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University of Saida Dr. Moulay Tahar.

Faculty of Sciences, Department of Mathematics

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Presented by

Manel Khaoula ADDAD [1]

Supervised by

Pr. Abdeljebbar **K**andouci and **Dr. O**mar **K**ebiri

Theme:

On the stochastic partial differential equations with Lévy noise

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[1] e-mail: mnladdad@gmail.com

Dedication

I would first like to thank God for giving me the strength and courage to do this work.

To my generous parents. To whom I owe all my gratitude. No words could possibly be poetic sufficient to convey what you deserve.

To those who encouraged me and raise my spirits, my sisters, my brother, my cousins, and my dear friends Asmaa, Boutheina, Khaoula, and Ferial

To everyone who supported me and drew hope in every step I took, I dedicate this work to you ...

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Introduction

Stochastic partial differential equations (SPDEs) constitute a powerful mathematical framework for modeling complex systems influenced by random fluctuations. The incorporation of stochastic terms into partial differential equations allows for the realistic representation of uncertainties and variability inherent in a wide range of phenomena. In recent years, considerable attention has been directed toward the analysis of SPDEs driven by Lévy noise. The last refers to a class of stochastic processes characterized by jumps of arbitrary magnitude and non-zero probability, enabling the modeling of extreme events and heavy-tailed distributions in various real-world systems.

The study of SPDEs driven by Lévy noise aims to provide a deeper understanding of the behavior and properties of these equations, as well as to develop analytical techniques and numerical methods to study their solutions. This class of equations presents unique challenges due to the complex nature of Lévy noise and its impact on the dynamics of the system. This thesis aims to provide a solid foundation for analyzing and characterizing SPDEs driven by Lévy noise and to explore their applications in different scientific domains.

Specifically, extensive research has been conducted on parabolic stochastic partial differential equations (SPDEs) driven by Gaussian white noise, mentioned in ([17], [6]) and for more recent work, see [5], [21] and references therein. The exploration of parabolic stochastic partial differential equations (SPDEs) driven by Poisson white noise is relatively less recognized, with initial investigations reported in reference [1]. Additionally, Mueller conducted a study on a heat equation driven by $\alpha-$stable Lévy noise in this context, as in [14]. It is worth noting that the formulation of a parabolic SPDE driven by Poisson random measure in [19] differs from the approach in [1].

Recently, the application of stochastic partial differential equations has expanded, leading to the discovery of new uses and implications in [20]. Notably, [12] and [9] provide further insights into this field.

The objective of this master thesis is to conduct a comprehensive analysis of SPDEs driven by Lévy noise. The research will encompass the investigation of the well-posedness, regularity, and long-time behavior of solutions to these equations. Additionally, numerical approximation methods will be explored to effectively compute and simulate the solutions. The analysis will be based on the integration of mathematical tools such as stochastic calculus and the theory of Lévy processes.

This work is presented in three chapters:

In chapter 1 we introduce the necessary mathematical foundations, including Lévy processes, and stochastic calculus. These tools will be essential for the subsequent analysis and understanding of SPDEs driven by Lévy noise.

In chapter 2 we focus on establishing the well-posedness and regularity properties of solutions to SPDEs driven by Lévy noise in finite-dimensional spaces and some examples of SPDE in infinite and finite dimensions. The existence, uniqueness, and regularity of results will be investigated under different types of SPDEs with Lévy noise in both dimensions. First, in the finite dimension, we will study the existence and uniqueness of SPDE driven by Lévy noise with two examples of heat and wave equations. Second, in the infinite dimension, we will discuss SPDEs with respect to a square-integrable Lévy martingale with a typical example, also we will work on SPDEs with respect to Lévy space-time white noise.

In chapter 3 we explore numerical analysis of SPDEs. First, we define the stochastic optimal control and explain the dynamic programming and the stochastic Hamilton-Jacobi-Bellman equation. After that, we will move to the derivation of the partial differential equation (PDE) corresponding to the stochastic optimal control, we study then numerical method for approximating solutions to SPDEs driven by Lévy noise in finite-dimensional spaces. The finite difference will be considered.

Contents

List Of Notations And Symbols

SPDEs	**S**tochastic **P**artial **D**ifferential **E**quations
SDE	**S**tochastic **D**ifferential **E**quations
PDE	**P**artial **D**ifferential **E**quations
ODE	**O**rdinary **D**ifferential **E**quations
RKHS	**R**eproducing **K**ernel **H**ilbert **S**pace
SOC	**S**tochastic **O**ptimal **C**ontrol
HJB	**H**amilton **J**acobi **B**ellman equation
$\overset{\mathrm{d}}{=}$	Equals by definition
$\mathcal{L}$	Second-order pseudo-differential operator with constant coefficients
L	Lévy white noise
$\mathcal{F}G_t$	Fourier transform of G_t
$\mathbb{E}$	Expected value
$\mathcal{X}_T$	Set of all adapted processes that are square-integrable
$\mathcal{F}_t$	Filtration
$UC_b(U)$	Space of uniformly continuous bounded functions
$C_b(U)$	Space of bounded continuous functions
$\mathcal{L}(X)$	Distribution of a random element X
$\hat{\pi}$	Compensated Poisson random measure
λ, μ, ν	Usually measures
M	Usually a martingale
$L(U, H)$	Space of bounded (i.e. continuous) linear operators
$L_{HS}(U, H)$	Space of Hilbert-Schmidt operators
$L_1^+(U)$	Space of nuclear symmetric positive-definite operators
ΔP	Increments of the Poisson process
ΔB	Increments of Brownian motion

Chapter 1

Generalities on Lévy processes

In general, Lévy processes are defined as functions of the Wiener and the Poisson processes.

1.1 Lévy processes

Definition 1.1. *Let $X = (X(t), t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that it has independent increments if for each $n \in \mathbb{N}$ and each $0 \leq t_1 \leq t_2 \leq ... < t_{n+1} < \infty$ the random variables $(X(t_{j+1}) - X(t_j), 1 \leq j \leq n)$ are independent and that it has stationary increments, we have*

$$(X(t_{j+1}) - X(t_j)) \overset{\mathrm{d}}{=} (X(t_{j+1} - t_j) - X(0)).$$

We say that X is a Lévy process if:

*(**L1**) X(0)=0 (a.s);*

*(**L2**) X has independent and stationary increments;*

*(**L3**) X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$*

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0.$$

Note that in the presence of (L1) and (L2), the third property (L3) is equivalent to the condition

$$\lim_{t \to 0} P(|X(t) > a|) = 0 \qquad \text{for all } a > 0.$$

Remark 1.1. *A stochastic process $L = (L(t), t > 0)$ taking values in E has independent increments if, for each $0 < t_0 < t_1 < ... < t_n$, the (E, ξ)-valued random variables*

$L(t_1) - L(t_0), L(t_2) - L(t_1), ..., L(t_n) - L(t_{n-1})$ are independent. If the law $\mathcal{L}(L(t) - L(s))$ of $L(t) - L(s)$ depends only on the difference $t - s$ then we say that L has stationary, or time-homogeneous, independent increments. if in addition, E is a Banach space, $L(0) = 0$, and the process L is stochastically continuous then L is called a Lévy process.

Lemma 1.1. *Let L be a Lévy process on a Banach space E and let μ_t be the law of the random variable $L(t)$. Then, denoting by $\mu * \nu$ the convolution of the measures μ and ν, we have*

1. *$\mu_0 = \delta_0$ and $\mu_{t+s} = \mu_t * \mu_s$ for all $t, s \geq 0$,*
2. *$\mu(x : \|x\| < r) \to 1$ as $t \to 0$ for every $r > 0$,*
3. *μ_t converges weakly to σ_0 as $t \to 0$.*

Theorem 1.1. *Every Lévy process has a càdlàg modification.*

Given a càdlàg process L we define the process of jumps of L by $\Delta L(t) := L(t) - L(t-), t \geq 0$. The following result of De Acosta (1980) is a special case of a more general theorem of Rosinski (1995).

Theorem 1.2. *[22] (De Acosta) Assume that $(L(t), t > 0)$ is a càdlàg Lévy process in a Banach space B with jumps bounded by a fixed number $c > 0$; that is, $|\Delta L(t)|_B < c$ for every $t > 0$. Then, for any $\beta > 0$ and $t > 0$,*

$$\mathbb{E}e^{\beta\|\Delta L(t)_B\|} < \infty. \tag{1.1}$$

1.2 Poisson processes

The Poisson processes $(N_t)_{t\geq 0}$ is a point process, which is defined as:

$$N_t = \sum_{n=0}^{\infty} \mathbb{1}_{(T_n \leq t)},$$

where $(T_n)_n$ is a sequence of random times such that the increments $T_{n+1} - T_n$ are independent with an exponential distribution.

Proposition 1.1. *[22] Assume that Z is a positive random variable such that, for all $t, s \geq 0$, $\mathbb{P}(Z > t + s/Z > t) = \mathbb{P}(Z > s)$. Then Z has an exponential distribution with parameter a, that is, there exists a constant $a > 0$ such that $\mathbb{P}(Z > t) = e^{-at}$ for $t > 0$.*

Lemma 1.2. *Let $\alpha \geq 0$, and let Z_n have a geometric distribution with parameter $p_n = \alpha/n$. and Let λ_n be the distribution of Z_n/n. Then (λ_n) converges weakly to an exponential distribution with parameter α.*

Lemma 1.3. *Assume that (X_n) is a sequence of independent random variables with Poisson distributions $P(a_n)$. Then $X = \sum_{n=1}^{\infty} X_n$ has Poisson distribution $\mathcal{P}_{a_n}$, with $a = \sum_{n=1}^{\infty} a_n$. Moreover, the Laplace transform of $\mathcal{P}(a)$ is equal to*

$$\sum_{k=0}^{\infty} e^{-rk} P(a)(k) = \sum_{k=0}^{\infty} e^{-rk} \frac{a^k}{k!} e^{-a} = exp\left\{a(e^{-r}-1)\right\}, \qquad r > 0,$$

if $a < \infty$ and 0 if $a = \infty$.

Definition 1.2. *A Poisson process with intensity a is a Lévy process $\Pi = (\Pi(t), t \geq 0)$ such that, for every $t \geq 0, \Pi(t)$ has the Poisson distribution $\mathcal{P}(at)$.*

The Poisson process's main characteristics and construction are given in the next proposition.

Proposition 1.2.

(i) Let (Z_n) be a sequence of independent exponentially distributed random variables with parameter a. Then the formula

$$\Pi(t) = \begin{cases} 0 & \text{if } t < Z_1 \\ k & \text{if } t \in [Z_1 + ... + Z_k, Z_1 + ... + Z_{k+1}], \end{cases} \tag{1.2}$$

defines a Poisson process with intensity a.

(ii) Conversely, given a Poisson process with intensity defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a sequence (Z_n) of independent random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ having an exponential distribution with parameter a such that formula (1.2) holds.

(iii) If Π is a Poisson process with intensity a then, for all $z \in \mathbb{C}$ and $t \geq 0$, we have

$$\mathbb{E}e^{z\Pi(t)} = exp\left\{at(e^z - 1)\right\}.$$

(iv) if Π is a Poisson process then it has only jumps of size 1, that is,

$$\mathbb{P}(\Delta\Pi(t) := \Pi(t) - \Pi(t-) \in \{0, 1\}) = 1, \quad t \geq 0. \tag{1.3}$$

Conversely, any $\mathbb{Z}_+$-valued Lévy process Π satisfying (1.3) is a Poisson process.

1.3 Wiener processes

Definition 1.3. *Let $q > 0$. A real-valued mean-zero Gaussian process $W = (W(t), t \geq 0)$ with continuous trajectories and covariance function*

$$\mathbb{E}W(t)W(s) = (t \wedge s)q,$$

is called a Wiener process with diffusion q. If the diffusion is equal to 1 then W is called standard.

Definition 1.4. *Assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space and that W is a Wiener process in $\mathbb{R}^d$ adapted to $(\mathcal{F}_t)$. Then W is a Wiener process with respect to $(\mathcal{F}_t)$ or an $(\mathcal{F}_t)-$Wiener process if, for all $t, h \geq 0, W(t+h) - W(t)$ is independent of $\mathcal{F}_t$.*

1.4 Compound Poisson processes in a Hilbert space

Definition 1.5. *Let ν be a finite measure on a Hilbert space U such that $\nu(\{0\}) = 0$. A compound Poisson process with the Lévy measure (also called the jump intensity measure) ν is a cádlág Lévy process L satisfying*

$$\mathbb{P}(L(t) \in \Gamma) = e^{-\nu(U)t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \nu^{*(\Gamma)}, \qquad \forall t \geq 0, \Gamma \in \mathcal{B}(U). \tag{1.4}$$

In the formula above, we use the convention that ν^0 is equal to the unit measure concentrated at 0, that is, $\nu^0 = \delta_0$.

The theorem that follows demonstrates how to build a compound Poisson process with given ν.

Theorem 1.3. *[22] Let ν be a finite measure supported on $U \setminus \{0\}$, and let $a = \nu(U)$.*

(i) Let $Z_1, Z_2, ...$ be independent random variables with identical distributions equal to $a^{-1}\nu$. In addition, let $(\Pi(t), t \geq 0)$ be a Poisson process with intensity a, independent of $Z_1, Z_2, ...$ Then

$$L(t) = \sum_{j=1}^{\Pi(t)} Z_j \tag{1.5}$$

is a compound Poisson process with jump intensity measure ν.

(ii) *Given a compound Poisson process L with jump intensity measure ν, one can find a sequence of independent random variables $Z_1, Z_2, ...$ with identical laws equal to $a^{-1}\nu$. and a Poisson process $(\Pi(t), t \geq 0)$ with intensity a, independent of $Z_1, Z_2, ...$, such that (1.5) holds.*

(iii) *For $z \in \mathbb{C}, t \geq 0$ and $x \in U$,*

$$\mathbb{E}e^{z\langle x,L(t)\rangle U} = exp\left\{-t\int_U (1 - e^{\langle x,y\rangle U})\nu(dy)\right\}.$$

Proposition 1.3. *[22]*

(i) *For each $\Gamma \in \mathcal{B}(U \setminus \{0\})$, the process $(\pi([0,t],\Gamma), t \geq 0)$ is a Poisson process with intensity $\nu(\Gamma)$.*

(ii) *if the sets $\Gamma_1, ..., \Gamma_M$ are disjoint then the random variables $\pi([0,t],\Gamma_j), j = 1, ..., M$, are independent.*

(iii) *For each $\Gamma \in \mathcal{B}(U \setminus \{0\})$, the process $(\hat{\pi}([0,t],\Gamma), t \geq 0)$ is a martingale with respect to the filtration $(\bar{\mathcal{F}}^{\pi}_{t+})$ where*

$$\mathcal{F}^{\pi}_{t+} := \sigma\left\{\pi([0,s],\Gamma) : s \leq t, \Gamma \in \mathcal{B}(U)\right\}.$$

Remark 1.2. *The proposition is true for the jump intensity measure of an arbitrary Lévy process and sets Γ that are separated from the origin, that is, satisfying $\Gamma \cap \{y : |y|_U \leq r\} = \emptyset$ for r sufficiently small.*

Proposition 1.4. *Let L be the compound Poisson process with jump intensity measure ν.*

(i) *The process L is integrable if and only if*

$$\int_U |y|_U \,\nu(dy) < \infty \tag{1.6}$$

Moreover, if (1.6) holds then

$$\mathbb{E}L(t) = t\int_U y\nu(dy) \tag{1.7}$$

and the compensated compound process $\widehat{L}(t) = L(t) - \mathbb{E}L(t), t \geq 0$, is a martingale with respect to $(\bar{\mathcal{F}}^{L}_{t+})$

(ii) For all $z \in \mathbb{C}, t \geq 0$ and $x \in U$,

$$\mathbb{E}e^{z\left\langle x,\widehat{L}(t)\right\rangle)_U} = exp\left\{-t\int_U (1 - e^{z\langle x,y\rangle_U} + z\left\langle x,y\right\rangle_U)\nu(dy)\right\}.$$

(iii) The process L, and hence $\widehat{L}$ is square integrable if and only if

$$\int_U |y|_U^2 \,\nu(dy) < \infty \tag{1.8}$$

Moreover $\mathbb{E}\left|\widehat{L}_t\right|_U^2 = t\int_U |y|_U^2 \,\nu(dy)$ and, for all $x, \tilde{x} \in U$ and $t \geq 0$,

$$\mathbb{E}\left\langle \widehat{L}(t), x\right\rangle_U \left\langle \widehat{L}(t), \tilde{x}\right\rangle_U = t\int_U \left\langle x,y\right\rangle_U \left\langle \tilde{x},y\right\rangle_U \nu(dy).$$

1.5 Lévy-Khinchin decomposition

Assume that L is a càdlàg Lévy process on a Hilbert space U. Given a Borel set A separated from 0 (see Remark 1.2), write

$$\pi_A(t) := \sum_{s\leq t} \mathbb{1}_A(\Delta L(s)), \qquad t \geq 0.$$

Note that the càdlàg property of L implies that π_A is $\mathbb{Z}_+$-valued. Clearly, it is a Lévy process with jumps of size 1. Thus, by the assertion (iv) of the proposition 1.2, π_A is a Poisson process. Note that $\mathbb{E}\pi_A(t) = t\mathbb{E}\pi_A(1) = t\nu(A)$, where ν is a measure that is finite on sets separated from 0. Write

$$L_A(t) := \sum_{s\leq t} \mathbb{1}_A(\Delta L(s))\Delta L(s).$$

Then L_A is a well-defined Lévy process. In the sequel, we need the following Lévy-Khinchin decomposition.

Theorem 1.4. *(Lévy-Khinchin decomposition)[22]*

(i) If ν is a jump intensity measure corresponding to a Lévy process then

$$\int_U (|y|)_U^2 \wedge 1)\nu(dy) < \infty \tag{1.9}$$

(ii) Every Lévy process has the following representation:

$$L(t) = at + W(t) + \sum_{k=1}^{\infty}(L_{A_k}(t) - t\int_{A_k} y\nu(dy)) + L_{A_0}(t),$$

where $A_0 := \{x : |x|_U > r_0\}$, $A_k := \{x : r_k \leq |x|_U < r_{k-1}\}$, *is an arbitrary sequence decreasing to 0, W is a Wiener process, all members of the representation are independent processes and the series converge* $\mathbb{P}$*-a.s. uniformly on each bounded subinterval of* $[0, \infty)$.

Remark 1.3. *It follows from the proof, that the processes*

$$L_n(t) := L_{A_n}(t) - t\int_{A_n} y\nu(dy), \tag{1.10}$$

are independent compensated compound Poisson processes.

Hence we have the decomposition

$$L(t) = at + W(t) + \sum_{n=1}^{\infty} L_n(t) + L_0(t), \tag{1.11}$$

of the Lévy process L, where the processes $W, L_n, n \geq 0$, *and* L_0 *are independent, W is a Wiener process,* L_0 *is a compound Poisson process with jump intensity measure* $\mathbb{1}_{\{|y|_U \geq r_0\}}(y)\nu(dy)$ *and each* L_n *is a compensated compound Poisson process with jump intensity measure*

$$\mathbb{1}_{\{r_{n+1} \leq |y|_U < r_n\}}(y)\nu(dy).$$

Remark 1.4. *A similar representation theorem holds not only for Hilbert spaces but also for Banach spaces (see [22])*

Lemma 1.4. *For any disjoint Borel sets* $A_1 ..., A_{n_t}$ *separated from zero, the processes* $L_{A_1}, ..., L_{A_m}, L - L_{A_l}, ..., L - L_{A_m}$ *are independent Lévy.*

Lemma 1.5. *For every Borel set A separated from 0 and for all* $u \in U$,

$$\mathbb{E}exp\left\{i\left\langle u, L_A(t)\right\rangle_U\right\} = exp\left\{-t\int_A (1 - e^{i\langle u,x\rangle_U})\nu(dx)\right\}.$$

Lemma 1.6. *If assumption (1.9) is satisfied then the series in (1.11) converge* $\mathbb{P}$*-a.s. uniformly on each bounded interval* $[0, T]$.

The following result is a direct consequence of the Lévy-Khinchin decomposition.

Definition 1.6. *(Convolution semigroup of measures)*

Let $\mathcal{M}_1(\mathbb{R}^d)$ denote the set of all Borel probability measures on $\mathbb{R}^d$. We define the convolution of two probability measures as follows:

$$(\mu_1 * \mu_2)(A) = \int_{\mathbb{R}^d} \mathbb{1}_A(x+y)\mu_1(dx)\mu_2(dy) \tag{1.12}$$

for each $\mu_i \in \mathcal{M}_1(\mathbb{R}^d), i = 1, 2,$ and each $A \in \mathcal{B}(\mathbb{R}^d)$.
By Fubini's theorem, we have

$$(\mu_1 * \mu_2)(A) = \int_{\mathbb{R}^d} \mu_1(A-x)\mu_2(dx) = \int_{\mathbb{R}^d} \mu_2(A-x)\mu_2(dx),$$

where $A - x = \{y - x, y \in A\}$ and we have used the fact that $\mathbb{1}_A(x+y) = \mathbb{1}_{A-x}(y)$.

Theorem 1.5. *(Lévy-Khinchin formula)[22] Let denote $L_1^+(U)$ space of nuclear symmetric positive-definite operators*

(i) Given $a \in U$, $Q \in L_1^+(U)$ and a non-negative measure ν concentrated on $U \setminus \{0\}$ satisfying (1.9), there is a convolution semigroup (μ_t) of measures such that

$$\int_U e^{i\langle x,y\rangle_U} \mu_t(dy) = e^{-t\psi(x)}, \tag{1.13}$$

where

$$\psi(x) = -i\langle a, x\rangle_U + \frac{1}{2}\langle Qx, x\rangle_U + \int_U (1 - e^{i\langle x,y\rangle_U} + \mathbb{1}_{\{|y|_U<1\}}(y)i\langle x, y\rangle_U)\nu(dy). \tag{1.14}$$

(ii) Conversely, for each convolution semigroup (μ_t) of measures, there exist $a \in U, Q \in L_1^+(t)$ and a non-negative measure ν concentrated on $U \setminus \{0\}$ satisfying (1.9) in such a way that (1.13) holds with ψ defined by (1.14).

Definition 1.7. *Let L be a Lévy process and let (μ_t) be the family of its distributions. We call the measure ν appearing in (1.14) the Lévy measure or the jump intensity measure of L or (μ_t). We call the triple (a, Q, ν) the characteristics of L.*
The Lévy-Khinchin formula gives the characteristic function of a Lévy process. It turns out that it is also useful for computing characteristic functionals of stochastic integrals. As an example of a simple application, we present the following result.

Corollary 1.1. *Let L be a real-valued Lévy process with exponent ψ, and let $f : \mathbb{R} \mapsto \mathbb{R}$. Assume that the Riemann-Stieltjes integrals $\int_0^t f(s)dL(s)$ and $\int_0^t \psi(xf(s))ds$ exist. Then*

$$\mathbb{E}\left[exp\left\{ix\int_0^t f(s)dL(s)\right\}\right] = exp\left\{-\int_0^t \psi(xf(s))ds\right\}, \qquad x \in \mathbb{R}.$$

1.5.1 Laplace transforms of convolution semigroups

In some situations, it is more convenient to determine convolution semigroups of measures in terms of Laplace rather than Fourier transforms.

Theorem 1.6. *Let (μ_t) be a convolution semigroup of measures on a Hilbert space U, with exponent given by (1.13) and (1.14).*

(i) Let $x \in U$ Then the Laplace transform $\int_U e^{-\langle x,y\rangle_U}\mu_t(dy)$ is finite for some $t > 0$ (equivalently for all $t > 0$) if and only if

$$\int_{\{|y|_U \geq 1\}} e^{-\langle x,y\rangle_U}\nu(dy) < \infty \tag{1.15}$$

(ii) If (1.15) holds then

$$\int_U e^{-\langle x,y\rangle_U}\mu_t(dy) = e^{-t\tilde{\psi}(x)}, \qquad \forall t > 0$$

where

$$\tilde{\psi}(x) = \langle a, x\rangle_U - \frac{1}{2}\langle Qx, x\rangle_U + \tilde{\psi}_0(x),$$
$$\tilde{\psi}_0(x) = \int_U (1 - e^{-\langle x,y\rangle_U} - \langle x,y\rangle_U\,\chi_{\{|y|_U \leq 1\}})\nu(dy).$$

Theorem 1.7. *[22] A family (λ_t) of measures on $[0, +\infty)$ is a convolution semigroup of measures if and only if their Laplace transforms $\widetilde{\lambda}_t$ are of the form*

$$\widetilde{\lambda}_t(r) = e^{-t\tilde{\psi}(r)}, \qquad \widetilde{\psi}(r) = \gamma r + \int_0^{+\infty}(1 - e^{-r\xi})\nu(d\xi), \qquad r > 0,$$

where γ is a positive constant and ν is a non-negative measure on $(0, +\infty)$ satisfying

$$\int_0^1 \xi\nu(d\xi) < \infty, \qquad \int_1^{+\infty}\nu(d\xi) < \infty.$$

Remark 1.5. *This theorem describes the family (λ_t) a subordinator. It is concerned with measures on the half line $[0, +\infty)$; however, it is possible to find Laplace transforms of some important families on $\mathbb{R}$.*

Theorem 1.8. *Assume that ν is a measure on $(0, +\infty)$ satisfying the following conditions:*

$$\int_0^1 \xi^2\nu(d\xi) < \infty \qquad \textit{and} \qquad \int_1^{+\infty}\xi\nu(d\xi) < \infty.$$

Then there exists a convolution semigroup (μ_t) *of measures on* $\mathbb{R}$ *such that*

$$\int_{\mathbb{R}} e^{ir\xi}\mu_t(d\xi) = e^{-t\psi(r)}, \qquad r \in \mathbb{R},$$

where the Lévy exponent is given by

$$\psi(r) = \int_0^{+\infty} (1 - e^{ir\xi} + ir\xi)\nu(d\xi), \qquad r \in \mathbb{R}.$$

Moreover, for all $r > 0$ *and* $t > 0$, $\int_{\mathbb{R}} e^{-r\xi}\mu_t(d\xi) < \infty$ *and*

$$\int_{\mathbb{R}} e^{-ir\xi}\mu_t(d\xi) = e^{-t\widetilde{\psi}(r)},$$

where

$$\widetilde{\psi}(r) = \int_0^{+\infty} (1 - e^{-r\xi} - r\xi)\nu(d\xi).$$

Remark 1.6. *Obviously, every Lévy process* L *associated with a convolution semigroup of measures* (λ_t) *whose support lies in the interval* $[0, +\infty)$ *exhibits upward trajectories. The process* L*, which corresponds to the semigroup constructed in Theorem 1.8, solely possesses positive jumps. However, due to a certain drift, it can attain strictly negative values with a non-zero probability.*

1.6 Square integrable Lévy processes

Consider L as a Lévy process in U, defined on a filtered probability space that meets the standard conditions. We make the assumption that for any $t > s$, the difference $L(t) - L(s)$ is independent of the $\mathcal{F}_s$.

Remark 1.7. *It is evident that for any* $t > s$*, the difference* $L(t) - L(s)$ *is independent of* $\overline{\mathcal{F}}^L_s$*. If* L *is a right-continuous process, then the increment* $L(t) - L(s)$ *is independent of the augmented sigma-algebra* $\overline{\mathcal{F}}^L_{s+}$*. Moreover, if* L *is both integrable and has a mean of zero, then* L *qualifies as a martingale with respect to the filtration* $(\mathcal{F}_t)$*.*

Definition 1.8. *Let's assume that* L *is a square-integrable process. Our initial finding presents precise expressions for the mean and covariance of* L*. We use the notation* $L_1^+(U)$ *to refer to the set of all symmetric non-negative-definite nuclear operators on* U*.*

Theorem 1.9. *[22] There exist $m \in U$ and a linear operator $Q \in L_1^+(U)$ such that, for all $t, s \geq 0$ and $x, y \in U$.*

$$\begin{aligned}\mathbb{E}\,\langle L(t), x\rangle_U &= \langle m, x\rangle_U\, t,\\ \mathbb{E}\,\langle L(t) - mt, x\rangle_U \,\langle L(s) - ms, y\rangle_U &= t \wedge s\,\langle Qx, y\rangle_U\,,\\ \mathbb{E}\,|L(t) - mt|_U^2 &= tTrQ\end{aligned}$$

Definition 1.9. *The theorem mentioned above introduces the vector m and the operator Q as the mean and covariance operator of the process L, correspondingly.*

Remark 1.8. *It should be noted that the covariance operator of the process L is identical to the covariance operator of $L(1)$.*

Theorem 1.10.

(i) A Lévy process L on a Hilbert space U is square integrable if and only if its Lévy measure satisfies

$$\int_U |y|_U^2\, \nu(dy) < \infty \tag{1.16}$$

(ii) Assume (1.16). Let L have the representation (1.11), let Q_0 be the covariance operator of the Wiener part of L, and let Q_1 be the covariance operator of the jump part. Then

$$\langle Q_1 x, z\rangle_U = \int_U \langle x, y\rangle_U \,\langle z, y\rangle_u\, \nu(dy), \qquad x, z \in U$$

$$\mathbb{E}L(t) = (a + \int_{(|y|_U \geq r_0)} y\nu(dy))t,$$

and the covariance Q of L is equal to $Q_0 + Q_1$.

1.7 Lévy semigroups

This section focuses on transition semigroups of Lévy processes and their generators. A collection $S = (S(t), t \geq 0)$ of bounded linear operators on a Banach space $(B, |.|_B)$ is referred to as a C_0-semigroup if

(i) $S(0)$ is the identity operator I,

(ii) $S(t)S(s) = S(t + s)$ for all $t, s \geq 0$,

(iii) $t \in [0, \infty) \mapsto S(t)z \in B$ is continuous for each $z \in B$.

Let's suppose that S is a C_0-semigroup on B. We define an element $z \in B$ to be in the domain of the generator of S if the limit $\lim_{t\to 0} t^{-1}(S(t)z - z) =: Az$ exists. The set of all such z is represented by $D(A)$, and Az, for $z \in D(A)$, is a linear operator known as the generator of S.

1.7.1 Basic properties of Lévy processes

Consider L as a Lévy process in a Hilbert space U, and let μ_t denote the distribution of $L(t)$. In this case, (μ_t) forms a convolution semigroup of measures, and L can be regarded as a Markov process with a transition function $P_t(x, \Gamma) = \mu_t(\Gamma - x)$. The associated semigroup can be expressed as follows:

$$P_t\varphi(x) = \int_U \varphi(x+y)\mu_t(dy) \tag{1.17}$$

The spaces $C_b(U)$ and $UC_b(U)$ represent the sets of all bounded continuous functions on U and bounded uniformly continuous functions on U, respectively. They are equipped with the supremum norm. The result mentioned here is attributed to Tessitore and Zabczyk in 2001 (Tessitore and Zabczyk, 2001b).

Theorem 1.11. *Let (P_t) be defined on $C_b(U)$ by (1.17). Then (P_t) is a C_0-semigroup on $C_b(U)$ if and only if either (μ_t) is the convolution semigroup of measures of a compound Poisson process or $\mu_t = \delta_0, t > 0$.*

Definition 1.10. *A semigroup P consisting of continuous linear operators on $UC_b(U)$ is considered translation invariant or spatially homogeneous if, for any $a \in U$ and $t \geq 0$, the following equality holds:*

$$\varphi \in UC_b(U), P_t(\tau_a\varphi) = \tau_a(P_t\varphi).$$

The space $UC_b(U)$ proves to be more advantageous when dealing with the transition semigroups of Lévy processes, as indicated by the following theorem.

Theorem 1.12. *[22]*

(i) if (P_t) is defined on $UC_b(U)$ by (1.17) then (P_t) is a C_0-semigroup on $UC_b(U)$.

(ii) A Markov semigroup (P_t) on $UC_b(U)$ is an invariant translation if and only if it is given by (1.17) for some convolution semigroup of measures.

1.7.2 Generators for Lévy processes

Within this section, we establish the expression of the generator (for regular functions) of an arbitrary Lévy process in a Hilbert space U. However, before delving into the general case, we first examine certain special processes.

1. **Compound Poisson Process**

   We demonstrate that the generator of a compound Poisson process L can be expressed as follows:

$$\varphi(x) = \int_U (\varphi(x+y) - \varphi(x))\nu(dy), \qquad \varphi \in UC_b(U), \tag{1.18}$$

   Where ν represents the Lévy measure of L. It should be noted that A is a bounded linear operator on $UC_b(U)$. Recall that the corresponding distributions μ_t are given by (1.4). In order to demonstrate that the operator A defined in (1.18) is indeed the generator of the process L, we need to establish that, for every $\varphi \in UC_b(U)$, the following condition holds:

$$\left\| \frac{P_t\varphi - \varphi}{t} - A\varphi \right\|_\infty = sup \left| \int_U (\varphi(x+y) - \varphi(x))(\frac{1}{t}\mu_t(dy) - \nu(dy)) \right| =: J(t) \to 0$$

   as $t \to 0$.

   Since

$$\frac{1}{t}\mu_t - \nu = \frac{e^{-at}}{t}\delta_0 + (e^{-at} - 1)\nu + e^{-at}\sum_{k=2}^{\infty} \frac{t^{n-1}}{n!}\nu^{*n},$$

   we have

$$J(t) \leq 2\left\|\varphi\right\|_\infty (\left|e^{-at} - 1\right| \nu(U) + e^{-at}\sum_{n=2}^{\infty} \frac{t^{n-1}}{n!}\nu^{*n}(U)),$$

   which gives the desired conclusion.

2. **Uniform motion**

   Let's consider the deterministic process $L(t) = at$, where $a \in U$ is a fixed parameter. This process is evidently a Lévy process, and its corresponding distributions are given by $\mu_t = \delta_{ta}$ for $t \geq 0$. The generator A, which operates on functions $\varphi \in UC_b^1(U)$, satisfies the following equation:

$$A\varphi(x) = \lim_{t\to 0} \frac{1}{t}(\varphi(x+ta) - \varphi(x)) = \langle a, D\varphi(x)\rangle_U .$$

3. **Arbitrary Lévy semigroup**

   Let $UC_b^1(U)$ and $UC_b^2(U)$ denote the spaces of uniformly continuous bounded functions along with all their derivatives up to order 1 and up to order 2, respectively. We present the following result.

Theorem 1.13. *Assume that (P_t) is the transition semigroup of a Lévy process L on a Hilbert space U with Lévy exponent (1.14). If $\varphi \in UC_b^2(U)$, then for each $x \in U$ we have*

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}(P_t\varphi(x)-\varphi(x)) &= A\varphi(x)\\ &= \langle a, D\varphi(x)\rangle_U + \frac{1}{2}TrQD^2\varphi(x)\\ &+ \int_U (\varphi(x+y)-\varphi(x)-\chi_{\{|y|_U<1\}}(y)\,\langle D\varphi(x), y\rangle_U)\nu(dy)\end{aligned}$$

where the convergence is uniform in x.

1.8 Stochastic integrals based on Lévy processes

The objective of this section is to investigate different forms of stochastic integration where the integrator is a Lévy process.

1.8.1 Poisson stochastic integrals

Consider the set E, which is obtained by removing the element 0 from the set $\hat{B}$. Here, $\hat{B}$ represents the set of all elements x in $\mathbb{R}^d$ such that the absolute value of x is less than one. We define a stochastic process $Y = (Y(t), t \geq 0)$ as a Lévy-type stochastic integral if it can be expressed in the following manner for every $1 \leq i \leq d$ and $t \geq 0$:

$$\begin{aligned}Y^i(t) = Y^i(0) + \int_0^t G^i(s)ds + \int_0^t F_j^i(s)dB^j(s)\\ + \int_0^t\int_{|x|<1} H^i(s,x)\tilde{N}(ds,dx)\\ + \int_0^t\int_{|x|\geq 1} K^i(s,x)N(ds,dx),\end{aligned} \tag{1.19}$$

For each combination of $1 \leq i \leq d$, $1 \leq j \leq m$, and $t \geq 0$, we have certain conditions: $|G^i|^{1/2}, F_j^i \in \mathcal{P}_2(T)$, $H^i \in \mathcal{P}_2(T, E)$, and K is predictable. Here, B represents an m-dimensional standard Brownian motion, and N is an independent Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ with a compensator $\tilde{N}$ and intensity measure ν, which we assume to be a Lévy measure.

Definition 1.11. *Consider a Lévy process L with characteristics (b, a, ν), where b belongs to $\mathbb{R}^d$. Let B_a be a Brownian motion with covariance matrix a, and let ν be an independent Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$. The Lévy-Itô decomposition of L is given by the equation:*

$$L(t) = bt + B_a(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx), \tag{1.20}$$

for each $t \geq 0$.

Let $L \in \mathcal{P}_2(t)$ for all $t \geq 0$ and in (1.19) we choose each $F_j^i = \sigma_j^i L, H^i = K^i = x^i L$, where $\sigma\sigma^T = a$. Then we can construct processes with the stochastic differential presentation

$$dY(t) = H(t)dL(t). \tag{1.21}$$

We call Y a Lévy stochastic integral.

Remark 1.9. *The Lebesgue-Stieltjes integral of the Lévy stochastic integral Y can also be constructed when X has finite variation. This construction is equivalent (except for a set of measure zero) to the recommendation (1.21); see [13].*

1.8.2 Square integrable integrators

To facilitate our analysis, we find it useful to introduce a particular class of martingales that adhere to the following condition:

$$\exists Q \in L_1^+(U) : 0 \leq s \leq t, \langle\langle M, M\rangle\rangle_T - \langle\langle M, M\rangle\rangle_s \leq (t-s)Q. \tag{1.22}$$

$\langle\langle M, M\rangle\rangle_s, s \geq 0$ is absolutely continuous and

$$Q_s = \frac{d}{ds}\langle\langle M, M\rangle\rangle_s; \qquad \forall s \geq 0, \mathbb{P} - a.s. \tag{1.23}$$

The following lemma offers a justification for the assumption made in (1.23).

Lemma 1.7. *Assume that Q and R are non-negative operators on a Hilbert space V and that $R \leq Q$. If Φ is a linear operator from V into a Hilbert space H then*

$$\left\|\Phi R^{\frac{1}{2}}\right\|_{L_{(HS)}(V,H)} \leq \left\|\Phi Q^{\frac{1}{2}}\right\|_{L_{(HS)}(V,H)}.$$

Proposition 1.5. *Assume (1.22), Then $L^2_{\mathcal{H},T}(H) \subseteq \mathcal{L}^2_{M,T}$ and for every $X \in L^2_{\mathcal{H},T}(H)$,*

$$\mathbb{E}\left|\int_0^t X(s)dM(s)\right| \leq \mathbb{E}\int_0^t \|X(s)\|^2_{L_{(HS)}(\mathcal{H},H)} ds.$$

In the most important case, where M is a Lévy process.

1.8.3 Itô's formula

Once we have a process Y with a stochastic differential, we can proceed to demonstrate Itô's formula for general Lévy-type stochastic integrals.

$$dY(t) = G(t)dt + F(t)dB(t) + \int_{|x|<1} H(t,x)\tilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)N(dt,dx), \quad (1.24)$$

For every combination of $1 \leq i \leq d$, $1 \leq j \leq m$, and $t \geq 0$, considering $|G^i|^{1/2}, F^i_j \in \mathcal{P}_2(T,E)$ and $H^i \in \mathcal{P}_2(T,E)$., along with K as predictable and $E = \hat{B} - \{0\}$, we will maintain the notation introduced earlier,

$$dY_c(t) = G(t)dt + F(t)dB(t),$$

Later on, we will require the discontinuous component of Y, denoted as Y_d, which can be expressed as follows:

$$dY_d(t) = \int_{|x|<1} H(t,x)\tilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)N(dt,dx)$$

such that for every $t \geq 0$

$$Y(t) = Y(0) + Y_c(t) + Y_d((t).$$

From this point forward, we will find it convenient to introduce the following local boundedness restriction on the small jumps.

Assumption. For all $t > 0$,

$$\sup_{0\leq s\leq t}\ \sup_{0<|x|<1} |H(s,x)| < \infty \quad a.s. \quad (1.25)$$

Theorem 1.14. *[2](Itô's theorem 1) If Y is a Lévy-type stochastic integral of the form (1.24), then, for each $f \in C^2(\mathbb{R}^d), t \geq 0$, with probability 1 we have*

$$\begin{aligned}
&f(Y(t)) - f(Y(0)) \\
&= \int_0^t \partial_i f(Y(s-))dY^i_c(s) + \frac{1}{2}\int_0^t \partial_i\partial_j f(Y(s-))d\left[dY^i_c dY^j_c\right](s) \\
&+ \int_0^t \int_{|x|\geq 1} \left[f(Y(s-)) + K(s,x) - f(Y(s-))\right] N(ds,dx) \\
&+ \int_0^t \int_{|x|<1} \left[f(Y(s-)) + H(s,x) - f(Y(s-))\right] \tilde{N}(ds,dx) \\
&+ \int_0^t \int_{|x|<1} \left[f(Y(s-)) + H(s,x) - f(Y(s-)) - H^i(s,x)\partial_i f(Y(s-))\right] \nu(dx)ds.
\end{aligned}$$

Proposition 1.6. *[2] If $H^i \in \mathcal{P}_2(t, E)$ for each $1 \le i \le d$ then*

$$\int_0^t \int_{|x|<1} \left| H^i(s,x) H^j(s,x) \right| N(ds, dx) < \infty \qquad a.s.$$

for each $1 \le i, j \le d, t \ge 0$.

Corollary 1.2. *If Y is a Lévy-type stochastic integral then for $1 \le i \le d, t \ge 0$,*

$$\sum_{0 \le s \le t} \Delta Y^i(s)^2 < \infty \qquad a.s.$$

In order to convert Itô's formula stated in Theorem 1.14 into a more comprehensive form, we will utilize Proposition 1.6.

Theorem 1.15. *[2](Itô theorem 2) If Y is a Lévy-type stochastic integral of the form 1.24 then, for each $f \in C^2(\mathbb{R}^d), t \ge 0$, with probability 1*
we have

$$f(Y(t)) - f(Y(0)) = \int_0^t \partial_i f(Y(s-)) dY^i(s) + \frac{1}{2} \int_0^t \partial_i \partial_j f(Y(s-)) d[Y_c^i, Y_c^j](s)$$

$$+ \sum_{0 \le s \le t} \left[f(Y(s)) - f(Y(s-)) - \Delta Y^i(s) \partial_i f(Y(s-)) \right].$$

Chapter 2

Stochastic analysis of SPDE driven by Lévy noise

2.1 Study of SPDE driven by Lévy process

In this section, we will examine the stochastic partial differential equation:

$$\mathcal{L}u(t,x) = \gamma(u(t,x))\dot{L}(t,x) + b(u(t,x)), \qquad t \geq 0, x \in \mathbb{R} \tag{2.1}$$

with prescribed deterministic initial conditions, we will focus on the following equation: $\mathcal{L}$ is a second-order pseudo-differential operator with constant coefficients, b and γ are real functions defined on $\mathbb{R}$, and L represents the Lévy white noise.

We'll prove that the equation (2.1) has a unique solution $u = \{u(t,x); t \geq 0, x \in \mathbb{R}\}$ which is continuous in $L^2(\Omega)$.

Definition 2.1. *Consider the solution $\omega = \{\omega(t,x); t \geq 0, x \in \mathbb{R}\}$ of the equation $\mathcal{L}u(t,x) = 0$, with the same initial conditions as (2.1). Additionally, let G_t be the fundamental solution of the same problem. We assume that G_t is a positive function in the intersection of $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. We denote the Fourier transform of G_t as $\mathcal{F}G_t$.*
Suppose that the applications b and γ are globally Lipschitzians of constant C_b and C_γ i.e:

$$|\gamma(x) - \gamma(y))| \leq C_\gamma |x-y| \qquad \textit{for all} \quad x, y \in \mathbb{R},$$

$$|b(x) - b(y)| \leq C_b |x-y| \qquad \textit{for all} \quad x, y \in \mathbb{R},$$

So, $|\gamma(x)| \leq |\gamma(x) - \gamma(0)| + |\gamma(0)| \leq C_\gamma |x| + |\gamma(0)|$, *it follows that:*

$$|\gamma(x)| \leq D_\gamma(1 + |x|) \qquad \text{for all} \quad x \in \mathbb{R},$$

where $D_\gamma = max(C_\gamma, |\gamma(0)|)$. *In the same way, if we note* $D_b = max(C_b, |d(0)|)$, *then:*

$$|b(x)| \leq D_b(1 + |x|) \qquad \text{for all} \quad x \in \mathbb{R}$$

We note that there is a constant $L_0 = max(C_\gamma, C_b)$ *such that:*

$$max\{|\gamma(x) - \gamma(y)|, |b(x) - b(y)|\} \leq L_0 |x - y| \qquad \text{for all} \quad x, y \in \mathbb{R}$$

By choosing $L_0 > max\{D_\gamma, D_b\}$, *we have:*

$$max\{|\gamma(x)|, |b(x)|\} \leq L_0(1 + |x|) \qquad \text{for all} \quad x \in \mathbb{R}$$

Now, we introduce the definition of the solution of (2.1).

Definition 2.2. *The process* $u = \{u(t, x); t \geq 0, x \in \mathbb{R}\}$ *is solution of (2.1). if* u *is predictable and for each* $t \geq 0$ *and* $x \in \mathbb{R}$ *we have:*

$$\begin{aligned} u(t,x) = \omega(t,x) &+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\gamma(u(s,y))L(ds,dy) \\ &+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)b(u(s,y))dyds \qquad a.s. \end{aligned} \tag{2.2}$$

The main result of this section is the following theorem.

Theorem 2.1. *[15] We assume that* $\omega(t, x)$ *is continuous relative to* (t, x) *and* $k := \sup_{(t,x)\in[0,T]\times\mathbb{R}} |\omega(t, x)|^2 < \infty$ *and* G *satisfies hypothesis* (H), *where*

$$(H) \begin{cases} a)\Gamma_T := \int_0^T \int_{\mathbb{R}} G_t(x)dxdt < \infty \text{ and } \nu_T := \int_0^T \int_{\mathbb{R}} G_t^2(x)dxdt < \infty, \\ b)t \mapsto \mathcal{F}G_t(\xi) \text{ is continuous for all } \xi \in \mathbb{R}, \\ c)\exists \varepsilon > 0 \text{ and a positive function } k_t(.), \text{ such as for all } t \geq 0 \text{ and } h \in [0, \varepsilon] \\ \qquad |\mathcal{F}G_{t+h}(\xi) - \mathcal{F}G_t(\xi)| \leq k_t(\xi), \\ \text{and} \\ \qquad \int_0^T \int_{\mathbb{R}} k_t^2 d\xi dt < \infty \end{cases}$$

The equation (2.1) admit an unique solution $u = \{u(t, x); t \geq 0, x \in \mathbb{R}\}$ *which is continuous on* $L^2(\Omega)$ *and satisfies the next condition: for each* $T > 0$ *we have :*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}|u(t,x)|^2 < \infty.$$

Proof. We will demonstrate that u is a solution. It is important to note that the mapping $(t,x) \to u_n(t,x) \in L^2(\Omega)$ is continuous. By employing

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \|u_n(t,x) - u(t,x)\|_{L^2(\Omega)} \underset{n\to\infty}{\to} 0,$$

we can extend Proposition 3.21 of [22] to random fields, which establishes that u undergoes a predictable change. Subsequently, we proceed with this modified version. Now, we establish that u satisfies (2.2) by taking the $L^2(\Omega)$ limit in relation to

$$\begin{aligned} u_{n+1}(t,x) &= w(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\gamma(u_n(s,y))L(ds,dy) \\ &+ \int_0^t \int_{\mathbb{R}} G_s(y)b(u_n(s,y))dyds, n \geq 0. \end{aligned} \tag{2.3}$$

After performing the calculation in [15], we determine that u satisfies (2.2).

By utilizing Lemma 15 from [8], we obtain the result stated in

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}\,|u_n(t,x)| < \infty.$$

To establish uniqueness, consider two solutions u and u' of (2.1). Let $d(t,x) = u(t,x) - u'(t,x)$:

$$\begin{aligned} \mathbb{E}\,|d(t,x)|^2 &\leq 2\mathbb{E}\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\left[\gamma(u(s,y)) - \gamma(u'(s,y))\right] L(ds,dy)\right|^2 \\ &+ 2\mathbb{E}\left|\int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\left[b(u(s,y)) - b(u'(s,y))\right] dy, ds\right|^2 \\ &= 2\nu \int_0^t \int_{\mathbb{R}} G^2_{t-s}(x-y)\mathbb{E}\,|\gamma(u(s,y)) - \gamma(u'(s,y))|^2\, dy, ds \\ &+ 2\Gamma_T \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathbb{E}\,|b(u(s,y)) - b(u'(s,y))|^2\, dy, ds \\ &\leq C_T \int_0^t \int_{\mathbb{R}} \left[G^2_{t-s}(x-y) + G_{t-s}(x-y)\right] \mathbb{E}\,|d(s,y)|^2\, dyds, \end{aligned}$$

where $C_T = max(\nu, \Gamma_T)$. Suppose then $H(t) = \sup_{x\in\mathbb{R}} \mathbb{E}\,|d(t,x)|^2$, we have :

$$\begin{aligned} \mathbb{E}\,|d(t,x)|^2 &\leq C_T \int_0^t H(s) \int_{\mathbb{R}} \left[G^2_{t-s}(x-y) + G_{t-s}(x-y)\right] dyds, \\ &= C_T \int_0^t H(s)g(t-s)ds. \end{aligned}$$

As a result

$$H(t) \leq C_T \int_0^t H(s)g(t-s)ds.$$

By applying Lemma 15 from [8] with $k_1 = k_2 = 0$. We conclude that $H(t) = 0$. So $u(t,x) = u'(t,x)$ almost surely for each $t \in [0,T]\,, x \in \mathbb{R}$. $\square$

2.2 Examples

We will present two examples of the application of the previous result.

2.2.1 Heat equation

Let's consider the stochastic heat equation from [7]:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \gamma(u(t,x))\dot{L}(t,x), & (t,x) \in (0,T) \times D, \\ u(t,x) = 0, & \text{for all } (t,x) \in (0,T) \times \partial D, \\ u(0,x) = u_0(x) & \text{for all } x \in D \end{cases} \tag{2.4}$$

we consider D to be the entire space $\mathbb{R}^d$ or a bounded domain in $\mathbb{R}^d$. The function $\gamma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, $u_0 : \overline{D} \to \mathbb{R}$ is a bounded continuous initial condition that vanishes on the boundary ∂D, and $\dot{L}$ represents a space-time white noise in the Lévy sense. If $D = \mathbb{R}^d$, the boundary conditions on u and u_0 are considered to be non-existent.

A predictable random field $u = (u(t,x) : (t,x) \in [0,T] \times D)$ is called a mild solution to (2.4) if for all $(t,x) \in [0,T] \times D$,

$$u(t,x) = V(t,x) + \int_0^t \int_D G_D(t-s;x,y)\gamma(u(s,y))L(ds,dy) \tag{2.5}$$

Where the homogeneous solution of (2.4) is

$$V(t,x) = \int_D G_D(t;x,y)u_0(y)dy, \quad (t,x) \in [0,T] \times D. \tag{2.6}$$

Remark 2.1. *In (2.5) and (2.6), G_D is the Green's function of the heat operator on D, for $D = \mathbb{R}^d$ equals the Gaussian density*

$$g(t,x) = (4\pi t)^{\frac{d}{2}} e^{-\frac{|x|^2}{4t}} \mathbb{1}_{t\geq 0}$$

while on a bounded domain D with a smooth boundary, it has the spectral representation

$$G_D(t,x,y) = \sum_{j\geq 1} \Phi_j(x)\Phi_j(y)e^{-\lambda_j t}\mathbb{1}_{t\geq 0}, \qquad \textit{for all} \quad x,y \in D,$$

where $(\lambda_j)_{j\geq 1}$ are the eigenvalues of $-\Delta$ with vanishing Dirichlet boundary conditions, and $(\Phi_j)_{j\geq 1}$ are the corresponding eigenfunctions forming a complete orthonormal basis of $L^2(D)$.

2.2.2 The wave equation

Let us now consider the stochastic wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \gamma(u(t,x))\dot{L}(t,x) + b(u(t,x)) & t \geq 0, x \in \mathbb{R} \\ u(0,x) = g(x), \qquad x \in \mathbb{R} \\ \frac{\partial u}{\partial t}(0,x) = h(x), \qquad x \in \mathbb{R}, \end{cases} \tag{2.7}$$

γ and b are assumed to be globally Lipschitz continuous with a Lipschitz constant denoted as L_0. Additionally, the function g is continuous, bounded, and h belongs to the space $L^1(\mathbb{R})$.

We note that the fundamental solution is given by:

$$G_t(x) = \frac{1}{2}\mathbb{1}_{\{|x|<t\}}$$

and its Fourier transformation by

$$\mathcal{F}G_t(\xi) = \frac{sin(t\,|\xi|)}{|\xi|}.$$

The solution of the homogeneous equation is given by:

$$\begin{aligned} w(t,x) &= \int_{\mathbb{R}} G_t(x-y)h(y)dy + \frac{\partial}{\partial t}\int_{\mathbb{R}} G_t(x-y)g(y)dy \\ &= \frac{1}{2}\int_{x-t}^{x+t} h(y)dy + \frac{1}{2}\left[g(x+t) + g(x-t)\right]. \end{aligned}$$

2.3 Some examples of SPDEs in infinite dimension

Over time there was a diversity of research for the study of the $SPDE$, in this chapter we study some different types of these equations while illustrating by numerical applications.

2.3.1 SPDE with respect to a square-integrable Lévy martingale

Within this particular section, we consider the real separable Hilbert spaces U, H, and V. We use the notation $L(U, H)$ to represent the space encompassing all bounded linear operators from U to H. Additionally, we denote the subspace of Hilbert-Schmidt operators within $L(U, H)$ as $L_{(HS)}(U, H)$.

Suppose L is a square-integrable Lévy process, where we have removed the large jumps, and it takes values in U, then

$$M(t) = L(t) - t\mathbb{E}L(1), \qquad t \geq 0.$$

where M is square integrable martingale.

Let's consider the assumption that a Hilbert space H is continuously embedded within a Hilbert space V. Now, let us examine a (SPDE):

$$du = (Au + F(u))dt + B(u)dM, \qquad u(0) = u_0 \in H, \tag{2.8}$$

where $(A, D(A))$ acts as a generator from a C_0-semigroup S on the Hilbert space H. The mapping F is defined from H to V, and for any $x \in H, B(x)$ represents a linear operator (which may not be bounded) from $\mathcal{H}$ to H. Here is a clear result of its existence.

Theorem 2.3.1. *[18] Let's assume that for any positive t, the semigroup $S(t)$ can be extended uniquely to a bounded linear mapping from V to H. Additionally, we assume that*

$$|S(t)(F(x) - F(y))|_H \leq b(t) |x - y|_H ,$$

$$\|S(t)(B(x) - B(y))\|_{L_{(H,S)}(\mathcal{H},H)} \leq a(t) |x - y|_H$$

and

$$|S(t)F(x)|_H \leq b(t)(1 + |x|_H),$$

$$\|S(t)B(x)\|_{L_{(H,S)}(\mathcal{H},H)} \leq a(t)(1 + |x|_H),$$

where

$$\int_0^T (b(t) + a^2(t))dt < \infty, \qquad \forall T > 0.$$

In that case, there exists only one adapted process u such that

$$\underset{0\leq t\leq T}{sup}\ \mathbb{E}\,|u(t)|_H^2 < \infty, \qquad \forall T > 0$$

and for all $t \geq 0$,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)F(u(s))dM(s), \qquad \mathbb{P} - a.s.$$

Proof. Let's set a specific finite time limit where $T > 0$. $\mathcal{X}_T$ represents the set of all adapted processes that are square-integrable $X : \Omega \times [0,T] \mapsto H$ such

$$t \in [0,T] \rightarrow \mathbb{E}\,|X(t)|_H^2 \in \mathbb{R}$$

is continuous. Consider the family of equivalent norms on $\mathcal{X}_T$

$$\|X\|_\beta := \underset{0\leq t\leq T}{sup}\ e^{-\beta t}\sqrt{\mathbb{E}\,|X(t)|_H^2}, \qquad \beta > 0.$$

Therefore, when $\mathcal{X}_T$ is equipped with the norm $\|.\|_\beta$ it forms a Banach space. Let's examine the mapping

$$\Psi(t) = S(t)u_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dM(s).$$

Hence $\Psi : \mathcal{X}_T \mapsto \mathcal{X}_T$. Furthermore, when β is sufficiently large it can be observed that Ψ is a contraction. Consequently, the desired conclusion can be derived from the Banach fixed point theorem. $\square$

2.3.2 Example

Let's look at the stochastic heat equation

$$du = (\Delta u + f(u))dt + b(u)dM, \qquad u(0) = u_0,$$

considered on a bounded region $\mathcal{O} \subset \mathbb{R}^d$ with zero-Dirichlet boundary conditions. Under the assumption that the reproducing kernel Hilbert space (RKHS) $\mathcal{H}$ of M is a subset of $H = L^2(\mathcal{O})$, and the functions f and b are mappings from $\mathbb{R}$ to $\mathbb{R}$, we can frame the equation (2.8) in the context where A represents the Laplace operator on $H = L^2(\mathcal{O})$ with Dirichlet boundary conditions. Additionally, the operators F and B correspond to Nemytskii type operators

$$F(\psi)(x) = f(\psi(x)), \quad B(\psi)\left[\phi\right](x) = b(\psi(x))\phi(x)$$

for $\psi \in L^2(\mathcal{O}), \phi \in \mathcal{H}, x \in \mathcal{O}$.
Note that if $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz then the corresponding $F : L^2(\mathcal{O}) \mapsto L^2(\mathcal{O})$ is Lipschitz as well. As far as concerned, then $B(u)$ is a bounded linear operator from $L^2(\mathcal{O})$ to $L^2(\mathcal{O})$ if and only if $b(u) \in L^\infty(\mathcal{O})$. Therefore B is an $L(L^2(\mathcal{O}), L^2(\mathcal{O}))-$valued if and only if b is bounded. Suppose that b is bounded. It is important to note that

$$B : L^2(\mathcal{O}) \mapsto L(L^2(\mathcal{O}), L^2(\mathcal{O}))$$

is continuous if and only if b is constant. For

$$\begin{aligned}
\|B(u) - B(\nu)\|^2_{L(L^2(\mathcal{O}),L^2(\mathcal{O}))} &= \sup_{|\psi|_{L^2(O)} \leq 1} \int_{\mathcal{O}} (b(u(x)) - b(\nu(x)))^2 \psi^2(x)dx \\
&= \|b(u) - b(\nu)\|^2_\infty .
\end{aligned}$$

Consider $a_1 \neq a_2 \in \mathbb{R}$ and let $\mathcal{O}\varepsilon$ be a subset of $\mathcal{O}$ with Lebesgue measure ε. Define $u\varepsilon(x) = a_1\chi_{\mathcal{O}\varepsilon}(x)$ and $\nu\varepsilon(x) = a_2\chi_{\mathcal{O}\varepsilon}(x)$ for $x \in \mathcal{O}$. Then $|b(u_\varepsilon) - b(\nu_\varepsilon)|_\infty = |b(a_1) - b(a_2)|$. However,

$$|u_\varepsilon - \nu_\varepsilon|_{L^2(\mathcal{O})} = |a_1 - a_2| \sqrt{\varepsilon}.$$

Note that $B(u)$ is Hilbert-Schmidt if and only if $b \equiv 0$.

Let G be the Green kernel. Then

$$\begin{aligned}
\|S(t)(B(u)-B(\nu))\|_{L(L^2(\mathcal{O}),L^2(\mathcal{O}))} &= \sup_{|\psi|_{L^2(O)}\leq 1} \int_{\mathcal{O}} \psi(x)S(t)(B(u)-B(\nu))(x)dx \\
&= |S(t)(B(u)-B(v))|_{L^\infty(\mathcal{O})} \\
&= \sup_{x\in\mathcal{O}} \int_{\mathcal{O}} G(t,x,y)\,|b(u(y))-b(\nu(y))|\,dy \\
&\leq |b(u)-b(\nu)|_{L^2(\mathcal{O})} \sup_{x\in\mathcal{O}} \left(\int_{\mathcal{O}} G^2(t,x,y)dy\right)^{1/2}.
\end{aligned}$$

Let's remember that d represents the dimension of the domain $\mathcal{O}$. By considering the Arronson estimates for the Green kernel, see [4]

$$G(t,x,y) \leq C_1 t^{1/2} exp\left\{-C_2\frac{|x-y|^2}{t}\right\}.$$

We can deduce the estimation

$$\sup_{x\in\mathcal{O}} \left(\int_{\mathcal{O}} G^2(t,x,y)dy\right)^{1/2} \leq C_3 t^{-d/4}.$$

Alternatively,

$$\begin{aligned}
\|S(t)(B(u)-B(\nu))\|^2_{L(L^2(\mathcal{O}),L^2(\mathcal{O}))} &= \int_{\mathcal{O}}\int_{\mathcal{O}} \left|G^2(t,x,y)b(u(y))-b(\nu(y))\right|^2 dydx \\
&\leq |b(u)-b(\nu)|^2_{L^2(\mathcal{O})} \sup_{y\in\mathcal{O}} \int_{\mathcal{O}} G^2(t,x,y)dx \\
&\leq C_3 t^{-d/2}\,|b(u)-b(\nu)|^2_{L^2(\mathcal{O})}.
\end{aligned}$$

Therefore, if $d=1$, then the existence of the solution follows from Theorem 2.3.1.

2.3.3 SPDEs with respect to Lévy space-time white noise

The proposed method, combines the techniques used for solving SPDEs driven by Gaussian noise in [6] with the method employed for solving ordinary SDEs driven by Lévy processes in [11].

Let's consider the following SPDEs of parabolic type with initial and Dirichlet boundary conditions

$$\begin{cases} (\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})u(t,x,w) = a(t,x,u(t,x,w)) + b(t,x,u(t,x,w))F_{t,x}(w), \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x,w) \in (0,\infty) \times [0,L] \times \Omega \\ u(t,0,w) = u(t,L,w) = 0, \qquad (t,w) \in [0,\infty) \times \Omega \\ u(0,x,w) = u_0(t,w), \qquad (x,w) \in [0,L] \times \Omega. \end{cases} \tag{2.9}$$

We have an arbitrarily fixed positive value L. The process F represents a white noise, while $a, b : [0,\infty) \times [0,L] \times \mathbb{R} \mapsto \mathbb{R}$ are measurable functions. The initial condition u_0 is $\mathcal{F}_0-$measurable and satisfies $u_0(0,w) = u_0(L,w) = 0$. Our aim here is to extend the conventional framework, where F is assumed to be Gaussian, to incorporate terms controlled by a "space-time Poisson white noise." As a result, the noise we consider exhibits a formal structure akin to that of a Lévy process. We refer to this noise as "the Lévy space-time white noise".

Expanding on the discussion in [3], we can simplify the analysis by considering the equations in the following form, without compromising generality

$$\begin{aligned} u(t,x,w) = &\int_0^L G_t(x,z)u_0(z,w)dz + \int_0^t \int_0^L G_{t-s}(x,z)f(s,z,u(s,z,w))dzds \\ &+ \int_0^t \int_0^L G_{t-s}(x,z)g(s,z,u(s,z,w))W(ds,dz) \\ &+ \int_0^{t+} \int_0^L \int_U G_{t-s}(x,z)h(s,z,u(s-,z,w);y)M(ds,dz,dy,w), \end{aligned} \tag{2.10}$$

where $f, g : [0,\infty) \times [0,L] \times \mathbb{R} \to \mathbb{R}$, $h : [0,\infty) \times [0,L] \times \mathbb{R} \times U \to \mathbb{R}$ are measurable, W is a Gaussian white noise, N is Poisson white noise and M is the associated (compensating) martingale measure.

We have the following main result

Theorem 2.3.2. *[3] Suppose that for every positive T, there exists a positive real function $K_T : [0,\infty] \times [0,L] \to [0,\infty)$ such that*

$$\int_0^L [G_{t-s}(x,z)]^2 K_T(s,z)dz \leq const.(t-s)^{-\alpha} \tag{2.11}$$

with $\alpha \in \left[\frac{1}{2}, 1\right)$ *such that we have*

$$|f(t,x,z)|^2 + |g(t,x,z)|^2 + \int_U |h(t,x,z;y)|^2 \nu(dy) \leq K_T(t,x)(1+|z|^2) \tag{2.12}$$

and

$$\begin{aligned} &|f(t,x,z_1) - f(t,x,z_2)|^2 + |g(t,x,z_1) - g(t,x,z_2)|^2 \\ &+ \int_U |h(t,x,z_1;y) - h(t,x,z_2;y)|^2 \nu(dy) \leq K_T(t,x)(|z_1 - z_2|^2 \end{aligned} \tag{2.13}$$

for all $(t,x) \in [0,T] \times [0,L]$ *and* $z, z_1, z_2 \in \mathbb{R}$. *Hence for every* $\mathcal{F}_0-$*measurable* $u_0 : [0,L] \times \Omega \to \mathbb{R}$ *with* $\int_0^L \mathbb{E}(|u_0(x,.)|^2)dx < \infty$, *there is a unique solution* u *to equation (2.10), characterized by the following property*

$$\sup_{x\in[0,L]} \mathbb{E}(|u(t,x,.)|^2) < \infty, \qquad \forall t \in [0,T].$$

Proof. First, we will establish the existence by initiating the following iterative scheme:

$$\begin{aligned} u_1(t,x,w) &:= \int_0^L G_t(x,z)u_0(z,w)dz \\ u_{n+1}(t,x,w) &:= u_1(t,x,w) + \int_0^t \int_0^L G_{t-s}(x,z)f(s,z,u_n(s,z,w))dzds \\ &+ \int_0^t \int_0^L G_{t-s}(x,z)g(s,z,u_n(s,z,w))W(ds,dz) \\ &+ \int_0^{t+} \int_0^L \int_U G_{t-s}(x,z)h(s,z,u_n(s-,z,w);y) \times M(ds,dz,dy,w) \end{aligned}$$

For every $n \in \mathbb{N}$, the function u_1 is continuous in $(t,x) \in (0,T] \times [0,L]$ almost surely. As a result, given our assumptions, the three integrals in u_2 are well-defined. There exists a cádlág modification of u_2 for $t \in (0,T]$ for all $x \in [0,L]$ and for almost all $w \in \Omega$. By induction, the same holds for u_n for each $n > 2$.

From now on, we can assume that u_n is cádlág in $t \in (0,T]$ for all $x \in [0,L]$ and for almost all $w \in \Omega$. Therefore, u_n is a cádlág process with $u_n(0,x,w) = u_0(x,w)$ for each $n \in \mathbb{N}$. It is worth noting that for any fixed $t > 0$, we can apply the Schwarz inequality, we have:

$$\begin{aligned} \mathbb{E}(|u_1(t,x,.)|^2) &\leq \int_0^L [G_t(x,z)]^2 dz \mathbb{E} \int_0^L [u_0(z,.)]^2 dz \\ &\leq C.t^{-\frac{1}{2}} \int_0^L \mathbb{E}([u_0(z,.)]^2)dz \end{aligned}$$

This implication signifies that

$$\sup_{x\in[0,L]} \mathbb{E}([u_1(t,x,.)]^2) < \infty, \quad \forall t \in [0,T] \tag{2.14}$$

Furthermore,

$$\int_0^t \underset{x\in[0,L]}{sup}\ \mathbb{E}([u_1(s,x,.)]^2)ds \leq cont.t^{\frac{1}{2}} < \infty, \quad \forall t \in [0,T]$$

Moreover, by (2.11) we have

$$\underset{x\in[0,L]}{sup}\ \mathbb{E}([u_1(t,x,.)]^2) \leq cont.t^{-\frac{1}{2}}(t^{\frac{1}{2}} + t^{\frac{3}{2}-\alpha}) < \infty, \quad \forall t \in [0,T]\,. \tag{2.15}$$

Now we have

$$H_n(t) := \underset{x\in[0,L]}{sup}\ \mathbb{E}\left\{|u_{n+1}(t,x,.) - u_n(t,x,.)|^2\right\}, \quad n \in \mathbb{N}.$$

Then by (2.13) and (2.14) we conclude that

$$H_1(t) < \infty.$$

Since $\{u_n(t,x,.)\}_{n\in\mathbb{N}}$ converges to $u(t,x,.)$ uniformly in $L^2(\Omega,\mathcal{F},P)$ for $(t,x)\in[0,T]\times[0,L]$, we can take the $L^2(\Omega,\mathcal{F},P)$-limit as $n\to\infty$ by integrating over the variable s in the interval $[0,T]$ and taking the supremum over the variable z in the interval $[0,L]$. Consequently, we obtain the following result:

$$\begin{aligned} u(t,x,w) &= u_1(t,x,w) + \int_0^t\int_0^L G_{t-s}(x,z)f(s,z,u(s,z,w))dzds \\ &+ \int_0^t\int_0^L G_{t-s}(x,z)g(s,z,u(s,z,w))W(ds,dz) \\ &+ \int_0^{t+}\int_0^L\int_U G_{t-s}(x,z)h(s,z,u(s-,z,w);y) \\ &\times\ M(ds,dz,dy,w). \end{aligned}$$

Which completes the proof of existence.

The uniqueness is proved as follows. Suppose u and u' are two solution of (2.10), let us set

$$H(t) := \underset{x\in[0,L]}{sup}\ \mathbb{E}\left[|u(t,x,.) - u'(t,x,.)|^2\right], \quad t\in[0,T]\,,$$

then clearly $H(t) < \infty, \forall t\in[0,T]$. Furthermore, we have

$$H(t) \leq \frac{C^n}{(n-1)!}\int_0^t H(s)(t-s)ds, \quad t\in[0,T]\,,$$

which implies that $H\equiv 0$ and from which we obtain the uniqueness.

□

Chapter 3

Numerical analysis of SPDEs

Stochastic optimal control is a field of study that deals with decision-making in dynamic systems under uncertainty. It combines principles from control theory and stochastic processes to determine the optimal control actions in situations where both the system dynamics and external influences are subject to random variations.

The goal of stochastic optimal control is to determine a control policy that optimizes a certain objective function, often represented as an expected value or a utility function. The control policy specifies the optimal actions to be taken at each point in time, based on available information and the underlying stochastic dynamics. The policy may be time- or history-dependent, meaning it can consider the entire history of observations.

To solve stochastic optimal control problems, various mathematical and computational techniques are employed. These include dynamic programming, stochastic calculus, optimization methods, and numerical approximation schemes. The optimal control problem is typically formulated as a mathematical optimization problem, and the solution can be obtained through analytical or numerical methods.

One of the cases that give us a SPDE is the study of stochastic optimal control (SOC) in which the coefficient depend on w. This will produce a Hamilton-Jacobi-Bellman equation (HJB) with random coefficients means we will have SPDE.

3.1 Stochastic optimal control

To initiate our discussion, we examine an ordinary differential equation (ODE) of the following form:

$$\begin{cases} \dot{x}(t) = f(x(t)) & (t > 0) \\ x(0) = x_0 \end{cases} \tag{3.1}$$

We are given an initial point $x_0 \in \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$. The objective is to find a curve $x : [0, \infty) \to \mathbb{R}^n$, which represents the dynamic evolution of the state of a "system".

To generalize the scenario, we introduce additional "control" parameters belonging to a set $A \subset \mathbb{R}^m$. This means that $f : \mathbb{R}^n \times A \to \mathbb{R}^n$. Thus, if we choose a specific value $a \in A$ and consider the corresponding dynamics, we have:

$$\begin{cases} \dot{x}(t) = f(x(t), a) & (t > 0) \\ x(0) = x_0, \end{cases}$$

This gives us the system's evolution when the parameter is fixed at the value a throughout the process.

Another possibility is to vary the parameter value as the system evolves. For example, let us consider the function $\alpha : [0, \infty) \to A$ defined as follows:

$$\alpha(t) = \begin{cases} a_1 & 0 \leq t \leq t_1 \\ a_2 & t_1 < t \leq t_2 \\ a_2 & t_2 < t \leq t_3 \qquad \text{etc.} \end{cases}$$

In general, we refer to a function $\alpha : [0, \infty) \to A$ as a control. For each control, we analyze the ODE given by:

$$\begin{cases} \dot{x}(t) = f(x(t), \alpha(t)) & (t > 0) \\ x(0) = x_0, \end{cases} \tag{3.2}$$

3.1.1 Cost function

The cost function is defined by:

$$J(x, t; u) := \int_0^T r(x(t), \alpha)dt + g(x(T)), \tag{3.3}$$

In the given problem, we consider the control $\alpha()$ and the corresponding solution $x(.)$ to the ordinary differential equation (ODE). The functions $r : \mathbb{R}^n \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ represent the running cost and the terminal cost, respectively. The time horizon is denoted by $T > 0$.

Definition 3.1. *[10] Suppose for given $\alpha^*(.)$, $\underset{\alpha \in A}{sup} J(\alpha(.)) = J(\alpha^*(.))$.*
We want

$$J[\alpha^*(.)] \geq J[\alpha(.)]$$

where, α^* called the optimal control of the problem (3.3).

3.1.2 Dynamic programming

Let consider the following SDE

$$(SDE) \qquad \begin{cases} dX(s) = f(X(s), A(s))ds + \sigma dW(s) & (t \leq s \leq T) \\ X(t) = x. \end{cases} \tag{3.4}$$

Then

$$X(\tau) = x + \int_t^\tau f(X(s), A(s))ds + \sigma [W(\tau) - W(t)]$$

for all $t \leq \tau \leq T$. Additionally, we introduce the expected cost function functional as follows:

$$J_{x,t}[A(.)] := \mathbb{E}\left\{ \int_t^T r(X(s), A(s))ds + g(X(T)) \right\}.$$

The value function is

$$v(x,t) := \underset{A(.) \in \mathcal{A}}{sup} J_{x,t}[A(.)].$$

Remark 3.1. *To use the dynamic programming method, we need to*

- *derive a partial differential equation (PDE) that is satisfied by the function v.*
- *use this PDE to construct an optimal control $A^*(.)$.*

3.1.3 A PDE for the value function

Consider any control $A(.)$, and assume that we use it for times $t \leq s \leq t+h$ for some $h > 0$. After that, we switch to the optimal control. In this case, we have the following :

$$v(x,t) \geq \mathbb{E}\left\{\int_t^{t+h} r(X(s),A(s))ds + v(X(t+h),t+h)\right\}. \tag{3.5}$$

Furthermore, the inequality in (3.5) becomes an equality if we choose $A(.) = A^*(.)$, which corresponds to an optimal control. From equation (3.5), it is evident that for any control $A(.)$, we have the following relationship:

$$\begin{aligned} 0 &\geq \mathbb{E}\left\{\int_t^{t+h} r(X(s),A(s))ds + v(X(t+h),t+h) - v(x,t)\right\} \\ &= \mathbb{E}\left\{\int_t^{t+h} r(X(s),A(s))ds\right\} + \mathbb{E}\left\{v(X(t+h),t+h) - v(x,t)\right\} \end{aligned}$$

By Itô's formula:

$$\begin{aligned} dv(X(s),s) &= v_t(X(s),s)ds + \sum_{i=1}^n v_{x_i}(X(s),s)dX^i(s) + \frac{1}{2}\sum_{i,j=1}^n v_{x_ix_j}(X(s),s)dX^i(s)dX^j(s) \\ &= v_t ds + \nabla_x v.(f(X(s),A(s))ds + \sigma dW(s))\frac{\sigma^2}{2}\Delta v ds. \end{aligned}$$

It follows that

$$\begin{aligned} v(X(t+h),t+h) - v(X(t),t) &= \int_t^{t+h} (v_t + \nabla_x v.f(X(s),A(s)) + \frac{\sigma^2}{2}\Delta v)ds \\ &+ \int_t^{t+h} \sigma\nabla_x v dW(s); \end{aligned}$$

and therefore, we can take expected values to deduce that

$$\mathbb{E}\left[v(X(t+h),t+h) - v(x,t)\right] = \mathbb{E}\left[\int_t^{t+h} (v_t + \nabla_x v.f(X(s),A(s)) + \frac{\sigma^2}{2}\Delta v)ds\right].$$

We can then derive the formula

$$\mathbb{E}\left[\int_t^{t+h} (r(X(s),A(s)) + v_t + \nabla_x v f(X(s),A(s)) + \frac{\sigma^2}{2}\Delta v)ds\right] \leq 0.$$

Divide by h:

$$\mathbb{E}\left[\frac{1}{h}\int_t^{t+h} r(X(s),A(s)) + v_t(X(s),s) + f(X(s),A(s)).\nabla_x v(X(s),s) + \frac{\sigma^2}{2}\Delta v(X(s),s))ds\right] \leq 0.$$

If $h \to 0$, recall that $X(t) = x$ and set $A(t) := a \in A$, we see that

$$r(x,a) + v_t(x,t) + f(x,a).\nabla_x v(x,t) + \frac{\sigma^2}{2}\Delta v(x,t) \leq 0.$$

The identity presented above holds for all x, t, and a, and it is indeed an equality when considering the optimal control. Consequently,

$$\sup_{a\in A}\left\{v_t(x,a) + f(x,a).\nabla_x v(x,t) + \frac{\sigma^2}{2}\Delta v(x,t) + r(x,a)\right\} = 0.$$

3.1.4 Stochastic Hamilton-Jacobi-Bellman equation

Theorem 3.1.1. *[10] The value function v for the stochastic control issue solves the following PDE.:*

$$(HJB) \qquad \begin{cases} v_t(x,t) + \frac{\sigma^2}{2}\Delta v(x,t) + \sup_{a\in A}\{f(x,a).\nabla_x v(x,t) + r(x,a)\} = 0, \\ v(x,T) = g(x) \qquad (x \in \mathbb{R}^n), \end{cases}$$

where $x \in \mathbb{R}^n$, $0 \leq t \leq T$, the stochastic Hamilton-Jacobi-Bellman equation corresponds to the above semilinear parabolic PDE.

Remark 3.2. *If the coefficients of the equation are random (depending on w) the PDE that we will get will be a SPDE.*

3.1.5 SDE driven by a Lévy process

Let's consider the stochastic differential equation (SDE):

$$\begin{cases} dY(t) = dY_u(t) = b(Y(t),u(t))dt + \sigma(Y(t),u(t))dB(t) + \int_{\mathbb{R}}\gamma(Y(t),u(t),\zeta)\widetilde{N}(dt,d\zeta), \\ Y(0) = y \in \mathbb{R}^k. \end{cases} \tag{3.6}$$

with $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$; $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $\gamma\,[0,T] \times \mathbb{R}^n \times \mathbb{R}^l_0 \to \mathbb{R}^{n\times l}$ are given functions, and let denote $\widetilde{N}(dt,d\zeta)$ the compensated jump measure of η defined:

$$\widetilde{N}(dt,d\zeta) \equiv N(dt,d\zeta) - \nu(d\zeta)dt$$

where $N(dt,d\zeta)$ is the differential notation of the random measure $N([0;t];U)$, and ζ can be considered as generic jump size.

The cost functional is :

$$J_u(y) = \mathbb{E}^y\left[\int_0^{\tau_s} f(Y(s),u(s))ds + g(Y(\tau_s))\mathbb{1}_{\{\tau_s<\infty\}}\right],$$

where $f(Y(s), u(s))$ is profit rate, $g(Y(\tau_s))$ is bequest function and $\tau_s = inf\,\{t \geq 0 : Y(t) \notin S\}$ with S is a given financial stability zone.
The problem is to find $u^* \in \mathcal{A}$ and $\Phi(y)$ such that

$$\Phi(y) = \sup_{u\in\mathcal{A}} J_u(y) = J_{u*}(y).$$

Theorem 3.1.2. *[16]*

1. *Assume that we can find a function $\varphi \in \mathcal{C}^2(\mathbb{R}^n)$ such that*

   (i) $A_v\varphi(y) + f(y, v) \leq 0$, for all $v \in \mathcal{V}$, where $\mathcal{V}$ is the set of possible control values, and

$$\begin{aligned} A_v\varphi(y) &= \sum_{i=1}^{k} b_i(y,v)\frac{\partial\varphi}{\partial y_i} + \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^T)_{ij}(y,v)\frac{\partial^2\varphi}{\partial y_i \partial y_j} \\ &+ \sum_m \int_{\mathbb{R}} \left\{\varphi(y+\gamma^{(k)}(y,v,\zeta)) - \varphi(y) - \nabla\varphi(y)\gamma^{(k)}(y,v,\zeta)\right\}\nu_k(d\zeta) \end{aligned}$$

   (ii) $\lim\limits_{t\to\tau_s} \varphi(Y(t)) = g(Y(\tau_s))\mathbb{1}_{\{\tau_s<\infty\}}$

   (iii) "Growth conditions:"

$$\begin{aligned} &\mathbb{E}^y\left[|\varphi(Y(\tau))| + \int_0^{\tau_s}\left\{|A\varphi(Y(t))| + \left|\sigma^T(Y(t))\nabla\varphi(Y(t))\right|^2\right\}dt\right] \\ +\ &\mathbb{E}^y\left[\int_0^{\tau_s}\left\{\sum_{j=1}^{l}\int_{\mathbb{R}}\left|\varphi(Y(t)+\gamma^{(j)}(Y(t),u(t),\zeta_j)) - \varphi(Y(t))\right|^2\nu_j(d\zeta_j)\right\}dt\right] < \infty \end{aligned}$$

   for all $u \in \mathcal{A}$ and all stopping time τ .

   (iv) $\{\varphi^-(Y(\tau))\}_{\tau\leq\tau_s}$ is uniformly integrable for all $u \in \mathcal{A}$ and $y \in \mathcal{S}$, where, in general, $x^- = max\,\{-x, 0\}$ for $x \in \mathbb{R}$.

   Therefore

$$\varphi(y) \geq \Phi(y).$$

2. *Suppose that for all $y \in \mathcal{S}$ we can find $v = \hat{u}(y)$ such that*

$$A_{\hat{u}(y)}\varphi(y) + f(y, \hat{u}(y)) = 0$$

   and $\hat{u}(y)$ is an admissible feedback control (Markov control), i.e. $\hat{u}(y)$ means $\hat{u}(Y(t))$. Then $\hat{u}(y)$ is an optimal control and

$$\varphi(y) = \Phi(y).$$

Remark 3.3. *This result is useful because, in a certain sense, it fundamentally reduces the highly complicated theoretical stochastic control problem to a classical problem of maximizing a function of (perhaps several) actual variable(s), namely the function* $v \mapsto A_v\varphi(y) + f(y,v); v \in \mathcal{V}$.

3.2 Derivation of the SPDE corresponding to the stochastic optimal control

Let's consider the stochastic optimal control

$$\begin{cases} dX_s = (AX_s + Bu_s)ds + BdW_s, \\ X_t = x \qquad\qquad x \in \mathbb{R} \end{cases} \tag{3.7}$$

A, B are two random variable in $\mathbb{R}$, and W is a $\mathbb{R}-$Brownian Motion. $u(.)$ is a stochastic process representing the control,

We consider the problem of minimizing, the cost functional

$$J(t,x;u) = \mathbb{E}[\int_t^T (X_s + \frac{1}{2}u_s^2)ds + X_T^2], \tag{3.8}$$

such that u is a stochastic process.

The value function for this problem is defined as follows:

$$v(t,x) = \underset{u\in A}{inf}\, J(t,x,u).$$

As usual, we say that the control $u(.) \in \mathcal{U}_s$ is optimal at (t,x) if $u(.)$ minimizes (3.8) one of the controls in $\mathcal{U}_s$.

We can write the (HJB) equation

$$\begin{cases} \partial_t v(t,x) + \underset{\alpha\in A}{inf}\left\{\frac{1}{2}B^2\partial_{xx}^2 v(t,x) + (Ax^2 + B\alpha)\partial_x v(t,x) + \frac{1}{2}\alpha^2 + x\right\} = 0, \\ v(x,T) = g(x). \end{cases} \tag{3.9}$$

We take the following equation:

$$\left\{\frac{1}{2}B^2\partial_{xx}^2 v(t,x) + (Ax^2 + B\alpha)\partial_x v(t,x) + \frac{1}{2}\alpha^2 + x\right\},$$

and we derive it with respect to α. After derivation we found $B\partial_x v(t,x) + \alpha = 0$, then we take out α and substitute it into the equation (3.9), Then we get the following (PDE) :

$$\partial_t v(t,x) + \frac{1}{2}B^2\partial_{xx}^2 v(t,x) + Ax^2\partial_x v(t,x) - \frac{1}{2}(B\partial_x v(t,x))^2 + x = 0. \tag{3.10}$$

3.3 Numerical result

3.3.1 Finite difference method

Finite difference methods are widely used for the numerical analysis of partial differential equations ($PDEs$). These methods approximate derivatives by using finite difference approximations, which convert the continuous PDE problem into a discrete problem that can be solved numerically on a grid.

To illustrate the process, let's consider a simple example of the one-dimensional heat equation:

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2} \tag{3.11}$$

where $v(t, x)$ is the unknown function to be solved, α is a constant diffusion coefficient, and x and t represent spatial and temporal variables, respectively.

To apply the finite difference method, we discretize the domain of the problem. Let's assume that we have a spatial domain $x \in [0, L]$ divided into N equally spaced grid points with a spacing $\Delta x = L/N$. Similarly, the time domain $t \in [0, T]$ is divided into M equally spaced time steps with a time increment $\Delta t = \frac{T}{M}$.

We approximate the derivatives using finite difference approximations. We can use forward, backward, or central difference schemes for the time derivative. Let's use the forward difference scheme:

$$\frac{\partial v}{\partial t} \approx \frac{v_{i+1,j} - v_{i,j}}{\Delta t}$$

where $v_{i,j}$ represents the value of v at the spatial grid point j and the time step i. For the spatial second derivative, we can use the central difference scheme:

$$\frac{\partial^2 v}{\partial x^2} \approx \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta x^2}$$

By substituting these approximations into the original PDE, we obtain a finite difference equation at each grid point:

$$\frac{v_{i+1,j} - v_{i,j}}{\Delta t} = \alpha \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta x^2}$$

This equation relates the values of v at the current time step $i + 1$ to the values at the previous time step i.

To solve the PDE numerically, we start from an initial condition $v(0,x)$ and iteratively compute the values of v at each time step using the finite difference equation. We can use methods like the explicit Euler method or the implicit Euler method to perform the time-stepping.

Once the time-stepping process is complete, we obtain the numerical solution of the PDE at different grid points over the spatial and temporal domains.

It's important to note that the choice of finite difference scheme (forward, backward, central), the grid spacing, and the time step size can affect the accuracy and stability of the numerical solution. Different $PDEs$ may require different types of finite difference approximations and numerical methods to ensure accurate and stable solutions. Additionally, more complex $PDEs$, such as nonlinear or higher-dimensional equations, may require advanced techniques beyond simple finite difference methods.

3.3.2 Examples

Example 1. Let's consider the partial differential equation (3.10) then we got from the SOC:

$$\partial_t v(t,x) + \frac{1}{2}B^2\partial^2_{xx}v(t,x) + Ax^2\partial_x v(t,x) - \frac{1}{2}(B\partial_x v(t,x))^2 + x = 0.$$

after using the previous numerical method 3.3.1 we obtain:

$$\frac{v_{i+1,j} - v_{i,j}}{\Delta t} + \frac{1}{2}B^2(\frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta x^2}) - \frac{1}{2}(B\frac{v_{i,j+1} - v_{i,j}}{\Delta x})^2 + Ax^2(\frac{v_{i,j+1} - v_{i,j}}{\Delta x}) + x = 0$$

Therefore

$$\begin{aligned} v_{i+1,j} &= v_{i,j} + \frac{\Delta t}{2\Delta x^2}\left[-B^2(v_{i,j+1} - 2v_{i,j} + v_{i,j-1}) + (B(v_{i,j+1} - v_{i,j}))^2\right] \\ &- \frac{\Delta t}{\Delta x}Ax^2(v_{i,j+1} - v_{i,j}) - \Delta t x = 0 \end{aligned}$$

where the initial condition of PDE (3.10) is $v(0,x) = g(x)$, for all $x \in \mathbb{R}$.

Assume that A is a random variable follows the normal distribution and B is a random variable follows the uniform distribution. for all $x \in [-10, 10]$, $\Delta x = 0.1$ and for all $t \in [0, 1]$, $\Delta t = 10^{-3}$, with the initial condition $g(x) = x^3 - 100$ and for five realizations we have:

Simulation of v for different realisations

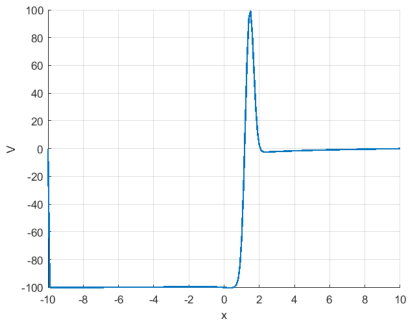

Figure 3.1: Plot of the solution of equation (3.10) for the first realisation of A and B.

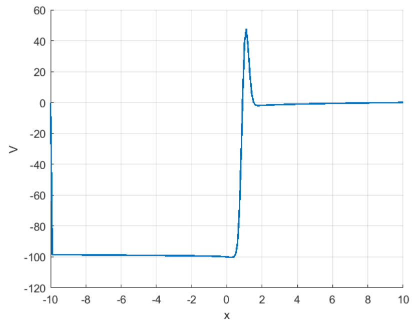

Figure 3.2: Plot of the solution of equation (3.10) for the second realisation of A and B.

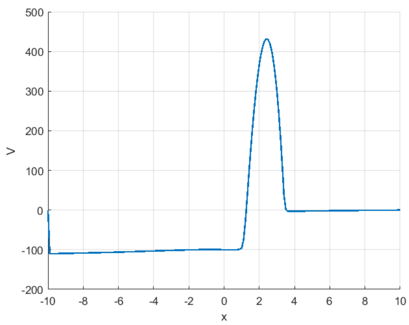

Figure 3.3: Plot of the solution of equation (3.10) for the third realisation of A and B.

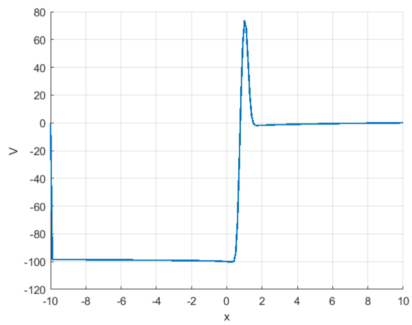

Figure 3.4: Plot of the solution of equation (3.10) for the forth realisation of A and B.

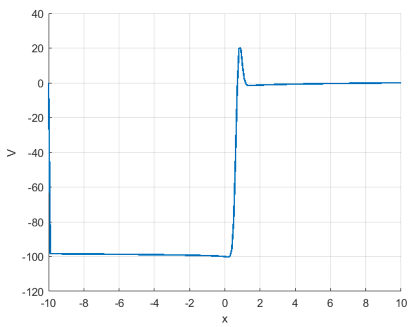

Figure 3.5: Plot of the solution ofequation (3.10) for the fifth realisationof A and B.

We remark that the five figures (3.1, 3.2, 3.3, 3.4 and 3.5) have the same form something almost linear then makes jump around $x = 1$ then decreasing then something almost linear.

Example 2. Let's consider the following $(SPDE)$:

$$\begin{cases} \partial_t v(t,x) = \partial^2_{xx} v(t,x) + a dL_t + b dB_t \\ v(0,x) = v_0, \end{cases} \tag{3.12}$$

where L is the Poisson process, B is the Brownian motion, a and b are constant coefficients.

To numerically analyze the stochastic partial differential equation (3.12) using finite difference methods (3.3.1), we will discretize the equation in both the spatial and temporal domains. Here's an overview of the numerical analysis using the finite difference method:

1. Discretize the spatial domain:
   - Divide the spatial domain Ω into a grid with N equally spaced points, resulting in a spatial grid spacing of Δx.
   - Denote the spatial grid points as $x_j = j \cdot \Delta x$ for $j = 0, 1, 2, \ldots, N$.
2. Discretize the temporal domain:
   - Divide the time domain $[0, T]$ into M equally spaced time steps, with a time step size of Δt.
   - Denote the time steps as $t_i = i \cdot \Delta t$ for $i = 0, 1, 2, \ldots, M$.
3. Approximate the stochastic terms:
   - For the Poisson process term dP, generate random numbers. The increments ΔP_i can be approximated using appropriate methods based on the properties of the Poisson process.
   - For the Brownian motion term dB, generate random numbers. The increments ΔB_i are $\mathcal{N}(0, \Delta t)$.
4. Construct the finite difference equation:
   - Substituting the spatial and stochastic approximations into the original SPDE, we obtain the following finite difference equation at each grid point:

$$\frac{v_{i+1,j} - v_{i,j}}{\Delta t} = \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta x^2} + a\Delta P_i + b\Delta B_i$$

This equation relates the values of v at the current time step t_{i+1} to the values at the previous time step t_i.

5. Solve the finite difference equation:
    - Start with an initial condition $v(0,x)$ and apply the finite difference equation iteratively in time.
    - For each time step t_i, solve the resulting system of algebraic equations to obtain the values of v at each grid point.

Here is the result of the simulation of v for different realizations with different initial conditions, where for all $x \in [-10, 10]$, $\Delta x = 0.1$ and for all $t \in [0, 1]$, $\Delta t = 10^{-2}$,

Simulation of v for different realisations with $a = 0$

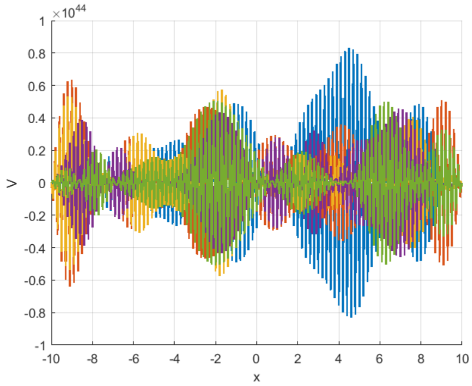

Figure 3.6: Simulation of v in (3.12) with $a = 0, b = 1$ and initial condition is $v(0, x) = 0.05$.

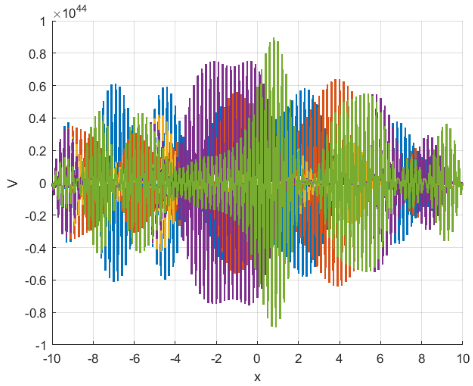

Figure 3.7: Simulation of v in (3.12) with $a = 0, b = 1$ and the initial condition is $v(0, x) = 0.1$.

The figures (3.6) and (3.7) are the plot of $v(T, x, w_i)$ where $i = 1 : 5$ with respect to $x \in \mathbb{R}$.

Simulation of v for different realisations with $b = 0$

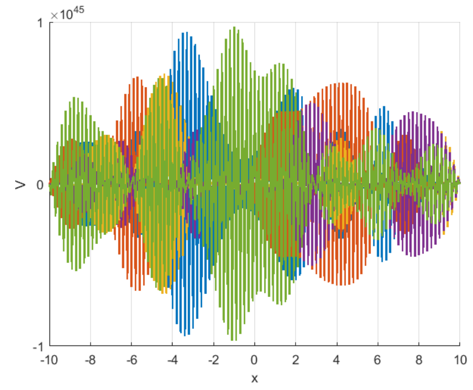

Figure 3.8: Simulation of v in (3.12) with $a = 1, b = 0$ and the initial condition is $v(0, x) = 0.05$.

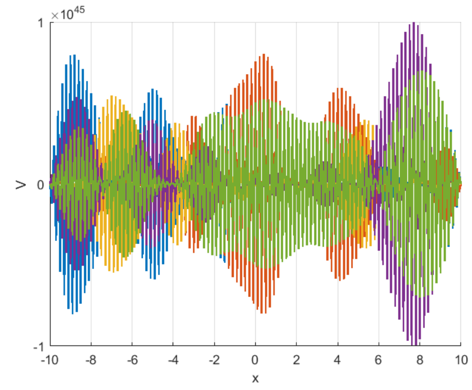

Figure 3.9: Simulation of v in (3.12) with $a = 1, b = 0$ and the initial condition is $v(0, x) = 0.1$.

The figures (3.8) and (3.9) are the plot of $v(T, x, w_i)$ where $i = 1 : 5$ with respect to $x \in \mathbb{R}$.

Simulation of v for different realisations with $a = 1$ and $b = 1$

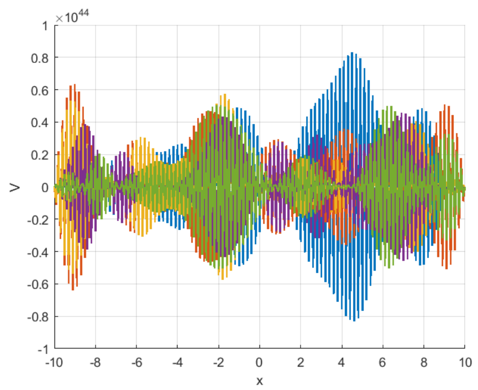

Figure 3.10: Simulation of v in (3.12) with $a = 1, b = 1$ and the initial condition is $v(0, x) = 0.05$.

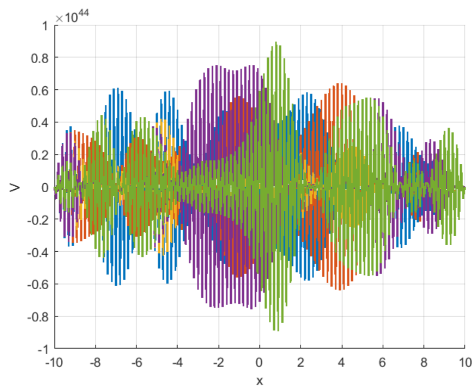

Figure 3.11: Simulation of v in (3.12) with $a = 1, b = 1$ and the initial condition is $v(0, x) = 0.1$.

The figures (3.10) and (3.11) are the plot of $v(T, x, w_i)$ where $i = 1 : 5$ with respect to $x \in \mathbb{R}$.

The behavior of the heat equation, namely equation (3.12) when $a = b = 0$. with the given initial condition, is that the temperature in the border changes a lot and became almost stable in the middle like in the case of the thermostat. Now when we added noise like the Poisson process or Brownian Motion, the temperature even in the middle will change. This is explained by: for example, the case where we added another liquid in the thermostat with different temperatures.

Conclusion

In this master thesis, we have embarked on a comprehensive analysis of stochastic partial differential equations (SPDEs) driven by Lévy noise. Our work has aimed to deepen our understanding of the behavior and properties of these equations, we finished our study with stochastic optimal control and a numerical simulation, and we presented two examples of SPDEs one that came from stochastic optimal control and another one with additive noises, Poisson process and Brownian Motion.

Furthermore, after a synthesis of most researchers carried out on SPDEs driven by Lévy, we deduce that it is really necessary to generalize these results with other non-Gaussian processes since there are currently more phenomena that are modeled by non-Gaussian processes.

We investigated the well-posedness, regularity, and numerical approximation methods of SPDEs, we have advanced our understanding of these equations and their practical applications. Our research contributes to the existing body of knowledge in this field and opens up avenues for further exploration, including the extension of these analyses to more complex systems and the development of new numerical methods. Overall, this research provides a solid foundation for future advancements in the analysis of SPDEs driven by Lévy noise in finite-dimensional spaces, facilitating the understanding of complex systems affected by random fluctuations and uncertainties.

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