Popular and Democratic Republic of Algeria
Ministry of the Higher Education and Scientific Research

Academic Year: 2022/2023

Fractional differential equations in b-metric spaces

Thesis Submitted of Requirements for the Degree of

Academic Masters

Dr Tahar Moulay University - SAIDA
Discipline : MATHEMATICS
Speciality : Mathematical Analysis

by

Saadia Saidi [1]

Under The Supervision of

Dr. Fatima Zohra **M**ostefai

Thesis presented and supported publicly on June 4, 2023
in front of the Jury made up of

Pr. OUAKKAS Sedik	University Dr Tahar Moulay - SAIDA	President
Dr. Fatima Zohra Mostefai	University Dr Tahar Moulay - SAIDA	Supervisor
Dr. Noria Bekkouche	University Dr Tahar Moulay - SAIDA	Jury
Dr. Lahcen Rabhi	University Dr Tahar Moulay - SAIDA	Jury

1. saidisaadia77@gmail.com

بسم الله الرحمن الرحيم

Dedication

I dedicate this modest work :

To my parents. I am grateful to my parents, who have given all the love and care,"Thank you mom, thank you dad";

To my husband Berrezoug;

To my sonssalah eddine,Khaled ben oualid and Lahcen;

To all my brothers and all my sister;

To soumia,Amina,Lahcen,youcef and ezine;

To all my friends;

All the relatives I mentioned and the others I forgot please excuse me.

Saadia Saidi

Acknowledgement

In the name of allah,the most gracious,the most merciful.

First of all,I would like to thank "ALLAH" almighty who provided me the power to complete this thesis.

I would like to express my sincere gratitude to my supervisor, Dr. Mostfai Fatima Zohra . Their helpful ,advice, patience,motivation,and their encouragement helped me to complete this research and write this thesis.

I also thank the members of the jury, Pr.OUAKKAS Seddik for agreeing to be the president of my thesis, Dr BEKKOUCHE Noria,Dr RABHI Lahcen for accepting to discuss my thesis.I also thank the members of the mathematics department of the university of Saida.

Many thanks go in particular to all my teachers Abess Hafida,Rahmani saadia Bouroumi Rekia and all. I will never forget my teachers,from the primary to the university.Finally,I would also like to thank my friends .To every one help me in my study. of course my thanks and gratitude to my family.

Saadia Saidi

Abstract

Fractional differential equations in b-metric spaces

The aim of this thesis is to introduce the concept of b-metric space which is a natural and novel extension of the standard metric space. We also introduce the notion of a generalized α-Geraghty contraction type mapping in b-metric spaces which is one of the interesting generalizations of Banach contraction principle.We state the existence and uniqueness of a fixed point theorems for this mapping. Then we apply the obtained theorem to study the existence of solutions to the fractional differential equations(in Caputo sense) in the setting of b-metric space.

This work is illustrated with examples of applications. The results of chapter 2 are found in the works [1].[2].[3].‘

Keywords : b-metric space,Geraghty contraction type mapping,fixed point,fractional differential equations(in Caputo sense).

Résumé

Équations différentielles fractionnaires .

dans les espaces b-métriques.

L'objectif de cette thèse est d'introduire le concept d'espace b-métrique qui est une extension naturelle et nouvelle de l'espace métrique standard. Nous introduisons également la notion d'application généralisée de type contraction α-Geraghty dans les espaces b-métriques qui est une des généralisations intéressantes du principe de contraction de Banach. Nous énonçons des theorémes d'existence et d'unicité d'un point fixe pour de telles applications. Ensuite, nous appliquons le théorème obtenu pour étudier l'existence de solutions aux équations différentielles fractionnaires (au sens de Caputo) dans le contexte de l'espace b-métrique.
On illustre ce travail par des exemples d'applications. Les résultats de chapitres 2 se trouvent dans les travaux [1].[2].[3].

Mots Clés : Espace b-métrique,l'applications de type contraction de Geraghty, Point fixe,équations différentielles fractionnaires (au sens de Caputo).

ملخص

المعادلات التفاضلية الكسرية في الفضاءات b–المترية

الهدف من هذه الأطروحة هو تقديم مفهوم الفضاء b–المتري والذي يعد امتدادًا طبيعيًا وجديدًا للفضاء المتري . وتقديم مفهوم التطبيق المعمم للتقلص من النـوع α جيرافتي في الفضاء b–المتري وهو أحد التعميمات المثيرة للاهتمام لمبدأ التطبيق المقلص في فضاء بناخ . بالإضافة ذكرنا وجود وتفرد نظرية النقطة الثابتة لهذا التطبيق .

بعد ذلك قمنا بدراسة وجود حلول للمعادلات التفاضلية الكسـرية (بمعنى كابتو) في سياق الفضاء b–المتري بتطبيق نظرية النقطة الثابتة .

أرفقنا هذا العمل بأمثلة .تطبيقية ، يمكن العثور على نتائج الفصل الثاني في الأعمال[3].[2].[1]

الكلمات المفتاحية : الفضاء b–المتري ، التطبيق المعمم للتقلص من النـوع α جيرافتي ، النقطة الثابتة ، المعادلات التفاضلية الكسرية (بمعن نكابتو)

Notations

* $d(.,.)$ **:** Distance on metric spaces .

* $d_s(.,.)$ **:** Distance on b-metric spaces.

* s **:**Coefficient of b-metric spaces

* $Lp([a,b])$ **:** Lebesgue spaces.

* $\mathbb{N}$:The set of natural numbers.

* $\mathbb{R}$,$\mathbb{R}^+$:The set of all real numbers, the set of non negative real numbers.

* $\| f \|_{L_p}$: Norm in $Lp([a,b])$.

* (x_n) :Sequence of elements.

* ε :Designates a parameter that is > 0 and approaches zero.

* $T, f : X \to X$:Self map on X.

* $Fix(T)$: The set of all fixed points of T.

* $\kappa(t, u, x(u))$:The kernel of the integral equation.

* $C([a,b])$:The set of all real valued continuous functions on [a, b].

* $\mathcal{F}$, $\mathcal{F}_f$ the set of Geraphty functions.

* $C(J,\mathbb{R})$: the Banach space of all continuous function from J into $\mathbb{R}$.

* $\| y \|_{\infty} = \sup\{| y(t) |; t \in J\}$:Norm in$C(J,\mathbb{R})$.

* ${}^{c}\mathrm{D}^{\alpha}, {}^{c}\mathrm{D}^{\beta}$: The Caputo derivatives of order α, β.

* D^{α} :The Riemann-Lioville derivative of order α.

* I^{α} :The fractional integral of order α for an integrable function.

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Introduction

The Banach contraction principle is a useful tool in the study of many branches of mathematics and mathematical sciences (economics, computer science and engineering). This principle was improved, generalized and extended in various ways and many fixed point results were obtained. One of the interesting generalizations of this basic principle was given by Geraghty [14]in 1973 by considering an auxiliary function. $\beta : [0, \infty) \to [0, 1)$ which satisfies the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \textit{ implies } \lim_{n\to\infty} t_n = 0$$

After that, Geraghty's result was generalized and many fixed point results were stated in many ways [17], [10],[16],[19]].

In 2004, Ran and Reurings stated a generalization of Banach contraction principle by using a partial order on a metric space.

In 2008, Suzuki [31] proved a generalization of Banach contraction principle by using a contraction condition depending on a non-increasing function Φ
the class of the function $\phi : [0, \infty) \to [0, \infty)$satisfying the following conditions

(a) ϕ is nondecreasing,

(b) ϕ is continuous,

(c) $\phi(t) = 0$ if and only if $t = 0$,

(d) ϕ is subadditive, that is $\phi(c + t) \leq \phi(c) + \phi(t)$

In 2012, Samet et al.[11] introduced the notion of α-admissible mappings. Karapinar et al. [10] defined the concept of triangular α-admissible mappings.

In 2013, Cho introduced the notion of α-Geraghty contraction type mappings and assured the unique fixed point theorems for such mappings in complete metric spaces.

Popescu [25] generalized the obtained results in [17] by using the concept of triangular α-orbital admissible mappings and studied other conditions to prove the existence and uniqueness of a fixed point of α-Geraghty contraction type mappings in complete metric spaces.
In 2015, Kumam et al. [22] introduced a new generalized quasi-contraction by adding four new values .

Another way to generalize Banach contraction principle, many authors was replaced the given metric space by some generalized metric space and stated analogues of fixed point theorem on metric spaces.

In 1989, Backhtin introduced the concept of b-metric space. In 1993, Czerwik extended the results of b-metric spaces, which is a natural and novel extension of the standard metric spaces.
The difference of b-metric from the standard metric is the triangle inequality. In the b-metric notion, the following inequality is used

$$d_s(x,z) \leq s[d_s(x,y) + d_s(y,z)], \text{ for some } s \geq 1 \text{and all } x,y,z \in X$$

Note that b-metric is a generalization of a metric. The first important difference between a metric and a b-metric is that the b-metric need not be a continuous function in its two variables. This led to many fixed point theorems on b-metric spaces being stated.

In 2011, Dukic and al. [12] generalized the class of Geraphty functions $\mathcal{F}$ to the class of functions $\mathcal{F}_s$ for some $s \geq 1$. By using the function $\beta : [0,\infty) \to [0, \frac{1}{s})$ which satisfies the condition

$$\lim_{n\to\infty} \beta(t_n) = \frac{1}{s} \textit{ implies } \lim_{n\to\infty} t_n = 0$$

the authors stated the existence and uniqueness of a fixed point for Geraghty contraction type mapping in b-metric spaces.

Recently, Samet et al. [27] introduced the notion of α-admissible mappings to combine some existing fixed point results in distinct setting. This idea was extended by Karapinar and Samet in[27] by introducing the notion of generalized α-contractive type mappings.

In the last few years, fractional calculus concepts were frequently applied to other disciplines, especially dealing with physical phenomena. In most of the available literature, fractional integral equations play an essential role in the qualitative analysis of the solutions for fractional differential equations [23]. Very recently, H. Afshari, S. Kalantari and E. Karapinar [1] investigated the existence of solutions for some fractional differential equations in metric and b-metric spaces.
Based on a fixed point theorem,the authors studied the following problem

$$\begin{cases} {}^cD^{\mu}x(t)) -^c D^{\nu}x(t)) = h(t, x(t)), \quad t \in J, \quad 0 < \nu < \mu < 1 \\ x(0) = x_0, \end{cases}$$

Where D^{μ}, D^{ν} are the Caputo derivatives of order μ, ν, respectively, and $h : J \times \mathbb{R} \to \mathbb{R}$ is continuous mapping.

The thesis contains two chapters organized as follows :

In Chapter 1, we throw light on basic definitions and introductory concepts. This chapter also includes many interesting results related to the b-metric spaces, some examples which satisfy the properties of above spaces, convergence, Cauchy sequence, completeness. We also stated some of the extended fixed point theorems for Geraphty contraction type mapping in b-metric space. We recall some definitions,notations and lemmas of fractional calculus.

in Chapter 2, we stated the existence and uniqueness of solutions for nonlinear integral equations by using some of the extended fixed point results for Geraghty contractions type mapping in b-metric spaces.At first,we considered the following integral equations

$$x(t) = h(t) + \int_0^1 \kappa(t,u) f(s, x(u)) du \quad \quad for \ all \ t \in [0,1]$$

Where $h : [0,1] \to \mathbb{R}$, $\kappa : [0,1] \times [0,1] \to \mathbb{R}$,$f : [0,1] \times \mathbb{R} \to \mathbb{R}$ are given functions satisfying some assumptions that will be specified later.

Secondly, by using some of the extended fixed point results for Geraghty contractions in b-metric spaces we apply these results to the fractional differential equations. We considered the following problem

$$\begin{cases} {}^cD^\mu x(t)) -^c D^\nu x(t)) = h(t, x(t)), \quad t \in J, \quad 0 < \nu < \mu < 1 \\ x(0) = x_0, \end{cases}$$

Where D^μ, D^ν are the Caputo derivatives of order μ, ν, respectively, and $h : J \times \mathbb{R} \to \mathbb{R}$ is continuous mapping.

The thesis concludes with a useful general **conclusion**.

Chapter 1

Preliminaries

1.1 Definition and examples of b-metric space

1.1.1 Metric space

First, we are going to recall the notion of metric space.

Definition 1.1.1. *[15] (Metric space)*
Let X be a nonempty set and let $d : X \times X \to \mathbb{R}_0^+$ be a function satisfying the conditions

(d1) $d(x, y) = 0$ *if and only if* $x = y$ *;*

(d2) $d(x, y) = d(y, x)$ *;*

(d3) $d(x, z) \leq d(x, y) + d(y, z)$*, for all* $x, y, z \in X$*.*

Then d is called metric on X and the pair (X, d) is called metric space.

Definition 1.1.2. *(Lipschitzian mapping)*
Let (X, d) be a metric space and T is a mapping from X to X. The mapping T is called a Lipschitz mapping if there exists a constant $k \geq 0$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. k is called the Lipschitz constant.

Example 1.1.1.
Let (X, d) a metric space such that $X = [1, 2]$ and $d(x, y) =\mid x - y \mid$ for all

$x, y \in X$. The mapping $T : X \times X \to \mathbb{R}^+$ defined by $T(x) = x^2$ is a lipschitz mapping. Indeed, since $x^2 - y^2 = (x + y)(x - y)$, we have

$$\begin{aligned} d(T(x), T(y)) &= \mid x^2 - y^2 \mid \\ &\leq \mid x + y \mid\mid x - y \mid \\ &\leq (\mid x \mid + \mid y \mid) \mid x - y \mid \\ &\leq (2 + 2) \mid x - y \mid \\ &\leq 4d(x, y). \end{aligned}$$

for all $x, y \in X$. This shows that T is a Lipschitz mapping, with Lipschitz constant $k = 4 \geq 0$.

Definition 1.1.3. *(Contraction mapping)*
Let (X, d) be a metric space and $T : X \to X$ a self mapping. T is called contraction mapping if there exists a constant $k < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq kd(x, y)$$

This contraction is also known as Banach[1] *contraction.*

Example 1.1.2.
Let (X, d) a metric space such that $X = [0, 1]$ and $d(x, y) =\mid x - y \mid$ for all $x, y \in X$. The function $T : X \to X$ defined by $T(x) = \ln\left(1 + \frac{x}{4}\right)$ is a contraction.

1.1.2 b-metric space

In the following definition we will recall the concept of b-metric space (introduced by Backhtin in 1989).

Definition 1.1.4. *[5] (b-Metric space) Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d_s : X \times X \to \mathbb{R}^+$ is said a b-metric if for all $x, y, z \in X$ the following conditions are satisfied :*

1. Stefan Banach (30 March 1892 – 31 August 1945) was a Polish mathematician who is generally considered one of the world's most important and influential 20th-century mathematicians.

(b1) $d_s(x,y)=0$ *if and only if* $x=y$ *;*

(b2) $d_s(x,y)=d_s(y,x)$ *;*

(b3) $d_s(x,z)\leq s[d_s(x,y)+d_s(y,z)]$, *for all* $x,y,z\in X$. *(b-triangular inequality).*

In this case, d_s is called b-metric on X, and the pair (X,d_s) is called a b-metric space (with constant s).

Next, we give some examples of b-metric spaces.

Example 1.1.3.
Let $X=\{\frac{-3}{2},0,\frac{1}{2}\}$ *and let* $d_s: X\times X\rightarrow[0,\infty)$ *defined by :*

$$d_s(\frac{-3}{2},0)=d_s(0,\frac{-3}{2})=2,$$

$$d_s(\frac{-3}{2},\frac{1}{2})=d_s(\frac{1}{2},\frac{-3}{2})=7,$$

$$d_s(0,\frac{1}{2})=d_s(\frac{1}{2},0)=3,$$

$$d_s(0,0)=d_s(\frac{1}{2},\frac{1}{2})=d_s(\frac{-3}{2},\frac{-3}{2})=0$$

It is clear that all the conditions $(b_1);(b_2)$ *and* (b_3) *of Definition 1.1.4 are satisfied ,*

(b1) $d_s(x,y)=0\Rightarrow x=y$ *;*

(b2) $d_s(x,y)=d_s(y,x)$ *;*

(b3) *For all* $x,y,z\in X$, *we have*

$$7=d_s(\frac{-3}{2},\frac{1}{2})\geq d_s(\frac{-3}{2},0)+d(0,\frac{1}{2})=2+3=5$$

so $s=\frac{7}{5}$ *Then :*

$$d_s(x,z)\leq\frac{7}{5}[d_s(x,y)+d_s(y,z)]$$

Then (X,d_s) *is a b-metric space with* $s=\frac{7}{5}$.

Example 1.1.4. *[29]*
Let (X, d) be a metric space and $\rho(x, y) = [d(x, y)]^p$, with $p > 1$ is a real number. We show that ρ is a b-metric with $s = 2^{p-1}$.

$$\begin{aligned} \rho(x,y) = 0 &\Leftrightarrow [d(x,y)]^p = 0 \\ &\Leftrightarrow d(x,y) = 0 \\ &\Leftrightarrow x = y \end{aligned}$$

$$\begin{aligned} \rho(x,y) &= [d(x,y)]^p \\ &= [d(y,x)]^p \\ &= \rho(x,y) \end{aligned}$$

If $1 < p < \infty$, then the convexity of the function $f(x) = x^p (x > 0)$ implies

$$\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}(a^p + b^p),$$

and hence, $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ holds. Thus, for each $x, y, z \in X$ we obtain

$$\begin{aligned} \rho(x,y) = (d(x,y))^p &\leq (d(x,z) + d(z,y))^p \\ &\leq 2^{p-1}(d(x,z)^p + d(z,y)^p) \\ &= 2^{p-1}(\rho(x,z) + \rho(z,y)). \end{aligned}$$

So conditions $(b_1); (b_2)$ and (b_3) of Definition 1.1.4 are satisfied and ρ is a b-metric.

However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.
For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x, y) =\mid x - y \mid$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$, but is not a metric on $\mathbb{R}$. We can see that on others examples.

Example 1.1.5.
Let $d : C(I) \times C(I) \longrightarrow \mathbb{R}^+$ be defined by

$$d(u,v) = \| (u-v)^2 \|_\infty = \sup \| u(\theta) - v(\theta) \|^2, \quad \text{for all } u, v \in C(I).$$

It is Clear that d is a b-metric with $s = 2$.

Example 1.1.6. *[20]*
Let X be the set of Lebesgue measurable functions on $[0,1]$ such that

$$\int_0^1 | f(x) |^2 \, dx < \infty.$$

Define $D : X \times X \to [0,\infty)$ by

$$D(f,g) = \int_0^1 | f(x) - g(x) |^2 \, dx.$$

Then D satisfies the following properties

(b_1) *D(f,g)=0 if and only if f=g,*

(b_2) *D(f,g)=D(g,f), for any $f, g \in X$,*

(b_3) $D(f,g) \leq 2(D(f,h) + D(h,g))$, *for any points $f, g, h \in X$.*

Remark 1.1.1.
Every metric space is a b-metric space with s=1, But in general,every b-metric space is not a metric space.

The following examples show that there exists a b-metric which is not a metric.

Example 1.1.7. *[1]*
Let $X = \{0,1,2\}$ and let $d : X \times X \to [0,\infty)$ defined by :

$$d(0,1) = 1, \quad d(0,2) = \frac{1}{2}, \quad d(1,2) = 2,$$
$$d(x,x) = 0, \quad d(x,y) = d(y,x), \textit{for all } x, y \in X.$$

Notice that d is not a metric, since we have

$$d(1,2) > d(1,0) + d(0,2)$$

However, it is easy to see that d is a b-metric with $s \geq \frac{4}{3}$.

Example 1.1.8. *[4]*
Let $X = [0,\infty), d : x \times X \to [0,\infty)$ defined by

$$d(x,y) = \begin{cases} 0 & \textit{if } x = y, \\ 4 & \textit{if } x, y \in [0,1], \\ \frac{9}{2} + \frac{1}{x+y} & \textit{if } x, y \in [1,\infty), \\ \frac{12}{5} & \textit{otherwise.} \end{cases}$$

(X, d) is b-metric space with $s = \frac{25}{24}$ but not a metric space. When $x = \frac{10}{9}, z = 1 \in [1, \infty)$ and $y \in (0, 1)$, We have

$$d(x, z) = \frac{9}{2} + \frac{1}{x + z} = \frac{9}{2} + \frac{9}{19} = \frac{189}{38},$$

and

$$d(x, y) + d(y, z) = \frac{12}{5} + \frac{12}{5} = \frac{24}{5}.$$

So that

$$d(x, z) \geq d(x, y) + d(y, z).$$

Hence d is a b-metric with $s = \frac{25}{24}$ but not a metric.

1.1.3 Properties of b-metric space

We need the following definitions and propositions in b-metric space. We recall the notion of b-convergence, b-Cauchy sequence and b-completeness in b-metric space. Let (x_n) a sequence in X.

Definition 1.1.5. *[7] ,[8](b-convergent, b-Cauchy)*
Let $(X, d_s), s \geq 1$ be a b-metric space . A sequence (x_n) in X is called :

(i) b-convergent if there exists $x^* \in X$ such that $d(x_n, x_*) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n \to \infty} x_n = x^*$.

(ii) b-cauchy if $d(x_m, x_n) \to 0$ as $n, m \to \infty$.

Proposition 1.1.1. *[9]*
In a b-metric space (X, d_s) the following assertions hold :

(P_1) A convergent sequence has a unique limit.

(P_2) Each convergent sequence is Cauchy sequence.

Proof. (P_1) By contradiction

We hope to prove "For all convergent sequences the limit is unique". The negation of this is "There exists at least one convergent sequence which does not have a unique limit".
This is what we assume. On the basis of this assumption let (x_n) denote a sequence with more than one limit, two of which are labelled as u_1 andu_2 with $u_1 \neq u_2$. Choose $\varepsilon = \frac{1}{3s}$ which is greater than zero since

$u_1 \neq u_2$.
Since u_1 is a limit of (x_n) we can apply the definition of limit with our choice of ε to find $N_1 \in \mathbb{N}$ such that

$$d_{s(x_n,u_1)}) < \varepsilon \quad \text{for all } n \neq N_1$$

Similarly, as u_2 is a limit of $(x_n)_{n\in\mathbb{N}}$ we can apply the definition of limit with our choice of ε to find $N_2 \in \mathbb{N}$ such that

$$d_{s(x_n,u_2)}) < \varepsilon \quad \text{for all } n \neq N_2$$

here is no reason to assume that in the two uses of the definition of limit we should find the same $N \in \mathbb{N}$ for the different u_1 and u_2. Choose any $m_0 > \max(N_1, N_2)$, then $d_s(x_{m_0}, u_1) < \varepsilon$ and $d_s(x_{m_0}, u_1) < \varepsilon$.
Using the b-triangle inequality, we have :

$$\begin{aligned} d_s(u_1,u_2) &\leq s[d_s(u_1,x_{m_0}) + d_s(x_{m_0},u_2)] \quad \text{(b-triangle inequality)} \\ &< s[\varepsilon+\varepsilon] \qquad \text{(by the choice of } m_0\text{)} \\ &= 2s\varepsilon \\ &= \frac{2s}{3s}d_s(u_1,u_2) \qquad \text{(by the definition of } \varepsilon\text{)} \\ &= \frac{2}{3}d_s(u_1,u_2) \end{aligned}$$

So we find that $d_s(x_n,u_2)$, which is not zero, satisfies $d_s(x_n,u_2) < \frac{2}{3}d_s(u_1,u_2)$, which is a contradiction.
Hence our assumption must be false, that is, there does not exists a sequence with more than one limit. Hence for all convergent sequences the limit is unique.

(P_2) Suppose (x_n) is a convergent sequence with limit u. For $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d_s(x_n,u_2) < \frac{\varepsilon}{2}$. We introduce x_m by $d_s(x_m,x_n)$. and use the b-triangle inequality

$$\begin{aligned} d_s(x_n,x_m) &\leq s[d_s(x_n,u) + d_s(u,x_m)] \quad \text{(b-triangle inequality)} \\ &< s[\frac{\varepsilon}{2}+\frac{\varepsilon}{2}] \\ &= s\varepsilon \\ &= \varepsilon^{'} \end{aligned}$$

whenever $n, m \geq N$. Thus the convergent (x_n) is Cauchy. □

Definition 1.1.6. *[9](Complete b-metric space)*
A b-metric space is called a complete b-metric space if every b-Cauchy sequence in X is b-convergent in X.

Definition 1.1.7. *(b-closed)*
A set $B \subset X$ is said to be b-closed if for any sequence (x_n) in B such that (x_n) is b-convergent to $z \in X$ then $z \in B$.

Remark 1.1.2.
We observe that the notions of convergent sequence, Cauchy sequence, and complete space are defined as in metric spaces.

Now, we consider the continuity of a mapping with respect to a b-metric defined as follows.

Definition 1.1.8. *[28](Continuity)*
Let (X, d_X) and (Y, d_Y) be two b-metric spaces with coefficient s and s', respectively. A mapping $T : X \to Y$ is a b-continuous at a point $x \in X$, if it is b-sequentially continuous at x. i.e, whenever $(x_n)$is b-convergent to x, $T(x_n)$ is b-convergent to $T(x)$.

Remark 1.1.3.
In the general case, a b-metric is not necessarily continuous.

The following example shows that a b-metric need not be continuous (see Boriceanu [8]).

Example 1.1.9. *[8]*
Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to [0, \infty)$ as follows.

$$d_s(m,n) = \begin{cases} 0 & \text{if } m = n \\ \left| \frac{1}{m} - \frac{1}{n} \right| & \text{if one of} m, n \text{ is even and the other is even or } \infty \\ 5 & \text{if one of m,n is odd and the other is odd (and} m \neq n) \text{ or } \infty \\ 2 & \text{otherwise} \end{cases}$$

Then (X, d_s) is a b-metric space with $s = \frac{5}{2}$. However, let $x_n = 2n$ for each $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} d(x_n, \infty) = \lim_{n\to\infty} d(2n, \infty) = \lim_{n\to\infty} \frac{1}{2n} = 0,$$

but

$$\lim_{n\to\infty} d(x_n, 1) = \lim_{n\to\infty} d(2n, 1) = 2 \neq d(\lim_{n\to\infty} x_n, 1) = d(\infty, 1) = 5$$

.

1.2 Geraphty Contraction

1.2.1 Introduction

It is known that the Banach principle is considered as a one of the most important theorems in the classical function analysis. There are many generalization of this theorem. The following generalizations is due to Geraphty. Geraphty generalized the Banach contraction mapping in metric spaces by using an auxiliary function instead of constant. In this section, we introduce first the notion of :

— Geraphty function.

— Generalized Geraphty contraction mapping.

— Generalized almost Geraphty contraction mapping,

— Generalized α Geraphty contraction mapping,

— Generalized $\alpha - \phi$ Geraphty contraction mapping in the setting of b-metric spaces.

After then, we give some definitions and examples.

1.2.2 Geraphty function

In 1973, Geraghty [14] introduced a class of functions to generalize the Banach contraction principle.

Definition 1.2.1. *[14]*
A function $\beta : [0, \infty) \to [0, 1)$ *which satisfies the condition :*

$$\lim_{n\to\infty} \beta(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0 \tag{1.1}$$

is called the Geraphty function.

We denote by $\mathcal{F}$ the set of Geraphty functions. In 2011,Dukic al used another set which they denoted by $\mathcal{F}_s$.

Definition 1.2.2. *[14] Let (X, d) be a complete b-metric space with $s > 1$. $\mathcal{F}_s$ denote the set of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ which satisfies the condition :*

$$\lim_{n\to\infty} \beta(t_n) = \frac{1}{s} \Rightarrow \lim_{n\to\infty} t_n = 0 \tag{1.2}$$

Some examples of Geraphty functions are given in the following.

Example 1.2.1.
$\beta : [0, \infty) \to [0, 1)$,

$$\beta(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

satisfies the condition of Geraphty (1.1)

Example 1.2.2.
$\beta : [0, \infty) \to [0, 1)$,

$$\beta(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{\ln(1+x)}{x} & \text{if } x > 0. \end{cases}$$

is a Geraphty operator.

Example 1.2.3.
The same conclusion is for $\beta : [0, \infty) \to [0, 1)$,

$$\beta(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{x}{e^x-1} & \text{if } x > 0. \end{cases}$$

Example 1.2.4.
The function $\beta : [0, \infty) \to [0, 1)$,

$$\beta(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{1+x} & \text{if } x > 0. \end{cases}$$

is a Geraphty operator.

Example 1.2.5.
The function $\beta : [0, \infty) \to [0, 1)$,

$$\beta(x) = \begin{cases} e^{-2x} & \text{if } x > 0, \\ \beta(0) \in [0, 1]. \end{cases}$$

Then $\beta \in \mathcal{F}$

1.2.3 Geraphty contraction

Definition 1.2.3. *[14]*
Let (X, d) be a metric space. An opeartor $T : X \to X$ is called a Geraphty contraction if there exist a function $\beta \in \mathcal{F}$ which satisfies for all $x, y \in X$, the condition :

$$d(Tx, Ty) \leq \beta(d(x, y)).d(x, y)$$

1.2.4 Generalized almost Geraphty contraction mapping

In the following, we introduce the notion of almost Geraphty contraction mapping in b-metric space as follows :

Definition 1.2.4. *[4]*
Let (X, d_s) be a b-metric space with coefficient $s \geq 1$ and let T be a self mapping of X. If there exists $\beta \in \mathcal{F}$ and $L \geq 0$ such that :

$$d(Tx, Ty) \leq \beta(M(x, y))M(x, y) + LN(x, y) \tag{1.3}$$

for all $x, y \in X$ where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}$$

and

$$N(x, y) = \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

Then we say that T is an almost Geraphty contraction mapping.

The importance of the class of almost Geraphty contraction type maps is that this class is properly includes the class of Geraphty contraction type maps studied by Faraji, Savic and Radenovic so that the class of almost Geraphty contraction type maps is larger than the class of Geraphty contraction type maps, which is illustrated in the following example.

Example 1.2.6. *[4]*
Let $X = [0, \infty)$ and let $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ (x + y)^2 & \text{if} \quad x \neq y. \end{cases}$$

Then clearly (X, d) *is a complete b-metric space with coefficient* $s = 2$.
Let $f : X \to X$ *a mapping defined by*

$$f(x) = \begin{cases} 1, & if \quad x \in [0, 1) \\ 2x - 1, & if \quad x \in [1, \infty). \end{cases}$$

We defined $\beta : [0, \infty) \to [0, \frac{1}{s})$ *by*

$$\beta(t) = \frac{1}{3 + t} \text{ for all } t > 0.$$

It's easy to see that $\beta \in \mathcal{F}_f$. *Without loss of generality, we assume that* $x \geq y$. *We have three cases.*

Case 1 *Let* $x, y \in [0, 1)$.

$$d(fx, fy) = d(1, 1) = 0$$

and the inequality (1.3) holds.

Case 2 *Let* $x, y \in [1, \infty)$.

$$d(fx, fy) = d(2x-1, 2y-1) = (2x-1+2y-1)^2 = 4(x+y-1)^2,$$
$$d(x, y) = (x + y)^2,$$
$$d(x, fx) = d(x, 2x - 1) = (3x - 1)^2,$$
$$d(y, fy) = d(y, 2y - 1) = (3y - 1)^2,$$
$$d(x, fy) = (x, 2y - 1) = (x + 2y - 1)^2,$$
$$d(y, fx) = (y, 2x - 1) = (y + 2x - 1)^2.$$

$$\begin{aligned} M(x, y) &= \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s}\right\} \\ &= \max\left\{(x + y)^2, (3x - 1)^2, (3y - 1)^2, \frac{(x + 2y - 1)^2 + (y + 2x - 1)^2}{4}\right\} \\ &= (3x - 1)^2 \end{aligned}$$

and

$$\begin{aligned} N(x, y) &= \min\{d(x, fx), d(x, fy), d(y, fx)\} \\ &= \min\{(3x - 1)^2, (x + 2y - 1)^2, (y + 2x - 1)^2\} \\ &= (x + 2y - 1)^2. \end{aligned}$$

We consider

$$\begin{aligned} d(fx, fy) &= 4(x+y-1)^2 \\ &\le \frac{1}{3+(3x-1)^2}(3x-1)^2 + \frac{11}{3}(x+2y-1)^2 \\ &\le \beta(M(x,y))M(x,y) + LN(x,y) \end{aligned}$$

Case 3 *Let $x \in [1, \infty)$ and $y \in [0, 1)$.*

$d(fx, fy) = d(2x-1, 1) = (2x-1+1)^2 = 4x^2,$
$d(x, y) = (x+y)^2,$
$d(x, fx) = d(x, 2x-1)^2 = (3x-1)^2,$
$d(y, fy) = d(y, 1) = (y+1)^2,$
$d(x, fy) = d(x, 1) = (x+1)^2,$
$d(y, fx) = d(y, 2x-1) = (y+2x-1)^2.$

$$\begin{aligned} M(x,y) &= \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2s}\right\} \\ &= \max\left\{(x+y)^2, (3x-1)^2, (y+1)^2, \frac{(x+1)^2+(y+2x-1)^2}{4}\right\} \\ &= (3x-1)^2 \end{aligned}$$

and

$$\begin{aligned} N(x,y) &= \min\{d(x,fx), d(x,fy), d(y,fx)\} \\ &= \min\{(3x-1)^2, (x+1)^2, (y+2x-1)^2\} \\ &= \min\{(x+1)^2, (y+2x-1)^2\}. \end{aligned}$$

We consider

$$\begin{aligned} d(fx, fy) &= 4x^2 \\ &\le \frac{1}{3+(3x-1)^2}(3x-1)^2 + \frac{11}{3}\min\{(x+1)^2, (y+2x-1)^2\} \\ &\le \beta(M(x,y))M(x,y) + LN(x,y) \end{aligned}$$

From all above cases f is an almost Geraphty contraction mapping with $L = \frac{11}{3}$.

Here, we observe that if $L = 0$ then the inequality (1.3) fails to hold. Indeed, if we choose $x = 3$ and $y = 2$, we have

$$d(fx, fy) = 64, d(x, y) = 25, d(x, fx) = 64,$$

$$d(y, fy) = 25, d(x, fy) = 36, d(y, fx) = 49.$$

Thus

$$M(x, y) = \max\left\{25, 64, 25, \frac{36 + 49}{4}\right\} = 64.$$

Here we note that $d(fx, fy) = 64 \nleq \beta(64)64 = \beta(M(x, y))M(x, y)$ for any $\beta \in \mathcal{F}_f$

In 2012, Samet introduced the notion of α admissible mapping

Definition 1.2.5. *[27](α-admissible) Let X be a nonempty set and let $\alpha : X \times X \to [0, \infty)$ is given function. A mapping $T : X \to X$ is called α admissible mapping if*

$$\forall x, y \in X, \alpha(x, y) \geq 1 \textit{ implies } \alpha(Tx, Ty) \geq 1.$$

Example 1.2.7.

Let $X = [0, \infty]$ and $d(x, y) =\mid x - y \mid$ for all $x, y \in X$. We define a mapping $f : X \to X$ by :

$$f(x) = \begin{cases} \frac{1}{8}x^2 & \textit{if } x \in [0, 1] \\ \ln x & \textit{if } x \in (1, \infty) \end{cases}$$

and we define also $\alpha : X \times X \to [0, \infty)$,

$$\alpha(x, y) = \begin{cases} 1 & \textit{if } \ x, y \in [0, 1] \\ 0, & \textit{otherwise.} \end{cases}$$

Then f is α-admissible. Indeed, for $x, y \in X$, we notice that if $d(x, y) \geq 1$ then $x, y \in [0, 1]$. So,

$$f(x) = \frac{1}{8}x^2 \leq \frac{1}{8}$$

and

$$f(y) = \frac{1}{8}y^2 \leq \frac{1}{8}$$

then $fx, fy \in [0, 1]$. It follows that

$$\alpha(fx, fy) \geq 1 \Rightarrow f \textit{ is } \alpha - \textit{admissible.}$$

Example 1.2.8.
Let $X = \mathbb{R}$, *define* $f : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$,

$$fx = \begin{cases} \ln(x), & \text{if } x \neq 0 \\ 3, & \text{otherwise.} \end{cases} \quad \text{and } \alpha(x, y) = \begin{cases} 3, & \text{if } x > y \\ 0, & \text{otherwise.} \end{cases}$$

Let $x, y \in \mathbb{R}$

1. *if* $x = y = 0$

$$\begin{aligned} x = y = 0, \quad &\Rightarrow \quad \alpha(fx, fy) = \alpha(3, 3) = 3 \geq 1 \\ &\Rightarrow \quad f \text{ is } \alpha \; - admissible. \end{aligned}$$

2. *if* $x, y \neq 0$,

$$\begin{aligned} \alpha(x, y) = 3, \quad &\Rightarrow \quad x > y \\ &\Rightarrow \quad \ln \mid x \mid > \ln \mid y \mid \\ &\Rightarrow \quad fx > fy \\ &\Rightarrow \quad \alpha(fx, fy) = 3 \geq 1 \\ &\Rightarrow \quad f \text{ is } \alpha\text{-admissible.} \end{aligned}$$

Karapinar and all[18] defined the concept of triangular $\alpha-$*admissible mapping*

Definition 1.2.6. *[18](triangular* α*-admissible)*
A mapping $T : X \to X$ *is said to be triangular* α*-admissible if for* $x, y, z \in X$

1. T *is* α*-admissible,*
2. $\alpha(x, z) \geq 1$ *and* $\alpha(z, y) \geq 1$ *implies that* $\alpha(x, y) \geq 1$..

Example 1.2.9.
Let $X = \mathbb{R}$ *and* $\alpha : X \times X \to [0, \infty)$, *such that*

$$\alpha(x, y) = e^{x-y} \quad \text{for all } x, y, z \in \mathbb{R}.$$

If $\alpha(x, z) \geq 1$ *and* $\alpha(z, y) \geq 1$ *then*

$$\begin{aligned} \alpha(x, y) &= e^{x-y} \\ &= e^{x-z}.e^{z-y} \\ &= \alpha(x, z).\alpha(z, y) \geq 1 \end{aligned}$$

Hence α *is triangular*

Example 1.2.10.
Let $X = (0, \infty)$ *and* $\alpha : X \to [0, \infty)$,

$$\alpha(x, y) = \frac{x}{y} \text{ for all } x, y, z \in (0, \infty)$$

if $\alpha(x, z) \geq 1$ *and* $\alpha(z, y) \geq 1$

$$\begin{aligned} \alpha(x, y) &= \frac{x}{y} \\ &= \frac{x}{z} . \frac{z}{y} \\ &= \alpha(x, y).\alpha(z, y) \geq 1 \end{aligned}$$

Hence α *is triangular*

The definitions of $\alpha-$ orbital admissible mapping and triangular $\alpha-$ orbital admissible mapping are defined by Popescu in 2014.

Definition 1.2.7. *[25](*$\alpha-$*orbital admissible)*
Let $T : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$ *mappings.* T *is said to be* $\alpha-$ *orbital admissible if :*

$$\alpha(x, Tx) \geq 1 \text{ implies } \alpha(Tx, T^2x) \geq 1$$

Definition 1.2.8. *[25](triangular*α*-orbital admissible)*
Let $T : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$. T *is said to be triangular* $\alpha-$ *orbital admissible if*

1. T *is* α*-orbital admissible,*
2. $\alpha(x, y) \geq 1$ *and* $\alpha(y, Ty) \geq 1$ *implies that* $\alpha(x, Ty) \geq 1$.

Remark 1.2.1. *As mentioned in [25],*

* *Every* $\alpha-$ *admissible mapping is a* $\alpha-$ *orbital admissible mapping.*
* *Every triangular* $\alpha-$ *admissible mapping is a triangular* $\alpha-$ *orbital admissible mapping.*
* *But there exists a triangular* $\alpha-$ *orbital admissible mapping which is not a triangular* $\alpha-$ *admissible mapping*

The following example shows that there exists a triagular α orbital admissible mapping which is not triagular α-admissible.

Example 1.2.11.
Let $X = \{0,1,2,3\}$ *a subset of* $\mathbb{R}$ *and* $d : X \times X \to \mathbb{R}$ *defined by* $d(x,y) = \mid x - y \mid$ *. A mapping* $T : X \to X$ *is an operator defined by*

$$T0 = 0,\ T1 = 2,\ T2 = 1,\ T3 = 3.$$

Let $\alpha : X \times X \to \mathbb{R}$, *is a mapping such that*

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \{(0,1),(0,2),(1,1),(2,2),(1,2),(2,1),(1,3),(2,3)\}, \\ 0 & \text{otherwise} \end{cases}$$

1. *Since*
$$\alpha(1,T1) = \alpha(1,2) = 1, \quad \alpha(2,T1) = \alpha(2,2) = 1$$
$$\alpha(1,T2) = \alpha(1,1) = 1, \quad \alpha(2,T2) = \alpha(2,1) = 1,$$
then T *is* α *orbital admissible.*

2. *Since*
$$\alpha(0,1) = \alpha(1,2) = \alpha(0,2) = 1$$
$$\alpha(0,2) = \alpha(2,1) = \alpha(0,1) = 1$$
then T *is triangular* α *orbital admissible.*

3. *But*
$$\alpha(0,1) = \alpha(1,3) = 1$$
and
$$\alpha(0,3) = 0$$
So T *is not triangular* α*-admissible.*

Definition 1.2.9. *[25](*α*-Regular)*
Let (X, d_s) *be a b-metric space and* $\alpha : X \times X \to [0,\infty)$. *We say that* X *is an* α*-regular if for each sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to \infty$, *there exists a subsequence* $(X_{n(k)})_{k\in\mathbb{N}}$ *of* $(x_n)_n$ *with* $\alpha(x_{n(k)}, x) \geq 1$, *for all* k.

1.2.5 Generalized α- Geraphty contraction mapping

Popescu gave the definition of generalized α Geraphty contraction type mapping and proved the fixed theorems for such mapping in complete metric spaces.
We start with some definitions which we use in the subsequent development.

Definition 1.2.10. *[25]*
Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be a Generalized α Geraphty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$.

$$\alpha(x, y).d(Tx, Ty) \leq \beta(M_T(x, y)M_T(x, y)$$

Where

$$M_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

Example 1.2.12. *[25]*
Let $X = [-2, -1] \cup \{0\} \cup [1, 2]$ and let $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) =| x - y |$$

We defined $T : X \to X$

$$Tx = \begin{cases} -x & if \quad x \in [-2, -1) \cup (1, 2] \\ 0 & if \quad x \in \{-1, 0, 1\}. \end{cases}$$

We defined $\beta : [0, \infty) \to [0, 1)$ by $\beta(t) = \frac{1}{2}$ and $\alpha : X \times X \to \mathbb{R}$ by

$$\alpha(x, y) = \begin{cases} 1 & if \quad xy \geq 0 \\ 0 & \quad otherwise. \end{cases}$$

Case 1 *Let $x, y \in [-2, -1)$, then $xy \geq 0$ and by definition of α, $\alpha(x, y) = 1$.*

$$d(Tx, Ty) = d(-x, -y) =| -(x - y) |=| x - y |\leq 1,$$

$$d(x, y) =| x - y |\leq 1,$$

$$d(x, Tx) = d(x, -x) = 2 \mid x |= -2x \geq 2$$

$$d(y,Ty) = d(y,-y) = 2 \mid y \mid,$$

$$d(x,Ty) = d(x,-y) = \mid x+y \mid$$

$$d(y,Tx) = (y,-x) = \mid x+y \mid.$$

$$\begin{aligned} M_T(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\ &= 2 \mid x \mid \geq -2x \\ &\geq 2. \end{aligned}$$

we consider

$$\begin{aligned} d(Tx,Ty) &= \mid x-y \mid \\ &\leq \frac{1}{2} \times 2 \mid x \mid \\ &\leq \beta(M_T(x,y))M_T(x,y) \end{aligned}$$

Thus

$$\alpha(x,y).d(Tx,Ty) \leq \beta(M_T(x,y)M_T(x,y).$$

Case 2 *Let* $x, y \in (1,2]$.

$$d(Tx,Ty) = d(-x,-y) = \mid -(x-y) \mid = \mid x-y \mid \leq 1,$$

$$d(x,y) = \mid x-y \mid \leq 1,$$

$$d(x,Tx) = d(x,-x) = 2 \mid x \mid = 2x \geq 2,$$

$$d(y,Ty) = d(y,-y) = 2 \mid y \mid,$$

$$d(x,Ty) = d(x,-y) = \mid x+y \mid,$$

$$d(y,Tx) = (y,-x) = \mid x+y \mid.$$

this case is similar as the first one.

$$\begin{aligned} M_T(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\ &= 2 \mid x \mid \geq 2x \\ &\geq 2. \end{aligned}$$

Case 3 *Let $x \in [-2,-1) \cup (1,2]$ and $y \in \{-1,0,1\}$.*

$$d(Tx,Ty) = d(-x,0) =\mid -x \mid=\mid x \mid= -x \leq 1,$$

$$d(x,y) =\mid x-y \mid\leq 1,$$

$$d(x,Tx) = d(x,-x) =\mid 2x \mid= 2 \mid x \mid\geq 2,$$

$$d(y,Ty) = d(y,0) =\mid y \mid,$$

$$d(x,Ty) = d(x,0) =\mid x \mid,$$

$$d(y,Tx) = (y,0) =\mid x \mid .$$

$$\begin{aligned} M_T(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\ &\geq 2 \mid x \mid . \end{aligned}$$

Case 4 *Let $x,y \in \{-1,0,1\}$.*

$$d(Tx,Ty) = d(0,0) = 0,$$

$$\begin{aligned} d(Tx,Ty) &= 0 \\ &\leq \frac{M_T(x,y)}{2} \\ &\leq \beta(M_T(x,y))M_T(x,y) \end{aligned}$$

We obtain that T is a generalized α- Geraphty contraction type mapping. However, since $\alpha(-2,0) = \alpha(0,2) = 1, \alpha(-2,2) = 0$, T is not a triangular α admissible mapping.

1.2.6 Generalized $\alpha - \phi$ Geraphty contraction mapping

Recently, Karapinar introduced the concept of $\alpha-\phi$ Geraphty contraction type mapping in complete metric spaces.
Let denote Φ the class of the function $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions :

(a) ϕ is nondecreasing,

(b) ϕ is continuous,

(c) $\phi(t) = 0$ if and only if $t = 0$,

(d) ϕ is subadditive, that is $\phi(c+t) \leq \phi(c) + \phi(t)$

Definition 1.2.11.
Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be a Generalized $\alpha - \phi$ Geraphty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$.

$$\alpha(x,y).\phi(d(Tx,Ty)) \leq \beta(\phi(M(x,y))\phi(M(x,y))$$

Where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$$

and $\phi \in \Phi$.

Definition 1.2.12. *[1]*
Let (X, d_s) be a b-metric space with coefficient $s \geq 1$ and a self mapping on X, $T : X \to X$. T is a generalized$\alpha - \phi$ Geraphty contraction type mapping whenever there exists $\alpha : X \times X \to [0, \infty)$ such that for

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s}\right\}$$

We have

$$\alpha(x,y)\phi(s^3 d(Tx,Ty)) \leq \beta(\phi(M(x,y)))\phi(M(x,y)),$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\phi \in \Phi$.

On paper[1], the authors defined the generalized $\alpha - \phi$ Geraphty contraction type mapping in (X, d_s) b-metric space.

Definition 1.2.13. *[1]*
Let (X, d_s) be a b-metric space with coefficient $s \geq 1$ and let T be a self-mapping of X. We say that T is a generalized Geraphty contraction type mapping whenever there exists $\alpha : X \times X \to [0, \infty)$ and some $L \geq 0$ such that for

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s}\right\}$$

$$\textit{and } N(x,y) = \min\{d(x,Ty), d(y,Tx)\}$$

We have

$$\alpha(x,y)\phi(s^3 d(Tx,Ty)) \leq \beta(\phi(M(x,y)))\phi(M(x,y)) + L\psi(N(x,y)),$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\phi, \psi \in \Phi$.

1.3 Fixed point theorems for Geraphty contraction type mapping

Definition 1.3.1. *[6](fixed point)*
Let X be a nonempty set and $T : X \to X$ a self-mapping. $x \in X$ is a fixed point of T if $Tx = x$.

We denote by $Fix(T)$ the set of all fixed points of T.

In [14], Geraghty presents the following theorem of existence and uniqueness in 1973 by considering an auxiliary function. .

Theorem 1.1. *[14] Let (X, d) be a complete metric space . Let $T : X \to X$ be a given mapping that satisfies the following condition :*

$$d(Tx,Ty) \leq \beta(d(x,y).d(x,y)), \quad x, y \in X$$

where $\beta \in \mathcal{F}$.Then T has a unique fixed point in X.

In 1989, Bakhtin introduced b-metric spaces as generalization of metric spaces. Since then, several papers have been published on the fixed point theory in such spaces. In 2011, Dukic and all [12] reconsidered Theorem 1.1 in the framework of b-metric spaces.
Let (X, d) be a complete b-metric space with $s > 1$, and $\mathcal{F}_s$ denote the set of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ which satisfies the condition :

$$\lim_{n\to\infty} \beta(t_n) = \frac{1}{s} \text{ implies } \lim_{n\to\infty} t_n = 0$$

Theorem 1.2. *[12]*
Let (X, d) be a complete b-metric space with coefficient $s > 1$. Let $T : X \to X$ be a self-mapping. Suppose that there exists $\beta \in \mathcal{F}_s$ such that the following condition is satisfied :

$$d(Tx, Ty) \leq \beta(d(x, y)).d(x, y), \quad x, y \in X.$$

Then T has a unique fixed point $x^* \in X$.

The results of Geraghty in the context of various metric spaces have been extended by many researchers (see [13],[12],[26]-[30]). In 2019, Hamid Faraji and. all.[13] obtained and published two fixed point theorems for Geraghty contraction in b-metric spaces and their application, which we present below.

Theorem 1.3. *[13].*
Let (X, d) be a complete b-metric space with coefficient $s > 1$. Let $T : X \to X$ be a self-mapping satisfying :

$$d(Tx, Ty) \leq \beta(M(x, y).M(x, y)), \quad x, y \in X$$

where :

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}$$

and $\beta \in \mathcal{F}_s$. Then T has a unique fixed point in X.

Proof.
see the Proof of Theorem 3 in ([13]). □

Example 1.3.1. *[13]*
Let $x = 1, 2, 3$and $d : X \times X \to [0, \infty)$ be defined as follows

$d(1, 2) = d(2, 1) = 1$
$d(1, 3) = d(3, 1) = \frac{1}{9}$
$d(2, 3) = d(3, 2) = \frac{6}{9}$
$d(1, 1) = d(2, 2) = d(3, 3) = 0$
It is easy to check that (X, d) is a b-metric space with constant $s = \frac{3}{2}$.
Let $T1 = T3 = 1, T2 = 3$

And $\beta(t) = \frac{2}{3}\exp(-t), t > 0, \beta(0) \in [0, \frac{2}{3})$.
Then we have :

$$\begin{aligned} d(T1, T2) &= d(1,3) \\ &= \frac{1}{9} \\ &\leq \frac{2}{3}\exp(-1) \\ &= \beta(M(1,2))(M(1,2)) \end{aligned}$$

So $\quad d(T1, T2) \leq \beta(M(1,2))(M(1,2))$

$$\begin{aligned} d(T1, T3) &= d(1,1) \\ &= 0 \\ &\leq \beta(M(1,3))(M(1,3)) \end{aligned}$$

So $\quad d(T1, T3) \leq \beta(M(1,3))(M(1,3))$

$$\begin{aligned} d(T2, T3) &= d(3,1) \\ &= \frac{1}{9} \\ &\leq \frac{2}{3}\exp(-\frac{6}{9})(\frac{6}{9}) \\ &= \beta(M(2,3))(M(2,3)) \end{aligned}$$

So $\quad d(T2, T3) \leq \beta(M(2,3))(M(2,3))$
Therefore,the conditions of Theorem 1.3 are satisfied.

Theorem 1.4. *[13].*
Let (X, d) *be a complete b-metric space with coefficient* $s > 1$ *. Let* $T, S : X \to X$ *be self-mappings which satisfy :*

$$sd(Tx, Sy) \leq \beta(M(x,y).M(x,y)), \quad x, y \in X$$

where :

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Sy)\},$$

and $\beta \in \mathcal{F}_s$*. If* T *or* S *are continuous, then* T *and* S *have a unique common fixed point.*

Proof.
See the Proof of Theorem 4 in ([13]). □

In Theorem 1.4, if $T = S$, an interesting result is obtained that we present below in corollary 1.3.1 [12].

Corollary 1.3.1.
Let (X, d) be a complete b-metric space with coefficient $s > 1$. Let $T : X \to X$ be self-mapping which satisfy :

$$sd(Tx, Ty) \leq \beta(M(x, y)).M(x, y), \quad x, y \in X$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

and T is continuous. Then T has a unique fixed point in X.

Example 1.3.2.
Let $X = [0, 1]$ and $d : X \times X \to [0, \infty)$ be defined as follows

$$d(x, y) =\mid x - y \mid^2 \qquad \textit{for all } x, y \in [0, 1].$$

It is easy to check that (X, d) is a b-metric space with parameter $s = 2$. Set

$$Tx = \frac{x}{4}, \qquad \textit{for all } x \in X$$

and

$$\beta(t) = \frac{1}{4} \qquad \textit{for all } t > 0$$

then,

$$\begin{aligned} 2d(Tx, Ty) &= 2\left|\frac{x}{4} - \frac{y}{4}\right|^2 \\ &\leq \frac{1}{4}|x - y|^2 \\ &\leq \beta(M(x, y)).M(x, y) \end{aligned}$$

Then, the conditions of Corollary 1.3.1 are satisfied.

Theorem 1.5.
Let (X, d) be a complete b-metric space with $s > 1$ and $T : X \to X$ be a generalized $\alpha - \phi$ Geraghty contraction type mapping with the following properties

(i) *T is triangular α-orbital admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(iii) *either T is continuous or X is $\alpha-regular$.*

Then T has a fixed point in X.

Proof.
See Proof of Theorem 2.3 in [1] □

If we put $\alpha(x, y) = 1$ for all $x, y \in X$,$L = 0$ and $\phi(t) = t$ in the previous Theorem we obtain the following corollary

Corollary 1.3.2. *Let (X, d_s) be a b-metric space with coefficient $s \geq 1$ and let $T : X \to X$. a mapping on X, such that for all $x, y \in X$,*

$$s^3 d(Tx, Ty) \leq \beta(M(x, y))M(x, y),$$

where $\beta \in \mathcal{F}$ and

$$M(x, y) = \max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\Big\}$$

Then T has a unique fixed point.

For the uniqueness of the fixed point of a generalized $\alpha - \phi-$Geraghty contractive mapping, we will consider the following hypothesis

$$\text{(H)} \qquad \text{for all } x, y \in Fix(T), \text{ either } \alpha(x, y) \geq 1 \text{ or } \alpha(y, x) \geq 1.$$

Here, $Fix(T)$ denotes the set of fixed points of T.

Theorem 1.6.
Adding condition (H) to the hypotheses of Theorem 1.5, we obtain the uniqueness of the fixed point of T.

Example 1.3.3. *[1] Let X be a set of Lebesgue measurable functions on $[0, 1]$ such that*

$$\int_0^1 | x(t) | dt < 1.$$

Define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \Big(\int_0^1 | x(t) - y(t) | d(t)\Big)^2.$$

Then d is a b-metric on X with $s = 2$. $T : X \to X$ *is an operator defined by*

$$Tx(t) = \frac{1}{4}\ln(1+ \mid x(t) \mid).$$

Consider the mapping $\alpha : X \times X \to [0, \infty)$, *defined by*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x(t) \geq y(t) \text{ for all } t \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Set the mapping $\beta : [0, \infty) \to [0, \frac{1}{2})$ *defined by*

$$\beta(t) = \frac{(\ln(1 + \sqrt{t}))^2}{2t}$$

And $\phi : [0, \infty) \to [0, \infty)$ *defined by*

$$\phi(t) = t$$

Notice that $\phi \in \Phi$ *and* $\beta \in \mathcal{F}_s$. *Moreover,*

1. *T is triangular* α*-orbital admissible. indeed, We have*

   (a) *T is* $\alpha -$ *orbital admissible i.e.*

   $$\alpha(x, Tx) \geq 1 \Longrightarrow \alpha(Tx, T^2x) \geq 1$$

   In fact,

   $$\begin{aligned} If \quad \alpha(x, Tx) \geq 1, then x(t) \geq Tx(t) &\Longrightarrow Tx(t) \geq T(Tx(t)). \\ &\Longrightarrow Tx(t) \geq T^2x(t). \\ &\Longrightarrow \alpha(Tx, T^2x) \geq 1. \end{aligned}$$

   (b) *T is triangular i.e.*

   $$\begin{cases} \alpha(x, y) \geq 1, \\ \alpha(y, Ty) \geq 1, \end{cases} \Longrightarrow \alpha(x, Ty) \geq 1.$$

   indeed, If $\alpha(x, y) \geq 1$ *and* $\alpha(y, Ty) \geq 1$ *then*

   $$x(t) \geq y(t) \text{ and } y(t) \geq Ty(t),$$

   it follows that

   $$x(t) \geq Ty(t) \Rightarrow \alpha(x, Ty) \geq 1.$$

Hence T is triangular α-orbital admissible.

2. *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$:*

$$\begin{aligned} x_0 = 1 \implies & Tx_0 = T1 = \frac{1}{4}\ln 2 \approx 0.173. \\ \implies & 1 \geq T1 \\ \implies & \alpha(1, T1) \geq 1 \end{aligned}$$

3. *Now we prove that T a generalized $\alpha-\phi$ Geraphty contraction mapping. We have for all $t \in [0.1], x(t) \geq y(t)$*

$$\begin{aligned} & \sqrt{\alpha(x(t), y(t))\phi(s^3 d(Tx(t), Ty(t)))} \\ & = \sqrt{s^3 d(Tx(t), Ty(t))} \\ & = \sqrt{2^3 \int_0^1 \mid Tx(t) - Ty(t) \mid^2 dt)} \\ & = 2\sqrt{2} \int_0^1 \mid \frac{1}{4}\ln(1+ \mid x(t) \mid) - \frac{1}{4}\ln(1+ \mid y(t) \mid) \mid dt \\ & = \frac{2\sqrt{2}}{4} \int_0^1 \mid \ln(\frac{1+ \mid x(t)}{1+ \mid y(t)} \mid dt \\ & = \frac{1}{\sqrt{2}} \int_0^1 \mid \ln(1 + \frac{\mid x(t) - \mid y(t) \mid}{1+ \mid y(t)}) \mid dt \\ & \leq \frac{1}{\sqrt{2}} \int_0^1 \mid \ln(1+ \mid x(t) - \mid y(t) \mid) \mid dt \end{aligned}$$

By lemma 1.4.5 that we present below we get :

$$\begin{aligned} & \int_0^1 \mid \ln(1+ \mid x(t) - \mid y(t) \mid) \mid dt \\ & \leq \ln(\int_0^1 (1+ \mid x(t) - y(t) \mid dt) \\ & \leq \ln(\int_0^1 1dt + \int_0^1 \mid x(t) - y(t) \mid dt) \\ & \leq \ln(1 + \int_0^1 \mid x(t) - y(t) \mid dt) \\ & \leq \ln(1 + \sqrt{d(x(t), y(t))}) \end{aligned}$$

Therefore,

$$\begin{aligned}&\sqrt{\alpha(x(t),y(t))\phi(s^3d(Tx(t),Ty(t)))}\\&\leq \ln\left(1+\int_0^1 \mid x(t)-y(t)\mid dt\right)\\&\leq \frac{1}{\sqrt{2}}\ln(1+\sqrt{d(x(t),y(t)})\end{aligned}$$

So we obtain,

$$\begin{aligned}&\alpha(x(t),y(t))\phi(s^3d(Tx(t),Ty(t)))\\&\leq \frac{1}{2}\Big(\ln(1+\sqrt{d(x(t),y(t)})\Big)^2\\&\leq \frac{1}{2}\Big(\ln(1+\sqrt{M(x(t),y(t))})\Big)^2\\&=\frac{\Big(\ln(1+\sqrt{M(x(t),y(t))})\Big)^2}{2M(x(t),y(t))}\times M(x(t),y(t))\end{aligned}$$

Since $\alpha(x(t),y(t))\phi(s^3d(Tx(t),Ty(t)))=\beta(\phi(M(x(t),y(t)))\phi(M(x(t),y(t)))$. *Then* T *is a generalized* $\alpha-\phi$ *Geraphty contraction type mapping.*

4. *Either$T$ is continuous : by definition*

Then by Theorem 1.6, we see that T *has a fixed point.*

1.4 Fractional Calculus

In this section,we recall some definitions,notations and lemmas of the fractional calculus. By J we denote the closed unit interval,ie $[0,1]$.And by $C(J,\mathbb{R})$ we denote the Banach space of all continuous function from J into $\mathbb{R}$ with the norm :

$$\| y \|_\infty=\sup\{\mid y(t)\mid ; t\in J\},$$

Where $\mid . \mid$ denotes a suitable complete norm on $\mathbb{R}$.

Definition 1.4.1. *([21];[24])(fractional integral) The fractional order integral of the function* $h\in L^1([a,b])$ *of order* $\alpha\in\mathbb{R}_+$ *is defined by*

$$I_a^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)ds$$

where Γ *is the gamma function. When* $a = 0$*, we write* $I^\alpha h(t) = [h * \varphi_\alpha](t)$*, where*

$$\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \text{ for } t > 0,$$

$\varphi_\alpha(t) = 0$ *for* $t \leq 0$*, and* $\varphi_\alpha \to \delta(t)$ *as* $\alpha \to 0$*.*
Here δ *is the delta function.*

Definition 1.4.2. *([21];[24])(Caputo Fractional Derivative) For a function* h *given on the interval* $[a, b]$*, the Caputo fractional-order derivative of* h*, of order* $\alpha > 0$ *is defined by*

$${}^cD^\alpha_{a^+} h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{h^{(n)}(s)ds}{(t-s)^{1-n+\alpha}}, \quad n - 1\alpha \leq n,$$

Here $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of* α*..*

For example for $0 < \alpha \leq 1$ and $h : [a, b] \to E$ an absolutely continuous function, then the fractional derivative of order α of h exists.

From the definition of Caputo derivative, we can obtain the following auxiliary results ([21],[24]).

Lemme 1.4.1. *([21];[24])*
Let $\alpha > 0$*, then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

$c_i \in E$*,* $i = 0, 1, \ldots, n-1$*,* $n = [\alpha] + 1$*.*

Lemme 1.4.2. *([21];[24])*
Let $\alpha > 0$*, then*

$$I^{\alpha c} D^r h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$*,* $i = 0, 1, ..., n-1$*,* $n = [\alpha] + 1$*.*

Definition 1.4.3. *([21];[24])(Caputo derivative) the Caputo derivative of order* α *for a* $\mathcal{C}^n$ *function* $h : [0, \infty) \to \mathbb{R}$ *is defined by :*

$${}^c\mathrm{D}^\alpha h(t) = I^{n-\alpha} h^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^1 (t-s)^{n-\alpha-1} h^{(n)}(s) ds$$

Where $n = [\alpha] + 1$

Definition 1.4.4. *([21];[24])(Riemann-Lioville derivative) the Riemann-Lioville derivative of order α for a continuous function $h : [0,\infty) \to \mathbb{R}$ is defined by :*

$$\mathrm{D}^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\Big(\frac{d}{dt}\Big)^{n}\int_{0}^{1}\frac{h(t)}{(t-s)^{n-\alpha-1}}ds, \quad n-1<\alpha<n, \quad n\in\mathbb{N}$$

The following lemmas give some properties of fractional integral :

Lemme 1.4.3.
Let $\alpha,\beta>0$ for $h(t)=t^{\beta}$, we the following relation hold :

$${}^{c}\mathrm{D}^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}, \quad \beta\geq n$$

Lemme 1.4.4.
For a function $h\in \rfloor^{n}([0,1])$ and $\alpha>0$ then the following relation hold :

$$\mathrm{I}^{\alpha}({}^{c}\mathrm{D}^{\alpha}h(t)) = h(t)+c_{0}+c_{1}t+...c_{n-1}t^{n-1}, \quad c_{i}\in\mathbb{R}.$$

A Appendix

Lemme 1.4.5.
Let (X, μ) *be a measure space such that* $\mu(X) = 1$.*Let* $f \in L^1(X, \mu)$ *with* $f(x) > 0$ *for all* $x \in X$. *Then* $\ln(f) \in L^1(X, \mu)$ *and*

$$\int \ln(f) d\mu \leq \ln(\int f d\mu)$$

Proof.
Put $g(t) = t - 1 - \ln(t)$ and $h(t) = 1 - \frac{1}{t} - \ln(t)$ for all $t > 0$.
Then $g^{'}(t) = 1 - \frac{1}{t}$ and $h^{'}(t) = \frac{1}{t^2} - \frac{1}{t}$
We have $g(t) \geq g(1) = 0$ and $h(t) \leq h(1) = 0$ for all $t > 0$.
We deduce $t - 1 \geq \ln(t) \geq 1 - \frac{1}{t}$ for all $t > 0$.
Since f is measurable and ln is continous,$\ln(f)$ is measurable.Now for all $x \in X$,
Let $t = \frac{f(x)}{\|f\|_1}$. So,we have

$$1 - \frac{\| f \|_1}{f(x)} \leq \ln(f(x)) - \ln(\| f \|_1) \leq \frac{f(x)}{\| f \|_1} - 1.$$

Since the right-hand and the left-hand expression in the above estimations are both integrable,we have that :

$$\int (\ln(f(x)) - \ln(\| f \|_1)) d\mu \leq \int (\frac{f(x)}{\| f \|_1} - 1) d\mu = 0$$

There for :

$$\int \ln(f) d\mu \leq \ln(\int f d\mu)$$

□

Chapter 2

Existence and Results

2.1 Application to nonlinear integral equations

In this chapter, H, Afshari and all in [1] , studied the existence of solutions for nonlinear integral equations, as an application to the fixed point theorems proved in the previous section.

First, the authors considered the following integral equation

$$x(t) = h(t) + \int_0^1 \kappa(t,u) f(u, x(u))du \qquad \text{for all } t \in [0,1]. \tag{2.1}$$

And they established the result of existence and uniqueness presented below.

Let $X = C([0,1])$ be the set of real continuous functions defined on $[0,1]$, with the standard metric given by

$$\rho(x,y) = \sup_{t\in[0,1]} \mid x(t) - y(t) \mid \;\; for\ all\ x, y \in C([0,1]).$$

Now, for $p \geq 1$ let $d : X \times X \to [0,\infty)$ defined by

$$\begin{aligned} d(x,y) &= (\rho(x,y))^p \\ &= \big(\sup_{t\in[0,1]} \mid x(t) - y(t) \mid \big)^p \\ &= \sup_{t\in[0,1]} \mid x(t) - y(t) \mid^p, \quad for\ x, y \in C([0,1]) \end{aligned}$$

It easy to prove that (X, d) is a complete b-metric space with $s = 2^{p-1}$. Let Ω denote the class of non-decreasing functions $\omega : [0, \infty) \to [0, \infty)$ satisfying this inequality

$$(\omega(t))^p \leq t^p \omega(t^p) \text{ } for\ all\ p \geq 1 \text{ } and\ t \geq 0.$$

Let analyze the equation(2.1) under the following assumptions :

(A1) $h : I \to \mathbb{R}$ is a continuous function.

(A2) $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function,$f(t, x) \geq 0$ and there exists $\omega \in \Omega$ such for $x, y \in \mathbb{R}$,

$$| f(t, x) - f(t, y) | \leq \omega(| x - y |),$$

with $\omega(t_n) \to \frac{1}{2^{p-1}}$ as $n \to \infty$ implying $\lim_{n\to\infty} t_n = 0$.

(A3) $\kappa : [0, 1] \times [0, 1] \to \mathbb{R}$ is a continuous in $t \in [0, 1]$ for every $u \in [0, 1]$ and is measurable in $u \in [0, 1]$ for all $t \in [0, 1]$ such that $\kappa \geq 0$ and

$$\int_0^1 \kappa(t, u) du \leq \frac{1}{2^{3-\frac{3}{p}}}.$$

Theorem 2.1.
Under assumptions $(A1) - (A3)$ *the nonlinear integral equation (2.1) has a unique solution in* $C([0, 1])$.

Proof.
We shall reduce the existence of solutions of (2.1) to a fixed point problem. To this end, we consider the operator $T : X \to X$ defined by

$$Tx(t) = h(t) + \int_0^1 \kappa(t, u) f(s, x(u)) du \quad for\ all\ t \in [0, 1]$$

Clearly, the fixed point of the operator T are solutions of the problem (2.1). By virtue of our assumptions, T is well defined (this means that if $x \in X$, then $Tx \in X$.) Also, for $x, y \in X$, we have

$$\begin{aligned}
&\mid T(x)(t) - T(y)(t) \mid \\
&= \mid h(t) + \int_0^1 \kappa(t,u) f(s, x(u)) du - h(t) - \int_0^1 \kappa(t,u) f(s, y(u)) du \mid \\
&\leq \int_0^1 \kappa(t,u) \mid f(s, x(u)) - f(s, y(u)) \mid du \\
&\leq \int_0^1 \kappa(t,u) \omega(\mid x(u) - y(u) \mid) du.
\end{aligned}$$

Since the function ω is non-decreasing, we get

$$\omega(\mid x(u) - y(u) \mid) \leq \omega(\sup_{t \in [0,1]} \mid x(u) - y(u) \mid) = \omega(\rho(x,y)).$$

Therefore

$$\mid T(x)(t) - T(y)(t) \mid \leq \frac{1}{2^{3-\frac{3}{p}}} \omega(\rho(x,y)).$$

Now, we have

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t \in [0,1]} \mid T(x)(t) - T(y)(t) \mid^p \\
&\leq \Big[\frac{1}{2^{3-\frac{3}{p}}} \omega(\rho(x,y))\Big]^p \\
&\leq \frac{1}{2^{3p-3}} [\omega(\rho(x,y))]^p \\
&\leq \frac{1}{2^{3p-3}} (\rho(x,y))^p \omega(\rho(x,y)^p) \\
&\leq \frac{1}{2^{3p-3}} d(x,y) \omega(d(x,y)) \\
&\leq \frac{1}{2^{3p-3}} M(x,y) \omega(M(x,y))
\end{aligned}$$

That is

$$s^3 d(Tx, Ty) \leq \beta(M(x,y)) M(x,y),$$

Where $s = 2^{p-1}$ and $\beta(t) = \omega(t)$. Notice that if $\omega \in \mathcal{F}_s$, then $\beta \in \mathcal{F}_s$. By Corollary 1.3.2, equation (2.1) has unique solution in $C([0,1])$. $\square$

An other results of the existence of solutions for nonlinear integral equations in [2]. Let $X = C([0,1],\mathbb{R})$ be the set of real continuous functions defined on $[0,1]$, and let $d : X \times X \to [0,\infty)$ defined by

$$\begin{aligned} d(x,y) &= \| (x-y)^2 \|_\infty \\ &= \sup_{t\in[0,1]} (x(t)-y(t))^2, \quad for\ x,y \in C([0,1]) \end{aligned}$$

we consider the following integral equation

$$x(t) = P(t) + \int_0^1 S(t,u)f(u,x(u))du \qquad \text{for all } t \in [0,1]. \tag{2.2}$$

where $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ and $P : [0,1] \to \mathbb{R}$ are two continuous function and$S : [0,1]\times[0,1] \to [0,\infty)$ is a function such that $S(t,.) \in L^1[0,1]$ for all $t \in [0,1]$.

Consider the operator $T : X \to X$ defined by

$$T(x)(t) = P(t) + \int_0^1 S(t,u)f(u,x(u))du \qquad \text{for all } t \in [0,1]. \tag{2.3}$$

Theorem 2.2.

Let $X = C([0,1],\mathbb{R})$.Suppose there exist $\eta : X\times X \to [0,\infty)$,$\alpha : X\times X \to [0,\infty)$, and $\beta : [0,\infty) \to [0,\frac{1}{4})$such that the following conditions are satisfied

(H1) for all $u \in [0,1]$ and for all $x,y \in X$;

$$0 \leq | f(u,x(u)) - f(u,y(u)) | \leq \eta(x,y) \mid x(u)-y(u) \mid,$$

and

$$\| \int_0^1 S(t,u)\eta(x,y) \|_\infty^2 \leq \frac{\beta\big(\| (x-y)^2 \|_\infty \big)}{\alpha(x,y)};$$

(H2) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(H3) $\alpha(x,Tx) \geq 1 \Rightarrow \alpha(Tx,T^2x) \geq 1$

(H4) if (x_n) is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $(x_{n(k)})$ of (x_n) such that $\alpha(x_{n(k)},x) \geq 1$ for all k.

Then,the integral equation (2.2) has a solution in X.

Proof.
Cleary,any fixed point of (2.3)is a solution of (2.2).
By condition $(H1)$, we obtain

$$\begin{aligned}
&\alpha(x,y) \mid T(x)(t) - T(y)(t) \mid^2 \\
&= \alpha(x,y)[\mid \int_0^1 S(t,u)[f(u,x(u)) - f(u,y(u))du] \mid]^2 \\
&\leq \alpha(x,y)[\int_0^1 S(t,u) \mid f(u,x(u)) - f(u,y(u)) \mid du]^2 \\
&\leq \alpha(x,y)[\int_0^1 S(t,u)\eta(x,y)\sqrt{\mid x(u) - y(u) \mid^2} du]^2 \\
&\leq \alpha(x,y)[\int_0^1 S(t,u)\eta(x,y)\sqrt{\| (x-y)^2 \|_\infty}^2 du]^2 \\
&= \alpha(x,y) \| (x-y)^2 \|_\infty [\int_0^1 S(t,u)\eta(x,y)du]^2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\alpha(x,y) \| T(x)(t) - T(y)(t) \|^2 \\
&\leq \alpha(x,y) \| (x-y)^2 \|_\infty \| \int_0^1 S(t,u)\eta(x,y)du \|_\infty^2 . \\
&\leq \beta(\| (x-y)^2 \|_\infty) \| (x-y)^2 \|_\infty
\end{aligned}$$

Thus,for all $x, y \in X$,we obtain

$$\alpha(x,y)d(T(x),T(y)) \leq \beta(d(x,y))d(x,y)$$

This implies that corollary 1.5 holds with $\phi(t) = t$ and $L = 0$. Hence,the operator T has a fixed point,that is,the integral equation (2.2) has a solution in X. □

The following example illustrates Theorem 2.2.

Example 2.1.1.
Take $X = C([0,1], \mathbb{R})$.Consider the following functional integral equation we consider the following integral equation

$$x(t) = \frac{t^2}{1+t^2} + \frac{1}{27}\int_0^1 \frac{u\cos t}{54(1+t)} \frac{\mid x(u) \mid}{1+ \mid x(u) \mid} du, \qquad \textit{for all } t \in [0,1]. \quad (2.4)$$

Observe that the equation (2.4)is a special case of (2.2)with

$$\begin{aligned} P(t) &= \frac{t^2}{1+t^2} \\ S(t,u) &= \frac{u}{3(1+t)} \\ f(t,x) &= \frac{\cos t \mid x \mid}{18(1+\mid x \mid)} \end{aligned}$$

Consider the operator $T : X \to X$ *defined by*

$$T(x)(t) = \frac{t^2}{1+t^2} + \frac{1}{27}\int_0^1 \frac{u\cos t}{54(1+t)}\frac{\mid x(u)\mid}{1+\mid x(u)\mid}du, \qquad \text{for all } t \in [0,1]. \quad (2.5)$$

Define the mapping $\alpha : X \times X \to [0,\infty)$ *as*

$$\alpha(x,y) = \begin{cases} 1 & if x(t) \geq y(t) \\ 0 & otherwise. \end{cases}$$

Take $\beta : [0,\infty) \to [0,\frac{1}{4})$ *as*

$$\beta = \frac{t^2+1}{4t^2+8}$$

Let $\eta(x,y) = 1$*.For arbitrary fixed* $x, y \in \mathbb{R}$ *such that* $x \geq y$*.*
we obtain

$$\begin{aligned} \mid f(t,x) - f(t,y) \mid &= \mid \frac{\cos t \mid x \mid}{18(1+\mid x \mid)} - \frac{\cos t \mid y \mid}{18(1+\mid y \mid)} \mid \\ &\leq \frac{1}{18} \mid x - y \mid \\ &\leq \eta(x,y) \mid x - y \mid \end{aligned}$$

and

$$\begin{aligned} \| \int_0^1 S(t,u)\eta(x,y)du \|_\infty^2 &= \frac{1}{36} \\ \leq \frac{(\| (x-y)^2 \|_\infty)^2 + 1}{4(\| (x-y)^2 \|_\infty)^2 + 8} &= \beta(\| (x-y)^2 \|_\infty). \end{aligned}$$

Again,by definition of $\alpha(x,y)$*,it follows that*

(H2) $\alpha(1, T1) \geq 1$ *;*

(H3) $\alpha(x, Tx) \geq 1$ *implies that* $\alpha(Tx, T^2x) \geq 1$

(H4) if (x_n) *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$*for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $(x_{n(k)})$ *of* (x_n) *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k*.*

Hence,by using Theorem 2.2. the integral equation (2.4) has a solution in X*.*

2.2 Application to fractional differential equation

In this section, we consider the problem

$$\begin{cases} {}^cD^{\mu}x(t) - {}^cD^{\nu}x(t) = h(t, x(t)), \quad t \in J, \quad 0 < \nu < \mu < 1 \\ x(0) = x_0, \end{cases} \tag{2.6}$$

Where ${}^cD^{\mu}$, ${}^cD^{\nu}$ are the Caputo derivatives of order μ, ν, respectively and $h : J \times \mathbb{R} \to \mathbb{R}$ is continuous mapping.

First of all, we define what we mean by a solution of (2.6)

Definition 2.2.1. *A function $x \in C(J, \mathbb{R})$ is said to be a solution of (2.6) if x satisfies the equation*

$${}^cD^{\mu}x(t) - {}^cD^{\nu}x(t) = h(t, x(t)), \quad t \in J, 0 < \nu < \mu < 1,$$

and the condition $x(0) = x_0$.

Lemme 2.2.1. *([32]) Let $h : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function. If $x(.) \in \mathcal{C}(J)$ is a solution of the following integral equation*

$$\begin{aligned} x(t) = x_0 \quad &+ \quad \frac{1}{\Gamma(\mu-\nu)} \int_0^1 (t-s)^{\mu-\nu-1}(x(s)-x_0)ds \\ &+ \quad \frac{1}{\Gamma(\mu)} \int_0^1 (t-s)^{\mu-1} h(t, x(s))ds, \end{aligned}$$

then $x(t)$ is a solution of the fractional equation (2.6).

Proof. Let $x(t)$ is a solution of the fractional equation (2.6). Applying the integral operator I^{μ} on the first equation of (2.6), we get

$$I^{\mu}({}^cD^{\mu}x(t)) = I^{\mu}({}^cD^{\nu}x(t)) + I^{\mu}h(t, x(t)). \tag{2.7}$$

As we know that the Caputo fractioanl derivative can be defined via the Riemann Liouville derivative, for $0 < \beta \leq 1$, by

$${}^cD^{\nu}x(t) = D^{\nu}(x(s) - x_0)(t).$$

Then, by using Lemma 1.4.2, we obtain

$$x(t) = c_0 + I^{\mu} D^{\nu}(x(s) - x_0)(t) + I^{\mu} h(t, x(t)).$$

Using the boundary condition, we have $c_0 = x_0$, thus

$$\begin{aligned} x(t) = x_0 \quad &+ \quad \frac{1}{\Gamma(\mu - \nu)} \int_0^1 (t - s)^{\mu - \nu - 1}(x(s) - x_0) ds \\ &+ \quad \frac{1}{\Gamma(\mu)} \int_0^1 (t - s)^{\mu - 1} h(t - x(s)) ds, \end{aligned}$$

□

We recall the existence theorem 1.5 seen in Chapter 1.

Theorem 2.3. *Let (X, d) be a complete b-metric space and $f : X \to X$ be a generalized α-ϕ- Geraghty contraction such that :*

(i) *f is α-admissible,*

(ii) *There exists $u_0 \in X$ with $\alpha(u_0, fu_0) \geq 1$,*

(iii) *If $u_n \subseteq X$, and $\alpha(u_n, u_{n+1}) \geq 1$, then $\alpha(u_n, u) \geq 1$, for all n.*

Then f has a fixed point.

Let denote by $J = [0, 1]$ the closed unit interval and by $X = C(J, \mathbb{R})$ the set of all continuous functions. $d : X \times X \to [0, \infty)$ a mapping given by

$$d(x, y) = \| (x - y)^2 \|_{\infty} = \sup_{t \in [0,1]} (x(t) - y(t))^2.$$

It can be checked that the pair (X, d) is b-metric space with $s = 2$.
We shall reduce the existence of solutions of the boundary value problem (2.6) to fixed point problem. To this end we consider the operator

$$T : C[I, \mathbb{R}] \to C[I, \mathbb{R}]$$

defined by

$$\begin{aligned} Tx(t) = x_0 \quad &+ \quad \frac{1}{\Gamma(\mu - \nu)} \int_0^1 (t - s)^{\mu - \nu - 1}(x(s) - x_0) ds \\ &+ \quad \frac{1}{\Gamma(\mu)} \int_0^1 (t - s)^{\mu - 1} h(t - x(s)) ds, \end{aligned} \tag{2.8}$$

Let us list some conditions on the functions involved in the problem (2.6). Assume that

(A1) There exists $\lambda : [0, \infty) \to [0, 1/s^2)$ a nondecreasing function such that

$$|h(t, x(t)) - h(t, y(t))| \leq |x - y|\Big(\frac{(\mu - \nu)\Gamma(\mu - \nu)}{2\sqrt{2}}\sqrt{\lambda(\|x - y\|_\infty^2)} - 1\Big)$$

for any $t \in J$,

(A2) The exists $\phi : \mathbb{R}^2 \to \mathbb{R}$, such that, for all $t \in J$, and $x, y \in \mathcal{C}(J)$,

$$\phi(x(t), y(t)) \geq 0 \text{ implies } \phi(Tx(t), Ty(t)) \geq 0,$$

(A3) there exists $x_0 \in \mathcal{C}(J)$ such that $\phi(x_0(t), Tx_0(t)) \geq 0$ for all $t \in J$, where the operator T is defined by (2.8).

(A4) If $(x_n) \subset \mathcal{C}(J)$, such that $(x_n) \to x$ in $\mathcal{C}(J)$, and $\phi(x_n, x_{n+1}) \geq 0$, then $\phi(x_n, x) \geq 0$, for all n.

Theorem 2.4. *Assume that assumptions* $(A1) - (A4)$ *hold. Then, the problem (2.6) has at least one solution.*

Proof. By lemma (2.2.1), $x \in \mathcal{C}(J)$ is a solution of (2.6) if it is solution of the following integral equation

$$\begin{aligned} x(t) = x_0 \quad &+ \quad \frac{1}{\Gamma(\mu - \nu)} \int_0^1 (t - s)^{\mu - \nu - 1}(x(s) - x_0)ds \\ &+ \quad \frac{1}{\Gamma(\mu)} \int_0^1 (t - s)^{\mu - 1} h(t - x(s))ds, \end{aligned}$$

Then,the problem (2.6) is equivalent to finding a fixed point of the operator T.

Let $x, y \in \mathcal{C}(J)$, be with $\phi(x(t), y(t)) \geq 0$ for all $t \in J$

Step 1 T is a generalized α-ψ- Geraghty contraction type mapping.

By condition, we get

$$
\begin{aligned}
|Tx(t)-Ty(t)|^2 &= \Big|x_0+\frac{1}{\Gamma(\mu-\nu)}\int_0^1(t-s)^{\mu-\nu-1}(x(s)-x_0)ds \\
&+ \frac{1}{\Gamma(\mu)}\int_0^1(t-s)^{\mu-1}h(t,x(s))ds \\
&- x_0-\frac{1}{\Gamma(\mu-\nu)}\int_0^1(t-s)^{\mu-\nu-1}(y(s)-y_0)ds \\
&- \frac{1}{\Gamma(\mu)}\int_0^1(t-s)^{\mu-1}h(t,y(s))ds\Big|^2 \\
&= \Big|\frac{1}{\Gamma(\mu-\nu)}\int_0^1(t-s)^{\mu-\nu-1}(x(s)-y(s))ds \\
&+ \frac{1}{\Gamma(\mu)}\int_0^1(t-s)^{\mu-1}\Big(h(t,x(s))-h(t,y(s))\Big)ds\Big|^2
\end{aligned}
$$

So,

$$
\begin{aligned}
|Tx(t)-Ty(t)|^2 &\leq \Big|\frac{1}{\Gamma(\mu-\nu)}\int_0^1(t-s)^{\mu-\nu-1}|x(s)-y(s)|ds \\
&+ \frac{1}{\Gamma(\mu-\nu)}\int_0^1(t-s)^{\mu-\nu-1}|h(t,x(s))-h(t,y(s))|ds\Big|^2
\end{aligned}
$$

From the properties of the integral, and using **(A1)** we have

$$
\begin{aligned}
&|Tx(t)-Ty(t)|^2 \\
&\leq |\frac{1}{\Gamma(\mu-\nu)}\int_0^1(t-s)^{\mu-\nu-1}\Big(|(x(s)-y(s))| \\
&+|x(t)-y(t)|\Big(\frac{(\mu-\nu)\Gamma(\mu-\nu)}{2\sqrt{2}}\sqrt{\lambda(\|x-y\|_\infty^2)}-1\Big)\Big)ds|^2 \\
&\leq \Big|\frac{1}{\Gamma(\mu-\nu)}\int_0^1(t-s)^{\mu-\nu-1}|x(t)-y(t)|\frac{(\mu-\nu)\Gamma(\mu-\nu)}{2\sqrt{2}}\sqrt{\lambda\|x-y\|_\infty^2}ds\Big|^2 \\
&\leq \Big(\frac{1}{\Gamma(\mu-\nu)}\Big)^2|x(t)-y(t)|^2\frac{(\mu-\nu)^2\Big(\Gamma(\mu-\nu)\Big)^2}{8}\lambda\|x-y\|_\infty^2 \\
&\Big(\int_0^1(t-s)^{\mu-\nu-1}ds\Big)^2
\end{aligned}
$$

Then,we obtain

$$\begin{aligned}
&|Tx(t)-Ty(t)|^2\\
&\leq \frac{(\mu-\nu)^2}{8}|x(t)-y(t)|^2\Big(\Big[\frac{-s^{\mu-\nu}}{\mu-\nu}\Big]_0^1\Big)^2\lambda(\|x-y\|_\infty^2)\\
&\leq \frac{(\mu-\nu)^2}{8}\mid x(t)-y(t)\mid^2\frac{1}{(\mu-\nu)^2}\lambda(\|x-y\;\|_\infty^2)\\
&\leq \frac{1}{8}\|x(t)-y(t)\|^2.\lambda(\|x-y\|_\infty^2)\\
&\leq \frac{1}{s^3}d(x,y)\lambda(d(x,y))
\end{aligned}$$

Therefore for $x, y \in C(J), t \in J$ with $\phi(x(t), y(t)) \geq 0$, we have

$$\|(Tx-Ty)^2\|_\infty \leq \frac{1}{8}\|x(t)-y(t)\|_\infty^2\lambda(\|x-y\|_\infty^2).$$

Define $\alpha : C(J) \times C(J) \to [0, +\infty)$ by

$$\alpha(x,y)=\begin{cases} 1, & \phi(x(t),y(t))\geq 0, \quad \text{for all } t\in J,\\ 0, & \text{otherwise.}\end{cases}$$

And $\psi(t) = t$ for all $t \in J$. So,

$$\alpha(x,y)8d(Tx,Ty) \leq 8d(Tx,Ty) \leq \lambda(d(x,y))d(x,y).$$

Thus, by Definition 1.2.12, T is $\alpha-\psi-$Geraphty contraction type mapping.

Step 2 T is α-admissible.
From (**A2**) and by the definition of α,

$$\begin{aligned}
\alpha(x,y)\geq 1 &\implies \phi(x(t),y(t))\geq 0\\
&\implies \phi(Tx(t),Ty(t))\geq 0\\
&\implies \alpha(Tx(t),Ty(t))\geq 1
\end{aligned}$$

Thus, T is α-admissible for all $x, y \in \mathcal{C}(J)$.

step 3 There exists $x_0 \in \mathcal{C}(J)$ with $\alpha(x_0, Tu_0) \geq 1$,
From **(A3)**, it follows that there exists $x_0 \in \mathcal{C}(J)$ such that

$$\alpha(x_0, Tx_0) \geq 1.$$

Step 4 If $(x_n \subset X$, and $\alpha(x_n, x_{n+1}) \geq 1$, then $\alpha(x_n, x) \geq 1$, for all n.
This last condition is fulfilled from the assumption **(A4)**.

Hence, we deduce that the operator T has a fixed point which is solution of the problem (2.6) □

By taking $\lambda(t) = \frac{t}{4t+1}$ (it is clear that$\lambda \in \mathcal{F}$) in theorem **??**, we obtain the following result.

Corollary 2.2.1.
Suppose that

(i) *There exists* $\lambda : [0, \infty) \to [0, 1/s^2)$ *a nondecreasing function such that* $| h(t, x(t)) - h(t, y(t)) | \leq \frac{\mu-\nu}{2\sqrt{2}} | x - y | (\Gamma(\mu - \nu)\sqrt{\frac{\|x-y\|_\infty^2}{4\|x-y\|_\infty^2+1}} - 1)$
for any $t \in J$,*and* $x, y \in \mathbb{R}$

(ii) *There exists* $\phi : \mathbb{R}^2 \to \mathbb{R}$, *such that, for all* $t \in J$, *and* $x, y \in \mathcal{C}(J)$,

$$\phi(x(t), y(t)) \geq 0 \text{ implies } \phi(Tx(t), Ty(t)) \geq 0,$$

(iii) *there exists* $x_0 \in \mathcal{C}(J)$ *such that* $\phi(x_0(t), Tx_0(t)) \geq 0$ *for all* $t \in J$, *where the operator* T *is defined by (2.8).*

(iv) *If* $(x_n) \subseteq \mathcal{C}(J)$, $x_n \to x$ *in* $\mathcal{C}(J)$, *and* $\omega(x_n, x_{n+1}) \geq 0$, *then* $\omega(x_n, x) \geq 0$, *for all* n.

Then the problem (2.6) has at least one solution.

Conclusion

This thesis is devoted to the study of the existence and uniqueness of solutions to the differential equations which are generated by the Caputo fractional derivatives by using some of the generalized fixed point results for Geraghty contractions in b-metric spaces

1. The idea of b-metric was initiated from the works Bakhtin [2]. Czerwik [3] generalized the concept of a distance. He gave an axiom which was the triangular inequality and defined a b-metric space with a view of generalizing the Banach contraction mapping theorem.
2. The important difference between a metric and a b-metric is that the b-metric need not be a continuous function in its two variables. This led to many fixed point theorems on b-metric spaces being stated .So the results obtained for such rich spaces become more viable in different applications
3. This work allowed us to know the importance of the fractional calculus in mathematics. Since the differential equations of fractional order (fractional differential equations) take the great interest of the researchers due to wide application potential in various disciplines.

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