{K}\left(\mathbb{K} = \mathbb{R}\, or\, \mathbb{C}\right)$ is mesurable and $\int_\Omega |f(x)|^p \mathrm{d}x < +\infty$, endowed with the norm

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

For $p = \infty$, then $\exists\, C \geq 0, |f(x)| \leq C$ for almost every $x \in \Omega$ and we notice

$$\|f\|_{L^\infty} = \inf\left\{|f(x)| \leq C \ \textit{for almost every}\ x \in \Omega\right\}$$

Theorem 1.1.1.
$(L^p(\Omega), \|\cdot\|)$ is a Banach space.

Definition 1.1.3. *[9]*
Let E, F be two normed spaces and the map (Operator) $T : E \to F$. We say that f is completely continuous if:

- *T is continuous,*
- *T is relatively compact .*

Definition 1.1.4. *(Dirichlet's Formula).*
Let $h(x, y)$be a continuous function and α, β two positive reals. The following expression is known as the Dirichlet formula:

$$\int_0^t (t-x)^{\alpha-1}\mathrm{d}x \int_0^x (x-y)^{\beta-1}h(x,y)\mathrm{d}y = \int_0^t \mathrm{d}y \int_y^t (t-x)^{\alpha-1}(x-y)^{\beta-1}h(x,y)\mathrm{d}x \tag{1.1}$$

Certain special cases of Dirichlet's formula are of particular interest. For example, if

$$h(x, y) = g(x)f(y) \text{ and } g(x) \equiv 1,$$

then (1.1) takes the form

$$\int_0^t (t-x)^{\alpha-1}\mathrm{d}x \int_0^x (x-y)^{\beta-1}h(x,y)\mathrm{d}y = B(\alpha,\beta) \int_0^t (t-y)^{\alpha+\beta-1}f(y)\mathrm{d}y \tag{1.2}$$

where $\mathcal{B}$ is the Beta function.

We will use Dirichlet's formula to prove the law of exponents for fractional integrals.

1.2 Green Function

Green's functions are involved in solving certain differential equations. (in particular, the case of fractional differential equations).

Definition 1.2.1. *(Green's function in one dimension).*
Let (a, b) be a finite interval, $q : (a, b) \to \mathbb{R}$ a bounded and continuous function. We consider the problem of the differential equation and the

homogeneous boundary conditions:

$$\begin{cases} \left(-\dfrac{d^2}{dx^2}+q(x)\right)f(x)=h(x), & 0<x<1, \\ \alpha_1 f(a)+\beta_1 f'(a)=0, & \\ \alpha_1 f(b)+\beta_2 f'(b)=0. & \end{cases} \tag{1.3}$$

where h *is a given function and* α_j, β_j $(j=1,2)$ *are given constants.*
The Green's function method consists in solving, for each $y \in]a,b[$ *fixed,*

$$\left[-\frac{d^2}{dx^2}+q(x)\right]G(x,y)=\delta(x-y) \tag{1.4}$$

Equation (1.4) must be in the sense of distributions.
Green's function satisfies the same boundary conditions at $x=a$ *and* $x=b$*, we obtain the solution* f *of (1.3) by:*

$$f(x)=\int_a^b G(x,y)h(x)\mathrm{d}x \tag{1.5}$$

Theorem 1.2.1.
Green's function has the following properties:

1. $\left[-\dfrac{d^2}{dx^2}+q(x)\right]G(x,y)=0$ *on* (a,y) *and on* (b,y).
2. G *satisfies the boundary conditions.*
3. G *is continuous at* $x=y$.
4. $G(x,y)=G(y,x)$.

1.3 Some important theorems

Theorem 1.3.1. *[25](Ascoli-Arzelà).*
Let $T \subset C([0,b],\mathbb{R}^n)T$ *is relatively compact if:*

1. T *is bounded, i.e. there exists* $M>0$

$$\|y(t)\| \leq M, \forall t \in [0,b] \text{ and } y \in T$$

2. T *is equicontinuous i.e. for all* $\varepsilon>0$ *there exists* $\delta(\varepsilon)>0$

$$\forall t_1,t_2 \in [0,b], |t_1-t_2|<\delta \Rightarrow \|y(t_1)-y(t_2)\|<\varepsilon \, \forall y \in T$$

1.3.1 Guo–Krasnoselskii fixed point theorem

One of the most important tools in fixed point theory is the cone expansion and compression theorem proved by Krasnoseliskii in 1964 (see, [24] or [13]). It has been proven to be efficient in showing existence of positive solutions to various boundary value problems.

Definition 1.3.1. *(cone).*

Let E be a real Banach space. A nonempty closed convex set $\mathcal{P} \in E$ is called a cone if it satisfies the following two conditions:

a) $x \in \mathcal{P}, \lambda \geqslant 0$ implies $\lambda x \in \mathcal{P}$;

b) $x \in \mathcal{P}, -x \in \mathcal{P}$ implies $x = 0$.

Theorem 1.3.1.1. *[13](Fixed point theorem of cone expansion and compression of norm type).*

Let E be a Banach space, and let $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are open and bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator such that

1. *Compressive form:*
   $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1, and \|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2$;
2. *Expansive form:*
   $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_1, and \|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2$.

Then operator T has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

An illustration of this result in dimension 2 is depicted in Figs (1.1) and (1.2), with $P \cap \partial\Omega_1 = K_a$, $P \cap \partial\Omega_2 = K_b$, and $P \cap (\overline{\Omega_2} \setminus \Omega_1) = K(a, b)$.

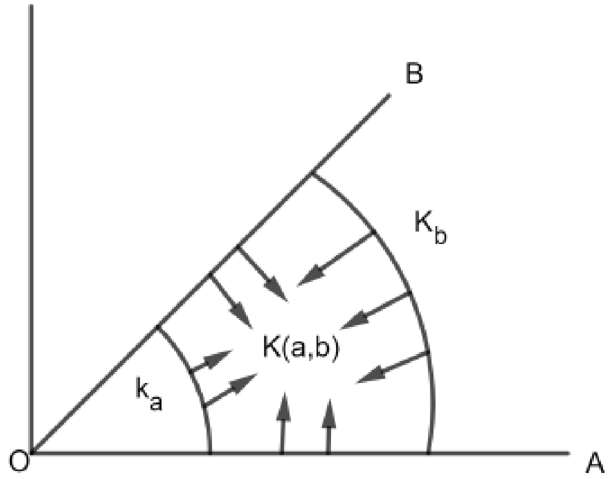

Figure 1.1: Compressive form.

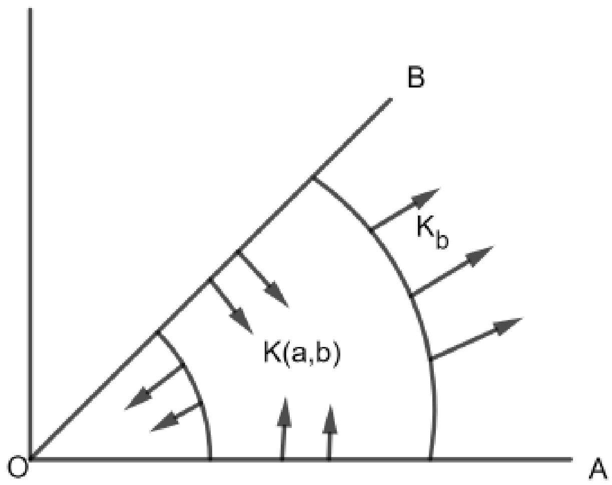

Figure 1.2: Expansive form.

Proof. (see [4]). □

1.4 Special function

In this section, some basic theory of the special functions that are used in the other chapters is given. We give here some information on the gamma and beta functions, these functions play the most important role in the theory of fractional differential equations.

1.4.1 The Gamma Function

Undoubtedly ,one the basic function of the fractional calculus is Euler's gamma function $\Gamma(\alpha)$, which generalizes the factorial $\alpha!$ and allows α to take also non-integer and even complex values . We will recall in this section some results on the gamma function which are important for other parts of this work.

Definition 1.4.1. *[11]*

The gamma function $\Gamma(\alpha)$ is defined by the following integral:

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} \mathrm{d}t, \quad \alpha \in \mathbb{R}. \tag{1.6}$$

Γ is continuous in $(0, +\infty)$.

Remark 1.4.1.1.

Let's put $t = x^2$ then

$$\mathrm{d}t = 2x\mathrm{d}x$$

so

$$\begin{aligned} \Gamma(\alpha) &= \int_0^{+\infty} e^{x^2} (x^2)^{\alpha-1} 2x\mathrm{d}x \\ &= 2\int_0^{+\infty} e^{x^2} x^{2\alpha-1} \mathrm{d}x \end{aligned} \tag{1.7}$$

(1.7)is another definition of Gamma function.

Lemma 1.4.1.1. *For all $\alpha > 0$, the Gamma function satisfies the properties following:*

1. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

2. $\Gamma(n) = (n-1)! \quad n \geq 1.$

3. $\Gamma(\alpha+n) = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)\Gamma(\alpha).$

Some particular values of $\Gamma(\alpha)$

- $\Gamma(1) = \Gamma(2) = \int_0^{+\infty} e^{-t}t^{1-1}dt = 1.$
- $\Gamma(\frac{1}{2}) = 2\int_0^{+\infty} e^{-t^2}dt = \sqrt{\pi}.$ (Gaussian integral)
- $\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}.$

1.4.2 The Beta Function

The other important function in fractional calculus is the function Euler beta

Definition 1.4.2. *[11]*
The beta function is usually defined by:

$$\mathcal{B}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t, \quad \alpha,\beta \in \mathbb{R}_+^* \tag{1.8}$$

There are also know as the Eulerian integral of first kind.

The beta function is related to the gamma function by:

$$\mathcal{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{1.9}$$

Chapter 2

Fractional Derivative and Integrals

In this chapter we present some of the defnitions, results, theories and main properties concerning the integral and the fractional derivative in the Riemann-Liouville sense. [see [11], [12]].

2.1 Riemann-Liouville fractionnal integral

Let $f : [a, b] \to \mathbb{R}$ be a continuous function a primitive of f givin by:

$$(I_a^1 f)(t) = \int_a^x f(\tau)\mathrm{d}\tau$$

For a double integrale we will have:

$$(I_a^2 f)(t) = \int_a^t \left(\int_a^s f(\tau)\mathrm{d}\tau \right) \mathrm{d}s.$$

According to Fubini's theorem, we find

$$\begin{aligned}(I_a^2 f)(t) &= \int_a^t \left(\int_\tau^t \mathrm{d}s \right) f(\tau)\mathrm{d}\tau. \quad (a \leqslant \tau \leqslant s \leqslant t \leqslant b) \\ &= \int_a^t (t-\tau) f(\tau)\mathrm{d}\tau.\end{aligned}$$

For the triple integral we will have:

$$(I_a^3 f)(t) = \frac{1}{2}\int_a^t (s-\tau)^2 f(\tau)\mathrm{d}\tau.$$

Repeating the same operation α times we get:

$$(I_a^\alpha f)(t) = \frac{1}{(\alpha-1)!}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau \tag{2.1}$$

for any integer α.
This formula is called the Cauchy formula and as we have $(\alpha-1)! = \Gamma(\alpha)$, Riemann realized that the last expression could make sense even when α taking non-integers values, so it was natural to define the fractional integration operator as following:

Definition 2.1.1.

Let $f \in L^1([a,b]), a \in \mathbb{R}, b$ can be finite or infinite the Riemann-Liouville fractional integral to the left of order $\alpha > 0$ of a function f is defined by:

$$(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \quad -\infty \leqslant a < t < +\infty \tag{2.2}$$

Remark 2.1.1.

$I_a^0 f(t) = f(t)$ *(I_a^0 is the identity operator).*

Remark 2.1.2.

By the simple change of variable $s = t - \tau$, we note that I_a^α can be written in the following form:

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^{t-a} s^{\alpha-1} f(t-s)\mathrm{d}s \tag{2.3}$$

(another definition of the integral of R-L).

2.1.1 Properties of the Fractional Integral in the sense of R-L

Lemma 2.1.1.1.

The integral operator I_a^α is linear.

Proof.

If f and g are two functions such that $I_a^{\alpha} f$ and $I_a^{\alpha} g$ exist, then for c_1 and c_2 two arbitrary real numbers we will have:

$$\begin{aligned} I_a^{\alpha}(c_1 f + c_2 g)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1}(c_1 f + c_2 g)(\tau)\mathrm{d}\tau \\ &= \frac{c_1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau + \frac{c_2}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} g(\tau)\mathrm{d}\tau. \end{aligned}$$

Hence

$$I_a^{\alpha}(c_1 f + c_2 g)(t) = c_1 I_a^{\alpha} f(t) + c_2 I_a^{\alpha} g(t)$$

□

Lemma 2.1.1.2.

$f \in C[a,b]$, the Riemann-Liouville fractional integral has the following semigroup property:

$$I_a^{\alpha}(I_a^{\beta} f)(t) = I_a^{\alpha+\beta} f(t) \tag{2.4}$$

to $\alpha > 0$ and $\beta > 0$.

Proof.

Let $f \in C[a,b], \alpha > 0, \beta > 0$, so:

$$\begin{aligned} I_a^{\alpha}(I_a^{\beta} f)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1}(I_a^{\beta} f)(\tau)\mathrm{d}\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \left[\frac{1}{\Gamma(\beta)} \int_a^{\tau} (\tau-s)^{\beta-1} f(s)\mathrm{d}s\right] \mathrm{d}\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-\tau)^{\alpha-1} \left[\int_a^{\tau} (\tau-s)^{\beta-1} f(s)\mathrm{d}s\right] \mathrm{d}\tau. \end{aligned}$$

According to the Dirichlet formula(1.1) we find:

$$\begin{aligned} I_a^{\alpha}(I_a^{\beta} f)(t) &= \frac{B(\beta,\alpha)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-s)^{\alpha+\beta-1} f(s)\mathrm{d}s \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-s)^{\alpha+\beta-1} f(s)\mathrm{d}s \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} f(s)\mathrm{d}s. \end{aligned}$$

Hence

$$I_a^\alpha(I_a^\beta f)(t) = I_a^{\alpha+\beta} f(t)$$

□

Lemma 2.1.1.3.
let $f \in C^0([a,b))$,So we have:

1. $\frac{\mathrm{d}}{\mathrm{dt}}\left(I_a^\alpha f\right)(t) = (I_a^{\alpha-1} f)(t) \qquad \alpha > 1.$
2. $\lim\limits_{\alpha\longrightarrow 0^+} (I_a^\alpha f)(t) = f(t) \qquad \alpha > 0.$

Proof.

1. From the reccurence formula of Euler's Gamma function $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ and as $\alpha > 1$, we can write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}(I_a^\alpha f)(t) &= \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau\right) \\
&= \frac{1}{(\alpha-1)\Gamma(\alpha-1)}\int_a^t \frac{\mathrm{d}}{\mathrm{dt}}(t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau \\
&= \frac{1}{(\alpha-1)\Gamma(\alpha-1)}\int_a^t (\alpha-1)(t-\tau)^{(\alpha-1)-1} f(\tau)\mathrm{d}\tau \\
&= \frac{1}{\Gamma(\alpha-1)}\int_a^t (t-\tau)^{(\alpha-1)-1} f(\tau)\mathrm{d}\tau.
\end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(I_a^\alpha f\right)(t) = (I_a^{\alpha-1} f)(t), \quad \alpha > 1$$

2. If $f \in C^1([a,b])$ then an integration by parts gives us

$$\begin{aligned}
(I_a^\alpha f)(t) &= \frac{(t-a)^\alpha f(a)}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\int_a^t (t-\tau)^\alpha f'(\tau)\mathrm{d}\tau \\
\lim_{\alpha\longrightarrow 0^+} (I_a^\alpha f)(t) &= f(a) + \int_a^t f'(\tau)\mathrm{d}\tau \\
&= f(a) + [f(t) - f(a)].
\end{aligned}$$

Hence

$$\lim_{\alpha\longrightarrow 0^+} (I_a^\alpha f)(t) = f(t) \quad \alpha > 0$$

□

Theorem 2.1.1.1. *[23]*
If $f \in L^1[a,b]$, and $\alpha > 0$ so $I_a^\alpha f(t)$exists for almost all $t \in [a,b]$ and we have $I_a^\alpha f \in L^1[a,b]$.

Proof. let $f \in L^1[a,b]$; we have

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau = \int_{-\infty}^{+\infty} g(t-\tau) h(\tau) \mathrm{d}\tau,$$

with $-\infty \leqslant a < t < +\infty$.
Such as

$$g(u) = \begin{cases} \dfrac{u^{\alpha-1}}{\Gamma(\alpha)}, & 0 < u \leqslant b-a \\ 0, & u \in \mathbb{R} - (0, b-a) \end{cases}$$

and

$$h(u) = \begin{cases} f(u), & a \leqslant u \leqslant b \\ 0, & u \in \mathbb{R} - [a,b] \end{cases}$$

So $g.h \in L^1(\mathbb{R})$, hence

$$I_a^\alpha f \in L^1[a,b]$$

□

Theorem 2.1.1.2. *[23]*
Let $\alpha > 0$ and let $(f_n)_{n=1}^\infty$ a sequence of uniformly continuous functions converging in $[a,b]$, then the sequence $(I_a^\alpha f_n)_{n=1}^\infty$ is uniformly convergent and we can invert the Riemann-Liouville fractional integral and the limit as follows:

$$\left(\lim_{n \longrightarrow +\infty} I_a^\alpha f_n \right)(t) = \left(I_a^\alpha \lim_{n \longrightarrow +\infty} f_n \right)(t)$$

Proof. For the first statement we utilize the well known fact, that if f denotes the limit of the sequence (f_n), the function f is continuous. For $\alpha = 0$ the stated result follows directly from the uniform convergence and for $\alpha > 0$ we can deduce

$$\begin{aligned} |I_a^\alpha f_n(x) - I_a^\alpha f(x)| &\leq \frac{1}{\Gamma(\alpha)} \int_a^x |f_n(x) - f(x)| (x-t)^{\alpha-1} \mathrm{d}t \\ &\leq \frac{1}{\Gamma(\alpha+1)} \|f_n - f\|_\infty (b-a)^\alpha. \end{aligned}$$

The last term converges uniformly to zero as $k \to \infty$ for all $x \in [a, b]$.

□

A direct consequence of this theorem points out the connection between fractional integrals and integer-order derivatives of an analytic function.

2.1.2 Examples of Fractional Integrals in the sense of R-L

Example 2.1.2.1.
let $f(t) = (t-a)^\beta$ *and calculat it's integral of order* $(\alpha > 0)$

$$\begin{aligned}
(I_a^\beta f)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau \\
I_a^\alpha (t-a)^\beta &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau-a)^\beta \, \mathrm{d}\tau
\end{aligned}$$

by using the change of variable $\tau = a + (t-a)u,$ *then the beta function we get*

$$\begin{aligned}
d\tau &= (t-a)du \\
\tau = a &\Rightarrow a + (t-a)u \Rightarrow u = 0 \\
\tau = t &\Rightarrow a + (t-a)u \Rightarrow u = 1
\end{aligned}$$

Then

$$\begin{aligned}
I_a^\alpha (t-a)^\beta &= \frac{1}{\Gamma(\alpha)} \int_a^t [t - (a + (t-a)u)]^{\alpha-1} [(a + (t-a)u) - a]^\beta (t-a) \mathrm{d}u \\
&= \frac{1}{\Gamma(\alpha)} \int_a^t (t-a)^{\alpha-1} (1-u)^{\alpha-1} (t-a)^\beta u^\beta (t-a) \mathrm{d}u \\
&= \frac{1}{\Gamma(\alpha)} \int_a^t (t-a)^{\alpha} (1-u)^{\alpha-1} (x-a)^\beta u^\beta \mathrm{d}u \\
&= \frac{1}{\Gamma(\alpha)} \int_a^t (t-a)^{\alpha+\beta} (1-u)^{\alpha-1} u^\beta \mathrm{d}u \\
&= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_a^t (1-u)^{\alpha-1} u^\beta \mathrm{d}u \\
&= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \cdot \mathcal{B}(\beta+1, \alpha)
\end{aligned}$$

$$= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

So

$$I_a^\alpha (t-a)^\beta = \frac{(t-a)^{\alpha+\beta}\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \tag{2.5}$$

For $a = 0$, we have

$$I_0^\alpha (t)^\beta = I^\alpha (t)^\beta = \frac{(t)^{\alpha+\beta}\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \tag{2.6}$$

Example 2.1.2.2.

Fractional integral of the constant function $f(t) = C$

$$\begin{aligned} I_a^\alpha C &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} C \mathrm{d}\tau \\ &= \frac{C}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \mathrm{d}\tau \\ &= \frac{C}{\Gamma(\alpha)} \left(\frac{-(t-\tau)\alpha}{\alpha} \right]_a^t \\ &= \frac{C}{\alpha\Gamma(\alpha)} (t-a)^\alpha \end{aligned}$$

Hence the result:

$$I_a^\alpha C = \frac{C}{\Gamma(\alpha+1)} (t-a)^\alpha \tag{2.7}$$

Example 2.1.2.3.

Fractional integral of the exponential function $f(t) = exp(nt)$ to $n > 0$ and $\alpha > 0$.

Using the formula (2.3)of the integral of R-L with $a = -\infty$, we obtain:

$$\begin{aligned} I_{-\infty}^\alpha exp(nt) &= \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} s^{\alpha-1} exp(n(t-s)) \mathrm{d}s \\ &= \frac{exp(nt)}{\Gamma(\alpha)} \int_0^{+\infty} s^{\alpha-1} exp(-ns) \mathrm{d}s \end{aligned}$$

By the change of variable $x = ns$,we deduce that

$$\begin{aligned} I_{-\infty}^\alpha exp(nt) &= \frac{exp(nt)}{\Gamma(\alpha)} \int_0^{+\infty} \left(\frac{x}{n}\right)^{\alpha-1} exp(-x) \frac{\mathrm{d}x}{n} \\ &= n^{-\alpha} \frac{exp(nt)}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} exp(-n) \mathrm{d}x \\ &= n^{-\alpha} \frac{exp(nt)}{\Gamma(\alpha)} \Gamma(\alpha), \end{aligned}$$

and from it we find

$$I^{\alpha}_{-\infty} exp(nt) = n^{-\alpha} exp(nt) \tag{2.8}$$

2.2 Riemann-Liouville fractionnal derivative

Definition 2.2.1.
The Riemann-Liouville fractional derivative of a continuous function $f : [a, b] \to \mathbb{R}$ *order* α *is defined by:*

$$D^{\alpha}_{a} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m} \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau) \mathrm{d}\tau = D^{m} \left[I^{m-\alpha} f)(t)\right] \tag{2.9}$$

with $x \in [a, b], m \in \mathbb{N} \setminus \{0\}$ *and* $-1 < \alpha < m$.
Where $D^{m} = \dfrac{\mathrm{d}}{\mathrm{d}t}$ *is derived of integer order* $m = [\alpha] + 1$.

Lemma 2.2.1. *[23]*
Let $\alpha > 0$ *and* $f \in L^{1}[a, b]$ *then the equality:*

$$D^{\alpha}_{a} I^{\alpha}_{a} f(t) = f(t)$$

is true for almost everything on $[a, b]$.

Proof.
Using the definition (2.9) and (2.4) we will have:

$$\begin{aligned} D^{\alpha}_{a} I^{\alpha}_{a} f(t) &= D^{m}_{a} I^{m-\alpha}_{a} \left(I^{\alpha}_{a} f(t)\right) \\ &= D^{m}_{a} \left(I^{m-\alpha}_{a} I^{\alpha}_{a}\right) f(t) \\ &= D^{m}_{a} I^{m}_{a} f(t). \end{aligned}$$

Hence

$$D^{\alpha}_{a} I^{\alpha}_{a} f(t) = f(t)$$

□

Lemma 2.2.2. *[19]*
Let $\alpha > 0$. *If we assume* $u \in C(0, 1) \cap L(0, 1)$, *then the fractional differential equation* $D^{\alpha} u(t) = 0$ *has the unique solution*

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}, \tag{2.10}$$

where $C_i \in \mathbb{R}, i = 1, 2, \cdots, n (n = [\alpha] + 1)$.

Lemma 2.2.3. *([19], [20])*
Assume that $u \in C(0,1) \cap L(0,1)$, with a fractional derivative of order $\alpha > 0$ that belongs to $u \in C(0,1) \cap L(0,1)$. Then

$$I_0^\alpha D_0^\alpha f(t) = f(t) + \sum_{k=1}^{n} C_k t^{\alpha-k}, \tag{2.11}$$

for some $C_k \in \mathbb{R}$, and $(n = [\alpha] + 1)$.

Remark 2.2.1.

1. *$D_a^0 f(t) = D^1[I_a^1 f(t)] = f(t)$ ($D_a^0$is the identity operator).*
2. *For $\alpha = n$ where n is an integer, the operator gives the same result as the differentiation classical of order n.*

$$D_a^n f(t) = D^{n+1} I_a^{n+1-n} f(t) = D^{n+1} I_a^1 f(t)$$

2.2.1 Main properties of R-L fractionnal derivative

Lemma 2.2.1.1. *[19]*
The R-L derivation operator has the following properties:

1. *It's a linear operator.*
2. *$D_a^\alpha \circ I_a^\alpha = Id$.*
3. *$D_a^\alpha \left(D_a^\beta f(t)\right) = D_a^{\alpha+\beta} f(t) - \sum_{j=1}^{n} \left[D_a^{\beta-j} f(t)\right]_{t=a} \dfrac{(t-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)}$ with $\alpha \in [m-1)$ and $\beta \in [n-1, n)$.*

Proof.

1. Let $f(x)$ and $g(x)$ be two functions defined on $[a, b]$ such that $D_a^\alpha f$ and $D_a^\alpha g$ exist almost everywhere. Moreover, let $\lambda, \mu \in \mathbb{R}$. Then

$D_a^\alpha\left(\lambda f(x)+\mu g(x)\right)$ exists almost everywhere.

$$\begin{aligned}
D_a^\alpha\left(\lambda f(x)+\mu g(x)\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left(I_a^{m-\alpha}\left(\lambda f(x)+\mu g(x)\right)\right)\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left[\frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1}\left(\lambda f(x)+\mu g(x)\right)\mathrm{d}t\right]\\
&= \frac{1}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left[\lambda\int_a^x (x-t)^{m-\alpha-1} f(x)\mathrm{d}t\right.\\
&\quad \left.+\mu\int_a^x (x-t)^{m-\alpha-1} g(x)\mathrm{d}t\right]\\
&= \lambda\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left[\frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1} f(x)\mathrm{d}t\right]\\
&\quad +\mu\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left[\frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1} g(x)\mathrm{d}t\right].
\end{aligned}$$

Hence

$$D_a^\alpha\left(\lambda f(x)+\mu g(x)\right)=\lambda D_a^\alpha f(x)+\mu D_a^\alpha g(x)$$

2. In order to prove the second property, we need to consider the case of an integer $\alpha=m\geq 1$

$$\begin{aligned}
\left[\left(D_a^m\circ I_a^m\right)f\right](x) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left[I_a^m f(x)\right]\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left[\frac{1}{\Gamma(m)}\int_a^x (x-t)^{m-1} f(t)\mathrm{d}t\right]\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{(m-1)!}\int_a^x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m-1} (x-t)^{m-1} f(t)\mathrm{d}t\right]\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{(m-1)!}\int_a^x (m-1)! f(t)\mathrm{d}t\right]\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x).
\end{aligned}$$

Now we take $\alpha\in]m-1,m[$ and using the integral composition property R-L we will have

$$\begin{aligned}
\left[\left(D_a^\alpha\circ I_a^\alpha\right)f\right](x) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left[\left(I_a^{m-\alpha}(I_a^\alpha f)\right)(x)\right]\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \left[I_a^m f(x)\right]\\
&= f(x).
\end{aligned}$$

Hence

$$[(D_a^m \circ I_a^m) f](x) = f(x)$$

3. To prove the third property using subsequently the definition of the R-L fractional derivative (2.9), we obtain:

$$\begin{aligned} D_a^\alpha \left(D_a^\beta f(t)\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m [I_a^{m-\alpha} D_a^\beta f(t)] \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \left[D^{-(m-\alpha)} D_a^\beta f(t)\right] \\ &= D_a^{\alpha+\beta} f(t) - \sum_{j=1}^{n} \left[D_a^{\beta-j} f(t)\right]_{t=a} \frac{(t-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)}. \end{aligned}$$

$$D_a^\alpha \left(D_a^\beta f(t)\right) = D_a^{\alpha+\beta} f(t) - \sum_{j=1}^{n} \left[D_a^{\beta-j} f(t)\right]_{t=a} \frac{(t-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)} \tag{2.12}$$

Interchanging α and β (and therfore m and n), we can write:

$$D_a^\beta \left(D_a^\alpha f(t)\right) = D_a^{\alpha+\beta} f(t) - \sum_{j=1}^{m} \left[D_a^{\alpha-j} f(t)\right]_{t=a} \frac{(t-a)^{-\beta-j}}{\Gamma(1-\beta-j)} \tag{2.13}$$

□

Corollary 2.2.1.1.
The comparison of the relationships (2.12) and (2.13) says that in the general case the R-L fractional derivative operators D_a^α and D_a^β do not commute.

2.2.2 Examples of Fractional Derivatives

We consider the same functions of the previous example, and we calculate their fractional derivatives in the R-L sense of order α.

Example 2.2.2.1.
$f(t) = (t-a)^\beta$ *with* $\beta > -1$

$$D_a^\alpha (t-a)^\beta = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m [I^{m-\alpha}(t-a))^\beta]$$

we have seen the expression of the integral of order α of this function, so for order $m-\alpha$ we obtain

$$I_a^{m-\alpha}(t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+m-\alpha)}(t-a)^{\beta+m-\alpha}$$

So

$$\begin{aligned} D_a^\alpha(t-a)^\beta &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+m-\alpha)}(t-a)^{\beta+m-\alpha}\right] \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m (t-a)^{\beta+m-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+m-\alpha)}\frac{\Gamma(\beta+m-\alpha+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha} \end{aligned}$$

Because

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m (t-a)^{\beta+m-\alpha} &= (\beta+m-\alpha).(\beta+m-\alpha-1).(\beta+m-\alpha-(m-1)).(t-a)^{\beta-\alpha} \\ &= \frac{(\beta+m-\alpha)!}{\beta+m-\alpha-m)!}(t-a)^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+m-\alpha+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}. \end{aligned}$$

Hence

$$D_a^\alpha(t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}$$

In the case where $a=0$ we have:

$$D_a^\alpha(t)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t)^{\beta-\alpha}$$

Example 2.2.2.2.

R-L fractional derivative of the constant function $f(t)=C$

$$D_a^\alpha = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m (I_a^{m-\alpha}C)$$

we have seen the expression of the integral of order α of this function, so for order $m-\alpha$ we obtain

$$I_a^{m-\alpha}C = \frac{C}{\Gamma(m-\alpha+1)}(t-a)^{m-\alpha}$$

So

$$
\begin{aligned}
D_a^{\alpha} C &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \left(\frac{C}{\Gamma(m-\alpha+1)}(t-a)^{m-\alpha}\right) \\
&= \frac{C}{\Gamma(m-\alpha+1)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m (t-a)^{m-\alpha} \qquad (2.14)
\end{aligned}
$$

We have

$$
\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m (t-a)^{m-\alpha} = (m-\alpha)(m-\alpha-1)\dots(1-\alpha)(t-a)^{-\alpha}, \qquad (2.15)
$$

and

$$
\Gamma(m-\alpha+1) = (m-\alpha)(m-\alpha-1)\dots(1-\alpha)\Gamma(1-\alpha). \qquad (2.16)
$$

By substituting(2.15) and (2.16) in (2.14) we get

$$
D_a^{\alpha} C = \frac{C(m-\alpha)(m-\alpha-1)\dots(1-\alpha)(t-a)^{-\alpha}}{(m-\alpha)(m-\alpha-1)\dots(1-\alpha)\Gamma(1-\alpha)},
$$

So

$$
D_a^{\alpha} C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}
$$

This means that the derivative in the R-L sense of the constant is not zero.

Example 2.2.2.3.
The exponential function $f(t) = \exp(nt)$*to* $n > 0$, *and* $\alpha > 0$.

Using the formula (2.9) with $a = -\infty$ *and the result (2.8) we get:*

$$
\begin{aligned}
D_{-\infty}^{\alpha} \exp(nt) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) I_{-\infty}^{m\alpha} \exp(nt) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \left(n^{\alpha-m} \exp(nt)\right) \\
&= n^{\alpha-m} n^m \exp(nt)
\end{aligned}
$$

So

$$
D_{-\infty}^{\alpha} \exp(nt) = n^{\alpha} \exp(nt)
$$

Chapter 3

Nonlinear fractional differential equations with integral boundary value conditions

In this chapter, we show some contributions of researchers to the finding of the existence of the solution for the different fractional differential equations. A. Cabada and Z. Hamdi [5] studied the existence of the solution of the following nonlinear fractional differntial equations with integral boundary value conditions .

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \lambda \int_0^1 u(s)\mathrm{d}s. \end{cases} \tag{3.1}$$

where $2 < \alpha < 3$, $0 < \lambda$, $\lambda \neq \alpha$, D^{α} is the Rieman-Liouville fractional derivative and f is a continuous function.

3.1 Green function

In this section, using Green's function and its properties, we will establish the positivity of the solution of the problem (3.1), for this we need the next theorem.

Theorem 3.1.1. *[5]*

Let $2 < \alpha \leqslant 3$ *and* $\lambda \neq \alpha$. *Assume* $y \in C[0,1]$, *then auxiliary problem*

$$\begin{cases} D^{\alpha}u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \lambda \int_0^1 u(s)\mathrm{d}s. \end{cases} \tag{3.2}$$

has a unique solution $u \in C^1[0,1]$, *given by the expression*

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leq 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leq 1. \end{cases} \tag{3.3}$$

Proof.

In view of Lemma (2.2.3), the equation $D^{\alpha}u(t) + y(t) = 0$, is equivalent to the integral equation

$$u(t) = \int_0^t \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3},$$

with $2 < \alpha \leqslant 3$.

The boundary condition $u(0) = 0$ implies that $C_3 = 0$,because $\alpha - 3 < 0$, thus

$$u(t) = \int_0^t \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + C_1 t^{\alpha-1} + C_2 t^{\alpha-2},$$

and $u'(0) = 0$ implies that $C_2 = 0$, Thus

$$u(t) = \int_0^t \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + C_1 t^{\alpha-1}.$$

In view of the boundary condition $u(1) = \lambda \int_0^1 u(s)\mathrm{d}s$, we conclude that

$$u(1) = -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + C_1 \quad = \lambda \int_0^1 u(s)\mathrm{d}s.$$

So

$$C_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + \lambda \int_0^1 u(s)\mathrm{d}s.$$

Therefore, the unique solution of problem (3.2) is

$$u(t)=\int_0^t \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+t^{\alpha-1}\lambda\int_0^1 u(s)\mathrm{d}s. \tag{3.4}$$

Suppose $A=\int_0^1 u(s)\mathrm{d}s$. From the previous equality, we deduce that

$$\begin{aligned}
A &= \int_0^1 u(t)\mathrm{d}t\\
&= \int_0^1\int_0^t \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\mathrm{d}t+\int_0^1 t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\mathrm{d}t\\
&\quad +\int_0^1 t^{\alpha-1}\lambda A\mathrm{d}t
\end{aligned}$$

According to fubini's theorem we find

$$\begin{aligned}
A &= \int_0^1\int_s^1 \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}t\,\mathrm{d}s+\int_0^1 t^{\alpha-1}\mathrm{d}t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\\
&\quad +\int_0^1 t^{\alpha-1}\,\lambda\,A\,\mathrm{d}t\\
&= -\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)}\,y(s)\mathrm{d}s+\int_0^1 \frac{(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}\,y(s)\mathrm{d}s+\frac{\lambda A}{\alpha}\\
A\left(\frac{\alpha-\lambda}{\alpha}\right) &= -\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)}\,y(s)\mathrm{d}s+\int_0^1 \frac{(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}\,y(s)\mathrm{d}s
\end{aligned}$$

So

$$A=-\int_0^1 \frac{(1-s)^{\alpha}}{(\alpha-\lambda)\Gamma(\alpha)}\,y(s)\mathrm{d}s+\int_0^1 \frac{(1-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}\,y(s)\mathrm{d}s$$

Replacing this value in (3.4), the unique solution of (3.2) is expressed as

$$\begin{aligned}
u(t) &= \int_0^t \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s-t^{\alpha-1}\lambda\int_0^1 \frac{(1-s)^{\alpha}}{(\alpha-\lambda)\Gamma(\alpha)}\,y(s)\mathrm{d}s\\
&\quad +t^{\alpha-1}\lambda\int_0^1 \frac{(1-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}\,y(s)\mathrm{d}s,\\
&= \int_0^t \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}[\alpha-\lambda(1-s)]}{(\alpha-\lambda)\Gamma(\alpha)}\,y(s)\mathrm{d}s,\\
&= \int_0^t \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s+\int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1}[\alpha-\lambda(1-s)]}{(\alpha-\lambda)\Gamma(\alpha)}\,y(s)\mathrm{d}s\\
&\quad +\int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}[\alpha-\lambda(1-s)]}{(\alpha-\lambda)\Gamma(\alpha)}\,y(s)\mathrm{d}s,
\end{aligned}$$

$$= \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1}[\alpha-\lambda(1-s)]-(\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)} y(s)\mathrm{d}s$$
$$+ \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}[\alpha-\lambda(1-s)]}{(\alpha-\lambda)\Gamma(\alpha)} y(s)\mathrm{d}s.$$

Hence

$$u(t) = \int_0^1 G(t,s)\, y(s)\mathrm{d}s$$

□

Lemma 3.1.0.1. *[5]*
If $\alpha \in (2,3]$ and $\lambda \geqslant 0$, then the function $G(t,s)$ defined by (3.3) satisfies the following properties

1. $G(0,s) = G(t,1) = G(t,0) = 0$ *for all* $t,s \in [0,1]$ *and* $\lambda \neq \alpha$.
2. $G(1,s) = 0$ *for all* $s \in (0,1)$ *if and only if* $\lambda = 0$.
3. $(\alpha-\lambda)G(1,s) > 0$ *for all* $t,s \in (0,1)$ *if and only if* $\lambda \neq \alpha$.
4. $G(t,s) \leqslant \dfrac{1}{(\alpha-\lambda)\Gamma(\alpha-1)}$ *for all* $t,s \in [0,1]$ *and* $\lambda \in [0,\alpha)$.
5. $G(t,s)$ *is a continuous function for all* $t,s \in [0,1]$ *and* $\lambda \neq \alpha$.

Proof.

1. To prove the first property we put $t = 0$ in expression of G, we get directly $G(t,s) = 0$ for all $t,s \in [0,1]$ and $\lambda \neq \alpha$, and we obtain the same result if we substitute s for 1 or 0.

2. From Green's function which is given by the expression (3.3)we have

$$G(1,s) = \frac{(1-s)^{\alpha-1}\lambda s}{(\alpha-\lambda)\Gamma(\alpha)}; \qquad \forall\, s \in (0,1)$$

If $\lambda = 0$ so, $(1-s)^{\alpha-1}\lambda s = 0$, then

$$G(1,s) = 0,\ \forall\, s \in (0,1) \text{ if and only if } \lambda = 0$$

3. We have

$$G(1,s) = \frac{(1-s)^{\alpha-1}\lambda s}{(\alpha-\lambda)\Gamma(\alpha)}; \qquad \forall\, s \in (0,1)$$
$$(\alpha-\lambda)G(1,s) = \frac{(1-s)^{\alpha-1}\lambda s}{\Gamma(\alpha)},$$

and $(1-s)^{\alpha-1}\lambda > 0$, where $\lambda > 0, 0 < s < 1$, and we have $\Gamma(\alpha) > 0$, which implies that

$$(\alpha-\lambda)G(1,s) > 0, \ \text{ for all } \ t,\ s \in (0,1) \ \text{ if and only if } \ \lambda \neq \alpha$$

4. We have $\lambda \in [0,\alpha)$ and $\alpha \in [2,3]$ implies $\lambda < 1$.
Consider the first situation, if $0 \leqslant s \leqslant t \leqslant 1$, we have

$$\begin{aligned} G(t,s) &= \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)-(\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)} \\ &\leqslant \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)} \\ &\leqslant \frac{t^{\alpha-1}(1-s)^{\alpha-1}[\alpha-\lambda(1-s)]}{(\alpha-\lambda)\Gamma(\alpha)} \end{aligned}$$

Since $1 > t > s$ we deduce that

$$\begin{aligned} &\leqslant \frac{t^{\alpha-1}(\alpha-\lambda)}{(\alpha-\lambda)\Gamma(\alpha)} \\ &\leqslant \frac{(\alpha-\lambda)}{(\alpha-\lambda)\Gamma(\alpha)} \\ &\leqslant \frac{(\alpha-1)}{(\alpha-\lambda)\Gamma(\alpha)} \end{aligned}$$

From the properties of Gamma function we conclude this

$$G(t,s) \leqslant \frac{1}{(\alpha-\lambda)\Gamma(\alpha-1)}$$

We do the same steps in the second case, if $0 \leqslant t \leqslant s \leqslant 1$, and we will get the same result.
So

$$G(t,s) \leqslant \frac{1}{(\alpha-\lambda)\Gamma(\alpha-1)} \ \text{ for all } \ t,s \in [0,1] \text{ and } \ \lambda \in [0,\alpha)$$

5. We have

$$G(t,s)=\begin{cases}\dfrac{k(t,s)-g(t,s)}{(\alpha-\lambda)\Gamma(\alpha)}, & 0\leqslant s\leqslant t\leq 1,\\ \dfrac{k(t,s)}{(\alpha-\lambda)\Gamma(\alpha)}, & 0\leqslant t\leqslant s\leq 1.\end{cases}$$

with $k(t,s)=t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)$ and
$g(t,s)=(\alpha-\lambda)(t-s)^{\alpha-1}$.
In view of the continuity of the gamma function, $g(t,s)$ and $k(t,s)$ in $[0,1]$ the Green function $G(t,s)$ is continuous for all $t,s\in[0,1]$ and $\lambda\neq\alpha$.

□

Now, we prove two additional inequalities of the Green's function G. Such properties, together with the previous ones given above, will be of fundamental interest to ensure the existence of solutions of problem (3.1) that will be proven in the next section.

Lemma 3.1.0.2. *[5]*
Fix $2<\alpha\leqslant 3$ and $0<\lambda<\alpha$. Let $G(t,s)$ be the Green's function related to problem (3.2) given by the expression (3.3). Then the following inequalities hold:

$$t^{\alpha-1}G(1,s)\leqslant G(t,s)\leqslant\frac{\alpha}{\lambda}G(1,s),\qquad \textit{for}\quad \textit{all}\ t,s\in(0,1). \tag{3.5}$$

Proof.
Assume in a first moment that $0<t\leqslant s<1$. In such a case:
Suppose the function $h(t,s)\equiv\dfrac{G(t,s)}{G(1,s)}$

$$\begin{aligned}h(t,s)&=\frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(1-s)^{\alpha-1}\lambda s}\\&=\frac{t^{\alpha-1}(\alpha-\lambda+\lambda s)}{\lambda s}\\&=t^{\alpha-1}\left(1+\frac{(\alpha-\lambda)}{\lambda s}\right).\end{aligned}$$

So

$$t^{\alpha-1} \leqslant t^{\alpha-1}\left(1+\frac{(\alpha-\lambda)}{\lambda s}\right) = h(t,s)$$

and we deduce that

$$\begin{aligned}
t^{\alpha-1} \leqslant h(t,s) &\leqslant t^{\alpha-1}\left(1+\frac{(\alpha-\lambda)}{\lambda s}\right)\\
&\leqslant t^{\alpha-1}+\frac{t^{\alpha-1}\alpha}{\lambda s}-\frac{t^{\alpha-1}\lambda}{\lambda s}\\
&\leqslant t^{\alpha-1}+\frac{t^{\alpha-1}\alpha}{\lambda s}-\frac{t^{\alpha-1}}{s}\\
&\leqslant t^{\alpha-1}+\frac{t^{\alpha-1}\alpha}{\lambda}-t^{\alpha-1}\\
&\leqslant \frac{t^{\alpha-1}\alpha}{\lambda}, \qquad \forall\, t \leqslant s < 1
\end{aligned}$$

Hence

$$t^{\alpha-1}G(1,s) \leqslant G(t,s) \leqslant \frac{\alpha}{\lambda}G(1,s), \quad \forall\, 0 < t \leqslant s < 1$$

On the other hand, if $0 < s \leqslant t < 1$ we have that

$$h(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)-(\alpha-\lambda)(t-s)^{\alpha-1}}{(1-s)^{\alpha-1}\lambda s}$$

and since $s \geqslant ts$ we deduce that

$$h(t,s) \geqslant \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)-(\alpha-\lambda)(t-ts)^{\alpha-1}}{(1-s)^{\alpha-1}\lambda s}$$

$$h(t,s) \geqslant \frac{t^{\alpha-1}(1-s)^{\alpha-1}\lambda s}{(1-s)^{\alpha-1}\lambda s} = t^{\alpha-1}$$

Now, we verify that $h(t,s) \leqslant \dfrac{\alpha}{\lambda}$ for all $0 < s \leqslant t < 1$.

$$\begin{aligned}
h(t,s) &= \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)-(\alpha-\lambda)(t-s)^{\alpha-1}}{(1-s)^{\alpha-1}\lambda s}\\
&\leqslant \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(1-s)^{\alpha-1}\lambda s}\\
&\leqslant t^{\alpha-1}\left(1+\frac{(\alpha-\lambda)}{\lambda s}\right),
\end{aligned}$$

Then, we have

$$\begin{aligned}
t^{\alpha-1} \leqslant h(t,s) &\leqslant t^{\alpha-1}\left(1+\frac{(\alpha-\lambda)}{\lambda s}\right)\\
&\leqslant t^{\alpha-1}+\frac{t^{\alpha-1}\alpha}{\lambda s}-\frac{t^{\alpha-1}\lambda}{\lambda s}\\
&\leqslant t^{\alpha-1}+\frac{t^{\alpha-1}\alpha}{\lambda s}-\frac{t^{\alpha-1}}{s}\\
&\leqslant t^{\alpha-1}+\frac{t^{\alpha-1}\alpha}{\lambda}-t^{\alpha-1}\\
&\leqslant \frac{t^{\alpha-1}\alpha}{\lambda}, \qquad \forall\, s \leqslant t < 1
\end{aligned}$$

So

$$h(t,s) \leqslant \frac{\alpha}{\lambda}, \quad \forall\, t < 1$$

Hence

$$t^{\alpha-1}G(1,s) \leqslant G(t,s) \leqslant \frac{\alpha}{\lambda}G(1,s), \quad \forall\, 0 < s \leqslant t < 1$$

Now, we have that the inequalities (3.5) are fulfilled. □

As a corollary of the previous result and Lemma (3.1.0.1), we deduce the following:

Corollary 3.1.0.1.
Let G be the Green's function related to problem (3.2), which is given by the expression (3.3). Then, for all $\alpha \in (2,3]$ and $\lambda \geqslant 0$, the following property holds:

$$G(t,s) > 0 \ \textit{for all} \ \ t,s \in (0,1) \ \textit{and all} \ \ \lambda \in [0,\alpha)$$

3.2 Existence of positive solutions

This section is devoted to prove the existence of a positive solution of the nonlinear boundary value problem (3.1). To this end, we use the Guo–Krasnoselskii fixed point theorem (1.3.1.1).
Let $E = C[0,1]$ be the Banach space endowed with the usual supremum norm $\|u\| = \max\limits_{t\in[0,1]} |u(t)|$. And suppose the following assumption:
(f) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

Lemma 3.2.1.
Let $2 < \alpha \leqslant 3$ and $\lambda \neq \alpha$. If $u \in E$, a solution of the fractional problem (3.1) then it satisfies

$$u(t) \geqslant 0, \forall\, t \in [0,1],\; u(t) \geqslant \frac{t^{\alpha-1}\lambda}{\alpha}\|u\|, \forall\, t \in [t_0, 1]\,. \tag{3.6}$$

with $t_0 \in (0,1]$ fixed.

Proof.
By lemma (3.1.0.2) we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)y(s)\mathrm{d}s \\ &\leqslant \frac{\alpha}{\lambda}\int_0^1 G(1,s)y(s)\mathrm{d}s \end{aligned}$$

it follows that

$$\|u\| \leqslant \frac{\alpha}{\lambda}\int_0^1 G(1,s)y(s)\mathrm{d}s \tag{3.7}$$

and we have

$$\begin{aligned} u(t) &\geqslant t^{\alpha-1}\int_0^1 G(1,s)y(s)\mathrm{d}s \\ &\geqslant t^{\alpha-1}\frac{\lambda}{\alpha}\|u\| \end{aligned}$$

from it we have

$$\min u(t) \geqslant \begin{cases} t^{\alpha-1}\dfrac{\lambda}{\alpha}\|u\| & \text{if} \quad t \in [t_0, 1], t_0 > 0 \\ 0 & \text{if} \quad t \in [0,1] \end{cases}$$

□

Let $t_0 = \frac{1}{2}$ and define a cone $\mathcal{P} \cap E$ as follows

$$\mathcal{P} = \left\{ u \in E, u(t) \geqslant 0,\;\; \forall\; t \in [0,1], \quad u(t) \geqslant \frac{t^{\alpha-1}\lambda}{\alpha}\|u\|,\;\; \forall\; t \in \left[\frac{1}{2}, 1\right] \right\} \tag{3.8}$$

it fulfills the above conditions (1.3.1).

Define now the operator $T : E \to E$ as

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \tag{3.9}$$

with G defined in (3.3) and the function $u \in E$, the solution of the problem (3.1) if and only if $Tu(t) = u(t), \forall\, t \in [0,1]$.

Lemma 3.2.2.

$T : \mathcal{P} \to \mathcal{P}$ is completely continuous.

Proof.

From the continuity and the non negativeness of functions G and f on their domains of definition, we have that if $u \in \mathcal{P}$ then $Tu \in E$ and $Tu(t) \geqslant 0$ for all $t \in [0,1]$.

1. Let us prove in first that $T(\mathcal{P}) \subset \mathcal{P}$.

Take $u \in \mathcal{P}$, then, for all $t \in [0,1]$, by using Lemmas (3.1.0.1) and (3.1.0.2), the following inequalities are satisfied

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \\ &\geqslant t^{\alpha-1} \int_0^1 G(1,s) f(s,u(s)) \mathrm{d}s \\ &\geqslant \frac{t^{\alpha-1}\lambda}{\alpha} \int_0^1 \max_{t\in[0,1]} \{G(t,s)\} f(s,u(s)) \mathrm{d}s \\ &\geqslant \frac{t^{\alpha-1}\lambda}{\alpha} \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \right\} \\ &\geqslant \frac{t^{\alpha-1}\lambda}{\alpha} \|Tu\|. \end{aligned}$$

In view of the continuity of functions G and f, the operator $T : \mathcal{P} \to \mathcal{P}$ is continuous.

2. Let $\Omega \cap \mathcal{P}$ be bounded, which is to say there exists a positive constant $M > 0$ such that $\|u\|_\infty \leqslant M$ for all $u \in \Omega$.

Define now

$$L = \max_{0\leqslant t\leqslant 1, 0\leqslant u\leqslant M} |f(t,u)| + 1$$

Then, for all $u \in \Omega$, it is satisfied that

$$\begin{aligned} |Tu(t)| &= \left| \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \right| \\ &\leqslant \int_0^1 |G(t,s) f(s,u(s))| \, \mathrm{d}s \\ &\leqslant L \int_0^1 G(t,s) \mathrm{d}s \end{aligned}$$

and we have $G(t,s) \leqslant \dfrac{1}{(\alpha-\lambda)\Gamma(\alpha-1)}$ for all $t,s \in [0,1]$ and $\lambda \in [0,\alpha)$.

So

$$\begin{aligned} |Tu(t)| &\leqslant \frac{L}{(\alpha-\lambda)\Gamma(\alpha-1)} \\ &\leqslant \frac{(\alpha-1)L}{(\alpha-1)(\alpha-\lambda)\Gamma(\alpha-1)} \\ &\leqslant \frac{\alpha L - L}{(\alpha-\lambda)\Gamma(\alpha)} \\ &\leqslant \frac{\alpha L}{(\alpha-\lambda)\Gamma(\alpha)} \qquad \forall t \in [0,1], \end{aligned}$$

Hence, the set $T(\Omega)$ is bounded in E.

3. Finally, we show that T is equicontinuous.

For each $u \in \Omega$, $\forall\, t,s \in [0,1]$ we have

$$\begin{aligned} |(Tu)'(t)| &= \left| \int_0^t \frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s)) \mathrm{d}s + \int_0^1 \frac{t^{\alpha-2}(1-s)^{\alpha-1}[\alpha-\lambda(1-s)]}{(\alpha-\lambda)\Gamma(\alpha-1)} f(s,u(s)) \mathrm{d}s \right| \\ &\leqslant \int_0^t \left| \frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right| |f(s,u(s))| \mathrm{d}s + \int_0^1 \left| \frac{\alpha t^{\alpha-2}(1-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha-1)} \right| |f(s,u(s))| \mathrm{d}s \\ &\quad - \int_0^1 \left| \frac{\lambda t^{\alpha-2}(1-s)^{\alpha}}{(\alpha-\lambda)\Gamma(\alpha-1)} \right| |f(s,u(s))| \mathrm{d}s \\ &\leqslant \frac{L}{\Gamma(\alpha-1)} \int_0^t \left| (t-s)^{\alpha-2} \right| \mathrm{d}s + \frac{\alpha L}{(\alpha-\lambda)\Gamma(\alpha-1)} \int_0^1 \left| t^{\alpha-2}(1-s)^{\alpha-1} \right| \mathrm{d}s \\ &\quad - \frac{\lambda L}{(\alpha-\lambda)\Gamma(\alpha-1)} \int_0^1 \left|{}^{\alpha-2}(1-s)^{\alpha}\right| \mathrm{d}s \\ &\leqslant \frac{L}{\Gamma(\alpha-1)} \int_0^t \left| (t-s)^{\alpha-2} \right| \mathrm{d}s + \frac{\alpha L}{(\alpha-\lambda)\Gamma(\alpha-1)} \int_0^1 \left| t^{\alpha-2}(1-s)^{\alpha-1} \right| \mathrm{d}s \\ &\leqslant \frac{L}{(\alpha-1)\Gamma(\alpha-1)} \left| \left[(t-s)^{\alpha-1} \right]_0^t \right| + \frac{\alpha L}{\alpha(\alpha-\lambda)\Gamma(\alpha-1)} \left| \left[t^{\alpha-2}(1-s)^{\alpha} \right]_0^1 \right| \end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{L}{\Gamma(\alpha)} + \frac{L}{(\alpha-\lambda)\Gamma(\alpha-1)} \\
&\leqslant \frac{L}{\Gamma(\alpha)} + \frac{(\alpha-1)L}{(\alpha-1)(\alpha-\lambda)\Gamma(\alpha-1)} \\
&\leqslant \frac{L}{\Gamma(\alpha)} + \frac{\alpha L}{(\alpha-\lambda)\Gamma(\alpha)} - \frac{L}{(\alpha-\lambda)\Gamma(\alpha)} \\
&\leqslant \frac{L}{\Gamma(\alpha)} + \frac{\alpha L}{(\alpha-\lambda)\Gamma(\alpha)} \\
&= N.
\end{aligned}$$

As consequence, for all $t_1, t_2 \in [0,1], t_1 < t_2$, we have

$$|(Tu)(t_2) - (Tu)(t_1)| \leqslant \int_1^2 |(Tu)'(s)|\mathrm{d}s \leqslant N(t_2 - t_1)$$

so the set $T(\Omega)$ is equicontinuous in E.
By means of the Arzela-Ascoli theorem (1.3.1), we have $T : \mathcal{P} \to \mathcal{P}$ is completely continuous.

□

Now, we are in position to prove the existence of positive solutions of the nonlinear boundary value problem. For this, we use the known Guo– Krasnoselskii fixed point theorem (1.3.1.1).
Let introduce some notations

$$f_0 = \lim_{u\to 0^+}\left\{\min_{t\in[1/2,1]}\left\{\frac{f(t,u)}{u}\right\}\right\} \quad \text{and} \quad f_\infty = \lim_{u\to\infty}\left\{\min_{t\in[1/2,1]}\left\{\frac{f(t,u)}{u}\right\}\right\},$$

$$f^0 = \lim_{u\to 0^+}\left\{\max_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\} \quad \text{and} \quad f^\infty = \lim_{u\to\infty}\left\{\max_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\}$$

Theorem 3.1.

Assume that condition (f) holds coupled with one of the two following conditions:

(i) (sublinear case)$f_0 = \infty$ and $f^\infty = 0$.

(ii) (superlinear case)$f^0 = 0$ and $f_\infty = \infty$.

Then for all $\alpha \in (2,3]$ and $\lambda \in (0,\alpha)$, the probleme (3.1) has a positive solution that belongs to the cone $\mathcal{P}$ defined in (3.8).

Proof.

(i) Consider now the first situation sublinear case ($f_0 = \infty$ and $f^\infty = 0$). Since $f_0 = \infty$, then there exists a constant $\rho_1 > 0$ such that $f(t, u) \geqslant \delta_1 u$ for all $0 < u \leqslant \rho_1$, where $\delta_1 > 0$ and

$$\delta_1 \frac{\lambda}{\alpha} \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^{1} s^{\alpha-1} G(t,s)\mathrm{d}s \right\} \geqslant 1 \tag{3.10}$$

Take $u \in \mathcal{P}$, such that $\|u\| = \rho_1$, then from expression (3.8), we get

$$\|Tu\| = \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f(s,u(s))\mathrm{d}s \right\}$$

$$\begin{aligned}
&\geqslant \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^{1} G(t,s) f(s,u(s))\mathrm{d}s \right\} \\
&\geqslant \delta_1 \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^{1} G(t,s) u(s)\mathrm{d}s \right\} \\
&\geqslant \delta_1 \|u\| \frac{\lambda}{\alpha} \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^{1} s^{\alpha-1} G(t,s)\mathrm{d}s \right\} \\
&\geqslant \|u\|.
\end{aligned}$$

On the other hand, since $f(t, \cdot)$ is a continuous function on $[0, \infty)$, we can define the following function :

$$\tilde{f}(t,u) = \max_{z\in[0,u]} f(t,z)$$

Clearly $\tilde{f}(t, \cdot)$ is nondecreasing on $[0, \infty)$, moreover, since $f^\infty = 0$ it is obvious that (see [22]).

$$\lim_{u\to\infty} \left\{ \max_{t\in[0,1]} \frac{\tilde{f}(t,u)}{u} \right\} = 0$$

Choos new $\delta_2 > 0$ satisfying the following property:

$$\frac{\delta_2}{(\alpha-\lambda)\Gamma(\alpha-1)} \leqslant 1 \tag{3.11}$$

Therefore there exists a constant $\rho_2 > \rho_1 > 0$ such that $\tilde{f}(t, u) \leqslant \delta_2 u$ for all $u \geqslant \rho_2$.
Consider $u \in \mathcal{P}$ be such that $\|u\| = \rho_2$, then, from the definition of $\tilde{f}$, Eq. (3.11) and property (4) in Lemma (3.1.0.1), we attain at the following inequalities:

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f(s, u(s)) \mathrm{d}s \right\} \\
&\leqslant \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) \tilde{f}(s, \|u\|) \mathrm{d}s \right\} \\
&\leqslant \delta_2 \|u\| \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) \mathrm{d}s \right\} \\
&\leqslant \frac{\delta_2}{(\alpha - \lambda)\Gamma(\alpha - 1)} \|u\|
\end{aligned}$$

Hence

$$\|Tu\| \leqslant \|u\|$$

Thus, by the first part of Guo–Krasnoselskii fixed point theorem(1.3.1.1), we conclude that problem (3.1) has at least one positive solution u such that

$$\rho_1 \leqslant \|u\| \leqslant \rho_2$$

(ii) Consider now superlinear case ($f^0 = 0$ and $f_\infty = \infty$).
Let $\delta_2 > 0$ be given as in Eq. (3.11). Since $f^0 = 0$, there exists a constant $r_1 > 0$ such that $f(t, u) \leqslant \delta_2 u$ for $0 \leqslant u \leqslant r_1$.
Take $u \in \mathcal{P}$, such that $\|u\| = r_1$. Then, from the previous calcul, we have

$$\|Tu\| = \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f(s, u(s)) \mathrm{d}s \right\} \leqslant \frac{\delta_2}{(\alpha - \lambda)\Gamma(\alpha - 1)} \|u\| \leqslant \|u\|.$$

Consider now $\delta_3 > 0$ satisfying

$$\delta_3 \frac{\lambda}{2^{\alpha-1}\alpha} \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) \mathrm{d}s \right\} \geqslant 1 \tag{3.12}$$

The fact that $f_\infty = \infty$ says us there exists a constant $r_2 > r_1 > 0$ with$\|u\| = r_1$ such that $f(t, u) \geqslant \delta_3 u$ for all $u \geqslant r_2$.

By definition of the cone $\mathcal{P}$ we have

$$\begin{aligned} r_2 &\geqslant \min_{t\in[1/2,1]} \left\{ \frac{t^{\alpha-1}}{\alpha} \lambda r_1 \right\} \\ &\geqslant \frac{\alpha}{2^{\alpha-1}} \lambda \|u\|. \end{aligned}$$

Let now $u \in \mathcal{P}$ be such that $\|u\| = r_2 \frac{\alpha}{\lambda} 2^{\alpha-1}$. As consequence, the following inequalities holds:

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \right\} \\ &\geqslant \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) f(s,u(s)) \mathrm{d}s \right\} \\ &\geqslant \delta_3 \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) u(s) \mathrm{d}s \right\} \\ &\geqslant \delta_3 \frac{t^{\alpha-1}\lambda}{\alpha} \|u\| \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) \mathrm{d}s \right\} \\ &\geqslant \delta_3 \frac{\lambda}{2^{\alpha-1}\alpha} \|u\| \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) \mathrm{d}s \right\}, \qquad \forall t \in [1/2,1] \\ &\geqslant \|u\|. \end{aligned}$$

Hence

$$\|Tu\| \geqslant \|u\|$$

Therefore, by the second part of Guo–Krasnoselskii fixed point theorem (1.3.1.1), we can conclude that problem (3.1) has at least one positive solution. □

Remark 3.2.1.
It is important to point out that, since $G(0,s) = 0$, in order to ensure the existence of r_2 in case 2 of the previous theorem, it is necessary to reduce the interval of definition $[0,1]$ to the smaller one $[1/2,1]$.

In fact, given two real constants $0 < a \leqslant b \leqslant 1$, by redefining

$$f_0 = \lim_{u\to 0^+} \left\{ \min_{t\in[a,b]} \left\{ \frac{f(t,u)}{u} \right\} \right\} \qquad \textit{and} \qquad f_\infty = \lim_{u\to\infty} \left\{ \min_{t\in[a,b]} \left\{ \frac{f(t,u)}{u} \right\} \right\},$$

it is immediate to verify that Theorem (3.1) remains true in the cone

$$P=\left\{u\in E, u(t)\geqslant 0\ \forall\ t\in[0,1],\quad u(t)\geqslant \frac{t^{\alpha-1}\lambda}{\alpha}\|u\|,\ \forall\ t\in\left[\frac{1}{2},1\right]\right\}.$$

3.3 Examples

We now give two examples to illustrate our results. The first example and the second one are chosen such that the conditions (i) and (ii) are satisfied, respectively.

Example 3.3.1. *[5]*
Let consider the fractional differential equation (3.1) with

$$f(t,u(t))=\sqrt{u(t)}+\log(tu^2(t)+2).$$

One can easily see that for all $u>0$

$$\min_{t\in[1/2,1]}\left\{\frac{f(t,u)}{u}\right\}=\frac{\sqrt{u}+\log\left(\frac{1}{2}u^2+2\right)}{u}$$

and

$$\max_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}=\frac{\sqrt{u}+\log\left(u^2+2\right)}{u}$$

Now, we calculate f_0

$$\begin{aligned}\lim_{u\to 0^+}\left\{\min_{t\in[1/2,1]}\left\{\frac{f(t,u)}{u}\right\}\right\}&=\lim_{u\to 0}\left(\frac{\sqrt{u}+\log\left(\frac{1}{2}u^2+2\right)}{u}\right)\\&=\lim_{u\to 0}\left(\frac{\sqrt{u}-\log(2)+\log(u^2+4)}{u}\right)\\&=\infty.\end{aligned}$$

Next, we calculate f^∞

$$\begin{aligned}\lim_{u\to\infty}\left\{\max_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\}&=\lim_{u\to\infty}\left(\frac{\sqrt{u}+\log\left(u^2+2\right)}{u}\right)\\&=\lim_{u\to\infty}\left(\frac{4u^{\frac{3}{2}}+u^2+2}{2\sqrt{u}(u^2+2)}\right)\\&=\lim_{u\to\infty}\left(\frac{2\left(3\sqrt{u}+u\right)\sqrt{u}}{5u^2+2}\right)\end{aligned}$$

$$
\begin{aligned}
&= 2 \lim_{u\to\infty} \left(\frac{(3\sqrt{u}+u)\sqrt{u}}{5u^2+2} \right) \\
&= 2 \left(\frac{\lim\limits_{u\to\infty} (3\sqrt{u}+u)\sqrt{u}}{\lim\limits_{u\to\infty} (5u^2+2)} \right) \\
&= 2 \left(\frac{\lim\limits_{u\to\infty} \left(\frac{3}{u} + \frac{1}{\sqrt{u}} \right)}{\lim\limits_{u\to\infty} \left(5 + \frac{2}{u} \right)} \right) \\
&= 2 \left(\lim_{u\to\infty} \left(\frac{3}{u} \right) + \lim_{u\to\infty} \left(\frac{1}{\sqrt{u}} \right) \right) \cdot \left(\frac{1}{\lim\limits_{u\to\infty} \left(5 + \frac{2}{u} \right)} \right) \\
&= 0.
\end{aligned}
$$

From the first part of Theorem (3.1), we get that the problem (3.1) has a positive solution.

Example 3.3.2. *[5]*
Let consider the fractional differential equation (3.1) with

$$f(t, u(t)) = u^2(t) - u(t) + t(e^{u(t)} - 1).$$

Then, for every $u > 0$ it is verified that

$$\min_{t\in[1/2,1]} \left\{ \frac{f(t,u)}{u} \right\} = u - 1 + \frac{e^u - 1}{2u}$$

and

$$\max_{t\in[0,1]} \left\{ \frac{f(t,u)}{u} \right\} = u - 1 + \frac{e^u - 1}{u}$$

Now, we calculate f_∞

$$
\begin{aligned}
\lim_{u\to\infty} \left(u - 1 + \frac{e^u - 1}{2u} \right) &= \lim_{u\to\infty} (u-1) + \frac{1}{2} \lim_{u\to\infty} \left(\frac{e^u - 1}{u} \right) \\
&= \lim_{u\to\infty} (u-1) + \frac{1}{2} \left(\frac{\lim\limits_{u\to\infty} (e^u - 1)}{\lim\limits_{u\to\infty} (u)} \right) \\
&= \lim_{u\to\infty} (u-1) + \frac{1}{2} \left(\frac{\lim\limits_{u\to\infty} (e^u - 1)'}{\lim\limits_{u\to\infty} (u)'} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{u\to\infty}(u-1) + \frac{1}{2}\lim_{u\to\infty}(e^u) \\
&= \infty.
\end{aligned}
$$

Then,we calculate f^0

$$
\begin{aligned}
\lim_{u\to 0}\left(u-1+\frac{e^u-1}{u}\right) &= \lim_{u\to 0}(u-1) + \lim_{u\to 0}\left(\frac{e^u-1}{u}\right) \\
&= \lim_{u\to 0}(u-1) + \left(\frac{\lim\limits_{u\to 0}(e^u-1)'}{\lim\limits_{u\to 0}(u)'}\right) \\
&= \lim_{u\to 0}(u-1) + \lim_{u\to 0}(e^u) \\
&= 0.
\end{aligned}
$$

Then, from the second part of theorem (3.1), we get that the problem (3.1) has at least one positive solution.

Chapter 4

Positive solution for fractional boundary value problems with integral boundary conditions and parameter dependence

In this chapter, we investigate the existence of positive solutions of the following fractional differential equation with integral boundary conditions (see [1]).

$$D^{\delta}u(t) + f(t, u(t)) = 0,\ 0 < t < 1,\ 1 < \delta \leq 2, \tag{4.1}$$

$$u(0) = 0, \quad u(1) = \lambda \int_0^1 h(r)u(r)\mathrm{d}r. \tag{4.2}$$

where D^{δ} is the Riemann–Liouville fractional derivative and f is a given function.

The boundary conditions (4.2) can be thought as a mechanism putted at the end point of an oscillator, which is characterized by theweighted function h and the parameter λ, that controls its displacement according to the feedback from devices measuring the displacements along different parts of the oscillator.

Our assumed problem will more complicated and general than the problems considered in chapter 3.

4.1 Green function

To get the expression for the Green's function of boundary value problem (4.1) and (4.2), we start by solving the following auxiliary problem:

$$D^{\delta}u(t)+\sigma(t)=0,\ 0<t<1,\ 1<\delta\leq 2, \tag{4.3}$$

$$u(0)=0, \quad u(1)=\lambda\int_0^1 h(r)u(r)\mathrm{d}r. \tag{4.4}$$

Lemma 4.1.1. *[1]*
Let $1<\delta\leq 2$. Suppose that $1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r\neq 0$. A function $u\in C[0,1]$ is a solution of the linear boundary value problem (4.3) and (4.4) if and only if it satisfies the integral equation

$$u(t)=\int_0^1 G(t,s)\sigma(s)\mathrm{d}s$$

where $G(t,s)$ is the Green's function given by

$$G(t,s)=G_1(t,s)+G_2(t,s)$$

with

$$G_1(t,s)=\begin{cases}\dfrac{t^{\delta-1}(1-s)^{\delta-1}-(t-s)^{\delta-1}}{\Gamma(\delta)}, & 0\leq s\leq t\leq 1;\\ \dfrac{t^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}, & 0\leq t\leq s\leq 1.\end{cases} \tag{4.5}$$

and

$$G_2(t,s)=\frac{\lambda t^{\delta-1}}{1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r}\int_0^1 h(r)G_1(r,s)\mathrm{d}r \tag{4.6}$$

Proof.
By lemma (2.2.3) we have that the u is a solution of the linear equation (4.3) if and only if it satisfies

$$u(t)=-\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)\mathrm{d}s+C_1t^{\delta-1}+C_2t^{\delta-2}$$

Condition $u(0)=0$ implies necessarily that $C_2=0$, so

$$u(t)=-\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)\mathrm{d}s+C_1t^{\delta-1}$$

and

$$u(1) = -\int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)\mathrm{d}s + C_1$$

Since $u(1) = \lambda\int_0^1 h(r)u(r)\mathrm{d}r$, we have that

$$\begin{aligned} u(1) &= \lambda\int_0^1 h(r)\left(-\int_0^r \frac{(r-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)\mathrm{d}s + C_1 r^{\delta-1}\right)\mathrm{d}r \\ &= \lambda C_1\int_0^1 h(r)r^{\delta-1}\mathrm{d}r - \frac{\lambda}{\Gamma(\delta)}\int_0^1 h(r)\int_0^1 (r-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r. \end{aligned}$$

From it we deduce

$$\begin{aligned} C_1\left(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r\right) = \frac{1}{\Gamma(\delta)}\Bigg(&\int_0^1 (1-s)^{\delta-1}\sigma(s)\mathrm{d}s - \lambda\int_0^1 h(r)\cdot \\ &\int_0^1 (r-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r\Bigg) \end{aligned}$$

Now, since $1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r \neq 0$, we have

$$\begin{aligned} C_1 = \frac{1}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\Bigg(&\int_0^1 (1-s)^{\delta-1}\sigma(s)\mathrm{d}s - \lambda\int_0^1 h(r)\cdot \\ &\int_0^r (r-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r\Bigg) \end{aligned}$$

Finally, we have the expression

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)\mathrm{d}s + \frac{t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 (1-s)^{\delta-1}\sigma(s)\mathrm{d}s \\ &- \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 h(r)\int_0^r (r-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r \\ = &-\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)\mathrm{d}s \\ &+ \frac{t^{\delta-1}(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r + \lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 (1-s)^{\delta-1}\sigma(s)\mathrm{d}s \\ &- \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 h(r)\int_0^r (r-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)\mathrm{d}s + \frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\sigma(s)\mathrm{d}s \\
&\quad + \frac{\lambda t^{\delta-1}\int_0^1 h(r)r^{\delta-1}\mathrm{d}r}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 (1-s)^{\delta-1}\sigma(s)\mathrm{d}s \\
&\quad - \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 h(r)\int_0^r (r-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r \\
&= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)\mathrm{d}s + \frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^t (1-s)^{\delta-1}\sigma(s)\mathrm{d}s + \frac{t^{\delta-1}}{\Gamma(\delta)}\int_t^1 (1-s)^{\delta-1}\sigma(s)\mathrm{d}s \\
&\quad + \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 h(r)r^{\delta-1}\mathrm{d}r\int_0^1 (1-s)^{\delta-1}\sigma(s)\mathrm{d}s \\
&\quad - \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 h(r)\int_0^r (r-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r \\
&= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)\mathrm{d}s + \frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^t (1-s)^{\delta-1}\sigma(s)\mathrm{d}s + \frac{t^{\delta-1}}{\Gamma(\delta)}\int_t^1 (1-s)^{\delta-1}\sigma(s)\mathrm{d}s \\
&\quad + \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 h(r)\int_0^1 r^{\delta-1}(1-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r \\
&\quad - \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 h(r)\int_0^r (r-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r \\
&= -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\sigma(s)\mathrm{d}s + \frac{t^{\delta-1}}{\Gamma(\delta)}\int_0^t (1-s)^{\delta-1}\sigma(s)\mathrm{d}s + \frac{t^{\delta-1}}{\Gamma(\delta)}\int_t^1 (1-s)^{\delta-1}\sigma(s)\mathrm{d}s \\
&\quad + \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 h(r)\int_0^r r^{\delta-1}(1-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r \\
&\quad - \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 h(r)\int_0^r (r-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r \\
&\quad + \frac{\lambda t^{\delta-1}}{\Gamma(\delta)(1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r)}\int_0^1 h(r)\int_r^1 r^{\delta-1}(1-s)^{\delta-1}\sigma(s)\mathrm{d}s\,\mathrm{d}r \\
&= \int_0^1 G_1(t,s)\sigma(s)\mathrm{d}s + \frac{\lambda t^{\delta-1}}{1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r}\int_0^1 h(r)\cdot\int_0^1 G_1(r,s)\sigma(s)\mathrm{d}s\,\mathrm{d}r \\
&= \int_0^1 G_1(t,s)\sigma(s)\mathrm{d}s + \frac{\lambda t^{\delta-1}}{1-\lambda\int_0^1 h(r)r^{\delta-1}\mathrm{d}r}\int_0^1 h(r)G_1(r,s)\sigma(s)\mathrm{d}s\,\mathrm{d}r.
\end{aligned}
$$

And on it

$$
u(t) = \int_0^1 G_1(t,s)\sigma(s)\mathrm{d}s + \int_0^1 G_2(t,s)\sigma(s)\,\mathrm{d}s
$$

$\square$

Lemma 4.1.0.3. *[1]*

The function $G_1(t,s)$ defined in Lemma (4.1.1) has the following properties:

1. *$G_1(t,s) \in \mathcal{C}([0,1]\times[0,1])$.*
2. *$G_1(t,s) > 0$ for $(t,s) \in (0,1)\times(0,1)$ and $G_1(0,s) = 0 = G_1(1,s)$ for $s \in [0,1]$.*
3. *$G_1(t,s) = G_1(1-s, 1-t)$, $\forall\, t,s \in [0,1]$.*

Proof.

1. We have

$$G_1(t,s) = \begin{cases} \dfrac{g_1(t,s) - g_2(t,s)}{\Gamma(\delta)}, & 0 \le s \le t \le 1; \\ \dfrac{g_1(t,s)}{\Gamma(\delta)}, & 0 \le t \le s \le 1. \end{cases}$$

with $g_1(t,s) = t^{\delta-1}(1-s)^{\delta-1}$ and $g_2(t,s) = (t-s)^{\delta-1}$.
Since $\Gamma(\delta)$, $g_1(t,s)$ and $g_2(t,s)$ are a continuous function in $[0,1]$ they follows that $G_1(t,s)$ is a continous function in $[0,1]\times[0,1]$.

2. We consider the first case $0 \le s \le t \le 1$ if $ts > s > 0$ implies that

$$t^{\delta-1}(1-s)^{\delta-1} > (t-s)^{\delta-1} > 0$$

and since $\Gamma(\delta) > 0$ we deduce that

$$G_1(t,s) > 0$$

and if $t = 1$ we have $t^{\delta-1}(1-s)^{\delta-1} = (t-s)^{\delta-1}$.
So

$$G_1(t,s) = 0$$

On the other hand, if $0 < t \le s < 1$ we have $0 < ts < t$ so
$t^{\delta-1}(1-s)^{\delta-1} > 0$ and on it $G_1(t,s) > 0$
and if $t = 0$ so $G_1(t,s) = 0$.
From it we conclude that

$$G_1(t,s) > 0 \;\; \forall\, t,s \in (0,1) \;\text{ and }\; G_1(0,s) = G_1(1,s) = 0, \forall\, s \in [0,1]$$

3. If $0 < s \le t < 1$ we have

$$G_1(t,s) = \frac{t^{\delta-1}(1-s)^{\delta-1} - (t-s)^{\delta-1}}{\Gamma(\delta)}$$

Let's put $t = 1-s$ then $s = 1-t$, so

$$\begin{aligned} G_1(1-s,1-t) &= \frac{(1-s)^{\delta-1}\left(1-(1-t)\right)^{\delta-1} - \left((1-s)-(1-t)\right)^{\delta-1}}{\Gamma(\delta)} \\ &= \frac{(1-s)^{\delta-1}t^{\delta-1} - (t-s)^{\delta-1}}{\Gamma(\delta)} \\ &= G_1(t,s). \end{aligned}$$

□

In next result, we deduce two inequalities that, as we will see, will be fundamental to ensure the existence of the solutions of the nonlinear problem (4.1) and (4.2).

Lemma 4.1.0.4. *[1]*
Let the function $G_1(t,s)$ be defined in Lemma 2.3 and fix $t_0 \in (0,1)$, then G_1 satisfies the following inequalities:

$$G_1(t,s) \le \frac{s^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}, \quad \forall\, t \in [0,1] \tag{4.7}$$

and

$$s^{\delta-1}(1-s)^{\delta-1}K(t,t_0) \ge G_1(t,s), \quad \forall t \in [0,1], \ s \in [t_0,1] \tag{4.8}$$

with

$$K(t,t_0) = \begin{cases} \dfrac{t^{\delta-1}}{\Gamma(\delta)}, & \text{if } 0 \le t \le t_0 < 1, \\ \min\left\{ \dfrac{t^{\delta-1}}{\Gamma(\delta)}, \dfrac{t^{\delta-1}(1-t_0)^{\delta-1}(t-t_0)^{\delta-1}}{\Gamma(\delta)t^{\delta-1}(1-t_0)^{\delta-1}} \right\}, & \text{if } 0 < t_0 < t \le 1. \end{cases} \tag{4.9}$$

Proof.
For $s > t$,

$$\frac{\partial G_1}{\partial t}(t,s) = \frac{\delta-1}{\Gamma(\delta)}(1-s)^{\delta-1}t^{\delta-2} > 0$$

For $s < t$, since $1 < \delta \leq 2$, we have

$$\begin{aligned}\frac{\partial G_1}{\partial t}(t,s) &= \frac{\delta-1}{\Gamma(\delta)}\left((1-s)^{\delta-1}t^{\delta-2} - (t-s)^{\delta-2}\right)\\ &\leq \frac{\delta-1}{\Gamma(\delta)}\left(t^{\delta-2} - (t-s)^{\delta-2}\right)\end{aligned}$$

hence

$$\frac{\partial G_1}{\partial t}(t,s) < 0$$

As a consequence, it is fulfilled that

$$G_1(t,s) \leq G_1(s,s) = \frac{s^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}, \qquad \forall t,s \in [0,1]$$

and inequality (4.7) holds.
On the other hand, we have that

$$\frac{\partial G_1}{\partial s}(t,s) = \frac{\left((t-s)^{\delta-2} - (1-s)^{\delta-2}t^{\delta-1}\right)(\delta-1)}{\Gamma(\delta)}0, \text{ for } 0 \leq s < t \leq 1$$

and

$$\frac{\partial G_1}{\partial s}(t,s) = \frac{-\left((1-s)^{\delta-2}t^{\delta-1}\right)(\delta-1)}{\Gamma(\delta)}, \text{ for } 0 \leq t < s \leq 1.$$

Then, we have

$$(1-s)^{\delta-2} > t^{\delta-1}(1-s)^{\delta-2} > 0 \quad \text{and} \quad (t-s)^{\delta-2} > (1-s)^{\delta-2}$$

So

$$(t-s)^{\delta-2} - t^{\delta-1}(1-s)^{\delta-2} > 0$$

As a consequence, we deduce that

$$\frac{\partial G_1}{\partial s}(t,s) > 0, \quad \text{for } 0 \leq s < t \leq 1$$

$$\frac{\partial G_1}{\partial s}(t,s) < 0, \quad \text{for } 0 \leq t < s \leq 1$$

Now, we introduce the following function:

$$F_1(t,s) = \frac{G_1(t,s)}{s^{\delta-1}(1-s)^{\delta-1}}, \qquad (t,s) \in [0,1] \times (0,1)$$

as a direct consequence of previous arguments, we deduce that

$$\frac{\partial F_1}{\partial t}(t,s) < 0 \text{ for } 0 \le s < t \le 1,$$

and

$$\frac{\partial F_1}{\partial t}(t,s) > 0 \text{ for } 0 \le t < s \le 1.$$

As a consequence, we have that

$$\frac{G_1(t,s)}{s^{\delta-1}(1-s)^{\delta-1}} \le \frac{G_1(s,s)}{s^{\delta-1}(1-s)^{\delta-1}} = \frac{1}{\Gamma(\delta)}$$

By the other hand,

$$\frac{\partial F_1}{\partial s}(t,s) = \begin{cases} -\dfrac{t^{\delta-1}}{\Gamma(\delta-1)s^{\delta}}, & 0 \le t < s \le 1, \\ \dfrac{(\delta-1)\left(t\left(s^2-2st+t\right)(t-s)^{\delta}-(t-s)^2 t^{\delta}(1-s)^{\delta}\right)}{t\,\Gamma(\delta)(t-s)^2(1-s)^{\delta}s^{\delta}}, & 0 \le s < t \le 1. \end{cases}$$

As a direct consequence, we deduce that

$$\frac{\partial F_1}{\partial s}(t,s) < 0 \text{ for } 0 \le t < s \le 1$$

On the other hand, for the case $0 \le s < t \le 1$, we have that

$$\frac{\partial F_1}{\partial s}(t,s) = \frac{(\delta-1)\left(h_1(t,s,\delta)-h_2(t,s)\right)}{t\,\Gamma(\delta)(t-s)^2(1-s)^{\delta}s^{\delta}}$$

with $h_1(t,s,\delta) = (t-s)^{2-\delta}t^{\delta-1}(1-s)^{\delta}$ and $h_2(t,s) = s^2-2st+t$.

So, we have that $\dfrac{\partial F_1}{\partial s}(t,s) > 0$ if and only if $h_1(t,s,\delta) > h_2(t,s)$.

Now, since

$$\frac{\partial h_1}{\partial \delta}(t,s,\delta) = (1-s)^{\delta}t^{\delta-1}(t-s)^{2-\delta}\log\left(\frac{t-ts}{t-s}\right)$$

we have that h_1 is strictly increasing on the δ interval $[1,2]$ for any $0 \le s < t \le 1$ given.

Thus, since $h_2(t,s) - h_1(t,s,2) = (1-t)s^2 > 0$, we conclude that

$$\frac{\partial F_1}{\partial s}(t,s) > 0 \text{ for all } 0 < s < t < 1$$

So, for any $t_0 \in (0,1)$ fixed, $\forall t \in [0,1],\ \ s \in [t_0, 1]$, we have that

$$\begin{aligned}\frac{G_1(t,s)}{s^{\delta-1}(1-s)^{\delta-1}} &\geq \min\left\{\lim_{s\to 1^-}\frac{G_1(t,s)}{s(1-s)^{\delta-1}}, \frac{G_1(t,t0)}{t_0^{\delta-1}(1-t_0)^{\delta-1}}\right\}\\ &= \min\left\{\frac{t^{\delta-1}}{\Gamma(\delta)}, \frac{G_1(t,t0)}{t_0^{\delta-1}(1-t_0)^{\delta-1}}\right\}\\ &= K(t,t_0).\end{aligned}$$

and the result is concluded. □

By virtue of this lemma, we can give now the main result of this section.

Lemma 4.1.0.5. *[1]*

Let $t_0 \in (0,1)$ be fixed and h introduced at boundary condition (3.2). Denote by $A = \int_0^1 h(r)r^{\delta-1}\mathrm{d}r$, $B = \int_0^1 h(r)\mathrm{d}r$ and $C_0 = \int_{t_0}^1 K(t,t_0)h(r)\mathrm{d}r$. Assume that $h \geq 0$ on $[0,1]$ and $1-\lambda A > 0$. Then the Green's function $G(t,s)$ defined in Lemma (4.1.1) satisfies the inequalities

$$\frac{\lambda C_0 t^{\delta-1}}{1-\lambda A}s^{\delta-1}(1-s)^{\delta-1} \leq G(t,s) \leq \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda B}{1-\lambda A}\right)s^{\delta-1}(1-s)^{\delta-1}, \quad \forall t,s \in [0,1] \tag{4.10}$$

Proof.

From the definition of G, the inequality (4.7) and the fact that $1 < \delta \leq 2$, we have the following inequalities for all $t,s \in [0,1]$:

$$\begin{aligned}G(t,s) &= G_1(t,s) + G_2(t,s)\\ &= G_1(t,s) + \frac{\lambda t^{\delta-1}}{1-\lambda A}\int_0^1 h(r)G_1(t,s)\mathrm{d}r\\ &\leq \frac{1}{\Gamma(\delta)}s^{\delta-1}(1-s)^{\delta-1} + \frac{\lambda t^{\delta-1}}{1-\lambda A}\int_0^1 \frac{1}{\Gamma(\delta)}s^{\delta-1}(1-s)^{\delta-1}h(r)\mathrm{d}r\\ &\leq \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda t^{\delta-1}}{1-\lambda A}\int_0^1 h(r)\mathrm{d}r\right)s^{\delta-1}(1-s)^{\delta-1}\\ &\leq \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda t^{\delta-1}B}{1-\lambda A}\right)s^{\delta-1}(1-s)^{\delta-1}\\ &\leq \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda B}{1-\lambda A}\right)s^{\delta-1}(1-s)^{\delta-1}.\end{aligned}$$

On the other hand, by Lemma (4.1.0.3), (4.2) and (4.8), we have for all $t, s \in [0,1]$:

$$\begin{aligned} G(t,s) &= G_1(t,s) + G_2(t,s) \\ &\geq G_2(t,s) \\ &\geq \frac{\lambda t^{\delta-1}}{1-\lambda A}\int_0^1 h(r)G_1(t,s)\mathrm{d}r \\ &\geq \frac{\lambda t^{\delta-1}}{1-\lambda A}\int_0^1 h(r)s^{\delta-1}(1-s)^{\delta-1}K(t,t_0)\mathrm{d}r \\ &\geq \frac{\lambda t^{\delta-1}}{1-\lambda A}C_0 s^{\delta-1}(1-s)^{\delta-1}, \end{aligned}$$

as we want to prove. □

As a direct consequence, we deduce the following Corollary.

Corollary 4.1.0.2.
If $h \geq 0$ on $[0,1]$ and $1-\lambda A > 0$ then the Green's function $G(t,s)$ defined in Lemma (4.1.1) satisfies the inequalities

$$\frac{\lambda\, t\, C_0}{1-\lambda A}s^{\delta-1}(1-s)^{\delta-1} \leq t^{2-\delta}G(t,s) \leq \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda B}{1-\lambda A}\right)s^{\delta-1}(1-s)^{\delta-1}, \quad \forall t,s \in [0,1]$$

4.2 Existence of positive solutions

In this section we study the existence of positive solution for Riemann fractianal boundary value problem with integral boundary condition and paramater dependance (4.1) and (4.2) with help of Guo-Krasnoselskii fixed point theorem (1.3.1.1).

Now for any $u : (0,1] \to \mathbb{R}$, we define function $\overline{u} : [0,1] \to \mathbb{R}$ as follows:

$$\overline{u}(t) = \begin{cases} t^{2-\delta}u(t) & \text{if } t \in (0,1], \\ \lim\limits_{t\to 0^+} t^{2-\delta}u(t) & \text{if } t = 0, \end{cases}$$

provided that such limit exists.
Consider the Banach space

$$E = C_\delta[0,1] := \{\overline{u} : [0,1] \to \mathbb{R},\ \text{is a continous functin in } [0,1]\}$$

endowed with the maximum norm $\|u\| = \max\limits_{0\leq t\leq 1}|\overline{u}(t)|$.

Lemma 4.2.1.
Let $\delta \in [1,2]$ and $\lambda \neq \alpha$. If $u \in E$ a solution of the fractional problem (4.3) and (4.4) then it satisfies

$$\min_{t\in[0,1]} \overline{u}(t) \geqslant t^{2-\delta} P(t,t_0)\|u\|, \quad \forall t \in [0,1] \tag{4.11}$$

with $P(t,t_0) = \dfrac{\left(\Gamma(\delta)\dfrac{\lambda t^{\delta-1}}{1-\lambda A} C_0\right)}{\left(1+\dfrac{\lambda B}{1-\lambda A}\right)}$

Proof.

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\sigma(s)\mathrm{d}s \\ &\leq \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda B}{1-\lambda A}\right)\int_0^1 s^{\delta-1}(1-s)^{\delta-1}\sigma(s)\mathrm{d}s \end{aligned}$$

and on it

$$\|u\| \leq \frac{1}{\Gamma(\delta)}\left(1+\frac{\lambda B}{1-\lambda A}\right)\int_0^1 s^{\delta-1}(1-s)^{\delta-1}\sigma(s)\mathrm{d}s$$

On the other hand, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\sigma(s)\mathrm{d}s \\ &\geq \frac{\lambda t^{\delta-1}}{1-\lambda A} C_0 \int_0^1 s^{\delta-1}(1-s)^{\delta-1}\sigma(s)\mathrm{d}s \\ &\geq \frac{\left(\Gamma(\delta)\dfrac{\lambda t^{\delta-1}}{1-\lambda A}C_0\right)}{\left(1+\dfrac{\lambda B}{1-\lambda A}\right)}\|u\| \end{aligned}$$

So

$$\min_{t\in[0,1]} u(t) \geq P(t,t_0)\|u\|$$

and we have $\overline{u}(t) = t^{2-\delta}u(t)$.

Then

$$\min_{t\in[0,1]} \overline{u}(t) = \min_{t\in[0,1]} t^{2-\delta}u(t) \geq t^{2-\delta}P(t,t_0)\|u\|$$

□

Notice that, provided that $h > 0$ on $[0,1]$ and $1 - \lambda A > 0$, we deduce from (4.10) that $0 \leq P(t, t_0) \leq 1$ for all $t \in [0,1]$ and $t_0 \in (0,1)$.

Define now the cone $P_0 \subset E$ by

$$P_0 = \{u \in E, \overline{u}(t) \geq t^{2-\delta} P(t, t_0) \|u\|, \forall t \in [0,1]\}$$

Now,we assume the following hypothesis on the nonlinear part of the equation:

(H_1) Function , $f : [0,1] \times \mathbb{R} \to [0, \infty)$ is continuous. So, we define the operator $T : P_0 \to E$ by

$$(Tu)(t) = \int_0^1 G(t,s) f(s, \overline{u}(s)) \mathrm{d}s, \quad t \in [0,1] \tag{4.12}$$

Lemma 4.2.2. *[1]*
$T : P_0 \to P_0$ is completely continuous.

Proof.
Let us prove in first that $T(P_0) \subset P_0$. Notice from the definition of T and Corollary (4.1.0.2) that for $u \in P_0$, $Tu(t) \geq 0$ for all $t \in [0,1]$ and

$$\begin{aligned}
t^{2-\delta}(Tu) &= \int_0^1 t^{2\delta} G(t,s) f(s, \overline{u}(s)) \mathrm{d}s \\
&\geq \int_0^1 t^{2-\delta} \frac{\lambda t^{\delta-1}}{1-\lambda A} C_0 s^{\delta-1} (1-s)^{\delta-1} f(s, \overline{u}(s)) \mathrm{d}s \\
&= t^{2-\delta} \frac{\Gamma(\delta) \dfrac{\lambda t^{\delta-1}}{1-\lambda A} C_0}{\left(1 + \dfrac{\lambda B}{1-\lambda A}\right)} \int_0^1 \frac{\left(1 + \dfrac{\lambda B}{1-\lambda A}\right)}{\Gamma(\delta)} s^{\delta-1} (1-s)^{\delta-1} f(s, \overline{u}(s)) \mathrm{d}s \\
&\geq t^{2-\delta} P(t, t_0) \int_0^1 \max_{0 \leq t \leq 1} \left\{ t^{2-\delta} G(t,s) \right\} f(s, \overline{u}(s)) \mathrm{d}s \\
&\geq t^{2-\delta} P(t, t_0) \max_{0 \leq t \leq 1} \left\{ \int_0^1 t^{2-\delta} G(t,s) f(s, \overline{u}(s)) \mathrm{d}s \right\} \\
&= t^{2-\delta} P(t, t_0) \|Tu\|.
\end{aligned}$$

Thus, $T(P_0) \subset P_0$.

In addition, since f is a continuous function it folows that T is a continuous operator.

Next, we show that T is uniformly bounded. Let $D \subset P$ be a bounded set, i.e. ther exists a constant $L > 0$ such that $\|u\| \leq L$, for all $u \in D$. Set

$$M = \max_{0 \leq s \leq 1, 0 \leq u \leq L} \{f(s, \overline{u}(s))\}$$

and denoting the Beta function as

$$\mathcal{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} \mathrm{d}t$$

Then, from Lemma (4.1.0.5), and for all $u \in D$, we have

$$\begin{aligned}
|t^{2-\delta}Tu(t)| &= \left| \int_0^1 t^{2-\delta} G(t,s) f(s, \overline{u}(s)) \mathrm{d}s \right| \\
&\leq \frac{M}{\Gamma(\delta)} \left(1 + \frac{\lambda B}{1 - \lambda A}\right) \int_0^1 s^{\delta-1}(1-s)^{\delta-1} \mathrm{d}s \\
&= \frac{M}{\Gamma(\delta)} \left(1 + \frac{\lambda B}{1 - \lambda A}\right) \mathcal{B}(\delta, \delta) \\
&= M \left(1 + \frac{\lambda B}{1 - \lambda A}\right) \frac{\Gamma(\delta)}{\Gamma(2\delta)}.
\end{aligned}$$

Hence, $T(D)$ is bounded.
Finally, we show that T is equicontinuous, as follows.
For all $\epsilon > 0$ and for each $u \in P$, let $t_1, t_2 \in [0,1]$, be such that $t_1 < t_2$.
We have to prove that there is $\eta > 0$ valid for all $u \in D$, such that
$|t^{2-\delta}Tu(t_2) - t^{2-\delta}Tu(t_1)| < \epsilon$, when $t_2 - t_1 < \eta$.

One has

$$\begin{aligned}
|t^{2-\delta}Tu(t_2) - t^{2-\delta}Tu(t_1)| &= \left| \int_0^1 \left[t_2^{2-\delta} G(t_2, s) - t_1^{2-\delta} G(t_1, s)\right] f(s, \overline{u}(s)) \mathrm{d}s \right| \\
&\leq \int_0^1 \left|t_2^{2-\delta} G(t_2, s) - t_1^{2-\delta} G(t_1, s)\right| f(s, \overline{u}(s)) \mathrm{d}s \\
&\leq M \int_0^1 \left|t_2^{2-\delta} G(t_2, s) - t_1^{2-\delta} G(t_1, s)\right| \mathrm{d}s.
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_0^1 \left|t_2^{2-\delta} G(t_2, s) - t_1^{2-\delta} G(t_1, s)\right| \mathrm{d}s \quad &\leq \int_0^1 \left|t_2^{2-\delta} G_1(t_2, s) - t_1^{2-\delta} G_1(t_1, s)\right| \mathrm{d}s \\
&+ \int_0^1 \left|t_2^{2-\delta} G_2(t_2, s) - t_1^{2-\delta} G_2(t_1, s)\right| \mathrm{d}s
\end{aligned}$$

From the expression of G_1, we get

$$
\begin{aligned}
&\int_0^1 \left|t_2^{2-\delta}G_1(t_2,s)-t_1^{2-\delta}G_1(t_1,s)\right|\mathrm{d}s=\int_0^{t_1}\left|t_2^{2-\delta}G_1(t_2,s)-t_1^{2-\delta}G_1(t_1,s)\right|\mathrm{d}s\\
&\quad+\int_{t_1}^{t_2}\left|t_2^{2-\delta}G_1(t_2,s)-t_1^{2-\delta}G_1(t_1,s)\right|\mathrm{d}s+\int_{t_2}^{1}\left|t_2^{2-\delta}G_1(t_2,s)-t_1^{2-\delta}G_1(t_1,s)\right|\mathrm{d}s\\
&=\int_0^{t_1}\left|\frac{t_2(1-s)^{\delta-1}-t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)}-\frac{t_1(1-s)^{\delta-1}-t_1^{2-\delta}(t_1-s)^{\delta-1}}{\Gamma(\delta)}\right|\mathrm{d}s\\
&\quad+\int_{t_1}^{t_2}\left|\frac{t_2(1-s)^{\delta-1}-t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)}-\frac{t_1(1-s)^{\delta-1}}{\Gamma(\delta)}\right|\mathrm{d}s\\
&\quad+\int_{t_2}^{1}\left|\frac{t_2(1-s)^{\delta-1}}{\Gamma(\delta)}-\frac{t_1(1-s)^{\delta-1}}{\Gamma(\delta)}\right|\mathrm{d}s\\
&=\int_0^{t_1}\left|\frac{(t_2-t_1)(1-s)^{\delta-1}+t_1^{2-\delta}(t_1-s)^{\delta-1}-t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)}\right|\mathrm{d}s\\
&\quad+\int_{t_1}^{t_2}\left|\frac{(t_2-t_1)(1-s)^{\delta-1}-t_2^{2-\delta}(t_2-s)^{\delta-1}}{\Gamma(\delta)}\right|\mathrm{d}s+\int_{t_2}^{1}\left|\frac{(t_2-t_1)(1-s)^{\delta-1}}{\Gamma(\delta)}\right|\mathrm{d}s\\
&=\left|\frac{-(t_2-t_1)\left[(1-s)^{\delta}\right]_0^{t_1}-t_1^{2-\delta}\left[t_1-s)^{\delta}\right]_0^{t_1}+t_2^{2-\delta}\left[(t_2-s)^{\delta}\right]_0^{t_1}}{\delta\,\Gamma(\delta)}\right|
\end{aligned}
$$

$$
\begin{aligned}
&\quad+\left|\frac{-(t_2-t_1)\left[(1-s)^{\delta}\right]_{t_1}^{t_2}+t_2^{2-\delta}\left[t_2-s)^{\delta}\right]_{t_1}^{t_2}}{\delta\,\Gamma(\delta)}\right|+\left|\frac{-(t_2-t_1)\left[(1-s)^{\delta}\right]_{t_2}^{1}}{\delta\,\Gamma(\delta)}\right|\\
&=\left|\frac{-(t_2-t_1)\left[(1-t_1)^{\delta}-1\right]+t_2^{2-\delta}(t_2-t_1)^{\delta}+t_1^2-t_2^2}{\delta\,\Gamma(\delta)}\right|\\
&\quad+\left|\frac{-(t_2-t_1)\left[(1-t_2)^{\delta}-(1-t_1)^{\delta}\right]-t_2^{2-\delta}\left[(t_2-t_1)^{\delta}\right]}{\delta\,\Gamma(\delta)}\right|+\frac{(t_2-t_1)(1-t_2)^{\delta}}{\delta\,\Gamma(\delta)}\\
&=\left|\frac{t_2^{2-\delta}(t_2-t_1)^{\delta}+(t_1-t_2)\left[(1-t_1)^{\delta}+t_1+t_2-1\right]}{\Gamma(\delta+1)}\right|\\
&\quad+\left|\frac{(t_2-t_1)\left[(1-t_1)^{\delta}-(1-t_2)^{\delta}\right]-t_2^{2-\delta}\left[(t_2-t_1)^{\delta}\right]}{\Gamma(\delta+1)}\right|+\frac{(t_2-t_1)(1-t_2)^{\delta}}{\Gamma(\delta+1)}.
\end{aligned}
$$

So, we have that there is $\eta>0$ valid for all $u\in D$, such that
$|t^{2-\delta}Tu(t_2)-t^{2-\delta}Tu(t_1)|<\epsilon$, when $t_2-t_1<\eta$.
Now, denote by $H(s)=\int_0^1 h(r)G_1(r,s)\mathrm{d}s$ and $h^*=\max\limits_{t\in[0,1]}\{h(t)\}$.

Then, from the expression of $G_2(t,s)$ and the inequality (4.7), using that

$$\begin{aligned}\int_0^1 H(s)\mathrm{d}s &\leq h^* \int_0^1\int_0^1 G_1(r,s)\mathrm{d}r\mathrm{d}s \\ &\leq h^* \int_0^1\int_0^1 \frac{r^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}\mathrm{d}r\mathrm{d}s \\ &\leq h^* \int_0^1 \frac{s^{\delta-1}(1-s)^{\delta-1}}{\Gamma(\delta)}\mathrm{d}s, \qquad \forall\, r\leq s \\ &= h^* \frac{1}{\Gamma(\delta)}\mathcal{B}(\delta,\delta) \\ &= h^* \frac{1}{\Gamma(\delta)}\frac{\Gamma(\delta)\Gamma(\delta)}{\Gamma(2\delta)} \\ &= h^* \frac{\Gamma(\delta)}{\Gamma(2\delta)}\end{aligned}$$

we get

$$\begin{aligned}\int_0^1 |t_2^{2-\delta}G_2(t_2,s) - t_1^{2-\delta}G_2(t_1,s)|\mathrm{d}s &= \int_0^1 \frac{\lambda(t_2-t_1)}{1-\lambda A}H(s)\mathrm{d}s \\ &\leq \frac{\Gamma(\delta)}{\Gamma(2\delta)}\frac{\lambda h^*}{1-\lambda A}(t_2-t_1).\end{aligned}$$

Thus, we obtain theat the set $T(D)$ is equicontinuous in E.
Now, from the Arzelà-Ascoli Theorem we conclude that $T: P\to P$ is completely continuous operator. □

Now, we are in position to prove the existence of positive solutions of the nonlinear boundary value problem. For this, we use the known Guo–Krasnoselskii fixed point theorem (1.3.1.1).
Let introduce some notations:

$$f_0 = \lim_{\overline{u}\to 0^+}\left\{\min_{t\in[0,1]}\left\{\frac{f(t,\overline{u})}{\overline{u}}\right\}\right\} \quad \text{and} \quad f_\infty = \lim_{\overline{u}\to \infty}\left\{\min_{t\in[0,1]}\left\{\frac{f(t,\overline{u})}{\overline{u}}\right\}\right\},$$

$$f^0 = \lim_{\overline{u}\to 0^+}\left\{\max_{t\in[0,1]}\left\{\frac{f(t,\overline{u})}{\overline{u}}\right\}\right\} \quad \text{and} \quad f^\infty = \lim_{\overline{u}\to \infty}\left\{\max_{t\in[0,1]}\left\{\frac{f(t,\overline{u})}{\overline{u}}\right\}\right\}.$$

Theorem 4.2.1. *[1]*
Assume that $h\geq 0$ on $[0,1]$, $1-\lambda A>0$ and (H_1) hold coupled with one of the following conditions:

1. *Sublinear case:* $f_0 = \infty$ *and* $f^\infty = 0$.

2. *Superlinear case:* $f^0 = 0$ *and* $f_\infty = \infty$.
   Then Problem (4.1) and (4.2) has at least one positive solution.

Proof.

1. Consider the first situation (1):
   Since $f_0 = \infty$, then there exists a constant $R_1 > 0$ such that $f(t, \overline{u}) \geq r_1 \overline{u}$ for all $0 < \overline{u} \leq R_1$ and $t \in [0, 1]$, where $r_1 > 0$ is defined as

$$r_1 = \frac{(1-\lambda A)(1-\lambda A+\lambda B)\Gamma(1+2\delta)}{\lambda^2\, C_0^2\, \delta\, \Gamma^3(\delta)} \tag{4.13}$$

   Take $u \in P_0$ such that $\|u\| = R_1$. Then from expression (4.13), we get

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta} G(t,s) f(s, \overline{u}(s)) \mathrm{d}s \right\} \\
&\geq r_1 \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta} G(t,s) \overline{u}(s) \mathrm{d}s \right\} \\
&\geq r_1 \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta} G(t,s)\, s^{2-\delta}\, P(s,t_0) \|u\| \mathrm{d}s \right\}
\end{aligned}$$

$$\begin{aligned}
&\geq r_1 \|u\| \max_{t\in[0,1]} \left\{ t \int_0^1 \frac{\lambda}{1-\lambda A}\, C_0\, s\, (1-s)^{\delta-1} P(s,t_0) \mathrm{d}s \right\} \\
&= r_1 \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)(1-\lambda A+\lambda B)} \int_0^1 s^\delta (1-s)^{\delta-1} \mathrm{d}s \\
&= r_1 \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)(1-\lambda A+\lambda B)} \mathcal{B}(\delta+1, \delta) \\
&= r_1 \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)(1-\lambda A+\lambda B)} \frac{\Gamma(\delta+1)\Gamma(\delta)}{\Gamma(1+2\delta)} \\
&= r_1 \|u\| \cdot \frac{\lambda^2\, C_0^2\, \delta\, \Gamma^3(\delta)}{(1-\lambda A)(1-\lambda A+\lambda B)\, \Gamma(1+2\delta)} \\
&= \|u\|.
\end{aligned}$$

   On the other hand, since $f(t, \cdot)$ is a continuous function on $[0, \infty)$, we define a new function:

$$\widehat{f}(t, \overline{u}) = \max_{y\in[0,\overline{u}]} \{f(t, y)\}$$

Cleary $\widehat{f}(t,\cdot)$ is nondecreasing on $[0,\infty)$. Moreover, since $f^{\infty}=0$ it is obvious that

$$\lim_{\overline{u}\to\infty}\left\{\max_{t\in[0,1]}\frac{\widehat{f}(t,\overline{u})}{\overline{u}}\right\}=0$$

Choos now $r_2>0$ defined as the following constant:

$$r_2=\frac{(1\lambda A)\,\Gamma(2\delta)}{(1-\lambda A+\lambda B)\,\Gamma(\delta)} \tag{4.14}$$

Therefore, there exists a constant $R_2>R_1>0$ such that $\widehat{f}(t,\overline{u})\le r_2\,\overline{u}$ for all $\overline{u}\ge R_2$ and $t\in[0,1]$. Consider $u\in P_0$ such that $\|u\|=R_2$. Then from the definition of $\widehat{f}$, inequality (4.14) and Lemma (4.1.0.4), we attain at the following inequalities:

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}\left\{\int_0^1 t^{2-\delta}\,G(t,s)f(s,\overline{u}(s))\mathrm{d}s\right\}\\
&\le \max_{t\in[0,1]}\left\{\int_0^1 t^{2-\delta}\,G(t,s)\widehat{f}(s,\|u\|)\mathrm{d}s\right\}\\
&\le r_2\|u\|\max_{t\in[0,1]}\left\{\int_0^1 t^{2-\delta}\,G(t,s)\mathrm{d}s\right\}
\end{aligned}$$

$$\begin{aligned}
&\le r_2\|u\|\frac{(1-\lambda A+\lambda B)}{(1-\lambda A)\,\Gamma(\delta)}\int_0^1 s^{\delta-1}(1-s)^{\delta-1}\mathrm{d}s\\
&= r_2\|u\|\frac{(1-\lambda A+\lambda B)}{(1-\lambda A)\,\Gamma(\delta)}\mathcal{B}(\delta,\delta)\\
&= r_2\|u\|\frac{(1-\lambda A+\lambda B)}{(1-\lambda A)\,\Gamma(\delta)}\frac{\Gamma^2(\delta)}{\Gamma(2\delta)}\\
&= r_2\|u\|\frac{(1-\lambda A+\lambda B)\,\Gamma(\delta)}{(1-\lambda A)\,\Gamma(2\delta)}\\
&= \|u\|.
\end{aligned}$$

Thus, by the first part of Guo–Krasnoselskii fixed point theorem (1.3.1.1), we conclude that the problem (4.1) and (4.2) has at least one positive solution u such that

$$R_1\le\|u\|\le R_2$$

2. Consider now the second case (2)
   Let $r_2 > 0$ be chosen as in Eq. (4.14). Since $f_0 = 0$, there exists a constant $\tau_1 > 0$ such that $f(t,\overline{u}) \leq r_2\,\overline{u}$ for $0 \leq \overline{u} \leq \tau_1$ and $t \in [0,1]$.

   Take $u \in P_0$ such that $\|u\| = \tau_1$. Then, arguing as in the previous case, we have

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]}\left\{\int_0^1 t^{2-\delta}G(t,s)f(s,\overline{u}(s))\mathrm{d}s\right\}\\ &\leq r_2\|u\|\frac{(1-\lambda A+\lambda B)\,\Gamma(\delta)}{(1-\lambda A)\,\Gamma(2\delta)}\\ &= \|u\|.\end{aligned}$$

   Now, by denoting the incomplete beta function as

$$\mathcal{B}_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\mathrm{d}t$$

   for any fixed $t_1 \in (0,1)$, we define $r_3 > 0$ as follows:

$$r_3 = \frac{(1-\lambda A)(1-\lambda A+\lambda B)}{\lambda^2\,C_0^2\,\Gamma(\delta)}\left(\frac{\sqrt{\pi}\,\Gamma(\delta)}{\Gamma\left(\delta+\dfrac{1}{2}\right)\,4^\delta} - \mathcal{B}_{t_1}(\delta+1,\delta)\right)^{-1} \tag{4.15}$$

   The fact that $f_\infty = \infty$ ensures that there exists a constant $\tau_2 > \tau_1 > 0$ such that $f(t,\overline{u}) \geq r_3\overline{u}$ for all $\overline{u} \geq \tau_2$ and $t \in [0,1]$.
   By the definition of $P(t,t_0)$ is clear that

$$P_1 = \min_{t\in[t_1,1]}\left\{t^{2-\delta}\,P(t,t_0)\right\} > 0$$

   Let now $u \in P_0$ be such that $\|u\| = \tau_2/P_1$. As consequence, since $u \in P_0$, the following inequality holds:

$$\overline{u}(t) \geq t^{2-\delta}\,P(t,t_0)\,\|u\| \geq P_1\,\|u\| = \tau_2, \quad \forall\, t \in [t_1,1]$$

So, condition (2) gives us the following properties:

$$
\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} \left\{ \int_0^1 t^{2-\delta} G(t,s) f(s,\overline{u}(s)) \mathrm{d}s \right\} \\
&\geq \max_{t\in[0,1]} \left\{ \int_{t_1}^1 t^{2-\delta} G(t,s) f(s,\overline{u}(s)) \mathrm{d}s \right\} \\
&\geq r_3 \max_{t\in[0,1]} \left\{ \int_{t_1}^1 t^{2-\delta}\, G(t,s)\, \overline{u}(s) \mathrm{d}s \right\} \\
&\geq r_3\, \|u\| \max_{t\in[0,1]} \left\{ \int_{t_1}^1 t^{2-\delta}\, G(t,s)\, s^{2-\delta}\, P(s,t_0) \mathrm{d}s \right\} \\
&\geq r_3\, \|u\| \max_{t\in[0,1]} \left\{ t \int_{t_1}^1 \frac{\lambda}{1-\lambda A}\, C_0\, s(1-s)^{\delta-1}\, P(s,t_0) \mathrm{d}s \right\} \\
&= r_3\, \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)(1-\lambda A+\lambda B)} \int_{t_1}^1 s^{\delta}(1-s)^{\delta-1} \mathrm{d}s \\
&= r_3\, \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)(1-\lambda A+\lambda B)} \left(\int_0^1 s^{\delta}(1-s)^{\delta-1} \mathrm{d}s - \int_0^{t_1} s^{\delta}(1-s)^{\delta-1} \mathrm{d}s \right) \\
&= r_3\, \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)(1-\lambda A+\lambda B)} \left(\mathcal{B}(\delta+1,\delta) - \mathcal{B}_{t_1}(\delta+1,\delta) \right) \\
&= r_3\, \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)(1-\lambda A+\lambda B)} \left(\frac{\Gamma(\delta+1)\Gamma(\delta)}{\Gamma(2\delta+1)} - \mathcal{B}_{t_1}(\delta+1,\delta) \right) \\
&= r_3\, \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)(1-\lambda A+\lambda B)} \left(\frac{\Gamma(\delta)^2}{2\Gamma(2\delta)} - \mathcal{B}_{t_1}(\delta+1,\delta) \right) \\
&= r_3\, \|u\| \frac{\lambda^2\, C_0^2\, \Gamma(\delta)}{(1-\lambda A)(1-\lambda A+\lambda B)} \left(\frac{\Gamma(\delta)\, \sqrt{\pi}}{4^{\delta}\, \Gamma\left(\delta+\frac{1}{2}\right)} - \mathcal{B}_{t_1}(\delta+1,\delta) \right) \\
&= \|u\|.
\end{aligned}
$$

Because according to the properties of gamma function we have

$$
\Gamma(2\delta) = \frac{\Gamma(\delta)\, \Gamma\left(\delta+\frac{1}{2}\right)}{2^{1-2\delta}\, \Gamma\left(\frac{1}{2}\right)} \quad \text{with } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}
$$

Therefore, by the second part of Guo–Krasnoselskii fixed point theorem (1.3.1.1), we conclude that the problem (4.1) and (4.2) has at least one positive solution.

$\square$

4.3 Examples

We now give two examples to illustrate our results. The first example and the second one are chosen such that the conditions of theorem (4.2.1) sublinear case (1) and superlinear case (2) are satisfied, respectively.

Example 4.3.1. *[1]*
The problem

$$\begin{cases} D^{\frac{3}{2}}\,u(t)\,+\,f(t,u(t))=0 \\ u(0)=0, \quad u(1)=\lambda\int_0^1 s^{\frac{1}{2}}u(s)\mathrm{d}s, \end{cases} \tag{4.16}$$

with

$$f(t,x)=\begin{cases} t+\sqrt{x}\ \arctan\left(\frac{1}{x}\right) & \textit{if} \quad x>0,\ t\in[0,1] \\ t & if \quad x=0,\ t\in[0,1] \end{cases}$$

Here, $\delta=\dfrac{3}{2}$ *and* $h(t)=t^{\frac{1}{2}}$.
Then

$$\begin{aligned} A &= \int_0^1 h(t)\,t^{\delta-1}\,\mathrm{d}t \\ &= \int_0^1 t^{\frac{1}{2}}\,t^{\frac{3}{2}-1}\,\mathrm{d}t \\ &= \left[\frac{t}{2}\right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$

$1-\lambda A>0$ *for any* $\lambda<2$.

It is clear that $f(t,u):[0,1]\times[0,+\infty)\to[0,+\infty)$ *is continuous and for every* $u>0$ *it is verified that*

$$\min_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}=\frac{\sqrt{u}\arctan\left(\frac{1}{u}\right)}{u}=\frac{\arctan\left(\frac{1}{u}\right)}{\sqrt{u}}$$

and

$$\max_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}=\frac{1+\sqrt{u}\arctan\left(\frac{1}{u}\right)}{u}=\frac{1}{u}+\frac{\arctan\left(\frac{1}{u}\right)}{\sqrt{u}}$$

Now, we calculate f_0

$$\begin{aligned}
\lim_{u\to 0^+}\left\{\min_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\} &= \lim_{u\to 0^+}\left\{\frac{\arctan\left(\frac{1}{u}\right)}{\sqrt{u}}\right\}\\
&= \frac{\lim\limits_{u\to 0^+}\left(\arctan\left(\dfrac{1}{u}\right)\right)}{\lim\limits_{u\to 0^+}\left(\sqrt{u}\right)}\\
&= \frac{\infty}{0}\\
&= \infty.
\end{aligned}$$

Then, we calculate this f^∞

$$\begin{aligned}
\lim_{u\to \infty}\left\{\max_{t\in[0,1]}\left\{\frac{f(t,u)}{u}\right\}\right\} &= \lim_{u\to \infty}\left\{\frac{1}{u}+\frac{\arctan\left(\frac{1}{u}\right)}{\sqrt{u}}\right\}\\
&= \lim_{u\to \infty}\left(\frac{1}{u}\right)+\lim_{u\to \infty}\left(\frac{\arctan\left(\frac{1}{u}\right)}{\sqrt{u}}\right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{u\to \infty}\left(\frac{1}{u}\right)+\frac{\lim\limits_{u\to \infty}\left(\arctan\left(\dfrac{1}{u}\right)\right)}{\lim\limits_{u\to \infty}\left(\sqrt{u}\right)}\\
&= 0+\frac{0}{\infty}\\
&= 0.
\end{aligned}$$

Then by the first part of Theorem (4.2.1), the problem (4.16) has at least one positive solution for any $0<\lambda<1$.

Example 4.3.2. *[1]*
The problem

$$\begin{cases} D^{\frac{3}{2}}u(t)+u^{\beta}(t)+(t-1)u(t)=0, & t\in(0,1)\\ u(0)=0, \quad u(1)=\lambda\int_0^1 e^s u(s)\mathrm{d}s. \end{cases} \tag{4.17}$$

Here, $\delta=\dfrac{3}{2}$, $h(t)=e^t$ *and* $f(t,u)=u^{\beta}+(t-1)u$.
A numerical calculation leads to $A=\int_0^1 e^t\,t^{\frac{1}{2}}\,\mathrm{d}t\approx 1.25563$ *and* $1-\lambda A>0$ *for any* $0<\lambda<\frac{1}{A}$.

$f(t,u) : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous function and for every* $u > 0$ *it's verified that*

$$\min_{t\in[0,1]} \left\{ \frac{f(t,u)}{u} \right\} = \frac{u^\beta - u}{u} = u^{\beta-1} - 1$$

and

$$\max_{t\in[0,1]} \left\{ \frac{f(t,u)}{u} \right\} = \frac{u^\beta}{u} = u^{\beta-1}$$

Now, we calculat f^0*, we find for any* $\beta > 1$

$$\lim_{u\to 0^+} \left\{ \max_{t\in[0,1]} \left\{ \frac{f(t,u)}{u} \right\} \right\} = \lim_{u\to 0^+} \left(u^{\beta-1}\right) = 0$$

Then, we count f_∞

$$\begin{aligned} \lim_{u\to +\infty} \left\{ \min_{t\in[0,1]} \left\{ \frac{f(t,u)}{u} \right\} \right\} &= \lim_{u\to +\infty} \left(u^{\beta-1} - 1\right) \\ &= \lim_{u\to +\infty} \left(u^{\beta-1}\right) + \lim_{u\to +\infty} (-1) \end{aligned}$$

So $f_\infty = +\infty, \quad \forall \beta > 1$
Then, from the second part of Theorem (4.2.1), the problem (4.17) has at least one positive solution for any $\beta > 1$ *and* $\lambda \in (0, 0796413)$.

Conclusion

In this work, we study the existence of solutions of fractional differential equations involving a Riemann-Liouville derivative of order Alpha by using some fixed point theorems notably, Guo-Krasnoselskii theorem of expansion and compression of cones.

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