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Linear equations with additive and multiplicative noise in infinite dimension and properties of solutions

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Dedication

All praise to **Allah**, today we fold the day's tiredness and the errand summing up between the cover of this humble work.

I dedicate my work to:

My great teacher and messenger, **Mohammed-peace and grace from Allah be upon him**, who taught us the purpose of life.

My dear **mother**, my raison d'etre, my life, the shiny lantern.

To the **family** of **Kadi Hanifi** and her kids. And to my beloved grandmother

To the sources of encouragement and support, companions, and to the one who was my companion in life To all those if the pen forgets them, the heart will not forget them.

To all the faithful to love God and love the religion of Muhammad, peace be upon him.

To all those who are looking for glory and pride in Islam and nothing else.

Aissaoui Roumaissa

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Notations

$\mathbb{N}$	**Set of natural numbers.**
$\mathbb{R}_+$	**Set of nonnegative real numbers.**
$\mathbb{R}^n$	**Real space of dimension n.**
$(\Omega, \mathcal{F}_t, \mathbb{P})$	**Probability space.**
$L_1(\mathbb{U}, \mathbb{H})$	**Space of all nuclear operators from** $\mathbb{U}$ *to* $\mathbb{H}$.
$\mathscr{N}_W^2(0, T; \mathbb{k})$	**Set of all stochastic processes.**
$\|.\|$	**Standard of norm .**
$\|\|\|_{L_2}$	**Hilbert–Schmidt norm.**
L^p	**Lebesgue spaces of integrable functions.**
$\mathcal{U}_s$	**The set of admissible controls.**
$\mathbb{1}$	**Indicator function.**
$\mathbb{H}, \mathbb{U}$	**Separable Hilbert spaces.**
W	**Wiener processes.**
trQ	**Trace of Q.**
$Q^{\frac{1}{2}}$	**Square root of** $Q \in L(\mathbb{U})$.
A^*	**Adjoint operator of** $A \in L(\mathbb{U}, \mathbb{H})$.
Q	**Set of rational numbers.**
$\mathbb{P}.a.s$	**Almost surely.**
$\equiv$	**Equivalence.**

$i.i.d.$	**Independent and identically distributed**.
(ONB)	**Orthonormal basis**.
$(PDEs)$	**Partial differential equations**.
$(HJBE)$	**Hamilton-Jacobi-Bellman equation**.
$(BSDEs)$	**Backward stochastic differential equations**.
$(FBSDEs)$	**Forward-backward stochastic differential equations**.

Introduction

Backward stochastic differential equation (BSDE, in short) is a stochastic differential equation with a terminal condition in which the solution is required to be adapted with respect to an underlying filtration. BSDEs naturally arise in various applications such as stochastic control, mathematical finance, and nonlinear Feynman-Kac formula.

For the earliest version of the linear BSDE's in finite dimensions has been introduced in 1973 by Bismut [1], in 1983 Bensoussan [2] used the martingale representation theorem to prove the wellposed-ness result of general linear BSDE's. Pardoux and Peng [19] generalized the notion in 1990 and were the first to consider general BSDE's and to solve the question of existence and uniqueness.

The first extension to the infinite dimensional case is due to Hu and Peng [10], who used the concept of mild solution to BSDE, give an existence and uniqueness result for the equation with an operator A, infinitesimal generator of a strongly continuous semigroup and the coefficient f Lipschitz in y and z. Further results can be found in Tessitore [32], Confortola [3], Pardou and Rascanu [28], the authors Fuhrman and Hu [6] proved the existence and uniqueness of the solution to BSDE assuming that the driver is uniformly Lipschitz with respect to (y, z). BSDEs have important applications in stochastic control, financial markets (see El Karoui, Peng and Quenez [27]) and partial differential equations PDEs. The equation for the adjoint process in infinite dimensional optimal stochastic control is a linear version. S. Peng [29] studied the existence and uniqueness of the following kind of BSDE. There are many people who studied the finite dimensions BSDEs. After that many authors deals with the infinite dimensional case. Here, we are interested in optimal control of SDEs in infinite dimensions which produces a BSDE in infinite dimension.

Our thesis is divided into three chapters:

In chapter 1, After an introductory part including the bibliographical context and the motivation section, we state the definitions and probabilistic tools that we have used throughout this thesis. We also give some background and some basic concepts. We have given a definition of the Q-wiener process and some of its properties, and we have defined the stochastic integral.

In chapter 2, We present two fundamental equations for the linear model. Equations with linear noise in both the additive and multiplicative directions We first define strong, weak, and mild solutions and establish their basic properties. Then we prove the existence of weak solutions, and we prove the existence and uniqueness of mild solutions. At the end, we derive the existence of strong solutions.

In chapter 3, We discuss our study on the infinite-dimensional backward stochastic differential equation, and an important motivation for BDEs is a connection with stochastic control and how to get it, with the use of stochastic Hamilton-Jacobi-Bellman (HJB) equations. We focus here on the reformulation of the underlying dynamic programming equation as a system of forward-backward stochastic differential equations that is solved by least squares Monte Carlo. We give examples of BSDE in infinite dimensions. In the end, we discuss the numerical simulation of the uncoupled FBSDE.

Chapter 1

Generalities on stochastic integral via Q-Wienner processes

In this chapter, we introduce some basic concepts, we gather some definitions and some tools which will be necessary for the definition of stochastic integral with respect to Wienner processes.

Let $\mathbb{H}$ and $\mathbb{U}$ be two separable Hilbert spaces.This chapter is devoted to the construction of the stochastic Itô integral

$$\int_0^t \Phi_s dW_s,$$

where W is a Wiener process on a Hilbert space $\mathbb{U}$ and Φ is a process with values that are linear but not necessarily bounded operators from $\mathbb{U}$ into a Hilbert space $\mathbb{H}$.
We will begin by collecting basic facts on Hilbert space valued Wiener processes, including cylindrical Wiener processes. Next, we define the stochastic integral in stages based on elementary processes. We also establish basic properties of the stochastic integral, including the It$\hat{o}$ formula.

Definitions 1.1. (Bochner integrable)[4] Let (X, Σ, μ) be a measure space, and $\mathbb{B}$ a Banach space. The Bochner integral of a function $f : X \to \mathbb{B}$ is defined in much the same way as the Lebesgue integral. First, define a simple function to be any finite sum of the form:

$$s(x) = \sum_{i=1}^{n} \mathbb{1}_{E_i}(x) b_i,$$

where the E_i are disjoint sets of the σ-algebra Σ, b_i are distinct elements of $\mathbb{B}$ and $\mathbb{1}_E$ is characteristic function of E.

If $\mu(E_i)$ is finite whenever $b_i \neq 0$, then the simple function is integrable, and the integral is then defined by:

$$\int_X \left[\sum_{i=1}^{n} \mathbb{1}_{E_i}(x) b_i\right] d\mu = \sum_{i=1}^{n} \mu(E_i) b_i,$$

exactly as it is for the ordinary Lebesgue integral.

Definitions 1.2. (Trace class operator) Assume that $\mathbb{H}$ is a Hilbert space and that $A : \mathbb{H} \to \mathbb{H}$ is a non-negative, self adjoint bounded linear operator on $\mathbb{H}$. The sum of the series is the trace of A, represented by $\operatorname{Tr} A$.

$$\operatorname{Tr} A = \sum_{k} \langle Ae_k, e_k \rangle,$$

where $(e_k)_k$ is an orthonormal basis (ONB) in $\mathbb{H}$.

The operator $A : \mathbb{H} \to \mathbb{H}$ is said to be **of trace class** if $\operatorname{Tr}(|A|) < \infty$, ($|A| := \sqrt{A^* A}$).

Definitions 1.3. [24](Hilbert–Schmidt operators) Let $(e_n)_{n\in\mathbb{N}}$ a Hilbert basis in $\mathbb{U}$. We call operator of Hilbert-Schmidt, a linear operator $\phi : \mathbb{U} \to \mathbb{H}$ as the sum $\sum_{n\in\mathbb{N}} \|\phi(e_n)\|^2$ converge. This sum is then independent of the choice of the orthonormal basis, and we note $L_2(\mathbb{U}, \mathbb{H})$ the space of the Hilbert-Schmidt operators, equipped with the standard norm

$$\|\phi\|_{L_2} = \sqrt{\sum_{n\in\mathbb{N}} \|\phi(e_n)\|^2}.$$

are independent of the choice of the basis. Moreover, the space $L_2(\mathbb{U}, \mathbb{H})$ of all Hilbert–Schmidt operators from $\mathbb{U}$ to $\mathbb{U}$ equipped with the inner product

$$\langle \phi, \phi' \rangle_{L_2} := \sum_{k\in\mathbb{N}} \langle \phi e_k, \phi' e_k \rangle, \qquad for\ \phi, \phi' \in L_2(\mathbb{U}, \mathbb{H}),$$

is a separable Hilbert space. As it is shown in [[24], Rem. B.0.6] Hilbert–Schmidt operators enjoy the following properties.

Definitions 1.4. (Eigenvalues) We consider $\mathbb{H}$ and $\mathbb{U}$ be two Hilbert spaces over the field $\mathbb{K}$. We say that a number $\lambda \in \mathbb{K}$ is an eigenvalue of the operator $T : \mathbb{H} \to \mathbb{U}$ if there is a vector

$$x \in \mathbb{H}, x \neq 0 : Tx = \lambda x,$$

so the vector x is then called an eigenvector associated with the eigenvalue λ.

Definitions 1.5. (Gaussian Measure) According to the general definition a probability measure μ on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$ is called Gaussian if for arbitrary $h \in \mathbb{H}$ there exist $m \in \mathbb{R}$, $q \geq 0$ such that,

$$\mu(\{x \in \mathbb{H}; \langle\, h, x\rangle \in \mathrm{A}\}) = \mathcal{N}(m, q)(\mathrm{A}), \quad \forall \mathrm{A} \in \mathcal{B}(\mathbb{R})$$

In particular, if μ is Gaussian, the following functional is

$$\mathbb{H} \to \mathbb{R}, \quad h \to \int_{\mathbb{H}} \langle\, h, x\rangle \mu(dx)$$
$$\mathbb{H} \times \mathbb{H} \to \mathbb{R}, \quad (h_1, h_2) \to \int_{\mathbb{H}} \langle\, h_1, x\rangle\langle\, h_2, x\rangle \mu(dx)$$

are well defined and continuous.

Definitions 1.6. (Covariance operator) For a probability measure $\mathbb{P}$ on a Hilbert space $\mathbb{H}$ with inner product $\langle\cdot,\cdot\rangle$, the covariance of $\mathbb{P}$ is the bilinear form $Cov : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ given by:

$$Cov(x, y) = \int_{\mathbb{H}} \langle\, x, z\rangle\langle\, y, z\rangle d\mathbb{P}(z),$$

for all x and y.
The covariance operator C is then defined by: $Cov(x, y) = \langle\, Cx, y\rangle$.

1.1 Hilbert-Space-Valued Process

In order to define the integrator part, we first introduce Wiener process and then define Q-Wiener processes and we give an example about cylindrical Wiener processes, and Hilbert space-valued Wiener process in a natural way. Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space, and $\Bbbk$ be a real separable Hilbert space with the scalar product denoted

by $t\langle .,.\rangle_{\mathbb{K}}$. We will always assume that $(\Omega, \mathcal{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is complete.
We consider a separable Hilbert space $\mathbb{U}$ with the inner product $\langle .,.\rangle$ and $\{W_t, t \geq 0\}$ is a $\mathbb{U}$-valued Wiener process. Then, for each $u \in \mathbb{U}$, the process $\langle W_t, u\rangle_{t\geq 0}$. is a real valued Wiener process. This means that $\mathscr{L}(W_t)$ is a Gaussian measure with mean vector 0. We also have that for arbitrary $u, v \in \mathbb{U}$, $t \geq 0$, $s \geq 0$

$$\mathbb{E}[\langle W_t, u\rangle\langle W_s, u\rangle] = (t \wedge s)\mathbb{E}[\langle W_1, u\rangle^2]$$

and

$$\mathbb{E}[\langle W_t, u\rangle\langle W_s, v\rangle] = \mathbb{E}[\langle W_1, u\rangle\langle W_1, v\rangle] = (t \wedge s)\langle Qu, v\rangle,$$

where Q is the covariance operator of the Gaussian measure (W_1), The operator Q is of trace class and it completely characterizes the distribution of $\{W_t, t \geq 0\}$.
Let Q be a trace class non-negative operator on a Hilbert space $\mathbb{U}$.

1.1.1 Wiener processes

Definitions 1.7. A real valued stochastic process $W = \{W_t, t \geq 0\}$ is called a wiener process if :

- C1: W has continuous trajectories and $W_0 = 0$,
- C2: W has independent increments and

$$\mathscr{L}(W_t - W_s) = \mathscr{L}(W_{t-s}), \quad t \geq s \geq 0,$$

- C3: $\mathscr{L}(W_t) = \mathscr{L}(-W_t), \quad t \geq 0.$

1.1.2 Q-Wiener processes

Definitions 1.8. A $\mathbb{U}$-valued stochastic process $W = \{W_t, t \geq 0\}$ is called a Q-Wiener process if:

- C1: $W_0 = 0$,
- C2: W has continuous trajectories,
- C3: W has independent increments,

- C4: $\mathscr{L}(W_t - W_s) = \mathcal{N}(0,(t-s)Q), \quad t \geq s \geq 0.$

Note that there exists a complete orthonormal system (e_k) in $\mathbb{U}$ and a bounded sequence of non-negative real number (λ_k) such that $Qe_k = \lambda_k e_k$ (according to the orthonormalization Process of Gramme Schmidt).

Definitions 1.9. (Brownian motion) An $\mathbb{U}$-valued process $W = \{W_t, t \in [0,T]\}$ is called an $\mathbb{U}$-valued Brownian motion if it enjoys the following properties:

- $W_0 = 0$ almost surely,
- $W_t - W_s$ is a normal random variable with mean 0 and variance $t-s$ whenever $s < t$,
- $W_t - W_s$ is independent of $\mathscr{F}_s$, s < t,
- W_t has continuous paths.

Propositions 1.1. *Assume that $\{W_t, t \geq 0\}$ is a Q-Wiener process. Then the following statements hold:*

- *C1: W is a Gaussian process on $\mathbb{U}$ and*

$$\mathbb{E}(W_t) = 0, \quad \mathrm{Cov}(W_t) = tQ, \quad t \geq 0.$$

- *C2: For arbitrary $t \geq 0$, W has the expansion*

$$W_t = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\beta_i e_i \tag{1.1}$$

where

$$\beta_i(t) = \frac{1}{\sqrt{\lambda_i}}\langle W_t, e_i\rangle, \quad i \in \mathbb{N},$$

are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ and the series in C2 are convergent in $L^2(\Omega, \mathcal{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$.

1.1.3 Generalized Wiener processes

Let $\{W_t, t \geq 0\}$ be a Wiener process on a Hilbert space $\mathbb{U}$ and let Q be its covariance operator. For each $a \in \mathbb{U}$, we define a real valued Wiener process $\{W_a(t), t \geq 0\}$ by the formula:

$$W_a(t) = \langle a, W_t \rangle, \qquad t \geq 0.$$

The transformation $a \rightarrow \mathrm{W}_a$ is linear from $\mathbb{U}$ to the space of stochastic processes. Additionally, it is continuous in the sense that:

$$t \geq 0, \{a_n\} \subset \mathbb{U}, \lim_{n\to\infty} a_n = a \Rightarrow \lim_{n\to\infty} \mathbb{E} \left| W_a(t) - W_{a_n}(t) \right|^2 = 0. \tag{1.2}$$

Any linear transformation $a \rightarrow \mathrm{W}_a$ whose values are real valued Wiener processes on $[a, +\infty)$ satisfying (1.2) is called a generalized Wiener processes.

Now we consider the following cylindrical Wiener processes.

1.1.4 Cylindrical Wiener processes

The generalized Wiener process is called a cylindrical Wiener process in $\mathbb{U}$, If the covariance Q is the identity operator $\mathbf{I}$.

Definitions 1.10. We call a family $\{\tilde{W}_t, t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ a cylindrical Wiener process in a Hilbert space $\mathbb{K}$ if:

- For an arbitrary $t \geq 0$, the mapping $\tilde{W}_t : \mathbb{K} \longrightarrow L^2(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is linear,
- For an arbitrary $k \in \mathbb{K}$, $\{\tilde{W}_t, t \geq 0\}$ is an $\mathbb{F}$-Brownian motion,
- For an arbitrary $k, k' \in \mathbb{K}$ and $t \geq 0$, $\mathbb{E}(\tilde{W}_t(k)\tilde{W}_t(k')) = t\langle k, k' \rangle_{\mathbb{K}}$.

For every $t \geq 0$, $\tilde{W}_t/\sqrt{t}$ is a standard cylindrical Gaussian random variable, so that for any $k \in \mathbb{K}$, $\tilde{W}_t(k)$ can be represented as a $\mathbb{P}$-a.s. convergent series

$$\tilde{W}_t(k) = \sum_{j=1}^{\infty} \langle k, k' \rangle_{\mathbb{K}} \tilde{W}_t(f_j),$$

where $\{f_j\}_{j=1}^{\infty}$ is an ONB in $\mathbb{K}$.

1.2 Stochastic Integral with Respect to a Wiener Process

We are given here a Q-Wiener process $(\Omega, \mathscr{F}, \mathbb{P})$ having values in $\mathbb{U}$. W_t is given by (1.1). For the sake of simplicity of fact, we require that $\lambda_k > 0$ for all $k \in \mathbb{N}$. Let a normal filtration $\{\mathscr{F}_t\}_{t\geq 0}$ in $\mathscr{F}$ and we assume that

- (i) W_t is $\mathscr{F}_t$- measurable,
- (ii) $W_{t+h} - W_t$ is independent of $\mathscr{F}_t, \quad h \geq 0,\, t \geq 0.$

If a Q-Wiener process W satisfies (i) and (ii), we say that W is a Q-Wiener process with respect to $\{\mathscr{F}_t\}_{t\geq 0}$. However, to shorten the formulation we usually avoid stressing the dependence on the filtration.

Definitions 1.11. Let us fix $T < \infty$. An $L = L(\mathbb{U}, \mathbb{H})$-valued process Φ_t, $t \in [0, T]$ taking only a finite number of values is said to be elementary if there exists a sequence $0 = t_0 < t_1 < ... < t_k = T$ and a sequence $\phi_0, \phi_1 .... \phi_{k-1}$ of L-valued random variables taking on only a finite number of values such that ϕ_m are $\mathscr{F}_{t_m}$-measurable and

$$\phi_t = \phi_m,\ for\ t \in (t_m, t_{m+1}], \quad m = 0, 1.....k-1.$$

1.2.1 Stochastic integral with respect to n elementary Processes

Definitions 1.12. (Elementary Processes) For elementary processes ϕ one defines the stochastic integral by the formula:

$$\int_0^t \Phi_s dW_s = \sum_{m=1}^{k-1} \phi_m \left(W_{t_{m+1}\wedge t} - W_{t_m\wedge t}\right),$$

and denote it by $\phi.W_t, t \in [0, T]$.

1. It is useful, at this moment, to introduce the subspace $\mathbb{U}_0 = Q^{1/2}(\mathbb{U})$ of $\mathbb{U}$ which, endowed with the inner product,

$$\langle u, v\rangle_0 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle u, e_k\rangle \langle v, e_k\rangle = \left\langle Q^{-1/2}u, Q^{-1/2}v\right\rangle, \quad u, v \in \mathbb{U}_0,$$

is a Hilbert space.

2. In the construction of the stochastic integral for more general processes, an important role will be played by the space of all Hilbert–Schmidt operators $L_2^0 = L_2(\mathbb{U}, \mathbb{H})$ from $\mathbb{U}$ into $\mathbb{H}$. The space L_2^0 is also a separable Hilbert space, equipped with the norm

$$\|x\|_{L_2^0}^2 = \sum_{h,k=1}^{\infty} |\langle xg_h, f_k\rangle|^2 = \sum_{h,k=1}^{\infty} \lambda_h |\langle xe_h, f_k\rangle|^2 = \left\|xQ^{1/2}\right\|_{L_2(\mathbb{U},\mathbb{H})}^2 = \mathrm{Tr}\left(\left(xQ^{1/2}\right)\left(xQ^{1/2}\right)^*\right),$$

Where $g_j = \sqrt{\lambda_j} e_j$, $\{e_j\}$ and $\{f_j, j \in \mathbb{N}\}$ are complete orthonormal bases in $\mathbb{U}_0$, $\mathbb{U}$ and $\mathbb{H}$ respectively.

Let ϕ_t, $t \in [0, T]$, be a measurable L_2^0 valued process, we define the norms
$$\|\phi\|_t = \left[\mathbb{E}\int_0^t \|\phi_s\|_{L_2^0}^2 \, ds\right]^{1/2} = \left[\mathbb{E}\int_0^t \mathrm{Tr}\left(\left(\phi_s Q^{1/2}\right)\left(\phi_s Q^{1/2}\right)^*\right) ds\right]^{1/2}, \quad t \in [0, T].$$

Propositions 1.2. *If a process ϕ is elementary and $\|\phi\|_t < \infty$ then the process $\phi \cdot W$ is a continuous, square integrable $\mathbb{H}$-valued martingale on $[0, T]$ and*

$$\mathbb{E}|\phi.W_t|^2 = \|\phi\|_t^2, \quad 0 \le t \le T.$$

Propositions 1.3. *The following statements hold:*

1. *If a mapping ϕ from Ω_T, into L is L-predictable then ϕ is also L_2^0 -predictable.In particular, elementary processes are L_2^0-predictable.*

2. *If ϕ is an L_2^0-predictable process such that $\|\phi\|_T < \infty$ then there exists a sequence $\{\phi_n\}$ of elementary processes such that $\|\phi - \phi_n\|_T \to 0$ as $n \to \infty$.*

As a final step we extend the definition of the stochastic integral to L_2^0-predictable processes satisfying,
$$\mathbb{P}\left(\int_0^T \|\Phi_s\|_{L_2^0} \, ds < \infty\right) = 1.$$
All such processes are called stochastically integrable on $[0, T]$. They form a linear space denoted by $\mathcal{N}_W(0, T, L_2^0)$, more simply $\mathcal{N}_W(0, T)$ or even $\mathcal{N}_W$. The extension can be accomplished by the so called localization procedure. To do so we need the following.

Lemma 1.4. *[34] Assume that $\phi \in \mathcal{N}_W(0, T, L_2^0)$ and that τ is an $\{\mathscr{F}_t\}$-stopping time such that $\mathbb{P}(\tau \le T) = 1$ Then*

$$\int_0^T I_{[0,\tau]}(s)\phi_s dW_s = \Phi.W(\tau \wedge t), \quad t \in (0, T], \mathbb{P} - a.s.$$

1.2.2 Properties of the stochastic integral

Theorem 1.5. *[34] Assume that $\phi \in \mathcal{N}_W^2(0,T,L_2^0)$, then the stochastic integral $\Phi \cdot W$ is a continuous square integrable martingale, and its quadratic variation is of the form:*

$$\ll \Phi.W_t \gg = \int_0^t Q_\phi(s)ds,$$

where

$$Q_\phi(s) = \left(\Phi_s Q^{1/2}\right)\left(\Phi_s Q^{1/2}\right)^*, \quad s,t \in [0,T].$$

Propositions 1.6. *Assume that Φ_1,$\Phi_2 \in \mathcal{N}_W^2(0,T,L_2^0)$. then*

$$\mathbb{E}(\phi_i.W_t) = 0, \quad \mathbb{E}(\|\phi_i.W_t\|^2) < \infty, \quad t \in [0,T], i = 1,2.$$

Moreover, the correlation operators

$$V(t,s) = Cor(\phi_1.W_t, \phi_2.W_s), \quad s,t \in [0,T]$$

based on the formula

$$V(t,s) = \mathbb{E}\int_0^{t\wedge s}\left(\Phi_2(r)Q^{1/2}\right)\left(\Phi_1(r)Q^{1/2}\right)^* dr. \tag{1.3}$$

Proof.
Note that $\Phi_2(r)Q^{1/2}$ and $\left(\Phi_1(r)Q^{1/2}\right)^*$, $r \in [0,T]$, are respectively $L_2(\mathbb{H},\mathbb{U})$ and $L_2(\mathbb{H},\mathbb{U})$-valued processes. Therefore the process

$$\Phi_2(r)Q^{1/2}\left(\Phi_r Q^{1/2}\right)^*, r \in [0,T],$$

is an $L_1(\mathbb{H},\mathbb{H})$-valued process and we have

$$\left\|\left(\Phi_2(r)Q^{1/2}\right)\left(\Phi_1(r)Q^{1/2}\right)^*\right\|_1 \leq \left\|\left(\Phi_2(r)Q^{1/2}\right)\right\|_{L_2(\mathbb{H},\mathbb{U})}\left\|\left(\Phi_1(r)Q^{1/2}\right)\right\|_{L_2(\mathbb{H},\mathbb{U})}, \quad r \in [0,T].$$

Consequently

$$\begin{aligned}
\mathbb{E}\int_0^T \left\|\left(\Phi_2(r)Q^{1/2}\right)\left(\Phi_1(r)Q^{1/2}\right)^*\right\|_1 dr &\leq \mathbb{E}\int_0^T \left\|\left(\Phi_2(r)Q^{1/2}\right)\right\|_{L_2(\mathbb{H},\mathbb{U})}\left\|\left(\Phi_1(r)Q^{1/2}\right)\right\|_{L_2(\mathbb{H},\mathbb{U})} dr \\
&\leq \mathbb{E}\left[\int_0^T \left\|\left(\Phi_1(r)Q^{1/2}\right)\right\|_{L_2(\mathbb{H},\mathbb{U})} dr\right] \\
&\quad \left[\int_0^T \left\|\left(\Phi_2(r)Q^{1/2}\right)\right\|_{L_2(\mathbb{H},\mathbb{U})}\right] \\
&\leq \|\Phi_1\| . \|\Phi_2\|_T < \infty,
\end{aligned}$$

and therefore the integral (1.3) exists as an $L_1(\mathbb{H}, \mathbb{H})$-valued Bochner integral, the operator $V(t,s)$ is defined by

$$\mathbb{E}\left\langle \phi_1.W_t, a\right\rangle \left\langle \phi_2.W_s, b\right\rangle = \left\langle V(t,s)a, b\right\rangle, \quad a, b \in \mathbb{H}.$$

One can easily see that if, in addition, Φ_1 and Φ_2 are simple processes then

$$\begin{aligned}\mathbb{E}\left\langle \phi_1.W_t, a\right\rangle \left\langle \phi_2.W_s, b\right\rangle &= \mathbb{E}\int_0^{t\wedge s} \left\langle \phi_1(r).dW_r, a\right\rangle \int_0^{t\wedge s} \left\langle \phi_2(r).dW_r, b\right\rangle \\ &= \left\langle Q^{1/2}\phi_1^*(r)a, Q^{1/2}\phi_2^*(r)b\right\rangle dr.\end{aligned}$$

1.2.3 The Itô Formula

Theorem 1.7. *[34] Assume that Φ is an L_2^0-valued process stochastically integrable in $[0,T]$, φ a $\mathbb{H}$-valued predictable process Bochner integrable on $[0,T]$, $\mathbb{P}$-a.s, and X_0 a $\mathscr{F}_0$-measurable $\mathbb{H}$-valued random variable. Then the following process*

$$X_t = X_0 + \int_0^t \varphi_s ds + \int_0^t \Phi_s dW_s, \quad t \in [0,T],$$

is well defined. Assume that a function $F : [0,T] \times \mathbb{H} \to \mathbb{R}^1$ and its partial derivatives F_t, F_x, F_{xx}, are uniformly continuous on bounded subsets of $[0,T] \times \mathbb{H}$. We have

$$\begin{aligned}F(t, X_t) =& F(0, X_0) + \int_0^t \left\langle F_x(s, X_s), \Phi_s dW_s\right\rangle_{\mathbb{H}} \\ &+ \int_0^t (F_t(s, X_s) + \left\langle F_x(s, X_s, \Psi(s)\right\rangle_{\mathbb{H}} \\ &+ \frac{1}{2}\operatorname{Tr}\left[F_{xx}(s, X_s)\left(\Phi(s)Q^{1/2}\right)\left(\Phi_s Q^{1/2}\right)^*\right])ds.\end{aligned}$$

Chapter 2

Analysis of linear SDEs driven by Q-Wiener processes

We first introduce some preliminary notation and concepts necessary for the chapter. Then we pass to linear equations with additive noise and multiplication noise. Then we can provide weak solutions to these equations. In the end, we provide sufficient conditions for the existence of strong solutions.

For studying stochastic differential equations, one has to differentiate between strong and weak solution. A strong solution is usually defined as a measurable functional of given Wiener process (on some path space) that satisfies equation in a classical or generalized sense [13]. Strong solution exists for many classes such as: It$\hat{o}$ equations with Lipschitz coefficients [30][16], stochastic evolution equations with monotone coefficients [23][18], Kushner's and Zakai's of nonlinear equations.

In what follows, we group some tools that will be necessary:

We are given a probability space $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ together with a normal filtration $(\mathscr{F}_t)_{t\geq 0}$. We consider two Hilbert spaces $\mathbb{H}$ and $\mathbb{U}$, and a Q-Wiener process W_t on $(\Omega, \mathbb{F} = (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, with the covariance operator $Q \in L(U)$. If $TrQ < +\infty$, then W is a genuine Wiener process, whereas if $Q = I, W$ is a cylindrical process and in this case it has continuous paths in another Hilbert space $\mathbb{U}_1$ larger than $\mathbb{U}$, see Chapter 1. We assume that there exists a complete orthonormal system e_k in $\mathbb{U}$, a bounded sequence $\{\lambda_k\}$ of nonnegative real numbers such that $\{Qe_k = \lambda_k e_k, k \in \mathbb{N}\}$, and a sequence $\{\beta_k\}$ of real independent Brownian motions such that

$$\langle W_t, u\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \langle u, e_k\rangle\, \beta_k(t), \quad u \in U, t \geq 0.$$

Definitions 2.1. (semi group)

A semi-group with one parameter of linear operators on a Banach X space is a family of bounded linear operators $T : [0, +\infty[\to \mathcal{B}(X)$, verifying:

- $T_{t+s} = T_t T_s, \quad \forall t, s > 0,$
- $T_0 = I.$

We have the fundamental properties of a $\mathcal{C}_0$-semi-group. Whether $(T_t)_{t\geq 0}$ a $\mathcal{C}$-semi-group on X of an infinitesimal generator A, then:

- The $D(A)$ domain is dense in X, i.e $\overline{D(A)} = X$.
- A is a closed linear operator, i.e. its graph is closed.
- For all $t \geq 0$ and all $x \in D(A), T_t x \in D(A)$, and $\dfrac{dT_t x}{dt} = AT_t x = T_t Ax$.
- For all $t \geq 0$ and all $x \in X$, $\displaystyle\int_0^t T_s x ds \in D(A)$ and we've $T_t x = x + A\left(\displaystyle\int_0^t T_s x ds\right)$.

Definitions 2.2. (Cauchy problems)

Linear evolution equations, as parabolic, hyperbolic or delay equations, can often be formulated as an evolution equation in a Banach space E:

$$\begin{cases} u'_t = A_0 u_t, \\ u_0 = x \in E. \end{cases} \tag{2.1}$$

with A_0 is a linear operator, in general unbounded, defined on a dense linear subspace $D(A_0)$ of E. In (2.1), $u'(t)$ stands for the strong derivative of u_t

$$u'_t = \lim_{h\to 0} \frac{u_{t+h} - u_t}{h}, \tag{2.2}$$

The limit is taken from the topology of E.

Problem (2.1) is the initial value problem or the Cauchy problem relative to the operator A_0.

Definitions 2.3. We say that the Cauchy problem (2.1) is well posed if:

- for arbitrary $x \in D(A_0)$ there exists exactly one strongly differentiable function $u(t,x)$, $t \in [0,+\infty)$, satisfying (2.1) for all $t \in [0,+\infty)$.
- if $x_n \in D(A_0)$ and $\lim_{n\to 0} x_n = 0$, then for all $t \in [0,+\infty)$ we have

$$\lim_{n\to 0} u(t,x_n) = 0. \tag{2.3}$$

If the limit in (2.3) is uniform in t on compact subsets of $[0,+\infty)$ we say that the Cauchy problem (2.1) is uniformly well posed.

We will now suppose that the Cauchy problem (2.3) is uniformly well posed and define operators $S_t : D(A_0) \to \mathbb{E}$ by the formula:

$$S_t x = u(t,x), \quad \forall x \in D(A_0), \quad \forall t \geq 0.$$

For all $t \geq 0$ the linear operator S_t can be uniquely extended to a linear bounded operator on the entire $\mathbb{E}$, which we still denote by S_t. We have clearly

$$S_0 = I, \tag{2.4}$$

moreover, by the uniqueness

$$S_{t+s} = S_t S_s, \quad \forall t,s \geq 0. \tag{2.5}$$

Finally, by the uniform boundedness theorem [34], it follows that:

$$S(.)x \quad \text{is continuous in} \quad [0,+\infty), \quad \forall x \in E. \tag{2.6}$$

In this way, we are led directly from the study of the uniformly well posed problem to the family. $S_t, t \geq 0$, of linear bounded operators in E satisfying (2.3), (2.4) and (2.5).

Any family $S(.)$ of bounded linear operators on E satisfying (2.3), (2.4) and (2.5). is called a $\mathcal{C}_0$-semigroup of linear operators. So the concept of $\mathcal{C}_0$-semigroup is in a sense equivalent to that of uniformly well posed Cauchy problem.

The infinitesimal generator A of $S(.)$ is a linear operator defined as follows

$$\begin{cases} D(A) = \left\{ x \in \mathbb{H} : \lim_{h\to 0^+} \dfrac{S_t x - x}{h} \right\} \textit{exists}, \\ Ax = \lim_{h\to 0^+} \dfrac{S_t x - x}{h}, \quad \forall x \in D(A). \end{cases}$$

It is easy to see that A is an extension of A_0, and moreover that the problem

$$\begin{cases} u'_t = Au_t, \quad t \geq 0. \\ u_0 = x \in \mathbb{E}. \end{cases} \tag{2.7}$$

is also uniformly well posed with the same associated semigroup $S(.)$. This is why in our following considerations we will consider the Cauchy problem (2.29) with the operator A being the infinitesimal generator of a $\mathcal{C}_0$-semigroup.

2.1 Linear equations with additive noise

In this part, we want to deal with the problem linear equations with additive noise. We are going to examine the general properties of the equation of this type in a general framework. We'll look into the next linear affine equation:

$$\begin{cases} dX_t = (AX_t + f_t)dt + BdW_t, \\ X_0 = \xi. \end{cases} \tag{2.8}$$

where $A : D(A) \subset \mathbb{H} \longrightarrow \mathbb{H}$ and $B : \mathbb{U} \longrightarrow \mathbb{H}$ are linear operators and f is an $\mathbb{H}$-valued stochastic process. We will assume that the deterministic Cauchy problem

$$\begin{cases} u'_t = Au_t, \\ u_0 = x \in \mathbb{H}. \end{cases} \tag{2.9}$$

is uniformly well posed.

Definitions 2.4. (Strong solutions) Given $B = (B_t)$ a $\mathscr{F}_t$-Brownian motion, Z a random vector on $(\Omega, \mathscr{F}_t)$ independent of B, b and σ are locally bounded measurable functions. We call strong solution of homogeneous SDEs

$$X_t = Z + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s, \tag{2.10}$$

or non-homogeneous SDEs

$$X_t = Z + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \tag{2.11}$$

the $(X, B, (\mathscr{F}_t))$ verifying

- X is $\mathscr{F}_t^{Z,B}$ -adapted; where $\mathscr{F}_t^{Z,B}$ is Brownian filtration augmented from the sigma algebra generated by Z.
- (X, B) verify (2.10) or (2.11).

Definitions 2.5. (weak solution) Given b and σ locally bounded measurable functions. We call a weak solution SDE (2.10) or (2.11) a triplet $(\tilde{X}, \tilde{B}, (\mathcal{H}_t))$ on a space $(\tilde{\Omega}, \mathbb{H}, \mathbb{P})$ where

- $\tilde{B}$ is an $\mathcal{H}_t$-Brownian motion.
- $\tilde{X}, \tilde{B}$ verify (2.10) or (2.11).

2.1.1 Concept of solutions

An $\mathbb{H}$-valued predictable process X_t, $t \in [0, T]$, is said to be a strong solution to (2.8) if X_t takes values in $D(A)$,

$$\int_0^T |AX_s| ds < +\infty, \quad \mathbb{P} - a.s.$$

and for $t \in [0, T]$

$$X_t = \xi + \int_0^T [AX_s + f_s] ds + BW_t, \quad \mathbb{P} - a.s.$$

This definition is meaningful only if W is a $\mathbb{U}$-valued process and therefore requires that $Tr[Q] < +\infty$. Note that a strong solution should necessarily have a continuous modification. An $\mathbb{H}$-valued predictable process $X_t, t \in [0, T]$, is said to be a weak solution of (2.8) if the trajectories of $X(.)$ are $\mathbb{P} - a.s.$ Bochner integrable and if for all $z \in D(A^*)$ and all $t \in [0, T]$ we have

$$\langle X_t, z\rangle = \langle \xi, z\rangle + \int_0^t [\langle X_s, A^* z\rangle + \langle f_s, z\rangle] ds + \langle BW_t, z\rangle, \quad \mathbb{P} - a.s.$$

This definition is appropriate for a cylindrical Wiener process because the scalar processes $\langle BW_t, z\rangle$, $t \in [0, T]$, are well defined random processes. It is clear that a strong solution is also a weak one.

Definitions 2.6. It is of great importance in our study of linear to establish first the basic properties of the process

$$W_A(t) = \int_0^t S_{t-s} B dW_s, \quad t \geq 0.$$

Theorem 2.1. *[34]*

1. *If the operator A generates a $\mathcal{C}_0$-semigroup $S(.)$ in $\mathbb{H}$ and $B \in L(\mathbb{U};\mathbb{H})$. It is also natural to require the following.*

Assume that Hypothesis (1) and

$$\int_0^T \|S_r B\|^2_{L_2^0}\, dr = \int_0^T Tr[S_r BQB^* S_r^*] dr < +\infty. \tag{2.12}$$

Then

- *the process $W_A(.)$ is Gaussian, continuous in mean square and has a predictable version.*

- *we have*

$$Cov(W_A(t)) = \int_0^t S_r BQB^* S_r^* dr, \quad t \in [0,T],$$

- *the trajectories of $W_A(.)$ are $\mathbb{P}$-a.s. square integrable and the law $\mathscr{L}(W_A(.))$ is a symmetric Gaussian measure on $\mathbb{H} = L^2(0,T;\mathbb{H})$ with the covariance operators*

$$\varphi_t = \int_0^T G(t,s)\varphi_s ds, \quad t \in [0,T],$$

where

$$G(t,s) = \int_0^{t\wedge s} S_{t-r} BQB^* S^*_{s-r} dr, \quad t,s \in [0,T]$$

and $t \wedge s = min\,\{t,s\}$.

2.1.2 Existence and uniqueness of weak solutions

We have the theorem (2.4) which is the main result of this section. For this proof, we need the following lemmas:

Propositions 2.2. *Let A be the infinitesimal generator of a $\mathcal{C}_0$-semigroup $S(.)$ in $\mathbb{E}$ and $f \in L^1(0,T;E)$. Then there exists a unique weak solution u of the equation following*

$$\begin{cases} u'(t) = Au(t) + f(t), \quad t \in [0,T]. \\ u(0) = x \in \mathbb{E}, \end{cases} \tag{2.13}$$

where A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup $S(.)$ in $\mathbb{E}$ and f in $L^p(0,T;\mathbb{E})$, $p \geq 1$.

and it is given by the variation of constants formula

$$u_t = S_t x + \int_0^t S_{t-s} f_s ds, \quad t \in [0, T]. \tag{2.14}$$

The function $u(.)$ defined by (2.14) is called the mild solution of problem (2.13). Before proving a sufficient condition for the existence of strict solutions, it is convenient to introduce the approximating problem

$$\begin{cases} u_n'(t) = A_n u_n(t) + f_t, \quad t \in [0, T]. \\ u_n(0) = x \in X, \end{cases} \tag{2.15}$$

where A_n are the Yosida approximations of A. Clearly problem (2.15) has a unique solution $u_n \in W^{1,1}(0, T; \mathbb{E})$, given by the variation of constants formula

$$u_n(t) = S_n(t)x + \int_0^t S_n(t-s) f_s ds, \quad t \in [0, T], \tag{2.16}$$

where $S_n(t) = e^{tA_n}, t > 0$, and moreover

$$\lim_{n \to 0} u_n = u \quad in \quad \mathcal{C}([0, T]; E).$$

Lemma 2.3. *Let X be a weak solution of problem (2.8) with $\xi = 0$, $f \equiv 0$. Then, for arbitrary function $\xi(.) \in \mathcal{C}^1([0, T]; D(A^*))$ and $t \in [0, T]$, we have*

$$\langle X_t, \zeta_t \rangle = \int_0^t [\langle X_s, \zeta_s' + A^* \zeta_s \rangle] ds + \int_0^t [\langle \zeta_s, B dW_s \rangle].$$

Theorem 2.4.

1. *If the operator A generates a $\mathcal{C}_0$-semigroup $S(.)$ in $\mathbb{H}$ and $B \in L(\mathbb{H}; \mathbb{U})$ It is also natural to require the following.*

2. (a) *f is a predictable process with Bochner integrable trajectories on an arbitrary finite interval $[0, T]$.*

   (b) *ξ is $\mathscr{F}_0$-measurable.*

Assume (2.12) and Hypotheses (1), (2). So equation (2.8) has precisely a weak solution that is given by the formula.

$$X_t = S_t \xi + \int_0^t S_{t-s} f_s ds + \int_0^t S_{t-s} B dW_s, \qquad t \in [0, T]. \tag{2.17}$$

Formula (2.17) is a stochastic generalization of the classical variation of constants formula.

Proof.

It easily follows from Proposition (2.2) that a process X is a weak solution to (2.8) if and only if the process $\tilde{X}$ given by the formula:

$$\tilde{X}_t = X_t - \left(S_t\xi + \int_0^t S_{t-s} f_s ds \right), \qquad t \in [0,T],$$

is a weak solution to

$$d\tilde{X}_t = A\tilde{X}_t dt + BdW_t, \qquad \tilde{X}_0 = 0.$$

So, we can assume, without any loss of generality, that $\xi = 0$ and $f \equiv 0$

To prove the existence, we show that equation (2.8) with $\xi = 0$ and $f \equiv 0$ is satisfied by the process $W_A(.)$. We fix $t \in [0,T]$ and let $\zeta \in D(A^*)$. We have

$$\int_0^t \langle A^*\zeta, W_A(s)\rangle ds = \int_0^t \left\langle A^*\zeta, \int_0^t \mathbb{1}_{[0,s]}(r) S(s-r) BdW_r \right\rangle ds$$

and consequently,

$$\begin{aligned}
\int_0^t \langle A^*\zeta, W_A(s)\rangle ds &= \int_0^t \left\langle \int_0^t \mathbb{1}_{[0,s]}(r) B^* S^*_{s-r} A^*\zeta ds, dW_r \right\rangle \\
&= \int_0^t \left\langle \int_0^t B^* S^*_{s-r} A^*\zeta ds, dW_r \right\rangle \\
&= \int_0^t \left\langle \int_0^t \left(\frac{d}{ds} B^* S^*_{s-r}\zeta \right) ds, dW_r \right\rangle \\
&= \int_0^t \langle B^* S^*_{s-r}\zeta, dW_r\rangle - \int_0^t \langle B^*\zeta, dW_r\rangle \\
&= \langle \zeta, W_A(t)\rangle - \langle \zeta, RW_t\rangle.
\end{aligned}$$

As a result, $W_A(.)$ is a weak solution.

To prove uniqueness, we need the lemma (2.3).

Consider first the functions of the form $\zeta = \zeta_0\varphi_s$, $s \in [0,T]$ where $\phi \in \mathcal{C}^1([0,T])$ and $\zeta_0 \in D(A^*)$. Let

$$F_{\zeta_0}(t) = \int_0^t \langle X_s, A^*\zeta_0\rangle ds + \langle BW_t, \zeta_0\rangle.$$

Applying Itô's formula to the process $F_{\zeta_0}(t)\varphi_s$ we get

$$d[F_{\zeta_0}(s)\varphi_s] = \varphi_s dF_{\zeta_0}(s) + \varphi'_s F_{\zeta_0}(s) ds.$$

Specifically

$$F_{\zeta_0}(t)\varphi_t = \int_0^t \langle \zeta_s, BdW_s\rangle + \int_0^t [\varphi_s \langle X_s, A^*\zeta_0\rangle + \varphi'_s \langle X_s, \zeta_0\rangle] ds.$$

Since $F_{\zeta_0}(.) = \langle X(.), \zeta_0\rangle$, $\mathbb{P}$-a.s, the lemma is proved for the special function $\zeta_t = \zeta_0\varphi_t$ Since these functions are linearly dense in $\mathcal{C}^1([0,T]; D(A^*))$ the lemma is true in general.

Let X be a weak solution and let $\zeta_0 \in D(A^*)$. Applying Lemma (2.3) to the function $\zeta_s = S^*_{t-s}\zeta_0$, $s \in [0,T]$, we have

$$\langle X_t, \zeta_0\rangle = \int_0^t \langle S_{t-s}BdW_s, \zeta_0\rangle$$

and, since $D(A^*)$ is dense in $\mathbb{H}$ we have $X = W_A$. □

To illustrate, let us give an example.

Exemple 2.1. (Heat equation). Let $\mathbb{U} = \mathbb{H} = L^2(\phi)$, where ϕ is a bounded open set in $\mathbb{R}^{\mathbb{N}}$ with a regular boundary $\partial\phi$. Consider the problem

$$\begin{cases} d_tX(t,\xi) = \Delta_\xi X(t,\xi)dt + dW(t,\xi), t \geq 0, \xi \in \phi, \\ X(t,\xi) = 0, \quad t \geq 0, \xi \in \partial\phi \\ X(0,\xi) = 0, \quad \xi \in \phi, \end{cases}$$

and let A be the realization of the Laplace operator in $L^2(\phi)$ with Dirichlet boundary conditions. If $TrQ < \infty, t \geq 0$ we have $TrQ_t < \infty$ and we have

$$\int_0^t \|S_rB\|^2_{L_2^0} dr = \int_0^t Tr[S_rBQB^*S_r^*]dr < \infty, \tag{2.18}$$

is fulfilled

Assume now that $Q = I$ and that, $Ae_k = -\mu_k e_k$, where $\{\mu_k > 0, k \in \mathbb{N}\}$. In this case, (2.18) is fulfilled if and only if $\sum_{k=1}^{\infty} \frac{1}{\mu_k} < \infty$. As easily seen, this conditions holds only for $N = 1$.

2.1.3 Existence of strong solutions

A strong solution is not easy and some additional conditions must be met. We are concerned now with the existence of strong solutions to (2.8). Thus we are concerned with the equation

$$X_t = x + \int_0^t (AX_s + f_s)ds + W_t, \quad t \in [0,T].$$

Since strong solutions are also weak solutions, we know that they should be in the form

$$X_t = S_tx + \int_0^t S_{t-s}f_sds + \int_0^t S_{t-s}dW_s, \quad t \in [0,T].$$

To prove the next theorem 2.8, we need the following lemmas.

Lemma 2.5. *Let $S(.)$ be a $\mathcal{C}_0$-semigroup in $\mathbb{E}$ and let A be its infinitesimal generator. Then A is closed and the domain $D(A)$ is dense in $\mathbb{E}$. Moreover, if $x \in D(A)$, then*

$$S(.)x \in \mathcal{C}^1([0, +\infty); \mathbb{E}) \cap \mathcal{C}([0, +\infty), D(A)),$$

and

$$\frac{d}{dt} S_t x = A S_t x = S_t A x, \quad t \geq 0.$$

Lemma 2.6. *Let A be the infinitesimal generator of a $\mathcal{C}_0$-semigroup $S(.)$ in $\mathbb{E}$.*

1. *If $x \in D(A)$ and $f \in W^{1,p}(0, T; \mathbb{E})$ with $p \geq 1$, then problem (2.13) has a unique solution u in $\mathcal{C}([0, T]; \mathbb{E})$.*

2. *If $x \in D(A)$ and $f \in L^p(0, T; D(A))$, then problem (2.13) has a unique solution u in $L^p(0, T; \mathbb{E})$.*

Lemma 2.7. *If $\phi_t(L_2^0(H)) \subset D(A)$, $\mathbb{P} - a.s.$ for all $t \in [0, T]$ and*

$$\mathbb{P}\left(\int_0^T ||\phi_s||^2_{L_0^2 D(A)} ds < \infty \right) = 1,$$

$$\mathbb{P}\left(\int_0^T ||A\phi_s||^2_{L_0^2 D(A)} ds < \infty \right) = 1,$$

then $\mathbb{P}\left(\int_0^T \phi_s dW_s \in D(A) \right) = 1$ and

$$A \int_0^T \phi_s dW_s = \int_0^T A\phi_s dW_s, \quad \mathbb{P} - a.s.$$

Theorem 2.8. *Assume that*

- *$Q^{\frac{1}{2}}(H) \subset D(A)$ and $AQ^{\frac{1}{2}}$ is a Hilbert–Schmidt operator,*
- *$x \in D(A)$, $f \in C^1([0, T]; H) \cap C([0, T]; D(A))$.*

Then problem (2.8) has a strong solution.

Proof:

The result is true if $f=0$, $W=0$, by lemma (2.5).Assume now that $x=0$, $W=0$. For every $t\in[0,T]$ we have

$$\int_0^t |AS_{t-\sigma}f_\sigma|\,d\sigma \le \int_0^t \|S_{t-\sigma}\|\,|Af_\sigma|\,d\sigma < +\infty.$$

and therefore, by lemma (2.6) $X_t \in \int_0^t S_{t-\sigma}f_\sigma d\sigma \in D(A)$ and

$$AX_t = \int_0^t AS_{t-\sigma}f_\sigma d\sigma, \quad t\in[0,T].$$

As well

$$\begin{aligned}
\int_0^t AX_s ds &= \int_0^t \left(\int_0^t AS_{t-\sigma}f_\sigma d\sigma\right) ds \\
&= \int_0^t \left(\int_0^{t-\sigma} \frac{d}{ds}S_s f_\sigma ds\right) d\sigma \\
&= \int_0^t AS_{t-\sigma}f_\sigma d\sigma - \int_0^t f_\sigma d\sigma \\
&= X_t - \int_0^t f_\sigma d\sigma, \quad t\in[0,T].
\end{aligned}$$

Assume finally that $x=0$, $f=0$. Note that

$$\int_0^t \left\|AS_sQ^{\frac{1}{2}}\right\|_{L_2}^2 ds = \int_0^t \left\|AS_sQ^{\frac{1}{2}}\right\|_{L_2}^2 ds \le \left\|AQ^{\frac{1}{2}}\right\|^2 \int_0^t \|S_s\|_{L_2}^2 ds < \infty.$$

Therefore, by lemma (2.7)

$$W_A(t) = \int_0^t S_{t-\sigma}dW_\sigma \in D(A), \quad \mathbb{P}-a.s.$$

and for $t\in[0,T]$, $\mathbb{P}-a.s.$

$$AW_A(t) = \int_0^t AS_{t-\sigma}dW_\sigma.$$

Since $W_A(.)$ is a weak solution of (2.8), for $t\in[0,T]$ and $\zeta\in D(A^*)$, $\mathbb{P}-a.s.$

$$\begin{aligned}
\langle W_A(t),\zeta\rangle &= \int_0^t \langle W_A(s),A^*\zeta\rangle\,ds + \langle W_t,\zeta\rangle \\
&= \int_0^t \langle AW_A(s),\zeta\rangle\,ds + \langle W_t,\zeta\rangle \\
&= \left\langle A\int_0^t W_A(s)ds,\zeta\right\rangle + \langle W_t,\zeta\rangle.
\end{aligned}$$

as a result

$$W_A(t) = \int_0^t AW_A(s)ds + W_t.$$

2.2 Linear equations with multiplicative noise

We will consider the following linear affine equation:

$$\begin{cases} dX_t = (AX_t + f_t)dt + B(X_t)dW_t, \\ X_0 = \xi. \end{cases} \tag{2.19}$$

on a time interval $[0, T]$, where $A : D(A) \subset \mathbb{H} \longrightarrow \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup $S(.)$, ξ is an $\mathscr{F}_0$-measurable $\mathbb{H}$-valued random variable, f is a predictable process with local integrable trajectories and B :$D(B) \subset \mathbb{H} \longrightarrow L_2^0 = L_2(\mathbb{H}, \mathbb{U})$ is a linear operator.

Just like with additive noise, we define a strong solution of problem (2.19) as an H-valued predictable process $X_t, t \in [0, T]$, which takes values in $D(A) \cap D(B)$, $\mathbb{P}$-a.s. such that

$$\begin{cases} \mathbb{P}\left(\int_0^T [|X_s| + |AX_s|]\, ds < +\infty\right) = 1, \\ \mathbb{P}\left(\int_0^T \|BX_s\|^2_{L_2^0}\, ds < +\infty\right) = 1, \end{cases}$$

and, for arbitrary $t \in [0, T]$ and $\mathbb{P}$-a.s.

$$X_t = \xi + \int_0^t (AX_s + f_s)ds + \int_0^t B(X_s)dW_s. \tag{2.20}$$

An $\mathbb{H}$-valued predictable process $(X_t, t \in [0, T])$, is said to be a weak solution to (2.19) if X takes values in D(B), $\mathbb{P}$-a.s.

$$\mathbb{P}\left(\int_0^T |X_s|\, ds < +\infty\right) = 1 \tag{2.21}$$

$$\mathbb{P}\left(\int_0^T \|BX_s\|^2_{L_2^0}\, ds < +\infty\right) = 1 \tag{2.22}$$

and for arbitrary $t \in [0, T]$ and $\varsigma \in D(A^*)$,

$$\langle X_t, \varsigma\rangle = \langle \xi, \varsigma\rangle + \int_0^t (\langle X_s, A^*\varsigma\rangle + \langle f_s, \varsigma\rangle)\, ds + \int_0^t \langle \varsigma, B(X_s dW_s\rangle\,, \quad \mathbb{P} - a.s.$$

We also need the concept of the so called mild solution of (2.19). An $\mathbb{H}$-valued predictable process $X_t, t \in [0, T]$, is said to be a mild solution to (2.19) if X takes values in D(B), $\mathbb{P}$-a.s., (2.21) and (2.22) hold and for arbitrary $t \in [0,T]$,

$$X_t = S_t\xi + \int_0^t S_{t-s} f_s ds + \int_0^t S_{t-s} B(X_s) dW_s.$$

It is clear that a strong solution is also a weak solution.

Theorem 2.9. *Assume that $A : D(A) \subset \mathbb{H} \longrightarrow \mathbb{H}$ is the infinitesimal generator of a C_0-semigroup $S(.)$ in $\mathbb{H}$ and that $\phi \subset \mathcal{N}_w^2$.Then a strong solution is a weak solution and a weak solution is always a mild solution of problem (2.19). Conversely, if X is a mild solution of (2.19) and $\mathbb{E}\int_0^t \|BX_s\|^2_{L_2^0} ds < +\infty$, then X is also a weak solution of (2.19).*

In the case when B is bounded we discuss it here briefly by way of illustration. We first consider mild solutions.

Theorem 2.10. *Assume that A is the infinitesimal generator of a C_0-semigroup $S(.)$ in $\mathbb{H}, \mathbb{E}|\xi|2 < +\infty$ and $B \in L(\mathbb{H}; L_2^0)$. Then equation (2.19) has a unique mild solution $X \in \mathcal{N}_W^2(0,T;\mathbb{H})$, identical with a weak solution.*

Proof:

Denote by $\mathcal{H}$ the space of all $\mathbb{H}$-valued predictable processes Y such that $|Y|_{\mathcal{H}} = \sup_{t\in[0,T]} \mathbb{E}|Y_t|^2 < +\infty$ and for any Y define

$$\mathcal{H}(Y)_t = S_t\xi + \int_0^t S_{t-s} f_s ds + \int_0^t S_{t-s} B(Y_s) dW_s, \quad t \in [0,T],$$

and

$$\mathcal{H}_1(Y)_t = \int_0^t S_{t-s} B(Y_s) dW_s, \quad t \in [0,T].$$

We might suppose that, $\|S_t\| \leq M, t \geq 0$, and we have

$$\begin{aligned} |\mathcal{H}_1(Y)|_{\mathcal{H}} &\leq \sup \mathbb{E}\left(\int_0^t \|S_{t-s}B(Y)_s\|^2_{L_2^0} ds\right)^{\frac{1}{2}} \\ &\leq M \|B\|_{L(H;L_2^0)} \sqrt{T}\, |Y|_{\mathcal{H}}, \quad t \in [0,T]. \end{aligned}$$

So, if T is sufficiently small, $\mathcal{H}$ is a contraction and it is easy to see that its unique fixed point can be identified as the solution to (2.19). The case of general T can be handled in a standard way. We are now providing an existence result for strong solutions.

Propositions 2.11. *Assume that the hypotheses of Theorem 2.10 hold, $\xi = x \in D(A)$ and $f \equiv 0$. Let moreover $0 \in \rho(A)$ and assume that B_A, given by*

$$B_A(x)u = AB(A^{-1}x)u, \quad x \in \mathbb{H}, u \in \mathbb{U},$$

belongs to $L(H; L_2^0)$. Then the equation (2.19) has a unique strong solution.

Proof:

Let $x \in D(A)$, and let X and Y be the mild solutions of (2.19) and

$$\begin{cases} dY_t = AY_t dt + AB(A^{-1})Y_t dW_t, \\ Y_0 = Ax, \end{cases}$$

which exist by Theorem (2.10). Consider the approximating problems

$$\begin{cases} dX_n(t) = A_n X_n(t)dt + B(X_n(t))dW_t, \\ X_n(0) = x, \end{cases}$$

and

$$\begin{cases} dY_n(t) = A_n Y_n(t)dt + A_n B(A_n^{-1} Y_n(t))dW_t, \\ Y_n(0) = Ax, \end{cases}$$

where A_n are the Yosida approximations of A. We have clearly

$$Y_n(t) = A_n X_n(t), \quad t \geq 0, n \in \mathbb{N}$$

and so

$$X_n(t) = x + \int_0^t Y_n(s)ds + \int_0^t B(X_s(t))dW_s, \quad t \geq 0, n \in \mathbb{N}. \tag{2.23}$$

Additionally, it is simple to verify that $X_n \to X$ and $Y_n \to Y$, as $n \to \infty$, in $\mathcal{N}_W^2(0,T;\mathbb{H})$. But the operator A is closed and this implies $X_t \in D(A)$, a.s. and $Y_t = AX_t$. Now, letting n tend to infinity in (2.23). There is no other way to get the solution than the mild solution.

We recall some background on analysis and necessary to the definition of the mild solution.

2.2.1 Existence of mild solutions in the analytic case

We will give results for a mild solution to the problem

$$\begin{cases} dX_t = AX_t dt + B(X_t)dW_t, \\ X_0 = x \in \mathbb{H}, \end{cases} \tag{2.24}$$

that is for the integral equation

$$X_t = S_t x + \int_0^t S_{t-s} B(X_s) dW_s.$$

Let us introduce some notations. We set $v_t = S_t x$, $t \geq 0$, and for any process y we denote by $\tau(y)$ the following process

$$\tau(y)(t) = \int_0^t S(t-s)B(y_s)dW_s, \quad t \in [0,T],$$

so that, solving equation (2.24) is equivalent to finding a fixed point for the problem $X_t = v_t + \tau(X)_t, t \in [0,T]$.

2.2.2 Existence of solutions in the analytic case

We are here concerned with problem (2.24) under the hypothesis that A is the infinitesimal generator of an analytic semigroup $S(.)$ in $\mathbb{H}$. For $x \in D(B)$ and $\theta \in (0,1)$ we denote by $\|B(x)\|_\theta$ the Hilbert-Schmidt norm of the operator $B(x)$ considered as operator from U_0 into $D_A(\theta, 2)$.

Lemma 2.12. *[34] The following statements hold.*

- *If $x \in D_A(\theta, 2))$ with $\theta \in (0, 1/2)$, then $u_1 \in L^2([0,\infty); D_A(\theta + 1/2, 2))$.*
- *If $x \in D_A(1/2, 2))$ then $u_1 \in L^2[0,\infty)$.*

Theorem 2.13. *Assume that there exists $\theta \in (0, \frac{1}{2})$, $\eta \in (0, 1-2\theta)$ and $K > 0$ such that*
$B \in L(D_A(\theta + \frac{1}{2}, 2), L_2(U_0; D_A(\theta, 2)))$ and

$$\|B(x)\|_\theta^2 \leq \eta \, |x|_{\theta+\frac{1}{2}}^2 + K \, |x|_\theta^2 , \quad x \in D_A(\theta + \frac{1}{2}, 2).$$

Then for any $x \in D_A(\theta, 2)$, equation (2.24) has a mild solution

$$X \in \mathscr{N}^2_W(0, T; D_A(\theta + \frac{1}{2}, 2)),$$

identical with a weak solution. Moreover the solution has a continuous modification as a process with values in $D_A(\theta, 2)$) .

Proof:
We introduce the space $Z_T = \mathscr{N}^2_W(0, T; D_A(\theta + \frac{1}{2}, 2))$ endowed with the norm

$$\|Y\|^2_Z = \mathbb{E}\int_0^T |Y|^2_{\theta+\frac{1}{2}}\, dt + L\mathbb{E}\int_0^T |Y_t|^2_\theta\, dt,$$

where L is a positive number to be chosen later. The mapping Γ, defined by

$$\Gamma(Y)_t = S_t x + \int_0^t S_{t-s} B(Y_s) dW_s,$$

is a well defined transformation from Z_T into Z_T and moreover

$$\mathbb{E}\int_0^T |\Gamma Y_t|^2_{\theta+\frac{1}{2}} dt \le \frac{1}{1-2\theta}\mathbb{E}\int_0^T |B(Y_t)|^2_\theta dt \le \frac{\eta}{1-2\theta}\mathbb{E}\int_0^T |Y_t|^2_{\theta+\frac{1}{2}} dt + k\frac{\eta}{1-2\theta}\mathbb{E}\int_0^T |Y_t|^2_\theta dt. \tag{2.25}$$

However, we also have

$$\mathbb{E}\int_0^t |\Gamma Y_s|^2_\theta ds \le \mathbb{E}\int_0^t |B(Y_s)|^2_\theta ds \le \eta\mathbb{E}\int_0^t |Y_s|^2_{\theta+\frac{1}{2}} dt + k\mathbb{E}\int_0^t |Y_s|^2_\theta ds. \tag{2.26}$$

Combining (2.25) and (2.26) we find

$$|\Gamma Y|^2_Z \le \left[\frac{\eta}{1-2\theta} + LT\eta\right]\mathbb{E}\int_0^T |Y_s|^2_{\theta+\frac{1}{2}} dt + k\left[\frac{1}{1-2\theta} + LT\right]\mathbb{E}\int_0^T |Y_s|^2_\theta ds. \tag{2.27}$$

Now choose $L > \dfrac{k}{1-2\theta}$ and $T < \dfrac{1-2\theta-\eta}{L\eta(1-2\theta)}$ Then by (2.26) it follows that Γ is a contraction.
Moreover $v \in Z_T$ in virtue of Lemma (2.12); therefore by the contraction principle, equation (2.24) has a unique solution X in Z_T . By standard arguments, the restriction on T can be removed. Since the process $\phi_t = B(X_t), t \in [0, T]$, belongs to $\mathscr{N}^2_w(0, T; \mathbb{H})$ therefore by Theorem (2.9) the process X is also a weak solution to (2.24).

Hypothesis 2.14. *[34]*

The operator A generates an analytic semigroup of negative type. We indicate by $D_A(\theta, 2)$ the real interpolation space between $D(A)$ and X. Let $\{e_k\}$ be an orthonormal basis in U_0 and set

$$k(s) = \phi_s e_k, \quad s \in [0, T], \quad k \in \mathbb{N}.$$

For any Hilbert space K let $\mathscr{N}_W^2(0, T; K)$ denote the space of all k-valued predictable processes X such that

$$\|X\|^2_{\mathscr{N}_W^2(0,T;\Bbbk)} = \int_0^T \mathbb{E} \|X_s\|_K^2 \, ds < +\infty.$$

If $K = D_A(\theta, 2)$ and $T = +\infty$ then we write shortly $\|.\|_\theta$ instead of $\|.\|_{\mathscr{N}_W^2((0,+\infty;D_A(\theta,2))}$ In a similar way $\|\pi\|_\theta$ stands for $\|\pi\|_{(0,+\infty;L_\theta^{0,\theta})}$. We notice that $\|.\|_\theta$ is given by the formula

$$\|\phi\|_\theta = \sum_{k=1}^{\infty} \int_0^\infty \mathbb{E}|\varphi_k(s)|_\theta^2 ds.$$

Important information is provided by the following theorem.

Theorem 2.15. *Assume Hypothesis 2.14 and that $\phi \in (0, +\infty; L_2^{0,\theta})$*

- *$\theta \in (0, \frac{1}{2})$ then $W_A^\phi \in \mathscr{N}_W^2(0, +\infty; D_A(\theta + \frac{1}{2}, 2))$ and*

$$\left\| W_A^\phi \right\|^2_{\theta+\frac{1}{2}} = \frac{1}{1 - 2\theta} \|\phi\|_\theta^2 .$$

- *If $\theta = \frac{1}{2}$ then $W_A^\phi \in \mathscr{N}_W^2(0, +\infty; D(A))$, and*

$$\left\| A W_A^\phi \right\|^2 = \|\phi\|^2_{\frac{1}{2}} .$$

- *$\phi \in (\frac{1}{2}, 1)$ then $AW_A^\phi \in \mathscr{N}_W^2(0, +\infty; D_A(\theta - \frac{1}{2}, 2))$*

$$\left\| A W_A^\phi \right\|^2_{\theta-\frac{1}{2}} = \frac{1}{3 - 2\theta} \|\phi\|_\theta^2 .$$

In particular, for $t \geq 0, W_A^\phi \in D(A)$, $\mathbb{P}$-a.s.

Propositions 2.16. *[34]*

Assume that A is a variational operator and let $\phi \in \mathcal{N}_W^2(0,T;L_2^0)$. Then $W_A^\phi \in \mathcal{N}_W^2(0,T;V)$ and

$$\left\|W_A^\phi\right\|^2_{\mathcal{N}_W^2(0,T;V)} \leq \frac{1}{2}\left\|\phi\right\|^2_{\mathcal{N}_W^2(0,T;L_2^0)}.$$

Theorem 2.17. *We assume that A is a variational operator and that there exists $\theta \in (0,1)$ and $K > 0$ such that*

$$\frac{1}{2}\left\|B_z\right\|^2_{L_2^0} + \theta a(z,z) \leq K|z|^2, \forall z \in V.$$

Then, for any $x \in \mathbb{H}$ there exists a unique weak (and mild) solution $X \in \mathcal{N}_W^2(0,T;V)$ of (2.24) .Moreover the solution has a continuous modification as a process with values in $\mathbb{H}$.

Proof:

The proof is similar to the previous one but instead of Theorem (2.14) one has to use Proposition 2.16. Take as Z the space $\mathcal{N}_W^2(0,T;V)$ and note that

$$-\int_0^T a(y_s,y_s)ds = \theta\int_0^T \left\|y_s\right\|^2_V ds.$$

2.2.3 Existence of strong solutions

Strong solutions exist very rarely. Here, we study a class of equations (2.24) for which this is the case. Their special feature is that they can be reduced to deterministic problems.

We are concerned with the problem

$$\begin{cases} dX_t = AX_t dt + \sum_{k=1}^N B_k X_t d\beta_k, \\ X_0 = x \in \mathbb{H}, \end{cases} \tag{2.28}$$

where $A : D(A) \subset \mathbb{H} \to \mathbb{H}, B_k : D(B_k) \subset \mathbb{H} \to \mathbb{H}, k = 1,2,...,N$ are generators of semigroups $S_t = \exp^{tA}$ and $S(t)_k = \exp^{tB_k}$ respectively. We concentrate on a finite number of independent, real Wiener processes $\beta_1(.),...,\beta_N(.)$ to simplify presentation, but generalizations to the case of infinitely many Wiener processes are possible. For the same reason we assume that $\xi = 0$ and $f \equiv 0$. We will need the following conditions.

Hypothesis 2.18.

1. *Operators B_1,..., B_N generate mutually commuting C_0-groups e^{B_1} ,..., e^{B_N} respectively.*
2. *For $k = 1, ..., N, D(B_k^2) \supset D(A)$ and $\bigcap_{k=1}^{N} D((B_k^*)^2)$ is dense in $\mathbb{H}$*
3. *The operator*
$$C = A - \frac{1}{2}\sum_{k=1}^{N} B_K^2, \quad D(C) = D(A),$$
is the infinitesimal generator of a C_0-semigroup $S_0(t) = \exp^{Ct}, t \geq 0$.

In order to solve (2.24), we define

$$U_t = \prod_{k=1}^{N} S_k(\beta_k(t)), v_t) = U_t^{-1}X_t, \quad t \in [0, T],$$

and introduce the equation

$$\begin{cases} v_t' = U_t^{-1}CU_tv_t, \\ v_0 = x, \end{cases} \tag{2.29}$$

Propositions 2.19. *Assume Hypothesis (2.18). If X is a strong solution to (2.24), then the process v satisfies (2.29). Conversely, if v is a predictable process whose trajectories are of class $\mathcal{C}^1$ and satisfies (2.29), then the process $X(.) = U(.)v(.)$ takes values in $D(C)$, $\mathbb{P}$-a.s. and it is a strong solution of (2.24).*

Proof. For fixed $\zeta \in \mathbb{H}$, we define

$$z_\zeta(t) = \prod_{i=1}^{N} S_i^*(-\beta_i(t))\zeta = (U^{-1})_t^*\zeta, \quad t \in [0, T].$$

We will show that if $\zeta \in \bigcap_{k=1}^{N} D(B_k^*)^2$, then

$$dz_\zeta = \frac{1}{2}\sum_{i=1}^{N}(B_i^*)^2 z_\zeta dt - \sum_{i=1}^{N} B_i^* z_\zeta d\beta_i. \tag{2.30}$$

To do so let us fix $\eta \in \mathbb{H}$ and apply $It\hat{o}'s$ formula to the process

$$\langle z_\zeta(t), \eta\rangle = \psi\left(\beta_1(t).....\beta_N(t)\right), \quad t \in [0,T],$$

where

$$\psi\left(x_1.....x_N\right) = \left\langle \zeta, \prod_{i=1}^{N} S_i(-x_i)\eta \right\rangle, \quad (x_1.....x_N) \in \mathbb{R}^N,$$

Let $\zeta \in \bigcap_{k=1}^{N} D(B_k^*)^2$. Since

$$\frac{\partial \psi}{\partial x_j} = -\left\langle B_j^* \prod_{i=1}^{N} S_i^*(-x_i)\zeta, \eta \right\rangle,$$

and

$$\frac{\partial^2 \psi}{\partial x_j^2} = -\left\langle (B_j^*)^2 \prod_{i=1}^{N} S_i^*(-x_i)\zeta, \eta \right\rangle,$$

therefore

$$d\left\langle z_\zeta, \eta\right\rangle = -\sum_{i=1}^{N} \left\langle B_i^2 z_\zeta, \eta \right\rangle d\beta_i + \frac{1}{2}\left\langle \sum_{i=1}^{N} (B_i^2) z_\zeta, \eta \right\rangle dt,$$

and consequently (2.30) holds. Taking into account 1 and 2 of the hypothesis (2.24) and that

$$\langle v_t, \zeta\rangle = \langle X_t, z_\zeta(t)\rangle, \quad t \in [0,T].$$

one obtains (applying again $It\hat{o}'s$ formula)

$$\begin{aligned} d\left\langle v_t, \zeta\right\rangle &= \left\langle dX_t, dz_\zeta(t)\right\rangle + \left\langle X_t, dz_\zeta(t)\right\rangle - \sum_{i=1}^{N} \left\langle B_i X, B_i^* z_\zeta(t)\right\rangle dt \\ &= \left\langle \left(A - \frac{1}{2}\sum_{i=1}^{N} B_i^2\right) X, z_\zeta \right\rangle dt \\ &= \left\langle U_t^{-1} C U_t v_t, \zeta \right\rangle. \end{aligned}$$

Now, we're assuming that Hypothesis (2.18) holds and formulate some condition implying solvability of the equation (2.29). We set

$$\begin{cases} D(C_t) = x \in \mathbb{H}; U_t x \in D(C), \\ C_t x = U_{-t} C U_t x, \forall x \in D(C_t). \end{cases} \tag{2.31}$$

Chapter 3

Backward stochastic differential equations in infinite dimension

This Chapter, we present the research work on stochastic optimal control for SDEs in infinite dimensions. We study BSDEs and Forward-backward stochastic differential equations (FBSDEs, in short).

3.1 Backward stochastic differential equations

A backward stochastic differential equation on a bounded interval $[0,T]$ is an equation of the form

$$\begin{cases} dY_t = Z_t dW_t - BY_t dt - f(t, Y_t, Z_t)dt, \\ Y_T = \xi. \end{cases} \tag{3.1}$$

Here W is a cylindrical Wiener process in a Hilbert space $\mathbb{U}$, with completed natural filtration denoted $(\mathscr{F}_t)_{t\geq 0}$. The unknown process is an $\mathscr{F}_t$-progressive pair (Y,Z), where Y takes values in another Hilbert space $\mathbb{U}$ and Z in the space $L_2(\mathbb{U},\mathbb{H})$ of Hilbert-Schmidt operators from $\mathbb{U}$ to $\mathbb{H}$. An $\mathscr{F}_T$-measurable terminal condition ξ is given for the process Y and the equation is solved backwards in time.
The coefficient f is called the generator, and for given $y \in \mathbb{U}, z \in L_2(\mathbb{U},\mathbb{H})$, the process $t \to f(t,y,z)$ is assumed to be $(\mathscr{F}_t)$-progressive. B denotes the infinitesimal generator of a strongly continuous semi group in $\mathbb{H}$. The occurrence of the stochastic

differential $Z_t dW_t$ and the addition of another unknown process Z makes the problem well posed in the class of progressive processes, under appropriate conditions. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with the natural filtration of the Brownian motion $(\mathscr{F}_t)_{t\geq 0}$, and ξ is a $\mathscr{F}_T$-measurable random variable. We aim to solve the following equation:

$$-\frac{dY_t}{dt} = g(Y_t), \quad t \in [0,T], \text{with} \quad Y_T = \xi$$

To solve a BSDE with a given terminal condition, it is necessary to find a couple (Y_t, Z_t) of processes adapted with respect to the filtration of the Brownian motion. For more details, see the work of E. Pardoux and S. Peng [25], [19]. The backward stochastic differential equations have the following form:

$$-dY_t = g(t, Y_t, Z_t)dt - Z_t dB_t, \quad Y_T = \xi, \tag{3.2}$$

equivalently

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + \int_t^T Z_s dB_s, \quad 0 \leq t \leq T. \tag{3.3}$$

The process $\{g(t,y,z)\}_{0\leq t\leq T}$ progressively measurable. Where the terminal condition ξ is $\mathscr{F}_T$ -measurable, square integrable random variable and $(B_t)_{t\geq 0}$ are d-dimensional Brownian motion process.

We consider the following notations:

1. $S^2(\mathbb{R}^K)$ is the space of progressively measurable processes Y such that

   $$\|Y\|_{S^2}^2 = \mathbb{E}(\sup_{0\leq t\leq T} |Y_t|^2) < \infty,$$

   and $S_c^2(\mathbb{R}_K)$ denotes the subspace of the continuous process. Two indistinguishable processes will always be identified, and we will keep the same ratings for the quotient spaces.

2. $\mathcal{M}^2(\mathbb{R}^{K\times d})$ is the space of progressively measurable processes Z such that

   $$\|Z\|_{\mathcal{M}^2}^2 = \mathbb{E}\left(\int_0^T \|Z_t\|^2 dt\right) < \infty,$$

   where, for $z \in \mathbb{R}^{K\times d}$, $\|z\|^2 = Tr(zz^*)$, and $M^2(\mathbb{R}^{K\times d})$ denotes the $\mathcal{M}^2(\mathbb{R}^{K\times d})$ equivalent classes.

Definitions 3.1. A solution of the equation (3.3) is the couple process $\{(Y_t, Z_t)\}_{0\leq t\leq T}$ that verify

- Y and Z are progressively measurable processes with values in $\mathbb{R}^K$ and $\mathbb{R}^{K\times d}$ respectively,
- $\mathbb{P}.a.s$ $\int_0^T \left\{|g(r, Y_r, Z_r)| + \|Z_r\|^2\right\} dr < \infty$,
- $\mathbb{P}.a.s$ we have $Y_t = \xi + \int_t^T |g(r, Y_r, Z_r)|\, dr - \int_t^T Z_r dB_r, \quad 0 \leq t \leq T.$

ξ is a random variable, $\mathcal{F}_T$-measurable with values in $\mathbb{R}^k$.
Consider the following assumptions:

1. Lipschitz condition: There exist $\lambda > 0$ such that, for any t, y_1, z_1, y_2, z_2 we have:
$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq \lambda\left(|y_1 - y_2| + \|z_1 - z_2\|\right).$$
2. Integrability condition:
$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(r, 0, 0)|^2\, dr\right] < \infty.$$

The role of Z: The role of the process Z is to make the trajectories of the process Y adapted.

Theorem 3.1. *[25] According to assumptions (1) and (2), the BSDE (3.3) has a unique solution* (Y, Z), *with* $Z \in \mathcal{M}^2$.

3.2 The optimal control problem

The idea here is to connect stochastic optimal control with BSDE by applying the non-linear Fymen-Kac formula. First, we write the Hamilton Jacobie Belmann equation corresponding to our stochastic optimal control, then we remove the infimum by applying the convexity propriety of the running cost with respect to the control variable, and then we end up with a PDE, which is a semi-linear second-order PDEs that has a link to a BSDEs using the non-linear Fymen-Kac formula (see [33]).

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathscr{F}_t)_{t>0}, \mathbb{P})$ satisfying the usual conditions on which an m-dimensional standard Brownian motion W is defined, consider the following controlled stochastic differential equation in finite dimensions

$$\begin{cases} dX_s = b(s, X_s, u_s)ds + \sigma(s, X_s, u_s)dW_s, \\ X_0 = x \in \mathbb{R}^n, \end{cases} \tag{3.4}$$

with, $b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$, U a given separate metric space and fix $T \in (0, \infty)$. The function $u(.)$ is called the control representing the action, decision, or policy of decision makers. This non-participative restriction in mathematical terms can be represented by $u(.)$ which must be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted. That is, $u(.)$ control is taken as

$$U[0,T] = \Big\{u : [0,T] \times \Omega \to U | u(.) \ and \ \{\mathcal{F}_t\}_{t\geq 0} - adapted.\Big\}$$

The following introduces cost function :

$$J(t, x; u) = \mathbb{E}\left[\int_t^T f(s, X_s, u_s)ds + g(X_T)\right], \tag{3.5}$$

such that $f(., X(.), u(.)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $g(X_T) \in L^1_{\mathcal{F}}(\Omega, \mathbb{R})$, $X(.)$ is the unique solution of the equation (3.4).

All admissible controls are listed $\mathcal{U} \in [0,T]$. Therefore our strong formulation of stochastic optimal control problem is the strong formulation problem Minimize the formula (3.5) on $\mathcal{U} \in [0,T]$. The value function for this problem is defined as usual as follows:

$$v(t, x) = \inf_{u\in\mathcal{U}[0,T]} J(t, x; u). \tag{3.6}$$

v verify partial differential equation with stop Hamilton, Jacobi Bellman which we will define in the next

3.2.1 Hamilton-Jacobi-Bellman equation

To sum up, we have the value function v for our stochastic control problem, which solves this partial differential equation, (PDE in short).

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \inf\limits_{\beta \in \mathbb{R}^n} \{b(t,x,\beta).\nabla v(t,x) + f(t,x,\beta)\} + \dfrac{1}{2}\left[\sigma\sigma^T(t,x,\beta).\Delta v(t,x)\right] = 0, \\ v(T,x) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3.7}$$

with, $b : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^n, \sigma : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^{n\times m}$, and $f(.,x,\beta) \in L^1_{\mathcal{F}}(0,T;\mathbb{R})$ This semi-linear parabolic PDE is the HJB equation.

Remark 3.1. We can derive from the HJB a PDE without infrimum if we have convexity with respect to β.

3.2.2 Optimal control in infinite dimension

We present a class of infinite dimensional optimal control problems. Let $\mathbb{H}$ and $\mathbb{U}$ be two separable Hilbert spaces, $Q \in \mathcal{L}(\mathbb{U}), \mathbb{U}_0 := Q^{1/2}(\mathbb{U})$. Assume that $W_Q = W_Q(t), s \leq t \leq T$ is an $\mathbb{U}$-valued $\mathscr{F}^t_s$-Q-Wiener process, as well as indicate by $\mathcal{L}_2(\mathbb{U}_0, \mathbb{H})$ the Hilbert space of the Hilbert-Schmidt operators from $\mathbb{U}_0$ to $\mathbb{H}$.
We denote by $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ the generator of the C_0-semigroup e^{tA}. The operator A^* denotes the adjoint of A. Recall that $D(A)$ and $D(A^*)$ are Banach spaces when endowed with the graph norm. Let Λ be a Polish space (see [5]).
Given an adapted process $\beta : [s,T] \times \Omega \to \Lambda$. We consider the state equation

$$\begin{cases} dX_t = (AX_t + b(t,X_t,\beta_t))\,dt + \sigma(t,X_t)dW_Q(t) \\ X_s = x, \end{cases} \tag{3.8}$$

in which $b : [0,T]\times\mathbb{H}\times\Lambda \to \mathbb{H}$ is a continuous function, and $\sigma : [0,T]\times\mathbb{H} \to \mathcal{L}_2(\mathbb{U}_0, \mathbb{H})$ is continuous, and for $C > 0$, we have

$$\|b(s,x,\beta) - b(s,y,\beta)\| \leq C\,\|x-y\|,$$
$$\|b(s,x,\beta)\| \leq C(1+\|x\|),$$

and

$$\|\sigma(s,x)-\sigma(s,y)\|_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H})} \leq C\|x-y\|,$$
$$\|\sigma(s,x)\|_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H})} \leq C(1+\|x\|),$$

for all $x, y \in \mathbb{H}, s \in [0,T]$, *and* $\beta \in \Lambda$. The solution of (3.8) is understood in the mild sense: an $\mathbb{H}$-valued adapted process $X(.)$ is a solution if

$$\mathbb{P}\left\{\int_s^T \left(\|X_r\| + \|b(r,X_r,\beta_r)\| + \|\sigma(r,X_r)\|^2_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H})}\right) dr < +\infty\right\} = 1$$

and

$$X_t = e^{(t-s)A} + \int_s^t e^{(t-s)A} b(r,x_r,a_r)dr + \int_s^t e^{(t-s)A}\sigma(r,x_r)dW_Q(r).$$

Let $l : [0,T]\times\mathbb{H}\times\Lambda \to \mathbb{R}$ be a measurable function and $g : \mathbb{H} \to \mathbb{R}$ is a continuous function.

We consider the class $\mathcal{U}_s$ of admissible controls constituted by the adapted processes $\beta : [s,T]\times\Omega \to \Lambda$ such that $(r,\omega) \to l(r, X(r,s,x,\beta(.)),\beta_r) + g(X(T,s,x,\beta(.)))$ is dr $\otimes$ $d\mathbb{P}$-is quasi-integrable. This means that, either its positive or negative part are integrable.

We consider the problem of minimizing, for $\beta(.) \in \mathcal{U}_s$, the cost functional

$$J(s,x;\beta(.)) = \mathbb{E}\left[\int_s^T l(r,X(r;s,x,\beta(.)),\beta(r))dr + g(X(T;s,x,\beta(.)))\right]. \tag{3.9}$$

The value function of this problem is defined, as usual, as

$$v(s,x) = \inf_{\beta(.)\in\mathcal{U}_s} J(s,x;\beta(.)).$$

As usual we say that the control $\beta^*(.) \in \mathcal{U}_s$ is optimal at (s,x) if $\beta^*(.)$ minimizes (3.9) among the controls in $\mathcal{U}_s$. if $J(s,x;\beta^*(.)) = v(s,x)$.

The HJB equation associated with the minimization problem above is

$$\begin{cases}\partial_s v + \langle A^*\partial_x v, x\rangle + \dfrac{1}{2}Tr[\sigma(s,x)\sigma^*(s,x)\partial^2_{xx}v] + \inf\limits_{\beta\in\Lambda}\{\langle\partial_x v, b(s,x,\beta)\rangle + l(s,x,\beta)\} = 0,\\ v(T,x) = g(x).\end{cases} \tag{3.10}$$

In the above equation $\partial_x v$ (respectively $\partial^2_{xx} v$) in the first derivatives of v with respect to the variable x. Let $(s, x) \in [0, T] \times \mathbb{H}, \partial_x v(s, x)$ is identified with elements of $\mathbb{H}$. $\partial^2_{xx} v(s, x)$ which is element of $(\mathbb{H} \otimes_\pi \mathbb{H})^*$ is naturally associated with a symmetric bounded operator on $\mathbb{H}$.

Remark 3.2. The interest of such formula is to be able to give a solution of PDE in the form of expectation. Using Monte Carlo method to approximate the expectations, one can simulate the PDE solution.

3.3 Forward-backward stochastic differential equations

We use FBSDEs to provide a probabilistic formula for a quasi-linear PDE of the parabolic type. We need the definition of viscosity solutions.

Definitions 3.2. (viscosity solutions) [20]

Let $d \in \mathbb{N}, T \in (0, \infty)$, let $M \subseteq \mathbb{R}^d$ be a non-empty open set, and let $G : (0, T) \times M \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be degenerate elliptic, (see Definitions 2.3 and 2.4 in [20]). We say that u is a viscosity solution of

$$\frac{\partial u}{\partial t}(t, x) + G(t, x, u(t, x), \nabla_x u(t, x), (Hess_x u)(t, x)) = 0 \tag{3.11}$$

for $(t, x) \times (0, T) \times M$ if and only if

- We have that u is a viscosity subsolution of (3.11) for $(t, x) \times (0, T) \times M$ and
- We have that u is a viscosity supersolution of 3.11 for $(t, x) \times (0, T) \times M$ (For the definitions of viscosity subsolution and viscosity supersolution, see [20]).

Definitions 3.3. (Viscosity subsolutions) [20]
Let $d \in \mathbb{N}, T \in (0, \infty)$, let $M \subseteq \mathbb{R}^d$ be a non-empty open set, and let $G : (0, T) \times M \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be degenerate elliptic, and G is linear in the Hess. We say that u is a viscosity solution of

$$\frac{\partial u}{\partial t}(t, x) + G(t, x, u(t, x), \nabla_x u(t, x), (Hess_x u)(t, x)) \geq 0 \tag{3.12}$$

for $(t, x) \times (0, T) \times M$ (we say that u is a viscosity subsolution of the equation (3.11) if and only if there exists a set A such that

- we have that $(0, T) \times M \subseteq A$,
- we have that $u : A \to \mathbb{R}$ is an upper semi-continuous function from A to $\mathbb{R}$, and
- $t \in (0, T), x \in M, \phi \in \mathcal{C}^{1,2}((0, T) \times M, \mathbb{R})$ with $\phi(t, x) = u(t, x)$ and $\phi \geq u$ that the equation (3.12).

Definitions 3.4. (Viscosity supersolutions) [20]
Let $d \in \mathbb{N}, T \in (0, \infty)$, let $M \subseteq \mathbb{R}^d$ be a non-empty open set, and let $G : (0, T) \times M \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be degenerate elliptic, and G is linear in the Hess. We say that u is a viscosity solution of

$$\frac{\partial u}{\partial t}(t, x) + G(t, x, u(t, x), \nabla_x u(t, x), (Hess_x u)(t, x)) \leq 0 \tag{3.13}$$

for $(t, x) \times (0, T) \times M$ (we say that u is a viscosity subsolution of the equation (3.11) if and only if there exists a set A such that

- we have that $(0, T) \times M \subseteq A$,
- we have that $u : A \to \mathbb{R}$ is a lower semi-continuous function from A to $\mathbb{R}$, and
- $t \in (0, T), x \in M, \phi \in \mathcal{C}^{1,2}((0, T) \times M, \mathbb{R})$ with $\phi(t, x) = u(t, x)$ and $\phi \geq u$ that the equation (3.13).

Hypothesis 3.2. *[26] Let the following conditions:*

1. *There exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that for all $t, x, x_1, x_2, y, y_1, y_2, z$ and*

$$\langle f(t, x_1, y, z) - f(t, x_2, y, z), x_1 - x_2 \rangle \leq \lambda_1 |x_1 - x_2|^2 ,$$

$$\langle g(t, x, y_1, z) - g(t, x, y_2, z), y_1 - y_2 \rangle \leq \lambda_2 |y_1 - y_2|^2 .$$

2. *The processes* $f(., x, y, z), g(., x, y, z)$ *and* $\sigma(., x, y, z)$ *are* $\mathscr{F}_t$*-adapted, and the random variable* $g(x)$ *is* $\mathscr{F}_t$*-measurable,* (x, y, z)*. Moreover, the following holds:*

$$\mathbb{E}\int_0^T |f(s,0,0,0)|^2\,ds + \mathbb{E}\int_0^T |g(s,0,0,0)|^2\,ds$$
$$+\mathbb{E}\int_0^T \|\sigma(s,0,0,0)\|^2\,ds + \mathbb{E}\,|h(0)|^2 < \infty$$

For every $(t,x) \in [0,T]\times\mathbb{R}^n$, let $\left\{(X^{t,x}(s), Y^{t,x}(s), Z^{t,x}(s)), t \le s \le T\right\}$ indicate the unique solution.

$$\begin{cases} X^{t,x}(s) = x + \displaystyle\int_t^s f(r, X^{t,x}(r))dr + \int_t^s \sigma(r, X^{t,x}(r))dB_r, \\ \\ Y^{t,x}(s) = h(X^{t,x}(T)) + \displaystyle\int_s^T g(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r))dr - \int_s^T Z^{t,x}(r)dB_r. \end{cases} \tag{3.14}$$

We assume that the functions f, g, σ, h are deterministic. We will see in this section that the function $u(t,x) := Y^{t,x}(t), (t,x) \in [0,T]\times\mathbb{R}^n$, is a viscosity solution of the following backward quasilinear second-order parabolic (PDE):

$$\begin{cases} \dfrac{1}{2}Tr(\sigma(t,x)\times\sigma^T(t,x))\Delta u(t,x) + \langle f(t,x), \nabla u(t,x)\rangle \\ \quad +g(t,x,u(t,x),\sigma(t,x)\nabla u(t,x)) = 0, \\ \\ u(T,x) = h(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3.15}$$

Theorem 3.3. *[26] Assume that the functions* f, σ, g, h *are deterministic, globally continuous, and that they satisfy the following conditions (1) in 3.2* and *The function* h *is uniformly Lipschitz continuous in* x*. That is, there exists* k *such that for all* x_1, x_2*, we have* $|h(x_1) - h(x_2)| \le k\,|x_1 - x_2|$.

Then, the function u *defined by* $u(t,x) := Y_t^{t,x}, (t,x) \in [0,T]\times\mathbb{R}^n$*, is continuous and it is a viscosity solution of the PDE (3.15).*

3.3.1 Least-squares Monte Carlo

In this section we study numerical analysis of FBSDE, (see [22], [31]) that is solved by a least squares Monte Carlo algorithm.

The least-squares Monte Carlo technique is based on the Euler approximation, a method studied in [17], especially

$$\begin{cases} \hat{X}_{n+1} = \hat{X}_n + \Delta t b(\hat{X}_n, t_n) + \sqrt{\Delta t}\sigma(\hat{X}_n)\xi_{n+1}, \quad X_0 = x, \\ \hat{Y}_{n+1} = \hat{Y}_n + \Delta t h(\hat{X}_n, \hat{Y}_n, \hat{Z}_n) + \sqrt{\Delta t}(\hat{Z}_n)\xi_{n+1}, \quad Y_T = g(X_T), \end{cases} \tag{3.16}$$

where $(\hat{X}_n, \hat{Y}_n, \hat{Z}_n)$ represents the joint process's numerical discretization (X_s, Y_s, Z_s), and $(\xi_i)_{i\geq 1}$ is an i.i.d. sequence of normalised Gaussian random variables.
By definition, the continuous-time process (X_s, Y_s, Z_s) is adapted to the filtration generated by $(B_t)_{0\leq t\leq s}$. For the discretised process, this implies

$$\hat{Y}_n = \mathbb{E}\left[\hat{Y}_n|\mathcal{F}_n\right] = \mathbb{E}\left[\hat{Y}_{n+1} + \Delta t h(\hat{X}_n, \hat{Y}_n, \hat{Z}_n)|\mathcal{F}_n\right],$$

using that $\hat{Z}_n$ is independent of ξ_{n+1}, and $\mathcal{F}_n = \sigma\left(\left\{\hat{B}_k : 0 \leq k \leq n\right\}\right)$ be the σ-algebra generated by the discrete Brownian motion $\hat{B}_k := \sqrt{\Delta t}\sum_{i\leq n}\xi_i$.
In order to compute $\hat{Y}_n$ from $\hat{Y}_{n+1}$, It is simple to swap out $\hat{Y}_n, \hat{Z}_n$ on the right hand side by $\hat{Y}_{n+1}, \hat{Z}_{n+1}$, so that we end up with the fully explicit time stepping scheme

$$\hat{Y}_n = \mathbb{E}\left[\hat{Y}_{n+1} + \Delta t h(\hat{X}_n, \hat{Y}_{n+1}, \hat{Z}_{n+1})|\mathcal{F}_n\right]. \tag{3.17}$$

The next section deals with how to calculate conditional expectations with regard to $\mathcal{F}_n$. Recall that the conditional expectation is the best estimate in the L^2 space:

$$\mathbb{E}\left[S|\mathcal{F}_n\right] = \underset{Y\in L^2, \mathcal{F}_n-measurable}{argmin} \mathbb{E}\left[|Y - S|^2\right],$$

Here measurability with respect to $\mathcal{F}_n$ means that $(\hat{Y}_n, \hat{Z}_n)$ can be represented as functions of $\hat{X}_n$. In view of the equation (3.17) and $V_K(x,t) = \sum_{k=1}^{K}\alpha_k(t)\phi_k(x)$, in which $\alpha_k \in \mathbb{R}$, and continuously differentiable basis functions ϕ_k, this suggests the approximation scheme

$$\hat{Y}_n \approx \underset{Y=Y(\hat{X}_n)}{argmin} \sum_{m=1}^{M}\left|Y - \hat{Y}_{n+1}^{(m)} - \Delta t h(\hat{X}_n, \hat{Y}_{n+1}^{(m)}, \hat{Z}_{n+1}^{(m)})|\mathcal{F}_n\right|^2,$$

where the data at time t_{n+1} is delivered in the form of M independent realisations of the forward process, $\hat{X}_n^{(m)}$ $n, m = 1, ..., M$, the resulting values for $\hat{Y}_{n+1}$,

$$\hat{Y}_{n+1}^{(m)} = \sum_{k=1}^{K} \alpha_k(t_{n+1})\phi_k(\hat{X}_{n+1}^{(m)}),$$

and

$$\hat{Z}_{n+1}^{(m)} = \sigma(\hat{X}_{n+1}^{(m)})^T \sum_{k=1}^{K} \alpha_k(t_{n+1})\nabla\phi_k(\hat{X}_{n+1}^{(m)}).$$

At time $T := N\Delta t$, the data are determined by the terminal cost:

$$Y_N^{(m)} = g(X_N^{(m)}), \quad Z_N^{(m)} = \sigma(X_N^{(m)})^T \nabla(X_N^{(m)})$$

We call $\hat{\alpha} = (\hat{\alpha}_1, ..., \hat{\alpha}_k)$ the vector of the unknowns,

$$\hat{\alpha}(t_n) = \underset{\alpha \in \mathbb{R}^k}{argmin} \left\| A_n\alpha - b_n \right\|^2, \tag{3.18}$$

with coefficients

$$A_n = (\phi_k(\hat{X}_n^{(m)})), \tag{3.19}$$

and

$$b_n = (\hat{Y}_{n+1}^{(m)} + \Delta t h(\hat{X}_n^{(m)}, \hat{Y}_{n+1}^{(m)}, \hat{Z}_{n+1}^{(m)}))_{m=1,...,M}. \tag{3.20}$$

By considering the coefficient matrix $A_n \in \mathbb{R}^{M \times K}, K \leq M$ defined by (3.19) has maximum rank K, then the solution to (3.18) is given by

$$\hat{\alpha}(t_n) = \left(A_n^T A_n\right)^{-1} A_n^T b_n.$$

We may create the numerical simulations component using the Least-squares Monte Carlo technique.

Algorithm 1 Least-squares Monte Carlo

- Give M the number of realisations, N the number of subdivision of t, and K is the number basis.
- Set $\Delta t = T/N$.
- Set the initial condition $x \in \mathbb{R}^d$.
- Generate M independent realisations $\hat{X}^{(1)}, ..., \hat{X}^{(M)}$ for the SDE with the initial condition $X_0 = x$, here we simulate three additional realisations.
- Selecting radial basis as Gaussian $\{\phi_k \in C^1(\mathbb{R}^d, \mathbb{R})\}_{k=1,...K}$.
- Set the terminal condition of the BSDE by: $\hat{Y}_N^{(m)} = g(\hat{X}_N^{(m)})$, $\hat{Z}_N^{(m)} = \sigma(\hat{X}_N^{(m)})^T \nabla g(\hat{X}_N^{(m)})$.
- Assemble linear system $A_n \hat{\alpha}(t_n) = b_n$ according to (3.18)-(3.20). Calculate $\hat{Y}_n^{(m)}$ and $\hat{Z}_n^{(m)}$ based on
$\hat{Y}_n^{(m)} = \sum_{k=1}^{K} \alpha_k(t_n)\phi_k(\hat{X}_n^{(m)})$, $\quad \hat{Z}_n^{(m)} = \sigma(\hat{X}_n^{(m)})^T \sum_{k=1}^{K} \alpha_k(t_n)\nabla\phi_k(\hat{X}_n^{(k)})$.

Exemple 3.1. Let stochastic optimal control in infinite dimensions.

$$\begin{cases} dX_s = (AX_s + Bu_s)ds + BdW_s, \\ X_t = x \in \mathbb{H}. \end{cases} \tag{3.21}$$

A and B are linear operators, where $(-A)$ is specifically a Laplace operator. $u(.)$ is a stochastic process that represents the control.

The problem of minimising, the cost functional following

$$J(t, x; u) = \mathbb{E}[\int_t^T (X_s^T A X_s + \frac{1}{2} \|u\|_s^2)ds + g(X_T)], \tag{3.22}$$

such that u is a stochastic process and $g(X_T) = X_T^T X_T$ in Hilbert spaces.

We make the projection in Hilbert space, with the finite dimension defined by its biases function $(\phi_j)_{j=1}^K$ can be defined as follows:

$$\phi_j(X) = \frac{1}{\sqrt{2\sigma^2\pi}} \times exp(-\frac{1}{2\sigma}(X - X_j)^T \times (X - X_j)),$$

in which X_j is an additional realisation of the forward.
$\hat{B}_n$ is a vector, and we define a matrix $\hat{A}_n$ following

$$\hat{A}_n = \frac{1}{h}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

than the stochastic optimal control became

$$\begin{cases} d\hat{X}_s = (\hat{A}_n\hat{X}_s + \hat{B}_n\hat{u}_s)ds + \hat{B}_n dW_s, \\ \hat{X}_t = \hat{x} \in \mathbb{R}^n. \end{cases} \tag{3.23}$$

We consider the problem of minimising, the cost functional

$$\hat{J}_n(t,\hat{x};\hat{u}) = \mathbb{E}[\int_t^T (\hat{X}_s^T\hat{A}_n\hat{X}_s + \frac{1}{2}\|\hat{u}\|_s^2)ds + g(\hat{X}_T)], \tag{3.24}$$

such that $\hat{u}$ is a stochastic process and $g(\hat{X}_T) = \hat{X}_T^T\hat{X}_T$ in $\mathbb{R}$.
The value function for this problem is defined as usual as follows:

$$\hat{v}(t,\hat{x}) = \inf_{\hat{u}(.)\in\mathcal{U}_s} \hat{J}_n(t,\hat{x};\hat{u}),$$

we say that the control $\hat{u}(.) \in \mathcal{U}_s$ is optimal at (t,x) if $\hat{u}(.)$ minimizes (3.24) one of the controls in $\mathcal{U}_s$.
We can write the HJB equation.

$$\begin{cases} \frac{\partial\hat{v}}{\partial t}(t,\hat{x}) + \inf_{\alpha\in\mathcal{A}}\left\{(\hat{A}_n\hat{x} + \hat{B}_n\alpha)\nabla\hat{v}(t,\hat{x}) + \frac{1}{2}Tr(\hat{B}_n\hat{B}_n^T\Delta\hat{v}(t,\hat{x})) + \frac{1}{2}\alpha^2 + \hat{x}^T\hat{x}\right\} = 0, \\ \hat{v}(T,\hat{x}) = g(\hat{x}). \end{cases} \tag{3.25}$$

The first derivative of $\hat{v}$ with respect to the x variable in the aforementioned equation is denoted by the symbol ∂v. We take the following equation:

$$\left\{(\hat{A}_n\hat{x} + \hat{B}_n\alpha)\nabla\hat{v}(t,\hat{x}) + \frac{1}{2}Tr(\hat{B}_n\hat{B}_n^T\Delta\hat{v}(t,\hat{x})) + \frac{1}{2}\alpha^2 + \hat{x}^T\hat{x}\right\},$$

and we derive this equation with respect to α. After derivation we find $\hat{B}_n\nabla\hat{v}(t,\hat{x})+\alpha$ and we write $\hat{B}_n\nabla\hat{v}(t,\hat{x})+\alpha=0$, Then we take out α and substitute it into the equation (3.25), then we get the following PDE

$$\frac{\partial\hat{v}}{\partial t}(t,\hat{x}) + [\hat{A}_n\hat{x}\nabla\hat{v}(t,\hat{x}) + \frac{1}{2}\hat{B}_n^2\nabla^2\hat{v}(t,\hat{x}) + \frac{1}{2}Tr(\hat{B}_n\hat{B}_n^T\Delta\hat{v}(t,\hat{x})) + \hat{x}^T\hat{x}].$$

Finally, we write the following FBSDE

$$\begin{cases} dY_t = -(\frac{1}{2}Z_s^2 + X_s^T\hat{A}_nX_s)dt + Z_sdW_s, \quad X_T = g(X_T), \\ dX_t = \hat{A}_nX_tdt + \hat{B}_ndW_t, \quad X_t = X_0. \end{cases} \tag{3.26}$$

We got a nonlinear BSDE, we transform it to linear one using the following transformation. Assume that, $K_t = e^{\alpha Y_t}$, and we write the following equation using the It$\hat{o}$ Formula

$$dK_t = \alpha e^{\alpha Y_t}dY_t + \frac{1}{2}\alpha^2e^{\alpha Y_t}\tilde{Z}_t^2dt, \tag{3.27}$$

We substitute dY_t into the equation (3.27), we obtain

$$dK_t = \alpha e^{\alpha Y_t}[-(\frac{1}{2}Z_t^2 + X_t^T\hat{A}_nX_t)dt + Z_tdW_t] + \frac{1}{2}\alpha^2\tilde{Z}_t^2e^{\alpha Y_t}dt, \tag{3.28}$$

Let's take $\alpha = 1$. It becomes (3.28) as follows:

$$dK_t = K_t(-X_t^TAX_tdt + Z_tdW_t). \tag{3.29}$$

3.3.2 Numerical simulations

We solve this example using the algorithm 1 by taking the time step equals 10^{-2}, and the number of realisations $M = 1000$. We put the number of basis equal to three in both examples.

We consider the following SDE:

$$dX_t = a(X_t,t)dt + b(X_t,t)dW_t.$$

We consider the forward equation by Euler Maruyama of the above SDE

$$Y_{n+1} = Y_n + a(Y_n, t_n)\,\Delta t + b(Y_n, t_n)\,\Delta W_n,$$

Solve the backward equation

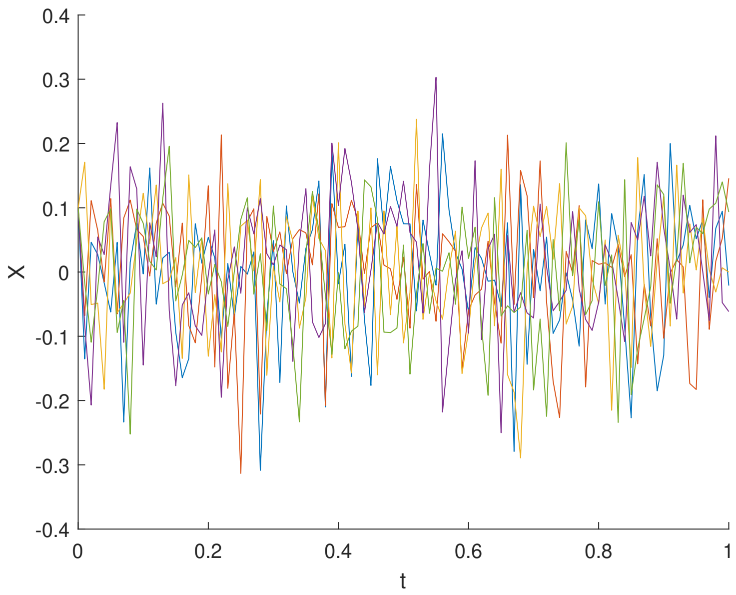

Figure 3.1: Trajectories of the SDE (3.26).

- Define the terminal condition $Y_T = g(X_T)$.
- Define the basis field as Gaussian density centred in extra trajectories.
- Find Y_{t_i} and Z_{t_i} until t_i for $i = 2$.
- We put $Y_1 = mean(Y_2(:))$ and $Z_1 = mean(Z_2(:))$.

The following are the trajectories of BSDE (3.29)

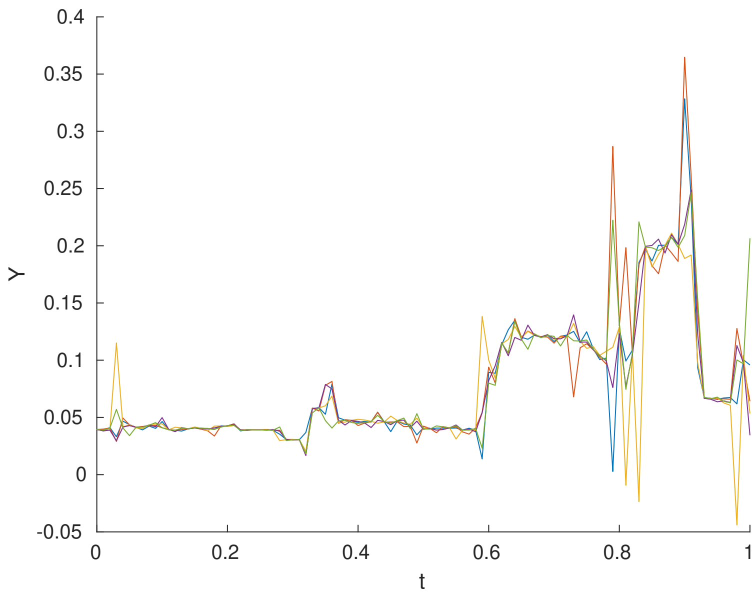

Figure 3.2: Trajectories of the BSDE (3.29).

Remark 3.3. In figure (3.3.2) we plot seven realisations. We compute a gradient. of the terminal condition manual.

Exemple 3.2. We consider now the following the FBSDE in infinite dimension, and we solve the backward equation through simulation.

$$\begin{cases} dY_t = (Y_t \left\langle X_t^2, X_t^4 \right\rangle + a \times Z_t)dt + Z_t dW_t, \quad X_T = g(X_T) \\ dX_t = AX_t dt + BW_t, \quad X_0 = x. \end{cases} \tag{3.30}$$

In this example, we have no linearity in the backward equation for the variable X,

but this doesn't change the linearity of the backward equation, which is that the backward equation in 3.31 is linear in Y and Z. We put a is a constant and B_n is a vector. From [14] we get the projection of the operator A on the finite subspace with dimension n as follows:

$$A_n = \frac{1}{h}\begin{pmatrix} 7 & -0.5 & & 0 \\ -0.5 & 7 & \ddots & \\ & \ddots & \ddots & -0.5 \\ 0 & & -0.5 & 7 \end{pmatrix}$$

We make the projection on a subspace of the Hilbert space with the finite dimension, and we get FBSDE in the finite dimension following

$$\begin{cases} d\hat{Y}_t = (\hat{Y}_t \left\langle \hat{X}_t^2, \hat{X}_t^4 \right\rangle + a \times \hat{Z}_t)dt + \hat{Z}_t dW_t, \quad \hat{X}_T = g(\hat{X}_T) \\ d\hat{X}_t = A_n \hat{X}_t dt + B_n W_t \quad , \hat{X}_0 = \hat{x}. \end{cases} \tag{3.31}$$

In this example, we put that the time step equals 10^{-3}, and the number of realisations $M = 1500$, and we solve the equation (3.31) using the algorithm 1.

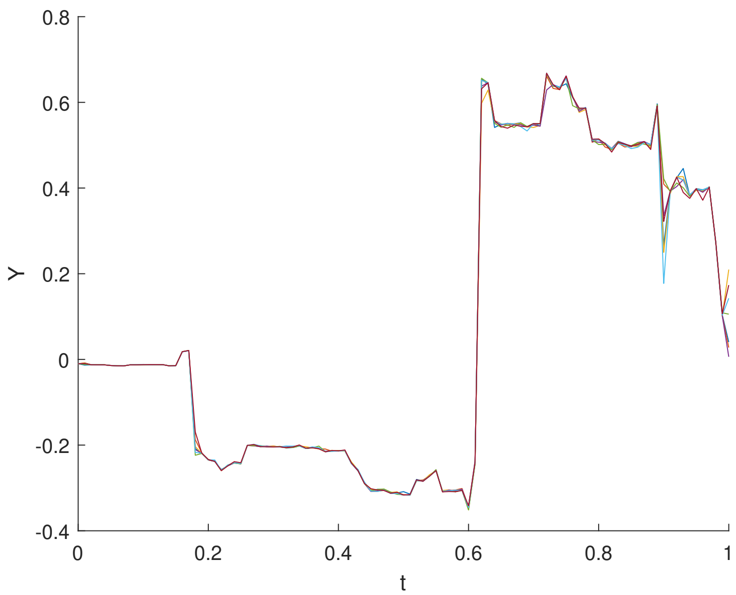

Figure 3.3: Trajectories of Y the solution of the BSDE (3.31).

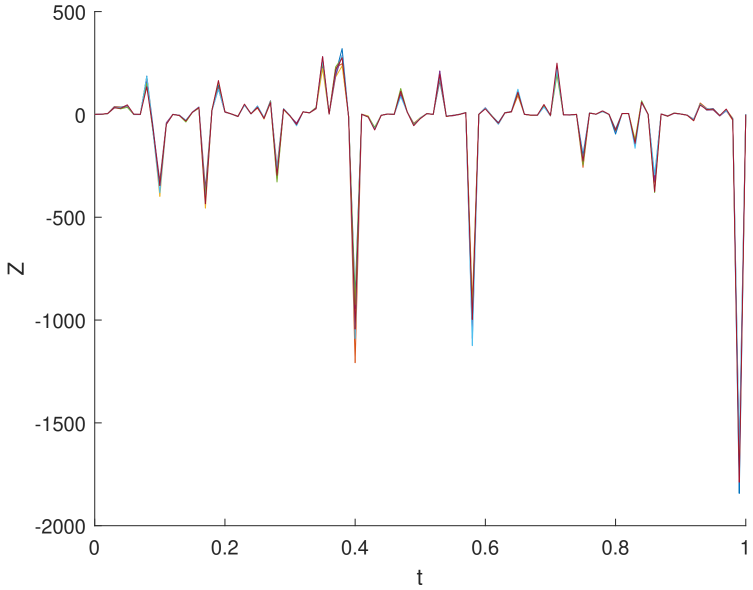

Figure 3.4: Trajectories of Z the solution of the BSDE (3.31).

For Z, we plot 7 realisations of the equation (3.31).

Remark 3.4. Z controls the trajectories of the backward to be adapted.

Conclusion

In this thesis, we have studied stochastic differential equations in infinite dimensions. We took as example the optimal control linear in infinite dimension and applied Hamilton-Jacobi-Bellman equations to get the forward-backward stochastic differential equation, which can be solved quite efficiently using the least squares Monte Carlo method. We have taken two examples of this, and we discussed the numerical simulation of the decoupled FBSDE.

In the end, we hope and predict that research on this subject will be active and promising since there are still different questions without any accurate answers. For example, take the nonlinear case, also study the convergence of the space-time discretization.

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