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STOCHASTIC INTEGRATION OF NON-ADAPTED PROCESSES WITH RESPECT TO PROCESSES NOT BEING SEMIMARTINGALES.

Thèse présentée en vue de l'obtention du grade de **Docteur**

Université de Saida

Discipline : MATHEMATIQUES

Spécialité: Modèles Stochastiques, Statistique et Applications

par

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Soutenue le 16/03/2023 devant le jury composé de

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Dedication

This thesis is dedicated to

my parents, who have sincerely raised me with their caring and have actively supported me in my determination to find and realize my potential.

my hasband Abdelhak: I am profoundly and eternally indebted to him for his love and encouragement, without forgetting a special dedicate to my girls Kawthar, Ines and Amira

my sisters Fatima, Amel, Chaimaa, Anfel and brother Khelifa, who have supported me all the way since the beginning of my study.

my second family Fakhat, from big to small, and the best sisters ***Asmaa***, ***Marwa*** *and* ***Hadjar***

those who have been deprived from their right to study and to all those who believe in the richness of learning.

Acknowledgments

First of all, I thank *Allah* for His help and blessing.

This thesis would not have been possible without the invaluable support and guidance of so many people, to whom I am greatly indebted:

I want to express my great appreciation and deepest gratitude to my supervisor, **Mr.** ***KANDOUCI Abdeldjebbar***, without whom this thesis would not have been emerged. It was a great honor and an immense pleasure to be able to accomplish this work under his direction. I could benefit from his sound advice and careful guidance, his availability, his kindness and his constant support to carry out this work. Again thank you !

I want to thank very warmly **Mlle.** ***RAHMANI Sâadia*** for chairing the jury. I am very honored that **Mr.** ***MECHAB Boubakeur***, **Mr.** ***BENAISSA Samir***, **Mr.** ***GUENDOUZI Toufik*** and **Melle.** ***BENZIADI Fatima*** have agreed to participate in my jury.

Many thanks go to all my friends and colleague: **A.Mordji, F.Ghazouani**, for their encouragement, backing, in particular during the time of preparation of this work. I hope to get the chance to return the favor.

The most important persons are saved for last: my parents deserve special mention for their inseparable support and prayers.

I would like to thank everybody who has contributed to the successful realization of this thesis, as well as express my apology that I could not mention every one by his or her name.

Contents

Introduction

As is well-known, the classical Brownian motion is a stochastic process which is selfsimilar of index $1/2$ and has stationary increments. It is actually the only continuous Gaussian process (up to a constant factor) to have these two properties that are often observed in the "real life", for instance in the movement of particles suspended in a fluid, or in the behavior of the logarithm of the price of a financial asset. More generally, it is natural to wonder whether there exists a stochastic process which would be at the same time Gaussian, with stationary increments and selfsimilar, but not necessarily with an index $1/2$ as in the Brownian motion case. Such a process happens to exist, and was introduced by Kolmogorov [34] in the early 1940s for modeling turbulence in liquids.
The name fractional Brownian motion, which is the terminology everyone uses nowadays, comes from the paper by Mandelbrot and Van Ness [42]. The law of fractional Brownian motion relies on a single parameter H between 0 and 1, the so-called Hurst parameter or selfsimilarity index. The fractional Brownian motion is interesting for modeling purposes, as it allows the modeler to adjust the value of H to be as close as possible to its observations. It is worthwhile noting at this stage, however, that the picture is not as rosy as it seems. Indeed, except when its selfsimilarity index is $1/2$, fractional Brownian motion is neither a semimartingale, nor a Markov process. As a consequence, its toolbox is limited, so that solving problems involving fractional Brownian motion is often a non-trivial task. On the positive side, the fractional Brownian motion offers new challenges for the specialists of stochastic calculus!

If $H \neq \frac{1}{2}$, the fractional Brownian motion is not a semimartingale and we cannot apply the stochastic calculus developed by Itô in order to define stochastic integrals with respect to fractional Brownian motion. Different approaches have been used in order to construct a stochastic calculus with respect to fractional Brownian motion and we can mention the following contributions to this problem:

- Lin [41] and Dai and Heyde [16] defined stochastic integrals with respect to the fractional Brownian motion with parameter $H > \frac{1}{2}$ using a pathwise Riemann-Stieltjes method. The integrator must have finite $p-$variation where $\frac{1}{p} + H > 1$.

- The stochastic calculus of variations (see [48]) with respect to the Gaussian process B is a powerful technique that can be used to define stochastic integrals. More precisely, as in the case of the Brownian motion, the divergence operator with respect to B can be interpreted as a stochastic integral. This idea has been developed by Decreusefond and Üstünel [18, 17], Carmona and Coutin [11], Alòs, Mazet and Nualart [4, 3], Duncan, Hu and Pasik-Duncan [22] and Hu and Øksendal [29]. The integral constructed by this method has zero mean, and can be obtained as the limit of Riemann sums defined using Wick products.

- Using the notions of the fractional integral and the derivative, Zähle has introduced in [43] a pathwise stochastic integral with respect to the fractional Brownian motion B with parameter $H \in (0, 1)$. If the integrator has $\lambda-$Hölder continuous paths with $\lambda > 1 - H$, then this integral can be interpreted as a Riemann-Stieltjes integral and coincides with the forward and Stratonovich integrals studied in [4] and [1].

There are some representations of the fractional Brownian motion as a Wiener integral (i.e., with respect to Brownian motion). We would like to have such Lévy Hida representation. We have that the natural filtration of the Brownian motion and of the fractional Brownian motion that is generated, coincides comparing to the Mandelbrot Van-Ness representation.

This thesis consists of four chapters. In the first chapter, we focus on the theory of stochastic integration. We devote Chapter 2 to a brief summary of the theory of stochastic and fractional calculus. In this chapter we give definitions and properties of the needed theory. We briefly recall some basic properties of the Brownian motion, semimartingales and the fractional Brownian motion, then we discuss integration with respect to Wiener processes.

In Chapter 3, we give a property of the instant independence and stochastic integration. The results presented in this chapter generalize those presented in Ayed and al[6].

In Chapter 4, we work at a new paper submitted (this paper contains a new approach for stochastic integration with respect to multifractional Brownian motion). The results presented in this chapter are based on the results obtained in the paper[25].

Chapter 1

Preliminary Background

In this chapter the basic concepts and results concerning the stochastic calculus of continuous stochastic processes in Euclidean spaces are established. We take some introductory facts from probability theory. For more details, we refer the reader to [9, 24, 62, 32, 33]. We first start with the stochastic process, the Wiener process and the fractional Brownian motion.

1.1 Basic Definitions

In this section the basic notations of the theory of stochastic calculus are considered.

1.1.1 Gaussian processes

Definition 1.1. A real-valued stochastic process $(X_t)_{t\geq 0}$ is a *Gaussian process* if every finite linear combination of $(X_t)_{t\geq 0}$ is a Gaussian $r.v, i.e.$

$$\forall n, \forall t, 1 \leq i \leq n, \forall a, \sum_{i=1}^{n} a_i X_{t_i} \ \ is \ a \ Gaussian \ r.v.$$

Definition 1.2. Let $X = (X_t)_{t\geq 0}$ et $Y = (Y_t)_{t\geq 0}$ be two stochastic processes defined on the same probability space. If $\mathbb{P}(X_t = Y_t) = 1$ for all $t \geq 0$, we say that X and Y are *modifications* of each other.

Definition 1.3. Let X and X' be defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then X and X' are *indistinguishable* if and only if

$$\mathbb{P}(\{w \in \Omega : X_t(w) = X'_t(w) \, \forall t \geq 0\}) = 1.$$

There is a chain of implications:

$$indistinguishable \;\Rightarrow\; modification.$$

Definition 1.4. let $X = (X_t)_{t\in\mathbb{T}}$ and $Y = (Y_t)_{t\in\mathbb{T}}$ be two stochastic processes, possibly defined on two different probability space. We say that X and Y have the *same law*, and we write $X \overset{law}{=} Y$, to indicate that $(X_{t_1}, ..., X_{t_d})$ and $(Y_{t_1}, ..., Y_{t_d})$ have the same law for all $d \geq 0$ and all $t_1, ..., t_d \in \mathbb{T}$.

Proposition 1.1.1. Two Gaussian processes have the same law if and only if they have the same mean and covariance functions.

Definition 1.5. A symmetric function $\Gamma : \mathbb{T}^2 \to \mathbb{R}$ is of *positive type* if

$$\sum_{k,l=1}^{d} a_k a_l \Gamma(t_k, t_l) \geq 0$$

for all $d \geq 1$, $t_1, ..., t_d \in \mathbb{T}$ and $a_1, ..., a_d \in \mathbb{R}$.

Theorem 1.1. ***(Kolmogrov)***
Consider a symmetric function $\Gamma : \mathbb{T}^2 \to \mathbb{R}$. Then, there exists a centered Gaussian process $X = (X_t)_{t\in\mathbb{T}}$ having Γ for covariance function if and only if Γ is of a positive type.

1.1.2 Continuity

Definition 1.6. A stochastic process $(X_t)_{t\geq 0}$ is said to be *continuous* if $\mathbb{P}(\{w \in \Omega : t \to X_t(w) \text{ is continuous}\}) = 1$, *i.e.* its sample paths are continuous *a.s.*

Definition 1.7. A stochastic process $(X_t)_{t\geq 0}$ is said to be *stochastically continuous* at t if $X_{t+h} \overset{\mathbb{P}}{\to} X_t$ as $h \to 0$.

Definition 1.8. A stochastic process is said to be *càdlàg* (resp. *càglàd*) if all sample paths are right-continuous with left-hand limits (resp. left-continuous with right-hand limits.

Lemma 1.1.1. ***(Kolmogrov-Čentsov)***
Fix a compact interval $\mathbb{T} = [0, T] \subset \mathbb{R}_+$, and let $X = (X_t)_{t\in\mathbb{T}}$ be a centered Gaussian process. Suppose that there exists $C, \eta > 0$ such that, for all $s, t \in \mathbb{T}$,

$$\mathbb{E}[(X_t - X_s)^2] \leq C \mid t - s \mid^\eta . \tag{1.1}$$

*Then, for all $\alpha \in (0, \eta/2)$, there exists a modification Y of X with α-*Hölder *continuous paths. In particular, X admits a continuous modification.*

Proof. Fix $t > s$. Since X is Gaussian and centered, we have that

$$X_t - X_s \overset{law}{=} \sqrt{\mathbb{E}[(X_t - X_s)^2]}\, G$$

where $G \sim \mathcal{N}(0, 1)$. We deduce from (1.1) that, for all $p \geq 1$,

$$\mathbb{E}[\mid X_t - X_s \mid^p] \leq C^{p/2}\mathbb{E}[\mid G \mid^p] \mid t - s \mid^{\eta p/2} .$$

Therefore, the general version of the classical Kolmogrov-Čentsov lemma applies and gives the desired result.

1.1.3 Filtration and measurability

Definition 1.9. A *filtration* on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family $(\mathcal{F}_t)_{t\in\mathrm{T}}$ of sub σ-field of $\mathcal{F}$.
A measurable space endowed with a filtration $(\mathcal{F}_t)_{t\in\mathrm{T}}$ is said to be a filtered space.

Definition 1.10. The filtration is said to be *right continuous* if $\mathcal{F}_{t_+} = \mathcal{F}_t$, $\forall t \geq 0$, where $\forall t > 0$ we set, $\mathcal{F}_{t_+} = \bigcap_{s>t} \mathcal{F}_s$

Definition 1.11. A filtration is said to be *complete* if the $\mathbb{P}$-negligible set of $\mathcal{F}_\infty$ are in $\mathcal{F}_0$ and if the probability space is complete.

Definition 1.12. A filtration satisfies the *usual condition* if it is right continuous and complete.

Remark 1.1. *The interests to work with filtrations which are satisfying the usual condition are that every kind of limit of the adapted processes is still adapted. Moreover, every modification of a progressively measurable process stays progressively measurable.*

Definition 1.13. ***(Measurable Process)***
A stochastic process $(X_t)_{t\geq 0}$ is *measurable* if the application $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is measurable w.r.t $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ i.e. if

$$\forall A \in \mathcal{B}(\mathbb{R}),\ \{(t,w) : X_t(w) \in A\} \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$$

The process $(X_t)_{t\geq 0}$ is said to be $(\mathcal{F}_t)_{t\geq 0}$ *adapted*, if X_t is $\mathcal{F}_t$ measurable for each $t \geq 0$.
The process $(X_t)_{t\geq 0}$ is obviously adapted with respect to the natural filtration.

Proposition 1.1.2. *A continuous stochastic process is measurable.*

Proof. Let $(X_t)_{t\geq 0}$ a continuous stochastic process. First, we show that for $A \in \mathcal{B}(\mathbb{R}_+)$, we have

$$\{(t,w) \in [0,1] \times \Omega,\ \ X_t(w) \in A\}\ \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}. \tag{1.2}$$

For $n \in \mathbb{N}$, let

$$X_t^n = X_{\frac{[2^n t]}{2^n}},\ \ t \in [0,1],$$

since the paths of X^n are piecewise constant, we have that

$$\{(t,w) \in [0,1] \times \Omega,\ \ X_t^n(w) \in A\}\ \ \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}.$$

Besides, $\forall t \in [0,1],\ \ w \in \Omega$, we have

$$\lim_{n\to\infty} X_t^n(w) = X_t(w).$$

Then we have (1.2). By the same argument, we can prove that $\forall k \in \mathbb{N}$,

$$\{(t,w) \in [k,k+1] \times \Omega,\ \ X_t(w) \in A\}\ \ \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}.$$

Since

$$\bigcup_{k\in\mathbb{N}} \{(t,w) \in [k,k+1]\times\Omega,\ \ X_t(w)\in A\} = \{(t,w)\in\mathbb{R}\times\Omega,\ \ X_t(w)\in A\},$$

we have the result.

Definition 1.14. ***(Progressively Measurable Process)***
A process is *progressively measurable* if for each t its restriction to the time interval $[0,t]$, is measurable with respect to $\mathcal{B}_{[0,t]}\otimes\mathcal{F}_t$, where $\mathcal{B}_{[0,t]}$ is the Borel σ-algebra of subsets of $[0,t]$.

Remark 1.2. *Note that every progressively measurable process is adapted (and measurable). Besides, as well as in the Proposition 1.1.2, a continuous process adapted to $(\mathcal{F}_t)$ is progressively measurable. More precisely, any càdlàg or càglàd process is progressively measurable.*

Definition 1.15. Let $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$ a filtered space. A process $(X_t)_{t\in\mathrm{T}}$ is said to be *predictable* (resp. *optional*) if it is an càglàd (resp. càdlàg) $\mathcal{F}_t$-adapted process.
We note the σ-field generated by càglàd (resp. càdlàg) $\mathcal{F}_t$-adapted process by $\mathcal{P}$ (resp. $\mathcal{O}$).

In fact, there is this inclusion chain

$$\underbrace{\mathcal{P}}_{\textit{predictable processes}} \subset \underbrace{\mathcal{O}}_{\textit{optional processes}} \subset \underbrace{\mathcal{P}rog}_{\textit{progressively measurable}} \subset \underbrace{\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}_\infty}_{\textit{measurable}}$$

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_s\}$ satisfying the *usual conditions*:

- $\mathcal{F}_s = \bigcap_{t>s}\mathcal{F}_t$ for all $s\geq 0$;
- All $A\in\mathcal{F}$ with $\mathbb{P}(A)=0$ are contained in $\mathcal{F}_t$.

A family $(X(t),t\geq 0)$ of $\mathbb{R}^d$-valued random variables on $(\Omega,\mathcal{F},\mathbb{P})$ is called a *stochastic process*; this process is adapted if all $X(t)$ are $\mathcal{F}_t$-measurable. We denote $\mathcal{B}$, the

Borel σ-field on $[0, \infty)$. The process X is measurable if $(t, \omega) \mapsto X(t, \omega)$ is a $\mathcal{B} \bigotimes \mathcal{F}$-measurable mapping. We say that $(X(t), t \geq 0)$ is *continuous* if the *trajectories* $t \mapsto X(t, \omega)$ are continuous for all $\omega \in \Omega$.

One can show that a process is measurable if it is right-continuous ([32], Prop. 1.13).

1.2 Semimartingales

The continuous semimartingales are important in stochastic calculus, because they are the most general class of integrators for which the classical stochastic integration is defined. Besides, as we know, the fractional Brownian motion is not a semimartingale for $H \neq \frac{1}{2}$. Therefore, its study is crucial for our project. Intuitively, a semimartingale is the sum of a local martingale with a process of finite variation. Note there are different types of semimartingales, but we only give a usual definition. For the deepest study of semimartingales, the reader could study [65] or in [44] VII.2.

Definition 1.16. Let $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ be an integrable process, then X is a:

i) Martingale if and only if $\mathbb{E}(X_t|\mathcal{F}_s) = X_s \quad a.s.$ for $0 \leq s \leq t < \infty$

ii) Supermartingale if and only if $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s \quad a.s.$ for $0 \leq s \leq t < \infty$

iii) Submartingale if and only if $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s \quad a.s.$ for $0 \leq s \leq t < \infty$

Definition 1.17. $M = \{M_t, \mathcal{F}_t, t \geq 0\}$ is a *local-martingale* if and only if there exists a sequence of stopping times T_n tending to infinity such that $M^{T_n}$are martingales for all n. The space of local martingales is denoted $\mathcal{M}_{loc}$, and the subspace of continuous local martingales is denoted $\mathcal{M}_{loc}^c$.

Definition 1.2. *A stochastic process is called a semimartingale if it can be written in the form*

$$X_t = X_0 + M_t + A_t,$$

where $(M_t)_{t\in\mathbb{R}^+}$ is a local martingale vanishing at 0 and $(A_t)_{t\in\mathbb{R}^+}$ is a right-continuous $(\mathcal{F}_t)$ -adapted process of finite variation vanishing at 0.

Remark 1.3. *Note that if we deal with the same filtration, this decomposition is unique.*

The next proposition will be on the basis of the integration with respect to semimartingales. Moreover, it is deeply linked with the non-semimartingale property of the fractional Brownian motion. Therefore, the Itô calculus cannot be applied for the fractional Brownian motion.

Proposition 1.2.1. *A continuous semimartingale $(X_t)_{t\geq 0} = (M_t + A_t)_{t\geq 0}$ has a finite quadratic variation and $\langle X, X\rangle_t = \langle M, M\rangle_t$.*

Proof. The proof is given in [55] p.128

1.3 Brownian Motion

In what follows, we will state a number of important facts regarding Brownian motion. Most of the proofs are skipped. For more details about the proofs, we refer the reader to ([32]).

1.3.1 Definition and properties

Definition 1.18. ([33]). *A stochastic process $(W_t)_{t\in\mathbb{R}_+}$ is called a standard Brownian motion if it satisfies the following conditions:*

1. $\mathbb{P}[W_t(\omega) = 0] = 1$, *for all* $\omega \in \Omega$,

2. *Independent increments. For each* $0 \leq t_1 < t_2 < \ldots < t_m$, *the real valued*

$$W(t_1), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1}),$$

are independent.

3. *Stationary increments. For each $0 \leq s < t, W(t) - W(s)$ is a centered real valued normally distributed with variance $t-s$, i.e.,*

$$W(t) - W(s) \sim \mathcal{N}(0, t-s).$$

4. *Almost all sample paths of W_t are continuous functions, i.e.,*

$$\mathbb{P}(\omega \in \Omega, t \to W_t(\omega) \quad is \quad continuous) = 1.$$

Remark 1.3. 1. Notice that the natural filtration of the Brownian motion is $\mathcal{F}_t^W = \sigma(W_s, s \leq t)$.

2. We can define the Brownian motion without the last condition of continuous paths, because with a stochastic process satisfying the second and the third conditions, by applying the Kolmogorov's continuity theorem, there exists a modification of $(W_t)_{t \in \mathbb{R}_+}$ which has continuous paths a.s.

3. A Brownian motion is also called a Wiener process since it is the canonical process defined on the Wiener space.

Proposition 1.3.1. ([24]). *The Brownian motion $(W_t)_{t \in \mathbb{R}_+}$ is a Gaussian process with mean 0 and covariance function $Cov(W_t, W_s) = s \wedge t$.*

Proof. We have that $W_t = W_t - W_0$. Thus, $W_t \sim \mathcal{N}(0, t)$ by definition. Moreover, without loss of generality, we assume $s < t$. Hence, we have

$$E(W_s W_t) = E(W_s(W_t - W_s) + W_s^2) = 0 + s = s.$$

□

Note that since the Brownian motion is a continuous Gaussian process, the proposition 1.3.1 characterizes uniquely the Brownian motion.

We will give here some properties of the standard Brownian motion.

Properties 1.3.1.1. ([33]). *Let $W(t)_{t \in \mathbb{R}_+}$ be a standard Brownian motion*

1. *Self-similarity. For any $T > 0$, $\{T^{-1/2}W(Tt)\}$ is a Brownian motion.*

2. *Symmetry. $\{-W(t), t \geq 0\}$ is also a Brownian motion.*

3. *$\{tW(1/t), t > 0\}$ is also a Brownian motion.*

4. *If $W(t)$ is a Brownian motion on $[0,1]$, then $(t+1)W(1/t+1) - W(1)$ is a Brownian motion on $[0, \infty)$.*

We have seen that each stochastic process is characterized by its finite-dimensional distribution. Hence, let us give the finite-dimensional distribution of the Brownian motion. In fact, as a Gaussian process, its finite-dimensional distributions are Gaussian. The finite-dimensional distributions of the Brownian motion are given by:
for $n \in \mathbb{N}, t_1 \leq ... \leq t_n$,

$$\mu_{B_{t_1},\ldots,B_{t_n}} = \frac{1}{\sqrt{2\pi t_1}} e^{\frac{-x_1^2}{2t_1}} \prod_{i=1}^{n-1} \frac{1}{\sqrt{2\pi(t_{i+1}-t_i)}} e^{-\frac{(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}}$$

So every process which has (1.3.1) as finite-dimensional distributions is a Brownian motion.

1.3.2 Quadratic variation and Brownian motion

Proposition 1.3.2. *Let $W(t)_{t\in\mathbb{R}+}$ be a Brownian motion. For $t \geq 0$, for all sequence of subdivisions $\Delta_n[0,t]$, such that $\lim_{n\to\infty} |\Delta_n[0,t]| = 0$ we have*

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} \left(W_{\frac{it}{2^n}} - W_{\frac{(i-1)t}{2^n}} \right)^2 = t, \qquad p.s.$$

Proof. The proof can be found in ([62], p.46).

1.3.3 Brownian paths

Proposition 1.3.3. ([32]). *A Brownian motion has its paths a.s., locally γ-Hölder continuous for $\gamma \in [0, 1/2)$.*

Proposition 1.3.4. ([32]). *The Brownian motion's sample paths are a.s., nowhere differentiable.*

1.3.4 Properties of Brownian motion paths

Continuity and differentiability

Almost surely, the sample paths of $W(t), 0 \leq t \leq T$

1. are continuous functions of t,

2. are not differentiable at any point.

Theorem 1.4. *For every t_0,*

$$\limsup_{t \to t_0} \left| \frac{W(t) - W(t_0)}{t - t_0} \right| = \infty \qquad a.s.,$$

which implies that for any t_0, almost every sample $W(t)$ is not differentiable at this point.

Proof. We refer the reader to ([33]).

1.3.5 Brownian motion and martingales

As a stochastic process, we could ask, knowing well all properties of martingales, if the Brownian motion is one.

Definition 1.19. *A stochastic process $\{X(t); t \geq 0\}$ is a martingale if for any t it is integrable, $E|X(t)| < \infty$, and for any $s > 0$ $E(X(t+s)|\mathcal{F}_t) = X(t)a.s.$ where $\mathcal{F}_t$ is the information about the process up to time t, that is, $\{\mathcal{F}_t\}$ is a collection of $\sigma-$algebras such that*

1. *$\mathcal{F}_u \subset \mathcal{F}_t$, if $u \leq t$;*

2. *$X(t)$ is $\mathcal{F}_t$ measurable.*

Proposition 1.3.5. ([24]). *Let $(W_t)_{t\in\mathbb{R}_+}$ be a Brownian motion. Then the following processes are $(\mathcal{F}_t^W)$-martingales:*

1. $(W(t)_{t\geq 0}$,

2. $(W^2(t) - t)_{t\geq 0}$,

3. For any u, $(e^{uW(t)-\frac{u^2}{2}t})_{t\geq 0}$.

Proof. We refer the reader to ([33]):

Markov Property

Definition 1.20. *Let $X(t); t \geq 0$ be a stochastic process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$. The process is called a Markov process if for any t and $s >$ 0, the conditional distribution of $X(t+s)$ given $\mathcal{F}_t$ is the same as the conditional distribution of $X(t+s)$ given $X(t)$, that is,*

$$\mathbb{P}\left(X(t+s) \leq y|\mathcal{F}_t\right) = \mathbb{P}\left(X(t+s) \leq y|X(t)\right)$$

or equivalently, if for any t and $s > 0$ and every non negative Borel-measurable function f, there is another Borel-measurable function g such that

$$E[f(X(t+s))|\mathcal{F}_t] = g\left(X(t)\right)$$

Theorem 1.5. *The Brownian motion $(W(t))$ has the Markov property.*

Example 1.6.

1. *For any $T > 0, \{T^{-1/2}W(Tt)\}$ is a Brownian motion(the self-similarity property).*

2. *The process*

$$\frac{t}{\sqrt{\pi}} + \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{sin(jt)}{j}\xi_j,$$

   where ξ_j are independent standard normal random variables, is a Brownian motion on $[0;\pi]$ (the random series representation).

3. $\{-W(t); t \geq 0\}$ *is also a Brownian motion (the symmetry property).*

4. $\{tW(1/t); t > 0\}$ *is also a Brownian motion (time reversal).*

5. *If* $W(t)$ *is a Brownian motion on* $[0,1]$, *then* $(t+1)W(1/t+1) - W(1)$ *is a Brownian motion on* $[0,\infty)$.*(property of the existence).*

1.4 Fractional Brownian Motion

The fractional Brownian motion (fbm for short) was originally defined and studied by Kolmogorov [34] within a Hilbert space framework.

1.4.1 Existence of the fractional Brownian Motion

The next proposition shows us the existence of the fractional Brownian motion.

Proposition 1.4.1. *Let* $H > 0$ *be a real parameter. Then, there exists a continuous centered Gaussian process* $B^H = (B_t^H)_{t\geq 0}$ *with a covariance function given by*

$$\Gamma_H(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - \mid t - s \mid^{2H}), \quad s,t \geq 0 \tag{1.3}$$

if and only if $H \leq 1$. *In this case, the sample paths of* B^H *are, for any* $\alpha \in (0,H)$ α*-Hölder continuous on each compact set.*

Proof. According to Kolmogrov's theorem 1.1, to get our first claim, we must show that Γ_H is of a positive type if and only if $H \leq 1$.
Assume first that $H > 1$. When $t_1 = 1$, $t_2 = 2$, $a_1 = -2$ and $a_2 = 1$, we have

$$a_1^2\Gamma_H(t_1,t_1) + 2a_1a_2\Gamma_H(t_1,t_2) + a_2^2\Gamma_H(t_2,t_2) = 4 - 2^{2H} < 0$$

As a consequence, Γ_H is not of a positive type when $H > 1$.
The function Γ_1 is of a positive type; indeed, $\Gamma_1(s,t) = st$ so that, for all $d \geq 1$,$t_1, ..., t_d \geq 0$ and $a_1, ..., a_d \in \mathbb{R}$,

$$\sum_{k,l=1}^{d} \Gamma_1(t_k,t_l)a_ka_l = (\sum_{k=1}^{d} t_ka_k)^2 \geq 0.$$

Consider now the case $H \in (0,1)$. For any $x \in \mathbb{R}$, the change of variable $v = u \mid x \mid$ (whenever $x \neq 0$) leads to the representation

$$\mid x \mid^{2H} = \frac{1}{C_H} \int_0^\infty \frac{1 - e^{-u^2 x^2}}{u^{1+2H}} du,$$

where $C_H = \int_0^\infty (1 - e^{-u^2}) u^{-1-2H} du < \infty$. Therefore, for any $s, t \geq 0$, we have

$$\begin{aligned} s^{2H} + t^{2H} - \mid t - s \mid^{2H} &= \frac{1}{C_H} \int_0^\infty \frac{(1 - e^{-u^2 t^2})(1 - e^{-u^2 s^2})}{u^{1+2H}} du \\ &\quad + \frac{1}{C_H} \int_0^\infty \frac{e^{-u^2 t^2}(e^{2u^2 ts} - 1) e^{-u^2 s^2}}{u^{1+2H}} du \\ &= \frac{1}{C_H} \int_0^\infty \frac{(1 - e^{-u^2 t^2})(1 - e^{-u^2 s^2})}{u^{1+2H}} du \\ &\quad + \frac{1}{C_H} \sum_{n=1}^\infty \frac{2^n}{n!} \int_0^\infty \frac{t^n e^{-u^2 t^2} s^n e^{-u^2 s^2}}{u^{1-2n+2H}} du \end{aligned}$$

so that, for all $d \geq 1$, $t_1, ..., t_d \geq 0$ and $a_1, ..., a_d \in \mathbb{R}$,

$$\begin{aligned} \sum_{k,l=1}^d \frac{1}{2}(t_k^{2H} + t_l^{2H} - \mid t_k - t_l \mid^{2H}) a_k a_l &= \frac{1}{2C_H} \int_0^\infty \frac{(\sum_{k=1}^d (1 - e^{-u^2 t_k^2}) a_k)^2}{u^{1+2H}} du \\ &\quad + \frac{1}{2C_H} \sum_{n=1}^\infty \frac{2^n}{n!} \int_0^\infty \frac{(\sum_{k=1}^d t_k^n e^{-u^2 t_k^2} a_k)^2}{u^{1-2n+2H}} du \end{aligned}$$

that is Γ_H is of a positive type when $H \in (0,1)$.

To conclude, in the second part of the proposition, we suppose that $H \in (0,1)$ and consider a centered Gaussian process B^H with a covariance function given by (1.3). Then, we have

$$\mathbb{E}[(B_t^H - B_t^H)^2] = \mid t - s \mid^{2H}, \quad s, t \geq 0,$$

so that Kolmogrov-Čentsov lemma 1.1.1 applies and shows that the sample paths of B^H are α-Hölder continuous.

The Hurst parameter H accounts not only for the sign of the correlation of the increments, but also for the regularity of the sample paths. Indeed, for $H > \frac{1}{2}$, the increments are positively correlated, and for $H < \frac{1}{2}$ they are negatively correlated. furthermore, for every $\beta \in (0, H)$, its sample paths are almost surely Hölder continuous with index β. Finally, it is worthy of note that for $H > \frac{1}{2}$, according to Beran's definition , it is a long memory process: the covariance of increments at distance u decreases as u^{2H-2}.
These significant properties make the fractional Brownian motion a natural candidate as a model of noise in mathematical finance (see Comte and Renault [14], Rogers [56]), and in communication networks (see, for instance, Leland, Taqqu et al. [52].
Recently, there has been numerous attempts at defining a stochastic integral with respect to fractional Brownian motion. Indeed, for $H \neq \frac{1}{2}$, $B^{(H)}$ is not a semimartingale, and the usual Itô's stochastic calculus may not be applied. However, the integral

$$\int_0^t a(s) dB^{(H)}(s) \tag{1.4}$$

may be defined for suitable a. In one hand, since $B^{(H)}$ has almost its sample paths Hölder continuous of index β, for any $\beta < H$, the integral (1.4) exists in the Riemann-Stieltjes sense (path by path) if almost every sample path of a has finite $p-$variation with $\frac{1}{p} + \beta > 1$ (see Young [66]): this is the approach used by Dai and Heyde [16] when $H > \frac{1}{2}$. Let us recall that the $p-$variation of a function f over an interval $[0, t]$ is the least upper bound of sums $\sum_k |f(x_k - f(x_{k-1}))|^p$ over all partitions $0 = x_0 < x_1 < \ldots < x_n = T$. A recent survey of the important properties of Riemann-Stieltjes integral is the concentrated advanced course of Dudley and Norvaisa [21]. An extension of the Riemann-Stieltjes integral has been defined by Zähle [52], by means of composition formulas, integration by parts formula, Weyl derivative formula concerning fractional integration/differentiation, and the generalized quadratic variation of Russo and Vallois [57, 58].
On the other hand, $B^{(H)}$ is a Gaussian process, and (1.4) can be defined for deterministic processes a by way of an L^2 isometry: see, for example, Norros, Valkeila

and Virtamo or Pipiras and Taqqu [52]. With the help of the stochastic calculus of variations (see [52]) this integral may be extended to random processes a. In this case, the stochastic integral (1.4) is a divergence operator, that is the adjoint of a stochastic gradient operator (see the pioneering paper of Decreusefond and Ustunel [18]). It must be noted that Duncan, Hu and Pasik-Duncan [23] have defined the stochastic integral in a similar way by using Wick product. Ciesielski, Kerkyacharian and Roynette [12] also used the Gaussian property of $B^{(H)}$ to prove that $B^{(H)}$ belongs to suitable function spaces and construct a stochastic integral.
Eventually, Alos, Mazet and Nualart [4] have established the following ideas introduced in a previous version of this paper, very sharp sufficient conditions that ensure the existence of the stochastic integral (1.4).
In a similar way, given a Hilbert space $\mathbb{V}$ we denote by $\mathbb{D}^{k,p}(\mathbb{V})$ the corresponding Sobolev space of $\mathcal{V}$-valued random variables. The divergence operator δ is the adjoint of the derivative operator, defined by means of the duality relationship.

$$E(F\delta(u)) = E(DF, u)_{\mathcal{H}}$$

where u is a random variable in $L^2(\Omega; \mathcal{H})$. We say that u belongs to the domain of the operator δ, denoted by Dom δ, if the above expression is continuous in the L^2 norm of F. A basic result says that the space $\mathbb{D}^{1,2}(\mathcal{H})$ is included in Dom δ.
The following are two basic properties of the divergence operator:

1. For any $u \in \mathbb{D}^{1,2}(\mathcal{H})$:

$$E\delta(u)^2 = E\|u\|^2_{\mathcal{H}} + E < D_u, (D_u)^* >_{\mathcal{H}\bigotimes\mathcal{H}} \tag{1.5}$$

   where $(D_u)^*$ is the adjoint of (D_u) in the Hilbert space $\mathcal{H}\bigotimes\mathcal{H}$

2. For any F in $\mathbb{D}^{1,2}$ and any u in the domain of δ such that Fu and $F\delta(u)+ < DF, u >_{\mathcal{H}}$ are square integrable, then Fu is in the domain of δ and

$$\delta(F_u) = F\delta(u) + < DF, u >_{\mathcal{H}} \tag{1.6}$$

We denote by $|\mathcal{H}|\bigotimes|\mathcal{H}|$ the space of measurable functions φ on $[0,T]^2$ such that

$$\|\varphi\|^2_{|\mathcal{H}|\bigotimes|\mathcal{H}|} = \alpha_H^2 \int_{[0,T]^4} |\varphi_{r,\theta}||\varphi_{u,\eta}||r-u|^{2H-2}|\theta-\eta| dr du d\theta d\eta < \infty$$

As we mentioned before, $|\mathcal{H}|\bigotimes|\mathcal{H}|$ is a Banach space with respect to the norm $\|.\|_{|\mathcal{H}|\bigotimes|\mathcal{H}|}$. Furthermore, equipped with the inner product

$$< \varphi, \psi >_{\mathcal{H}\bigotimes\mathcal{H}} = \alpha_H^2 \int_{[0,T]^4} \varphi_{r,\theta}\varphi_{u,\eta}|r-u|^{2H-2}|\theta-\eta|^{2H-2} drdud\theta d\eta$$

The space $|\mathcal{H}|\bigotimes|\mathcal{H}|$ is isometric to a subspace of $\mathcal{H}\bigotimes\mathcal{H}$ and it will be identified with this subspace.
We will recall briefly some of the basic results on fractional Brownian motion. For more details about this process, we can refer the reader to [9, 18, 15].

Definition 1.7. ([9]). *The (two-sided) fractional Brownian motion with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, having the properties:*

1. $B_0^H = 0$.
2. $\mathbb{E}[B_t^H] = 0, \qquad t \in \mathbb{R}$.
3. $\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big), \qquad s,t \in \mathbb{R}$.

Remark 1.8. Since $\mathbb{E}[B_t^H - B_s^H]^2 = |t-s|^{2H}$ and B^H is a Gaussian process, it has a continuous modification, according to the Kolmogorov theorem.

1.4.2 Correlation between two increments

For $H = \frac{1}{2}$, B^H is a standard Brownian motion. Hence, in this case the increments of the process are independent.
For $H \neq \frac{1}{2}$ the increments are not independent. More precisely, by Definition 1.7 the covariance between $B^H(t+h) - B^H(t)$ and $B^H(s+h) - B^H(s)$ with $s+h \leq t$ and $t-s = nh$ is

$$\rho_H(n) = \frac{1}{2}h^{2H}\Big[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\Big].$$

In particular, we obtain that two increments of the form $B^H(t+h) - B^H(t)$ and $B^H(t+2h) - B^H(t+h)$ are positively correlated for $H > \frac{1}{2}$, while they are negatively correlated for $H < \frac{1}{2}$.

1.4.3 Hölder continuity

We have seen that a Brownian motion is locally Hölder continuous of order strictly less than 1/2. Hence, we have the following proposition which generalizes this result to the fBm.

Proposition 1.4.2. ([9]). *Let $H \in (0,1)$. The fBm B^H admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*

Proof. It follows from the Kolmogorov's continuity criterion and the fact that for any $\alpha > 0$, we have

$$\mathbb{E}\Big(|B_t^H - B_s^H|^\alpha\Big) = \mathbb{E}\Big(|B_1^H|^\alpha\Big)|t-s|^{\alpha H}. \qquad \square$$

1.4.4 Basic properties

We will first define the self-similarity and long-range dependence in the framework of general stationary stochastic processes.

Definition 1.21. [19]. *A stochastic process $\{X(t) : t \geq 0\}$ is said to be self-similar if there exists $H \geq 0$ such that for any $a > 0$, $\{X(at)\}$ and $\{a^H X(t)\}$ have identical finite dimensional distributions.*

Proposition 1.4.3. *Let B^H be a fractional Brownian motion of hurst parameter $H \in (0,1)$. Then:*

1. *[Selfsimilarity] For all $a > 0$, $(B_{at}^H) \stackrel{d}{=} (a^H B_t^H)$.*

2. *[Stationarity of increments] For all $h > 0$, $(B_{t+h}^H - B_h^H) \stackrel{d}{=} B_t^H$.*

3. [*Hölder continuity*] *For each* $0 < \varepsilon < H$ *and each* $T > 0$ *there exists a random variable* $K_{\varepsilon,T}$ *such that*

$$\mid B^H(t) - B^H(s) \mid \leq K_{\varepsilon,T} \mid t-s \mid^{H-\varepsilon}$$

4. [*Differentiability*] *The sample paths of fBm are nowhere differentiable.*

Proof. First, let us prove the selfsimilarity property. We have that

$$\begin{aligned}
\mathbb{E}(B^H_{at}B^H_{as}) &= \frac{1}{2}((at)^{2H} + (as)^{2H} - (a \mid t-s \mid)^{2H}) \\
&= a^{2H}\mathbb{E}(B^H_t B^H_s) \\
&= \mathbb{E}((a^H B^H_t)(a^H B^H_s))
\end{aligned}$$

Thus, since all processes are centered and Gaussian, it implies that

$$(B^H_{at}) \stackrel{d}{=} (a^H B^H_t).$$

Second, we show that it has stationary increments. Note that for all $h > 0$, we have

$$\begin{aligned}
\mathbb{E}((B^H_{t+h} - B^H_h)(B^H_{s+h} - B^H_h)) &= \mathbb{E}(B^H_{t+h}B^H_{s+h}) - \mathbb{E}(B^H_{t+h}B^H_h) - \mathbb{E}(B^H_{s+h}B^H_h) + \mathbb{E}((B^H_h)^2) \\
&= \frac{1}{2}\left[((t+h)^{2H} + (s+h)^{2H} - \mid t-s \mid^{2H})\right. \\
&\quad \left. - \left((t+h)^{2H} + h^{2H} - t^{2H}\right) - \left((s+h)^{2H} + h^{2H} - s^{2H}\right) + 2h^{2H}\right] \\
&= \frac{1}{2}(t^{2H} + s^{2H} - \mid t-s \mid^{2H}) = \mathbb{E}(B^H_t B^H_s).
\end{aligned}$$

Therefore, the fBm is a stationary increment.

For the Hölder continuity, it follows from Kolmogrov-Čentsov lemma 1.1.1 and the fact that for any $\alpha > 0$, we have

$$\mathbb{E}(\mid B^H_t - B^H_s \mid^\alpha) = \mathbb{E}(\mid B^H_1 \mid^\alpha) \mid t-s \mid^{2H}$$

Finally, let's prove the differentiability, indeed for every $t_0 \in [0, \infty]$,

$$\mathbb{P}\left(\limsup_{t \to t_0} \mid \frac{B^H_t - B^H_{t_0}}{t - t_0} \mid = \infty\right) = 1.$$

Let us denote by $\mathfrak{B}_{t,t_0} = \frac{B_t^H - B_{t_0}^H}{t-t_0}$, using the selfsimilarity property, we have

$$\mathfrak{B}_{t,t_0} \stackrel{d}{=} (t-t_0)^{H-1} B_1^H$$

We define $\mathfrak{u}(t,\omega) = \{\sup_{0 \le s \le t} \mid \frac{B_s^H}{s} \mid > d\}$. Then, for any sequence $(t_n)_{n\in\mathbb{N}}$ decreasing to 0,

we have $\mathfrak{u}(t_n,\omega) \supseteq \mathfrak{u}(t_{n+1},\omega)$. Thus,

$$\mathbb{P}(\lim_{n\to\infty} \mathfrak{u}(t_n)) = \lim_{n\to\infty} \mathbb{P}(\mathfrak{u}(t_n))$$

and

$$\mathbb{P}(\mathfrak{u}(t_n)) \ge \mathbb{P}(\mid \frac{B_{t_n}^{(H)})}{t_n} \mid > d) = \mathbb{P}(\mid B_1^{(H)} \mid > t_n^{1-H} d) \stackrel{n\to\infty}{\to} 1.$$

Chapter 2

Stochastic integration with respect to fractional Brownian motion

Here we will study the simplest stochastic integral, where the integrand and the integrator are random variables. The first who defined this integral was K. Itô in 1944 (see [37]). Therefore, we utilized this integral after Itô's definition. In fact, the integrand will be an adapted stochastic process with respect to the natural filtration of the Brownian motion.

2.1 Fractional calculus

Another way to handle Young's integrals is to use the so-called fractional calculus. Let $f \in L^1(a,b)$ and $\alpha > 0$. The left-sided and the right-sided fractional integrals of f of order α are defined respectively by:

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I^{\alpha}_{b-} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy,$$

where $(-1)^{\alpha} = e^{i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1} e^{-u} du$ is the Gamma function. Let us denote by $I^{\alpha}_{a+}(L^p)$ (respectively $I^{\alpha}_{b-}(L^p)$) the image of $L^p(a,b)$ by the operator I^{α}_{a+}

(respectively I^{α}_{b-}). If $f \in I^{\alpha}_{a+}(L^p)$ (respectively $f \in I^{\alpha}_{b-}(L^p)$) and $0 < \alpha < 1$, we define for $x \in (a, b)$ the left and the right derivatives by:

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right) \mathbb{1}_{(a,b)}(x)$$

and respectively,

$$D^{\alpha}_{b-}f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy \right) \mathbb{1}_{(a,b)}(x)$$

We have the following property:

$$D^{\alpha}_{a+}D^{\beta}_{a+} = D^{\alpha+\beta}_{a+}, \qquad D^{\alpha}_{b-}D^{\beta}_{b-} = D^{\alpha+\beta}_{b-}$$

and for $f \in I^{\alpha}_{a+}(L^p), \quad g \in I^{\alpha}_{b-}(L^p)$

$$\int_a^b D^{\alpha}_{a+}f(t)g(t)dt = (-1)^{-\alpha} \int_a^b f(t) D^{\alpha}_{b-}g(t)dt$$

The key point that allows to use fractional calculus to study Young's integrals is the following proposition which is due to M. Zähle [43].

Proposition 2.1.1. *[43]Let $f \in C^{\lambda}([a, b])$ and $g \in C^{\beta}([a; b])$ with $\lambda + \beta > 1$: Let $1 - \beta < \alpha < \lambda$. Then, the Young's integral exists and it can be expressed as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D^{\alpha}_{a+}f(t) D^{1-\alpha}_{b-} g_{b-}(t)dt;$$

where $g_{b-}(t) = g(t) - g(b)$.

2.1.1 Bounded variation

As we could expect, the fBm is, as the Brownian motion, a process of unbounded variation.

Proposition 2.1.2. ([9]). *The fBm is of unbounded variation, i.e.,*

$$\sup_{t_i} \sum_i |B^H_{t_{i+1}} - B^H_{t_i}| = \infty.$$

Proof. It is clear by equation (2.1).

$\square$

2.1.2 Stochastic integral representation

Here we discuss some of the integral representations for the fBm. In ([9]) it is proved that the process

$$\begin{aligned} Z(t) &= \frac{1}{\Gamma(H+\frac{1}{2})} \int_{\mathbb{R}} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) dW(s) \\ &= \frac{1}{\Gamma(H+\frac{1}{2})} \Big(\int_{-\infty}^{0} ((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}) dW(s) \\ &\quad + \int_0^t (t-s)^{H-\frac{1}{2}} dW(s) \Big), \end{aligned}$$

where $W(t)$ is a standard Brownian motion and Γ represents the gamma function, is a fBm with Hurst index $H \in (0,1)$.

We can also represent the fBm over a finite interval, i.e.,

$$B^H_t = \int_0^t K_H(t,s) dW_s, \quad t \geq 0,$$

where:

1. For $H > \frac{1}{2}$,

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right]^{\frac{1}{2}}$ and $t > s$.

2. For $H < \frac{1}{2}$,

$$K_H(t,s) = c_H\left[(\frac{t}{s})^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right],$$

with $c_H = \left[\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}\right]^{\frac{1}{2}}$ and $t > s$.

2.1.3 Differentiability

As in the Brownian case, the fBm is a.s., nowhere differentiable. Effectively, we have the following proposition.

Proposition 2.1.3. *Let $H \in (0,1)$. The fBm sample path $B^H(\cdot)$ is not differentiable. Indeed, for every $t_0 \in [0,\infty)$*

$$\limsup_{t\to t_0}\left|\frac{B^H(t)-B^H(t_0)}{t-t_0}\right| = \infty,$$

with probability one.

proof. We refer the reader to ([9]).

2.1.4 The fBm is not a semimartingale for $H \neq \frac{1}{2}$

This is a crucial result of this section. Indeed, the fact that the fBm is not a semimartingale implies that we are not able to integrate with respect to it as we usually do in the classical stochastic calculus. Effectively, the most general class of integrators are semimartingales.
Let us now prove this result (fBm is not a semimartingale).
Proof. In fact, it is sufficient to compute p-variation of B^H. More precisely, we assert that the index of $p-$variation of a fBm is $\frac{1}{H}$. Indeed, let us consider for fixed $p > 0$,

$$Y_{n,p} := \sum_{i=1}^{n} |B^H_{\frac{i}{n}} - B^H_{\frac{i-1}{n}}|^p n^{p(H-1)}.$$

Now if we consider

$$\tilde{Y}_{n,p} := \sum_{i=1}^{n} |B_i^H - B_{i-1}^H|^p \frac{1}{n}$$

We have, by the self-similar property of the fBm, that $Y_{n,p} \stackrel{d}{=} \tilde{Y}_{n,p}$. Besides, remark that the sequence $(B_n^H - B_{n-1}^H)_{n\in\mathbb{Z}}$ is stationary and ergodic. Therefore, we can use the ergodic theorem and obtain that

$$\tilde{Y}_{n,p} \xrightarrow{L^1} \mathbb{E}\big(|B^H(1)|^p\big) \qquad a.s., \qquad as \qquad n \to \infty.$$

So that $Y_{n,p} \xrightarrow{\mathbb{P}} \mathbb{E}\big(|B^H(1)|^p\big)$ which implies $Y_{n,p} \xrightarrow{\mathcal{D}} \mathbb{E}\big(|B^H(1)|^p\big)$. Therefore,

$$V_{n,p} := \sum_{i=1}^{n} |B^H_{\frac{i}{n}} - B^H_{\frac{i-1}{n}}|^p \xrightarrow{\mathbb{P}} \begin{cases} 0, & if \quad pH > 1 \\ \infty & if \quad pH < 1 \end{cases} \quad as \quad n \to \infty. \tag{2.1}$$

Then, we showed that the index of $p-$variation is $\frac{1}{H}$. However, for a semimartingale, the index must be either in $[0,1]$ or equal to 2, i.e., $\frac{1}{H} \in [0,1] \cup \{2\}$. But since $H \in (0,1)$, $H^{-1} \notin [0,1]$. Therefore, the fBm is a semimartingale only for $H = \frac{1}{2}$.

□

$V_{n,p}$ converges in probability to zero as n tends to infinity if $pH > 1$, and to infinity if $pH < 1$. Consider the following two cases:

(i) If $H < \frac{1}{2}$, we can choose $p > 2$ such that $pH < 1$, and we obtain that the $p-$variation of fBm (defined as the limit in probability $\lim_{n\to\infty} V_{n,p}$) is infinite. Hence, the quadratic variation (p=2) is also infinite.

(ii) If $H > \frac{1}{2}$, we can choose p such that $\frac{1}{H} < p < 2$. Then, the $p-$variation is zero, and as a consequence, the quadratic variation is also zero. On the other hand, if we choose p such that $1 < p < \frac{1}{H}$ we deduce that the total variation is infinite.

Therefore, we have proved that for $H \neq \frac{1}{2}$ fBm cannot be a semimartingale. In the paper [13], Cheridito has introduced the notion of the weak semimartingale as a stochastic process $\{X_t, t \geq 0\}$ such that for each $T > 0$, the set of random variables

$$\left\{ \sum_{j=1}^{n} f_i(B_{t_i}^{(H)} - B_{t_{i-1}}^{(H)}), n \geq 1, 0 \leq t_0 < \ldots < t_n \leq T, |f_i| \leq 1, f_i \text{ } is \text{ } \mathcal{F}_{t_{i-1}}^X - mesurable \right\}$$

is bounded in L^0, where for each $t \geq 0$, $\mathcal{F}_t^X$ is the σ-field generated by the random variables $\{X_s, 0 \leq s \leq t\}$. It is important to remark that this σ-field is not completed with the null sets. Then, in [13] it is proved that fBm is not a weak semimartingale if $H \neq \frac{1}{2}$.

2.1.5 Generalized Stieltjes integral

Let $\alpha \in (0, \frac{1}{2})$. For any measurable function $f : [0,T] \to \mathbb{R}$ we introduce the following notation

$$\|f\|_\alpha := |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds. \tag{2.2}$$

Denote by $W^{\alpha,\infty}$ the space of measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t \in [0,T]} \|f(t)\|_\alpha < \infty. \tag{2.3}$$

An equivalent norm can be defined by

$$\|f\|_{\alpha,\mu} := \sup_{t \in [0,T]} e^{-\mu t} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds \right), \qquad \mu \geq 0. \tag{2.4}$$

Note that for any ϵ, $(0 < \epsilon < \alpha)$, we have the inclusions

$$\mathcal{C}^{\alpha+\epsilon}([0,T]; \mathbb{R}) \subset W^{\alpha,\infty}([0,T]; \mathbb{R}) \subset \mathcal{C}^{\alpha-\epsilon}([0,T]; \mathbb{R}).$$

In particular, both the fractional Brownian motion B^H, with $H > \frac{1}{2}$, and the standard Brownian motion W, have their trajectories in $W^{\alpha,\infty}$. We refer the reader to ([27], [45]) for further details on this topics. We denote by $W_T^{1-\alpha,\infty}([0,T]; \mathbb{R})$ the space of continuous functions $g : [0,T] \to \mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0<s<t<T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty.$$

Clearly, for all $\epsilon > 0$ we have

$$\mathcal{C}^{1-\alpha+\epsilon}([0,T];\mathbb{R}) \subset W_T^{1-\alpha,\infty}([0,T];\mathbb{R}) \subset \mathcal{C}^{1-\alpha}([0,T];\mathbb{R}).$$

Denote

$$\Lambda_\alpha(g;[0,T]) = \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |(D_{t^-}^{1-\alpha} g_{t^-})(s)|,$$

where $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ is the Euler function and

$$(D_{t^-}^{1-\alpha} g_{t^-})(s) = \frac{e^{i\pi(1-\alpha)}}{\Gamma(\alpha)} \left(\frac{g(s)-g(t)}{(t-s)^{1-\alpha}} + (1-\alpha) \int_s^t \frac{g(s)-g(y)}{(y-s)^{2-\alpha}} dy \right) \mathbf{1}_{(0,t)}(s).$$

We also define the space $W^{\alpha,1}([0,T];\mathbb{R})$ of measurable functions f on $[0,T]$ such that

$$\|f\|_{\alpha,1;[0,T]} = \int_0^T \left[\frac{|f(t)|}{t^\alpha} + \int_0^t \frac{|f(t)-f(y)|}{(t-y)^{\alpha+1}} dy \right] dt < \infty.$$

We have $W^{\alpha,\infty}([0,T];\mathbb{R}) \subset W^{\alpha,1}([0,T];\mathbb{R})$ and $\|f\|_{\alpha,1;[0,T]} \leq \left(T + \frac{T^{1-\alpha}}{1-\alpha} \right) \|f\|_{\alpha,\infty;[0,T]}$.

In [43], Zähle introduced the generalized Stieltjes integral as follows.

Definition 2.1. *The generalized (fractional) Stieltjes integral* $\int_0^T f(x) dg(x)$ *is defined in terms of the fractional derivative operators*

$$(D_{0+}^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{t^\alpha} + \alpha \int_0^t \frac{f(t)-f(y)}{(t-y)^{\alpha+1}} dy \right) \mathbf{1}_{(0,T)}(t),$$

and

$$(D_{T_-}^{1-\alpha} g_{T_-})(t) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left(\frac{g_{T-}(t)}{(T-t)^{1-\alpha}} + (1-\alpha) \int_t^T \frac{g_{T-}(t)-g_{T-}(y)}{(y-t)^{2+\alpha}} dy \right) \mathbf{1}_{(0,T)}(t),$$

as

$$\int_0^T f(x) dg(x) := (-1)^\alpha \int_0^T (D_{0_+}^\alpha f)(t) (D_{T_-}^{1-\alpha} g_{T_-})(t) dt. \tag{2.5}$$

The following proposition is the estimate of the generalized Stieltjes integral.

Proposition 2.1.4. ([45]). *Fix* $0 < \alpha < \frac{1}{2}$. *There are two functions* $g \in W_T^{1-\alpha,\infty}(0,T)$ *and* $f \in W^{\alpha,1}(0,T)$*; we set*

$$G_s^t(f) = \int_s^t f_r dg_r.$$

Then, for all $r < t \leq T$ *we have*

$$\begin{aligned}\left|\int_s^t f_r dg_r\right| &\leq \sup_{s\leq r<\tau\leq t} |(D_{\tau-}^{1-\alpha} g_{\tau-})(r)| \int_s^t |(D_{\tau-}^{\alpha} g_{s+})(\tau)| d\tau \\ &\leq \Lambda_\alpha(g,[s,t]) \|f\|_{\alpha,1;[0,T]} \\ &\leq c_{\alpha,T}\Lambda_\alpha(g,[s,t])\|f\|_{\alpha,\infty},\end{aligned} \tag{2.6}$$

$c_{\alpha,T} = \left(T + \frac{T^{1-\alpha}}{1-\alpha}\right)$.

As follows from [59], for any $1 - H < \alpha < 1$, we can define the integral w.r.t. the fBm according to (2.5).

Definition 2.2. ([46]). *The integral with respect to the fBm is defined as*

$$\int_0^T f dB^H := (-1)^\alpha \int_0^T (D_{0_+}^{\alpha} f)(t)(D_{T_-}^{1-\alpha} B_{T_-}^H)(t)dt. \tag{2.7}$$

2.1.6 Lack of Markov property

Theorem 2.1. *Let* B^H *be a fractional Brownian motion of Hurst index* $H \in (0,1) - \{\frac{1}{2}\}$. *Then,* B^H *is not a Markov process.*

Since the fBm is a Gaussian centered process, we need the next lemma to prove this result.

Lemma 2.1.1. If X is a Gaussian centered Markovian process, then for all $s < t < u$

$$\mathbb{E}(X_t X_s)\mathbb{E}(X_t X_u) = \mathbb{E}(X_t X_t)\mathbb{E}(X_u X_s)$$

Proof. Note that $R_{st} = cov(X_s, X_t)$. Since X is a Markov process, then $\forall\ s < t < u$

$$\mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u/X_t) = \mathbb{E}(X_u) + \frac{cov(X_t, X_u)}{var(X_t)}(X_t - \mathbb{E}(X_t))$$

Therefore,

$$\begin{cases} \mathbb{E}(X_u/X_t) = \dfrac{R_{ut}}{R_{tt}} X_t, \\ \mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u) + \theta_{uv}\theta_v^{-1}(v - \mathbb{E}(v)) \end{cases}$$

where $v = \begin{pmatrix} X_t \\ X_s \end{pmatrix}$ and $\theta_{uv} = \mathbb{E}[X_u v^t]$, $\theta_v = \mathbb{E}(v^t v)$

We have that,

$$\theta_{uv} = (R_{ut} R_{us}) \ and \ \theta_v = \begin{pmatrix} R_{tt} & R_{ts} \\ R_{st} & R_{ss} \end{pmatrix}$$

$$\theta_v^{-1} v = \frac{1}{R_{tt}R_{ss} - R_{ts}^2} \begin{pmatrix} R_{ss}X_t - R_{ts}X_s \\ R_{tt}X_s - R_{st}X_t \end{pmatrix}$$

We observe that,

$$\begin{aligned} \mathbb{E}(X_u/X_t, X_s) &= \theta_{uv}\theta_v^{-1} v \\ &= \frac{1}{R_{tt}R_{ss} - R_{ts}^2}(R_{ut}R_{ss}X_t - R_{ut}R_{ts}X_s - R_{us}R_{st}X_t + R_{us}R_{tt}X_s). \end{aligned}$$

Hence, $\mathbb{E}(X_u/X_t, X_s) = \mathbb{E}(X_u/X_t)$ we have

$$\frac{R_{ut}}{R_{tt}} X_t = \frac{1}{R_{tt}R_{ss} - R_{ts}^2}(R_{ut}R_{ss}X_t - R_{ut}R_{ts}X_s - R_{us}R_{st}X_t + R_{us}R_{tt}X_s)$$

Moreover,

$$X_t(R_{tt}R_{ut}R_{ss} - R_{tt}R_{ut}R_{ss} - R_{ut}R_{st}^2 + R_{tt}R_{us}R_{st}) + X_s(R_{tt}R_{ut}R_{st} - R_{tt}^2 R_{us}) = 0$$

$$R_{st}X_t(R_{tt}R_{us} - R_{ut}R_{st}) - R_{tt}X_s(R_{tt}R_{us} - R_{ut}R_{st}) = 0$$

or,

$$(R_{tt}R_{us} - R_{ut}R_{st})(R_{st}X_t - R_{tt}X_s) = 0,$$

then,

$$R_{tt}R_{us} - R_{ut}R_{st} = 0$$

which is the result.

Proof of theorem 2.1 We proceed by contradiction. Assume that B^H is a Markov process. Since it is a Gaussian process as well, by the previous lemma we have, for $s = 1 < t = 2 < u = 3$

$$\mathbb{E}(B_1^H B_2^H)\mathbb{E}(B_2^H B_3^H) = \mathbb{E}(B_2^H B_2^H)\mathbb{E}(B_1^H B_3^H)$$

so,

$$\frac{1}{4}(1 + 2^{2H} - 1)(2^{2H} + 3^{2H} - 1) = 2^{2H}\frac{1}{2}(1 + 3^{2H} - 2^{2H})$$

$$2^{2H}(2^{2H} + 3^{2H} - 1) = 2^{2H}[2(1 + 3^{2H} - 2^{2H})]$$

by differentiating

$$3 + 3^{2H} + 3(2^{2H}) = 0$$

,

$$1 + 3^{2H-1} + 2^{2H} = 0$$

we deduce that, $1 + 3^{2H-1} + 2^{2H} = 0$ only if $H = \frac{1}{2}$ which leads to a contradiction.

2.1.7 Long and short-range dependence

The Process with long-range dependence has many applications, such as in telecommunication especially in the internet traffic problems. Basically, the notion of long-range dependence is that the variance of the sum of a stationary sequence grows non-linearly with respect to n.

Definition 2.3. A stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits a long-range dependence if $\rho(n) = cov(X_k, X_{k+n})$ satisfies

$$\lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1$$

for $\alpha \in (0,1)$ and some constant c.

Remark 2.1. *If a stationary sequence $(X_n)_{n\in\mathbb{N}}$ is long-range dependent, then the dependence between X_k and X_{k+1} decays slowly as n tends to infinity and $\sum_{n=1}^{\infty} \rho(n) = \infty$.*

Proposition 2.1.5. The fBm is one of the simplest processes which exhibit long-range dependency.

Proof. Let us consider its increments

$$X_k = B_k^H - B_{k-1}^H, \quad X_{k+1} = B_{k+n}^H - B_{k+n-1}^H.$$

Since the fBm is centered, then

$$\begin{aligned}
\rho(n) &= \mathbb{E}(X_k, X_{k+n}) = \mathbb{E}\left[(B_k^H - B_{k-1}^H)(B_{k+n}^H - B_{k+n-1}^H)\right] \\
&= \mathbb{E}\left[(B_{n+1}^H - B_n^H)B_1^H\right] = \mathbb{E}(B_{n+1}^H B_1^H) - \mathbb{E}(B_n^H B_1^H) \\
&= \frac{1}{2}\left[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right] \\
&= \frac{1}{2}n^{2H}\left[(1+\frac{1}{n})^{2H} - 2 + (1-\frac{1}{n})^{2H}\right] \\
&= \frac{n^{2H}}{2}\left[1 + \frac{2H}{n} + \frac{H(2H-1)}{n^2} - 2 + 1 - \frac{2H}{n} + \frac{H(2H-1)}{n^2} + o(\frac{1}{n^2})\right] \\
&= H(2H-1)n^{2H-2} + o(n^{2H-2})
\end{aligned}$$

It follows that for $H > \frac{1}{2}$, we have

$$\rho(n) > 0 \;\; and \;\; \sum_n \rho(n) = \infty.$$

and for $H < \frac{1}{2}$, we have

$$\rho(n) < 0 \quad and \quad \sum_n \rho(n) < \infty.$$

Therefore, we say that the fBm has a long-range dependence property if and only if $H > \frac{1}{2}$ and for the other case, it has a short-range dependence.

2.2 General construction of the space of integrands using integral representation

In this section, we will explain the reasoning we adopt to construct suitable spaces of integrands in order to have a well-defined integral. Note that it is a heuristic approach; recall that we can represent an fBm by an integral over $\mathbf{T}$ of a kernel with respect to the Brownian motion. Since the fBm is a particular case of the so-called Volterra process, we say that X_t is a Volterra process, if we can write

$$X_t = \int_0^t K(t,s)dB_s,$$

where K is the Volterra kernel and B is a Brownian motion.(see [7] and [31]). Now let us focus on the fBm, which can be represented by

$$(B_t^{(H)})_{t\in\mathbf{T}} \equiv (\int_{\mathbf{T}} k_H(t,s)dB_s)_{t\in\mathbf{T}},$$

with $k_H(t,s) = \mathbf{k}_H\mathbf{1}_{[0,t)}(s)$ where the kernel is in fact the image of the indicator function through the operator $\mathbf{k}_H$. Without going deeply in the theory of operator, it is in fact the Hilbert-Schmidt operator. Thus, heuristically,

$$I^H(f) \equiv \int_{\mathbf{T}} \mathbf{k}_H f(s)dB_s.$$

So, to get it well defined, we must have as a space of integrands

$$\mathcal{S}^H = \{f : \int_{\mathbf{T}} (\mathbf{k}_H f(s))^2 ds < \infty\},$$

with a satisfying inner product ,

$$\langle f, g \rangle_{\mathcal{S}^H} = \mathbb{E}(I^H(f) I^H(f))$$

This is the general construction in [52] for the case $\mathbf{T} = \mathbb{R}$ and [51] for the case $\mathbf{T} = [0, T]$. Besides, as we shall see, for example in subsection 10.5.4 in [40], even if the approach is different, we will use this idea to construct the integral.

Riemann-Stieltjes integral

Riemann-Stieltjes integral is an important notion to understand the stochastic integration. But first, let us recall the basic Riemann integral.

Definition 2.4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. We define the *Riemann integral* over $[a, b] \subset \mathbb{R}$ by

$$\int_a^b f(t)dt = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1}),$$

if the limit exists, where $\Delta_n = \{t_0, t_1, ..., t_n\}$ is a partition of $[a, b]$ such that $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$, $\|\Delta_n\| = \max_{1 \le i \le n}(t_i - t_{i-1})$ and τ_i is an evaluation point in the interval $[t_{i-1}, t_i]$.

Definition 2.5. The *p-variation* of a function $f : [a, b] \to \mathbb{R}$ is defined as

$$\sum_{i=1}^{n} (f(t_k^n) - f(t_{k-1}^n))^p,$$

if the limit exists, where $\Delta_n = \{t_0, t_1, ..., t_n\}$ is a partition of $[a, b]$ and the mesh goes to 0 as $n \to \infty$.

Definition 2.6. A function of a *bounded variation* is a function $g : [a, b] \to \mathbb{R}$ such that $\forall t > 0$,

$$\sup_{\pi \in \mathcal{P}} \sum_{i=1}^{nP} |g(t_i) - g(t_{i-1})| < \infty,$$

where the supremum is taken over the set $\mathcal{P} = \{\pi = \{t_0, ..., t_{nP}\}, \pi$ is a partition of $[a, b]\}$.
We denote by BV the set of functions of the bounded variations.

Definition 2.7. Let $f : [a, b] \to \mathbb{R}$ be continuous and $g : [a, b] \to \mathbb{R}$ be a function of a bounded variation. We define the *Riemann-Stieltjes* integral as follows:

$$\int_a^b f(t)dg(t) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(\tau_i)(g(t_i) - g(t_{i-1})),$$

if the limit exists, where $\Delta_n = \{t_0, t_1, ..., t_n\}$ is a partition of $[a, b]$ and the mesh goes to 0 as $n \to \infty$.

Remark 2.2. *Note that if $g(t) = t$, then the Riemann-Stieltjes integral is the Riemann integral.*

Proposition 2.2.1. *[62] If f is continuous and $g \in \mathcal{C}^1$, then*

$$\int_a^b f(t)dg(t) = \int_a^b f(t)g'(t)dt$$

and if $f, g \in BV$ then

$$\int_a^b f(t)dg(t) = f(b)g(b) - f(a)g(a) - \int_a^b g(t)df(t). \tag{2.8}$$

Wiener integral

The Wiener integral is an integral where we have deterministic integrands and a Gaussian process as an integrator. It generalizes the theory of Riemann-Stieltjes integral. Let us define the integral:

$$I(f) = \int_a^b f(t)dB_t^H \tag{2.9}$$

In fact, we could think of applying the integration by parts of the formula of the Riemann-Stieltjes integral (2.8), and obtain

$$\int_a^b f(t)dB_t^H = f(b)B_b^H - f(a)B_a^H - \int_a^b B_t^H df(t), \tag{2.10}$$

where the integrals are Riemann-Stieltjes integrals. But the problem is, as we saw, that $B_t^H \notin BV$. Hence, the equation (2.10) is not well defined as a Riemann-Stieltjes integral in this case. Therefore, we need a new approach to define the integral (2.9): the so-called Wiener integral.

2.2.1 Construction of the Wiener integral w.r.t FBm

The basic idea is to extend the isometry map from the set of step functions $\mathcal{E}$ into the space $L^2(\Omega)$ generated by the integrator, to an isometry defined on a larger space of integrands, usually noted $\tilde{\mathcal{H}}$ and such that $\bar{\mathcal{E}} = \tilde{\mathcal{H}}$. Let us recall that the Wiener integral (w.r.t. a Gaussian process) of a function $f \in \mathcal{H}$ is a random variable. More explicitly, it is a centered Gaussian random variable. With variance $\int_T f(t)^2 dt$ in the case of a standard Brownian motion. Therefore, the Wiener integral generates a Gaussian space. Let us denote this subspace of $L^2(\Omega, \mathcal{F}^{(Z)}, (\mathcal{F}_t^{(Z)})_{t\in T}, \mathbb{P}^Z)$ by $\overline{S_P(Z)}$ (Note that if $f \in \mathcal{E}$, $\int_T f(t)dZ_t$ generates $S_P(Z)$.) In our case, we take the Gaussian process $Z = B^{(H)}$, as an fBm, so we obtain $\overline{S_P(Z)} = \overline{S_{PT}(B^{(H)})} \subset L^2(\Omega, \mathcal{F}^{(H)}, (\mathcal{F}_t^{(H)})_{t\in T}, \mathbb{P}^H)$.

2.3 Russo-Vallois integral

Definition 2.8. Let X, Y be two real continuous processes defined on $[0, T]$. The symmetric integral (in the sense of *Russo-Vallois*) is defined by

$$\int_0^T Y_u d^\circ X_u = \mathbb{P}\text{-}\lim_{\varepsilon\to 0} \int_0^T \frac{Y_{u+\varepsilon} + Y_u}{2} \frac{X_{u+\varepsilon} - X_u}{\varepsilon} du, \tag{2.11}$$

provided that the limit exists and with the convention that $Y_t = Y_T$ and $X_t = X_T$ when $t > T$.

Theorem 2.2. *([26], page793) The symmetric integral $\int_0^T f(B_u^H) d^\circ B_u^H$ exists for any $f : \mathbb{R} \to \mathbb{R}$ of class C^5 if and only if $H \in (\frac{1}{6}, 1)$. In this case, we have, for any* *primitive F of f:*

$$F(B_T^H) = F(0) + \int_0^T f(B_u^H) d^\circ B_u^H.$$

When $H \leq 1/6$, one can consider the so-called m-order Newton-Côtes integral:

Definition 2.9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, let X, Y be two continuous processes on $[0, T]$ and let $m \geq 1$ be an integer. The *m-order Newton-Côtes integral* (in the sense of Russo-Vallois) of $f(Y)$ with respect to X is defined by

$$\int_0^T f(Y_u) d^{NC,m} X_u = \mathbb{P}\text{-}\lim_{\varepsilon \to 0} - \int_0^T \left(\int_0^1 f(Y_s + \beta(Y_{s+\varepsilon} - Y_s)) \mu_m(d\beta) \right) \frac{X_{u+\varepsilon} - X_u}{\varepsilon} du,$$

provided that the limit exists and with the same convention above with $\mu_1 = \frac{1}{2}(\delta_0 + \delta_1)$ and, for $m \geq 2$,

$$\mu_m = \sum_{j=0}^{2(m-1)} \left(\int_0^1 \prod_{j \neq k} \frac{2(m-1)u - k}{j - k} du \right) \delta_{\frac{j}{(2m-2)}},$$

δ being the Dirac measure.

Theorem 2.3. *([26], page793) Let $m \geq 1$ be an integer. The m-order Newton-Côtes integral $\int_0^T f(B_u^H) d^{NC,m} B_u^H$ exists for any $f : \mathbb{R} \to \mathbb{R}$ of class C^{4m+1} if and only if $H \in (\frac{1}{4m+2}, 1)$. In this case, we have for any primitive F of f:*

$$F(B_T^H) = F(0) + \int_0^T f(B_u^H) d^{NC,m} B_u^H.$$

2.4 Skorohod integral

In this section, we focus on the Skorohod integral. This stochastic integral, introduced for the first time by A. Skorohod in 1975, may be regarded as an extension of the Itô's integral to integrands that are not necessarily $\mathbb{F}$-adapted. The Skorohod integral is also connected to the Malliavin derivative, which is introduced with full details in [49, Chap. 3].

Let $u = u(t,\omega)$, $t \in [0,T]$, $\omega \in \Omega$, be a measurable stochastic process such that, for all $t \in [0,T]$, $u(t)$ is a $\mathcal{F}_T$ -measurable random variable and

$$\mathbb{E}[u^2(t)] < \infty.$$

Then, for each $t \in [0,T]$, we can apply the Wiener-It$\hat{o}$'s chaos expansion to the random variable $u(t) = u(t,\omega)$, $\omega \in \Omega$, and thus there exist symmetric functions $f_{n,t} = f_{n,t}(t_1,...,t_n)$, $(t_1,...,t_n) \in [0,T]^n$, in $\tilde{L}^2([0,T]^n)$, $n = 1,2,...$, such that

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}),$$

where

$$I_n(f) = \int_{[0,T]^n} f(t_1,...,t_n)dW(t_1)...dW(t(n),$$

$(W_t)_{t\in[0,T]}$ is a Wiener process and $f \in \tilde{L}^2([0,T]^n)$, and the convergence takes place in $L^2(\mathbb{P})$. Moreover, we have the isometry

$$\| u \|^2_{L^2(\mathbb{P})} = \sum_{n=0}^{\infty} n! \| f_n \|^2_{L^2([0,T]^n)} . \tag{2.12}$$

For more details see [49]. Note that the functions $f_{n,t}$, $n = 1,2,...$, depend on the parameter $t \in [0,T]$, and so we can write

$$f_n(t_1,...,t_n,t_{n+1}) = f_n(t_1,...,t_n,t) := f_{n,t}(t_1,...,t_n)$$

and we may regard f_n as a function of $n+1$ variables. Since this function is symmetric with respect to its first n variables, its symmetrization $\tilde{f}_n$ is given by

$$\tilde{f}_n(t_1,...,t_{n+1}) = \frac{1}{n+1}\left[f_n(t_1,...,t_{n+1}) + f_n(t_2,...,t_{n+1},t_1) + ... + f_n(t_1,....,t_{n-1},t_{n+1},t_n)\right] \tag{2.13}$$

Definition 2.10. Let $u(t)$, $t \in [0,T]$, be a measurable stochastic process such that for all $t \in [0,T]$ the random variable $u(t)$ is $\mathcal{F}_T$ -measurable and $\mathbb{E}[u^2(t)] < \infty$. Let its Wiener-Itô's chaos expansion be

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}) = \sum_{n=0}^{\infty} I_n(f_n(.,t)).$$

Then, we define the Skorohod integral of u by

$$\delta(u) = \int_0^T u(t)\delta W(t) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$$

when the sum is convergent in $L^2(\mathbb{P})$. Here $\tilde{f}_n$, $n = 1,2,...$, are the symmetric functions (2.13) derived from $f_n(.,t)$, $n = 1,2,....$ We say that u is a *Skorohod integrable*, and we write $u \in Dom(\delta)$ if the series $\delta(u)$ converges in $L^2(\mathbb{P})$.

Remark 2.3. By (2.12) a stochastic process u belongs to $Dom(\delta)$ if and only if

$$\mathbb{E}[\delta(u)^2] = \sum_{n=0}^{\infty}(n+1)! \parallel f_n \parallel^2_{L^2([0,T]^{n+1})} < \infty.$$

2.4.1 The Skorohod integral for fBm

The stochastic Integrals with respect to fBm were defined mostly for deterministic or linear integrands, but in other cases it was much more complicated to establish such integral, since the path regularity of the fBM varies with the Hurst parameter H. In particular, if $H > \frac{1}{2}$, then the paths of B^H are essentially α-Hölder continuous for all $\alpha < H$, hence a pathwise stochastic integral approach is quite effective likewise Young (see [54]). In the general case, especially when $H < \frac{1}{2}$, the path of fBm becomes rather "rough" and the pathwise approach for stochastic integrals; therefore other definitions of stochastic integrals have been introduced. The most notable is the divergence-type integration (or Skorohod integral), which is based on the idea of Malliavin calculus (see for example [49, 30, 64]). For this case we briefly introduce Malliavin derivative with respect to certain Gaussian processes; in particular, for

fractional Brownian motion.
Let W be a standard Brownian motion and assume $G = (G_t)_{t\in[0,T]}$ is a continuous centred Gaussian process of the form

$$G_t = \int_0^t K(t,s)dW_s \tag{2.14}$$

where the kernel K satisfies $\sup\limits_{t\in[0,T]} \int_0^t K(t,s)^2 ds < \infty$. In particular, the fractional Brownian motion is of this form by representation (3.14). First we recall some definitions.

Definition 2.11. We denote by $\mathcal{E}_G$ the set of simple random variables of the form

$$F = \sum_{k=1}^{n} a_k G_{t_k}$$

where $n \in \mathbb{N}$, $a_k \in \mathbb{R}$ and $t_k \in [0,T]$ for $k = 1, ..., n$.

Definition 2.12. The Gaussian space $\mathcal{H}_1$ associated to G is the closure of $\mathcal{E}_G$ in $L^2(\Omega)$.

Definition 2.13. *The reproducing Hilbert space $\mathcal{H}_G$ of G is the closure of $\mathcal{E}_G$ with respect to the inner product*

$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_{\mathcal{H}} = R_G(t,s).$$

In what follows, we will drop G in the notation.

The mapping $\mathbb{1}_{[0,t]} \to G_t$ can be extended to an isometry between the Hilbert space $\mathcal{H}$ and the Gaussian space $\mathcal{H}_1$. The image of $\varphi \in \mathcal{H}$ in this isometry is denoted by $G(\varphi)$. In particular, we have $G(\mathbb{1}_{[0,t]}) = G_t$.

Definition 2.14. Denote by $\mathcal{S}$ the space of all smooth random variables of the form

$$F = f(G(\varphi_1), ..., G(\varphi_n)), \qquad \varphi_1, ..., \varphi_n \in \mathcal{H},$$

where $f \in C_b^\infty(\mathbb{R}^n)$ i.e. f and all its derivatives are bounded. The *Malliavin derivative* $D = D(G)$ of F is an element of $L^2(\Omega, \mathcal{H})$ defined by

$$DF = \sum_{i=1}^{n} \partial_i f(G(\varphi_1), ..., G(\varphi_n))\varphi_i.$$

In particular, $DG_t = \mathbb{1}_{[0,t]}$.

Definition 2.15. We denote $\mathbb{D}_G^{1,2} = \mathbb{D}^{1,2}$ be the Hilbert space of all square integrable Malliavin derivative random variables defined as the closure of $\mathcal{S}$ with respect to the norm

$$\| F \|_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}(\| DF \|_{\mathcal{H}}^2).$$

Now we are ready to define the *divergence operator* δ as the adjoint operator of the Malliavin derivative D.

Definition 2.16. The domain Dom δ of the operator δ is the set of random variables $u \in L^2(\Omega, \mathcal{H})$ satisfying

$$|\mathbb{E}(\langle DF, u\rangle_{\mathcal{H}})| \leq c_u \| F \|_{L^2}$$

for any $F \in \mathbb{D}^{1,2}$ and some constant c_u depending only on u. For $u \in$ Dom δ the divergence operator $\delta(u)$ is a square integrable random variable defined by the duality relation

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(\langle DF, u\rangle_{\mathcal{H}}), \qquad \forall F \in \mathbb{D}^{1,2}$$

for any $F \in \mathbb{D}^{1,2}$.

We use the notation

$$\delta(u) = \int_0^T u_s \delta G_s.$$

Recall now the special form of G given by (2.14) which is clearly the fractional Brownian motion, and define a linear operator K^* from $\mathcal{E}$ to $L^2[0,T]$ by

$$(K^*\varphi)(s) = \varphi(s)K(T,s) + \int_s^T [\varphi(t) - \varphi(s)]K(dt,s).$$

With the help of this operator according to [64], the Hilbert space $\mathcal{H}$ generated by G can be represented as $\mathcal{H} = (K^*)^{-1}(L^2[0,T])$. Furthermore, $\mathbb{D}^{1,2}_G(\mathcal{H}) = (K^*)^{-1}(\mathbb{D}^{1,2}_W(L^2[0,T]))$. Moreover, we can represent $\delta^{(G)}$ with $\delta^{(W)}$ by the relation

$$\int_0^t u_s \delta G_s = \int_0^t (Ku)_s \delta W_s$$

provided that $Ku \in \mathrm{Dom}\ \delta^{(W)}$.

Chapter 3

Stochastic integration for non-adapted processes with respect to fractional Brownian motion

This chapter is the subject of a publication in the thesis
Let $B(t)$ be a Brownian motion and let $\{\mathcal{F}_t\}$ be a filtration such that

- $B(t)$ is adapted to $\{\mathcal{F}_t\}$.
- $B(t) - B(s)$ and $\{\mathcal{F}_t\}$ are independent for $s \leq t$.

Suppose $f(t)$ is a stochastic process satisfying the following conditions:

1. $f(t)$ is adapted to $\{\mathcal{F}_t\}$,
2. $E \int_a^b |f(t)|^2 dt < \infty$

Then, the Itô integral $\mathcal{I} = \int_a^b f(t) dB(t)$ is defined (see, e.g., Chapter 4 of the book [28]) and we have the equalities:

$$E(\mathcal{I}) = 0, E(|\mathcal{I}|^2) = E \int_a^b |f(t)|^2 dt$$

Moreover, we have the next theorem (see, e.g., Theorems 4.6.1 and 4.6.2 in the book [28].)

Theorem 3.1. *Let $f(t)$ be a stochastic process satisfying the above conditions (1) and (2). Then, the stochastic process*

$$X_t = \int_a^t f(s)dB(s), a \le t \le b,$$

is a continuous martingale.

More generally, suppose $f(t)$ is a stochastic process satisfying the following conditions:

(a) $f(t)$ is adapted to $\{\mathcal{F}_t\}$,

(b) $\int_a^b |f(t)|^2 dt < \infty$ almost surely.

Then, the Itô integral $\int_a^b f(t)dB(t)$ is defined (see, e.g., Chapter 5 of the book [28]) and we have the next theorem (see, e.g., Theorems 5.5.2 and 5.5.5 in the book [28].)

Theorem 3.2. *Let $f(t)$ be a stochastic process satisfying the above conditions (a) and (b). Then, the stochastic process*

$$X_t = \int_a^t f(s)dB(s), a \le t \le b,$$

is a continuous local martingale.

3.1 Itô's formula for fBm

In this section, we will show the Itô's formula for the indefinite Skorohod integral.

Theorem 3.3. *[60] Let f be a function of class $C^2(\mathbb{R})$. For each $t \in [0, T]$ the following formula holds*

$$f(B^H(t)) = f(0) + \int_0^t f'(B^H(s))\delta B^H(s) + H \int_0^t f''(B^H(s))s^{2H-1}ds$$

3.2 Itô integral

Let us start with the simplest case of random integrands. For convenience, we will use $L^2_{ad}([a,b]\times\Omega)$ to denote the space of all stochastic processes $f(t,\omega), a \le t \le b, \omega \in \Omega$, satisfying the following conditions:

(1) $f(t,\omega)$ is adapted to the filtration $\{\mathcal{F}_t\}$

(2)$\int_a^b \mathbb{E}|f(t)|^2 dt < \infty$. We will use Itô's original ideas to define the stochastic integral

$$\int_a^b f(t)dW(t) \tag{3.1}$$

for $f \in L^2_{ad}([a,b]\times\Omega)$. For clarity, we divide the discussion into three steps.

Step 1. *f is a step stochastic process in $L^2_{ad}([a,b]\times\Omega)$.*

Suppose f is a step stochastic process given by

$$f(t,\omega) = \sum_{i=1}^{n} \xi_{i-1}(\omega)\mathbf{1}_{(t_{i-1},t_i]}(t), \tag{3.2}$$

where ξ_{i-1} is $\mathcal{F}_{t_{i-1}}$-measurable and $\mathbb{E}(\xi_{i-1}^2) < \infty$. In this case we define

$$I(f) = \sum_{i=1}^{n} \xi_{i-1}(W(t_i) - W(t_{i-1})). \tag{3.3}$$

Obviously, $I(af+bg) = aI(f) + bI(g)$ for any $a, b \in \mathbb{R}$ and any such step stochastic processes f and g. Moreover, we have the next lemma.

Lemma 3.2.1. *Let $I(f)$ be defined by equation (3.3). Then, $\mathbb{E}I(f) = 0$ and*

$$\mathbb{E}(|I(f)|^2) = \int_a^b \mathbb{E}(|f(t)|^2)dt. \tag{3.4}$$

Proof. The proof can be found in ([28]).

Step 2. *An approximation lemma.*

We need to prove an approximation lemma in this step in order to be able to define the stochastic integral $\int_a^b f(t)dW(t)$ for general stochastic processes $f \in L^2_{ad}([a,b]\times\Omega)$.

Lemma 3.2.2. *Suppose $f \in L^2_{ad}([a,b]\times\Omega)$. Then, there exists a sequence $\{f_n(t); n \geq 1\}$ of step stochastic processes in $f \in L^2_{ad}([a,b]\times\Omega)$ such that*

$$\lim_{n\to\infty} \int_a^b \mathbb{E}\{|f(t) - f(t_n)|^2\}dt = 0. \tag{3.5}$$

Proof. We refer the reader to [28]

Step 3. *Stochastic integral $\int_a^b f(t)dW(t)$ for $f \in L^2_{ad}([a,b]\times\Omega)$.*
We use Steps 1 and 2 to define the stochastic integral

$$\int_a^b f(t)dW(t), \qquad f \in L^2_{ad}([a,b]\times\Omega).$$

Apply Lemma 3.2.2 to get a sequence $\{f_n(t,\omega); n \geq 1\}$ of adapted step stochastic processes such that equation (3.5) holds. For each $n, I(f_n)$ is defined by Step 1. By Lemma 3.2.1 we have

$$\mathbb{E}(|f_n(t) - f_m(t)|^2) = \int_a^b \mathbb{E}(|f_n(t) - f_m(t)|^2)dt \qquad \to 0, as \quad n, m \to \infty.$$

Hence, the sequence $\{I(f_n)\}$ is a Cauchy sequence in $L^2(\Omega)$. Define

$$I(f) = \lim_{n\to\infty} I(f_n), \quad in\ L^2(\Omega). \tag{3.6}$$

We can use arguments similar to those in [28] for the Wiener integral to show that the above $I(f)$ is well-defined.

Definition 3.1. ([28]). *The limit $I(f)$ defined in equation (3.6) is called the Itô integral of f and is denoted by $\int_a^b f(t)dW(t)$.*

Thus, the Itô integral $I(f)$ is defined for $f \in L^2_{ad}([a,b] \times \Omega)$ and the mapping I is linear, namely, for any $a, b \in \mathbb{R}$ and $f, g \in L^2_{ad}([a,b] \times \Omega)$,

$$I(af + bg) = aI(f) + bI(g).$$

We clearly see that Lemma 3.2.1 remains valid for $f \in L^2_{ad}([a,b] \times \Omega)$. We state this fact as the next theorem.

Theorem 3.2.1. ([28]). *Suppose $f \in L^2_{ad}([a,b] \times \Omega)$. Then, the Itô integral $I(f) = \int_a^b f(t)dW(t)$ is a random variable with $\mathbb{E}\{I(f)\} = 0$ and*

$$E(|I(f)|^2) = \int_a^b \mathbb{E}(|f(t)|^2)dt.$$

By this theorem, the Itô integral $I : L^2_{ad}([a,b] \times \Omega) \to L^2(\Omega)$, is an isometry. Since I is also linear, we have the following corollary.

Corollary 3.2.0.1. ([28]). *For any $f, g \in L^2_{ad}([a,b] \times \Omega)$, the following equality holds:*

$$\mathbb{E}(\int_a^b f(t)dW(t) \int_a^b g(t)dW(t)) = \int_a^b f(t)g(t)dW(t).$$

To construct the integral with respect to the fractional Brownian motion, we use the generalized (fractional) Stieltjes integral (see [54]-[43]).

3.3 K. Itô's idea

Suppose a stochastic process $f(t)$ is not adapted to this filtration. Then $\int_a^b f(t)dB(t)$ cannot be defined as an Itô integral.

1. Stochastic integral: $\int_0^1 B(1)dB(t) =?$ (See equations (3.7) and (3.10).)

We first describe K. Itô's ideas to define the stochastic integral $\int_0^1 B(1)dB(t)$ in his lecture at the 1976 Kyoto Symposium on SDE's [35]. Enlarge the filtration in order for the integrand $B(1)$ to be adapted, namely, let

$$\mathcal{G}_t = \sigma\{\mathcal{F}_t, B(1)\}.$$

Although $B(t)$ is not a Brownian motion with respect to the larger filtration $\{\mathcal{G}_t\}$, it can be decomposed as

$$B(t) = \left(B(t) - \int_0^t \frac{B(1) - B(u)}{1-u} du\right) + \int_0^t \frac{B(1) - B(u)}{1-u} du,$$

which shows that $B(t)$ is a quasimartingale with respect to the filtration $\{\mathcal{G}_t\}$. Then the stochastic integral $\int_0^1 B(1)dB(t)$ can be defined as a stochastic integral with respect to a quasimartingale and

$$\int_0^1 B(1)dB(t) = B(1)^2. \tag{3.7}$$

Our new viewpoint in [5] comes from the simple observation that the anticipating integrand $B(1)$ has the following obvious decomposition

$$B(1) = (B(1) - B(t)) + B(t). \tag{3.8}$$

Note that the integral for the second term $B(t)$ is within the Itô theory. Thus, we only need to define the stochastic integral $\int_0^1 (B(1) - B(t))dB(t)$. This leads to the question: "What is so special about the integrand $B(1) - B(t)$?" To find out the answer, consider another anticipating integrand $B(1)^2$. This integrand can be decomposed as

$$B(1)^2 = [B(1) - B(t)]^2 + 2B(t)[B(1) - B(t)] + B(t)^2.$$

Observe that the last term $B(t)^2$ and the factor $B(t)$ in the second term are adapted stochastic processes, while the first term $[B(1) - B(t)]^2$ and the factor $B(1) - B(t)$ in the second term have the same property (which is to be defined below) as that of the first term in Equation (3.8). We can also try to decompose integrands such as $B(1)^n$ and $e^{B(1)}$ to discover the common property stated in the next definition.

Definition 3.2. *A stochastic process $\varphi(t)$ is said to be instantly independent with respect to a filtration $\{\mathcal{F}_t\}$ if $\varphi(t)$ and $\mathcal{F}_t$ are independent for each t.*

Clearly, $[B(1)-B(t)]^n, e^{B(1)-B(t)}$, and $\int_t^1 h(s)dB(s)$ are all instantly independent for $0 \leq t \leq 1$, where $h(s)$ is a deterministic function in $L^2([0;1])$.

Lemma 3.4. *If a stochastic process $\varphi(t)$ is both adapted and instantly independent with respect to a filtration $\{\mathcal{F}_t\}$, then $\varphi(t)$ is a deterministic function.*

Proof. Since $\varphi(t)$ is adapted, we have $\mathbb{E}(\varphi(t)|\mathcal{F}_t) = \varphi(t)$. On the other hand, since $\varphi(t)$ is instantly independent, we also have $\mathbb{E}(\varphi(t)|\mathcal{F}_t) = \mathbb{E}(\varphi(t))$. Hence, $\varphi(t) = \mathbb{E}(\varphi(t))$, which shows that $\varphi(t)$ is a deterministic function. □

In view of lemma (3.4), we can regard the collection of instantly independent stochastic processes as a counterpart of the Itô theory. Namely, the Itô part consists of adapted stochastic processes and the counterpart consists of instantly independent stochastic processes. Moreover, we observe from the above discussion that many anticipating stochastic processes can be decomposed into sums of the products of an Itô part and a counterpart.

Remark 3.5. *Note that the Brownian motion $(W_t)_{t\in[0,T]}$ and the FBM $(B_t^H)_{t\in[0,T]}$ generate the same filtration. More precisely, the natural filtration of the Brownian motion and of the FBM that generates through the Levy-Hida representation coincide.*

Thus, our viewpoint in fact stems from Itô's ideas. We simply reverse the roles of the integrand and the integrator, i.e.,

- Keep the filtration $\{\mathcal{F}_t\}$ and the Brownian motion $B(t)$.

- Decompose an integrand as a sum of terms, each being the product of an adapted stochastic process and an instantly independent stochastic processes.

This leads to the question : " How do we define a stochastic integral $\int_a^b f(t)dB(t)$ for an adapted stochastic process $f(t)$ (in the Itô part) and an instantly independent stochastic process $\varphi(t)$ (in the counterpart)? " The answer is in the next section.

3.4 Stochastic integration with respect to Brownian motion

H. Kuo and W. Ayed [6] are going to propose a new approach for stochastic integration with respect to Brownian motion. The key idea in our approach to anticipating stochastic integration is the evaluation points for the integrand. Consider the instantly independent stochastic process $B(1) - B(t)$ in the right-hand side of equation (3.8). How do we " define" the stochastic integral $\int_0^1 (B(1) - B(t))dB(t)$?

Let $\triangle = \{0 = t_0, t_1, t_2, \cdots, t_n = 1\}$ be a partition of the interval $[0, 1]$. On the subinterval $[t_{i-1}, t_i]$, we take the " right endpoint " t_i as the evaluation point for the integrand $B(1) - B(t)$ to form a Riemann like sum.

Then, we define the integral

$$\int_0^1 (B(1) - B(t))dB(t) = \lim_{\|\triangle\|\to 0} \sum_{i=1}^{n} (B(1) - B(t_i))(B_{t_i} - B_{t_{i-1}})$$

$$= B(1)^2 - \lim_{\|\triangle\|\to 0} \sum_{i=1}^{n} B(t_i)(B_{t_i} - B_{t_{i-1}})$$

$$= B(1)^2 - \lim_{\|\triangle\|\to 0} \sum_{i=1}^{n} \{[B(t_i) - B(t_{i-1})] + B(t_{i-1})\}(B(t_i) - B(t_{i-1}))$$

$$= B(1)^2 - 1 - \int_0^1 B(t)dB(t) \tag{3.9}$$

where the last integral is an Itô integral. It follows from equations (3.8) and (3.9) that we have a new stochastic integral

$$\int_0^1 B(1)dB(t) = B(1)^2 - 1 \ , \tag{3.10}$$

which is different from the one in Equation (3.7) defined by K.Itô [35].

Note that our new stochastic integral has expectation 0, a property that we want to keep for our new stochastic integral. The above discussion leads to the following definition of a new stochastic integral of a stochastic process which is the product of an adapted stochastic process (in the Itô part) and an instantly independent stochastic process (in the counterpart).

Definition 3.3. *For an adapted stochastic process $f(t)$ and an instantly independent stochastic process $\varphi(t)$, we define the stochastic integral of $f(t)\varphi(t)$ to be the limit*

$$\int_a^b f(t)\varphi(t)dB(t) = \lim_{\|\triangle\|\to 0} \sum_{i=1}^n f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1}))$$

provided that the limit in probability exists.

In general, for a stochastic process $F(t) = \sum_{n=1}^N f_n(t)\varphi_n(t)$ with $f_n(t)$'s being adapted and $\varphi_n(t)$ instantly independent, we define

$$\int_a^b F(t)dB(t) = \sum_{n=1}^N \int_a^b f_n(t)\varphi_n(t)$$

This stochastic integral is in fact well-defined. Obviously, there is a natural question: "What is the class of stochastic processes for which the new stochastic integral is defined?" Unfortunately, we do not have the answer yet.

Example 3.6. *We mention to two stochastic integrals from [5]*

$$\int_0^t B(1)B(s)dB(s) = \begin{cases} \frac{1}{2}B(1)(B(t)^2 - t) - \int_0^t B(s)ds, 0 \le t \le 1; \\ \frac{1}{2}B(1)(B(t)^2 - t) - \int_0^t B(s)ds, t > 1. \end{cases}$$

In general, for a continuous function $f(x)$, we have

$$\int_0^t B(1)f(B(s))dB(s) = \begin{cases} B(1)\int_0^t f(B(s))dB(s) - \int_0^t f(B(s))ds, 0 \le t \le 1; \\ B(1)\int_0^t f(B(s))dB(s) - \int_0^t f(B(s))ds, t > 1. \end{cases}$$

Example 3.7. *Let $f(t)$ and $g(t)$ be two deterministic functions in $L^2([0,1])$. Then,*

$$\int_0^1 g(t)\left(\int_0^1 f(s)dB(s)\right)dB(t) = \int_{[0,1]^2} f(s)g(t)dB(s)dB(t),$$

where the right-hand side is a double Wiener-Itô integral (see Chapter 9 in [28]). To prove this equality, note that the Wiener integral of $f(s)$ in the left-hand side has the decomposition

$$\int_0^1 f(s)dB(s) = \int_0^t f(s)dB(s) + \int_t^1 f(s)dB(s)$$

where the first integral is in the Itô part and the second integral is in the counterpart. For convenience, let $\triangle B_i = B(t_i) - B(t_{i-1})$. By definition (3.3), we have

$$\int_0^1 g(t)\left(\int_0^1 f(s)dB(s)\right)dB(t) = \lim_{\|\triangle\|\longrightarrow 0}\sum_{i=1}^{n} g(t_{i-1})\left(\int_0^{t_{i-1}} f(s)dB(s) + \int_{t_i}^1 f(s)dB(s)\right)\triangle B_i$$

$$= \lim_{\|\triangle\|\longrightarrow 0}\sum_{i=1}^{n} g(t_{i-1})\left(\int_0^1 f(s)dB(s) - \int_{t_i}^{t_{i-1}} f(s)dB(s)\right)\triangle B_i$$

$$= \int_0^1 f(s)dB(s)\int_0^1 g(t)dB(t) - \lim_{\|\triangle\|\longrightarrow 0}\sum_{i=1}^{n} f(t_{i-1})g(t_{i-1})(\triangle B_i)^2$$

$$= \int_0^1 f(s)dB(s)\int_0^1 g(t)dB(t) - \int_0^1 f(t)g(t)d(t)$$

which is exactly the Wiener-Itô double integral in the right-hand side of equation (3.7). In [36] K. Itô proved the following well-known theorem on multiple Wiener-Itô integral (see also theorem 9.6.7 in the book [28].)

Theorem 3.8. *(K. Itô 1951) Let $f \in L^2([a;b]^n)$ and $\hat{f}$ its symmetrization. Then,*

$$\int_{[a;b]^n} f(t_1, t_2, ..., t_n)dB(t_1)dB(t_2)...dB(t_n)$$

$$= n!\int_a^b ... \int_a^{t_{n-2}}\left[\int_a^{t_{n-1}} \hat{f}(t_1, ..., t_{n-1}, t_n)dB(t_n)\right]dB(t_{n-1})...dB(t_1)$$

Note that the restriction to the region $a \leq t_n \leq t_{n-1} \leq ... \leq t_2 \leq t_1 \leq b$ for the iterated integrals is to ensure that in each step of the iteration the integrand is adapted so that the integral is defined as an Itô integral. However, as seen from example (3.7), there is no need to impose this restriction since in each step the integral is defined as a stochastic integral in definition (3.3). By using the similar arguments as those in example (3.7), we can prove the next theorem.

Theorem 3.9. *Let $f \in L^2([a;b]^n)$. Then,*

$$\int_{[a;b]^n} f(t_1, t_2, ..., t_n) dB(t_1) dB(t_2) ... dB(t_n) \tag{3.11}$$

$$= \int_a^b ... \int_a^b \left[\int_a^b f(t_1, ..., t_{n-1}, t_n) dB(t_n) \right] dB(t_{n-1}) ... dB(t_1)$$

Observe that we do not have to use the symmetrization $\hat{f}$ in the right-hand side of equation (3.11). In fact, it is obvious that the iterated new stochastic integrals for f and $\hat{f}$ are equal. In view of this theorem, a multiple Wiener-Itô integral can be evaluated as an iterated stochastic integral, just like multiple integrals and iterated integrals in ordinary calculus.

3.5 Representation of the FBm

The fractional Brownian motion can be expressed as a Wiener integral with respect to the Wiener process in several ways. Let us recall two of them.

3.5.1 Levy-Hida representation

Note that the FBM is a particular case of Volterra processes. Following Decreusfond and Üstünel, we have this kernel :

$$K_H(t,s) = \frac{(t-s)_+^{H-1/2}}{\Gamma(H+1/2)} F(1/2-H, H-1/2, H+1/2, 1-\frac{t}{s}), 0 < s < t < \infty,$$

where F is the Gauss hypergeometric function.

For the case $H \in (1/2, 1)$, we have that the kernel is

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t |u-s|^{H-3/2} u^{H-1/2} du, \quad t > s,$$

where

$$c_H = (\frac{H(2H-1)}{\mathbf{B}(2-2H, H-1/2)})^{1/2}$$

with $\mathbf{B}$ the Beta function, i.e. $\mathbf{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$.

We have

$$\frac{\partial K_H(t,s)}{\partial t} = c_H(\frac{t}{s})^{H-1/2}(t-s)^{H-3/2}.$$

Now, we introduce a linear operator $K_H^* : \mathcal{E} \to L^2([0,T])$, defined by

$$(K_H^*\phi)(s) = \int_s^T \phi(t)\frac{\partial K_H(t,s)}{\partial t}dt, \tag{3.12}$$

where $\phi \in \mathcal{E}$.

For the case $H \in (0, 1/2)$, we have that the kernel is given by

$$K_H(t,s) = b_H((\tfrac{t}{s})^{H-1/2}(t-s)^{H-1/2} - (H-1/2)s^{1/2-H} \textstyle\int_s^t (u-s)^{H-1/2}u^{H-3/2}du),$$

where

$$b_H = (\frac{2H}{(1-2H)\mathbf{B}(1-2H, H+1/2)})^{1/2}.$$

We have

$$\frac{\partial K_H(t,s)}{\partial t} = c_H(H-1/2\frac{t}{s})^{H-1/2}(t-s)^{H-3/2}.$$

Now, we introduce a linear operator $K_H^* : \mathcal{E} \to L^2([0,T])$, defined by

$$(K_H^*\phi)(s) := K_H(T,s)\phi(s) + \int_s^T (\phi(t) - \phi(s))\frac{\partial K_H(t,s)}{\partial t} dt. \tag{3.13}$$

Case $H = 1/2$ It is obvious that $K_{1/2}(t,s) = 1_{[0,t]}(s)$. Indeed, we obtain

$$B_t^{1/2} = \int_0^t K_{1/2}(t,s) dW_s = \int_0^t 1_{[0,t]}(s) dW_s = W_t$$

We have

$$(K_H^* 1_{[0,t]})(s) = K_H(t,s) 1_{[0,t]}(s).$$

Thus, the operator K_H^* is an isometry between $\mathcal{E}$ and $L^2([0,T])$ that can be extended to an isometry between the closure of $\mathcal{E}$, namely the Hilbert space $S^{(H)}$ and $L^2([0,T])$. Indeed, we have

$$\begin{aligned}
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{S^{(H)}} &= R_H(t,s) \\
&= \int_0^{t \wedge s} K_H(t,u) K_H(s,u) du \\
&= \langle K_H(t,.) 1_{[0,t]}, K_H(s,.) 1_{[0,s]} \rangle_{L^2([0,T])} \\
&= \langle K_H^* 1_{[0,t]}, K_H^* 1_{[0,s]} \rangle_{L^2([0,T])}
\end{aligned}$$

3.5.2 Moving average representation

The fBm can be represented as an integral with respect to a standard Brownian motion on the whole real line. Let $(B_s)_{s \in \mathbb{R}}$ be a standard Brownian motion. Then,

$$B_t^H = \frac{1}{C(H)} \int_{\mathbb{R}} \left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right] dB_s, \tag{3.14}$$

with $C(H) > 0$ an explicit normalizing constant, is a fractional Brownian motion.

3.5.3 Harmonizable representation

There is another representation which uses the complex-valued Brownian motion (but the fBm is real-valued). In fact, for a fBm $(B_t^H)_{t\in\mathbb{R}}$, we obtain

$$B_t^H = \frac{1}{C_2(H)} \int_{\mathbb{R}} \frac{e^{itx}-1}{ix} |x|^{-(H-\frac{1}{2})} d\tilde{B}_x, \quad t \in \mathbb{R},$$

where $(\tilde{B}_t)_{t\in\mathbb{R}}$ is a complex Brownian measure and

$$C_2(H) = \left(\frac{\pi}{H\Gamma(2H) sin(H\pi)} \right)^{1/2}.$$

Let us note that the complex Brownian measure on $\mathbb{R}$ can be splitted as $\tilde{B} = B_1 + iB_2$ and is such that $B_1(A) = B_1(-A)$, $B_2(A) = -B_2(-A)$ and $\mathbb{E}(B_1(A))^2 = \frac{|A|}{2}$, $\forall A \in \mathcal{B}(\mathbb{R})$.
We also call this representation, the spectral representation.

3.6 Stochastic integration for non-adapted processes with respect to fBm

In this section, we have introduced a new approach on stochastic integration for non-adapted processes with respect to processes having irregular trajectories, based on the Levy -Hida representation. Our approach is used to solve stochastic differential equations driven by a fractional Brownian motion for integrants not necessarily adapted. Hoping that these results will serve to other processes such as sub fractional Brownian motion, mixed fractional Brownian motion or Gaussian processes in general. Using the previous theory, we could define the Wiener integration using the operator K_H^* as

$$\int_0^T \phi(s) dB_s^H = \int_0^T (K_H^* \phi)(s) dB_s,$$

for $\phi \in S^{(H)}$. But, for the right-hand side of equation to be well-defined, we must have that $K_H^*\phi \in L^2([0,T])$.

Theorem 3.10 (Definition)**.** *Let* $\triangle = \{0 = t_0, t_1, t_2, \cdots, t_n = T\}$ *be a partition of the interval* $[0, T]$*. On the subinterval* $[t_{i-1}, t_i]$*, we take the " right endpoint "*t_i *as the evaluation point for the integrand. For an adapted stochastic process* $f(t)$ *and an instantly independent stochastic process* $g(t)$*, we define the stochastic integral of* $f(t)g(t)$ *to be the limit*

$$\int_0^T f(t)g(t)dB^H(t) = \lim_{\|\triangle\|\to 0} \sum_{i=1}^{n} \psi_1^H(f)(t_{i-1})\psi_2^H(g)(t_i)(B(t_i) - B(t_{i-1}))$$

Proof.

Write $K_H^*(f.g)$ like in (3.12) and in (3.13) then develop a sequence of calculus (based on the results obtained by Joachim [39]) applied to the kernel in two cases $H < 1/2$ and $H > 1/2$.

Namely, when one wants to compute $\int_0^1 wvdx$ with $v(x) = \int_0^x v'(y)dy$, we obtain by a classical integration by parts (including the trace terms in the integral) or by Fubini's theorem,

$$\int_0^1 wvdx = \int v'(x) \int_x^1 w(y)dydx,$$

For H $>$ $\mathbf{\frac{1}{2}}$:

$$\begin{aligned}
\int_0^T f(t)g(t)dB_t^H &= \int_0^T (K_H^*(f.g)(t))dB_t \\
&= \int_0^T \int_t^T (f.g)(u)\frac{\partial K_H(u,t)}{\partial t}dudB_t \\
&= C_H \int_0^T \int_t^T (f.g)(u)(\frac{u}{t})^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}}dudB_t \\
&= C_H \int_0^T t^{\frac{1}{2}-H} \int_t^T f(u).g(u)u^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}}dudB_t \\
&= C_H \int_0^T t^{\frac{1}{2}-H} \left[\int_t^T \left(g(u)u^{H-\frac{1}{2}}\right)' \int_u^T f(y)(y-t)^{H-\frac{3}{2}}dydu\right] dB_t \\
&= C_H \int_0^T t^{\frac{1}{2}-H}[\Gamma(H-\frac{1}{2}) \int_t^T \left(g(u)u^{H-\frac{1}{2}}\right)' [-\frac{1}{\Gamma(H-\frac{1}{2})} \int_t^u f(y)(y-t)^{H-\frac{3}{2}}dy \\
&+ \frac{1}{\Gamma(H-\frac{1}{2})} \int_t^T f(y)(y-t)^{H-\frac{3}{2}}dy]du]dB_t
\end{aligned}$$

$$= C_H \int_0^T t^{\frac{1}{2}-H} \left[\Gamma(H-\frac{1}{2}) \int_t^T \left(g(u)u^{H-\frac{1}{2}}\right)' (-(I_{u^-}^{H-\frac{1}{2}}f)(t) + (I_{T^-}^{H-\frac{1}{2}}f)(t))du\right] dB_t.$$

Let

$$J = \Gamma(H-\frac{1}{2}) \int_t^T \left(g(u)u^{H-\frac{1}{2}}\right)' (-(I_{u^-}^{H-\frac{1}{2}}f)(t)+(I_{T^-}^{H-\frac{1}{2}}f)(t))du = \int_t^T f(u).g(u)u^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}}du.$$

Then,

$$\begin{aligned}
J &= -\Gamma(H-\frac{1}{2}) \left[g(t)t^{H-\frac{1}{2}}(I_{T^-}^{H-\frac{1}{2}}f)(t))\right] - \Gamma(H-\frac{1}{2}) \int_t^T g(u)u^{H-\frac{1}{2}}.f(u)(u-t)^{H-\frac{3}{2}}du \\
&= \int_t^T f(u).g(u)u^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}}du.
\end{aligned}$$

It means that $J = \frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})} \left[g(t) t^{H-\frac{1}{2}} (I_{T^-}^{H-\frac{1}{2}} f)(t) \right]$. Then,

$$\begin{aligned}
\int_0^T f(t)g(t)dB_t^H &= \tfrac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})} C_H \int_0^T g(t)(I_{T^-}^{H-\frac{1}{2}} f)(t)dB_t \\
&= \frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})} C_H \lim_{\|\triangle\|\to 0} \sum_{i=1}^{n} (I_{T^-}^{H-\frac{1}{2}} f)(t_{i-1}) g(t_i)(B(t_i) - B(t_{i-1})) \\
&= \lim_{\|\triangle\|\to 0} \sum_{i=1}^{n} (I_{T^-}^{H-\frac{1}{2}} f)(t_{i-1}) \frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})} C_H g(t_i)(B(t_i) - B(t_{i-1})) \\
&= \lim_{\|\triangle\|\to 0} \sum_{i=1}^{n} \psi_1^H(f)(t_{i-1}) \psi_2^H(g)(t_i)(B(t_i) - B(t_{i-1})),
\end{aligned}$$

where $\psi_1^H(f)(t_{i-1}) = (I_{T^-}^{H-\frac{1}{2}} f)(t_{i-1})$ and $\psi_2^H(g)(t_i) = \frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})} C_H g(t_i)$

For $\mathbf{H < \frac{1}{2}}$:

$$
\begin{aligned}
\int_0^T f(t)g(t)dB_t^H &= \int_0^T (K_H^*(f.g)(t))dB_t \\
&= \int_0^T \int_t^T (f.g)(u)\frac{\partial K_H(u,t)}{\partial t} dudB_t \\
&= \int_0^T \int_t^T (f.g)(u)\frac{\partial K_H(u,t)}{\partial t} dudB_t \\
&= C_H(H-\frac{1}{2})\int_0^T \int_t^T (f.g)(u)(\frac{u}{t})^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}} dudB_t \\
&= C_H(H-\frac{1}{2})\int_0^T t^{\frac{1}{2}-H} \int_t^T f(u).g(u)u^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}} dudB_t \\
&= C_H(H-\frac{1}{2})\int_0^T t^{\frac{1}{2}-H} \frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}\left[g(t)t^{H-\frac{1}{2}}(I_{T^-}^{H-\frac{1}{2}}f)(t)\right] dB_t \\
&= C_H(H-\frac{1}{2})\int_0^T \frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}\left[g(t)(I_{T^-}^{H-\frac{1}{2}}f)(t)\right] dB_t
\end{aligned}
$$

$$
\begin{aligned}
&= C_H(H-\frac{1}{2})\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}\int_0^T g(t)(I_{T^-}^{H-\frac{1}{2}}f)(t)dB_t \\
&= \frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}C_H(H-\frac{1}{2})\lim_{\|\triangle\|\to 0}\sum_{i=1}^n (I_{T^-}^{H-\frac{1}{2}}f)(t_{i-1})g(t_i)(B(t_i)-B(t_{i-1})) \\
&= \lim_{\|\triangle\|\to 0}\sum_{i=1}^n (I_{T^-}^{H-\frac{1}{2}}f)(t_{i-1})\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}C_H(H-\frac{1}{2})g(t_i)(B(t_i)-B(t_{i-1})) \\
&= \lim_{\|\triangle\|\to 0}\sum_{i=1}^n \psi_1^H(f)(t_{i-1})\psi_2^H(g)(t_i)(B(t_i)-B(t_{i-1})),
\end{aligned}
$$

where

$$\psi_1^H(f)(t_{i-1}) = (I_{T^-}^{H-\frac{1}{2}} f)(t_{i-1}) \text{ and } \psi_2^H(g)(t_i) = \frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})} C_H(H-\frac{1}{2}) g(t_i).$$

Chapter 4

New approach for stochastic integration w.r.t multifractional Brownian motion

This chapter is the subject of a paper submitted for publication.

4.1 Motivation and background

Multifractional Brownian motion was introduced to overcome the following limitations. The basic idea is to replace the real H by a function $t \mapsto h(t)$ ranging in $(0, 1)$. Several definitions of the multifractional Brownian motion exist. The first ones were proposed in [50] and in [8]. A more general approach was introduced in [61]. We first need to define a fractional Brownian field:

Definition 4.1. *(Fractional Brownian field). Let $(\Omega, \mathcal{F}, P)$ be a probability space. A fractional Brownian field on $\mathbb{R} \times (0, 1)$ is a Gaussian field, noted $(\mathbf{B}(t, H))_{(t,H)\in\mathbb{R}\times(0,1)}$, such that, for every H in $(0, 1)$, the process $(B_t^H)_{t\in\mathbb{R}}$ defined by $B_t^H := \mathbf{B}(t, H)$ is a fractional Brownian motion with Hurst parameter H*[1]

[1] Alternatively, one might start from a family of fBms $(B^H)_{H\in(0,1)}$ (i.e. $B^H := (B_t^H)_{t\in\mathbb{R}}$ is an fBm for every H in $(0, 1)$) and define from it the field $(\mathbf{B}(t, H))_{(t,H)\in\mathbb{R}\times(0,1)}$ by $\mathbf{B}(t, H) := B_t^H$. However it is not true, in general, that the field $(\mathbf{B}(t, H))_{(t,H)\in\mathbb{R}\times(0,1)}$ obtained in this way is Gaussian.

A multifractional Brownian motion is simply a "path" traced on a fractional Brownian field. More precisely, it is defined as follows:

Definition 4.2. *(Multifractional Brownian motion). Let $h : \mathbb{R} \to (0,1)$ be a deterministic continuous function and $\mathbf{B}$ be a fractional Brownian field. A multifractional Brownian motion (mBm in short) on $\mathbf{B}$ with functional parameter h is the Gaussian process $B^h := (B^h_t)_{t\in\mathbb{R}}$ defined by $B^h_t := \mathbf{B}(t, h(t))$ for all t in $\mathbb{R}$.*

$(\mathcal{H}_1)$: $\forall[a,b] \subset \mathbb{R}, \forall[c,d] \subset (0,1), \exists(\Lambda,\delta) \in (\mathbb{R}^*_+)^2$, such that $\mathbb{E}[(\mathbf{B}(t,H)-\mathbf{B}(t,H'))^2] \leq \Lambda|H-H'|^\delta$, for all (t,H,H') in $[a,b]\times[c,d]^2$.
Using the equality $\mathbb{E}[(\mathbf{B}(t,H)-\mathbf{B}(s,H))^2] = |t-s|^{2H}$ and the triangular inequality for the L^2-norm, Assumption $(\mathcal{H}_1)$ is seen to be equivalent to the following one:
$(\mathcal{H})$: $\forall[a,b]\times[c,d] \in \mathbb{R}\times(0,1), \exists(\Lambda,\delta) \in (\mathbb{R}^*_+)^2, s.t. \mathbb{E}[(\mathbf{B}(t,H)-\mathbf{B}(s,H'))^2] \leq \Lambda|t-s|^{2c} + |H-H'|^\delta$, for all $(t,s,H,H') \in [a,b]^2\times[c,d]^2$.
Thus, we will refer either to assumption $(\mathcal{H}_1)$ or $(\mathcal{H})$ in the sequel.

Remark 4.1. *Assumption $(\mathcal{H})$ entails that the map $(t,s,H,H') \mapsto \mathbb{E}[\mathbf{B}(t,H)\mathbf{B}(s,H')]$ is continuous on $\mathbb{R}^2\times(0,1)^2$.*

4.2 Approximation of multifractional Brownian motion

Since an mBm is just a continuous path traced on a fractional Brownian field, a natural question is to enquire whether it may be approximated by patching adequately chosen fBms, and in which sense.
Heuristically, for $a < b$, we divide $[a,b)$ into "small" intervals $[t_i, t_{i+1})$, and replace on each of these B^h by the fBm B^{H_i} where $H_i := h(t_i)$. It seems reasonable to expect that the resulting process $\sum_i B^{H_i}_t \mathbf{1}_{[t_i,t_{i+1})}(t)$ will converge, in a sense to be made precise, to B^h when the sizes of the intervals $[t_i, t_{i+1})$ go to 0.
Our aim in this section is to make this line of thought rigorous.

4.2.1 Approximation of mBm by piecewise fBms

In the sequel, we fix a fractional Brownian field $\mathbf{B}$ and a continuous function h, thus an mBm, noted B^h. We aim to prove that this mBm can be approximated on every compact interval $[a,b]$ by patching together fractional Brownian motions defined on a sequence of partitions of $[a,b]$. In that view, we choose an increasing sequence $(q_n)_{n\in\mathbb{N}}$ of integers such that $q_0 := 1$. For a compact interval $[a,b]$ of $\mathbb{R}$ and $n \in \mathbb{N}$, let $x^{(n)} := \{x_k^{(n)}; k \in [[0,q_n]]\}$ where $x_k^{(n)} := a + k\frac{(b-a)}{q_n}$ (for integers p and q with $p < q$, $[[p,q]]$ denotes the set $\{p, p+1, ..., q\}$). Define, for $n \in \mathbb{N}$, the partition $\mathcal{A}_n := \{[x_k^{(n)}, x_{k+1}^{(n)}); k \in [[0, q_n - 1]]\} \cup \{x_{q_n}^{(n)}\}$. Thus, $\mathcal{A} := (\mathcal{A}_n)_{n\in\mathbb{N}}$ is a sequence of partitions of $[a,b]$ with mesh size that tends to 0 as n tends to $+\infty$. For t in $[a,b]$ and $n \in \mathbb{N}$, there exists a unique integer p in $[[0, q_n - 1]]$ such that $x_p^{(n)} \le t < x_{p+1}^{(n)}$. We will note $x_t^{(n)}$ the real $x_p^{(n)}$ in the sequel. The sequence $(x_t^{(n)})_{n\in\mathbb{N}}$ converges to t as n tends to $+\infty$. Besides, define for $n \in \mathbb{N}$, the function $h_n : [a,b] \to (0,1)$ by setting $h_n(b) = h(b)$ and, for any t in $[a,b)$, $h_n(t) := h(x_t^{(n)})$. The sequence of step functions $(h_n)_{n\in\mathbb{N}}$ converges pointwise to h on $[a,b]$. Define, for t in $[a,b]$ and $n \in \mathbb{N}$, the process

$$B_t^{h_n} := \mathbf{B}(t, h_n(t)) = \sum_{k=0}^{q_n - 1} \mathbf{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t)\mathbf{B}(t, h(x_k^{(n)}) + \mathbf{1}_{\{b\}}(t)\mathbf{B}(b, h(b)). \tag{4.1}$$

Note that, despite the notation, the process B^{h_n} is not an mBm, as h_n is not continuous. We believe however there is no risk of confusion in using this notation. B^{h_n} is almost surely càdlàg and discontinuous at times $x_k^{(n)}, k \in [[0, q_n]]$.
The following theorem shows that the mBm appears as a limit of sums of fBms:

Theorem 4.1. *(Approximation theorem). Let $\mathbf{B}$ be a fractional Brownian field, $h : \mathbb{R} \to (0,1)$ be a continuous deterministic function and B^h be the associated mBm. Let $[a,b]$ be a compact interval of $\mathbb{R}$, $\mathcal{A}$ be a sequence of partitions as defined above, and consider the sequence of processes defined in (4.1). Then,*

1. *If $\mathbf{B}$ is such that the map $C : (t, s, H, H') \mapsto \mathbf{E}[\mathbf{B}(t,H)\mathbf{B}(s,H')]$ is continuous on $[a,b]^2 \times h([a,b])^2$ then the sequence of processes $(B^{h_n})_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$*

to B^h,i.e.,

$$\forall t \in [a,b],\ \lim_{n\to+\infty} \mathbf{E}\left[(B_t^{h_n} - B_t^h)^2\right] = 0.$$

2. *If $\mathbf{B}$ satisfies the assumption $(\mathcal{H})$ and if h is β-Hölder continuous for some positive real β, then the sequence of processes $(B^{h_n})_{n\in\mathbb{N}}$ converges*

   (i) *in law to B^h, i.e., $\{B_t^{h_n}; t \in [a,b]\} \longrightarrow^{law}_{n\to+\infty} \{B_t^h; t\in[a,b]\}$.*

   (ii) *almost surely to B^h, i.e., $P\left(\{\forall t \in [a,b],\ \lim_{n\to+\infty} B_t^{h_n} = B_t^h\}\right) = 1$.*

Before we proceed to the proof, we note that point 2(i) is a statement different from the well-known localisability of mBm, i.e., the fact that the moving average (see[50]), harmonizable (see [8]) and Volterra mBms (see [10]) are all "tangents" to fBms in the following sense: for every real u,

$$\left\{\frac{B^h_{u+rt} - B^h_u}{r^{h(u)}}; t\in[a,b]\right\} \longrightarrow^{law}_{r\to0^+} \{B_t^{h(u)}; t\in[a,b]\}$$

Proof:

1. Let $t \in [a,b]$. For any $n\in\mathbb{N}$, one computes

   $$\mathbf{E}\left[(B_t^{h_n} - B_t^h)^2\right] = C(t,t,h(x_t^{(n)}),h(x_t^{(n)})) - 2C(t,t,h(x_t^{(n)}),h(t)) + C(t,t,h(t),h(t)).$$

   The continuity of the maps $h, (t,H,H') \mapsto C(t,t,H,H')$ and the fact that $\lim_{n\to+\infty} x_t^{(n)} = t$ entails that $\lim_{n\to\infty} \mathbf{E}\left[(B_t^{h_n} - B_t^h)^2\right] = 0$

2. By assumption, there exists (η,β) in $\mathbb{R}_*^+ \times \mathbb{R}_*^+$ such that for all (s,t) in $[a,b]$,

   $$|h(s) - h(t)| = \eta|s-t|^\beta. \tag{4.2}$$

   (i) We proceed as usual in two steps (see for example [20, 55]), **a)**:finite-dimensional convergence and **b)**:tightness of the sequence of probability measures $(PoB^{h_n})_{n\in\mathbb{N}}$.

a) **Finite dimensional convergence**

Since the processes B^h and B^{h_n} defined by (4.1) are centred and Gaussian, it is sufficient to prove that $\lim_{n\to\infty} \mathbf{E}\left[B_t^{h_n} B_s^{h_n}\right] = \mathbf{E}\left[B_t^h B_s^h\right]$ for every (s,t) in $[a,b]^2$.

The cases where $t = b$ or $s = t$ are consequences of point 1. above. We now assume that $a \leq s < t < b$. One computes

$$\mathbf{E}\left[B_t^{h_n} B_s^{h_n}\right] = \sum_{(k,j)\in[[0,q_n-1]]^2} \mathbf{1}_{[x_k^{(n)},x_{k+1}^{(n)})}(t)\mathbf{1}_{[x_j^{(n)},x_{j+1}^{(n)})}(s)\mathbf{E}\left[\mathbf{B}(t,h_n(t))\mathbf{B}(s,h_n(s))\right].$$

Hence, $\mathbf{E}\left[B_t^{h_n} B_s^{h_n}\right] = \mathbf{E}\left[\mathbf{B}(t,h(x_t^{(n)}))\mathbf{B}(s,h(x_s^{(n)}))\right]$ for all large enough integers n (i.e., such that $x_s^{(n)} \leq s < x_t^{(n)} \leq t$). The continuity of h, and the fact that $\lim_{n\to\infty}(x_t^{(n)}, x_s^{(n)}) = (t,s)$ entail that $\lim_{n\to\infty} \mathbf{E}\left[B_t^{h_n} B_s^{h_n}\right] = \mathbf{E}\left[B_t^h B_s^h\right]$.

b) **Tightness of the sequence of probability measures** $(PoB^{h_n})_{n\in\mathbb{N}}$.

We are in the particular case where a sequence of càdlàg processes converges to a continuous one. The theorem on page 92 of [53] applies to this situation: it is sufficient to show that, for every positive reals ε and τ, there exists an integer m and a grid $\{t_i\}_{i\in[[0,m]]}$, such that $a = t_0 < t_1 < ... < t_m = b$,

$$\limsup_{n\to+\infty} P\left(\left\{\max_{0\leq i\leq m} \sup_{t\in[t_i;t_{i+1})} |B_t^{h_n} - B_{t_i}^{h_n}| > \tau\right\}\right) < \varepsilon. \tag{4.3}$$

Denote $c : \mathbb{R}_+^* \to \mathbb{R}$ the modulus of continuity of the map $(t,u) \mapsto \mathbf{B}(t,h(u))$, defined on $[a,b]\times[a,b]$, that is:

$$c(\delta) := \sup_{|t_1-t_2|<\delta, |u_1-u_2|<\delta} |\mathbf{B}(t_1,h(u_1)) - \mathbf{B}(t_2,h(u_2))|$$

Since the map $(t,u) \mapsto \mathbf{B}(t,h(u)$ is almost surely uniformly continuous on $[a,b]^2, c(\delta)$ tends almost surely to 0 when δ tends to 0.

Let us now fix (ε,τ) in $(\mathbb{R}_+^*)^2$ Choose $\delta>0$ such that $P(c(\delta)>\tau)<\varepsilon, m:=m(\tau,\varepsilon)$ and $\Delta'_m:=\{t_i; i\in[[0,m]]\}$, with $a=t_0<t_1<...<t_m=b$, such that $|\Delta'_m|<\frac{\delta}{2}$, where $|\Delta'_m|:=\max_{i=0,\dots,m-1}|t_{i+1}-t_i|$. Finally, denote N the smallest positive integer n such that $|\Delta_n|:=\frac{(b-a)}{q_n}<\frac{\delta}{4}$.

Define $A_{(m,n)}:=\max\limits_{0\le i\le m}\sup\limits_{t\in[t_i;t_{i+1})}|B_t^{h_n}-B_{t_i}^{h_n}|$.

Since $|t-x_t^{(n)}|\le|\Delta_n|$ for every t in $[a,b]$, the following inequalities hold almost surely:

$$A_{(m,n)}:=\max_{0\le i\le m}\sup_{t\in[t_i;t_{i+1})}|\mathbf{B}(t,h(x_t^{(n)}))-\mathbf{B}(t_i,h(x_{t_i}^{(n)}))|$$

$$\le\max_{0\le i\le m}\sup_{t\in[t_i;t_{i+1})}\sup_{(u,u'):|t-u|<|\Delta_n|,|t_i-u'|<|\Delta_n|}|\mathbf{B}(t,h(u))-\mathbf{B}(t_i,h(u'))|$$

$$\le\sup_{|s_1-s_2|<|\Delta'_m|,|u_1-u_2|<2|\Delta_n|+|\Delta'_m|}|\mathbf{B}(s_1,h(u_1))-\mathbf{B}(s_2,h(u_2))|$$

$$\le c(2|\Delta_n|+|\Delta'_m|)$$

$$\le c(\delta)$$

We have proved that $P(A_{(m,n)}>\tau)\le P(c(\delta)>\tau)$. This establishes (4.3)

(ii) Almost sure convergence

Denote $\tilde{\Omega}$ the measurable subset of Ω, verifying $P(\tilde{\Omega})=1$, such that for every $\omega\in\tilde{\Omega},(t,H)\mapsto B(t,H)(\omega)$ is continuous on $[a,b]\times[H_1,H_2]$. Then, for every ω in Ω, we get:

$|B_t^{h_n}(\omega')-B_t^{h}(\omega')|=|B(t,h_n(t)))(\omega')-B(t,h(t))(\omega')|=|B(t,h(x_t{}^{(n)}))(\omega')-B(t,h(t))(\omega')|\rightarrow_{n\to+\infty}0$

This ends the proof.

Remark 4.2. *With some additional work, one may establish the almost sure convergence of $(B^{h_n})n\in\mathbb{N}$ under the sole condition of the continuity of h.*

4.3 Stochastic integration w.r.t. mBm as limits of integral w.r.t fBm

The results of the previous section, especially 2 (i) of theorem 4.1, suggest that one may define stochastic integrals with respect to mBm as limits of integrals with respect to approximating fBms. We formalize this intuition in the present section.
We consider as above a fractional field $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$, but assume in addition that the field is C^1 in H on $(0,1)$ in the $L^2(\Omega)$ sense, i.e., we assume that the map $H \mapsto \mathbf{B}(t,H)$, from $(0,1)$ to $L^2(\Omega)$, is C^1 for every real t. We will denote $\frac{\partial \mathbf{B}}{\partial H}(t,H')$ the $L^2(\Omega)$-derivative at point H' of the map $H \mapsto \mathbf{B}(t,H)$. The field $\left(\frac{\partial \mathbf{B}(t,H)}{\partial H}\right)_{(t,H)\in\mathbb{R}\times(0,1)}$ is of course Gaussian. We will need that the derivative field satisfies the same assumption $(\mathcal{H}_1)$ as $\mathbf{B}(t,H)$. More precisely, from now on, we assume that $\mathbf{B}(t,H)$ satisfies $(\mathcal{H}_2)$:
$(\mathcal{H}_2)$: For all $[a,b]\times[c,d]\subset\mathbb{R}\times(0,1), H\mapsto\mathbf{B}(t,H)$ is C_1 in the $L^2(\Omega)$ sense from $(0,1)$ to $L^2(\Omega)$ for every t in $[a,b]$, and there exists $(\Delta,\alpha,\lambda)\in(R_+^*)^3$ such that, for all (t,s,H,H') in $[a,b]^2\times[c,d]^2$,

$$\mathbf{E}\left[(\frac{\partial \mathbf{B}}{\partial H}(t,H)-\frac{\partial \mathbf{B}}{\partial H}(s,H'))^2\right]\leq\Delta\left(|t-s|^\alpha+|H-H'|^\lambda\right)$$

Proposition 4.3.1. *The fractional Brownian fields* $\mathbf{B}_i := (\mathbf{B}_i(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}, i\in [[1,4]]$, *verify Assumption* $(\mathcal{H}_2)$.

Proof 4.3.1. *: The proof of this proposition in the case of* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *may be found in Appendix B in [38]. The ones for* $\mathbf{B}_3$ *and* $\mathbf{B}_4$ *are easily obtained using results from [50] and [10] and are left to the reader.*

In the remaining of the paper (except in theorem 4.2), we consider a C_1 deterministic function $h:\mathbb{R}\to(0,1)$, a fractional field $\mathbf{B}$ which fulfills the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, and the associated mBm $B_t^h := \mathbf{B}(t,h(t))$.
We now explain in a heuristic way how to define an integral with respect to mBm

using approximating fBms. Write the "differential" of $\mathbf{B}(t, H)$:

$$d\mathbf{B}(t,H) = \frac{\partial \mathbf{B}}{\partial t}(t,H)dt + \frac{\partial \mathbf{B}}{\partial H}(t,H)dH$$

Of course, this is only formal as $t \mapsto \mathbf{B}(t,H)$ is not differentiable in the L^2-sense nor almost surely with respect to t. It is, however, in the sense of Hida distributions, but we are not interested in this fact at this stage. With a differentiable function h in place of H, this (again formally) yields

$$d\mathbf{B}(t,h(t)) = \frac{\partial \mathbf{B}}{\partial t}(t,h(t))dt + h'(t)\frac{\partial \mathbf{B}}{\partial H}(t,h(t))dt \tag{4.4}$$

The second term on the right-hand side of (4.4) is defined for almost every ω and every real t by assumption. Moreover, it is almost surely continuous as a function of t and thus Riemann integrable on compact intervals.
On the other hand, the first term of (4.4) has no meaning a priori since mBm is not differentiable with respect to t. However, since stochastic integrals with respect to fBm do exist, we are able to give a sense to $t \mapsto \frac{\partial \mathbf{B}}{\partial t}(t,H)$ for every fixed H in $(0,1)$. Continuing with our heuristic reasoning, we then approximate $\frac{\partial \mathbf{B}}{\partial t}(t,h(t))$ by $\lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \mathbf{1}_{[x_k^{(n)},x_{k+1}^{(n)})}(t)\frac{\partial \mathbf{B}}{\partial t}(t,h_n(t))$. This formally yields:

$$d\mathbf{B}(t,h(t) \approx \lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \mathbf{1}_{[x_k^{(n)},x_{k+1}^{(n)})}(t)\frac{\partial \mathbf{B}}{\partial t}(t,h_n(t))dt + h'(t)\frac{\partial \mathbf{B}}{\partial H}(t,h(t))dt \tag{4.5}$$

Assuming we may exchange integrals and limits, we would thus like to define, for suitable processes Y,

$$\int_0^1 Y_t d\mathbf{B}(t,h(t)) = \lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_t dB_t^{h(x_k^{(n)})} + \int_0^1 Y_t h'(t)\frac{\partial \mathbf{B}}{\partial H}(t,h(t))dt \tag{4.6}$$

where the first term of the right-hand side of (4.6) is a limit, in a sense to be made precise depending on the method of integration, of a sum of integrals with respect to fBms and the second term is a Riemann integral or an integral in a weaker sense (see section 4 in the paper[38]).

In order to make the above ideas more precise, let us fix some notations. $(\mathcal{M})$ will denote a given method of integration with respect to fBm (e.g Skorohod, white noise, pathwise,...). For the sake of notational simplicity, we will consider integrals over the interval $[0,1]$. For H in $(0,1)$, denote $\int_0^1 Y_t d^{(\mathcal{M})} B_t^H$ the integral of $Y := (Y_t)_{t\in[0,1]}$ on $[0,1]$ with respect to the fBm B^H, in the sense of method $(\mathcal{M})$, assuming it exists. The following notation will be useful:

Notation (integral with respect to lumped fBms) Let $Y := (Y_t)_{t\in[0,1]}$ be a real-valued process on $[0,1]$ which is integrable with respect to all fBms of index H in $h([0,1])$ in the sense of method $(\mathcal{M})$. We denote the integral with respect to lumped fBms in the sense of method $(\mathcal{M})$ by:

$$\int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n} := \sum_{k=0}^{q_n-1} \int_0^1 \mathbf{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) Y_t d^{(\mathcal{M})} B_t^{h(x_k^{(n)})}, n \in \mathbb{N} \tag{4.7}$$

(we use the same notations as in section 2: $(q_n)_{n\in\mathbb{N}}$ is an increasing sequence of integers with $q_0 = 1$ and the family $x^{(n)} := \{x_k^{(n)}; k \in [[0, q_n]]\}$ is defined by $x^{(n)} := \frac{k}{q_n}$ for k in $[[0, q_n]]$). With this notation, our tentative definition of an integral w.r.t. to mBm (4.7) reads:

$$\int_0^1 Y_t d\mathbf{B}(t, h(t)) := \lim_{n\to+\infty} \int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n} + \int_0^1 Y_t h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \tag{4.8}$$

The interest of (4.6) is that it allows to use any of the numerous definitions of stochastic integrals with respect to fBm, and automatically obtain a corresponding integral with respect to the mBm. It is worthwhile to note that, with this approach, an integral with respect to the mBm is a sum of two terms: the first one seems to depend only on the chosen method for integrating with respect to fBm (for instance, a white noise or pathwise Riemann integral), while the second is an integral which appears to depend only on the field used to define the chosen mBm, i.e., essentially on its correlation structure. This second term will imply that the integral with respect to the moving average mBm, for instance, is different from the one with respect to the harmonisable mBm. As the example of simple processes in the next subsection will show, the second term does however also depend on the integration method with respect to fBm.

Note that the nature of $\int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n}$ depends on $(\mathcal{M})$. For example, $\int_0^1 Y_t d^{(\mathcal{M})} B_t^H$ and hence $\int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n}$ will belong to $L^2(\Omega)$ if $(\mathcal{M})$ denotes the Skorohod integral, whereas $\int_0^1 Y_t d(\mathcal{M}) B_t^H$ and hence $\int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n}$ belong to the space $(\mathcal{S})^*$ of stochastic distributions when $(\mathcal{M})$ denotes the integral in the sense of white noise theory.

We will write $\int_0^1 Y_t d(\mathcal{M}) B_t^h$ for the integral of Y on $[0,1]$ with respect to mBm in the sense of $(\mathcal{M})$ (which is yet to be defined). When we do not want to specify a particular method but instead wish to refer to all methods at the same time, we will write $\int_0^1 Y_t dB_t^{h_n}$ and $\int_0^1 Y_t dB_t^h$ instead of $\int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n}$ and $\int_0^1 Y_t d^{(\mathcal{M})} B_t^h$. In order to gain a better understanding of our approach, we explore in the following subsection the particular cases of simple deterministic and then random integrands.

4.3.1 Integral with respect to mBm through approximating fBms

We now define in a precise way our integral with respect to the mBm. Let $(E, \|\|_E)$ and $(F, \|\|_F)$ be two normed linear spaces, endowed with their Borel σ-field $\mathcal{B}(E)$ and $\mathcal{B}(F)$. Let $Y := (Y_t)_{t\in[0,1]}$ be an E-valued process (i.e., Y_t belongs to E for every real t in $[0,1]$ and $t \mapsto Y_t$ is measurable from $(0,1)$ to $(E, \mathfrak{B}(E))$). Fix an integration method $(\mathcal{M})$. As explained in the previous subsection, we wish to define the integral w.r.t. an mBm B^h in the sense of $(\mathcal{M})$ by a formula of the kind:

$$\int_0^1 Y_t d^{(\mathcal{M})} B_t^h := \lim_{n\to+\infty} \int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n} + \int_0^1 h'(t) Y_t * \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \tag{4.9}$$

where the meaning of the limit depends on $(\mathcal{M})$ and where $*$ denotes the ordinary product (in the case of pathwise integrals) or Wick product (in other cases) depending on $(\mathcal{M})$. For this formula to make sense, it is certainly necessary that Y be $(\mathcal{M})$-

integrable w.r.t. fBm of all exponents α in $h([0,1])$. We thus define, for $\alpha \in (0,1)$,

$$\mathcal{H}_E^\alpha := \{Y \in E^{[0,1]} : \int_{[0,1]} Y_t d^{(\mathcal{M})} B_t^\alpha \text{ exists and belongs to } F\},$$

and

$$\mathcal{H}^E = \cap_{\alpha \in h([0,1])} \mathcal{H}_E^\alpha$$

We will always assume that there exist a subset Λ_E of $\mathcal{H}_E$ (maybe equal to $\mathcal{H}_E$) which may be endowed with a norm $\|\|_{\Lambda_E}$ such that $(\Lambda_E, \|\|_E)$ is complete and which satisfies the following property: there exist $M > 0$ and a real χ such that for all partitions of $[0,1]$ in intervals $A_1, ..., A_n$ of equal size $\frac{1}{n}$,

$$\|Y.\mathbf{1}_{A_1}\|_{\Lambda_E} + ... + \|Y.\mathbf{1}_{A_n}\|_{\Lambda_E} \leq M n^\chi \|Y\|_{\Lambda_E}. \tag{4.10}$$

When Y belongs to Λ_E, definition (4.10) will be a valid one as soon as the limit and the last term on the right hand side exist. It turns out that a simple sufficient condition guarantees the existence of the limit of the integral w.r.t. lumped fBms. Define, for $n \in \mathbb{N}$, the map

$$L_n : \Lambda_E \to F$$

$$Y \mapsto \int_{[0,1]} Y_t d^{(\mathcal{M})} B_t^{h_n}. \tag{4.11}$$

Before giving the main result of this section, namely theorem (4.2), we indicate that the spaces E and F are $E(\omega) := \mathbb{R}$ and $F(\omega) := \mathbb{R}$ for a pathwise integral. The following theorem provides a sufficient condition under which $(L_n(Y))_{n\in\mathbb{N}}$ converges in F. We use again the notations of section 2.

Theorem 4.2. *Let h be a β-Hölder function. Assume that the function $\mathcal{I} : \Lambda_E \times (0,1) \to F$ defined by:*

$$\mathcal{I}(Y, \alpha) := \int_{[0,1]} Y_t dB_t^\alpha,$$

is θ-Hölder continuous with respect to α uniformly in Y for a real number $\theta > 0$, i.e., there exists $K > 0$ such that:

$$\forall Y \in \Lambda_E, \forall(\alpha, \alpha') \in (0,1)^2, \|\mathcal{I}(Y,\alpha) - \mathcal{I}(Y,\alpha')\|_F \leq K|\alpha - \alpha'|^\theta \|Y\|_{\Lambda_E} \tag{4.12}$$

Choose an increasing sequence $(q_n)_{n\in\mathbb{N}}$ *of positive integers such that* $\sum_{n=0}^{+\infty}\frac{q_{n+1}^{\chi}}{q_n^{\beta\theta}} < +\infty$.

Then, the sequence of functions $(L_n)_{n\in\mathbb{N}}$ *defined in (4.11) converges pointwise to a function* $L : \Lambda_E \to F$.

Proof 4.3.2. *For the sake of simplicity, we will establish the result only in the case where the sequence* $\mathcal{A} = (\mathcal{A}_n)_{n\in\mathbb{N}}$ *of partitions of* $[0,1]$ *is nested. Thus, for any point* $x_i^{(n)}$ *there exists a unique integer, denoted* k_i*, such that* $x_i^{(n)} = x_{k_i}^{(n+1)}$*. For* n *in* $\mathbb{N}$ *and* Y *in* Λ_E, $L_n(Y)$ *may be written as*

$$L_n(Y) = \sum_{p=0}^{q_n-1}\sum_{l=k_p}^{k_{p+1}-1}\int_{[x_l^{(n+1)},x_{l+1}^{(n+1)})} Y_t dB_t^{h(x_{k_p}^{(n+1)})}$$

while $L_{n+1}(Y)$ *may be decomposed as:*

$$L_{n+1}(Y) = \sum_{p=0}^{q_n-1}\sum_{l=k_p}^{k_{p+1}-1}\int_{[x_l^{(n+1)},x_{l+1}^{(n+1)})} Y_t dB_t^{h(x_l^{(n+1)})}$$

Setting $\Phi_n := \|L_n(Y) - L_{n+1}(Y)\|_F$*, and using (4.12), (4.2) and then (4.10), one gets:*

$$\Phi_n = \|\sum_{p=0}^{q_n-1}\sum_{l=k_p}^{k_{p+1}-1}(\mathcal{I}(Y.\mathbf{1}_{[x_l^{(n+1)},x_{l+1}^{(n+1)})}, h(x_l^{(n+1)})) - \mathcal{I}(Y.\mathbf{1}_{[x_l^{(n+1)},x_{l+1}^{(n+1)})}, h(x_{k_p}^{(n+1)})))\|_F$$

$$\leq \sum_{p=0}^{q_n-1}\sum_{l=k_p}^{k_{p+1}-1}\|(\mathcal{I}(Y.\mathbf{1}_{[x_l^{(n+1)},x_{l+1}^{(n+1)})}, h(x_l^{(n+1)})) - \mathcal{I}(Y.\mathbf{1}_{[x_l^{(n+1)},x_{l+1}^{(n+1)})}, h(x_{k_p}^{(n+1)})))\|_F$$

$$\leq K\sum_{p=0}^{q_n-1}\sum_{l=k_p}^{k_{p+1}-1}|h(x_l^{(n+1)}) - h(x_{k_p}^{(n+1)})|^{\theta}\|Y.\mathbf{1}_{[x_l^{(n+1)},x_{l+1}^{(n+1)})}\|_{\Lambda_E}$$

$$\leq K\eta^{\theta}q_n^{-\beta\theta}\sum_{p=0}^{q_n-1}\sum_{l=k_p}^{k_{p+1}-1}\|Y.\mathbf{1}_{[x_l^{(n+1)},x_{l+1}^{(n+1)})}\|_{\Lambda_E} \leq KM\eta^{\theta}\|Y\|_{\Lambda_E}q_n^{-\beta\theta}q_{n+1}^{\chi}.$$

Since by assumption $\sum_{n=0}^{+\infty} \frac{q_{n+1}^{\chi}}{q_n^{\beta\theta}} < +\infty$*, the series* $\sum_{n\in\mathbb{N}}(L_{n+1}(Y) - L_n(Y))$ *converges absolutely and consequently* $(L_n(Y))_{n\in\mathbb{N}}$ *converges to a limit* $L(Y)$ *as* n *goes to infinity.*

Remark 4.3. 1. *When the sequence of partitions* $(\mathcal{A}_n)_{n\in\mathbb{N}}$ *is nested and in the typical case* $q_n := 2^n$*, the condition* $\theta > \frac{\chi}{\beta}$ *entails the convergence of* $\sum_{n=0}^{+\infty} \frac{q_{n+1}^{\chi}}{q_n^{\beta\theta}}$.

If, for instance, $q_n := 2^{2^n}$*, one needs that* $\theta > \frac{2\chi}{\beta}$

2. *In our applications below, we will always assume that* h *is a* C_1 *function, and thus* $\beta = 1$.

For a process Y *in* Λ_E*, we will say that* $t \mapsto h'(t)Y_t\frac{\partial \mathbf{B}}{\partial H}(t, h(t))$ *is integrable on* $[0, 1]$ *if* $\int_{[0,1]} h'(t)Y_t\frac{\partial \mathbf{B}}{\partial H}(t, h(t))dt$ *exists for almost every* ω *in the case of a pathwise integral.*

Definition 4.3. *[38] Let* $\mathbf{B}$ *be a fractional field fulfilling the assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$. *Let* $B^h := \mathbf{B}(.,h(.))$ *be an mBm traced on* $\mathbf{B}$ *with* h *a* C_1 *function. Assume moreover that the pathwise integral fulfills the condition (4.12) and let* $Y := (Y_t)_{t\in[0,1]}$ *be an element of* Λ_E *such that the map* $t \mapsto h'(t)Y_t\frac{\partial \mathbf{B}}{\partial H}(t,h(t))$ *is integrable. The integral of* Y *with respect to* B^h *is defined as:*

$$\int_0^1 Y_t dB_t^h := \lim_{n\to\infty} \int_0^1 Y_t dB_t^{h_n} + \int_0^1 h'(t)Y_t \frac{\partial \mathbf{B}}{\partial H}(t,h(t))dt,$$

where the limit and equality both hold in F.

4.4 New approach for stochastic integration w.r.t multifractional Brownian motion

Theorem 4.3 (Definition)**.** *Let* $\triangle = \{x_k^{(n)}, x_{k+1}^{(n)}; k = \overline{0, q_n - 1}\}$ *be a partition of the subinterval* $[x_i, x_{i+1}]$. *We take the " right endpoint "*$x_i^{(n)}$ *as the evaluation point for the integrand. For an adapted stochastic process* $f(t)$ *and an instantly independent stochastic process* $g(t)$, *we define the stochastic integral of* $f(t)g(t)$ *to be the limit*

$$\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} f(t)g(t)dB^{h(x_k^{(n)})}(t) = \lim_{\|\triangle\|\to 0} \sum_{i=k}^{k+1} \psi_1^{h(x_k^{(n)})}(f)(x_i^{(n)})\psi_2^{h(x_k^{(n)})}(g)(x_{i+1}^{(n)})(B(x_{i+1}^{(n)})-B(x_i^{(n)}))$$

For $h(x_k^{(n)}) > \frac{1}{2}$: for each $k = \overline{0, q_n - 1}$

$$\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} f(t)g(t)dB_t^{h(x_k^{(n)})} = \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} (K^*_{h(x_k^{(n)})}(f.g)(t))dB_t,$$
$$= \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \int_t^{x_{k+1}^{(n)}} (f.g)(u) \frac{\partial K_{h(x_k^{(n)})}}{\partial t}(u,t) du dB_t,$$
$$= C_{h(x_k^{(n)})} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \int_t^{x_{k+1}^{(n)}} (f.g)(u) (\tfrac{u}{t})^{h(x_k^{(n)})-\frac{1}{2}} (u-t)^{h(x_k^{(n)})-\frac{3}{2}} du dB_t,$$

$$= C_{h(x_k^{(n)})} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\frac{1}{2}-h(x_k^{(n)})} \int_t^{x_{k+1}^{(n)}} f(u).g(u) u^{h(x_k^{(n)})-\frac{1}{2}} (u-t)^{h(x_k^{(n)})-\frac{3}{2}} du dB_t,$$

$$= C_{h(x_k^{(n)})} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\frac{1}{2}-h(x_k^{(n)})} \left[\int_t^{x_{k+1}^{(n)}} \left(g(u) u^{h(x_k^{(n)})-\frac{1}{2}} \right)' \int_u^{x_{k+1}^{(n)}} f(y)(y-t)^{h(x_k^{(n)})-\frac{3}{2}} dy du \right] dB_t,$$

$$= C_{h(x_k^{(n)})} \times$$
$$\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\frac{1}{2}-h(x_k^{(n)})} [\Gamma(h(x_k^{(n)}) - \tfrac{1}{2}) \int_t^{x_{k+1}^{(n)}} \left(g(u) u^{h(x_k^{(n)})-\frac{1}{2}} \right)' [- \tfrac{1}{\Gamma(h(x_k^{(n)})-\frac{1}{2})} \int_t^u f(y)(y-t)^{h(x_k^{(n)})-\frac{3}{2}} dy$$

$$+ \tfrac{1}{\Gamma(h(x_k^{(n)})-\frac{1}{2})} \int_t^{x_{k+1}^{(n)}} f(y)(y-t)^{h(x_k^{(n)})-\frac{3}{2}} dy] du] dB_t,$$

$$= C_{h(x_k^{(n)})} \times$$
$$\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\frac{1}{2}-h(x_k^{(n)})} \left[\Gamma(h(x_k^{(n)}) - \tfrac{1}{2}) \int_t^{x_{k+1}^{(n)}} \left(g(u) u^{h(x_k^{(n)})-\frac{1}{2}} \right)' (-(I_{u^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(t) + (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(t)) du \right] dB_t.$$

Let

$$J = \Gamma(h(x_k^{(n)}) - \frac{1}{2}) \int_t^{x_{k+1}^{(n)}} \left(g(u) u^{h(x_k^{(n)})-\frac{1}{2}} \right)' (-(I_{u^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(t) + (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(t)) du$$

$$= \int_t^{x_{k+1}^{(n)}} f(u).g(u) u^{h(x_k^{(n)})-\frac{1}{2}} (u-t)^{h(x_k^{(n)})-\frac{3}{2}} du.$$

Then ,

$$J = -\Gamma(h(x_k^{(n)}) - \tfrac{1}{2}) \left[g(t) t^{h(x_k^{(n)})-\frac{1}{2}} (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(t) \right] - \Gamma(h(x_k^{(n)}) - \tfrac{1}{2}) \times$$
$$\int_t^{x_{k+1}^{(n)}} g(u) u^{h(x_k^{(n)})-\frac{1}{2}} . f(u)(u-t)^{h(x_k^{(n)})-\frac{3}{2}} du$$
$$= \int_t^{x_{k+1}^{(n)}} f(u).g(u) u^{h(x_k^{(n)})-\frac{1}{2}} (u-t)^{h(x_k^{(n)})-\frac{3}{2}} du.$$

It means that $J = \frac{-\Gamma(h(x_k^{(n)})-\frac{1}{2})}{1+\Gamma(h(x_k^{(n)})-\frac{1}{2})} \left[g(t) t^{h(x_k^{(n)})-\frac{1}{2}} (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(t) \right]$. Then,

$$\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} f(t) g(t) dB_t^{h(x_k^{(n)})} = \frac{-\Gamma(h(x_k^{(n)})-\frac{1}{2})}{1+\Gamma(h(x_k^{(n)})-\frac{1}{2})} C_{h(x_k^{(n)})} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} g(t) (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(t) dB_t,$$

$$= \frac{-\Gamma(h(x_k^{(n)})-\frac{1}{2})}{1+\Gamma(h(x_k^{(n)})-\frac{1}{2})} C_{h(x_k^{(n)})} \lim_{\|\triangle\|\to 0} \sum_{i=k}^{k+1} (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(x_i^{(n)}) g(x_{i+1}^{(n)}) (B(x_{i+1}^{(n)}) - B(x_i^{(n)})),$$

$$= \lim_{\|\triangle\|\to 0} \sum_{i=k}^{k+1} (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(x_i^{(n)}) \frac{-\Gamma(h(x_k^{(n)}) - \frac{1}{2})}{1 + \Gamma(h(x_k^{(n)}) - \frac{1}{2})} C_{h(x_k^{(n)})} g(x_{i+1}^{(n)}) (B(x_{i+1}^{(n)}) - B(x_i^{(n)})),$$

$$= \lim_{\|\triangle\|\to 0} \sum_{i=k}^{k+1} \psi_1^{h(x_k^{(n)})}(f)(x_i^{(n)}) \psi_2^{h(x_k^{(n)})}(g)(x_{i+1}^{(n)}) (B(x_{i+1}^{(n)}) - B(x_i^{(n)})), \tag{4.13}$$

where
$\psi_1^{h(x_k^{(n)})}(f)(x_i^{(n)}) = (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(x_i^{(n)})$ and $\psi_2^{h(x_k^{(n)})}(g)(x_{i+1}^{(n)}) = \frac{-\Gamma(h(x_k^{(n)})-\frac{1}{2})}{1+\Gamma(h(x_k^{(n)})-\frac{1}{2})} C_{h(x_k^{(n)})} g(x_{i+1}^{(n)})$

For $h(x_k^{(n)}) < \frac{1}{2}$: for each $k = \overline{0, q_n - 1}$

$$\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} f(t)g(t)dB_t^{x_k^{(n)}} = \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} (K^*_{h(x_k^{(n)})}(f.g)(t))dB_t$$

$$= \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \int_t^{x_{k+1}^{(n)}} (f.g)(u)\frac{\partial K_{h(x_k^{(n)})}}{\partial t}(u,t)dudB_t$$

$$= \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \int_t^{x_{k+1}^{(n)}} (f.g)(u)\frac{\partial K_{h(x_k^{(n)})}}{\partial t}(u,t)dudB_t$$

$$= C_{h(x_k^{(n)})}(h(x_k^{(n)}) - \tfrac{1}{2}) \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \int_t^{x_{k+1}^{(n)}} (f.g)(u)(\tfrac{u}{t})^{h(x_k^{(n)})-\frac{1}{2}}(u-t)^{h(x_k^{(n)})-\frac{3}{2}}dudB_t$$

$$= C_{h(x_k^{(n)})}(h(x_k^{(n)}) - \tfrac{1}{2}) \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\frac{1}{2}-h(x_k^{(n)})} \int_t^{x_{k+1}^{(n)}} f(u).g(u)u^{h(x_k^{(n)})-\frac{1}{2}}(u-t)^{h(x_k^{(n)})-\frac{3}{2}}dudB_t$$

$$= C_{h(x_k^{(n)})}(h(x_k^{(n)}) - \tfrac{1}{2}) \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} t^{\frac{1}{2}-h(x_k^{(n)})} \tfrac{-\Gamma(h(x_k^{(n)})-\frac{1}{2})}{1+\Gamma(h(x_k^{(n)})-\frac{1}{2})} \left[g(t)t^{h(x_k^{(n)})-\frac{1}{2}}(I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}}f)(t)\right] dB_t$$

$$= C_{h(x_k^{(n)})}(h(x_k^{(n)}) - \tfrac{1}{2}) \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \tfrac{-\Gamma(h(x_k^{(n)})-\frac{1}{2})}{1+\Gamma(h(x_k^{(n)})-\frac{1}{2})} \left[g(t)(I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}}f)(t)\right] dB_t$$

$$= C_{h(x_k^{(n)})}(h(x_k^{(n)}) - \frac{1}{2})\frac{-\Gamma(h(x_k^{(n)}) - \frac{1}{2})}{1+\Gamma(h(x_k^{(n)}) - \frac{1}{2})} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} g(t)(I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}}f)(t)dB_t$$

$$= \frac{-\Gamma(h(x_k^{(n)}) - \frac{1}{2})}{1+\Gamma(h(x_k^{(n)}) - \frac{1}{2})}C_{h(x_k^{(n)})}(h(x_k^{(n)}) - \frac{1}{2}) \lim_{\|\triangle\|\to 0} \sum_{i=k}^{k+1} (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}}f)(x_i^{(n)})g(x_{i+1}^{(n)})(B(x_{i+1}^{(n)}) - B(x_i^{(n)}))$$

$$= \lim_{\|\triangle\|\to 0} \sum_{i=k}^{k+1} (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}}f)(x_i^{(n)})\frac{-\Gamma(h(x_k^{(n)}) - \frac{1}{2})}{1+\Gamma(h(x_k^{(n)}) - \frac{1}{2})}C_{h(x_k^{(n)})}(h(x_k^{(n)}) - \frac{1}{2})g(x_{i+1}^{(n)})(B(x_{i+1}^{(n)}) - B(x_i^{(n)}))$$

$$= \lim_{\|\triangle\|\to 0} \sum_{i=k}^{k+1} \psi_1^{h(x_k^{(n)})}(f)(x_i^{(n)})\psi_2^{h(x_k^{(n)})}(g)(x_{i+1}^{(n)})(B(x_{i+1}^{(n)}) - B(x_i^{(n)})), \tag{4.14}$$

where

$$\psi_1^{h(x_k^{(n)})}(f)(x_i^{(n)}) = (I^{h(x_k^{(n)})-\frac{1}{2}}_{[x_{k+1}^{(n)}]^-} f)(x_i^{(n)})$$

and

$$\psi_2^{h(x_k^{(n)})}(g)(x_{i+1}^{(n)}) = \frac{-\Gamma(h(x_k^{(n)})-\frac{1}{2})}{1+\Gamma(h(x_k^{(n)})-\frac{1}{2})} C_{h(x_k^{(n)})}(h(x_k^{(n)})-\frac{1}{2}) g(x_{i+1}^{(n)}).$$

4.4.1 Stochastic integraion with respect to mBm

We are finally able to define our integral:

Theorem 4.4 (Definition)**.** *Let* $\mathbf{B}$ *be a fractional field fulfilling the assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$*. Let* $B^h := \mathbf{B}(.,h(.))$ *be an mBm traced on* $\mathbf{B}$ *with* h *a* C_1 *function. Assume moreover that the pathwise integral fulfills the condition (4.12) and let* $f \times g := ((f \times g)(t))_{t\in[0,1]}$ *be an element of* Λ_E *such that*

$$\int_0^t f(s)g(s)dB_s^{h(s)} = \lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} f(s)g(s)dB_s^{h(x_k^{(n)})} + \int_0^t f(s)g(s)h'(s)\frac{\partial \mathbf{B}}{\partial H}(s,h(s))ds,$$

Proof 4.4.1. *Write* $\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} f(s)g(s)dB_s^{x_k^{(n)}}$ *like in (4.13) and in (4.14) (based on the results obtained by the paper [25]) applied to the kernel in two cases* $h(x_k^{(n)}) > 1/2$ *and* $h(x_k^{(n)}) < 1/2$*, for each* $k = \overline{0, q_n - 1}$*, then develop a sequence of calculus (based on the results obtained by [38])*

For $h(x_k^{(n)}) > \frac{1}{2}$: for each $k = \overline{0, q_n - 1}$

$$\int_0^t f(s)g(s)dB_s^{h(s)} = \lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \lim_{\|\triangle\|\to 0} \sum_{i=k}^{k+1} \psi_1^{h(x_k^{(n)})}(f)(x_i^{(n)})\psi_2^{h(x_k^{(n)})}(g)(x_{i+1}^{(n)})(B(x_{i+1}^{(n)})-B(x_i^{(n)}))+$$

$$\int_0^t f(s)g(s)h'(s)\frac{\partial B}{\partial H}(s,h(s))ds$$

where

$\psi_1^{h(x_k^{(n)})}(f)(x_i^{(n)}) = (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(x_i^{(n)})$ and $\psi_2^{h(x_k^{(n)})}(g)(x_{i+1}^{(n)}) = \frac{-\Gamma(h(x_k^{(n)})-\frac{1}{2})}{1+\Gamma(h(x_k^{(n)})-\frac{1}{2})} C_{h(x_k^{(n)})} g(x_{i+1}^{(n)})$

since the second integral on the right-hand side of the above equality exists.

For $h(x_k^{(n)}) < \frac{1}{2}$: for each $k = \overline{0, q_n - 1}$

$\int_0^t f(s)g(s)dB_s^{h(s)} = \lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \lim_{\|\triangle\|\to 0} \sum_{i=k}^{k+1} \psi_1^{h(x_k^{(n)})}(f)(x_i^{(n)}) \psi_2^{h(x_k^{(n)})}(g)(x_{i+1}^{(n)})(B(x_{i+1}^{(n)}) - B(x_i^{(n)})) +$
$\displaystyle\int_0^t f(s)g(s)h'(s)\frac{\partial B}{\partial H}(s, h(s))ds$

where

$$\psi_1^{h(x_k^{(n)})}(f)(x_i^{(n)}) = (I_{[x_{k+1}^{(n)}]^-}^{h(x_k^{(n)})-\frac{1}{2}} f)(x_i^{(n)})$$

and

$$\psi_2^{h(x_k^{(n)})}(g)(x_{i+1}^{(n)}) = \frac{-\Gamma(h(x_k^{(n)}) - \frac{1}{2})}{1 + \Gamma(h(x_k^{(n)}) - \frac{1}{2})} C_{h(x_k^{(n)})}(h(x_k^{(n)}) - \frac{1}{2}) g(x_{i+1}^{(n)}).$$

since the second integral on the right-hand side of the above equality exists.

Conclusion

In this thesis, we present the property of instant independence and we give a new approach on stochastic integration with respect to the fractional Brownian motion for processes not necessarily adapted based on Levy -Hida representation. Moreover, we have introduced a new approach on stochastic integration for non-adapted processes with respect to the multifractional Brownian motion, based on the results obtained by the paper [25]). Our approach is used to solve stochastic differential equations driven by a fractional and multifractional Brownian motion for integrants not necessarily adapted, hoping that these results will serve to other processes such as sub fractional and sub multifractional Brownian motions, mixed fractional and mixed multifractional Brownian motions or Gaussian processes in general.

Perspectives

For further work, there are many interesting issues to address such as :

- Using our approach to solve stochastic differential equations driven by a fractional and multifractional Brownian motion for integrants not necessarily adapted.
- Hoping that these results will serve to other processes such as sub multifractional Brownian motions and mixed multifractional Brownian motions.
- Extending the study deal with more general Gaussian processes.
- The fulfilment of numerical simulations related to the results of our research.
- Contribute to resolve accurate problems, in Finance and hydraulics particularly.

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