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A Generalization of b-Metric Space and Some Fixed Point Theorems

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Dedicace

First of all thanks to God Almighty my creator, my strong pillar, my source of inspiration, wisdom, knowledge and understanding.

I dedicate this thesis to two beloved people who have meant and continue to mean so much to me. Although they are no longer of this world, their memories continue to regulate my life.
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May you find peace and happiness in Paradise.
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Introduction

In the last few decades, a significant interest in fixed point theory has been directed to transposing classical metric fixed point result from metric spaces to certain generalized metric spaces.

Historically, the starting line in this field was well-defined by the creation of Banach's fixed point theorem, formulated and proved in 1920, which was published in 1922, it is one of the most important theorems in classical functional analysis. After that more results involving fixed point with different contractive mappings in metric spaces came into view for example Kannan [27], introduced Kannan contractive theorem to find fixed points of mappings which are not continuous. The Banach and Kannan fixed point theorems have been improved by various successful attempts. One such attempt is due to Reich [32].

On the other hand in 1989, Backhtin [12] introduced the concept of b-metric space. In 1993 Czerwik [29] first presented a generalization of Banach fixed point theorem in b-metric spaces. Many researchers including Aydi [11], Boriceanu [21], Chug [24], Bota [18] studied the extension of fixed point theorems in b-metric space.

Later on, Fagin et al. [26] discussed some kind of relaxation in triangular inequality and called this new distance measure as non-linear elastic mathing (NEM). Similar type of relaxed triangle inequality was also used for trade measure [7] and to measure ice floes [25]. All these applications intrigued and pushed Kamran et al. in [33] to introduce the concept of extended b-metric spaces as a generalization of b-metric spaces.

In this thesis we will study some fixed point theorems for single-valued mappings and operators that satisfy a cyclic-type contraction condition in complete extended b-metric spaces.

The thesis contains three chapters organized as follows:

In the first chapter we will discuss some basic definitions and properties that we will use later, This chapter also includes many interesting results related to the b-metric spaces, some examples which satisfy the properties of above spaces, some fixed-point theories, and convergence, Cauchy sequence, completeness.

In the second chapter we will throw light on our main topic, which is the extended b-metric space. Also, we will see some examples in different spaces, some important definitions that we need, some fixed-point theories especially for operators that satisfy a cyclic-type contraction.

In the last chapter we will move on to the applications of some fixed-point theorems that we saw in the second chapter on The Integral equations, Differential equations and Fractional differential equations.

Chapter 1

Preliminaries.

In this chapter we throw light on basic definitions and many interesting results related to the b-metric spaces, some examples which satisfy the properties of above spaces, convergence, Cauchy sequence, completeness and some fixed point theories.

1.1 Definitions

Definition 1.1.1. *[23]****(Metric space)***
Let X be a non-empty set and let $d : X \times X \longrightarrow \mathbb{R}^+$ be a function satisfying the conditions,

1. *$d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
3. *$d(x, y) \leq [d(x, z) + d(z, y)]$, for all $x, y, z \in X$.*

Then d it is called metric on X and the pair (X, d) is called metric space.

Definition 1.1.2. ***(Lipschitzian mapping)***
Let (X, d) be a metric space and T is a mapping from X to X. The mapping T is called a Lipschitz mapping if there exists a constant $k \geq 0$ such that

$$d(Tx, Ty) \leq kd(x, y),$$

for all $x, y \in X$. Where k is called the Lipschitz constant.

Example 1.1.1. *Consider* $X = [1,2]$ *and* $d : X \times X \longrightarrow \mathbb{R}^+$ *defined by* $d(x,y) = |x-y|$. *Define* $T : X \longrightarrow X$ *by* $T(x) = x^2$.
Since $x^2 - y^2 = (x+y)(x-y)$ *It follows that*

$$\begin{aligned} d(T(x),T(y)) &= |T(x) - T(y)| \\ &= |x^2 - y^2| \\ &\leq |x+y||x-y| \\ &\leq (|x| + |y|)|x-y| \\ &\leq (2+2)|x-y| \\ &= 4d(x,y), \end{aligned}$$

for all $x, y \in \mathbb{R}$. *This shows that* T *is a Lipschitz mapping, with Lipschitz constant* $k = 4$.

Definition 1.1.3. ***(Contraction mapping)***
A mapping $T : X \longrightarrow X$ *where* (X,d) *is a metric space, is said to be a contraction if there exists* $k \in [0,1)$ *such that for all* $x, y \in X$*;*

$$d(Tx, Ty) \leq kd(x,y) \tag{1.1}$$

Example 1.1.2. *Let* $X = [0,1]$ *and* $d : X \times X \longrightarrow \mathbb{R}^+$ *defined by*

$$d(x,y) = |x-y|.$$

Clearly (X,d) *is metric space. The function* $T : X \longrightarrow X$ *where*

$$T(x) = \ln(1 + \frac{x}{4})$$

is a contraction.

Definition 1.1.4. ***(ℓ^p spaces)***
Let $\mathbb{K}$ *the field either of real or complex numbers. We denote by* $\mathbb{K}^{\mathbb{N}}$ *the set of all sequences of elements of* $\mathbb{K}$ *which is a vector space.*
For $0 < p < \infty$, ℓ^p *is the subspace of* $\mathbb{K}^{\mathbb{N}}$ *consisting of all sequences* $(x_n)_{n\in\mathbb{N}}$ *satisfying*

$$\sum_n |x_n|^p < \infty$$

If $p \geq 1$, then the real-valued function $\|\cdot\|_p$ on ℓ^p defined by

$$\|x\|_p \;=\; \left(\sum_n |x_n|^p\right)^{1/p} \qquad \text{for all } x \in \ell^p$$

defines a norm on ℓ^p. In fact, ℓ^p is a complete metric space with respect to this norm, and therefore is a Banach space.
If $p = 2$ then ℓ^2 is also a Hilbert space when endowed with its canonical inner product, called the Euclidean inner product, defined for all $x, y \in \ell^p$ by

$$\langle x, y\rangle \;=\; \sum_n \overline{x_n} y_n.$$

For $0 < p < 1$, then ℓ^p does not carry a norm, but rather a metric defined by

$$d(x, y) = \sum_n |x_n - y_n|^p \,. \tag{1.2}$$

Definition 1.1.5. *Let (S, Σ, μ) be a measure space. If $0 < p < 1$, then $L^p(\mu)$ it is the quotient vector space of those measurable functions f such that*

$$N_p(f) = \int_S |f|^p \, d\mu < \infty.$$

We may introduce the $p-$ *norm* $\|f\|_p = N_p(f)^{1/p}$, but $\|\cdot\|_p$ does not satisfy the triangle inequality in this case, and defines only a quasi-norm. and so the function

$$d_p(f, g) = N_p(f - g) = \|f - g\|_p^p.$$

is a metric on $L^p(\mu)$.

Example 1.1.3. *For $0 < p < 1$, let $L^P[0,1]$ be the set of all functions $f : [0,1] \longrightarrow \mathbb{R}$ that are measurable and satisfy*

$$\int_0^1 |f(x)|^p dx < \infty,$$

with functions equal almost everywhere identified. We define a metric on $L^P[0,1]$ by

$$d_p(f, g) = \int_0^1 |f(x) - g(x)|^p dx. \tag{1.3}$$

Definition 1.1.6. ***(Convex function)***
A function f from a real interval I to $\mathbb{R}$ is said to be convex when for all x_1 and x_2 of I and all $t \in [0,1]$ we have:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Example 1.1.4. *The power function $\mathbb{R}_+^* \longrightarrow \mathbb{R}, x \longrightarrow x^p$ is convex if $p > 1$.*

Proposition 1.1.1. *A continuous function f on I is convex on I if (and only if) whatever the elements x_1 and x_2 of I:*

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

1.2 b-metric space

Definition 1.2.1. *[28]**(b-metric space)***
A b-metric on a nonempty set X is a function $d_s : X \times X \longrightarrow [0, \infty)$ satisfying the conditions

(d_s1) *$d(x,y) = 0 \Leftrightarrow x = y$;*

(d_s2) *$d(x,y) = d(y,x)$;*

(d_s3) *$d(x,y) \leq s[d(x,z) + d(z,y)]$;*

for all $x, y, z \in X$, and for some fixed number $s \geq 1$. The pair (X, d_s) is called a b-metric space.

The condition (d_s3) is called the s-relaxed triangle inequality and that give us for all $x_0, x_1 \cdots x_n \in X \quad \forall n \in \mathbb{N}$;

$$\begin{aligned} d_s(x_0, x_n) &\leq sd_s(x_0, x_1) + s^2 d_s(x_1, x_2) + \cdots + s^{n-1} d_s(x_{n-2}, x_{n-1}) \\ &\quad + s^{n-1} d_s(x_{n-1}, x_n) \end{aligned}$$

Indeed. we obtain successively

$$\begin{aligned} d_s(x_0, x_n) &\leq sd_s(x_0, x_1) + sd_s(x_1, x_n) \\ &\leq sd_s(x_0, x_1) + s^2 d_s(x_1, x_2) + s^2 d_s(x_2, x_n) \\ &\leq sd_s(x_0, x_1) + \cdots + s^{n-1} d_s(x_{n-2}, x_{n-1}) + s^{n-1} d_s(x_{n-1}, x_n) \end{aligned}$$

Example 1.2.1. *[33] Let* $X = \ell^p(\mathbb{R})$ *with* $0 < p < 1$ *where*

$$\ell^p(\mathbb{R}) = \{\{x_n\} \subset \mathbb{R} : \sum_n |x_n|^p < \infty\}.$$

Define $d : X \times X \longrightarrow \mathbb{R}^+$ *as:*

$$d(x, y) = \left(\sum_n |x_n - y_n|^p\right)^{1/p} < \infty$$

where $x = \{x_n\}$, $y = \{y_n\}$*. then* d *is a b-metric with coefficient* $s = 2^{1/p}$*.*

Indeed, (X, δ) is a metric space with $\delta(x, y) = \sum_n |x_n - y_n|^p < \infty$ from (1.2) by the triangle inequality we have

$$\delta(x, y) \leq \delta(x, z) + \delta(z, y) \text{ for all } x, y, z \in X$$

then

$$\begin{aligned}\delta(x, y)^{1/p} &\leq (\delta(x, z) + \delta(z, y))^{1/p} \\ &\leq (2 \max\{\delta(x, z), \delta(z, y)\})^{1/p} \\ &\leq 2^{1/p}(\delta(x, z)^{1/p} + \delta(x, z)^{1/p})\end{aligned}$$

So

$$d(x, y) \leq 2^{1/p}(d(x, z) + d(z, y))$$

The conditions (d_s1) and (d_s2) are satisfied then (X, d_s) is a b-metric space with $s = 2^{1/p}$.

Example 1.2.2. *[33] Let* $L^P[0, 1]$ *be the space of all functions* $x : [0, 1] \longrightarrow \mathbb{R}$ *such that* $\int_0^1 |x(t)|^p dt < \infty$*, with* $0 < p < 1$*, Define* $d_s : X \times X \longrightarrow \mathbb{R}^+$ *as :*

$$d_s(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{1/p}$$

We can prove it by the same way using (1.3) (X, d_s) is a b-metric space with coefficient $s = 2^{1/p}$.

Example 1.2.3. *[1] Let (X, d) be a metric space and $\sigma_d : X \times X \longrightarrow \mathbb{R}$ be defined by*

$$\sigma_d(x, y) = [d(x, y)]^p$$

for all $x, y \in X$ where $p > 1$ is a fixed real number. Then σ_d is a b-metric with $s = 2^{p-1}$

Indeed, conditions (d_s1) and (d_s2) in Definition 1.2.1 are satisfied and thus we only have to show that condition (d_s3) holds for σ_d.
If $p > 1$ the convexity of the function $f(x) = x^p$ (see Example 1.1.4) implies for each $x, y, z \in X$ we get

$$\begin{aligned} \sigma_d(x, y) = [d(x, y)]^p &\leq [d(x, z) + d(z, y)]^p \\ &\leq 2^{p-1}[d(x, z)^p + d(z, y)^p] \\ &\leq 2^{p-1}[\sigma_d(x, z) + \sigma_d(z, y)]. \end{aligned}$$

So condition (d_s3) holds and σ_d is a b-metric with $s = 2^{p-1}$.

We present an easy example to show that in general a b-metric need not be a metric.

Example 1.2.4. *If $X = \mathbb{R}$ is the set of real numbers and $|x - y|$ is the usual euclidean metric then $|x - y|^2$ is a b-metric on $\mathbb{R}$ with $s = 2$ but it is not a metric on $\mathbb{R}$.*

Indeed, conditions (d_1) and (d_2) in Definition 1.2.1 are satisfied and condition (d_3) holds by the identity of parallelogram, for all $x, y, z \in \mathbb{R}$

$$|x - y|^2 = |\underbrace{x - z}_{a} + \underbrace{z - y}_{b}|$$

$$\begin{aligned} |a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2 &\Rightarrow |a + b|^2 = 2|a|^2 + 2|b|^2 - |a - b|^2 \\ &\Rightarrow |a + b|^2 \leq 2|a|^2 + 2|b|^2 \\ &\Rightarrow |x - y|^2 \leq 2(|x - z|^2 + |z - y|^2) \end{aligned}$$

$|x - y|^2$ is a b-metric with $s = 2$.
If $x = 5, y = 1, z = 4$ the tringle inequality is not satisfied then $|x - y|^2$ is not a metric.

Remark 1.2.1. *The above examples show that the class of b-metric spaces is larger than the class of metric spaces. When s = 1, the concept of b-metric space coincides with the concept of metric space.*

The notions of convergent sequence, Cauchy sequence and complete space are defined as in metric spaces.

Definition 1.2.2. *[21](**Cauchy sequence**)*
Let (X, d_s) be a b-metric space. Then a sequence (x_n) in X is called Cauchy sequence if and only if for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

Definition 1.2.3. *[21](**Convergent sequence**)*
Let (X, d_s) be a b-metric space. Then a sequence (x_n) in X is called convergent sequence if and only if there exists $x \in X$ such that for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ we have $d(x_n, x) < \varepsilon$. In this case we write $\lim\limits_{n \to \infty} x_n = x$.

Definition 1.2.4. *[21](**Complete b-metric space**) The b-metric space is complete if every Cauchy sequence convergent.*

Remark 1.2.2. *For all $m, n \in \mathbb{N}$ let $m, n \longrightarrow \infty$ the inequality*

$$d_s(x_n, x_m) \leq s[d_s(x_n, x) + d_s(x, x_m)]$$

shows that every convergent sequance is cauchy.

Definition 1.2.5. *[34](**Fixed point**)*
Let X be a nonempty set and $T : X \longrightarrow X$ a selfmap. We say that $x \in X$ is a fixed point of T if $T(x) = x$ and denote by FT or $Fix(T)$ the set of all fixed points of T.

Definition 1.2.6. *[34](**The Picard iteration**)*
Let X be any set and $T : X \longrightarrow X$ a selfmap. For any given $x \in X$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we recall $T^n(x)$ the nth iterative of x under T. For any $x_0 \in X$, the sequence $\{x_n\}_{n \geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n(x_0), n = 1, 2 \cdots \tag{1.4}$$

Is called the sequence of successive approximations with the initial value x_0. It is also known as The Picard iteration starting at x_0.

Definition 1.2.7. *[37](Continuity)*
Let (X, d_b) and (X', d'_b) be two b-metric spaces with coefficient s and s', respectively. A mapping $T : X \longrightarrow X'$ is called continuous if each sequence $\{x_n\}$ in X, which converges to $x \in X$ with respect to d_b, then Tx_n converges to Tx with respect to d'_b.

Definition 1.2.8. *[28](Continuity of a b-metric)*
Let (X, d_s) be a b-metric space with constant $s \geq 1$. A b-metric is said to be

- *continuos if*

$$d_s(x_n, x) \longrightarrow 0, d_s(y_n, y) \longrightarrow 0 \Rightarrow d_s(x_n, y_n) \longrightarrow d_s(x, y)$$

- *separately continuous if the function $d_s(x, \cdot)$ is continuous on X for every $x \in X$, i.e.,*

$$d_s(y_n, y) \longrightarrow 0 \Rightarrow d_s(x, y_n) \longrightarrow d_s(x, y),$$

for all $(x_n), (y_n)$ in X and all $x, y \in X$.

Lemma 1.2.1. *Let (X, d_s) be a b-metric space, If d is continuos, then every convergent sequence has a unique limit.*

Proof. Let $\{x_n\}$ a convergent sequence $x_n \longrightarrow x$. Assume that there exist an other limit x^* such that $x_n \longrightarrow x^*$.
By continuity of d_s we have

$$d_s(x_n, x) \longrightarrow 0, d_s(x_n, x^*) \longrightarrow 0 \Rightarrow d_s(x_n, x_n) = 0 \longrightarrow d_s(x, x^*).$$

So

$$d_s(x, x^*) = 0 \Rightarrow x = x^*.$$

□

Definition 1.2.9. *[29](Topological notions)*
If (X, d_s) is a b-metric space then a subset $Y \subset X$ is called

(i) Compact if for every sequence of elements of Y there exists a subsequence that converges to an element of Y.

(ii) Closed if for each sequence $\{x_n\}$ in Y which converges to an element x, we have $x \in Y$.

1.3 Fixed point theorems

Theorem 1.1. ***(Banach contraction principle)***
Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a contraction on X. Then T has a unique fixed point $x \in X$ (such that $T(x) = x$).

Proof. Let us choose any $x_0 \in X$, and define the sequence (x_n), where

$$x_{n+1} = T(x_n), n = 1, 2, \cdots \tag{1.5}$$

Our proof strategy will be to show

1. This sequence is Cauchy.
2. This limit is a fixed point of T.
3. The fixed point is unique.

step 1 By using (1.5) and the fact that T is a contraction we have

$$\begin{aligned} d(x_{m+1}, x_m) &= d(T(x_m), T(x_{m-1})) \\ &\leq K d(x_m, x_{m-1}) \\ &\leq K d(T(x_{m-1}), T(x_{m-2})) \\ &\leq K^2 d(x_{m-1}, x_{m-2}) \\ &\leq \vdots \\ &\leq K^m d(x_1, x_0). \end{aligned}$$

Hence by the triangle inequality we get (for $n \geq m$) that

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq (K^m + K^{m+1} + \cdots + K^{n-1}) d(x_1, x_0) \\ &\leq K^m \frac{1 - K^{n-m}}{1 - K} d(x_1, x_0). \end{aligned}$$

where in the last equality we have used the summation formula for a geometric series. Since $0 < K < 1$, we have $1 - K^{n-m} < 1$, and consequently

$$d(x_m, x_n) \leq \frac{K^m}{1 - K} d(x_1, x_0)$$

Since $0 < K < 1$ and $d(x_0, x_1)$ are fixed, it is clear that we can make $d(x_m, x_n)$ as small as we please by choosing m sufficiently large (and $n > m$). This proves that (x_n) is Cauchy. Finally, since (X, d) is complete, there exists an $x \in X$ such that $x_n \longrightarrow x$.

step 2 To show that x is a fixed point, we consider the distance $d(x, T(x))$. From the triangle inequality we get

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_m) + d(x_m, T(x)) \\ &\leq d(x, x_m) + d(T(x_{m-1}), T(x)) \\ &\leq d(x, x_m) + Kd(x_{m-1}, x) \end{aligned}$$

and since $x_n \longrightarrow x$ it is clear that we can make this distance as small as we please by choosing m sufficiently large. We conclude that

$$d(x, T(x)) = 0 \Rightarrow T(x) = x.$$

so $x \in X$ is a fixed point of T.

step 3 Suppose there are two fixed points $x = T(x)$ and $x^* = T(x^*)$. Since T is a contraction with constant $K \in [0, 1)$ then, we obtain

$$d(x, x^*) = d(T(x), T(x^*)) \leq Kd(x, x^*)$$

which implies $d(x, x^*) = 0$ since $0 < K < 1$. Hence $x = x^*$, and the fixed point x of T is unique.

□

Lemma 1.3.1. *[30] Let (X, d_s) be a b-metric space and (x_n) a sequence in X such that*

$$d_s(x_{n+1}, x_{n+2}) \leq qd_s(x_n, x_{n+1}), n = 0, 1, \cdots$$

where $0 \leq q < 1$. Then the sequence (x_n) is Cauchy sequence in X provided that $sq < 1$.

Proof. For any n,

$$\begin{aligned} d_s(x_{n+1}, x_{n+2}) &\leq q d_s(x_n, x_{n+1}) \\ &\leq q^2 d_s(x_{n-1}, x_n) \leq \cdots \leq q^{n+1} d_s(x_0, x_1). \end{aligned}$$

For $n < m$, by the triangle inequality,

$$\begin{aligned} d_s(x_n, x_m) &\leq s d_s(x_n, x_{n+1}) + s^2 d_s(x_{n+1}, x_{n+2}) + \cdots \\ &\quad + s^{m-n-1}[d_s(x_{m-2}, x_{m-1}) + d_s(x_{m-1}, x_m)] \\ &\leq s q^n (1 + sq + s^2 q^2 + \cdots) d_s(x_0, x_1) \\ &\leq \frac{sq^n}{(1 - sq)} d_s(x_0, x_1) \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \end{aligned}$$

and (x_n) is Cauchy. □

In the following we recollect the extension of Banach contraction principle in case of b-metric spaces.

Theorem 1.2. *Let (X, d_s) be a complete b-metric space with constant s such that b-metric is a continuous functional. Let $T : X \longrightarrow X$ be a contraction having contraction constant $k \in [0, 1)$ such that $ks < 1$. Then T has a unique fixed point.*

Proof. Let $x_0 \in X$ and $\{x_n\}_{n \geq 1}$ be a sequence in X defined as following

$$x_n = T x_{n-1} = T^n x_0, \quad n = 1, 2, \cdots$$

step 1 Let's prove that x_n is a Cauchy sequence
Since T is a contraction with constant $k \in [0, 1)$ then, we obtain

$$\begin{aligned} d_s(T x_n, T x_{n+1}) &\leq k d_s(x_n, x_{n+1}) \quad n = 0, 1, \cdots \\ d_s(x_{n+1}, x_{n+2}) &\leq k d_s(x_n, x_{n+1}) \end{aligned}$$

then (x_n) is a Cauchy sequence by Lemma 1.3.1.

step 2 Now, we show that x^* is a fixed point of T.
Indeed, in view of completeness of X; we consider that $\{x_n\}_{n \geq 1}$ convergent to $x^* \in X$.

$$T x^* = \lim_{n \to \infty} T x_n = \lim_{n \to \infty} x_{n+1} = x^*$$

Therefore, x^* is a fixed point of T.

step 3 Finally we have to show that the fixed point is unique. Assume that x' is an other fixed point of T. Then, $Tx' = x'$.

$$d_s(x^*, x') = d_s(Tx^*, Tx') \leq kd_s(x^*, x')$$

So the fixed point is unique. This completes the proof.

□

If the metric space (X, d) is complete then the mapping satisfying (1.1) has a unique fixed point. Inequality (1.1) implies continuity of T. A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity. Kannan in [27] established the following result in which the above question has been answered in the affirmative. If $T : X \longrightarrow X$ where (X, d) is a complete metric space, satisfies the inequality

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] \tag{1.6}$$

where $a \in \left[0, \frac{1}{2}\right)$ and, $x, y \in X$ then T has a unique fixed point. The mappings satisfying (1.6) are called Kannan type mappings.

Theorem 1.3. *Let (X, d_s) be a complete b-metric space with constant $s \geq 1$ and define the sequence $\{x_n\}_{n=1}^{\infty} \subset X$ by the recursion (1.4).*
Let $T : X \longrightarrow X$ be a mapping for which there exists $\mu \in \left[0, \frac{1}{2}\right)$ such that

$$d_s(Tx, Ty) \leq \mu[d_s(x, Tx) + d_s(y, Ty)] \tag{1.7}$$

for all, $x, y \in X$ and $(s + 1)\mu < 1$. Then, there exists $x^* \in X$ such that $x_n \longrightarrow x^*$ and x^* is unique fixed point of T.

Proof. Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined as

$$x_n = Tx_{n-1} = T^n x_0, n = 1, 2 \cdots .$$

By using (1.7) and (1.4) we obtain that

$$\begin{aligned} d_s(x_n, x_{n+1}) &= d_s(Tx_{n-1}, Tx_n) \\ &\leq \mu[d_s(x_{n-1}, x_n) + d_s(x_n, x_{n+1})] \end{aligned}$$

and we obtain

$$d_s(x_n, x_{n+1}) \leq \frac{\mu}{1-\mu} d_s(x_{n-1}, x_n)$$

Similarly, we have

$$d_s(x_n, x_{n+1}) \leq \left(\frac{\mu}{1-\mu}\right)^n d_s(x_0, x_1) \tag{1.8}$$

Note that $\mu \in \left[0, \frac{1}{2}\right)$ then $\frac{\mu}{1-\mu} \in [0, 1)$. Thus, T is a contraction mapping. We deduce, in similar manner to that in the proof of Theorem 1.2 that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and hence, a convergent sequence, too. We consider that $\{x_n\}_{n=1}^{\infty}$ convergent to $x^* \in X$ then we have

$$\begin{aligned} d_s(x^*, Tx^*) &\leq s[d_s(x^*, x_n) + d_s(x_n, Tx^*)] \\ &\leq sd_s(x^*, x_n) + s\mu[d_s(x_{n-1}, x_n) + d_s(x_*, Tx^*)] \end{aligned}$$

and we arrive at

$$d_s(x^*, Tx^*) \leq \frac{s}{1-s\mu} d_s(x^*, x_n) + \frac{s\mu}{1-s\mu} d_s(x_{n-1}, x_n)$$

Also, thanks to (1.8), we obtain that

$$d_s(x^*, Tx^*) \leq \frac{s}{1-s\mu} d_s(x^*, x_n) + \frac{s\mu}{1-s\mu} \left(\frac{\mu}{1-\mu}\right)^n d_s(x_0, x_1)$$

Letting $n \longrightarrow \infty$ $\lim_{n \longrightarrow \infty} d_s(x^*, Tx^*) = 0$ Therefore, $x^* = Tx^*$ and implies that x^* is a fixed point of T. It is easy to see the fixed point is unique. □

The following theorem is given by Reich [32]:

Theorem 1.4. *Let X be a complete metric space with metric d and let $T : X \longrightarrow X$ be a function with the following property*

$$d(T(x), T(y)) \leq ad(x, T(x)) + bd(y, T(y)) + cd(x, y)$$

for all $x, y \in X$ where a, b, c are non-negative and satisfy $a+b+c < 1$. Then T has a unique fixed point.

Theorem 1.5. *Let M be a complete b-metric space with metric d_s and let $T : M \longrightarrow M$ be a function with the following*

$$d_s(T(x), T(y)) \leq ad_s(x, T(x)) + bd_s(y, T(y)) + cd_s(x, y)$$

for all $x, y \in M$ where a, b, c are non-negative real numbers and satisfy $a + s(b + c) < 1$ for $s \geq 1$ then T has a unique fixed point.

Proof. Let $x_0 \in M$ and $\{x_n\}$ be a sequence in M, such that

$$x_n = Tx_{n-1} = T^n x_0$$

step 1 First, we prove that x_n is a Cauchy sequence

$$\begin{aligned} d_s(x_{n+1}, x_n) &= d_s(Tx_n, Tx_{n-1}) \\ &\leq ad_s(x_n, Tx_n) + bd_s(x_{n-1}, Tx_{n-1}) \\ &+ cd_s(x_n, x_{n-1}) \\ &\leq ad_s(x_n, x_{n+1}) + bd_s(x_{n-1}, x_n) + cd_s(x_n, x_{n-1}) \\ \Rightarrow (1 - a)d_s(x_{n+1}, x_n) &\leq (b + c)d_s(x_n, x_{n-1}) \\ d_s(x_{n+1}, x_n) &\leq \frac{b + c}{1 - a} d_s(x_n, x_{n-1}) = pd_s(x_n, x_{n-1}) \end{aligned}$$

continuing this process we can easily say that

$$d_s(x_{n+1}, x_n) \leq p^n d_s(x_0, x_1)$$

Using Lemma 1.3.1 we get $\{x_n\}$ is a Cauchy sequence in M since M is complete .we consider that $\{x_n\}$ converge to x^*.

step 2 Now we show that x^* is a fixed point of T.
we have

$$\begin{aligned} d_s(x^*, Tx^*) &\leq s[d_s(x^*, x_n) + d_s(x_n, Tx^*)] \\ &\leq s[d_s(x^*, x_n) + d_s(Tx_{n-1}, Tx^*)] \\ &\leq s[d_s(x_n, x^*) + ad_s(x^*, Tx^*) + bd_s(Tx_{n-1}, x_{n-1}) \\ &+ cd_s(x_{n-1}, x^*)] \\ \Rightarrow (1 - as)d_s(x^*, Tx^*) &\leq s[d_s(x^*, x_n) + bd_s(x_{n-1}, x_n) + cd_s(x_{n-1}, x^*)] \\ d_s(x^*, Tx^*) &\leq \frac{s}{1 - as}[d_s(x^*, x_n) + bd_s(x_{n-1}, x_n) \\ &+ cd_s(x_{n-1}, x^*)] \end{aligned}$$

Taking lim $n \longrightarrow \infty$, we get

$$\lim_{n \longrightarrow \infty} d_s(x^*, Tx^*) = 0 \Rightarrow x^* = Tx^*$$

x^* is the fixed point of T.

step 3 Now, for the uniqueness of fixed point. Let x and y be two fixed points of T

$$\begin{aligned} d_s(x, y) &= d_s(Tx, Ty) \\ &\leq ad_s(x, T(x)) + bd_s(y, T(y)) + cd_s(x, y) \\ \Rightarrow d_s(x, y) &\leq cd_s(x, y) \end{aligned}$$

which is a contradiction. The proof is complete.

$\square$

Chapter 2

Extended b-metric space.

In this chapter, we will discuss the definition of the extended b-metric space, some examples, the difference between it and the b-metric space, and finally some fixed point theorems.

2.1 Definitions

Definition 2.1.1. *[33]**(Extended b-metric space)***
Let X be a non empty set and $\theta : X \times X \longrightarrow [1, \infty)$. A function $d_\theta : X \times X \longrightarrow [0, \infty)$ is called an extended b-metric if for all $x, y, z \in X$ it satisfies:

$(d_\theta 1)$ $d_\theta(x, y) = 0$ *iff* $x = y$;

$(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$;

$(d_\theta 3)$ $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The pair (X, d_θ) is called an extended b-metric space.

Remark 2.1.1. *[33] If $\theta(x, y) = s$ for $s \geq 1$ then we obtain the definition of a b-metric space.*

Example 2.1.1. *[33] Let $X = \{1, 2, 3\}$. Define $\theta : X \times X \longrightarrow \mathbb{R}^+$and $d_\theta : X \times X \longrightarrow \mathbb{R}^+$ as:*

$$\theta(x, y) = 1 + x + y$$

$$d_\theta(1,1) = d_\theta(2,2) = d_\theta(3,3) = 0$$

$$d_\theta(1,2) = d_\theta(2,1) = 80, d_\theta(1,3) = d_\theta(3,1) = 1000, d_\theta(2,3) = d_\theta(3,2) = 600$$

Indeed, $(d_\theta 1)$ and $(d_\theta 2)$ trivially hold. For $(d_\theta 3)$ we have:

$$d_\theta(1,2) = 80, \theta(1,2)[d_\theta(1,3) + d_\theta(3,2)] = 4(1000 + 600) = 6400$$

$$d_\theta(1,3) = 1000, \theta(1,3)[d_\theta(1,2) + d_\theta(2,3)] = 5(80 + 600) = 3400$$

Similar calculations hold for $d_\theta(2,3)$. Hence for all $x, y, z \in X$

$$d_\theta(x,z) \leq \theta(x,z)[d_\theta(x,y) + d_\theta(y,z)]$$

Hence (X, d_θ) is an extended b-metric space.

Example 2.1.2. *Let $X = [0, +\infty)$. Define two mappings $\theta : X \times X \longrightarrow [1, +\infty)$ and $d_\theta : X \times X \longrightarrow [0, +\infty)$ as follows:*

$$\theta(x,y) = 1 + x + y,$$

for all $x, y \in X$, and

$$d_\theta(x,y) = \begin{cases} x + y, & x \neq y, \\ 0, & x = y. \end{cases} \tag{2.1}$$

Then, (X, d_θ) is an extended b-metric space.

Indeed, $(d_\theta 1)$ and $(d_\theta 2)$ in Definition 2.1.1 are clear. Let $x, y, z \in X$. We prove that $(d_\theta 3)$ in Definition 2.1.1 is satisfied.

(i) If $x = y$, then $(d_\theta 3)$ is clear.

(ii) If $x \neq y$, $x = z$, then

$$\begin{aligned} \theta(x,y)[d_\theta(x,z) + d_\theta(z,y)] &= (1 + x + y)[0 + (z + y)] \\ &= (1 + x + y)(x + y) \\ &\geq x + y = d_\theta(x,y). \end{aligned}$$

(iii) If $x \neq y$, $y = z$, then

$$\begin{aligned}\theta(x,y)[d_\theta(x,z)+d_\theta(z,y)] &= (1+x+y)[(x+z)+0]\\ &= (1+x+y)(x+y)\\ &\geq x+y = d_\theta(x,y).\end{aligned}$$

(iv) If $x \neq y$, $y \neq z$, $x \neq z$, then

$$\begin{aligned}\theta(x,y)[d_\theta(x,z)+d_\theta(z,y)] &= (1+x+y)[(x+z)+(z+y)]\\ &\geq x+2z+y\\ &\geq x+y = d_\theta(x,y).\end{aligned}$$

Consider the above cases, it follows that $(d_\theta 3)$ holds. Hence, the claim holds.

Example 2.1.3. *[33] Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real valued functions define on $[a,b]$. Note that X is complete extended b-metric space by considering*

$$d_\theta(x,y) = \sup_{t\in[a,b]} |x(t)-y(t)|^2,$$

with

$$\theta(x,y) = |x(t)| + |y(t)| + 2,$$

where $\theta : X \times X \longrightarrow [1,\infty)$.

$(d_\theta 1)$ and $(d_\theta 2)$ in Definition 2.1.1 are clear. Let $f,h,g \in X$. We prove that $(d_\theta 3)$ in Definition 2.1.1 is satisfied.

(i) If $f = h$, then $(d_\theta 3)$ is clear.

(ii) If $f \neq h$, $f = g$, then

$$\begin{aligned}\theta(f,h)[d_\theta(f,g)+d_\theta(g,h)] &= (2+|f(t)|+|h(t)|)[\sup_{t\in[a,b]}|f(t)-g(t)|^2\\ &\quad+\sup_{t\in[a,b]}|g(t)-h(t)|^2]\\ &= (2+|f(t)|+|h(t)|)\sup_{t\in[a,b]}|f(t)-h(t)|^2\\ &\geq \sup_{t\in[a,b]}|f(t)-h(t)|^2 = d_\theta(f,h).\end{aligned}$$

(iii) If $f \neq h$, $h = g$, then

$$\begin{aligned}\theta(f,h)[d_\theta(f,g)+d_\theta(g,h)] &= (2+|f(t)|+|h(t)|)[\sup_{t\in[a,b]}|f(t)-g(t)|^2 \\ &+ \sup_{t\in[a,b]}|g(t)-h(t)|^2] \\ &= (2+|f(t)|+|h(t)|)_s up_{t\in[a,b]}|f(t)-h(t)|^2 \\ &\geq \sup_{t\in[a,b]}|f(t)-h(t)|^2 = d_\theta(f,h).\end{aligned}$$

(iv) If $f \neq h$, $h \neq g$, $f \neq g$, then for all $t \in [a,b]$ we have

$$|f(t)-h(t)| \leq |f(t)-g(t)|+|g(t)-h(t)|$$

Using Proposition 1.1.1

$$\begin{aligned}|f(t)-h(t)|^2 &\leq 2(|f(t)-g(t)|^2+|g(t)-h(t)|^2) \\ \sup_{t\in[a,b]}|f(t)-h(t)|^2 &\leq 2(\sup_{t\in[a,b]}|f(t)-g(t)|^2+\sup_{t\in[a,b]}|g(t)-h(t)|^2)\end{aligned}$$

Therefore

$$\frac{\sup_{t\in[a,b]}|f(t)-h(t)|^2}{\sup_{t\in[a,b]}|f(t)-g(t)|^2+\sup_{t\in[a,b]}|g(t)-h(t)|^2} \leq 2 \leq \theta(f,h)$$

Then

$$d_\theta(f,h) \leq \theta(f,h)[d_\theta(f,g)+d_\theta(g,h)]$$

So $(d_\theta 3)$ holds for all $f,g,h \in X$.

The concepts of convergence, Cauchy sequence and completeness can easily be extended to the case of an extended b-metric space.

Definition 2.1.2. *[33](**Convergent sequence**)*
Let (X,d_θ) be an extended b-metric space a sequence $\{x_n\}$ in X is called convergent sequence to $x \in X$, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d_\theta(x_n,x) < \varepsilon$, for all $n \geq N$. In this case, we write $\lim\limits_{n\longrightarrow\infty} x_n = x$.

Definition 2.1.3. *[33](**Cauchy sequence**)*
Let (X,d_θ) be an extended b-metric space a sequence $\{x_n\}$ in X is said to be Cauchy, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$d_\theta(x_m,x_n) < \varepsilon, \text{ for all } m,n \geq N.$$

Definition 2.1.4. *[33](**Complete extended b-metric space**)*
An extended b-metric space (X, d_θ) is complete if every Cauchy sequence in X is convergent.

Remark 2.1.2. *[33] Note that, in general a b-metric is not a continuous functional and thus so is an extended b-metric.*

Example 2.1.4. *[22] Let $X = \mathbb{N} \cup \infty$ and let $d_s : X \times X \longrightarrow \mathbb{R}$ be defined by:*

$$d_s(m, n) = \begin{cases} 0 & \text{if } m = n \\ |\frac{1}{m} - \frac{1}{n}| & \text{if } m, n \text{ are even or } mn = \infty \\ 5 & \text{if } m, n \text{ are odd and } m \neq n \\ 2 & \text{otherwise} \end{cases} \tag{2.2}$$

Then (X, d_s) is a b-metric with $s = 3$ but it is not continuous.

It is easy to see that for all $m, n, p \in X$, we have

$$d_s(m, p) \leq 3[d_s(m, n) + d_s(n, p)].$$

Let $x_n = 2n$ for each $n \in \mathbb{N}$ then $d_s(2n, \infty) = \dfrac{1}{2n} \longrightarrow 0 \quad as \quad n \longrightarrow \infty$. and let $y_n = 1$ then $d_s(y_n, 1) = 0 \longrightarrow 0 \quad as \quad n \longrightarrow \infty$.
But $d_s(x_n, y_n) = 2 \nrightarrow d_s(\infty, 1) \quad as\ n \longrightarrow \infty$.

Lemma 2.1.1. *[33] Let (X, d_θ) be an extended b-metric space. If d_θ is continuous, then every convergent sequence has a unique limit.*

Definition 2.1.5. *[19](**continuity**)*
Let (X, d_θ) and (X', d'_θ) be two extended b-metric spaces. A mapping $T : X \longrightarrow X'$ is called continuous if, for each sequence $\{x_n\}$ in X, which converges to $x \in X$ with respect to d_θ, $\{Tx_n\}$ converges to Tx with respect to d'_θ.

Definition 2.1.6. *[15](**The Orbit of a mapping**)*
Suppose that the pair (X, d_θ) is an extended b-metric space For a self-mapping $T : X \longrightarrow X$, for each $\xi \in X$ and $n \in \mathbb{N}$, we define

$$\mathcal{O}(\xi; n) = \{\xi, T\xi, \cdots, T^n\xi\} \text{ and } \mathcal{O}(\xi; \infty) = \{\xi, T\xi, \cdots, T^n\xi, \cdots\}$$

We say that the set $\mathcal{O}(\xi; \infty)$ is the orbit of T.

Definition 2.1.7. *[3] Suppose that the pair (X, d_θ) is an extended b-metric space. A self-mapping $T : X \longrightarrow X$ is called orbitally continuous if $\lim_{i\longrightarrow\infty} T^{n_i}(\xi) = \xi$ for some $\xi \in X$ implies that $\lim_{i\longrightarrow\infty} T(T^{n_i}(\xi)) = T\xi$. Moreover, if every Cauchy sequence of the form $\{T^{n_i}(\xi)\}_{i=1}^{\infty}, \xi \in X$ converges in (X, d_θ), then we say that an extended b-metric space (X, d_θ) is called T-orbitally complete.*

Remark 2.1.3. *[3] It is evident that the orbital continuity of T yields orbital continuity of any iterative power of T, that is, orbital continuity of T^m for any $m \in \mathbb{N}$.*

Definition 2.1.8. *[17](α-**admissible**)*
Let $\alpha : X \times X \longrightarrow [0, +\infty)$. A self-map $T : X \longrightarrow X$ is said to be α-admissible if

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1, \text{ for every } x, y \in X.$$

Definition 2.1.9. *[3](α-**orbital admissible**)*
Suppose that T is a self-mapping on a non-empty set X. Let

$$\alpha : X \times X \longrightarrow [0, \infty)$$

Then T is called an α-orbital admissible if, for all $\xi \in X$, we have

$$\alpha(\xi, T\xi) \geq 1 \Rightarrow \alpha(T\xi, T^2\xi) \geq 1 \tag{2.3}$$

Lemma 2.1.2. *Let $T : X \longrightarrow X$ be an α-orbital admissible mapping and $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then we have*

$$\alpha(x_{n-1}, x_n) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Proof. By assumption, there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. On account of the definition of $\{x_n\} \subset X$ and owing to the fact that T is α-orbital admissible, we derive

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, T^2x_0) = \alpha(x_1, x_2) \geq 1.$$

Recursively, we have

$$\alpha(x_{n-1}, x_n) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

□

Definition 2.1.10. *[17](Cyclic operator)*
Let (X, d_s) be a b-metric space. Let p be a positive integer; $p \geq 2$, $A_1, \cdots, A_p$ be nonempty and closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \longrightarrow Y$. Then, T is called a cyclic operator if

1. *$A_i, i \in \{1, 2, ...p\}$ are nonempty subsets;*

2. *$T(A_1) \subseteq A_2, \cdots, T(A_{p-1}) \subseteq A_p, T(A_p) \subseteq A_1$.*

A function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which is increasing and satisfies the property $\lim\limits_{n \longrightarrow +\infty} \varphi^n(t) = 0$ for all $t \geq 0$ is said to be a comparison function (see Matkowski [13]).

Example 2.1.5. *The mapping $\varphi(t) = \alpha t$, $t \in \mathbb{R}_+$, where $0 \leq \alpha < 1$, is a comparison function.*

The notion of b-comparison function was first given by Berinde in [35]. Regarding this he stated the following remark.

Remark 2.1.4. *[17] Let (X, d_s) be a b-metric with $s \geq 1$. A function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a b-comparison function if it is increasing and satisfies the property that $\sum_{n=0}^{+\infty} s^n \varphi^n(t)$ converges for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$.*

Definition 2.1.11. *[17](Extended comparison function)*
Let (X, d_θ) be an extended b-metric space. We say that a function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an extended comparison function if for all $t \in \mathbb{R}_+$;

(i) φ is monotone increasing;

(ii) There exists a cyclic operator $T : Y \subset X \longrightarrow X$, where $Y = \bigcup_{i=1}^{p} A_i$ such that for some $x_0 \in X$, $\mathcal{O}(x_0) \subset Y$, the sum

$$\sum_{n=0}^{+\infty} \varphi^n(t) \prod_{i=1}^{n} \theta(x_i, x_m)$$

converges for every $m \in \mathbb{N}$. We notice that $x_n = T^n x_0$ with $n = 1, 2, \cdots$

Lemma 2.1.3. *[17] Let (X, d_θ) be an extended b-metric space, $Y = \bigcup_{i=1}^{p} A_i \subset X$,*
$T : Y \subset X \longrightarrow X$ a cyclic operator, $x_0 \in X$ and

$$\lim_{n,m \longrightarrow +\infty} \theta(x_n, x_m) < \frac{1}{\lambda}$$

where $\lambda \in (0,1)$ and $x_n = T^n x_0$ for $n = 1, 2, \cdots$. Assume that Ψ is a comparison function. Then $\varphi(t) = \lambda\Psi$ is an extended b-comparison function for T at x_0.

Next we will give the definition of orbital lower semicontinuity with respect to a cyclic operator T.

Definition 2.1.12. *[17] Let X a nonempty set and $T : Y \longrightarrow Y$ be a cyclic operator where $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, and for some $x_0 \in X$ such that the orbit of x_0, $\mathcal{O}(x_0) \subset Y$. A function $S : X \longrightarrow \mathbb{R}$ is T-orbitally lower semicontinuous at $t \in X$ if $\{x_n\} \subset \mathcal{O}(x_0)$ and $x_n \longrightarrow t$ implies $S(t) \leq \liminf_{n \longrightarrow +\infty} S(x_n)$.*

2.2 Fixed point theorems

Lemma 2.2.1. *Let (X, d_θ) be an extended b-metric space. If there exists $q \in [0, 1)$ such that the sequence $\{x_n\}$, for an arbitrary $x_0 \in X$, satisfies*

$$\lim_{n,m \longrightarrow \infty} \theta(x_n, x_m) < \frac{1}{q}, \tag{2.5}$$

and

$$0 < d_\theta(x_n, x_{n+1}) \leq q d_\theta(x_{n-1}, x_n) \tag{2.6}$$

for any $n \in \mathbb{N}$, then the sequence $\{x_n\}$ is Cauchy in X.

Proof. Let $\{x_n\}_{n \in \mathbb{N}}$ be a given sequence. By employing Inequality (2.6) , recursively, we derive that

$$0 < d_\theta(x_n, x_{n+1}) \leq q^n d_\theta(x_0, x_1). \tag{2.7}$$

Since $q \in [0, 1)$, we find that

$$\lim_{n \longrightarrow \infty} d_\theta(x_n, x_{n+1}) = 0. \tag{2.8}$$

On the other hand, by $(d_\theta 3)$, together with triangular inequality, for $p \geq 1$, we derive that

$$\begin{aligned}
d_\theta(x_n, x_{n+p}) &\leq \theta(x_n, x_{n+p}) \cdot [d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_{n+p})] \\
&\leq \theta(x_n, x_{n+p}) d_\theta(x_n, x_{n+1}) + \theta(x_n, x_{n+p}) d_\theta(x_{n+1}, x_{n+p}) \\
&\leq \theta(x_n, x_{n+p}) q^n d_\theta(x_0, x_1) + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) \\
&\cdot [d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2}, x_{n+p})] \\
&\leq \theta(x_n, x_{n+p}) \cdot q^n d_\theta(x_0, x_1) + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) \\
&\cdot q^{n+1} d_\theta(x_0, x_1) + \cdots + \theta(x_n, x_{n+p}) \cdot \ldots \\
&\cdot \theta(x_{n+p-1}, x_{n+p}) \cdot q^{n+p-1} d_\theta(x_0, x_1) \\
&= d_\theta(x_0, x_1) \sum_{i=n}^{n+p-1} q^i \prod_{j=0}^{i-n} \theta(x_{n+j}, x_{n+p}).
\end{aligned}$$

Notice that the inequality above is dominated by

$$\sum_{i=n}^{n+p-1} q^i \prod_{j=0}^{i-n} \theta(x_{n+j}, x_{n+p}) \leq \sum_{i=1}^{n+p-1} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p})$$

On the other hand, by employing the ratio test, we conclude that the series $\sum_{i=1}^{\infty} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p})$ converges some $S \in (0, \infty)$.
Indeed, $\lim\limits_{i \longrightarrow \infty} \dfrac{a_{i+1}}{a_i} = \lim\limits_{i \longrightarrow \infty} q\theta(x_i, x_{i+p}) < 1$, which is why we obtain the desired result. Thus, we have

$$S = \sum_{i=1}^{\infty} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p}) \text{ with the partial sum } S_n = \sum_{i=1}^{n} q^i \prod_{j=1}^{i} \theta(x_j, x_{n+p}).$$

Consequently, we observe for $n \leq 1, p \leq 1$ that

$$d_\theta(x_n, x_{n+p}) \leq q^n d_\theta(x_0, x_1)[S_{n+p-1} - S_{n-1}]. \tag{2.9}$$

Letting $n \longrightarrow \infty$ in Equation (2.9), we conclude that the constructive sequence $\{x_n\}$ is Cauchy in the extended b-metric space (X, d_θ). □

Remark 2.2.1. *Note that if* $\lim\limits_{n,m \longrightarrow \infty} \theta(x_m, y_n)$ *does not exists then Lemma 2.2.1 is valid if, instead of condition* (2.5)*, we use the condition*

$$\limsup_{n,m \longrightarrow \infty} \theta(x_m, y_n) < \frac{1}{q}.$$

Note that in [6], the authors, instead of using condition (2.5) , used a weaker condition

$$\limsup_{n,m\longrightarrow\infty} \theta(x_n, x_m) < \infty$$

to prove the following theorem:

Theorem 2.1. *Let (X, d_θ) be a complete extended b-metric space such that d_θ is a continuous functional and the condition*

$$\lim_{n,m\longrightarrow\infty} \theta(x_m, x_n) < \infty \tag{2.10}$$

is fulfilled. Let $T : X \longrightarrow X$ be a mapping satisfying:

$$d_\theta(Tx, Ty) \leq \alpha d_\theta(x, y) + \beta d_\theta(x, Tx) + \gamma d_\theta(y, Ty), \text{ for all } x, y \in X, \tag{2.11}$$

where α, β, γ, non-negative real numbers with $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point $u^* \in X$.
Moreover, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in X converges to u^* such that $u_{n+1} = Tu_n$ for every $n \in \mathbb{N}$.

Proof. Let $u_0 \in X$ and $\{x_n\}$ be a sequence satisfying (2.10) such that

$$x_n = Tx_{n-1} = T^n x_0.$$

step 1 Prove that $\{x_n\}$ is a Cauchy sequence, From condition (2.11), we have

$$d_\theta(x_{n+1}, x_n) \leq \alpha d_\theta(x_n, x_{n-1}) + \beta d_\theta(x_n, x_{n+1}) + \gamma d_\theta(x_{n-1}, x_n).$$

Therefore,

$$d_\theta(x_{n+1}, x_n) \leq \frac{\alpha + \gamma}{1 - \beta} d_\theta(x_{n-1}, x_n)$$

Set $\lambda = \dfrac{\alpha + \gamma}{1 - \beta}$. Then, we have that $\lambda \in [0, 1)$. Hence, by Lemma 2.2.1, we obtain that $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d_θ), there exists $z \in X$ such that

$$\lim_{n\longrightarrow\infty} x_n = z$$

step 2 Now, we claim that z is the unique fixed point of T. For that,

$$\begin{aligned} d_\theta(Tz, x_{n+1}) &= d_\theta(Tz, Tx_n) \\ &\leq \alpha d_\theta(z, x_n) + \beta d_\theta(z, Tz) + \gamma d_\theta(x_n, x_{n+1}) \\ &\leq \alpha d_\theta(z, x_n) + \beta d_\theta(z, Tz) + \gamma \theta(x_n, x_{n+1}) \\ &\cdot [d_\theta(x_n, z) + d_\theta(z, x_{n+1})]. \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, we deduce

$$d_\theta(Tz, z) \leq \beta d_\theta(Tz, z)$$

which implies that $d_\theta(Tz, z) = 0$ and so $z = Tz$. Moreover the uniqueness can easily be obtained by using inequality (2.11).

□

If we take $\beta = \gamma = 0$ in (2.11), from Theorem 2.1, we obtain the following result, wich is an analogue of Banach contraction principle in the setting of extended b-metric space.

Corollary 2.2.1. *Let (X, d_θ) be a complete extended b-metric space such that d_θ is a continuous functional and the condition (2.10) is fulfilled. Let $T : X \longrightarrow X$ be a mapping satisfying:*

$$d_\theta(Tx, Ty) \leq \alpha d_\theta(x, y), \ for\ all\ x, y \in X,$$

where $\alpha \in [0, 1)$. Then T has a unique fixed point.
Moreover, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to x^* such that $x_{n+1} = Tx_n$ for every $n \in \mathbb{N}$.

If we take $\alpha = 0$ in (2.11), by Theorem 2.1, we obtain the following variant of Kannan theorem in extended b-metric spaces.

Corollary 2.2.2. *Let (X, d_θ) be a complete extended b-metric space such that d_θ is a continuous functional and the condition (2.10) is fulfilled. Let $T : X \longrightarrow X$ be a mapping satisfying:*

$$d_\theta(Tx, Ty) \leq \beta d_\theta(x, Tx) + \gamma d_\theta(y, Ty), \ for\ all\ x, y \in X,$$

where β, γ non-negative real numbers with $\beta + \gamma < 1$. Then T has a unique fixed point $x^* \in X$.
Moreover, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to x^* such that $x_{n+1} = Tx_n$ for every $n \in \mathbb{N}$.

Theorem 2.2. *Suppose that T is an orbitally continuous self-mapping on the T-orbitally complete extended b-metric space (X, d_θ). Assume that there exists $k \in [0, 1)$ and $a \geq 1$ such that*

$$\alpha(x, y)\mathcal{K}(x, y) - a\,\mathcal{N}(x, y) \leq kd_\theta(x, y) \tag{2.12}$$

With

$$\mathcal{K}(x, y) = \min\{d_\theta(Tx, Ty), d_\theta(x, Tx), d_\theta(y, Ty)\};$$
$$\mathcal{N}(x, y) = \min\{d_\theta(x, Ty), d_\theta(Tx, y)\};$$

for all $x, y \in X$. Furthermore, we presume that

(i) T is α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) $\lim\limits_{n,m\longrightarrow\infty} \theta(x_n, x_m) < \dfrac{1}{k}$.

Then, for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. By assumption (ii), there exists a point $x_0 \in X$ such that

$$\alpha(x_0, Tx_0) \geq 1.$$

We construct the sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n \ \forall n \in \mathbb{N}.$$

If $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}$, then $x^* = x_{n_0}$ forms a fixed point for T that the proof finishes. Hence, from now on, we assume that

$$x_n \neq x_{n+1} \text{ for all } n \in \mathbb{N}.$$

On account of the assumptions (i) and (ii), together with Lemma 2.1.2, Inequality (2.4) is yielded,that is,

$$\alpha(x_{n-1}, x_n) \geq 1 \text{ for all } n \in \mathbb{N}.$$

By replacing $x = x_{n-1}$ and $y = x_n$ in Inequality (2.12) and taking Equation (2.4) into account, we find that

$$\begin{aligned}\mathcal{K}(x_{n-1}, x_n) - a\,\mathcal{N}(x_{n-1}, x_n) &\leq \alpha(x_{n-1}, x_n)\mathcal{K}(x_{n-1}, x_n) - a\,\mathcal{N}(x_{n-1}, x_n)\\ &\leq kd_\theta(x_{n-1}, x_n)\end{aligned}$$

or,

$$\min\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n)\} \leq kd_\theta(x_{n-1}, x_n).$$

Since $k \in [0, 1)$, the case $d_\theta(x_{n-1}, x_n) \leq kd_\theta(x_{n-1}, x_n)$ is impossible. Thus, we conclude that

$$d_\theta(x_n, x_{n+1}) \leq kd_\theta(x_{n-1}, x_n).$$

On account of Lemma 2.2.1, we find that the sequence $\{x_n\}$ is a Cauchy sequence. By completeness of (X, d_θ), the sequence x_n converges to some point $u \in X$ as $n \longrightarrow \infty$. Owing to the construction $x_n = T^n x_0$ and the fact that (X, d_θ) is T-orbitally complete, there is $u \in X$ such that $x_n \longrightarrow u$. Since T is orbitally continuous, we deduce that $x_n \longrightarrow Tu$. Accordingly, we conclude that $u = Tu$. □

Example 2.2.1. *Let $X = \{1, 2, 3, 4\}$ be endowed with extended b-metric $d_\theta : X \times X \longrightarrow [0, \infty)$, defined by*

$$d_\theta(x, y) = (x - y)^2,$$

where $\theta : X \times X \longrightarrow [1, \infty)$,

$$\theta(x, y) = x + y + 1.$$

Let $k = \frac{1}{4}$, $a = 4$ and $T : X \longrightarrow X$ such that

$$T1 = T3 = 1, T2 = 4, T4 = 3.$$

Define also $\alpha : X \times X \longrightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 0 & if,\ (x, y) \in \{(3, 4), (4, 3)\} \\ 1 & otherwise. \end{cases} \tag{2.13}$$

Let us first notice that for any $x \in \{1,2,3,4\}$*, the sequence* $\{T^n x\}$ *tends to 1 when* $n \longrightarrow \infty$*. For this reason, we can conclude that the mapping* T *is orbitally continuous and* $\lim\limits_{n,m\longrightarrow\infty} \theta(T^n x, T^m x) = 3 < 4 = \frac{1}{k}$*, so (iii) is satisfied. It can also be easily verified that* T *is orbital admissible.*
If $x = 1$ *or* $y = 1$*, then* $d(1, T1) = 0$ *so Inequality* (2.12) *holds. We have to consider the following cases.*

case 1 *For* $x = 2$ *and* $y = 3$*, we have*
$d_\theta(2,3) = 1,\ d_\theta(T2, T3) = 9, d_\theta(2, T2) = 4,\ d_\theta(3, T3) = 4,$
$d_\theta(2, T3) = 1,\ d_\theta(3, T2) = 1$
and Inequality (2.12) *yields*

$$0 = \min\{9,4,4\} - 4\min\{1,1\} \leq \frac{1}{4} = \frac{1}{4} d_\theta(2,3).$$

case 2 *For* $x = 2$ *and* $y = 4$*, we have*
$d_\theta(2,4) = 4, d_\theta(T2, T4) = 1, d_\theta(2, T2) = 4, d_\theta(4, T4) = 1,$
$d_\theta(2, T4) = 1, d_\theta(4, T2) = 0$
and

$$1 = \min\{1,4,1\} - 4\min\{1,0\} \leq 1 = \frac{1}{4} d_\theta(2,4).$$

case 3 *For* $x = 3$ *and* $y = 4$*, because* $\alpha(3,4) = 0$*, Inequality* (2.12) *holds.*

Therefore, all the conditions of Theorem 2.2 are satisfied and T *has a fixed point,* $x = 1$*.*

Theorem 2.3. *Let* (X, d_θ) *be a complete extended b-metric space with* d_θ *a continuous functional. Let* $Y = \bigcup_{i=1}^{p} A_i \subseteq X$*, where* p *is a positive integer, be the set of all nonempty closed subsets of* X *and suppose* $T : Y \longrightarrow Y$ *be a cyclic operator such that*

(i) $T(A_i) \subseteq A_{i+1}$*, for all* $i \in \{1, 2, ...p\}$*;*

(ii) $d_\theta(Tx, Ty) \leq \varphi(d_\theta(x, y))$ *where* φ *is a b-extended comparison function for all* $x, y \in X$*.*

Then $T^n x_0 \longrightarrow x^* \in \bigcap_{i=1}^{p} A_i$*, as* $n \longrightarrow \infty$*. Moreover,* x^* *is a unique fixed point of* T *if and only if* $S = d_\theta(x, Tx)$ *is* T*-orbitally lower semicontinuous at* x^**.*

Proof. Let $x_0 \in Y$. Then there exists $i \in \{1, 2, \cdots, p\}$ such that $x_0 \in A_i$. From hypothesis (i) we have $x_1 = Tx_0 \in A_{i+1}$. Thus, we define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$.

step 1 First we shall show that $\{x_n\}$ is a Cauchy sequence.
If $x_n = x_{n+1}$ then x_n is a fixed point of T. We suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. From hypothesis (ii) it follows

$$d_\theta(x_n, x_{n+1}) = d_\theta(Tx_{n-1}, Tx_n) \leq \varphi d_\theta(x_{n-1}, x_n).$$

Applying (ii) successively we get

$$d_\theta(x_n, x_{n+1}) \leq \varphi^n d_\theta(x_0, x_1). \tag{2.14}$$

Furthermore we assume that x_0 is a non periodic point of T. If $x_0 = x_n$ we have for any $n \geq 2$

$$d_\theta(x_0, x_1) = d_\theta(x_0, Tx_0) = d_\theta(x_n, Tx_n).$$

Then $d_\theta(x_0, x_1) = d_\theta(x_n, x_{n+1})$. Thus, $d_\theta(x_0, x_1) \leq \varphi^n d_\theta(x_0, x_1)$.
Since $\varphi(t) < t$ we get a contradiction.
Therefore $d_\theta(x_0, x_1) = 0$ i.e., $x_0 = x_1$. Then x_0 is a fixed point of T.
Thus, we assume that $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $m \neq n$.
For any m, n with $m > n$ using the triangular inequality, we get

$$\begin{aligned}
d_\theta(x_n, x_m) \leq{}& \theta(x_n, x_m)\varphi^n d_\theta(x_0, x_1) + \cdots \\
& + \theta(x_n, x_m)\theta(x_{n+1}, x_m) \cdots \theta(x_{m-1}, x_m)\varphi^{m-1} d_\theta(x_0, x_1) \\
& \leq \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_{n-1}, x_m)\theta(x_n, x_m)\varphi^n(d_\theta(x_0, x_1)) \\
& + \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_n, x_m)\theta(x_{n+1}, x_m)\varphi^{n+1}(d_\theta(x_0, x_1)) \\
& + \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_{n-1}, x_m)\theta(x_n, x_m)\theta(x_{n+1}, x_m) \\
& \cdots \theta(x_{m-1}, x_m)\varphi^{m-1}(d_\theta(x_0, x_1))].
\end{aligned}$$

The series $\sum_{n=1}^{+\infty} \varphi^n \prod_{r=1}^{n} \theta(x_r, x_m)$ converges for each $m \in \mathbb{N}$ by ratio test. Let

$$\mathcal{G} = \sum_{n=1}^{+\infty} \varphi^n \prod_{r=1}^{n} \theta(x_r, x_m), \ \mathcal{G}_n = \sum_{j=1}^{n} \varphi^j \prod_{r=1}^{j} \theta(x_r, x_m).$$

Thus for $m > n$, we have

$$d_\theta(x_n, x_m) \leq [\mathcal{G}_{m-1} - \mathcal{G}_n].$$

Letting $n \longrightarrow \infty$ we conclude that $\{x_n\}$ is a Cauchy sequence in the subspace Y.

step 2 Since X is complete, Y is complete too. Therefore there exists $x^* \in Y$ such that $d_\theta(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$. Then $x_n = T^n x_0 \longrightarrow x^*$. The sequence $\{x_n\}$ has an infinite number of terms in each A_i for all $i \in \{1, 2, \cdots p\}$. Therefore $x^* \in \bigcap_{i=1}^p A_i$.

step 3 Now, we prove that x^* is a fixed point of T.
since $S = d_\theta(x, Tx)$ is T-orbitally lower semicontinuous at $x^* \in \bigcap_{i=1}^p A_i$ we obtain

$$\begin{aligned} d_\theta(x^*, Tx^*) &\leq \liminf_{n \longrightarrow +\infty} d_\theta(x_n, x_{n+1}) \\ &\leq \liminf_{n \longrightarrow \infty} \varphi^n(d_\theta(x_0, x_1)) = 0. \end{aligned}$$

Then $d_\theta(x^*, Tx^*) = 0$; results $x^* = Tx^*$.
Inversely, let $x^* = Tx^*$ and $x_n \in \mathcal{O} \subseteq Y$ with $x_n \longrightarrow x^*$. Then we have

$$S(x^*) = d_\theta(x^*, Tx^*) = 0 \leq \liminf_{n \longrightarrow \infty} d_\theta(T^n x_0, T^{n+1} x_0).$$

step 4 For the uniqueness of fixed point we suppose that there exists another fixed point $z = T(z) \in \bigcap_{i=1}^p A_i$.
By (ii) we get

$$d_\theta(x^*, z) = d_\theta(Tx^*, Tz) \leq \varphi(d_\theta(x^*, z)) < d_\theta(x^*, z).$$

Therefore $d_\theta(x^*, z) = 0$ which implies that $x^* = z$.

$\square$

Next let us give another general fixed point result.

Theorem 2.4. *Let (X, d_θ) be a complete extended b-metric space with d_θ a continuous functional. Let $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, where p is a positive integer, be the set of all nonempty closed subsets of X and suppose $T : Y \longrightarrow Y$ be a cyclic operator such that*

(i) $T(A_i) \subseteq A_{i+1}$, for all $i \in \{1, 2, \cdots p\}$;

(ii) $d_\theta(Tx, Ty) \leq \varphi(d_\theta(x, y))$ where φ is a b-extended comparison function for all $x, y \in X$ with $\varphi(0) = 0$.

Then T has a unique fixed point.

Proof. As in the proof of Theorem 2.3 we prove the existence of a Cauchy sequence $\{x_n\}$. Since Y is a complete subspace there exists $x^* \in Y$ such that $d_\theta(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow +\infty$.
Since, the sequence $\{x_n\}$ has an infinite number of terms in each A_i for all $i \in \{1, 2, \cdots p\}$ we have $x^* \in \bigcap_{i=1}^{p} A_i$. We must show that x^* is a fixed point for T. For any $n \in \mathbb{N}$ we have

$$\begin{aligned} d_\theta(Tx^*, x^*) &\leq \theta(Tx^*, x^*)[d_\theta(Tx^*, x_n) + d_\theta(x_n, x^*)] \\ &\leq \theta(Tx^*, x^*)[d_\theta(Tx^*, Tx_{n-1}) + d_\theta(x_n, x^*)] \\ &\leq \theta(Tx^*, x^*)[\varphi(d_\theta(x^*, x_{n-1})) + d_\theta(x_n, x^*)]. \end{aligned}$$

Since $\varphi(0) = 0$ for $n \longrightarrow +\infty$ we get that $d_\theta(Tx^*, x^*) = 0$. This implies $Tx^* = x^*$, i.e., x^* is a fixed point of T.
For uniqueness we follow the same steps as in Theorem 2.3.
We suppose that there exists another fixed point $z = T(z) \in \bigcap_{i=1}^{p} A_i$.
By (ii) we get

$$d_\theta(x^*, z) = d_\theta(Tx^*, Tz) \leq \varphi(d_\theta(x^*, z)) < d_\theta(x^*, z).$$

Therefore $d_\theta(x^*, z) = 0$ which implies that $x^* = z$. □

Example 2.2.2. *Let $X = \mathbb{R}^+$ endowed with $d_\theta : X \times X \longrightarrow \mathbb{R}^+$ defined by*

$$d_\theta = |x - y|^3,$$

and let$\theta : X \times X \longrightarrow [1, \infty)$ defined by

$$\theta(x, y) = x + y + 1.$$

It is easy to check that (X, d_θ) is a complete extended b-metric space.
Let $A_1 = [0, \frac{1}{2}], A_2 = [0, \frac{1}{3}], A_3 = [0, \frac{1}{5}]$ *be three subsets of* $X = \mathbb{R}^+$.

- *Define* $T : \bigcup_{i=1}^3 A_i \longrightarrow \bigcup_{i=1}^3 A_i$ *by* $Tx = \frac{x}{2}$.
  Obviously, $T(A_1) \subseteq A_2, T(A_2) \subseteq A_3, T(A_3) \subseteq A_1$. *Thus,* $\bigcup_{i=1}^3 A_i$ *is a cyclic representation with respect to* T.

- *Define* $\Psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *a comparison function by* $\Psi(t) = \frac{1}{2}t$.

- *We verify the contraction condition.*

$$d_\theta(Tx, Ty) = \left|\frac{x}{2} - \frac{y}{2}\right|^3 = \left|\frac{1}{2}(x-y)\right|^3 \leq \frac{1}{8}|x-y|^3 = \frac{1}{4}\Psi(d_\theta(x,y)).$$

  Taking into account for each $x \in \bigcup_{i=1}^3 A_i, T^n x = \frac{x}{2^n}$, *we obtain*

$$\begin{aligned}\lim_{n,m\longrightarrow\infty} \theta(x_n, x_m) &= \lim_{n,m\longrightarrow\infty} \theta(\frac{x}{2^n}, \frac{x}{2^m}) \\ &= \lim_{n,m\longrightarrow\infty} (\frac{x}{2^n} + \frac{x}{2^m} + 1) = 1 < 4.\end{aligned}$$

  Then $d_\theta(Tx, Ty) \leq \frac{1}{4}\Psi(d_\theta(x,y)) \leq \varphi(d_\theta(x,y)), where \varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, *defined by* $\varphi = \frac{1}{4}\Psi$, *is a b-extended comparison function.*

Therefore, all conditions of Theorem 2.3 (respectively Theorem 2.4) are satisfied. Then $0 \in \bigcap_{i=1}^3 A_i$ *is the unique fixed point of* T.

Theorem 2.5. *Let* (X, d_θ) *be a complete extended b-metric space such that* d_θ *be a continuous function and* $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be an extended b-comparison function. Let* $T : Y \longrightarrow Y$ *be a cyclic operator where* $Y = \bigcup_{i=1}^p A_i \subseteq X$, *with* p *integer* $i = \{1, 2, \cdots, p\}$, *be the set of nonempty closed subsets of* X, *such that*

(i) $\alpha(x,y)d_\theta(Tx,Ty) \leq \varphi(d_\theta(x,y))$, *for every* $x, y \in Y$;

(ii) T *is* α*-admissible;*

(iii) There exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$.

Then there exists a fixed point x^* of T.
Moreover, the fixed point x^* is unique, provides that

$$(H) \quad \forall x, y \in X, \exists z \in X \text{ such that } \alpha(x,z) \geq 1 \text{ and } \alpha(y,z) \geq 1.$$

Proof. Our proof strategy will be to show:

1. The existence of a fixed point.

2. The uniqueness of the fixed point.

step 1 First we prove the existence of a fixed point. By conditions (ii) and (iii) we obtain

$$\alpha(T^n x_0, T^{n+1} x_0) \geq 1.$$

For $n \in \mathbb{N}$ using (i) we get

$$\begin{aligned} d_\theta(T^n x_0, T^{n+1} x_0) &\leq \alpha(T^n x_0, T^{n+1} x_0) d_\theta(T^n x_0, T^{n+1} x_0) \\ &\leq \varphi(d_\theta(T^{n-1} x_0, T^n x_0)). \end{aligned}$$

Since φ is an increasing function, we have

$$d_\theta(T^n x_0, T^{n+1} x_0) \leq \varphi^n(d_\theta(x_0, T x_0))$$

This inequality is equivalent with (2.14) in Theorem 2.3. Thus all the hypotheses of Theorem 2.3 are satisfied. Thus there exists a fixed point.

step 2 The uniqueness of the fixed point.
In order to prove the uniqueness of the fixed point let us suppose that x^* and y^* are two fixed points of T.
From the hypothesis (H), there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1 \text{ and } \alpha(y^*, z) \geq 1.$$

Since T is α-admissible, we obtain

$$\alpha(x^*, T^n z) \geq 1 \text{ and } \alpha(y^*, T^n z) \geq 1. \tag{2.15}$$

Using (2.15) and the hypothesis (i), we have

$$\begin{aligned} d(x^*, T^n z) &= d_\theta(Tx^*, T(T^{n-1}z)) \\ &\leq \alpha(x^*, T^{n-1}z) d_\theta(Tx^*, T(T^{n-1}z)) \\ &\leq \varphi(d_\theta(x^*, T^{n-1}z)). \end{aligned}$$

This implies that

$$d(x^*, T^n z) \leq \varphi^n(d_\theta(x^*, z)), \text{ for all } n \in \mathbb{N}.$$

Then, letting $n \longrightarrow +\infty$, we have

$$T^n z \longrightarrow x^*. \tag{2.16}$$

Similarly, using (2.15) and hypothesis (i), we obtain

$$T^n z \longrightarrow y^* \textit{ as } n \longrightarrow +\infty. \tag{2.17}$$

Using (2.16) and (2.17), the uniqueness of the limit gives us $x^* = y^*$. The conclusion follows.

□

Theorem 2.6. *Let (X, d_θ) be a complete extended b-metric space with d_θ, a continuous functional. Let $\{A_i\}_{i=1}^p$, where p is a positive integer, be nonempty closed subsets of X, and suppose $T : \bigcup_{i=1}^p A_i \longrightarrow \bigcup_{i=1}^p A_i$, is a cyclic operator that satisfies the following conditions:*

(i) $T(A_i) \subseteq A_{i+1}$, *for all* $i \in \{1, 2, \cdots, p\}$;

(ii) $d(Tx, Ty) \leq \lambda d(x, y)$ *for all* $x \in A_i$, $y \in A_{i+1}$ *where* $\lambda \in [0, 1)$ *such that for each* $x \in X$,

$$\lim_{n,m \longrightarrow \infty} \theta(x_n, x_m) < \frac{1}{\lambda} \textit{ where } x_n = T^n(x), n = 1, 2, \cdots$$

Thus, T has a fixed point x^*. Moreover, for each $y \in X$, $T^n y \longrightarrow x^*$.

Proof. Let $x_0 \in \bigcup_{i=1}^p A_i$ if $i \in \{1, 2, \cdots, p\}$ exists such that $x_0 \in A_i$.
From hypothesis, (i) we have $x_1 = T(x_0) \in A_{i+1}$.
Thus, we define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$.

step 1 First we will show that $\{x_n\}$ is a Cauchy sequence.
If $x_n = x_{n+1}$, then x_n is a fixed point of T. We suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. From (ii), it follows that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda d(x_{n-1}, x_n).$$

If we repeat the process we obtain

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1). \tag{2.18}$$

Additionally, we assume that x_0 is a nonperiodic point of T. If $x_0 = x_n$ using (2.18), for any $n \geq 2$, we obtain

$$d(x_0, T(x_0)) = d(x_n, Tx_n).$$

Thus, $d(x_0, x_1) = d(x_n, x_{n+1})$ and $d(x_0, x_1) \leq \lambda^n d(x_0, x_1)$, a contradiction. Therefore, $d(x_0, x_1) = 0$, i.e., $x_0 = x_1$, and x_0 is a fixed point of T. Thus, we assume that $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $m \neq n$.
For any m, n with $m > n$ we obtain

$$\begin{aligned}
d(x_n, x_m) \leq{}& \theta(x_n, x_m)\lambda^n d_\theta(x_0, x_1) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\lambda^{n+1} d_\theta(x_0, x_1) \\
&+ \theta(x_n, x_m)\theta(x_{n+1}, x_m) \cdots \theta(x_{m-1}, x_m)\lambda^{m-1} d_\theta(x_0, x_1) \\
&\leq d_\theta(x_0, x_1)[\theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_{n-1}, x_m)\theta(x_n, x_m)\lambda^n + \\
&+ \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_n, x_m)\theta(x_{n+1}, x_m)\lambda^{n+1} + \cdots \\
&+ \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_{n-1}, x_m)\theta(x_n, x_m)\theta(x_{n+1}, x_m) \\
&\cdots \theta(x_{m-1}, x_m)\lambda^{m-1}].
\end{aligned}$$

Since $\lim\limits_{n,m \longrightarrow \infty} \theta(x_{n+1}, x_m)\lambda < 1$, The series

$$\sum_{n=1}^{\infty} \lambda^n \prod_{r=1}^{n} \theta(x_r, x_m)$$

converges by ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty} \lambda^n \prod_{r=1}^{n} \theta(x_r, x_m), \;\; S_n = \sum_{j=1}^{n} \lambda^j \prod_{r=1}^{j} \theta(x_r, x_m).$$

Thus, for $m > n$ we have

$$d_\theta(x_n, x_m) \leq d_\theta(x_0, x_1)[S_{m-1} - S_n].$$

Letting $n \longrightarrow \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence in $\bigcup_{i=1}^{p} A_i$, a subspace of the complete extended b-metric space X.
Therefore, there exists $x^* \in \bigcup_{i=1}^{p} A_i$ such that $d_\theta(x_n, x^*) \longrightarrow 0$, as $n \longrightarrow \infty$. Then, $\lim_{n \longrightarrow \infty} x_n = x^*$.
The sequence $\{x_n\}$ has an infinite number of terms in each A_i for all $i \in \{1, 2, \cdots, p\}$. Therefore, $x^* \in \bigcap_{i=1}^{p} A_i$.

step 2 We shall now show that x^* is a fixed point of T.
For any $n \in \mathbb{N}$ we have

$$\begin{aligned} d_\theta(Tx^*, x^*) &\leq \theta(Tx^*, x^*)[d_\theta(Tx^*, x_n) + d_\theta(x_n, x^*)] \\ &\leq \theta(Tx^*, x^*)[\lambda d_\theta(x^*, x_{n-1}) + d_\theta(x_n, x^*)]. \end{aligned}$$

We note that $d_\theta(Tx^*, x^*) \leq 0$ as $n \longrightarrow \infty$. Hence, $d_\theta(Tx^*, x^*) = 0$, which is equivalent to $x^* = Tx^*$. Thus, we proved that x^* is the fixed point of T.

step 3 Finally the uniqueness, let x' be another fixed point of T.
By hypothesis (ii), we obtain

$$d_\theta(x^*, x') = d_\theta(Tx^*, Tx') \leq \lambda d_\theta(x^*, x') < d_\theta(x^*, x'),$$

which is a contradiction.
Then, $d_\theta(x^*, x') = 0$ and $x^* = x'$. The fixed point is unique.

$\square$

In order to support our results, let us present the following example:

Example 2.2.3. *Let $X = \mathbb{R}^+$ endowed with $d_\theta : X \times X \longrightarrow \mathbb{R}^+$ defined by*

$$d_\theta = |x - y|^3,$$

and let $\theta : X \times X \longrightarrow [1, \infty)$ defined by

$$\theta(x, y) = x + y + 2.$$

It is easy to check that (X, d_θ) is a complete extended b-metric space.
Let $A_1 = [0,1], A_2 = \left[0, \frac{1}{2}\right], A_3 = \left[0, \frac{1}{3}\right]$ *be three subsets of* $X = \mathbb{R}^+$.
Define $T : \bigcup_{i=1}^{3} A_i \longrightarrow \bigcup_{i=1}^{3} A_i$ *by* $Tx = \frac{x}{2}$.
Obviously, $T(A_1) \subseteq A_2, T(A_2) \subseteq A_3, T(A_3) \subseteq A_1$. *Thus,* $\bigcup_{i=1}^{3} A_i$ *is a cyclic representation with respect to* T.
The contraction condition is also verified.

$$d_\theta(Tx, Ty) = \left|\frac{x}{2} - \frac{y}{2}\right|^3 = \left|\frac{1}{2}(x-y)\right|^3 \leq \frac{1}{8}|x-y|^3 = \frac{1}{8}d_\theta(x,y).$$

Taking into account for each $x \in \bigcup_{i=1}^{3} A_i, T^n x = \frac{x}{2^n}$, *we obtain*

$$\begin{aligned}\lim_{n,m\longrightarrow\infty} \theta(x_n, x_m) &= \lim_{n,m\longrightarrow\infty} \theta(\frac{x}{2^n}, \frac{x}{2^m}) \\ &= \lim_{n,m\longrightarrow\infty} (\frac{x}{2^n} + \frac{x}{2^m} + 2) = 2 < 8.\end{aligned}$$

Therefore, all conditions of Theorem 2.6 are satisfied, meaning that $0 \in \bigcup_{i=1}^{3} A_i$ *is the unique fixed point of* T.

Theorem 2.7. *Let* $T : \bigcup_{i=1}^{p} A_i \longrightarrow \bigcup_{i=1}^{p} A_i$, *be a cyclic operator defined as in Theorem 2.6. Then,* T *has the limit shadowing property, i.e., if a convergent sequence* $\{y_n\} \in \bigcup_{i=1}^{p} A_i$ *with* $d(y_{n+1}, Ty_n) \longrightarrow 0$, *as* $n \longrightarrow \infty$ *exists, then there exist* $x \in \bigcup_{i=1}^{p} A_i$ *such that* $d(y_n, T^n x) \longrightarrow 0$, *as* $n \longrightarrow \infty$.

Proof. As in the proof of Theorem 2.6, for any initial value $x \in \bigcup_{i=1}^{p} A_i$, $x^* \in \bigcup_{i=1}^{p} A_i$ is the unique fixed point of T. Thus, $d(y_n, x^*)$ and $d(y_{n+1}, x^*)$ are well defined.
Let $y \in X$ exist as the limit of the convergent sequence $\{y_n\} \in \bigcup_{i=1}^{p} A_i$.
We consider the following estimation:

$$\begin{aligned}d_\theta(y_{n+1}, x^*) &\leq \theta(y_{n+1}, x^*)[d_\theta(y_{n+1}, Ty_n) + d_\theta(Ty_n, x^*)] \\ &= \theta(y_{n+1}, x^*)[d_\theta(y_{n+1}, Ty_n) + d_\theta(Ty_n, Tx^*)] \\ &\leq \theta(y_{n+1}, x^*)[d_\theta(y_{n+1}, Ty_n) + \lambda(d_\theta(y_n, x^*)].\end{aligned}$$

Letting $n \longrightarrow \infty$, from the hypothesis, we have $d_\theta(y_{n+1}, Ty_n) \longrightarrow 0$. Thus, $d_\theta(y, x^*) \leq \lim_{n\longrightarrow\infty} \theta(y_{n+1}, x^*)\lambda d_\theta(y, x^*)$.

Since $\lim_{n \longrightarrow \infty} \theta(y_{n+1}, x^*)\lambda < 1$, this inequality is true only for the case of $d_\theta(y, x^*) = 0$. Thus, $y = x^*$ and we have $d_\theta(y_n, T^n x) \longrightarrow d(y, x^*) = 0$ as $n \longrightarrow \infty$.

$\square$

Chapter 3

Applications

In this chapter, we will discuss the application of some fixed point theorems to integral equations and fractional differential equations.

3.1 Applications to integral equations

in [17], the authors, establish the existence of a solution to the following integral equation

$$x(t) = p(t) + \int_0^t P(t,u)g(u,x(u))du, \ t \in [0,1], \tag{3.1}$$

where $g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $p : [0,1] \longrightarrow \mathbb{R}$ are two bounded continuous functions. $P : [0,1] \times [0,1] \longrightarrow [0,+\infty)$ is a function such that $P(t,\cdot) \in L^1([0,1])$ for all $t \in [0,1]$.
Consider the operator $T : Y \longrightarrow Y$, where $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, given by

$$T(x)(t) = p(t) + \int_0^t P(t,u)g(u,x(u))du. \tag{3.2}$$

Observe that each fixed point of T is a solution of integral Eq (3.1). Also, T is well defined since g and p are two bounded continuous functions.
Then let us give the following theorem on the existence of a fixed point for (3.2), which in turn reduces to the result for the existence of a solution to (3.1).

Theorem 3.1. *Let* $x = C([0,1],\mathbb{R})$ *the space of all continuous real valued functions defined on* $[0,1]$*. and* $Y = \bigcup_{i=1}^{3} A_i \subseteq X$*, where* A_1, A_2, A_3 *are non empty subsets of* X*. Let* $T : Y \longrightarrow Y$*, be a cyclic integral operator given by* (3.2)*. Suppose that the following conditions hold:*

(i) for $x, y \in Y$ *and for every* $u \in [0,1]$ *we have*

$$0 \leq g(u, x(u)) - g(u, y(u)) \leq \frac{1}{2}\sqrt{e^{|x(u)-y(u)|^2}}$$

(ii) for every $u \in [0,1]$ *we have*

$$\left|\left|\int_0^1 P(t,u)du\right|\right|_{+\infty} < 1$$

Then T *has a fixed point.*

Proof. We have $X = C([0,1],\mathbb{R})$. Then (X, d_θ) is a complete extended b-metric space with respect to

$$d_\theta(x,y) = ||x-y||_{+\infty} = \sup_{t\in[a,b]} |x(t)-y(t)|^2,$$

where $\theta : X \times X \longrightarrow [1, +\infty)$ is defined by

$$\theta(x,y) = |x(t)| + |y(t)| + 1.$$

step1 T is a cyclic operator on $\bigcup_{i=1}^{3} A_i$.

Let $A_1 = A_2 = A_3 = X = C([0,1],\mathbb{R})$ are non empty subsets of X. It is obvious $A_1 = A_2 = A_3$ are closed subsets of (X, d_θ). Clearly $T(A_1) \subset A_2, T(A_2) \subset A_3$ and $T(A_3) \subset A_1$. Then T is a cyclic operator on $Y = \bigcup_{i=1}^{3} A_i$.

step2 Let us prove the contraction condition

By condition (ii), we get for $x \in Y = \bigcup_{i=1}^{3} A_i$

$$\begin{aligned}
|T(x)(t) - T(y)(t)|^2 &= \left|\int_0^t P(t,u)[g(u,x(u)) - g(u,y(u))]du\right|^2 \\
&\leq \int_0^t |P(t,u)|^2 |g(u,x(u)) - g(u,y(u))|^2 du \\
&\leq \frac{1}{4}\int_0^t |P(t,u)|^2 e^{|x(u)-y(u)|^2} du \\
&\leq \frac{1}{4} e^{||x(u)-y(u)||_{+\infty}}.
\end{aligned}$$

Then we get

$$||Tx - Ty||_{+\infty} \leq \frac{1}{4} e^{||x(u) - y(u)||_{+\infty}}.$$

Hence

$$d_\theta(Tx, Ty) \leq \lambda \varphi(d_\theta(x, y))$$

where

$$\varphi(t) = \frac{1}{4} e^t \text{ is a comparison function.}$$

For $x \in Y$, $\lim\limits_{n,m \longrightarrow \infty} \theta(x_n, x_m) = 1 < 2$. Then, by Lemma 2.1.3, we get that $\lambda\varphi$ is an extended b-comparison function, with $\lambda = \dfrac{1}{4}$.

Thus all the conditions of Theorem 2.3 are satisfied. Then the cyclic integral operator T has a fixed point. □

As a result of the previous theorem, we have

Corollary 3.1.1. *Let $T : Y \longrightarrow Y$, where $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, be a cyclic integral operator given by*

$$T(x)(t) = p(t) + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} g(u, x(u)) du, \text{ with } t \in [0,1] \text{ and } \alpha \in (0,1)$$

where Γ is the Euler gamma function given by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

Suppose that for $x \in \bigcup_{i=1}^{3} A_i$ we have

$$0 \leq g(u, x(u)) - g(u, y(u)) \leq \frac{\Gamma(\alpha+1)}{4} \sqrt{e^{|x(u)-y(u)|^2}} \text{ for every } u \in [0,1].$$

Then T has a fixed point.

In Theorem 3.1 there is an error because the function $\dfrac{1}{4} e^t$ is not a comparison function, so we will take the function $\dfrac{1}{2} \ln(1+t)$ which fulfills the conditions of a comparison function and Theorem 3.1 becomes as follows:

Theorem 3.2. *Let $x = C([0,1], \mathbb{R})$ the space of all continuous real valued functions defined on $[0,1]$. and $Y = \bigcup_{i=1}^{3} A_i \subseteq X$, where A_1, A_2, A_3 are non empty subsets of X. Let$T : Y \longrightarrow Y$, be a cyclic integral operator given by (3.2). Suppose that the following conditions hold:*

(i) for $x, y \in Y$ and for every $u \in [0,1]$ we have

$$0 \leq g(u, x(u)) - g(u, y(u)) \leq \frac{1}{4}\sqrt{\ln(1 + |x(u) - y(u)|^2)}$$

(ii) for every $u \in [0,1]$ we have

$$\left|\left|\int_0^1 P(t,u)du\right|\right|_{+\infty} < 1$$

Then T has a fixed point.

Proof. We have $X = C([0,1], \mathbb{R})$. Then (X, d_θ) is a complete extended b-metric space with respect to

$$d_\theta(x,y) = ||x-y||_{+\infty} = \sup_{t\in[a,b]} |x(t) - y(t)|^2,$$

where $\theta : X \times X \longrightarrow [1, +\infty)$ is defined by

$$\theta(x,y) = |x(t)| + |y(t)| + 1.$$

step1 T is a cyclic operator on $\bigcup_{i=1}^{3} A_i$.

Let $A_1 = A_2 = A_3 = X = C([0,1], \mathbb{R})$ are non empty subsets of X. It is obvious $A_1 = A_2 = A_3$ are closed subsets of (X, d_θ). Clearly $T(A_1) \subset A_2, T(A_2) \subset A_3$ and $T(A_3) \subset A_1$. Then T is a cyclic operator on $Y = \bigcup_{i=1}^{3} A_i$.

step2 Let us prove the contraction condition

By condition (ii), we get for $x \in Y = \bigcup_{i=1}^{3} A_i$

$$\begin{aligned}
|T(x)(t) - T(y)(t)|^2 &= \left|\int_0^t P(t,u)[g(u,x(u)) - g(u,y(u))]du\right|^2 \\
&\leq \int_0^t |P(t,u)|^2 |g(u,x(u)) - g(u,y(u))|^2 du \\
&\leq \frac{1}{16}\int_0^t |P(t,u)|^2 \ln(1 + |x(u) - y(u)|^2) du \\
&\leq \frac{1}{16}\ln(1 + ||x(u) - y(u)||_{+\infty}).
\end{aligned}$$

Then we get

$$||Tx - Ty||_{+\infty} \leq \frac{1}{16} \ln(1 + ||x(u) - y(u)||_{+\infty}).$$

Hence

$$d_\theta(Tx, Ty) \leq \lambda\varphi(d_\theta(x, y))$$

where

$$\varphi(t) = \frac{1}{2} \ln(1 + t)$$

is a comparison function and.
For $x \in Y$, $\lim\limits_{n,m \longrightarrow \infty} \theta(x_n, x_m) = 1 < 2$. Then, by Lemma 2.1.3, we get that $\lambda\varphi$ is an extended b-comparison function, with $\lambda = \dfrac{1}{8}$.

Thus all the conditions of Theorem 2.3 are satisfied. Then the cyclic integral operator T has a fixed point. $\square$

Also, the Theorem 3.1.1 changes its form as follows:

Corollary 3.1.2. *Let $T : Y \longrightarrow Y$, where $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, be a cyclic integral operator given by*

$$T(x)(t) = p(t) + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} g(u, x(u))du, \text{ with } t \in [0, 1] \text{ and } \alpha \in (0, 1)$$

where Γ is the Euler gamma function given by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

Suppose that for $x \in \bigcup_{i=1}^{3} A_i$ we have

$$0 \leq g(u, x(u)) - g(u, y(u)) \leq \frac{\Gamma(\alpha+1)}{8} \sqrt{\ln(1 + |x(u) - y(u)|^2)} \text{ for every } u \in [0, 1].$$

Then T has a fixed point.

3.2 Applications to Nonlinear Ordinary Differential Equations

This section is devoted to the existence of a solution of a boundary value problem given by Nieto and Lopez in [14].

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(0) = u(a), \end{cases} \tag{3.3}$$

where $a > 0$ and $f : [0, a] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. A solution to (3.3) is the function $u \in C^1([0, a], \mathbb{R})$ satisfying (3.3), where $C^1([0, a], \mathbb{R})$ is the set of all continuous differentiable functions on $[0, a]$. We suggest that (3.3) has a lower solution if function $u \in C^1([0, a], \mathbb{R})$ exists, satisfying

$$\begin{cases} u'(t) \leq f(t, u(t)) \\ u(0) \leq u(a). \end{cases} \tag{3.4}$$

It is well known [8] that the existence of a lower solution A and an upper solution B with $A \leq B$ implies the existence of a solution of the boundary value problem between A and B.
In [14], we find the following results:

Theorem 3.3. *Let $a > 0$. Let $f : [0, a] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous mapping. Assume that $\alpha > 0, \beta > 0$ with $\beta < \alpha$ exist such that for any $x, y \in \mathbb{R}$ such that $x > y$,*

$$0 \leq f(t, x) + \alpha x - (f(t, y) + \alpha y) \leq \beta(x - y).$$

Thus, the existence of a lower solution of (3.3) provides the existence of a unique solution of (3.3).

Furthermore, let us provide a generalisation of Theorem 3.3 using cyclic operators for the case of extended b-metric spaces.

Theorem 3.4. *Let $a > 0$. Let $f : [0, a] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous mapping. Assume that $\alpha > 0, \beta > 0$ with $\beta < \alpha$ exist such that for any $x, y \in \mathbb{R}$,*

$$0 \leq f(t, x) + \alpha x - (f(t, y) + \alpha y) \leq \beta(x - y).$$

Thus, problem (3.3) has a unique solution.

Proof. We can rewrite problem (3.3) as follows:

$$\begin{cases} u'(t) + \alpha u(t) = f(t, u(t)) + \alpha u(t) \\ u(0) = u(a). \end{cases} \tag{3.5}$$

This problem is equivalent to the following integral equation:

$$u(t) = \int_0^a \mathcal{Q}(t,s)(f(s,u(s)) + \alpha u(s))ds,$$

where

$$\mathcal{Q}(t,s) = \begin{cases} \dfrac{e^{\alpha(a+s-t)}}{e^{\alpha a} - 1}, & 0 \leq s < t \leq a \\ \dfrac{e^{\alpha(s-t)}}{e^{\alpha a} - 1}, & 0 \leq t < s \leq a. \end{cases} \tag{3.6}$$

and $u \in C^1([0,1], \mathbb{R})$.

Let $X = C([0,a], \mathbb{R})$. Then, X is a complete extended b-metric space considering

$$d_\theta(x,y) = \sup_{t \in [a,b]} |x(t) - y(t)|^2,$$

with

$$\theta(x,y) = 2|x(t)| + |y(t)| + 1,$$

where $\theta : X \times X \longrightarrow [1, \infty)$.

Let $A_1 = A_2 = A_3 = X = C([0,a], \mathbb{R})$ three closed subsets of the space (X, d_θ).

Let us define the operator $T : \bigcup_{i=1}^3 A_i \longrightarrow \bigcup_{i=1}^3 A_i$ as follows:

$$Tu(t) = \int_0^a \mathcal{Q}(t,s)(f(s,u(s)) + \alpha u(s))ds$$

for $u \in C([0,a], \mathbb{R})$ and $t \in [0,a)$.

For $x, y \in C([0,a], \mathbb{R})$ and $t \in [0,a]$ we have

$$|f(t,x(t)) + \alpha x(t) - f(t,y(t)) - \alpha y(t)| \leq \sqrt{\beta}|x(t) - y(t)|.$$

Clearly, $T(A_1) \subseteq A_2, T(A_2) \subseteq A_3, T(A_3) \subseteq A_1$. Thus, T is a cyclic operator on $\bigcup_{i=1}^3 A_i$.

For any $x, y \in \bigcup_{i=1}^{3} A_i$ we have the following estimation:

$$\begin{aligned}|Tx(t) - Ty(t)|^2 &\le \int_0^a \mathcal{Q}(t,s)|f(s,x(s)) + \alpha x(s) - f(t,y(s)) - \alpha y(s)|^2 ds \\ &\le \int_0^a \mathcal{Q}(t,s)\beta|x(s) - y(s)|^2 ds \\ &\le \beta d_\theta(x,y) \sup_{0\le t\le a} \int_0^a \mathcal{Q}(t,s)ds \\ &= \frac{\beta}{\alpha} d_\theta(x,y).\end{aligned}$$

Thus, for $x, y \in C([0,a],\mathbb{R})$, we have

$$d_\theta(Tx, Ty) \le \frac{\beta}{\alpha} d_\theta(x,y).$$

Since $\lim_{n,m\longrightarrow\infty} \theta(x_n(t), x_m(t)) = 1 < \frac{\alpha}{\beta}$, we fulfilled all of the conditions of Theorem 2.6. Hence, we obtained the existence and uniqueness of fixed points of T. □

3.3 Applications to Nonlinear Fractional Differential Equations

In the last part of this chapter, M.Bota and L.Guran in [20] present an application of Theorem 2.6 for nonlinear fractional differential equations. The definition of the Caputo derivative of functional $g : [0,\infty) \longrightarrow \mathbb{R}$ of order $\beta > 0$ is given, where g is a continuous function as follows:

$$D^\beta(g(t)) := \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s)ds \ (n-1 < \beta < n,\ n = [\beta]+1),$$

where $[\beta]$ represents the integer part of the positive real number β, and Γ is a gamma function. Let us recall the Caputo type nonlinear fractional differential equation

$$D^\beta(x(t)) = f(t, x(t)) \tag{3.7}$$

with the integral boundary conditions

$$x(0) = 0, \quad x(1) = \int_0^\eta x(s)ds,$$

where $1 < \beta \le 2,\ 0 < \eta < 1, x \in C[0,1]$ and $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous given function (see [5]). Since f is continuous, Equation (3.7) is inverted as the following integral equation:

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds \\ & - \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^1 (1-s)^{\beta-1} f(s, x(s)) ds \\ & + \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} f(p, x(p)) dp \right) ds. \end{aligned}$$

In addition, we provide an existence theorem.

Theorem 3.5. *Taking into account the nonlinear fractional differential Equation (3.7), for every $x, y \in C[0,1]$ and $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ a given continuous mapping, we obtain*

$$|f(s, x(s)) - f(s, y(s))| \le \frac{\Gamma(\beta+1)}{\sqrt{50}} |x(s) - y(s)|, \qquad \textit{for all } s \in [0,1].$$

Thus, the Caputo type nonlinear fractional differential Equation (3.7) has a unique solution.
Moreover, for each $x_0 \in C[0,1]$, the sequence of the successive approximation $\{x_n\}$ defined by

$$\begin{aligned} x_n(t) = & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x_{n-1}(s)) ds \\ & - \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^1 (1-s)^{\beta-1} f(s, x_{n-1}(s)) ds \\ & + \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} f(p, x_{n-1}(p)) dp \right) ds. \end{aligned}$$

for all $n \in N$, converges to a unique solution of the nonlinear fractional differential equation of Caputo type (3.7).

Proof. Let $X = C[0,1]$. Thus, (X, d_θ) is a complete extended b-metric space with respect to

$$d_\theta(x, y) = ||x - y||_\infty = \sup_{t \in [a,b]} |x(t) - y(t)|^2,$$

where $\theta : X \times X \longrightarrow [1, \infty)$ is defined by

$$\theta(x, y) = |x(t)| + |y(t)| + 1.$$

And let $A_1 = A_2 = A_3 = X = C[0, 1]$ three nonempty subsets of X. $T : \bigcup_{i=1}^{3} A_i \longrightarrow \bigcup_{i=1}^{3} A_i$ is an operator defined as follows:

$$\begin{aligned} Tx(t) = &\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds \\ &- \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^1 (1-s)^{\beta-1} f(s, x(s)) ds \\ &+ \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} f(p, x(p)) dp \right) ds. \end{aligned}$$

Obviously, A_1, A_2, A_3 are closed subsets of (X, d_θ). Clearly, we have $T(A_1) \subset A_2, T(A_2) \subset A_3$ and $T(A_3) \subset A_1$. Thus, T is a cyclic operator on $\bigcup_{i=1}^{3} A_i$. Assuming $x, y \in \bigcup_{i=1}^{3} A_i$ and $t \in [0, 1]$, we obtain

$$\begin{aligned} &d_\theta(Tx, Ty) = |Tx(t) - Ty(t)|^2 \\ &= |\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds \\ &- \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^1 (1-s)^{\beta-1} f(s, x(s)) ds \\ &+ \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} f(p, x(p)) dp \right) ds \\ &- \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, y(s)) ds \\ &+ \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^1 (1-s)^{\beta-1} f(s, y(s)) ds \\ &- \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} f(p, y(p)) dp \right) ds|^2. \end{aligned}$$

Using the properties of the module, we obtain

$$
\begin{aligned}
d_\theta(Tx,Ty) &\leq |\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[f(s,x(s))-f(s,y(s))]ds \\
&- \frac{2t}{(2-\eta^2)(\Gamma(\beta))}\int_0^1 (1-s)^{\beta-1}[f(s,x(s))-f(s,y(s))]ds \\
&+ \frac{2t}{(2-\eta^2)(\Gamma(\beta))}\int_0^\eta \left(\int_0^s (s-p)^{\beta-1}[f(p,x(p))-f(s,y(s))]dp\right) ds|^2 \\
&\leq \frac{1}{\Gamma^2(\beta)}\int_0^t |t-s|^{2(\beta-1)}|f(s,x(s))-f(s,y(s))|^2 ds \\
&+ \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))}\int_0^1 (1-s)^{2(\beta-1)}|f(s,x(s))-f(s,y(s))|^2 ds \\
&+ \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))}\int_0^\eta \left(\int_0^s (s-p)^{2(\beta-1)}|f(p,x(p))-f(s,y(s))|^2 dp\right) ds
\end{aligned}
$$

Taking the supremum over $s \in [0,1]$, we obtain

$$
\begin{aligned}
d_\theta(Tx,Ty) &\leq \frac{1}{\Gamma^2(\beta)}\int_0^t |t-s|^{2(\beta-1)}\frac{\Gamma^2(\beta+1)}{50}\sup_{s\in[0,1]}|x(s)-y(s)|^2 ds \\
&+ \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))}\int_0^1 (1-s)^{2(\beta-1)}\frac{\Gamma^2(\beta+1)}{50} ds \\
&+ \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))}\int_0^\eta \left(\int_0^s (s-p)^{2(\beta-1)}\frac{\Gamma^2(\beta+1)}{50} dp\right) ds \\
&\leq \frac{\Gamma^2(\beta+1)}{50} d_\theta(x,y) \times \sup_{s\in[0,1]}\Big[\int_0^t |t-s|^{2(\beta-1)} ds \\
&+ \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))}\int_0^1 (1-s)^{2(\beta-1)} ds \\
&+ \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))}\int_0^\eta \int_0^s (s-p)^{2(\beta-1)} dp ds\Big] \\
&\leq \frac{1}{2} d_\theta(x,y).
\end{aligned}
$$

Since $\lim_{n,m\longrightarrow\infty}\theta(x_n,x_m) = 1 < 2$, we fulfilled all of the conditions of Theorem 2.6. Thus, a unique solution of the Caputo-type nonlinear fractional differential Equation (3.1) exists. □

Conclusion

It is well known that for proving the existence and uniqueness of the solution of different type of equations one can use the fixed point theory technique.
In this thesis we used the concept of extended b-metric spaces to formulate and prove some fixed point theories. An example and application of some Theorems is included to show their usability.
As a future work, here are some fixed-point theories of single valued maps that we propose the reader to formulate and prove in extended b-metric space:

1. The fixed-point theorem of Hardy-Rogers, see [10];
2. The theorem of $\acute{C}iri\acute{c}$, see [16];
3. The results of the common fixed points, see for example [4] and [9];
4. The Meir-Keeler result of fixed points, see [2];
5. The Sehgal theorem, see [36].

At the end of our thesis, and after collecting a huge amount of information from trusted sources and references, we can say that it was necessary to shed light on extended b-metric space to get these important results. And it is also necessary to do more research on the same topic in order to uncover more helpful information for science students and for all humanity.

Bibliography

[1] A. Aghajani, M. Abbas and J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Math. Slovaca*,(in Press).

[2] A. Meir and E. Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.*, 28 (1969), 326-329.

[3] B.Alqahtani, A.Fulga and E.Karapınar, Non-Unique Fixed Point Results in Extended B-Metric Space, *Mathematics* 2018, 6, 68.

[4] B. Fisher, Four mappings with a common fixed point, *The Journal of the University of Kuwait Science*, 8 (1981),131-139.

[5] D. Baleanu, S. Rezapour, M. Mohammadi, Some existence results on nonlinear fractional differential equations. *Philos. Trans. R. Soc. A, Math. Phys. Eng. Sci.* 2013, 371, 20120144.

[6] D. Zoran. Mitrovic, H.Isık, R.Stojan, The new results in extended b-metric spaces and applications, *Int. J. Nonlinear Anal. Appl.* 11 (2020) No. 1, 473-482.

[7] G. Cortelazzo, G. Mian, G. Vezzi, P. Zamperoni, Trademark shapes description by string matching techniques. *Pattern Recognit.* 1994, 27, 1005–1018.

[8] G.S.Ladde, V. Lakshmikantham, A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman: Boston, MA, USA, 1985.

[9] G. Jungck, Commuting mappings and fixed points, *Am. Math. Mon.*, 83 (1976), 261-263.

[10] G. E. Hardy and T. D. Rogers, A generalization of a fixed point theorem of Reich, *Can. Math. Bull.*, 16 (1973),201-206.

[11] Hassen Aydi, Monica-Felicia Bota, Erdal Karapinar, and Slobodanka Mitrović, A fixed point theorem for set-valued quasicontractions in b-metric spaces, *Fixed Point Theory and Application* (2012), no. 88.

[12] I.A. Bakhtin, The contraction mapping principle in almost metric spaces. *Funct. Anal.* 1989, 30, 26–37.

[13] J. Matkowski, Integrable solutions of functional equations, Warszawa: Instytut Matematyczny Polskiej Akademi Nauk, 1975.

[14] J.J.Nieto, R.R. Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order 2005, 22, 223–239.

[15] L.B. Ciric, . A generalization of Banach contraction principle. *Proc. Am. Math. Soc.* 1974, 45, 267–273.

[16] Lj. B.*Ćirić*, A generalization of Banach's contraction principle, *Proc. Am. Math. Soc.*, 45 (1974), 267-273.

[17] M.Bota, L.Guran, Existence of a solution of fractional differential equations using the fixed point technique in extended b-metric spaces, *AIMS Mathematics*, Volume 7, Issue 1, 518–535.

[18] M. Bota, A. Molnar, and C. Varga, On ekeland's variational principle in b-metric spaces, *Fixed Point Theory* 12 (2011), no. 2, 21-28.

[19] M.Aiman, M.Nabil, A.Kamal, S.Wasfi, New theorems on extended b-metric spaces under new contractions,*Nonlinear Anal. Model. Control*, 24(6):870–883.

[20] M.Bota, L.Guran, Existence of the Solutions of Nonlinear Fractional Differential Equations Using the Fixed Point Technique in Extended b-Metric Spaces, *Symmetry* 2021, 13, 158.

[21] M. Boriceanu, Fixed point theory for multivalued generalized contraction on a set with two b-metrics, *Studia Univ Babes-Bolyai Math. LIV* (3), (2009),1-14.

[22] N. Hussain, D. Doric, Z. Kadelburg, S. Radenovic, Suzuki-type fixed point results in metric type spaces. *Fixed Point Theory Appl.* 2012, 2012, 126.

[23] P. Hitzler, Generalized Metrics and Topology in Logic Programming Semantics, Ph.D. thesis, National University of Ireland, University College Cork, (2001).

[24] Renu Chugh, Vivek kumar, and Tamanna Kadian, Some fixed point theorems for multivalued mappings in generalized b-metric spaces, *International Journal of Mathematical Archive* (2012), no. 3, 1198-1210.

[25] R. McConnell, R. Kwok, J. Curlander, W. Kober, S. Pang, Ψ-S correlation and dynamic time warping: Two methods for tracking ice floes. *IEEE Trans. Geosci. Remote Sens.* 1991, 29, 1004–1012.

[26] R.Fagin, L. Stockmeyer, Relaxing the triangle inequality in pattern matching. *Int. J. Comput. Vis.* 1998, 30,219–231

[27] R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, 60, 1968, 71-76.

[28] S.Cobzas, b-metric space fixed point and Lipschitz functions, arXiv:1802.02722v3[math.FA] 2018-2019.

[29] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis* 1 (1993), 5-11.

[30] Shyam Lal Singh, Stephen Czerwik, and Krzysztof Król, Coincidences and fixed points of hybrid contractions, *Tamsui Oxford Journal of Mathematical Sciences* 24 (2008), no. 4, 401-416.

[31] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces. *Atti Sem. Mat. Univ. Modena* 1998, 46, 263–276.

[32] Simeon Reich, Some remarks concerning contraction mappings, *Canadian Mathematical Bulletins*, 14 (1971), no. 1, 121-124.

[33] T.Kamran, M.Samreen and Q.UL.Ain, A Generalization of b-Metric Space and Some Fixed Point Theorems. *Mathematics*, (2017), 5, 19.

[34] V. Berinde, *Iterative Approximation of Fixed Points*, Springer, (2006).

[35] V. Berinde, Contractii generalizate si aplicatii, Baia Mare: Editura Cub Press 22, 1997.

[36] V. M. Sehgal, A fixed point theorem for mappings with a contractive iterate, *Proc. Amer. Math. Soc.*, 23 (1969),631-634.

[37] W. Shatanawi, A. Pitea and R. Lazovic, Contraction conditions using comparison functions on b-metric spaces, *Fixed Point Theory Appl.*, (2014), 1-11.