République Algérienne Démocratique et Populaire
Ministère de l'enseignement supérieur et de la recherche scientifique

Université de Saida - Dr Moulay Tahar.
Faculté des Sciences.
Département de Mathématiques.

Mémoire présenté en vue de l'obtention du diplôme de

Master Académique

Filière : Mathématiques
Spécialité: Analyse Mathématiques (AM)

par

Cherifi **F**atima [1]

Sous la direction de

Dr. H. Abbas

Thème:

Extentions of some special functions and applications to fractional calculus

Soutenu le 05/06/2022 devant le jury composé de

Pr. G. Djellouli	Université de Saïda Dr. Moulay Tahar	Président
Dr. H. Abbas	Université de Saïda Dr. Moulay Tahar	Encadreur
Dr. F. Charif	Université de Saïda Dr. Moulay Tahar	Examinateur
Dr. A. Halimi	Université de Saïda Dr. Moulay Tahar	Examinateur

Année univ.: 2022/2023

[1]e-mail: fatimamath1980@gmail.com

Dedication

To you reader, I dedicate this work.

Acknowledgments

I want to start by thanking my classmates for their encouragement and support throughout this master thesis.

Above all, I would like to thank my supervisor, Dr. H. Abbas, for the time she has set aside for me, for her seemingly endless patience, and for the wealth of knowledge that she has shared with me during the preparation of this master thesis.

Contents

Introduction

There are many special functions that arise in the solution of various classical problems of physics. Some examples include Bessel functions, which are solutions to differential equations and popular in problems involving circular or cylindrical symmetry or wave propagation; beta functions, which are definite integrals and related to the gamma function; and hypergeometric functions, which are solutions to a second-order linear differential equation called the hypergeometric equation1. Other examples include Airy functions, elliptic functions, gamma functions, parabolic cylinder functions, Mathieu functions, spheroidal wave functions, Struve functions, and Kelvin functions.

A Bessel function is a solution to a second-order linear differential equation called the Bessel equation. They are named after Friedrich Bessel who first introduced them in 1817. They are used in problems involving circular or cylindrical symmetry (type of symmetry where a three-dimensional object is invariant under a rotation about an axis. For example, a cylinder has cylindrical symmetry because it looks the same when rotated about its central axis) or wave propagation.

The gamma function is one commonly used extension of the factorial function to complex numbers. It is defined for all complex numbers except the non-positive integers. The gamma function has many applications in mathematics, physics, and engineering. Some of the most popular extensions of the gamma function include the beta function and hypergeometric functions.

The gamma function has many applications in mathematics, physics, and engineering. Some of the most common applications of the gamma function include:

- Integration problems
- Calculating products
- Analytic number theory
- Probability theory
- Statistics
- ....

The gamma function is also used in many other fields such as quantum mechanics, fluid dynamics, and signal processing.

This report is organized as follow:

The first chpater is devoted to preliminaries of special functions. Second chapter concerns the extensions of fractional derivative based on some extensions of eulerian functions, properties are given and integral transforms among other. The last chapter is about a recent work on these extensions, namely Riemann liouville extended fractional derivative.

Chapter 1

On spectial Functions

1.1 Gamma functions

The Gamma function is a generalization of the factorial function to non-integer numbers.

It is often used in probability and statistics, as it shows up in the normalizing constants of important probability distributions such as the Chi-square and the Gamma.

In this lecture we define the Gamma function, we present and prove some of its properties, and we discuss how to calculate its values.

1.1.1 Definition

Definition 1.1.1. For z a complex number with $R(z) > 0$

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$$

Theorem 1.1.1. (Recursive formula) Given the above definition, it is straightforward to prove that the Gamma function satisfies the following recursion :

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

Proof. The recursion can be derived by using integration by parts:

$$\begin{aligned}\Gamma(z) &= \int_0^{+\infty} t^{z-1} e^{-t} dt \\ &= \left[-t^{z-1} e^{-t}\right]_0^{+\infty} + \int_0^{+\infty} (z-1) t^{z-2} e^{-t} dt \\ &= (z-1) \int_0^{+\infty} t^{(z-1)-1} e^{-t} dt \\ &= (z-1)\Gamma(z-1)\end{aligned}$$

■

Theorem 1.1.2. (Relation to the factorial function)

When the argument of the Gamma function is a natural number $n \in \mathbb{N}$ then its value is equal to the factorial of $n-1$:

$$\Gamma(n) = (n-1)!$$

Proof. First of all, we have that

$$\begin{aligned}\Gamma(1) &= \int_0^{+\infty} t^{1-1}e^{-t}dt \\ &= \int_0^{+\infty} e^{-t}dt \\ &= \left[-e^{-t}\right]_0^{+\infty} \\ &= 1\end{aligned}$$

Using the recursion

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

we obtain

$$\begin{aligned}\Gamma(1) &= (1-1)! = 1 \\ \Gamma(2) &= (2-1)\Gamma(2-1) = 1\cdot\Gamma(1) = 2! \\ \Gamma(3) &= (3-1)\Gamma(3-1) = 2\cdot\Gamma(2) = 1\cdot 2\cdot 3 = 3! \\ &\vdots \\ \Gamma(n) &= (n-1)\Gamma(n-1) = 1\cdot 2\cdot 3\cdot ...\cdot(n-1) = (n-1)!\end{aligned}$$

■

Remark. The integer form of the factorial function can be considered as a special case of two widely used functions for computing factorials of non-integer arguments, namely the Pochhammer's symbol, given as

$$(z)_n = \begin{cases} z(z+1)(z+2)...(z+n-1) = \dfrac{\Gamma(z+n)}{\Gamma(z)} = \dfrac{(z+n-1)!}{(z-1)!} & \text{n>0} \\ 1 = 0! & \text{n=0} \end{cases} \tag{1.1}$$

and the gamma function (Euler's integral of the second kind).

$$\Gamma(z) = (z-1)! \tag{1.2}$$

Another definition

In another letter written in October 13, 1729 also to his friend Goldbach, Euler gave another equivalent definition for $\Gamma(z)$
Euler Let $x > 0$ and define

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)...(x+p)} = \frac{p^x}{x(1+x/1)...(1+x/p)} \tag{1.3}$$

Gauss

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)(z+2)...(z+n)} \quad , z \neq 0, -1, -2, -3, ... \tag{1.4}$$

Weierstrass

$$\frac{1}{\Gamma(z)} = ze^{\gamma.z}\prod_{n=1}^{\infty}(1+\frac{z}{n})e^{-z/n} \tag{1.5}$$

where γ is the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - ln(n) = 0.57721566490 \tag{1.6}$$

An excellent approximation of γ is given by the very simple formula

$$\gamma = \frac{1}{2}(\sqrt[3]{10} - 1) = 0.5772173...$$

1.1.2 Some known values

Proposition 1.1.1. A well-known formula, which is often used in probability theory and statistics, is the following:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Proof. By using the definition and performing a change of variable, we obtain

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{\frac{1}{2}-1} e^{-t} dt$$

$$= \int_0^{+\infty} \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}} dt$$

$$= 2\int_0^{+\infty} e^{-u^2} du$$

$$= 2\frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi}$$

■

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(-\frac{1}{2}+1)}{-\frac{1}{2}}$$

$$= -2\Gamma(\frac{1}{2})$$

$$= -\sqrt{\pi}$$

By using this fact and the recursion formula previously shown, it is immediate to prove that

Proposition 1.1.2.

$$\Gamma\left(n+\frac{1}{2}\right)=\sqrt{\pi}\prod_{j=0}^{n-1}\left(j+\frac{1}{2}\right)$$

for $n \in \mathbb{N}$.

Proof. The result is obtained by iterating the recursion formula:

$$\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \left(n-1+\frac{1}{2}\right)\Gamma\left(n-1+\frac{1}{2}\right)\\
&= \left(n-1+\frac{1}{2}\right)\left(n-2+\frac{1}{2}\right)\Gamma\left(n-2+\frac{1}{2}\right)\\
&\vdots\\
&= \left(n-1+\frac{1}{2}\right)\left(n-2+\frac{1}{2}\right)\cdots\left(n-n+\frac{1}{2}\right)\Gamma\left(n-n+\frac{1}{2}\right)\\
&= \left(n-1+\frac{1}{2}\right)\left(n-2+\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\\
&= \sqrt{\pi}\prod_{j=0}^{n-1}\left(j+\frac{1}{2}\right)
\end{aligned}$$

■

Proposition 1.1.3. : Duplication Formula

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})=\sqrt{\pi}\Gamma(2z) \tag{1.7}$$

Proof. An easy proof can lie on the expression of $\Gamma_p(x)$ and $\Gamma_p(x+1/2)$ from

$$\Gamma_p(x)=\frac{p!p^x}{x(x+1)...(x+p)}=\frac{p^x}{x(1+x/1)...(1+x/p)}$$

then make the product and find the limit as $p\to\infty$.
Notice that by applying the duplication formula for $x=1/2$, , we retrieve the value of $\Gamma(1/2)$, while $x=1/6$ permits to compute

$$\Gamma(\frac{1}{6})=2^{-\frac{1}{3}}\sqrt{\frac{3}{\pi}}\Gamma^2(\frac{1}{3})$$

■

Proposition 1.1.4. : The complement formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z} \tag{1.8}$$

Proof. There is an important identity connecting the gamma function at the complementary values x and $1-x$. One way to obtain it is to start with Weierstrass formula

$$\frac{1}{\Gamma(x)}=xe^{\lambda x}\prod_{p=1}^{\infty}(1+\frac{x}{p})e^{-x/p}$$

which yields:

$$\frac{1}{\Gamma(x)}\frac{1}{\Gamma(-x)} = -x^2 e^{\gamma x} e^{-\gamma x} \prod_{p=1}^{\infty} [(1+\frac{x}{p})e^{-x/p}(1-\frac{x}{p})e^{x/p}]$$

But the functional equation gives $\Gamma(-x) = -\Gamma(1-x)/x$ and the equality writes as

$$\frac{1}{\Gamma(x)\Gamma(1-x)} = x\prod_{p=1}^{\infty}(1-\frac{x^2}{p^2})$$

and using the well-known infinite product :

$$\sin(\pi x) = \pi x \prod_{p=1}^{\infty}(1-\frac{x^2}{p^2})$$

finally gives

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{1.9}$$

Relation above is called the complement (or reflection) formula and is valid when x and $1-x$ are not negative or null integers and was discovered by Euler.

1.2 incomplete Gamma functions

Definition 1.2.1.

$$\gamma_L(z,x) = \int_0^x t^{z-1}e^{-t}dt \tag{1.10}$$

Some special values, integrals and series are listed below for convenience [11], [29].

1.2.1 Special Values of $\gamma(z,x)$ and $\Gamma(z)$ for "z" Integer (let $z=n$)

$$\gamma(1+n,x) = n![1-e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}] \quad n = 0,1,2... \tag{1.11}$$

$$\Gamma(1+n,x) = n!e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{k!} \quad n = 0,1,2... \tag{1.12}$$

$$\Gamma(-n,x) = \frac{(-1)^n}{n!}[\Gamma(0,x) - e^{-x}\sum_{x=0}^{n-1}(-1)^k\frac{k!}{x^{k+1}}] \quad n = 1,2,3... \tag{1.13}$$

1.2.2 Integral Representations of the Incomplete Gamma Functions

$$\gamma(z,x) = x^z cosec(\pi z)\int_0^{\pi} e^{x\cos(\theta}\cos(z\theta + x\sin\theta)d\theta \quad x \neq 0, z > 0, z \neq 1,2,... \tag{1.14}$$

$$\Gamma(z,x) = \frac{e^{-x}x^z}{\Gamma(1-z)}\int_0^{\infty}\frac{e^{-t}t^{-z}}{x+t}dt \quad z < 1, x > 0 \tag{1.15}$$

$$\Gamma(z,xy) = y^z e^{-xy}\int_0^{\infty} e^{-ty}(t+x)^{z-1}dt \quad y > 0, x > 0, z > 0 \tag{1.16}$$

1.2.3 Series Representations of the Incomplete Gamma Functions

$$\gamma(z,x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{z+n}}{n!(z+n)} \tag{1.17}$$

$$\Gamma(z,x) = \Gamma(z) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{z+n}}{n!(z+n)} \tag{1.18}$$

$$\Gamma(z+x) = e^{-x} x^z \sum_{n=0}^{\infty} \frac{L_n^z(x)}{n+1} \quad x > 0 \tag{1.19}$$

where $L_n^z(x)$ is the associated Laguerre polynomial (one can refer to [29] for more details).

1.2.4 Functional Representations of the Incomplete Gamma Functions

$$\gamma(z+1,x) = z\gamma(z,x) - x^z e^{-x} \tag{1.20}$$

$$\Gamma(z+1,x) = z\Gamma(z,x) + x^z e^{-x} \tag{1.21}$$

$$\frac{\Gamma(z+1,x)}{\Gamma(z+n)} = \frac{\Gamma(z,x)}{\Gamma(z)} + e^{-x} \sum_{k=0}^{n-1} \frac{x+\Bbbk}{\Gamma(z+k+1)} \tag{1.22}$$

$$\frac{d\gamma(z,x)}{dx} = -\frac{d\Gamma(z,x)}{dx} = x^{z-1} e^{-x} \tag{1.23}$$

1.2.5 Asymptotic Expansion of $\Gamma(z,x)$ for Large x

$$\Gamma(z,x) = x^{z-1} e^{-x} [1 + \frac{(z-1)}{x} + \frac{(z-1)(z-2)}{x^2} + ...] \quad x \to \infty \tag{1.24}$$

1.2.6 Relationships with Other Special Functions

$$\Gamma(0,x) = -Ei(-x) \tag{1.25}$$

$$\Gamma(0, ln\, 1/x) = -\ell i(x) \tag{1.26}$$

$$\Gamma(1/2, x^2) = \sqrt{\pi}(1 - erf(x)) = \sqrt{\pi} erfc(x) \tag{1.27}$$

where complementary error function defined as $erfc1 - erf$

$$\gamma(1/2, x^2) = \sqrt{\pi} erf(x) \tag{1.28}$$

where $erf(x)$ is the error function (also called the Gauss error function), often denoted by erf, is a complex function of a complex variable.

$$\gamma(z,x) = z^{-1} x^z e^{-x} M(z, 1+z, x) \tag{1.29}$$

$$\gamma(z,x) = z^{-1} x^z M(z, 1+z, -x) \tag{1.30}$$

Recall that $M(z, 1+z, -x)$ is the Mittag-Leffler function, also written as $E_{\alpha,\beta}$ is a special function, a complex function which depends on two complex parameters α, and β . It may be defined by the following series when the real part of α is strictly positive:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

More reading for Mittag-Leffler function can be found in [16], [15].

1.3 Beta Function B(x,y)

Another integral which is related to the Γ-function is the Beta function $B(x, y)$ which is defined as

Definition 1.3.1.

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \tag{1.31}$$

Remark. Other forms of the beta function are obtained by changes of variables. Thus

$$B(x, y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} du \quad \textit{by letting } \ t = \frac{u}{1-u} \tag{1.32}$$

From the definition it is easily seen that $B(x, y) = B(y, x)$.

$$B(x, y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2x-1}\theta d\theta \quad \textit{by letting } \ t = \sin^2\theta \tag{1.33}$$

Beta Function Properties

The important properties of beta function are as follows:

- This function is symmetric which means that the value of beta function is irrespective to the order of its parameters, i.e $B(p, q) = B(q, p)$.
- $B(p, q) = B(p, q+1) + B(p+1, q)$
- $B(p+1, q) = B(p, q).[p/(p+q)]$
- $B(p, q+1) = B(p, q).[q/(p+q)]$
- $B(p, q).B(p+q, 1-q) = \pi/p\sin(\pi q)$

1.4 Incomplete Beta Function B$_r(x, y)$

Like as one can define an incomplete gamma function, so can one define the incomplete beta function by the variable limit integral

$$B_r(x, y) = \int_0^r t^{x-1}(1-t)^{y-1} dt \quad 0 \le r \le 1 \tag{1.34}$$

with $a > 0$ and $b > 0$ if $r \neq 1$. One can also define

$$I_r(x,y) = \frac{B_r(x,y)}{B(x,y)} \tag{1.35}$$

Clearly when $x = 1$, $B_r(x,y)$ becomes the complete beta function and

$$I_1(x,y) = 1$$

The incomplete beta function and $I_r(x,y)$ satisfies the following relationships:

Proposition 1.4.1.

1. $I_r(x,y) = 1 - I_{1-r}(x,y)$, (Symmetry).
2. $I_r(x,y) = rI_r(x-1,y) + (1-r)I_r(x,y-1)$, (First recurrence formula).
3. $(x+y-xr)I_r(x,y) = x(1-r)I_r(x+1,y-1) + yI_r(x,y+1)$, (Second recurrence formula)
4. $(x+y)I_r(x,y) = xI_r(x+1,y) + yI_r(x,y+1)$, (Third recurrence formula)

1.4.1 Relationship Between the Gamma and Beta Functions

We know that there are two types of Euler integral functions. One is a beta function, and another one is a gamma function. Gamma is a single variable function, whereas Beta is a two-variable function. The relation between beta and gamma function will help to solve many problems in maths.

Proposition 1.4.2. The gamma and beta functions are related as

$$B(x,y) = \frac{\Gamma(x).\Gamma(y)}{\Gamma(x+y)}$$

Proof. Since we have another expression of the beta function defined as

$$\int_0^\infty e^{-pt}t^{z-1}dt = \frac{\Gamma(z)}{p^z}$$

which is obtained from the definition of the Γ-function with the change of variable $s = pt$.
Setting $p = 1+u$ and $z = x+y$, we get

$$\frac{1}{(1+u)^{x+y}} = \frac{1}{\Gamma(x+y)}\int_0^\infty e^{-(1+u)t}t^{x+y-1}dt \tag{1.36}$$

and substituting this result into the Beta function in

$$\begin{aligned} B(x,y) &= \frac{1}{\Gamma(x+y)}\int_0^\infty e^{-t}t^{x+y-1}dt\int_0^\infty e^{-ut}u^{z-1}du \\ &= \frac{\Gamma(x)}{\Gamma(x+y)}\int_0^\infty e^{-t}t^{y-1}dt \\ &= \frac{\Gamma(x).\Gamma(y)}{\Gamma(x+y)} \end{aligned} \tag{1.37}$$

■

1.5 Hypergeometric Functions

In this part, we give definitions and some properties of the hypergeometric functions, We refer to ([30], [12], [31], [18] for more details.
In 1769, Euler formed the second order linear differential equation

$$z(1-z)\frac{d^2y}{dz^2}+[c-(a+b+1)z]\frac{dy}{dz}-aby=0$$

(1.38)

where a, b and c are complex parameters, also known as the hypergeometric differential equation . The solutions (as series expansion) of the hypergeometric equation are valid in the neighborhood of $z = 0$, 1 or ∞. The hypergeometric differential equation is a prototype: every ordinary differential equation of second-order with at most three regular singular points can be brought to the hypergeometric differential equation by means of a suitable change of variables. The solutions of hypergeometric differential equation include many of the most interesting special functions of mathematical physics. Solutions to the hypergeometric differential equation are built out of the hypergeometric series. The solution of Euler?s hypergeometric differential equation is called hypergeometric function or Gaussian function ${}_2F_1$ introduced by Gauss (more details of finding solutions are available in [13] and [18] for series expansions).
Thus, if c is not an integer, the general solution of differential equation is valid in a neighborhood of the origin and can be given by :

$$y = A{}_2F_1(a, b; c; z) + Bz^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; z)$$

where A and B are arbitrary constants, and

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1+\frac{ab}{c.1}z+\frac{a(a+1)b(b+1)}{c(c+1).1.2}z^2+..... \\ &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c_n)}\frac{z^n}{n!} \end{aligned}$$

$$(c \neq 0, -1, -2, ...)$$

where $(\lambda)_v$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 \equiv 1 \; and \; (\lambda)_v := \frac{\Gamma(\lambda+v)}{\Gamma(\lambda)}$$

Hence

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

is called Gauss hypergeometric function. This series is convergent for $|z| < 1$ where $Re(c) > Re(b) > 0$ and $|z| = 1$ where $Re(c-a-b) > 0$.

Definition 1.5.1. The Gauss hypergeometric function can be given by Euler?s integral representation as follows:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(y)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt $$
$$ (|z|<1;\ Re(c)>Re(b)>0) $$

Remark. Replacing $z = \frac{z}{b}$ and by letting $|b| \to \infty$, in Gauss's hypergeometric equation eq. (1.5), we obtain

$$ z\frac{d^2y}{dz^2} + (c-z)\frac{dy}{dz} - ay = 0 $$

This equation has a regular singularity at $z = 0$, The simplest solution of the equation is

$$ \begin{aligned} \phi(a;c;z) &= 1 + \frac{a}{c.1}z + \frac{a(a+1)}{c(c+1).1.2}z^2 + ... \\ &= \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!} \end{aligned} $$
$$ (c \neq 0, -1, -2, ...) $$

Hence, we get

$$ \phi(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!} $$

which is called confluent hypergeometric function.

1.5.1 Confluent hypergeometric function

Definition 1.5.2. The confluent hypergeometric function can be given by an integral representation as follows:

$$ \phi(a;c;z) = \frac{\Gamma(c)}{\Gamma(x)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}exp(zt)dt $$
$$ (Re(c)>Re(a)>0) $$

1.5.2 Generalized hypergeometric function

A generalized form of the hypergeometric function is

Definition 1.5.3.

$$ {}_pF_q(\alpha_1,...,\alpha_p;\gamma_1,....,\gamma_q;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n.....\alpha_p)_n}{(\gamma_1)_n.....\gamma_q)_n}\frac{z^n}{n!} \tag{1.39} $$

$$ (p,q = 0,1,....) $$

Remark. 1. Setting $p = 2$, $q = 1$ in (1.39), we get the Gauss hypergeometric function,

$$F(\alpha_1, \alpha_2; \gamma_1; z) := {}_2F_1(\alpha_1, \alpha_2; \gamma_1; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n}{(\gamma_1)_n} \frac{z^n}{n!}$$

2. Setting $p = q = 1$ in 1.39, we get confluent hypergeometric function

$$\phi(\alpha_1; \gamma_1; z) =_1 F_1(\alpha_1; \gamma_1; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n}{(\gamma_1)_n} \frac{z^n}{n!}$$

1.5.3 Euler integral

and establish the Euler type integral representations:

Definition 1.5.4.

$$F_p(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} exp\left[-\frac{p}{t(1-t)}\right] dt,$$

(p>0; p=0 and |arg(1-z)|<$\pi < p$, $Re(c) > Re(b) > 0$)and

$$\phi_p(b; c; z) = \frac{exp(z)}{B(b, c-b)} \int_0^1 t^{c-b-1}(1-t)^{b-1} exp\left[-zt - \frac{p}{t(1-t)}\right] dt$$

$$(p > 0;\ p = 0 \ and \ Re(c) > Re(b) > 0)$$

Remark. They called these functions as extended Gauss hypergeometric function (EGHF) and extended confluent hypergeometric function, respectively, since $F_0(a, b; c; z) =_2 F_1(a, b; c; z)$ and $\phi_0(b; c; z) =_1 F_1(b; c; z)$

1.5.4 The Extended Appell's Functions

In mathematics, Appell series are a set of four hypergeometric series F_1, F_2, F_3, F_4 of two variables that were introduced by Paul Appell (1880) and that generalize Gauss's hypergeometric series ${}_2F_1$ of one variable. Appell established the set of partial differential equations of which these functions are solutions, and found various reduction formulas and expressions of these series in terms of hypergeometric series of one variable.

Definition 1.5.5. The extensions of the Appell's functions $F_1(a, b, c; d; x, y; p)$, $F_2(a, b, c; d, c; x, y; p)$, and the extended Lauricella's hypergeometric function $F^3_{D,p}(a, b, c, d; e; x, y, z)$ by

$$F_1(a, b, c; d; x, y; p) := \sum_{n,m=0}^{\infty} \frac{B_p(a+m+n, d-a)}{B(a, d-a)} (b)_n (c)_m \frac{x^n}{n!} \frac{y^m}{m!}$$

$$(max\{|x|, |y|\} < 1)$$

$$F_2(a, b, c; d, e; x, y; p) := \sum_{n,m=0}^{\infty} \frac{B_p(b+n, d-b) B_p(c+m, e-c)}{B(b, d-b) B(c; e-c)} \frac{x^n}{n!} \frac{y^m}{m!}$$

$$(|x| + |y| < 1)$$

and

$$F^3_{D,p}(a,b,c,d;e;x,y;z) := \sum_{m,n,r=0}^{\infty} \frac{B_p(a+m+n+r, e-a)(b)_m(c)_n(d)_r}{B(a,e-a)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^r}{r!}$$

$$(\sqrt{|x|} + \sqrt{|y|} + |z| < 1)$$

respectively.

Remark. Notice that the case $p = 0$ give the original functions. Like the Gauss hypergeometric series ${}_2F_1$, the Appell double series entail recurrence relations among contiguous functions.

For example, a basic set of such relations for Appell's F1 is given by:

Theorem 1.5.1. We have:

1. $(a-b_1-b_2)F_1(a,b_1,b_2,c;x,y) - a\,F_1(a+1,b_1,b_2,c;x,y) + b_1F_1(a,b_1+1,b_2,c;x,y) + b_2F_1(a,b_1,b_2+1,c;x,y) = 0$
2. $c\,F_1(a,b_1,b_2,c;x,y) - (c-a)F_1(a,b_1,b_2,c+1;x,y) - a\,F_1(a+1,b_1,b_2,c+1;x,y) = 0,$
3. $c\,F_1(a,b_1,b_2,c;x,y) + c(x-1)F_1(a,b_1+1,b_2,c;x,y) - (c-a)x\,F_1(a,b_1+1,b_2,c+1;x,y) = 0,$
4. $c\,F_1(a,b_1,b_2,c;x,y) + c(y-1)F_1(a,b_1,b_2+1,c;x,y) - (c-a)y\,F_1(a,b_1,b_2+1,c+1;x,y) = 0.$

Functions defined by Appell's double series can be represented in terms of double integrals involving elementary functions only [14]. However, Émile Picard (1881) discovered that Appell's F_1 can also be written as a one-dimensional Euler-type integral.

1.5.5 Integral representations of the functions $F_1(a,b,c;d;x,y;p)$ and $F_2(a,b,c;d,e;x,y;p)$

Now we proceed by obtaining the integral representations of the functions $F_1(a,b,c;d;x,y;p)$ and $F_2(a,b,c;d,e;x,y;p)$

Theorem 1.5.2. [25]
For the extended Appell's functions $F_1(a,b,c;d;x,y;p)$, we have the following integral representation:

$$F_1(a,b,c;d;x,y;p) = \frac{\Gamma(d)}{\Gamma(x)\Gamma(d-a)} \times \int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c} exp\left[-\frac{p}{t(1-t)}\right] dt$$

$$(p > 0;\ p = 0 \quad and \quad |arg(1-x)| < \pi,\ |arg(1-y)| < \pi;\ Re(d) > Re(a) > 0)$$

$$(Re(b) > 0,\ Re(c) > 0)$$

Theorem 1.5.3. [25]
For the function $F_2(a,b,c;d,e;x,y;p)$, we have the following integral representation:

$$F_2(a,b,c;d,e;x,y;p) = \frac{1}{B(b,d-b)B(c,e-c)}. \int_0^1\int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a}$$

$$\times \quad exp\left[-\frac{p}{t(1-t)} - \frac{p}{s(1-s)}\right] dtds$$

$$(p > 0; p = 0 \ and \ |x| + |y| < 1)$$

$$Re(d) > Re(b),\ Re(e) > Re(c) > 0,\ Re(a) > 0$$

Chapter 2

Extended special functions and Riemann-Liouville type fractional derivative operator

Introduction

Extensions and generalizations of some known special functions are important both from the theoretical and applied point of view. Also many extensions of fractional derivative operators have been developed and applied by many authors (see [3], [24],[7],[2],[17]...). These new extensions have proved to be very useful in various fields such as physics, engineering, statistics, actuarial sciences, economics, finance, survival analysis, life testing and telecommunications. The above-mentioned applications have largely motivated our present study.

Definition 2.0.1. $K_\alpha(z)$ is the modifed Bessel function of the third kind, or the Macdonald function with its integral representation given by:

$$K_\alpha(z) = \frac{1}{2}\int_0^\infty \exp[-k(z|t)]\frac{dt}{t^{\alpha+1}} \tag{2.1}$$

where $R(z) > 0$ and

$$k(z|t) = \frac{z}{2}(t + \frac{1}{t}) \tag{2.2}$$

For $\alpha = \frac{1}{2}$, we have

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z} \tag{2.3}$$

2.1 Extended special functions

2.1.1 Extended gamma and Beta functions

In 1994, Chaudhry and Zubair [9] introduced the following extension gamma function.

Definition 2.1.1. The extended gamma function is defined by :

$$\Gamma_p(x) := \int_0^\infty t^{x-1} exp(-t - pt^{-1})dt \tag{2.4}$$

$$(\Re(x) > 0,\ \Re(p) > 0)$$

In 1997, Chaudhry et al. [5] presented the following extension of Euler's beta function.

Definition 2.1.2. The extended beta function is defined by :

$$B_p(x) := \int_0^1 t^{x-1}(1-t)^{y-1} exp(-\frac{p}{t(1-t)})dt \tag{2.5}$$

$$(\Re(p) > 0,\ \Re(x) > 0,\ \Re(y) > 0)$$

Obviously we see that $\Gamma_0(x) = \Gamma(x)$ and $B_0(x,y) = B(x,y)$. All the properties of the Beta function can be derived from the relationships linking the Γ-function and the Beta function.

2.1.2 Extended incomplete gamma and incomplete Beta functions

Definition 2.1.3. (Extended incomplete gamma functions) The extended incomplete gamma functions constructed by using the exponential function are defined by

$$\gamma(\alpha, z; p) = \int_0^z t^{\alpha-1} \exp(-t - \frac{p}{t})dt \quad (R(p) > 0; p = 0, R(\alpha) > 0) \tag{2.6}$$

$$\Gamma(\alpha, z; p) = \int_0^\infty t^{\alpha-1} \exp(-t - \frac{p}{t})dt \quad (R(p) \geq 0) \tag{2.7}$$

with $|\arg z| < \pi$, which have been studied in detail by Chaudhry and Zubair (see, for example, [10] and [11]).

Remark. The extended incomplete gamma functions $\gamma(\alpha, z; p)$ and $\Gamma(\alpha, z; p)$ satisfy the following decomposition formula

$$\gamma(\alpha, z; p) + \Gamma(\alpha, z; p) = \Gamma_p(\alpha) = 2p^{\alpha/2} K_\alpha(2\sqrt{\pi}) \quad (R(p) > 0) \tag{2.8}$$

where $\Gamma_p(\alpha)$ is called extended gamma function.
Furthermore, the extension of incomplete beta function $B_r(a, b)$ is given by [?]

Definition 2.1.4. (Extended incomplete beta functions)

$$B_p(a, b; r) = \int_0^r t^{a-1}(1-t)^{b-1} \exp\left(\frac{-p}{t(t-1)}\right) dt \quad , R(p) > 0, R(a), R(b) > 0 and 0 < r < 1. \tag{2.9}$$

2.1.3 Extended hypergeometric function

Definition 2.1.5. The extended Gauss hypergeometric function $F_\mu(a,b;c;z;p;m)$ is defined by

$$F_\mu(a,b;c;z;p;m) = \sum_{n=0}^{\infty}(a)_n \frac{B_\mu(b+n,c-b;p;m)}{B(b,c-b)}\frac{z^n}{n!} \tag{2.10}$$

where $R(p) > 0, R(\mu) \geq 0$, $0 < R(b) < R(c)$ and $|z| < 1$.

Definition 2.1.6. The extended Appell hypergeometric function $F_{1,\mu}$ is defined by

$$F_{1,\mu}(a,b,c;d;x;y;p;m) = \sum_{n,k=0}^{\infty}(b)_n(c)_k \frac{B_\mu(a+n+k,d-a;p;m)}{B(a,d-a)}\frac{x^n}{n!}\frac{y^k}{k!} \tag{2.11}$$

where $R(p) > 0, R(\mu) \geq 0, 0 < R(a) < R(d)$ and $|x| < 1$, $|y| < 1$.

Definition 2.1.7. The extended Appell hypergeometric function $F_{2,\mu}$ is defined by

$$F_{2,\mu}(a,b,c;d,e;x,y;p;m) = \sum_{n,k=0}^{\infty}(a)_{n+k}\frac{B_\mu(b+n,d-b;p;m)}{B(b,d-b)}\frac{B_\mu(c+k,e-c;p;m)}{B(c,e-c)}\frac{x^n}{n!}\frac{y^k}{k!} \tag{2.12}$$

$R(p) > 0, R(\mu) \geq 0, 0 < R(b) < R(d)$, $0 < R(c) < R(e)$ and $|x| + |y| < 1$.

Definition 2.1.8. The extended Lauricella hypergeometric function $F^3_{D,\mu}$ is

$$F^3_{D,\mu}(a,b,c,d;e,x,y,z;p;m) = \sum_{n,k,r=0}^{\infty}(b)_n(c)_k(d)_r\frac{B_\mu(a+n+k+r,e-a;p;m)}{B(a,e-a)}\frac{x^n}{n!}\frac{y^k}{k!}\frac{z^r}{r!} \tag{2.13}$$

where $R(p) > 0, R(\mu) \geq 0, 0 < R(a) < R(e)$ and $|x| < 1, |y| < 1, |z| < 1$.
Here, it is important to mention that when we take $m = 1, \mu = 0$ and then letting $p \to 0$, function 2.10 reduces to the ordinary Gauss hypergeometric function defined by

$${}_2F_1(a,b;c;z) = {}_2F_1\left[\begin{matrix} a,b & z \\ c & \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{2.14}$$

where $(x)_n$ denotes the Pochhammer symbol defined previously.

For conditions of convergence and other related details of this function, see [4],[21] and [23]. Similarly, we can reduce the functions 2.11,2.12 and 2.13 to the well-known Appell functions F_1,F_2 and Lauricella function F^3_D, respectively (see [23]).
Now, we establish the integral representations of the extended hypergeometric functions given by (2.10- 2.13) as follows.

Theorem 2.1.1. The following integral representation for the extended Gauss hypergeometric function $F_\mu(a,b;c;z;p;m)$ is valid

$$F_\mu(a,b;c;z;p;m) = \sqrt{\frac{2p}{\pi}}\frac{1}{B(b,c-d)}\int_0^1 t^{b-\frac{3}{2}}(1-t)^{c-b-\frac{3}{2}}(1-zt)^{-a}K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right)dt \tag{2.15}$$

where $|\arg(1-z)| < \pi$,$R(p) > 0$,$m > 0$ and $R(\mu) \geq 0$.

Proof. By using **??** and employing the binomial expansion

$$(1 - zt)^{-a} = \sum_{n=0}^{\infty} (a)_n \frac{(zt)^n}{n!} \quad (|zt| < 1) \tag{2.16}$$

■

we get the above integral representation.

Theorem 2.1.2. The following integral representation for the extended hypergeometric function $F_{1,\mu}$ is valid

$$F_{1,\mu}(a, b, c; d; x, y; p; m) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(a, d-a)}$$

$$\times \int_0^1 t^{a-\frac{3}{2}} (1-t)^{d-a-\frac{3}{2}} (1-xt)^{-b} (1-yt)^{-c} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) dt \tag{2.17}$$

proof

For simplicity, let $\mathfrak{I}$ denote the left-hand side of (2.17). Then, using (2.10) yields

$$\mathfrak{I} = \sum_{n,k=0}^{\infty} (b)_n (c)_k \frac{B_\mu(a+n+k, d-a; p; m)}{B(a, d-a)} \frac{x^n}{n!} \frac{y^k}{k!} \tag{2.18}$$

By applying **??** to the integrand of 2.17, after a little simplification, we have

$$\mathfrak{I} = \sum_{n,k=0}^{\infty} \left\{ \sqrt{\frac{2p}{\pi}} \int_0^1 t^{a+n+k-\frac{3}{2}} (1-t)^{d-a-\frac{3}{2}} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) dt \right\} \frac{(b)_n (c)_k}{B(a, d-a)} \frac{x^n}{n!} \frac{y^k}{k!} \tag{2.19}$$

By interchanging the order of summation and integration in 2.19, we get

$$\mathfrak{I} = \sqrt{\frac{2p}{\pi}} \frac{1}{B(a, d-a)} \int_0^1 t^{a-\frac{3}{2}} (1-t)^{d-a-\frac{3}{2}} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right)$$

$$\times \left\{ \sum_{n=0}^{\infty} \frac{(b)_n}{n!} (xt)^n \right\} \left\{ \sum_{k=0}^{\infty} \frac{(c)_k}{k!} (yt)^k \right\} dt$$

$$= \sqrt{\frac{2p}{\pi}} \frac{1}{B(a, d-a)} \int_0^1 t^{a-\frac{3}{2}} (1-t)^{d-a-\frac{3}{2}}$$

$$\times (1-xt)^{-b} (1-yt)^{-c} K_{\mu+\frac{1}{2}} \left(\frac{p}{t^m (1-t)^m} \right) dt \tag{2.20}$$

which proves the integral representation 2.17.

To establish Theorem 2.1.6, we need to recall the following elementary series identity involving the bounded sequence of $\{f(N)\}_{N=0}^{\infty}$ stated in the following result.

Lemma 2.1.1. For a bounded sequence $\{f(N)\}_{N=0}^{\infty}$ of essentially arbitrary complex numbers, we have

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(n+k) \frac{x^n}{n!} \frac{y^k}{k!} \tag{2.21}$$

Theorem 2.1.3. The following integral representation for the extended hypergeometric function $F_{2,\mu}$ is valid

$$F_{2,\mu}(a,b,c;d,e;x,y;p;m) = \frac{2p}{\pi}\frac{1}{B(b,d-b)B(c,e-c)}$$

$$\times \int_0^1\int_0^1 t^{b-\frac{3}{2}}(1-t)^{d-b-\frac{3}{2}}w^{b-\frac{3}{2}}(1-w)^{e-c-\frac{3}{2}}(1-xt-yw)^{-a}$$

$$\times K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right)K_{\mu+\frac{1}{2}}\left(\frac{p}{w^m(1-w)^m}\right)dtdw \tag{2.22}$$

proof

Let $\mathcal{L}$ denote the left-hand side of 2.22. Then, using 2.12 yields

$$\mathcal{L} = \sum_{n,k=0}^{\infty}(a)_{n+k}\frac{B_\mu(b+n,d-b;p;m)}{B(b,d-b)}\frac{B_\mu(c+k,e-c;p;m)}{B(c,e-c)}\frac{x^n}{n!}\frac{y^k}{k!} \tag{2.23}$$

By applying **??** to the integrand of 2.25, we have

$$\mathcal{L} = \frac{2p}{\pi}\sum_{n,k=0}^{\infty}\left\{\int_0^1 t^{b+n-\frac{3}{2}}(1-t)^{d-b-\frac{3}{2}}K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right)dt\right\}$$

$$\times\left\{\int_0^1 w^{b+n-\frac{3}{2}}(1-w)^{e-c-\frac{3}{2}}K_{\mu+\frac{1}{2}}\left(\frac{p}{w^m(1-w)^m}\right)dw\right\}$$

$$\times\frac{(a)_{n+k}}{B(b,d-b)B(c,e-c)}\frac{x^n}{n!}\frac{y^k}{k!} \tag{2.24}$$

Next, interchanging the order of summation and integration in [?], which is guaranteed, yields

$$\mathcal{L} = \frac{2p}{\pi}\frac{1}{B(b,d-b)B(c,e-c)}\int_0^1\int_0^1 t^{b-\frac{3}{2}}(1-t)^{d-b-\frac{3}{2}}w^{b-\frac{3}{2}}(1-w)^{e-c-\frac{3}{2}}$$

$$\times K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right)K_{\mu+\frac{1}{2}}\left(\frac{p}{w^m(1-w)^m}\right)$$

$$\times\left(\sum_{n,k=0}^{\infty}(a)_{n+k}\frac{(xt)^n}{n!}\frac{(yw)^k}{k!}\right)dtdw \tag{2.25}$$

Finally, applying 2.21 to the double series in 2.25, we obtain the right-hand side of 2.22.

Theorem 2.1.4. The following integral representation for the extended hypergeometric function $F^3_{D,\mu}$ is valid

$$F^3_{D,\mu}(a,b,c,d;e,x,t,z;p;m) = \frac{1}{B(a,e-a)}\sqrt{\frac{2p}{\pi}}$$

$$\times\int_0^1\frac{t^{a-\frac{3}{2}}(1-t)^{e-a-\frac{3}{2}}}{(1-xt)^b(1-yt)^c(1-zt)^d}K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right)dt \tag{2.26}$$

Proof. A similar argument in the proof of Theorem 2.1.7 will be able to establish the integral representation in 2.26. Therefore, details of the proof are omitted. ■

Theorem 2.1.5. Let $x, y \in \mathbb{C}, m > 0, R(\mu) \geq 0$ and

$$R(s) > \max\{R(\mu), -\frac{1}{2} + \frac{1}{2m} - \frac{R(x)}{m}, -\frac{1}{2} + \frac{1}{2m} - \frac{R(y)}{m}\}$$

Then we have the following relation

$$\begin{aligned}
&\mathcal{M}\{B_\mu(x,y;p;m) : p \to s\} \\
&= \frac{1}{2^\mu} \frac{\Gamma(s+\mu)\Gamma(\frac{s}{2}-\frac{\mu}{2})\Gamma(x+ms+\frac{m-1}{2})\Gamma(y+ms+\frac{m-1}{2})}{\Gamma(\frac{s}{2}+\frac{\mu}{2})\Gamma(x+y+2ms+m-1)} \\
&= \frac{1}{2^\mu} \frac{\Gamma(s+\mu)\Gamma(\frac{s}{2}-\frac{\mu}{2}}{\Gamma(\frac{s}{2}+\frac{\mu}{2}} B(x+ms+\frac{m-1}{2}, y+ms+\frac{m-1}{2})
\end{aligned} \tag{2.27}$$

Proof. First, we have

$$\begin{aligned}
&\mathcal{M}\{B_\mu(x,y;p;m) : p \to s\} \\
&= \int_0^\infty \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}} K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right) dt \\
&= \sqrt{\frac{2}{\pi}} \int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}} \left[\int_0^\infty p^{s+\frac{1}{2}-1} K_{\mu+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right) dp\right] dt \\
&= \sqrt{\frac{2}{\pi}} \int_0^1 t^{x+m(s+\frac{1}{2})-\frac{3}{2}}(1-t)^{y+m(s+\frac{1}{2})-\frac{3}{2}} dt \int_0^\infty u^{s+\frac{1}{2}} - 1K_{\mu+\frac{1}{2}}(u)du \\
&= \sqrt{\frac{2}{\pi}} \frac{\Gamma(x+ms+\frac{m-1}{2})\Gamma(y+ms+\frac{m-1}{2})}{\Gamma(x+y+2ms+m-1)} \int_0^1 u^{s+\frac{1}{2}-1} K_{\mu+\frac{1}{2}}(u)du
\end{aligned} \tag{2.28}$$

Since the Mellin transform of the Macdonald function $K_v(z)$ is given by [[?],p. 37, Eq.(1.7.41)]:

$$\mathcal{M}\{K_v(z) : z \to s\} = 2^{s-2}\Gamma(\frac{s}{2}+\frac{v}{2})\Gamma(\frac{s}{2}-\frac{v}{2}) \tag{2.29}$$

the last integral in 2.28 can be evaluated as

$$\int_0^\infty u^{s+\frac{1}{2}-1} K_{\mu+\frac{1}{2}}(u)du = 2^{s-\frac{3}{2}}\Gamma(\frac{s}{2}+\frac{v}{2}+\frac{1}{2})\Gamma(\frac{s}{2}-\frac{\mu}{2}) = 2^{-\frac{1}{2}-\mu}\sqrt{\pi}\frac{\Gamma(s+\mu)\Gamma(\frac{s}{2}-\frac{\mu}{2})}{\Gamma(\frac{s}{2}+\frac{\mu}{2})} \tag{2.30}$$

where we have used

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\frac{1}{2})$$

Finally, we get

$$\mathrm{M}\{\mathrm{B}_\mu(x,y;p;m) : p \to s\} = \frac{1}{2^\mu} \frac{\Gamma(s+\mu)\Gamma(\frac{s}{2}-\frac{\mu}{2})\Gamma(x+ms+\frac{m-1}{2})\Gamma(y+ms+\frac{m-1}{2})}{\Gamma(\frac{s}{2}+\frac{\mu}{2})\Gamma(x+y+2ms+m-1)}$$

■

2.2 Fractional calculus

Fractional calculus concerns integrals and derivatives of (real or multivalued) functions at the non-integer order integrals and derivatives. These are called fractional derivatives and fractional integrals, which can be of real or complex orders and may include integer orders. During the last century, fractional differential equations have been proved to be powerful tools in the modelling of many phenomena in various fields of engineering, biology, physics and economics. We refer for more details to monograph of Kilbas et al. [16]. In this section we shall give some basic formulas and techniques which are necessary to better understand the rest of this thesis. The Riemann-Liouville approach will be explored by means of Euler-gamma and beta functions connected with this function.

Fractional integral and fractional derivative

The fractional derivatives are defined using fractional integrals. We present only one type of fractional integral operators, but there are several known forms of the fractional integrals.
For every $\delta > 0$ and a given local integrable function f.

Definition 2.2.1. [16, 22] (Fractional Integral of order δ)
The right Riemann-Liouville fractional integral of order α is defined by:

$$ {}_aI_t^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-u)^{\delta-1} f(u)du \quad -\infty \leq a < t < \infty \tag{2.31} $$

Alternatively, it can be defined also the left fractional integral as :

$$ {}_tI_b^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_t^b (u-t)^{\delta-1} f(u)du \quad -\infty < t < b < \infty \tag{2.32} $$

For particular values of the a and b, the following cases are known :

(i) Riemann: $a = 0, \quad b = +\infty$

(ii) Liouville: $a = -\infty, \quad b = 0$

Theorem 2.2.1. [16, 22]
We have

$$ {}_aI_t^\delta [C_1 f(t) + C_2 g(t)] = C_{1a}I_t^\delta f(t) + C_{2a}I_t^\delta g(t) $$

where C_1 and C_2 are constants and f, g are two arbitrary functions.

Definition 2.2.2. [16, 22] (Fractional derivative of order δ)
For every δ, Let $n = [\delta]$. The Riemann-Liouville derivative of order δ can be defined as :

$$ {}_aD_t^\delta f(t) = \left(\frac{d}{dt}\right)^n {}_aI_t^{n-\delta} f(t) \tag{2.33} $$

$$ = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dx}\right)^n \int_a^t (t-u)^{n-\delta-1} f(u)du \tag{2.34} $$

Theorem 2.2.2. [16, 22]
The following integro-derivation rules are valid :

$$\int_a^b \phi(x)_a I_x^\delta \psi(x)dx = \int_a^b \psi(x)_x I_b^\delta f(x)dx \tag{2.35}$$

$$\int_a^b f(x)_a D_x^\delta g(x)dx = \int_a^b g(x)_x D_b^\delta f(x)dx \tag{2.36}$$

Also, notice that ${}_aI_x^\delta {}_aD_x^\delta f(x) = f(x)$ for $0 < \delta < 1$.

Theorem 2.2.3. [16, 22]
The following integration and derivation rules are valid :

(a) ${}_aI_t^{\delta+1}[Df(t)] =_a I_t^\delta f(t) - \dfrac{(t-a)^\delta}{\Gamma(\delta+1)} f(a)$

(b) ${}_aI_t^\delta[{}_aD_t^\delta f(t)] = f(t) - \sum_{k=1}^{n} {}_aD_t^{\delta-k} f(t)|_{t=a} \dfrac{(t-a)^{\delta-k}}{\Gamma(\delta-k+1)}$

(c) $D[{}_aI_t^\delta f(t)] =_a I_t^\delta[Df(t)] + \dfrac{(t-a)^{\delta-1}}{\Gamma(\delta)} f(a)$

(d) ${}_aI_t^\delta f(t) =_a I_t^{\delta+p}[D^p f(t)] + \sum_{k=0}^{p-1} \dfrac{D^k f(a)(t-a)^{\delta+k}}{\Gamma(\delta+k+1)},$ where $Re(p)$ is positive.

(e) $D^p[{}_aI_t^\delta f(t)] =_a I_t^\delta[D^p f(t)] + \sum_{k=0}^{p-1} \dfrac{D^k f(a)(t-a)^{\delta+k}}{\Gamma(\delta+k+1)},$ where $Re(p)$ is positive.

Theorem 2.2.4. [16, 22]

1. ${}_aI_t^\delta {}_aI_t^\beta f(t) =_a I_t^{\delta+\beta} f(t)$
2. ${}_aD_t^\delta[{}_aI_t^\beta f(t)] =_a D_t^{\delta-\beta} f(t)$
3. ${}_aI_t^\delta[{}_aD_t^\beta f(t)] =_a I_t^{\delta-\beta} f(t) - \sum_{k=1}^{m} \dfrac{(t-a)^{\delta-k}}{\Gamma(\delta+1-k)} f(t)|_{t=a}$

   where: $m = [\beta] + 1$
4. ${}_aD_t^\delta[{}_aD_t^\beta f(t)] =_a D_t^{\delta+\beta} f(t) - \sum_{k=1}^{m} {}_aD_t^{\beta-k} f(t)|_{t=a} \dfrac{(t-a)^{-\delta-k}}{\Gamma(1-\delta-k)}$

Example 2.2.1. To solve the following FDE with initial value:

$$D^{\frac{1}{2}}y(t) = y(t),$$

$$D^{-\frac{1}{2}}y(0) = -2\sqrt{\pi}$$

We transforme it in first order differential equation.
Using theorem 2.2.4 (4), we obtain:

$$D^{\frac{1}{2}}[D^{\frac{1}{2}}y(t)] = y'(t) - D^{\frac{1}{2}}y(0)\frac{t^{-\frac{1}{2}-1}}{\Gamma(1-\frac{1}{2}-1)} = D^{\frac{1}{2}}y(t) = y(t)$$

$$y'(t) - t^{-\frac{3}{2}} = y(t)$$

Theorem 2.2.5. [16, 22]
If the function $f(t)$ possess continuous derivative, then for $\delta > 0$, $n = [\delta] + 1$:

$$ {}_aI_t^\delta f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(t-a)^{k-\delta}}{\Gamma(k+1-\delta)} + \frac{1}{\Gamma(n-\delta)} \int_a^t (t-y)^{n-\delta-1} f^{(n)}(y)dy $$

Theorem 2.2.6. [16, 22] We denote ${}_0I_t^\delta$ with I^δ, for $p \in N$, $\delta > 0$. It can be proved that:

(a) $I^\delta[t^p f(t)] = \displaystyle\sum_{k=0}^{p} \binom{-\delta}{k} \frac{d^k}{dt^k} t^p I^{\delta+k} f(t)$

(b) $D^\delta[t^p f(t)] = \displaystyle\sum_{k=0}^{p} \binom{\delta}{k} \frac{d^k}{dt^k} t^p D^{\delta-k} f(t)$

Example 2.2.2. Let calculate the Riemann-Liouville fractional derivative of the function $f(t) = t^\beta$ for $\delta > 0$, $n-1 < \delta < n$, $\beta > n-1$.
For instance, we can write:

$$ I = D^\delta t^\beta = \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t u^\beta (t-u)^{n-\delta-1} du $$

and we take :

$$ u = vt, \quad du = tdv $$

It follows:

$$ \begin{aligned} I &= \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t (vt)^\beta [(1-v)t]^{n-\delta-1} t dv \\ &= \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t (1-v)^{n-\delta-1} v^\beta t^{n-\delta+\beta} dv \\ &= \frac{1}{\Gamma(n-\delta)} \int_0^t (1-v)^{n-\delta-1} v^\beta \frac{d^n}{dt^n} t^{n-\delta+\beta} dv \end{aligned} $$

but

$$ \frac{d^n}{dt^n} t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)} t^{\lambda-n} $$

Recall that:

$$ B(p,q) = \int_0^1 v^{p-1}(1-v)^{q-1} dv $$

so that it results :

$$ I = \frac{1}{\Gamma(n-\delta)} \frac{\Gamma(n-\delta+\beta+1)}{\Gamma(-\delta+\beta+1)} t^{-\delta+\beta} \int_0^1 (1-v)^{n-\delta-1} v^\beta dv $$

$$ \int_0^1 (1-v)^{n-\delta-1} v^\beta dv = B(n-\delta, \beta+1) = \frac{\Gamma(n-\delta)\Gamma(\beta+1)}{\Gamma(n-\delta+\beta+1)} $$

$$ D^\delta t^\beta = I = \frac{\Gamma(\beta+1)}{\Gamma(-\delta+\beta+1)} t^{\beta-\delta} $$

Example 2.2.3. We can also find the Riemann-Liouville fractional integral and fractional derivative of

$$f(t) = (t-a)^{\beta}$$

For the fractional integral we apply the Riemann-Liouville definition:

$$I =_a I_t^{\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-u)^{\delta-1}(u-a)^{\beta} du$$

The following change of variables

$$\frac{u-a}{t-a} = v$$

$$du = (t-a)dv$$

allows us to calculate:

$$I = \frac{(t-a)^{\delta+\beta}}{\Gamma(\delta)} \int_0^1 (1-v)^{\delta-1} v^{\beta} dv = \frac{(t-a)^{\delta+\beta}}{\Gamma(\delta)} B(\delta, \beta+1)$$

$$I = \frac{\Gamma(\beta+1)}{\Gamma(\delta+\beta+1)}(t-a)^{\delta+\beta}$$

For the fractional derivative we apply the Riemann-Liouville definition:

$$Df =_a D_t^{\delta}(t-a)^{\beta} = \frac{d^n}{dt^n} {}_a I^{n-\delta}(t-a)^{\beta}$$

and finally :

$$Df = \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\delta+1)} \frac{d^n}{dt^n}(t-a)^{\beta+n-\delta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\delta+1)}(t-a)^{\beta-\delta}$$

2.3 Extended Riemann-Liouville type fractional derivative operator

We first recall that the classical Riemann-Liouville fractional derivative is defined by(see [[16], p. 286])

$$D_z^{\nu} f(z) = \frac{1}{\Gamma(-\nu)} \int_0^z (z-t)^{-\nu-1} f(t) dt$$

where $R(\nu) < 0$ and the integration path is a line from 0 to z in the complex t-plane. It coincides with the fractional integral of order $-\nu$. In case $m-1 < R(\nu) < m$, $m \in \mathbb{N}$, it is customary to write

$$D_z^{\nu} f(z) = \frac{d^m}{dz^m} D_z^{\nu-m} f(z) = \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(m-\nu)} \int_0^z (z-t)^{m-\nu-1} f(t) dt\right\}$$

We present the following new extended Riemann-Liouville-type fractional derivative operator.

Definition 2.3.1. The extended Riemann-Liouville fractional derivative is defined as

$$D_z^{\nu,\mu,p;m} f(z) = \frac{1}{\Gamma(-\nu)}\sqrt{\frac{2p}{\pi}} int_0^z (z-t)^{-\nu-1} f(t) K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right) dt \tag{2.37}$$

where $R(\nu) < 0$, $R(p) > 0$, $R(m) > 0$ and $R(\mu) \geq 0$.
For $n-1 < R(\nu) < n$, $n \in \mathbb{N}$, we write

$$D_z^{\nu,\mu,p;m} f(z) = \frac{d^n}{dz^n} D_z^{\nu-n,\mu,p;m} f(z) = \frac{d^n}{dz^n}\left\{\frac{1}{\Gamma(n-\nu)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{n-\nu-1} f(t) K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right) dt\right\} \tag{2.38}$$

If we take $m = 0$, $\mu = 0$, and $p \to 0$, then the above extended Riemann-Liouville fractional derivative operator reduces to the classical Riemann-Liouville fractional derivative operator.
Now, we begin our investigation by calculating the extended fractional derivatives of some elementary functions. For our purpose, we first establish two results involving the extended Riemann-Liouville fractional derivative operator.

Lemma 2.3.1. Let $R(\nu) < 0$, then we have

$$D_z^{\nu,\mu,p;m}\{z^\lambda\} = \frac{z^{\lambda-\nu}}{\Gamma(-\nu)} B_\mu(\lambda + \frac{3}{2}, -\nu + \frac{1}{2}, p; m) \tag{2.39}$$

proof

Using Definition 2.1.6 and 1, we have

$$D_z^{\nu,\mu,p;m}\{z^\lambda\} = \frac{1}{\Gamma(-\nu)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\nu-1} t^\lambda K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right)$$

$$= \frac{z^{\lambda-\nu}}{\Gamma(-\nu)}\sqrt{\frac{2p}{\pi}}\int_0^1 (1-u)^{(-\nu+\frac{1}{2})-\frac{3}{2}} u^{(\lambda+\frac{3}{2})-\frac{3}{2}} K_{\mu+\frac{1}{2}}\left(\frac{p}{u^m(1-u)^m}\right) du$$

$$= \frac{z^{\lambda-\nu}}{\Gamma(-\nu)} B_\mu(\lambda + \frac{3}{2}, -\nu + \frac{1}{2}; p; m)$$

Next, we apply the extended Riemann-Liouville fractional derivative to a function $f(z)$ analytic at the origin.

Lemma 2.3.2. Let $R(\nu) < 0$ and suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n (|z| < \rho)$ for some $\rho \in \mathbb{R}_+$. Then we have

$$D_z^{\nu,\mu,p;m}\{f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\nu,\mu,p;m}\{z^n\}$$

Proof. Using Definition 2.1.6 to the function $f(z)$ with its series expansion, we have

$$D_z^{\nu,\mu,p;m}\{f(z)\} = \frac{1}{\Gamma(-\nu)}\sqrt{\frac{2p}{\pi}}\int_0^1 (z-t)^{-\nu-t} K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right)\sum_{n=0}^{\infty} a_n t^n dt$$

Since the power series converges uniformly on any closed disk centered at the origin with its radius smaller than ρ, so does the series on the line segment from 0 to a fixed z for $|z| < \rho$. This fact guarantees term-by-term integration as follows:

$$D_z^{\nu,\mu,p;m}\{f(z)\} = \sum_{n=0}^{\infty} a_n \left\{ \frac{1}{\Gamma(-\nu)} \sqrt{\frac{2p}{\pi}} \int_0^1 (z-t)^{-\nu-t} K_{\mu+\frac{1}{2}} \left(\frac{pz^{2m}}{t^m(z-t)^m} \right) t^n dt \right\} = \sum_{n=0}^{\infty} a_n D_z^{\nu,\mu;p;m}\{z^n\}$$

■

As a consequence we have the following result.

Theorem 2.3.1. Let $R(\nu) < 0$ and suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n (|z| < \rho)$ for some $\rho \in \mathbb{R}_+$. Then we have

$$D_z^{\nu,\mu,p;m}\{z^{\lambda-1}f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\nu,\mu,p;m}\{z^{\lambda+n-1}\} = \frac{z^{\lambda-\nu-1}}{\Gamma(-\nu)} \sum_{n=0}^{\infty} a_n B_\mu(\lambda+n+\frac{1}{2}, -\nu+\frac{1}{2}; p; m) z^n$$

We present two subsequent theorems which may be useful to nd certain generating function.

Theorem 2.3.2. For $R(\nu) > R(\lambda) > -\frac{1}{2}$ we have

$$D_z^{\lambda-\nu,\mu,p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha}\right\} = \frac{z^{\nu-1}}{\Gamma(\nu-\lambda)} B(\lambda+\frac{1}{2}, \nu-\lambda+\frac{1}{2}) F_\mu(\alpha, \lambda+\frac{1}{2}; \nu+1; z; p; m) \quad (|z| < 1; \alpha \in \mathbb{C}) \tag{2.40}$$

Proof. Using 2.16 and applying Lemmas 2.1.2 and 2.1.3 we obtain

$$D_z^{\lambda-\nu,\mu,p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha}\right\} = D_z^{\lambda-\nu,\mu,p;m}\left\{z^{\lambda-1}\sum_{l=0}^{\infty}(\alpha)_l \frac{z^l}{l!}\right\}$$

$$= \sum_{l=0}^{\infty} \frac{(\alpha)_l}{l!} D_z^{\lambda-\nu,\mu,p;m}\{z^{\lambda+l-1}\}$$

$$= \sum_{l=0}^{\infty} \frac{(\alpha)_l}{l!} \frac{B_\mu(\lambda+l+\frac{1}{2}, \nu-\lambda+\frac{1}{2}; p; m)}{\Gamma(\nu-\lambda)} z^{\nu+l-1}$$

By using 2.10, we can get

$$D_z^{\lambda-\nu,\mu,p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha}\right\} = \frac{z^{\nu-1}}{\Gamma(\nu-\lambda)} F_\mu(\alpha, \lambda+\frac{1}{2}, \nu+1; z; p; m)$$

■

Theorem 2.3.3. Let $R(\nu) > R(\lambda) > -\frac{1}{2}$,$R(\alpha) > 0, R(beta) > 0$; $|az| < 1$ and $|bz| <$ 1.Then we have

$$D_z^{\lambda-\nu,\mu,p;m}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\}$$

$$= \frac{z^{\nu-1}}{\Gamma(\nu-\lambda)} B(\lambda+\frac{1}{2}, \nu-\lambda+\frac{1}{2}) F_{1,\mu}(\lambda+\frac{1}{2}, \alpha, \beta; \nu+1, az, bz; p; m) \tag{2.41}$$

proof

Use the binomial theorem for $(1-az)^{-\alpha}$ and $(1-bz)^{-\beta}$. Apply Lemmas 2.1.2 and 2.1.3 to obtain

$$D_z^{\lambda-\nu,\mu,p;m}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\}$$

$$= D_z^{\lambda-\nu,\mu,p;m}\left\{z^{\lambda-1}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(\alpha)_l(\beta)_k\frac{(az)^l}{l!}\frac{(bz)^k}{k!}\right\}$$

$$= \sum_{l,k=0}^{\infty}(\alpha)_l(\beta)_k D_z^{\lambda-\nu,\mu,p;m}\left\{z^{\lambda+l+k-1}\frac{(a)^l}{l!}\frac{(b)^k}{k!}\right\}$$

$$= z^{v-1}\sum_{l,k=0}^{\infty}(\alpha)_l(\beta)_k\frac{B_\mu(\lambda+l+k+\frac{1}{2},\nu-\lambda+\frac{1}{2};p;m)}{\Gamma(\nu-\lambda)}\frac{(az)^l}{l!}\frac{(bz)^k}{k!}$$

By using 2.11, we get

$$D_z^{\lambda-\nu,\mu,p;m}\left\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\}$$

$$= \frac{z^{\nu-1}}{\Gamma(\nu-\lambda)}B(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2})F_{1,\mu}(\lambda+\frac{1}{2},\alpha,\beta;\nu+1,az,bz;p,m)$$

Theorem 2.3.4. Let $R(\nu) > R(\lambda) > -\frac{1}{2}$, $R(\alpha) > 0, R(\beta) > 0, R(\alpha) > 0, R(\gamma) > 0$,$|az| < 1, |bz| < 1$ and $|cz| < 1$.Then we have

$$D_z^{\lambda-\nu,\mu,p;m}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\} = \frac{z^{\nu-1}}{\Gamma(\nu-\lambda)}$$

$$\times B(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2})F_{D,\mu}^3(\lambda+\frac{1}{2},\alpha,\beta,\gamma;\nu+1;az;bz,cz;p;m) \tag{2.42}$$

proof

As in the proof of Theorem 2.1.11 taking the binomial theorem for $(1-az)^{-\alpha}, (1-bz)^{-\beta}$ and $(1-cz)^{-\gamma}$ and applying Lemmas 2.1.2 and 2.1.3 and taking Definition 5 into account, one can easily prove Theorem 2.1.12.

Theorem 2.3.5. Let $R(\nu) > R(\lambda) > -\frac{1}{2}$, $R(\alpha) > 0, R(\beta) > 0, R(\alpha) > 0, R(\gamma) > 0$, $\frac{x}{1-z}$ and $|x|+|z| < 1$. . Then we have

$$D_z^{\lambda-\nu,\mu,p;m}\{z^{\lambda-1}(1-z)^{-\alpha}F_\mu(\alpha,\beta;\gamma;\frac{x}{1-z};p;m)\}$$

$$= z^{\nu-1}\frac{B(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2})}{\Gamma(\nu-\lambda)}F_{2,\mu}(\alpha,\beta,\lambda+\frac{1}{2};\gamma,\nu+1,x,z;p;m) \tag{2.43}$$

proof

By using 2.17 and applying the Definition 2.1.2 for F_μ, we get

$$D_z^{\lambda-\nu,\mu,p;m}\{z^{\lambda-1}(1-z)^{-\alpha}F_\mu(\alpha,\beta;\gamma;\frac{x}{1-z};p;m)\}$$

$$D_z^{\lambda-\nu,\mu,p;m}\left\{z^{\lambda-1}(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\frac{B_\mu(\beta+n,\gamma-\beta;p;m)}{B(\beta,\gamma-\beta)}\left(\frac{x}{1-z}\right)^n\right\}$$

$$=\sum_{n=0}^{\infty}(\alpha)_n\frac{B_\mu(\beta+n,\gamma-\beta;p;m)}{B(\beta,\gamma-\beta)}D_z\{z^{\lambda-1}(1-z)^{-\alpha-n}\}\frac{x^n}{n!}$$

Using Theorem 2.1.10 for $D_z\{z^{\lambda-1}(1-z)^{-\alpha-n}\}$ and interpreting the extended hypergeometric function F_μ as its series representation, we get

$$D_z^{\lambda-\nu,\mu,p;m}\{z^{\lambda-1}(1-z)^{-\alpha}F_\mu(\alpha,\beta;\gamma;\frac{x}{1-z};p;m)\}$$

$$\frac{z^{\nu-1}}{\Gamma(\nu-\lambda)}B(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2})\sum_{n,k=0}^{\infty}\left\{(\alpha)_{n+k}\frac{B_\mu(\beta+n,\gamma-\beta;p;m)}{B(\beta,\gamma-\beta)}\right.$$

$$\times\frac{B_\mu(\lambda+k+\frac{1}{2},\nu-\lambda+\frac{1}{2};p;m)}{B(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2})}\frac{x^n}{n!}\frac{z^k}{k!}$$

$$\frac{z^{\nu-1}}{\Gamma(\nu-\lambda)}B(\lambda+\frac{1}{2},\nu-\lambda+\frac{1}{2})F_{2,\mu}(\alpha,\beta,\lambda+\frac{1}{2};\gamma,\nu+1;x;z,p;m)$$

This completes the proof.

2.3.1 Mellin transforms and further results

In this section, we first obtain the Mellin transform of the extended Riemann-Liouville fractional derivative operator.

Definition 2.3.2. Let $f(t)$ be a function defined on the positive real axis $0 < t < \infty$. The Mellin transformation $\mathcal{M}$ is the operation mapping the function f into the function F defined on the complex plane by the relation:

$$\mathcal{M}[f;s]\equiv F(s))=\int_0^{\infty}f(t)t^{s-1}dt \tag{2.44}$$

The function $F(s)$ is called the Mellin transform of f. In general, the integral does exist only for complex values of $s=a+jb$ such that $a_1 < a < a_2$, where a_1 and a_2 depend on the function $f(t)$ to transform. This introduces what is called the strip of definition of the Mellin transform that will be denoted by $S(a_1,a_2)$. . In some cases, this strip may extend to a half-plane $(a_1=-\infty \quad or \quad a_2=+\infty)$ or to the whole complex s-plane $(a_1=-\infty \quad or \quad a_2=+\infty)$.

Theorem 2.3.6. Let $R(\nu)<0$,$m>0, R(\mu)\geq 0$ and

$$R(\nu)>\max\{R(\mu),-\frac{1}{2}-\frac{1}{m}-\frac{R(\lambda)}{m},-\frac{1}{2}+\frac{R(\nu)}{m}\}$$

Then we have the following relation

$$\mathcal{M}[D_z^{\nu,\mu;p;m}\{z^\lambda\}:s] = \frac{z^{\lambda-\nu}}{2^\mu\Gamma(-\nu)}\frac{\Gamma(s+\mu)\Gamma(\frac{s}{2}+\frac{\mu}{2})\Gamma(\lambda+ms+\frac{m}{2}+1)\Gamma(-\mu+ms+\frac{m}{2})}{\Gamma(\frac{s}{2}+\frac{\mu}{2})\Gamma(\lambda-\nu+2ms+m+1)}$$

$$= \frac{z^{\lambda-\nu}}{2^\mu\Gamma(-\nu)}\frac{\Gamma(s+\mu)\Gamma(\frac{s}{2}-\frac{\mu}{2})}{\Gamma(\frac{s}{2}+\frac{\mu}{2})}B(\lambda+ms+\frac{\mu}{2}+1;-\nu+ms+\frac{\mu}{2})$$

proof

Taking the Mellin transform and using Lemma 2.1.2, we have

$$\mathcal{M}[D_z^{\nu,\mu;p;m}\{z^\lambda\}:s] = \int_0^\infty p^{s-1}D_z^{\nu,\mu;p;m}\{z^\lambda\}dp = \frac{z^{\lambda-\nu}}{\Gamma(-\nu)}\int_0^\infty p^{s-1}B_\mu(\lambda+\frac{3}{2},-\nu+\frac{1}{2};p;m)dp$$

Applying Theorem 2.1.3 to the last integral yields the desired result.

Theorem 2.3.7. Let $R(\nu)<0, m>0, R(\mu)\geq 0, |z|<1$ and

$$R(\nu) > \max\{R(\mu), -\frac{1}{2}-\frac{1}{m}, -\frac{1}{2}+\frac{R(\nu)}{m}\}$$

Then we have the following relation

$$\mathcal{M}[D_z^{\nu,\mu;p;m}\{(1-z)^{-\alpha}\}:s] = \frac{z^{-\nu}}{2^\mu\Gamma(-\nu)}\frac{\Gamma(s+\mu)\Gamma(\frac{s}{2}-\frac{\mu}{2})}{\Gamma(\frac{s}{2}+\frac{\mu}{2})}B(ms+\frac{m}{2}+1,-\nu+ms+\frac{m}{2})$$

$$\times_2 F_1(\alpha, ms+\frac{\mu}{2}+1;-\nu+2ms+m+1;z) \tag{2.45}$$

where ${}_2F_1$ is a well known Gauss hypergeometric function given by 2.14

proof

Using the binomial series for $(1-z)^{-\lambda}$ and Theorem 2.1.16 $\lambda = n$ yields

$$\mathcal{M}[D_z^{\nu,\mu;p;m}\{(1-z)^{-\alpha}\}:s] = \mathcal{M}\left[D_z^{\nu,\mu;p;m}\left\{\sum_{n=0}^\infty \frac{(\alpha)_n}{n!}z^n\right\}:s\right]$$

$$= \sum_{n=0}^\infty \frac{(\alpha)_n}{n!}\mathcal{M}\left[D_z^{\nu,\mu;p;m}\{z^n\}:s\right]$$

$$= \sum_{n=0}^\infty \frac{(\alpha)_n}{n!}\frac{z^{n-\nu}}{2^\mu\Gamma(-\nu)}\frac{\Gamma(s+\mu)\Gamma(\frac{s}{2}+\frac{\mu}{2})\Gamma(\lambda+ms+\frac{m}{2}+1)\Gamma(-\mu+ms+\frac{m}{2})}{\Gamma(\frac{s}{2}+\frac{\mu}{2})\Gamma(\lambda-\nu+2ms+m+1)}$$

Then the last expression is easily seen to be equal to the desired one.

Chapter 3

Generalized Extended Riemann-Liouville type fractional derivative operator

Introduction

In recent years, incomplete gamma functions have been used in many problems in applied mathematics, statistics, engineering and many other fields including physics and biology. Most generally, special functions became powerful tools to treat all these areas. Classical gamma and Euler?s beta functions are defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt, \quad (R(\alpha) > 0) \tag{3.1}$$

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt \tag{3.2}$$

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad (R(x) > 0, R(y) > 0) \tag{3.3}$$

Using an exponential regulazing term, Chaudhry et al.[11] extended the incomplete gamma function as follows

$$\gamma(\alpha, x; p) = \int_0^x t^{\alpha-1} e^{-t-\frac{p}{t}} dt \quad (R(p) > 0; p = 0, R(\alpha) > 0) \tag{3.4}$$

$$\Gamma(\alpha, x; p) = \int_x^\infty t^{\alpha-1} e^{-t-\frac{p}{t}} dt \tag{3.5}$$

They proved the following recurrence formula

$$\gamma(\alpha, x; p) + \Gamma(\alpha, x; p) = 2p^{\alpha/2} K_\alpha(2\sqrt{p}) \quad (R(p) > 0)$$

where $K_\alpha(z)$ is the Macdonald function, known also as modified bessel function of the third kind, defined for any $Re(z) > 0$ by

$$K_\alpha(z) = \frac{(z/2)^\alpha}{2} \int_0^\infty t^{-\alpha-1} e^{-t-z^2/4t}$$

A first extension of Euler?s beta function is given by Chaudhry et al. [9] as follows:

$$B(x,y,p) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{\frac{-p}{t(1-t)}}dt, (R(p)>0; p=0, R(x)>0, R(y)>0) \tag{3.6}$$

These extensions are useful and provide new connections with error and Whittaker functions. For $p = 0$,3.4, 3.5 and 3.6 will be reduced to known incomplete gamma and beta functions 3.1, 3.2 and 3.3 respectively.
Instead of using the exponential function, Chaudhry and Zubair [11] proposed a generalized extension of 3.4, 3.5 in the following form

$$\gamma_\mu(\alpha,x;p) = \sqrt{\frac{2p}{\pi}} \int_0^x t^{\alpha-\frac{3}{2}}e^{-t}K_{\mu+\frac{1}{2}}(\frac{p}{t})dt \tag{3.7}$$

$$\Gamma_\mu(\alpha,x;p) = \sqrt{\frac{2p}{\pi}} \int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}K_{\mu+\frac{1}{2}}(\frac{p}{t})dt, \quad (R(x)>0, R(p)>0, -\infty<\alpha<\infty) \tag{3.8}$$

Nowadays, many authors are developing new extensions of Euler's gamma, beta and hypergeometric functions based on the paper of Chaudhry and Zubair [10] by considering exponential and certain modified special functions (see for more details [[26], [27], [19], [20],[6], [7]]). Very recently, Agarwal et al. [3] developed an extension of the Euler's beta function as follows

$$B_\mu(x,y;p;m) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}K_{\mu+\frac{1}{2}}(\frac{p}{t^m(1-t)^m})dt \tag{3.9}$$

where $x, y \in \mathbb{C}, \quad m > 0$ and $R(p) > 0$.

From now, we present a new generalized incomplete gamma and Euler?s beta functions developped by Abbas et al. [1] by substituting in 3.7, 3.8 and 3.9 the Macdonald function $K_\alpha(z)$ by it?s extended one developed by Boudjelkha [8], namely

$$R_K(z,\alpha,q,\lambda) = \frac{(z/2)^\alpha}{2} \int_0^\infty t^{-\alpha-1}\frac{e^{-qt-z^2/4t}}{1-\lambda e^{-t}} \tag{3.10}$$

where $|\arg z^2| < \pi/2$, $0 < q \leq 1$ and -1$\leq \lambda \leq 1$.
$Clearly, when \lambda = 0$ and $q = 1$, $\quad R_K(z,\alpha,q,\lambda)$ is reduced to $K_\alpha(z)$. Moreover, Boudjelkha proved that the $R_K(z,-\alpha,q,\lambda)$ function can be expanded in terms of $K_\alpha(z)$ as follows

$$R_K(z,-\alpha,q,\lambda) = \sum_{n=0}^\infty \lambda^n \frac{K_\alpha(z\sqrt{q+n})}{(q+n)^{\alpha/2}}, \quad R(z^2)>0, \quad 0<q\leq 1, \quad -1\leq\lambda\leq 1 \tag{3.11}$$

and showed that the behavior of the function $R_K(z,-\alpha,q,\lambda)$ for small values of z is described by the asymptotic formulas:

$$R_K(z,-\alpha,q,\lambda) \sim \begin{cases} \frac{1}{2}\frac{\Gamma(-z)}{(z/2)^{-\alpha}}(1-\lambda)^{-1}, z\to 0, -1<\lambda<1, R(\alpha)<0 \\ \frac{1}{2}\frac{\Gamma(-z)}{(z/2)^{\alpha}}\Phi(\lambda,\alpha,q), z\to 0, -1\leq\lambda\leq 1, R(\alpha)>1. \end{cases}$$

(3.12)

where $\Phi(\lambda, \alpha, q)$ stands for the Lerch function. As for the asymptotic behavior of this function, when $z \to \infty$, it is given by

$$R_K(z, -\alpha, q, \lambda) \sim \sqrt{\frac{\pi}{2z}} \frac{e^{-z\sqrt{q}}}{q^{\alpha/2+1/4}} \quad as \quad z \to \infty, |\arg z| < \frac{\pi}{4}, -1 \leq \lambda \leq 1. \tag{3.13}$$

In particular, when $q = 1$, we have

$$R_K(z, -\alpha, q, \lambda) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \quad asz \to \infty, |\arg z| < \frac{\pi}{4} \tag{3.14}$$

which is the same asymptotic formula as that of K_α.
Further, by using the generalized extended beta function we get other extensions of Gauss hypergeometric, confluent hypergeometric, Appell and Lauricella hypergeometric functions and we investigate some of their properties.

3.1 The generalized extended incomplete Gamma and Euler's beta functions

3.1.1 The generalized extended incomplete Gamma function

Definition 3.1.1. The generalized extended incomplete gamma functions are given by

$$\gamma_\mu(\alpha, x; q; \lambda; p) = \frac{2p}{\pi} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} R_K(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda) dt \tag{3.15}$$

$$\Gamma_\mu(\alpha, x; q; \lambda; p) = \frac{2p}{\pi} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} R_K(\frac{p}{t}, -\mu - \frac{1}{2}, q, \lambda) dt \tag{3.16}$$

where $R(x) > 0$, $0 < q \leq 1$,$-1 \leq \lambda \leq 1$, and $R(p) > 0$. When $\lambda = 0$ and $q = 1$, 3.15 and 3.16 are respectively reduced to the extended incomplete gamma

Proposition 3.1.1. (Decomposition theorem)

$$\begin{aligned}\Gamma_\mu(\alpha, x; q; \lambda; p) + \gamma_\mu(\alpha, x; q; \lambda; p) &= \frac{\Gamma(\alpha+\mu)}{\sqrt{\pi}} (\frac{p}{2})^{-\mu} \Phi_{1-\frac{\alpha+\mu}{2}, \frac{1}{2}-\frac{\alpha+\mu}{2}}(\lambda, \mu + \frac{1}{2}, q, \frac{p^2}{16}) \\ &+ \frac{\Gamma(\frac{-\alpha+\mu}{2})}{2\sqrt{\pi}} (\frac{p}{2})^{\alpha} \Phi_{\frac{1}{2}, \frac{\alpha+\mu+2}{2}}(\lambda, \frac{\mu-\alpha+1}{2}, q, \frac{p^2}{16}) \\ &- \frac{\Gamma(\frac{-\alpha+\mu+1}{2})}{2\sqrt{\pi}} (\frac{p}{2})^{\alpha+1} \Phi_{\frac{3}{2}, \frac{\alpha+\mu+3}{2}}(\lambda, \frac{\mu-\alpha}{2}, q, \frac{p^2}{16})\end{aligned} \tag{3.17}$$

with $R(p) > 0$, $-\infty < \alpha < \infty$ and

$$\begin{aligned}\Phi_{b_1,b_2}(\lambda, s, q, \xi) &= \int_0^\infty \frac{t^{s-1} e^{-qt}}{1 - \lambda e^{-t}} {}_0F_2\left(\begin{matrix} - \\ b_1, b_2 \end{matrix} ; -\frac{\xi}{t}\right) dt \\ &= \int_0^\infty \frac{t^{s-1} e^{-qt}}{1-^{-t}} {}_0F_2\left(\begin{matrix} - \\ b_1, b_2 \end{matrix} ; -\frac{\xi}{t}\right) dt\end{aligned}$$

Proof. We have

$$\Gamma_\mu(\alpha,x;q;\lambda;p)+\gamma_\mu(\alpha,x;q;\lambda;p)=\sqrt{\frac{2p}{\pi}}\int_0^\infty t^{\alpha-\frac{3}{2}}e^{-t}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda)dt$$

$$=\frac{1}{\sqrt{\pi}}(\frac{p}{2})^{-\mu}\int_0^\infty t^{\alpha+\mu-1}e^{-t}\left(\int_0^\infty \tau^{\mu-\frac{1}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau\right)dt$$

$$=\frac{1}{\sqrt{\pi}}(\frac{p}{2})^{-\mu}\int_0^\infty \tau^{\mu-\frac{1}{2}}\frac{e^{-q\tau}}{1-\lambda e^{-\tau}}\left(\int_0^\infty t^{\alpha+\mu-1}e^{-t}e^{-\frac{p^2}{4t^2\tau}}dt\right)d\tau \tag{3.18}$$

Using the integral [[28], pp. 31, formula 6], we obtain

$$\int_0^\infty t^{\alpha+\mu-1}e^{-t}e^{-\frac{p^2}{4t^2\tau}}dt=\Gamma(\alpha+\mu)\,{}_0F_2\left(\begin{matrix}-\\ 1-\frac{\alpha+\mu}{2},\frac{1}{2}-\frac{\alpha+\mu}{2}\end{matrix};\frac{p^2}{16\tau}\right)$$

$$+\frac{\Gamma(-\frac{\alpha+\mu}{2})}{2}\left(\frac{p^2}{4\tau}\right)^{\frac{\alpha+\mu}{2}}{}_0F_2\left(\begin{matrix}-\\ \frac{1}{2},\frac{\alpha+\mu+2}{2},\end{matrix};\frac{p^2}{16\tau}\right)$$

$$-\frac{\Gamma(-\frac{\alpha+\mu+1}{2})}{2}\left(\frac{p^2}{4\tau}\right)^{\frac{\alpha+\mu+1}{2}}{}_0F_2\left(\begin{matrix}-\\ \frac{3}{2},\frac{\alpha+\mu+3}{2},\end{matrix};\frac{p^2}{16\tau}\right) \tag{3.19}$$

■

Proposition 3.1.2. (Recurrence relation)

$$\Gamma_\mu(\alpha+1,x;q;\lambda;p)=(\alpha+\mu)\Gamma_\mu(\alpha,x;q;\lambda;p)+p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)+\sqrt{\frac{2p}{\pi}}x^{\alpha-\frac{1}{2}}e^{-x}R_K(\frac{p}{x},-\mu-\frac{1}{2},q,\lambda) \tag{3.20}$$

$R(p)>0,-\infty<\alpha<\infty$

Proof.

$$\frac{d}{dt}\left[R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right]=\frac{d}{dt}\left[\frac{(\frac{p}{2t})^{-\mu-\frac{1}{2}}}{2}\int_0^\infty \tau^{\mu-\frac{1}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau\right]$$

$$=\frac{\mu+\frac{1}{2}}{t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)+\frac{p}{t^2}R_K\left(\frac{p}{t},-\mu+\frac{1}{2},q,\lambda-\right) \tag{3.21}$$

Differentiating $t^{\alpha-\frac{1}{2}}e^{-t}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda)$ with respect to t and by using 3.21, we get

$$\frac{d}{dt}\left[t^{\alpha-\frac{1}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda\right)\right]=(\alpha+\mu)t^{\alpha-\frac{3}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu+\frac{1}{2},q,\lambda\right)$$

$$+pt^{\alpha-\frac{5}{2}}e^{-t}R_K\left(\frac{p}{t},-\mu+\frac{1}{2},q,\lambda\right)-t^{\alpha-\frac{1}{2}}e^{-t}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda) \tag{3.22}$$

Multiplying both sides of 3.22 by $\sqrt{\frac{2p}{\pi}}$ and integrating from x to ∞ and using 3.16, we find

$$0-\sqrt{\frac{2p}{\pi}}x^{\alpha-\frac{1}{2}}e^{-x}R_K(\frac{p}{x},-\mu-\frac{1}{2},q,\lambda)=(\alpha+\mu)\Gamma_\mu(\alpha,x;q;\lambda;p)+p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)-\Gamma_\mu(\alpha+1,x;q;\lambda;p)$$

which can be also written as

$$\Gamma_\mu(\alpha+1,x;q;\lambda;p)=(\alpha+\mu)\Gamma_\mu(\alpha,x;q;\lambda;p)+p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)+\sqrt{\frac{2p}{\pi}}x^{\alpha-\frac{1}{2}}e^{-x}R_K(\frac{p}{x},-\mu-\frac{1}{2},q,\lambda)$$

■

Proposition 3.1.3. The following formula holds

$$\Gamma_{\mu-1}(\alpha,x,1;\lambda;p)-\Gamma_{\mu+1}(\alpha,x,1;\lambda;p)+\frac{2\mu+1}{p}\Gamma_\mu(\alpha+1,x,1;\lambda;p)=\lambda\frac{\partial}{\partial\lambda}\Gamma_{\mu+1}(\alpha,x,1;\lambda;p) \tag{3.23}$$

$(R(p)>0,-\infty<\alpha<\infty)$

Proof. By using 3.16, for q = 1 and the following relation [[8],(22)], we get

$$R_K(z,-\alpha+1,1,\lambda)-R_K(z,-\alpha-1,1,\lambda)+\frac{2\alpha}{z}R_K(z,-\alpha,1,\lambda)=\lambda\frac{\partial}{\partial\lambda}R_K(z,-\alpha-1,1,\lambda) \tag{3.24}$$

■

Proposition 3.1.4. (Laplace transform) Let

$$H(\tau)\begin{cases}1 & \tau>0\\ 0 & \tau<0\end{cases}$$

be the Heaviside unit step function and Lbe the Laplace transform operator. Then

$$\mathcal{L}\left\{t^{\alpha-\frac{3}{2}}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda)H(t-x);s\right\}=\sqrt{\frac{\pi}{2p}}s^{-\alpha}\Gamma_\mu(\alpha,sx;q;\lambda;sp) \tag{3.25}$$

$$\mathcal{L}\left\{t^{\alpha-\frac{3}{2}}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda)H(t-x)H(t);s\right\}=\sqrt{\frac{\pi}{2p}}s^{-\alpha}\gamma_\mu(\alpha,sx;q;\lambda;sp) \tag{3.26}$$

$(x>0,R(p)>0,-\infty<\alpha<\infty)$

Proof. We have

$$\mathcal{L}\left\{t^{\alpha-\frac{3}{2}}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda)H(t-x);s\right\}=\int_0^\infty t^{\alpha-\frac{3}{2}}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda)e^{-st}H(t-s)dt$$

$$=\int_x^\infty t^{\alpha-\frac{3}{2}}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda)e^{-st}dt$$

Substituting $t=\frac{\tau}{s},dt$, we get

$$\int_x^\infty t^{\alpha-\frac{3}{2}}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda)e^{-st}dt=s^{-\alpha+\frac{1}{2}}\int_{sx}^\infty \tau^{\alpha-\frac{3}{2}}e^{-\tau}R_K(\frac{sp}{\tau},-\mu-frac12,q,\lambda)dt$$

$$\sqrt{\frac{\pi}{2p}}s^{-\alpha}\Gamma_\mu(\alpha,sx;q;\lambda;sp)$$

The proof of 3.26 is omitted since it is quite similar as that of 3.25. ■

Proposition 3.1.5. (Parametric differentiation)

$$\frac{\partial}{\partial p}(\Gamma_\mu(\alpha,x;q;\lambda;p)) = -\frac{1}{p}\left[\mu\Gamma_\mu(\alpha,x;q;\lambda;p) + p\Gamma_{\mu-1}(\alpha-1,x;q;\lambda;p)\right] \quad (3.27)$$

Proof.

$$\frac{\partial}{\partial p}(\Gamma_\mu(\alpha,x;q;\lambda;p)) = \frac{1}{2p}\sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda)dt$$
$$+\sqrt{\frac{2p}{\pi}}\int_x^\infty t^{\alpha-\frac{3}{2}}e^{-t}\frac{\partial}{\partial p}(R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda))dt \quad (3.28)$$

We have

$$\frac{\partial}{\partial p}(R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda)) = -\frac{\mu+\frac{1}{2}}{p}\frac{(p/2t)^{-\mu-\frac{1}{2}}}{2}\int_0^\infty \Gamma^{\mu-\frac{1}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d\tau - \frac{1}{t}\frac{(p/2t)^{-\mu+\frac{1}{2}}}{2}\int_0^\infty \tau^{\mu-\frac{3}{2}}\frac{e^{-q\tau-\frac{p^2}{4t^2\tau}}}{1-\lambda e^{-\tau}}d$$
$$= -\frac{\mu+\frac{1}{2}}{p}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda) - \frac{1}{t}R_K(\frac{p}{t},-\mu-\frac{1}{2},q,\lambda) \quad (3.29)$$

Finally, by Substituting 3.29 into 3.28 we get the desired result. ■

3.1.2 The generalized extended beta function

Definition 3.1.2. The generalized extended beta function is given by

$$B_\mu(x,y;q;\lambda;p;m) = \sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt \quad (3.30)$$

where $x,y\in\mathbb{C}, 0<q\leq 1; -1\leq\lambda\leq 1, m>0$ and $R(p)>0$

Taking $\lambda = 0$ and $q = 1$ 3.30 is reduced to the extended Euler's beta function 3.9 defined by Agarwal et al [?]

Proposition 3.1.6. (Functional relations)

1. The following formula holds

$$B_\mu(x,y;q;\lambda;p;m) = B_\mu(x+1,y;q;\lambda;p;m) + B_\mu(x,y+1;q;\lambda;p;m) \quad (3.31)$$

2. Let $n\in\mathbb{N}$. Then, the following summation formula holds

$$B_\mu(x,y;q;\lambda;p;m) = \sum_{K=0}^{n} B_\mu(x+K,y+n-K;q;\lambda;p;m) \quad (3.32)$$

Proof. 1. The right-hand side of 3.31 yields to

$$\sqrt{\frac{2p}{\pi}}\int_0^1\left\{t^{x-\frac{1}{2}}(1-t)^{y-\frac{3}{2}} + t^{x-\frac{3}{2}}(1-t)^{y-\frac{1}{2}}\right\}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt$$

which, after simplification, implies

$$\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)$$

which is equal to the left-hand side of 3.31.

2. The case $n = 0$ of 3.32 holds easily. The case $n = 1$ of 3.32 is just 3.31. For the other cases we can easily proceed by induction on n.

Proposition 3.1.7. The following formula holds

$$B_\mu(x, 1-y; q; \lambda; p; m) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_\mu(x+n, 1; q; \lambda; p; m) \tag{3.33}$$

By substituting the formula

$$(1-t)^{-y} = \sum_{n=0}^{\infty} (y)_n \frac{t^n}{n!}, (|t| < 1, y \in \mathbb{C}) \tag{3.34}$$

in the right-hand of 3.33 and after interchanging the order of integral and summation, we get

Proposition 3.1.8. The following formula holds

$$B_\mu(x, y; q; \lambda; p; m) = \sum_{n=0}^{\infty} B_\mu(x+n, y+1; q; \lambda; p; m) \tag{3.35}$$

Proof. By substituting again the formula

$$(1-t)^{y-1} = (1-t)^y \sum_{n=0}^{\infty} t^n, (|t| < 1)$$

in the right-hand of 3.30 and similarly as in the proof of Proposition 2.2.2 we get the desired result. ■

Lemma 3.1.1. Let $\mathcal{M}$ be the Mellin transform operator. Then

$$\mathcal{M} R_K(z, -\alpha, q, \lambda), z \to s = 2^{s-2} \Gamma(\frac{s-\alpha}{2}) \Gamma(\frac{s+\alpha}{2}) \Phi(\lambda, \frac{s+\alpha}{2}, q) \tag{3.36}$$

where $0 < q \leq 1$, or $-1 \leq \lambda < 1, R(s) > |R(\alpha)|$ or $\lambda = 1, R(s) > \max(R(\alpha), 2 - R(\alpha))$ and $\Phi(\lambda, \frac{s+\alpha}{2}, q)$
stands for the Lerch function(see [?],[?]).

Proof.

$$\mathcal{M}\{R_K(z, -\alpha, q, \lambda), z \to s\} = \int_0^\infty z^{s-1} R_K(z, -\alpha, q, \lambda) dz = 2^{\alpha-1} \int_0^\infty z^{s-\alpha-1} (\int_0^\infty t^{\alpha-1} \frac{e^{-qt-z^2/4t}}{1-\lambda e^{-t}} dt) dz$$

$$= 2^{\alpha-1} \int_0^\infty t^{\alpha-1} \frac{e^{-qt}}{1-\lambda e^{-t}} (\int_0^\infty z^{s-\alpha-1} e^{-z^2/4t} dz) dt$$

$$= 2^{s-2} \Gamma\left(\frac{s-\alpha}{2}\right) \int_0^\infty t^{\frac{s+\alpha}{2}-1} \frac{e^{-qt}}{1-\lambda e^{-t}} dt$$

$$= 2^{s-2} \Gamma\left(\frac{s-\alpha}{2}\right) \Gamma\left(\frac{s+\alpha}{2}\right) \Phi\left(\lambda, \frac{s+\alpha}{2}, q\right)$$

■

Proposition 3.1.9. (Mellin transform) The following expression holds true

$$\mathcal{M}\left\{B_\mu(x,y;q;\lambda;p,m),p\to s\right\}=\frac{2^{s-1}}{\sqrt{\pi}}B(x+ms+\frac{m-1}{s},y+ms+\frac{m-1}{s})\Gamma\left(\frac{s-\mu}{2}\right)\Gamma\left(\frac{s+\mu+1}{2}\right)$$

$$\Phi\left(\lambda,,\frac{s+\mu+1}{2},q\right)\quad(3.37)$$

where $x,y\in\mathbb{C},m>0$ and $0<q\leq 1$, or $-1\leq\lambda<1$,
$R(s)>\max\left\{R(\mu),-1-R(\mu),-\frac{1}{2}+\frac{1}{2m}-\frac{R(x)}{m},-\frac{1}{2}+\frac{1}{2m}-\frac{R(y)}{m}\right\}$
or $\lambda=1,\ R(s)>\max\left\{R(\mu),1-R(\mu),-\frac{1}{2}+\frac{1}{2m}-\frac{R(x)}{m},-\frac{1}{2}+\frac{1}{2m}-\frac{R(y)}{m}\right\}$

Proof. $\mathcal{M}\left\{B_\mu(x,y;q;\lambda;p;m),p\to s\right\}=\int_0^\infty p^{s-1}B_\mu(x,y;q;\lambda;p;m)dp$

$$=\int_0^\infty p^{s-1}\sqrt{\frac{2p}{\pi}}\left(\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt\right)dp$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}\left(\int_0^\infty p^{s+\frac{1}{2}-1}R_K\left(\frac{p}{t^m(1-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dp\right)dt$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 t^{x+m(s+\frac{1}{2})-\frac{3}{2}}(1-t)^{y+m(s+\frac{1}{2})-\frac{3}{2}}dt\int_0^\infty u^{s+\frac{1}{2}-1}R_K(u,-\mu-\frac{1}{2},q,\lambda)du$$

$$=\sqrt{\frac{2}{\pi}}B(x+ms+\frac{m-1}{2},y+ms+\frac{m-1}{2})\int_0^\infty u^{s+\frac{1}{2}-1}R_K(u,-\mu-\frac{1}{2},q,\lambda)du$$

Finally, by using Lemma 2.0.1 we get the desired result. ■

3.2 Extended Gauss hypergeometric and confluent hypergeometric functions

We use the generalized extended beta function 3.30 to extend hypergeometric and confluent hypergeometric functions, respectively, as follows:

Definition 3.2.1. The extended Gauss hypergeometric function $F_\mu(a,b;c;z;q;\lambda;p;m)$ and the confluent hypergeometric function $\Phi_\mu(b;c;z;q;\lambda;p;m)$ are respectively defined by

$$F_\mu(a,b;c;z;q;\lambda;p;m)=\sum_{n=0}^{\infty}(a)_n\frac{B_\mu(b+n,c-b;q;\lambda;p;m)}{B(b,c-b)}\frac{z^n}{n!}\qquad(3.38)$$

$(|z|<1,R(c)>R(b)>0,0<q\leq 1,-1\leq\lambda\leq 1,m>0,R(p)>0)$.

$$\Phi_\mu(b;c;z;q;\lambda;p;m)=\sum_{n=0}^{\infty}\frac{B_\mu(b+n,c-b;q;\lambda;p;m)}{B(b,c-b)}\frac{z^n}{n!}\qquad(3.39)$$

$(z\in\mathbb{C},R(c)>R(b)>0,-1\leq\lambda\leq 1,m>0,R(p)>0)$. Taking $\lambda=0$ and $q=1$, 3.38 reduces to the extended Gauss hypergeometric function defined by Agarwal et al. ([?],Definition 2.8).

Proposition 3.2.1. Integral representation

1. The following integral representation for the extended Gauss hypergeometric function F_μ is valid

$$F_\mu(a,b;c;z;q;\lambda;p;m) = \sqrt{\frac{2p}{\pi}}\frac{1}{B(b,c-b)}\int_0^1 t^{b-\frac{3}{2}}(1-t)^{c-b-\frac{3}{2}}(1-zt)^{-a}R_K(\frac{p}{t^m(1-t)^m}, -\mu-\frac{1}{2}, q, \lambda) \tag{3.40}$$

$\arg(1-z) < \pi, R(c) > R(b) > 0, 0 < q \leq 1, -1 \leq \lambda \leq 1, m > 0, R(p) > 0)$.

2. The following integral representation for the extended confluent hypergeometric function Φ is valid

$$\Phi_\mu(b;c;z;q;\lambda;p;m) = \sqrt{\frac{2p}{\pi}}\frac{1}{B(b,c-b)}\int_0^1 t^{b-\frac{3}{2}}(1-t)^{c-b-\frac{3}{2}}e^{zt}R_K(\frac{p}{t^m(1-t)^m}, -\mu-\frac{1}{2}, q, \lambda)dt \tag{3.41}$$

$R(c) > R(b) > 0, 0 < q \leq 1, -1 \leq \lambda \leq 1, m > 0, R(p) > 0)$.

Proof. 1. By using 3.30 and the generalized binomial expansion

$$(1-zt)^{-a} = \sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!}, (|zt| < 1) \tag{3.42}$$

we get the required result.

2. Similarly as in the proof of 1.

■

Proposition 3.2.2. Differentiation formula (a) For $n \in \mathbb{N}$

$$\frac{d^n}{dz^n}\{F_\mu(a,b;c;z;q;\lambda;p;m)\} = \frac{(a)_n(b)_n}{(c)_n}F_\mu(a+n,b+n;c+n;z;q;\lambda;p;m) \tag{3.43}$$

$(|z| < 1), R(c) > R(b) > 0, 0 < q \leq 1, -1 \leq \lambda \leq 1, m > 0, R(p) > 0)$

(b) For $n \in \mathbb{N}$

$$\frac{d^n}{dz^n}\{\Phi_\mu(b;c;z;q;\lambda;p;m)\} = \frac{(b)_n}{(c)_n}\Phi_\mu(b+n;c+n;z;q;\lambda;p;m) \tag{3.44}$$

$z \in \mathbb{C}, R(c) > R(b) > 0, 0 < q \leq 1, -1 \leq \lambda \leq 1, m > 0, R(p) > 0)$

Proof. (a) For $n = 1$, we have

$$\frac{d}{dz}\{F_\mu(a,b;c;z;q;\lambda;p;m)\} = \sum_{n=1}^{\infty}(a)_n\frac{B_\mu(b+n,c-b;q;\lambda;p;m)}{B(b,c-b)}\frac{z^{n-1}}{(n-1)!}$$

$$= \sum_{0}^{\infty}(a)_{n+1}\frac{B_\mu(b+n+1,c-b;q;\lambda;p;m)}{B(b,c-b)}\frac{z^n}{n!} \tag{3.45}$$

Using identities $B(b,c-b) = \frac{c}{b}B(b+1,c-b)$ and $(a)_{n+1} = a(a+1)_n$ in 3.45, we get

$$\frac{d}{dz}\{F_\mu(a,b;c;z;q;\lambda;p;m)\}\frac{ab}{c}\sum_{n=0}^{\infty}(a+1)_n\frac{B_\mu(b+n+1,c-b;q;\lambda;p;m)}{B(b,c-b)}\frac{z^n}{n!}$$

$$= \frac{ab}{c} F_\mu(a+1, b+1; c+1; z; q; \lambda; p; m) \tag{3.46}$$

and hence

$$\frac{d}{dz}\{F_\mu(a, b; c; z; q; \lambda; p; m)\} = \frac{ab}{c} F_\mu(a+1, b+1; c+1; z; q; \lambda; p; m) \tag{3.47}$$

Then, by using 3.47 repeatedly, we get3.43.
The proof of part (b) is similar as that of part (a) ■

Proposition 3.2.3. (Transformation formulas)

1. For $\arg(1-z) < \pi$, we have

$$F_\mu(a, b; c; z; q; \lambda; p; m) = (1-z)^{-a} F_\mu(a, c-b; c; \frac{z}{z-1}; q; \lambda; p; m)$$

R(c)>R(b)>0,0<q$\leq 1, -1 \leq \lambda \leq 1, m > 0, R(p) > 0$ (3.48)

2.

$$\Phi_\mu(b; c; z; q; \lambda; p; m) = e^z \Phi_\mu(c-b; c; -z; q; \lambda; p; m) \tag{3.49}$$

$z \in \mathbb{C}, R(c) > R(b) > 0, 0 < q \leq 1, -1 \leq \lambda \leq 1, m > 0, R(p) > 0)$

Proof. Replacing t by $1-t$ in the integral representations 3.40 and 3.41. ■

3.3 Extended Appell and Lauricella hypergeometric functions

Definition 3.3.1. Extended Appell hypergeometric functions $F_{1,\mu}$, $F_{2,\mu}$ and the Lauricella hypergeometric function $F^3_{D,\mu}$ are, respectively, defined by

$$F_{1,\mu}(a, b, c; d; x; y; q; \lambda; p; m) = \sum_{n,k=0}^{\infty} (b)_n (c)_k \frac{B_\mu(a+n+k, d-a; q : \lambda; p; m)}{B(a, d-a)} \frac{x^n}{n!} \frac{y^k}{k!} \tag{3.50}$$

$|x| < 1, |y| < 1, R(d) > R(a) > 0, 0 < q \leq 1, -1 \leq \lambda \leq 1, m > 0, R(p) > 0$

$$F_{2,\mu}(a, b, c; d; e; x; y; q; \lambda; p; m) = \sum_{n,k=0}^{\infty} (a)_{n+k} \frac{B_\mu(b+n, d-b; q; \lambda; p; m)}{B(b, d-b)} \frac{B_\mu(c+k, e-c; q; \lambda; p; m)}{B(c, e-c)} \frac{x^n}{n!} \frac{y^k}{k!}$$

(3.51)

$(|x|+|y| < 1, R(d) > R(b) > 0, R(e) > R(c) > 0, 0 < q \leq 1, -1 \leq \lambda \leq 1, m > 0, R(p) > 0)$

$$F^3_{D,\mu}(a, b, c; d; e; x; y, z; q; \lambda; p; m) = \sum_{n,k,r=0}^{\infty} (b)_n (c)_k (d)_r \frac{B_\mu(a+n+k+r, e-a; q; \lambda; p; m)}{B(a, e-a)} \frac{x^n}{n!} \frac{y^k}{k!} \frac{z^r}{r!} \tag{3.52}$$

$|x| < 1, |y| < 1, |z| < 1, R(e) > R(c) > 0, 0 < q \leq 1, -1 \leq \lambda \leq 1, m > 0, R(p) > 0)$ Taking $\lambda = 0$ and $q = 1$, 3.50,3.3 and 3.52 are reduced to extended Appell hypergeometric functions $F_{1,\mu}, F_{2,\mu}$ and the Lauricella hypergeometric function $F^3_{D,\mu}$, defined by Agarwal et al[[?],Definitions 2.9, 2.10,2.11].

Proposition 3.3.1. (Integral representation) The following integral representations for the extended Appell hypergeometric functions $F_{1,\mu}$,$F_{2,\mu}$ and the Lauricella hypergeometric function $F^3_{D,\mu}$ are, respectively, valid

$$F_{1,\mu}(a,b,c;d;x;y;q;\lambda;p;m) = \sqrt{\frac{2p}{\pi}}\frac{1}{B(a,d-a)}\int_0^1 t^{a-\frac{3}{2}}(1-t)^{d-a-\frac{3}{2}}(1-xt)^{-b}(1-yt)^{-c}$$

$$\mathrm{R}_K\left(\frac{p}{t^m(1-t)^m}-\mu-\frac{1}{2},q,\lambda\right)dt \ (3.53)$$

$$F_{2,\mu}(a,b,c;d;e;x;y;q;\lambda;p;m) = \frac{2p}{\pi}\frac{1}{B(b,d-b)B(c,e-c)}\int_0^1\int_0^1 t^{b-\frac{3}{2}}(1-t)^{d-b-\frac{3}{2}}w^{b-\frac{3}{2}}(1-w)^{e-c-\frac{3}{2}}$$

$$\text{(1-xt-yw)}^{-a}_K\left(\frac{p}{t^m(1-t)^m}-\mu-\frac{1}{2},q,\lambda\right)R_K\left(\frac{w}{t^m(1-w)^m}-\mu-\frac{1}{2},q,\lambda\right)dtdw \ (3.54)$$

$$F^3_{D,\mu}(a,b,c;d;e;x;y,z;q;\lambda;p;m) = \sqrt{\frac{2p}{\pi}}\frac{1}{B(a,e-a)}\int_0^1 t^{a-\frac{3}{2}}(1-t)^{e-a-\frac{3}{2}}(1-xt)^{-b}(1-yt)^{-c}(1-zt)^{-d}$$

$$_K\left(\frac{p}{t^m(1-t)^m}-\mu-\frac{1}{2},q,\lambda\right) \ (3.55)$$

3.4 The generalized extended Riemann-Liouville fractional derivative operator

The classical Riemann-Liouville fractional derivative operator is defined by

$$D^\delta_z f(z) = \frac{1}{\Gamma(-\delta)}\int_0^z (z-t)^{-\delta-1}f(t)dt \tag{3.56}$$

where $R(\delta) < 0$. It coincides with the fractional integral of order $-\delta$. In the case $n-1 < R(\delta) < n, n \in \mathbb{N}$, we write

$$D^\delta_z f(z) = \frac{d^n}{dz^n}\left\{\frac{1}{\Gamma(n-\delta)}\int_0^z (z-t)^{n-\delta-1}f(t)dt\right\} \tag{3.57}$$

Definition 3.4.1. The generalized extended Riemann-Liouville fractional derivative is defined as follows

$$D^{\delta,\mu,p;q;\lambda;m}_z f(z) = \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1}f(t)R_K\left(\frac{pz^{2m}}{t^m(z-t)^m},-\mu-\frac{1}{2},q,\lambda\right)dt \tag{3.58}$$

where $R(\delta) > 0$,$R(p) > 0$,$R(m) > 0$,$R(\mu) \geq 0$ and $n-1 < R(\delta) < n$,$n \in \mathbb{N}$ we have

$$D_z^{\delta,\mu,p;q;\lambda;m} f(z) = \frac{d^n}{dz^n} D_z^{\delta-n,\mu,p;q;\lambda;m} f(z) = \frac{d^n}{dz^n}\left\{\frac{1}{\Gamma(n-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{n-\delta-1} f(t)\right.$$

$$\mathrm{R}_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right) dt(3.59)$$

1. Taking $\lambda = 0$ and $q = 1$ the generalized extended Riemann-Liouville fractional derivative operator 3.58 is reduced to the extended Riemann-Liouville fractional derivative operator given by Agarwal et al.[?]

$$D_z^{\delta,\mu,p;m} f(z) = \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1} f(t) K_{\mu+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(z-t)^m}\right) dt \qquad (3.60)$$

   where $R(\delta) < 0$,$R(p) > 0$,$R(m) > 0$,$R(\mu) > 0$

2. If $\lambda = 0$, $q = 1$, $m = 0$, $\mu = 0$ and $p \to 0$, then the generalized extended Riemann-Liouville fractional derivative operator 3.58 reduces to the classical Riemann-Liouville fractional derivative operator 3.56.
   In order to calculate generalized extended fractional derivatives for some functions, we give two results concerning the generalized extended Riemann-Liouville fractional derivative operator of some elementary functions which will be useful in the sequel.

Lemma 3.4.1. Let $R(\delta) < 0$. Then, we have

$$D_z^{\delta,\mu,p;q;\lambda;m}\{z^\beta\} = \frac{z^{\beta-\delta}}{\Gamma(-\delta)} B_\mu(\beta+\frac{3}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m) \qquad (3.61)$$

Proof. Using Definition 2.4.1 and a local setting $t = zu$, we obtain

$$D_z^{\delta,\mu,p;q;\lambda;m}\{z^\beta\} = \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1} t^\beta R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right) dt$$

$$= \frac{z^{\beta-\delta}}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^1 (1-u)^{(-\delta+\frac{1}{2})-\frac{3}{2}} u^{(\beta+\frac{3}{2})-\frac{3}{2}} R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right) du$$

$$= \frac{z^{\beta-\delta}}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}} B_\mu(\beta+\frac{3}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m)$$

More generally, we give the generalized extended Riemann-Liouville fractional derivative of an analytic function $f(z)$ at the origin. ■

Lemma 3.4.2. Let $R(\delta) < 0$. If a function $f(z)$ is analytic at the origin, then

$$D_z^{\delta,\mu,p;q;\lambda;m}\{f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu,p;q;\lambda;m}\{z^n\}$$

Proof. Since f is analytic at the origin, its Maclaurin expansion is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (for $|z| < \rho$ with $\rho \in \mathbb{R}^+$ is the convergence radius). By substituting entire power series in Definition 2.4.1, we obtain

$$D_z^{\delta,\mu,p;q;\lambda;m}\{f(z)\} = \frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1} R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right)\sum_{n=0}^{\infty} a_n t^n dt$$

By virtue of the uniform continuity on the convergence disk, we can do integration term by term in the equation above. Thus

$$\begin{aligned} D_z^{\delta,\mu,p;q;\lambda;m}\{f(z)\} &= \sum_{n=0}^{\infty} a_n \left\{\frac{1}{\Gamma(-\delta)}\sqrt{\frac{2p}{\pi}}\int_0^z (z-t)^{-\delta-1} R_K\left(\frac{pz^{2m}}{t^m(z-t)^m}, -\mu-\frac{1}{2}, q, \lambda\right) t^n dt\right\} \\ &= \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu,p;q;\lambda;m}\{z^n\} \end{aligned}$$

■

Corollary 3.4.1.

$$D_z^{\delta,\mu,p;q;\lambda;m}\{(1-z)^{-\alpha}\} = \frac{z^{-\delta}}{\Gamma(-\delta)} B(\frac{3}{2}, -\delta+\frac{1}{2}) F_\mu(\alpha, \frac{3}{2}, -\delta+2; z; q; \lambda; p; m)$$

where $R(\alpha) > 0$ and $R(\delta) < 0$.

Proof. Using binomial Theorem for $(1-z)^{-\alpha}$ and Lemma 2.4.1, we obtain:

$$\begin{aligned} D_z^{\delta,\mu,p;q;\lambda;m}\{(1-z)^{-\alpha}\} &= D_z^{\delta,\mu,p;q;\lambda;m}\left\{\sum_{n=0}^{\infty}(a)_n \frac{z^n}{n!}\right\} = \sum_{n=0}^{\infty}\frac{(a)_n}{n!} D_z^{\delta,\mu,p;q;\lambda;m}\{z^n\} \\ &= \frac{z^{-\delta}}{\Gamma(-\delta)}\sum_{n=0}^{\infty}(\alpha)_n B_\mu(n+\frac{3}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m)\frac{z^n}{n!} \end{aligned}$$

Hence the result. ■

Combining previous Lemmas, we obtain the generalized extended derivative of the product of analytic function with a power function.

Theorem 3.4.1. Let $R(\delta) < 0$. Suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $(|z| < \rho)$ for some $\rho \in \mathbb{R}^+$. Then we have

$$D_z^{\delta,\mu,p;q;\lambda;m}\{z^{\beta-1}f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\delta,\mu,p;q;\lambda;m}\{z^{\beta+n-1}\} = \frac{z^{\beta-\delta-1}}{\Gamma(-\delta)}\sum_{n=0}^{\infty} a_n \beta_\mu(\beta+n+\frac{1}{2}, -\delta+\frac{1}{2}; p; q; \lambda; m) z^n \tag{3.62}$$

A subsequent result can be given as follows

Theorem 3.4.2. For $R(\delta) > R(\beta) > -\frac{1}{2}$, we have

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} = \frac{z^{\delta-1}}{\Gamma(\delta-\beta)} B(\beta+\frac{1}{2}, \delta-\beta+\frac{1}{2}) F_\mu(\alpha, \beta+\frac{1}{2}; \delta+1; z; q; \lambda; p; m)(|z| < 1; \alpha \in \mathbb{C}) \tag{3.63}$$

Proof. The result is easily established by taking $f(z) = (1-z)^{-\alpha}$, so we have

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} = D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}\sum_{k=0}^{\infty}(a)_k\frac{z^k}{k!}\}$$

$$= \sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta+k-1}\}$$

$$= \sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}\frac{B_\mu(\beta+k+\frac{1}{2},\delta-\beta+\frac{1}{2};p;q;\lambda;m)}{\Gamma(\delta-\beta)}z^{\delta+k-1}$$

By the expression 3.38, we get

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha}\} = \frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})F_\mu(\alpha,\beta+\frac{1}{2};\delta+1;z;q;\lambda;p;m)$$

■

Theorem 3.4.3. For$R(\delta) > R(\beta) > -\frac{1}{2}$, $R(\alpha) > 0$,$R(\gamma) > 0$,$|az| < 1$ and $|bz| < 1$. Then, the following generating relation holds true

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\} = \frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})F_{1,\mu}(\beta+\frac{1}{2},\alpha,\gamma,\delta+1;az,bz;q;\lambda;p;m \tag{3.64}$$

Proof. By applying the binomial Theorem to $(1-az)^{-\alpha}$ and $(1-bz)^{-\gamma}$ and making use of Lemmas 2.4.1and 2.4.2 we obtain

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\} = D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}\sum_{k=0}^{\infty}\}\sum_{r=0}^{\infty}(a)_k(\gamma)_r\frac{(az)^k}{k!}\}\frac{(bz)^r}{r!}\}$$

$$= \sum_{k,r=0}^{\infty}(\alpha)_t(\gamma)_r D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta+k+r-1}\frac{a^k}{k!}\frac{b^r}{r!}\}$$

$$= z^{\delta-1}\sum_{k,r=0}^{\infty}(\alpha)_t(\gamma)_r\frac{B_\mu(\beta+k+r+\frac{1}{2},\delta-\beta+\frac{1}{2};p;q;\lambda;m)}{\Gamma(\delta-\beta)}\frac{az^k}{k!}\frac{bz^r}{r!}$$

By using 3.50 we can get
$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}\} = \frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})F_{1,\mu}(\beta+\frac{1}{2},\alpha,\gamma;\delta+1;az;bz;q;\lambda;p;m)$ ■

Theorem 3.4.4. For $R(\delta) > R(\beta) > -\frac{1}{2}$, $R(\alpha) > 0$, $R(\gamma) > 0$, $R(\tau) > 0$, $|az| < 1$, $|bz| < 1$ and $|cz| < 1$ then we have

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}(1-bz)^{-\gamma}(1-cz)^{-\tau}\}$$
$$= \frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})F_{D,\mu}^2(\beta+\frac{1}{2},\alpha,\gamma,\tau;\delta+1;az;bz;q;\lambda;p;m) \tag{3.65}$$

Proof. The proof is similar to that of Theorem 2.4.3 it is sufficient to use the binomial Theorem for $(1-az)^{-\alpha}$, $(1-bz)^{-\gamma}$, $(1-cz)^{-\tau}$ then applying Lemmas 2.4.1 and 2.4.2. ■

Theorem 3.4.5. For $R(\delta) > R(\beta) > -\frac{1}{2}$, $R(\alpha) > 0$, $R(\tau) > R(\gamma) > 0$, $\left|\frac{x}{1-z}| < 1\right|$, and $|x| + |z| < 1$, we have

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-az)^{-\alpha}F_\mu(\alpha,\gamma,\tau;\frac{x}{1-z};q;\lambda;p;m)\}$$
$$= z^{\delta-1}\frac{B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})}{\Gamma(\delta-\beta)}F_{2,\mu}(\alpha,\gamma,\beta+\frac{1}{2},\tau;\delta+1;x;z;q;\lambda;p;m) \tag{3.66}$$

Proof. By the binomial formula and according to Definition 2.2.1, we expand $z^{\beta-1}(1-z)^{-\alpha}F_\mu(\alpha,\gamma,\tau;\frac{x}{1-z};q;\lambda;p;m)$ to get

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}F_\mu(\alpha,\gamma,\tau;\frac{x}{1-z};q;\lambda;p;m)\right\}$$
$$= D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}\sum_{n=0}\infty\frac{(\alpha)_n}{n!}\frac{B(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}\left(\frac{x}{1-z}\right)^n\right\}$$
$$= \sum_{n=0}^{\infty}(\alpha)_n\frac{B(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha-n}\}\frac{x^n}{n!}$$

In order to exhibit $F_{2,\mu}$ we apply Theorem 2.4.2 for $D_z^{\beta-\delta,\mu;p;q;\lambda;m}\{z^{\beta-1}(1-z)^{-\alpha-n}\}$ and substitute the extended hypergeometric function F_μ by its series representation, we obtain

$$D_z^{\beta-\delta,\mu;p;q;\lambda;m}\left\{z^{\beta-1}(1-z)^{-\alpha}F_\mu(\alpha,\gamma,\tau;\frac{x}{1-z};q;\lambda;p;m)\right\}$$
$$= \frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})\sum_{n,k=0}^{\infty}(\alpha)_{n+k}\frac{B_\mu(\gamma+n,\tau-\gamma;q;\lambda;p;m)}{B(\gamma,\tau-\gamma)}\times\frac{B_\mu(\beta+k+\frac{1}{2},\delta-\beta+\frac{1}{2},q,\lambda;p;m}{B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})}$$
$$= \frac{z^{\delta-1}}{\Gamma(\delta-\beta)}B(\beta+\frac{1}{2},\delta-\beta+\frac{1}{2})F_{2,\mu}(\alpha,\gamma,\beta+\frac{1}{2},\tau;\delta+1;x;z;q;\lambda;q;m)$$

This completes the proof. ■

Proposition 3.4.1. (Mellin transform)
The following expression holds true

$$\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}z^\beta,p\to s\} = 2^{s-1}z^{\beta-\delta}\frac{1}{\sqrt{\pi}}B(\beta+m(s+\frac{1}{2})+1,-\delta+m(s+\frac{1}{2}))$$
$$\times\Gamma\left(\frac{s-\mu}{2}\Gamma\right)\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda,\frac{s+\mu+1}{2},q\right) \tag{3.67}$$

for $R(\mu) \geq 0$, $m > 0$ and $R(s) > \max\{R(\mu), -\frac{1}{2}-\frac{1}{m}-\frac{R(\beta)}{m}, \frac{R(\delta)}{m}-\frac{1}{2}\}$.

Proof. We can prove this result by applying Mellin transform and using Lemma 2.4.1.

$$\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}z^\beta,p\to s\} = \frac{1}{\Gamma(-\delta)}\int_0^\infty p^{s-1}z^{\beta-\delta}B_\mu(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m)dp$$
$$= \frac{z^{\beta-\delta}}{\Gamma(-\delta)}\int_0^\infty p^{s-1}B_\mu(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m)dp$$

As the last integral is the Mellin transform of $B_\mu(\beta+\frac{3}{2},-\delta+\frac{1}{2};p;q;\lambda;m)$ the result immediately follows via Proposition 2.4.1. ■

Proposition 3.4.2. The following expression holds true

$$\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}(1-z)^{-\beta}, p \to s\} = 2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}B(+m(s+\frac{1}{2})+1, -\delta+m(s+\frac{1}{2}))\Gamma\left(\frac{s-\mu}{2}\Gamma\right)\left(\frac{s+\mu+1}{2}\right)$$

$$\times\Phi\left(\lambda, \frac{s+\mu+1}{2}, q\right)F_2^1(\beta, m(s+\frac{1}{2})+1, -\delta+m(2s+1)+1; z) \qquad (3.68)$$

where $R(\mu) \geq 0$,$R(\delta) < 0$,$m > 0$,$|z| < 1$, $R(s) > \max\{R(\mu), -\frac{1}{2}+\frac{1}{m}, \frac{\delta}{m}-\frac{1}{2}\}$ and F_2^1 is the well-known Gauss hypergeometric function.

Proof. The result can be proved using the binomial Theorem for $(1-z)^{-\alpha}$ and the Mellin transform of the general term. Indeed,

$$\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}(1-z)^{-\beta}, p \to s\} = \mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}\sum_{n=0}^{\infty}(\alpha)_n\frac{z^n}{n!}, p \to s\} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\mathcal{M}\{D_z^{\delta,\mu;p;q;\lambda;m}z^n, p \to s$$

$$= \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^{n-\delta}\frac{1}{\sqrt{\pi}}B(n+m(s+\frac{1}{2})+1, -\delta+m(s+\frac{1}{2}))\Gamma\left(\frac{s-\mu}{2}\Gamma\right)\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda, \frac{s+\mu+1}{2}, q\right)$$

$$= 2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{s-\mu}{2}\Gamma\right)\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda, \frac{s+\mu+1}{2}, q\right)\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}B(n+m(s+\frac{1}{2})+1, -\delta+m(s+\frac{1}{2}))z^n$$

$$= 2^{s-1}z^{-\delta}\frac{1}{\sqrt{\pi}}B\left(n+m(s+\frac{1}{2})+1, -\delta+m(s+\frac{1}{2})\right)\Gamma\left(\frac{s-\mu}{2}\Gamma\right)\left(\frac{s+\mu+1}{2}\right)\Phi\left(\lambda, \frac{s+\mu+1}{2}, q\right)$$

$$\times {}_2F_1(\beta, m(s+\frac{1}{2})+1; -\delta+m(s+1)+1; z) \quad \blacksquare$$

3.5 Generating function involving the extended generalized Gauss hypergeometric function

In this section, we establish some generating functions for the generalized Gauss hypergeometric functions.

Theorem 3.5.1. Let $R(\beta) > 0$ and $R(\gamma) > R(\alpha) > -\frac{1}{2}$. Then we have

$$\sum_{n=0}^{\infty}F_\mu(\beta+n, \alpha+\frac{1}{2}; \gamma+1; \frac{z}{1-t}; q; p; \lambda; m), \quad where|z| < \min\{1, |1-t|\} \qquad (3.69)$$

Proof. By considering the following elementary identity

$$(1-z)^{-\beta}\left(1-\frac{t}{1-z}\right)^{-\beta} = (1-t)^{-\beta}\left(1-\frac{z}{1-t}\right)^{-\beta}$$

and expanding its left-hand side to give

$$(1-z)^{-\beta}\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}\left(\frac{t}{1-t}\right) = (1-t)^{-\beta}\left(1-\frac{z}{1-t}\right)^{-\beta} \quad for|t| < |1-z| \qquad (3.70)$$

Multiplying both sides of 3.70 by $z^{\alpha-1}$ and applying the extended Riemann-Liouville fractional derivative operator $D^{\alpha-\gamma;\mu;q;p;\lambda;m}$ we find

$$D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{\sum_{n=0}^{\infty}\frac{(\beta)_n t^n}{n!}z^{\alpha-1}(1-z)^{-\beta-n}\right\} = D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{(1-t)^{-\beta}z^{\alpha-1}\left(1-\frac{z}{1-t}\right)^{-\beta}\}$$

Uniform convergence of the involved series allows us to permute the summation and fractional derivative operator to get

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{z^{\alpha-1}(1-z)^{-\beta-n}\}t^n = (1-t)^{-\beta}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\{z^{\alpha-1}(1-\frac{z}{1-t})^{-\beta}\} \tag{3.71}$$

The result easily follows using Theorem 2.4.2. ■

Theorem 3.5.2. Let $R(\beta)>0$, $R(\tau)>0$ and $R(\gamma)>R(\alpha)>-\frac{1}{2}$ then we have

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_\mu(\beta-n,\alpha+\frac{1}{2};\gamma+1;z;q;p;\lambda;m)t^n = (1-\beta)^{-\beta}F_{1,\mu}(\alpha+\frac{1}{2},\tau;\beta;\gamma+1;z;\frac{-zt}{1-t};q;p;\lambda;m)$$

where $|z|<1$,$|t|<|1-z|$ and |z||t|<|1-t|

Proof. By considering the following identity

$$[1-(1-z)t]^{-\beta} = (1-t)^{-\beta}(1+\frac{zt}{1-t})^{-\beta}$$

and expanding its left-hand side as power series, we get

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}(1-z)^n t^n = (1-t)^{-\beta} \quad for|t|<|1-z|$$

Multiplying both sides by $z^{\alpha-1}(1-z)^{-\tau}$ and applying the definition of the extended Riemann-Liouville fractional derivative operator $D^{\alpha-\gamma;\mu;q;p;\lambda;m}$ on both sides, we find

$$D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}z^{\alpha-1}(1-z)^{-\tau}(1-z)^n t^n\right\} = D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{(1-t)^{-\beta}z^{\alpha-1}(1-z)^{-\tau}(1-\frac{-zt}{1-t})^{-\beta}\right.$$

Interchanging the order of the summation and fractional derivative under the given conditions, we obtain

$$\sum_{n=0}^{\infty}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{z^{\alpha-1}(1-z)^{-\tau+n}\right\}t^n = (1-t)^{-\beta}D^{\alpha-\gamma;\mu;q;p;\lambda;m}\left\{z^{\alpha-1}(1-z)^{-\tau+}(1-\frac{z}{1-t})^{-\beta}\right\}$$

Finally, the desired result follows by Theorems 2.4.2 and 2.4.3. ■

Theorem 3.5.3. Let $R(\xi)>R(\upsilon)>-\frac{1}{2}$, $R(\gamma)>R(\alpha)>-\frac{1}{2}$ and $R(\beta)>0$ then we have

$$\sum_{n=0}^{\infty}F_\mu(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m)F_\mu(-n,\upsilon+\frac{1}{2};\xi+1;u;q;\lambda;p;m)t^n$$

$$= (1-t)^{-\beta}F_{2,\mu}(\beta,\alpha+\frac{1}{2},\upsilon+\frac{1}{2};\gamma+1,\xi+1,\frac{z}{1-t},\frac{-ut}{1-t},q,;\lambda;p;m)$$

where $|z|<1$, $|\frac{1-u}{1-z}t|<1$ and $\left|\frac{z}{1-t}\right|+\left|\frac{ut}{1-t}\right|<1$.

Proof. By replacing t by $(1-u)t$ in 3.69 and multiplying both sides of the resulting identity by $u^{\upsilon-1}$, we get

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_{\mu}(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m)u^{\upsilon-1}(1-u)^n t^n$$

$$=u^{\upsilon-1}[1-(1-z)t]^{-\beta}F_{\mu}(\beta,\alpha+\frac{1}{2};\gamma+1;\frac{z}{1-(1-u)t};q;\lambda;p;m) \qquad (3.72)$$

where $R(\beta)>0$ and $R(\gamma)>R(\alpha)>-\frac{1}{2}$.

Next, applying the fractional derivative $D^{\upsilon-\xi,\mu,q,\lambda,p,m}$ to both sides of 3.72 and changing the order of the summation and the fractional derivative under conditions $|z|<1$, $|\frac{1-u}{1-z}t|<1$ and $|\frac{z}{1-t}|+|\frac{ut}{1-t}|<1$, , yields

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_{\mu}(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m)D^{\upsilon-\xi,\mu,q,\lambda,p,m}\{u^{\upsilon-1}(1-u)^n\}t^n$$

$$=D^{\upsilon-\xi,\mu,q,\lambda,p,m}\left\{u^{\upsilon-1}[1-(1-u)t]^{-\beta}F_{\mu}(\beta,\alpha+\frac{1}{2};\gamma+1;\frac{z}{1-(1-u)t},q,\lambda;p;m)\right\}$$

The last identity can be written as follows:

$$\sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}F_{\mu}(\beta+n,\alpha+\frac{1}{2};\gamma+1;z;q;\lambda;p;m)D^{\upsilon-\xi,\mu,q,\lambda,p,m}\{u^{\upsilon-1}(1-u)^n\}t^n$$

$$=(1-t)^{-\beta}D^{\upsilon-\xi,\mu,q,\lambda,p,m}\left\{u^{\upsilon-1}[1-(1-u)t]^{-\beta}F_{\mu}\left(\beta,\alpha+\frac{1}{2};\gamma+1,\frac{\frac{z}{1-t}}{1-\frac{-ut}{1-t}};q;\lambda;p;m\right)\right\}$$

Thus, by using Theorems 2.4.2 and 2.4.5 in the resulting identity, we obtain the desired result.

■

Bibliography

[1] H. Abbas, A. Azzouz, M. B. Zahaf and M. Belmekki, *Generalized extended riemann-liouville type fractional derivative operator*, Kragujevac Journal of Mathematics **47** (2023), 57–80.

[2] P. Agarwal, J. Choi and R. B. Paris, *Extended riemann-liouville fractional derivative operator and its applications*, Journal of Nonlinear Science and Applications (JNSA) **8** (2015), 451–466.

[3] P. Agarwal, J. J. Nieto and M.-J. Luo, *Extended riemann-liouville type fractional derivative operator with applications*, Open Mathematics **15** (2017), 1667–1681.

[4] G. Andrews, R. Askey and R. Roy, *Special functions (cambridge university press, cambridge, 1999).* .

[5] M. Aslam Chaudhry, A. Qadir, M. Rafique and S. Zubair, *Extension of euler's beta function*, Journal of Computational and Applied Mathematics **78** (1997), 19–32.

[6] D. Baleanu, P. Agarwal, R. Parmar, M. Alqurashi and S. Salahshour, *Extended riemann–liouville fractional derivative operator and its applications*, J. Nonlinear Sci. Appl. **10** (2015), 2914–2924.

[7] M. Bohner, G. Rahman, S. Mubeen and K. NISAR, *A further extension of the extended riemann-liouville fractional derivative operator*, Turkish Journal of Mathematics **42** (2018), 2631–2642.

[8] M. Boudjelkha, *Extended riemann bessel functions*, in: *Conference Publications*, vol. 2005, American Institute of Mathematical Sciences, 2005, p. 121.

[9] M. A. Chaudhry and S. M. Zubair, *Generalized incomplete gamma functions with applications*, Journal of Computational and Applied Mathematics **55** (1994), 99–123.

[10] M. A. Chaudhry and S. M. Zubair, *On an extension of generalized incomplete gamma functions with applications*, The ANZIAM Journal **37** (1996), 392–405.

[11] M. A. Chaudhry and S. M. Zubair, *On a class of incomplete gamma functions with applications*, Chapman and Hall/CRC, 2001.

[12] M. W. Coffey and S. J. Johnston, *Some results involving series representations of hypergeometric functions*, Journal of computational and applied mathematics **233** (2009), 674–679.

[13] A. R. Forsyth, *Theory of differential equations*, vol. 2, CUP Archive, 1959.

[14] I. S. Gradshteyn and I. Ryzhik, *Table of integrals, series, and products. seventh*, Elsevier/Academic Press, Amsterdam **48** (2007), 1171.

[15] H. J. Haubold, A. M. Mathai and R. K. Saxena, *Mittag-leffler functions and their applications*, Journal of applied mathematics **2011**.

[16] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204, elsevier, 2006.

[17] I. Kıymaz, P. Agarwal, S. Jain and A. Çetinkaya, *On a new extension of caputo fractional derivative operator*, Advances in Real and Complex Analysis with Applications (2017), 261–275.

[18] N. Lebedev, *Special functions and their applications (translated from the russian and edited by ra silverman) prentice-hall*, Englewood Cliffs, New Jersey .

[19] M.-J. Luo, G. V. Milovanovic and P. Agarwal, *Some results on the extended beta and extended hypergeometric functions*, Applied Mathematics and Computation **248** (2014), 631–651.

[20] M.-J. Luo, G. V. Milovanovic and P. Agarwal, *Some results on the extended beta and extended hypergeometric functions*, Applied Mathematics and Computation **248** (2014), 631–651.

[21] A. M. Mathai and R. K. Saxena, *The H-function with applications in statistics and other disciplines*, John Wiley & Sons, 1978.

[22] C. Milici, G. Drăgănescu and J. T. Machado, *Introduction to fractional differential equations*, vol. 25, Springer, 2018.

[23] F. W. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *NIST handbook of mathematical functions hardback and CD-ROM*, Cambridge university press, 2010.

[24] M. A. Özarslan and E. ÖzergiÌn, *Some generating relations for extended hypergeometric functions via generalized fractional derivative operator*, Mathematical and Computer Modelling **52** (2010), 1825–1833.

[25] E. Özergin, *Some properties of hypergeometric functions*, Ph.D. thesis, Eastern Mediterranean University (EMU), 2011.

[26] E. Özergin, M. A. Özarslan and A. Altın, *Extension of gamma, beta and hypergeometric functions*, Journal of Computational and Applied Mathematics **235** (2011), 4601–4610.

[27] R. K. Parmar and R. Raina, *On the extended incomplete pochhammer symbols and hypergeometric functions*, arXiv preprint arXiv:1701.04159 .

[28] A. Prudnikov, *Integrals and series: direct laplace transforms.*

[29] R. A. Silverman et al., *Special functions and their applications*, Courier Corporation, 1972.

[30] N. M. Temme, *Special functions: An introduction to the classical functions of mathematical physics*, John Wiley & Sons, 1996.

[31] F. Whipple, *On series allied to the hypergeometric series with argument- 1*, Proceedings of the London Mathematical Society **2** (1930), 81–94.