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Dedication

To "MY MOTHER".

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Contents

Abstract:
In this thesis, we study some classes of stochastic differential equations and inclusions with impulsion and delay, and we prove the existence and unicity of mild solution in Hilbert space by using the fixed point theory and approximations methods with illustrative applications.

The research circulated in this thesis loads with the problem of fractional stochastic differential equations and inclusions in Hilbert space. We proved the existence results of a mild solution of fractional stochastic evolution inclusion involving the Caputo derivative in Hilbert space driven by a fBm, our desired results were obtained by using different tools such as; fractional calculation, operator semigroups, and fixed point theory. Also, we have studied the existence result of mild solution of Hilfer fractional stochastic differential equation with impulses driven by sub-fBm, the results are obtained by using fixed point theorem. Then, we have studied the time fractional stochastic heat equation dealing with additive noise. we found explicit solution formula in the sense of distributions under which the solution is a random field in $L^2(\mathbb{P})$. Finally, sufficient conditions are given to prove the existence and unicity of integral solution of non-densely defined fractional stochastic differential equation with non-instantaneous impulses driven by fBm.

The list of works:

Publications

1. R. Y. Moulay Hachemi.; T. Guendouzi., Impulsive stochastic differential equations involving Hilfer fractional derivative, Bulletin of the Institute of Mathematics. Academia Sinica (New series) **17** (4), pp. 417–438(2022).

2. R. Y. Moulay Hachemi.; B. Øksendal., The fractional stochastic heat equation driven by time-space white noise. Fractional Calculus and Applied Analysis, https://doi.org/10.1007/s13540-023-00134-7 (2023).

3. R. Y. Moulay Hachemi.; T. Guendouzi., Stochastic fractional differential inclusion driven by fractional Brownian motion., Random Oper. Stoch. Equ. **31**(2): 1–11 (2023).

4. R. Y. Moulay Hachemi.; T. Guendouzi., Non densely defined fractional stochastic evolution equations driven by fractional Brownian motion (submited).

Communications

- R. Y. Moulay Hachemi. Communication in October 2022. The fractional stochastic heat equation driven by time-space white noise, Hammamet conference-Tunis.
- R. Y. Moulay Hachemi. Communication in March 2022. Impulsive stochastic differential equations involving Hilfer fractional derivative, IDP'2022 The international day of probability, Saida, Algeria.
- R. Y. Communication in May 2022. Stochastic differential equations with non instantenous impulses, Mathématiques et informatiques appliquées aux sciences 11-12 Mai 2022, Tamanrasset, Algérie.
- R. Y. Presentation in April 2020. Stochastic fractional differential inclusion driven by fractional Brownian motion, within Workshop IDEPT 2020, Bejaia, Algeria.

Other training

- Participation in the CIMPA 2019 school on "Stochastic Analysis and Applications" organized by de Dr. Moulay Tahar-Saida University.
- Participation in Spring school on "Backward Stochastic Differential Equation & Stochastic Control", organized by LMSSA (April, $22^{th} - 26^{th}, 2018$).
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Introduction

The main purpose of this introduction is to give a general overview of the theory of stochastic differential equations and inclusions, also to provide the history of the most main results furnished by researchers; and to present the plan of our thesis.

Differential equations and inclusions with fractional derivatives have recently proved to be strong tools in the modeling of many phenomenas in various fields of engineering economics, physics, biology, ecology, aerodynamics, and fluid dynamic traffic models [6, 92, 114, 122]. For some fundamental results in the theory of differential equations involving Caputo and Riemann-Liouville fractional derivatives, please see [4, 5, 82, 126, 128, 129, 138] and the references therein. Since Hilfer [59] proposed the generalized Riemann-Liouville fractional derivative, there has been shown some interest in studying differential equations involving Hilfer fractional derivatives see [59]. The two-parameter family of Hilfer fractional derivative $D_{a^+}^{\alpha,\beta}$ of order α and type β permits to combine between the Caputo and Riemann derivatives. the two parameters give an extra degree of freedom on the initial conditions and produce more types of stationary states. Models with Hilfer fractional derivatives are discussed in [52][126].

Many systems in physics, mechanics, biology, and medicine use the concept of differential inclusions to model there phenomena. Also the fractional differential inclusions plays an important role in description of the memory and genetic properties see [31], for this reason, many kinds of research have been dedicated to the existence of mild solution for fractional differential inclusions.

-Yuri et al. [54] introduced the theory of equations and inclusions with mean derivatives and investigated a special type of such inclusions called inclusions of geometric Brownian motion type.

-Hu et al. [61] provided the necessary backgrounds material study fractional evolution equations and inclusions with Hille-Yosida operators, also he studied the existence of mild solution.

-Boudaoui et al. [20] studied the existence of mild solutions to stochastic differential equations with non-intantaneous impulses driven by fractional Brownian motion by using Banach fixed point theorem. Also, they proved in [19] the existence of mild solutions for the first-order impulsive semilinear stochastic functional differential inclusions driven by a fractional Brownian motion.

This thesis is divided into seven chapters. Where in the first one we focus on stochastic calculus and their precessus, at the end of this chapter we give the necessary definitions of tempered distribution and properties of semigroup.

Secondly, we will provide an overview of derivatives and integrals that have been studied in fractional calculus in more general settings, we start with some history of fractional calculus, and we recall some definitions of how to define derivatives and integrals of arbitrary order.

The third chapter is devoted to studying the different tools to understand the meaning of differential inclusion, where we introduce some basic definitions and results of multivalued maps and we give an example of the study of differential inclusion in the deterministic case.

In chapter 4, we proved the existence results of a mild solution of fractional stochastic evolution inclusion involving the Caputo derivative in Hilbert space driven by a fractional Brownian motion, our desired results were obtained by using different tools such as; fractional calculation, operator semigroups, and fixed point theory. The work is accepted and published.

In chapter 5, we have studied the time fractional stochastic heat equation dealing with additive noise and more special classes of fractional heat equations. we found explicit solution formula in the sense of distributions under which the solution is a random field in $L^2(\mathbb{P})$. The work is accepted and published.

Next, the chapter 6 we have studied the existence result of mild solution of Hilfer fractional stochastic differential equation with impulses driven by sub-fractional Brownian motion, the results are obtained by using fixed point theorem. We illustrated at the end by giving an application. Our work is accepted and published.

Finally, sufficient conditions are given in chapter 7 to prove the existence and unicity of integral solution of non-densely defined fractional stochastic differential equation with non-instantaneous impulses driven by fractional Brownian motion. The work is submitted and we wait for the positive reply.

Chapter 1

Preliminaries

In this chapter, we give important concepts that we will use in the sequel of our work. Then we begin to cite the useful tools of stochastic calculus.

1.1 Brownian motion

Definition 1.1. *[107] A Brownian process is a stochastic process $(B_t,\ t \geq 0)$, which satisfies*

1. *The process starts at the origin, $B_0 = 0$;*
2. *B_t has stationary, independent increments;*
3. *The process B_t is continuous in t;*
4. *The increments $B_t - B_s$ are normally distributed with mean zero and variance $\mid t-s \mid,\ \ B_t - B_s \sim N(0, \mid t-s \mid)$.*

The process $X_t = x + B_t$ has all the properties of a Brownian motion that starts at x. Since $B_t - B_s$ is stationary, its distribution function depends only on the time interval $t-s$, i.e.

$$P\left(B_{t+s} - B_s \leq a\right) = P\left(B_t - B_0 \leq a\right) = P\left(B_t \leq a\right);$$

from condition (4) we get that B_t is normally distributed with the mean $\mathbb{E}[B_t] = 0$ and $var[B_t] = t$.

$$B_t \sim N(0,t).$$

This implies also that the second moment is $\mathbb{E}[B_t^2] = t$. Let $0 < s < t$, since the increments are independent, we write
$\mathbb{E}[B_sB_t] = \mathbb{E}\left[(B_s - B_0)(B_t - B_s + B_s^2\right] = \mathbb{E}[B_s - B_0]\mathbb{E}[B_t - B_s] + \mathbb{E}[B_s^2] = s.$

As consequence, B_s and B_t are not independent. Condition (4) has a physical explanation; a pollen grain suspended in water is kicked by a very large numbers of water molecules.

These effects are average out into a resultant increment of the grain coordinate.

Proposition 1.1. *[107] A Brownian motion process B_t is a martingale with respect to the information set $\mathcal{F}_t = \sigma(B_s; s \leq t)$.*

Now, we will give the most principal properties of the Brownian motion.

Simple invariance properties of Brownian motion

The simple scaling invariance property of Brownian motion play a crucial role to define a transformation on the space of functions, which changes the individual Brownian random functions but their distribution stays unchanged.

Lemma 1.1. *[112](**Scaling invariance**) Suppose $\{B_t : t \geq 0\}$ is a standard Brownian motion and let $a > 0$. Then the process $\{X(t) : t \geq 0\}$ defined by $X(t) = \frac{1}{a}B(a^2 t)$ is also a standard Brownian motion.*

Theorem 1.1. *[112](**Time inversion**) Suppose $\{B(t) :\ t \leq 0\}$ is a standard Brownian motion. Then the process $\{X(t) :\ t \leq 0\}$ defined by*

$$X(t) = \begin{cases} 0 & \text{for } t \in [0,T] \\ tB(\frac{1}{t}) & \text{for } t > 0; \end{cases}$$

is also a standard Brownian motion.

Corollary 1.1. *[112](**Law of large numbers**)*
Almost surely, $\lim\limits_{t\to\infty} \frac{B_t}{t} = 0$.

Now, the question is; there is a nonrandom modulus of continuity for the Brownian motion? We find the answer in the next theorems.

Theorem 1.2. *[112] There exists a constant $c > 0$ such that, almost surely, for every sufficiently small $h > 0$ and all $0 \leq t \leq 1-h$*

$$\mid B(t+h) - B(t) \mid \leq c\sqrt{h\log(\frac{1}{h})}.$$

Definition 1.2. *A function $f : [0,\infty) \to \mathbb{R}$ is said to be locally α-Hôlder continuous at $x \leq 0$, if there exists $\varepsilon > 0$, and $c > 0$ such that;*

$$| f(x) - f(y) | \leq c \mid x - y \mid^{\alpha}$$

for all $y \leq 0$ with $| y - x | < \varepsilon$.
We refer to $\alpha > 0$ as the Hölder exponent and to $c > 0$ as the Hölder constant. α-Hölder continuity gets stronger as the exponent α gets larger.

Corollary 1.2. *[44](**Hölder continuity**)*
If $\alpha < \frac{1}{2}$, then, almost surely, Brownian motion is everywhere locally α-Hölder continuous.

Non differentiability of Brownian motion

Even if the Brownian motion is everywhere continuous, its randomness allows it to be not differentiable.

Proposition 1.2. *[112] Almost surely, for all $0 < a < b < \infty$, Brownian motion is not monotone on the interval $[a,b]$.*

Proposition 1.3. *[112] Almost surely,*
$\limsup_{n\to\infty} \frac{B(n)}{\sqrt{n}} = \infty$, *and* $\liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} = \infty$.

Definition 1.3. *[112] For a function f, we define the upper and lower right derivatives*

$$D^* f(t) = \limsup_{h\downarrow 0} \frac{f(t+h) - f(t)}{h},$$

and

$$D_* f(t) = \liminf_{h\downarrow 0} \frac{f(t+h) - f(t)}{h}.$$

Theorem 1.3. *[112] Fix $t \leq 0$. Then, almost surely, Brownian motion is not differentiable at t. Moreover, $D^* f(t) = +\infty$ and $D_* f(t) = -\infty$.*

Theorem 1.4. *[108] Almost surely, Brownian motion is nowhere differentiable. Furthermore, almost surely for all t, either $D^* f(t) = +\infty$ or $D_* f(t) = -\infty$ or both.*

Another important regularity property, which Brownian motion does not possess is to be of bounded variation.

Theorem 1.5. *[81] Suppose that the sequence of partitions*

$$0 = t_0^{(n)} \leq t_1^{(n)} \leq ... \leq t_{k(n)-1}^{(n)} \leq t_{k(n)}^{(n)} = t$$

is nested, i.e. at each step one or more partition points are added, and the mesh

$$\Delta(n) := sup_{1 \leq j \leq k(n)} \left\{ t_j^{(n)} - t_{j-1}^{(n)} \right\}$$

converges to zero. Then almost surely,

$$\lim_{n \to \infty} \sum_{j=1}^{k(n)} \left(B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right)^2 = t,$$

and therefore Brownian motion is of unbounded variation.

Definition 1.4. *[81] For a sequence of partitions as above, we define*

$$V^{(2)}(t) := \lim_{n \to \infty} \sum_{j=1}^{k(n)} \left(B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right)^2$$

to be the quadratic variation of Brownian motion.

The strong Markov property and the reflection principle

The Markov property states that Brownian motion is started anew at each deterministic time instance. It is a crucial property of Brownian motion that is hold also for an important class of random times. These random times are called stopping times.
The strong Markov property for Brownian motion was established by [63] and [46].

Theorem 1.6. *[63]**Strong Markov property***
For every almost surely finite stopping time T, the process $\{B(t+T) - B(T) : t \leq 0\}$ is a standard Brownian motion independent of $\mathcal{F}^+(T)$.

Proposition 1.4. *[46] A Brownian motion process B_t is a martingale with respect to the information set $\mathcal{F}_t = \sigma(B_s;\ s \leq t)$.*

1.2 Fractional Brownian motion

The theoretical study of fractional Brownian motion was motivated by new problems in mathematical finance and telecommunication networks.

We present theoretical results on the fractional Brownian motion including different useful definitions for our work.

Definition 1.5. *[101] Let H be a constant belonging to* $(0,1)$. *A fractional Brownian motion (fBm)* $(B^H(t))_{t\geq 0}$ *with Hurst index H is a continuous and centered Gaussian process with covariance function,*

$$\mathbb{E}\left[B^{(H)}(t)B^{(H)}(s)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - \mid t-s \mid^{2H}\right). \tag{1.1}$$

The fractional Brownian motion B^H *has the following properties:*

(1)- $B^{(H)}(0) = 0$ *and* $\left[\mathbb{E}B^{(H)}(t)\right] = 0$ *for all* $t \geq 0$.

(2)- $B^{(H)}$ *has homogeneous increments, i.e.,* $B^{(H)}(t+s) - B^{(H)}(s)$ *has the same law of* $B^{(H)}(t)$ *for* $s,t \geq 0$.

(3)- $B^{(H)}$ *is a Gaussian process and* $\mathbb{E}\left[B^{(H)}(t)\right]^2 = t^{2H}$, $t \geq 0$, *for all* $H \in (0,1)$.

(4)- $B^{(H)}$ *has continuous trajectories.*

Remark 1.1. *Since* $\mathbb{E}\left(B_t^{(H)} - B_s^{(H)}\right)^2 = \mid t-s \mid^{2H}$ *and* B^H *is a Gaussian process, it has a continuous modification, according to the Kolmogorov Theorem. Indeed for all* $n \leq 1$ *it holds that*

$$\mathbb{E} \mid B_t^H - B_s^H \mid^n = \frac{2^{\frac{n}{2}}}{\pi^{\frac{1}{2}}}\Gamma\left(\frac{n+1}{2}\right) \mid t-s \mid^{n+1}.$$

Remark 1.2. *[101] For* $H = 1$, *we set* $B_t^H = B_t^1 = t\xi$, *where* ξ *is a standard normal random variable. Moreover for* $H = \frac{1}{2}$, *the covariance function is* $\mathbb{E}\left[B_t^{\frac{1}{2}} B_s^{\frac{1}{2}}\right] = t \wedge s$, *i.e.* $B^{\frac{1}{2}} = W$ *a standard Wiener process, or a Brownian motion. This justifies the name "fractional Brownian motion ".* B^H *is a generalization of Brownian motion obtained by allowing the Hurst parameter to differ from* $\frac{1}{2}$.

Definition 1.6. *[101] A stochastic process* $X = \{X_t, t \in \mathbb{R}\}$ *is called b-self similar if* $\{X_{at},\, t \in \mathbb{R}\} =^d \left\{a^b X_t,\, t \in \mathbb{R}\right\}$ *in the sense of finite-dimensional distributions.*

1.2.1 Correlation between two increments

Proposition 1.1. *[12] For* $H = \frac{1}{2}$, B^H *is a standard Brownian motion, in this case, the increments are independent.*

Now the question is, are the increments independent in the case where
$H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$?
From (1.1) we obtain easily the following representation for the covariance of increments of fBm

$$\mathbb{E}\left[\left(B^H_{t_1} - B^H_{s_1}\right)\left(B^H_{t_2} - B^H_{s_2}\right)\right] = \frac{1}{2}\left(\mid t_1 - s_2 \mid^{2H} + \mid t_2 - s_1 \mid^{2H} - \mid t_2 - t_1 \mid^{2H} - \mid s_2 - s_1 \mid^{2H}\right). \tag{1.2}$$

Suppose that $s_1 < t_1 < s_2 < t_2$ so that the intervals $[s_1, t_1]$ and $[s_2, t_2]$ do not intersect. Moreover (1.2)can be expressed as

$$\frac{1}{2}\left(f(a_1) - f(a_2 - (f(b_1) - f(b_2))\right),$$

where $a_1 = t_2 - s_1,\ a_2 = t_2 - t_1,\ b_1 = s_2 - s_1,\ b_2 = s_2 - t_1,\ f(x) = x^{2H}$.
Clearly, $a_1 - a_2 = b_2 - b_1 = t_1 - s_1$.

Therefore,
$\mathbb{E}\left[\left(B^H_{t_1} - B^H_{s_1}\right)\left(B^H_{t_2} - B^H_{s_2}\right)\right] < 0\ for\ H \in (0, \frac{1}{2})$ in view of the concavité of f.

The increments are negatively correlated for $H \in (0, \frac{1}{2})$, is useful to model sequences with antipersistance.
$\mathbb{E}\left[\left(B^H_{t_1} - B^H_{s_1}\right)\left(B^H_{t_2} - B^H_{s_2}\right)\right] > 0\ for\ H \in (\frac{1}{2}, 1)$.

Therefore, the increments are positively correlated for $H \in (\frac{1}{2}, 1)$, here the process presents an aggregation behavior, this property is used in order to describe systems with memory and persistence, in other terms the property of long range memory.

1.2.2 Long range dependence

Definition 1.1. *[12] A stationary sequence $(X_n)_{n \in \mathbb{N}}$ exhibits long-range dependence if the autocovariance functions $\rho(n) = cov(X_k, X_{k+n})$, satisfy*

$$\lim_{n \to \infty} \frac{\rho(n)}{cn^{-\alpha}} = 1,$$

for some constant c and $\alpha \in (0, 1)$. In this case, the dependence between X_k and X_{k+n} decays slowly as n tends to infinity and $\sum_{n=1}^{\infty} \rho(n) = \infty$.

Hence, we obtain immediately that the increment to $B^H(k) - B^H(k-1)$ of B^H and $X_{k+n} := B^H(k+n) - B^H(k+n-1)$ of B^H have the long-range dependence property for $H > \frac{1}{2}$ since

$$\rho_H(n) = \frac{1}{2}\left[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right] \sim H(2H-1)n^{2H-2},$$

as n goes to infinity. In particular

$$\lim_{n\to\infty} \frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1.$$

Summarizing, we obtain

1- For $H > \frac{1}{2}$, $\sum_{n=1}^{\infty} \rho_H(n) = \infty$.

2- For $H < \frac{1}{2}$, $\sum_{n=1}^{\infty} \mid \rho_H(n) \mid < \infty$.

1.2.3 Self-similarity

Definition 1.2. *[120] We say that an $\mathbb{R}^d$-valued random process $X = (X_t)_{t\geq 0}$ is self-similar or satisfies the property of self-similarity if for every $a > 0$ there exists $b > 0$ such that*

$$Law(X_{at},\, t \geq 0) = Law(bX_t,\, t \geq 0).$$

i.e., for every choice $t_0, \ldots, t_n \in \mathbb{R}$,

$$\mathbb{P}(X_{at_0} \leq x_0, \ldots, X_{at_n} \leq x_n) = \mathbb{P}(bX_{t_0} \leq x_0, \ldots, bX_{t_n} \leq x_n)$$

for every $x_0, \ldots, x_n \in \mathbb{R}$.

Because the covariance function of the fBm is homogenous of order 2H, we obtain that B^H is a self-similar process with Hurst index H, i.e., for any constant $a > 0$ the process $B^H(at)$ and $a^{-H}B^H(t)$ have the same distribution law.

1.2.4 Hölder continuity

Theorem 1.1. *[12] Let $H \in (0,1)$. The fBm $B^{(H)}$ admits a version whose sample paths are almost surely Hölder continuous of order strictly less than H.*

1.2.5 Path differentiability

Now the question is; is the process B^H mean square differentiable?

Proposition 1.2. *[91] Let $H \in (0,1)$. The fBm sample path $B^{(H)}(.)$ is not differentiable. In fact, for every $t_0 \in [0,\infty)$*

$$\limsup_{t\to t_0} \mid \frac{B^{(H)}(t) - B^{(H)}(t_0)}{t - t_0} \mid = \infty$$

with probability 1.

1.2.6 Representation of the fBm on $\mathbb{R}$

There exist some representations of fBm as a Wiener integral such

$$B_t^{(H)} = c\int_{\mathbb{R}} K_H(t,u)dB_u, \tag{1.3}$$

where c is a standardized constant.

Mandelbrot-Van Ness representation of fBm

Let $W = \{W_t,\, t \in \mathbb{R}\}$ be the two-sided Wienner process.

Denote

$$K_H(t,u) := (t-u)_+^{\alpha} - (-u)_+^{\alpha},$$

where

$$\alpha = H - \frac{1}{2}$$

.

Theorem 1.2. *[91] The process $\bar{B}^H = \left\{\bar{B}_t^H,\, t \in \mathbb{R}\right\}$ defined by*

$$\bar{B}_t^H := c_H^{(1)} \int_{\mathbb{R}} K_H(t,u)dB_u, \quad H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1), \tag{1.4}$$

where

$$c_H^{(1)} = \left(\int_{\mathbb{R}_+} ((1+s)^{\alpha} - s^{\alpha})^2 ds + \frac{1}{2H}\right)^{-\frac{1}{2}} = \frac{(2H\sin(\pi H)\Gamma(2H))^{\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}.$$

Has a continuous modification which is normalized two-sided fBm.

Definition 1.3. *[101] Define the operator*

$$M_{+,-}^H := \begin{cases} c_H^{(2)} I_{+,-}^{\alpha} f & \text{for } H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1), \\ f & \text{for } H = \frac{1}{2}, \end{cases} \tag{1.5}$$

where

$$c_H^{(2)} = c_H^{(1)} \Gamma(H + \frac{1}{2}).$$

Corollary 1.1. *[101] It follows that for any $H \in (0,1)$ the process*

$$B_t^H = \int_{\mathbb{R}} \left(M_-^H \mathbf{1}_{(0,t)}\right)(s) dW_s,$$

is a normalized fractional Brownian motion.

Proposition 1.3. *[101] The domain $\mathcal{D}\left(M_-^H\right)$ of the operator M_-^H has the form*

$$\mathcal{D}\left(M_-^H\right) = \begin{cases} \bigcup_{1\le p\le \frac{1}{\alpha}} L_p(\mathbb{R}) & \text{for } H \in (\frac{1}{2},1)\ ,\ \alpha = H-\frac{1}{2}, \\ \bigcup_{p\ge 1} I_{+,-}^{\alpha}(L_p(\mathbb{R})) & \text{for } H\in(0,\frac{1}{2}), \\ \text{all measurable functions} & \text{for } H=\frac{1}{2}. \end{cases} \tag{1.6}$$

1.3 Cylindrical fractional Brownian motion

The purpose of this section is a study of cylindrical fractional Brownian motion in Banach spaces and, starting from this, to build up a related stochastic calculus in Banach spaces with respect to cylindrical fBm. Here U is a Banach space.
If Q is a non-negative, definite symmetric trace class operator on $\mathcal{K}$, then a $\mathcal{K}$-valued Q-fractional Brownian motion can be defined.

Definition 1.7. [42] *Let $\mathcal{K}$ be a separable Hilbert space and Q be a non-negative, nuclear, self-adjoint operator on $\mathcal{K}$. A continuous, zero mean, $\mathcal{K}$-valued Gaussian process $(B_Q^H(t), t\in\mathbb{R}^+)$ is said to be Q-fractional Brownian motion with Hurst parameter $H\in(0,1)$ and associated with the covariance operator Q if:*

1. $\mathbb{E}\langle k, B_Q^H(t)\rangle_{\mathcal{K}} = 0$*, for all* $k\in\mathcal{K}$ *and* $t\in\mathbb{R}^+$.
2. $\mathbb{E}\langle k, B_Q^H(s)\rangle_{\mathcal{K}}\langle k', B_Q^H(t)\rangle_{\mathcal{K}} = \frac{1}{2}\langle Qk, k'\rangle_{\mathcal{K}}(t^{2H}+s^{2H}-|t-s|^{2H})$*, for any* $s,t\in\mathbb{R}^+$ *and* $k,k'\in\mathcal{K}$.
3. $(B_Q^H(t), t\ge 0)$ *has* $\mathcal{K}$*-valued continuous sample path* $\mathbb{P}$*.a.s.*

Definition 1.8. [42] *Let Q be a non- negative definite symetric-class operator on a separable Hilbert space $\mathcal{K}$, $\{e_n\}_{n=1}^{\infty}$ be an ONB in $\mathcal{K}$ diagonalizing Q and the corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Let $\beta_n^H(t)$ be a sequence of real, independent standard fractional Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$ for $n=1,2,\ldots$ and $t\in\mathbb{R}$. The process*

$$W_t = \sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)e_n,$$

is called a Q-fractional Brownian motion in $\mathcal{K}$.

Remark 1.3. [42] *If Q is a nuclear operator, then a cylindrical fractional Brownian motion is a Q-fractional Brownian motion.*

1.4 Cylindrical and Q-Sub-Fractional Brownian motion

As an extension of Brownian motion, recently, Bojdecki et al. [16] introduced and studied a rather special class of self-similar Gaussian process. This process arises from occupation time fluctuations of branching particle systems with Poisson's initial condition. This process is called Sub-fractional Brownian motion.

1.4.1 Cylindrical sub fractional Brownian motion

Definition 1.4. *[16] Let $\mathcal{K}$ be a separable Hilbert space. A continuous, zero mean, $\mathcal{K}$-valued Gaussian process $\left(S_I^H(t), t \geq 0\right)$ is said to be cylindrical sub-fractional Brownian motion with Hurst parameter $H \in (0,1)$ if his covariance is given by*

$$E\left\langle k, S_I^H(s)\right\rangle\left\langle k', S_I^H(t)\right\rangle = \left\langle k, k'\right\rangle\left[s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right]\right] \text{ for all } s,t \in \mathbb{R}^+ \text{and } k,k' \in \mathcal{K}.$$

Definition 1.5. *[16] Let Q be a non-negative, self-adjoint bounded linear operator that is not nuclear, then a cylindrical sub-fractional Brownian motion is defined by the formal series*

$$S_I^H(t) = \sum_{n=1}^{\infty} S_n^H(t) e_n \quad t \geq 0,$$

where $\{S_n^H(t)\}_{n=1}^{\infty}$ is a sequence of independent, real-valued standard sub fractional Brownian motion with Hurst parameter $H \in (0,1)$ and $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal basis in the Hilbert space $\mathcal{K}$.

1.4.2 Q-sub fractional Brownian motion

Let $(U, \|.\|_U, \langle . \rangle_U)$ and $\left(\mathcal{K}, \|.\|_{\mathcal{K}}, \langle . \rangle_{\mathcal{K}}\right)$ be two separable Hilbert space. Let $\mathcal{L}(\mathcal{K}, U)$ denote the space of all bounded linear operator from $\mathcal{K}$ to U and $Q \in \mathcal{L}(\mathcal{K}, U)$ be a non-negative self-adjoint operator.

Definition 1.6. *[16] Let $\mathcal{K}$ be a separable Hilbert space and Q be a non- negative self-adjoint operator on $\mathcal{K}$. A continuous, zero mean $\mathcal{K}$-valued Gaussian process $\left(S_Q^H(t), t \geq 0\right)$ is said to be Q-sub fractional Brownian motion with Hurst parameter $H \in (0,1)$ associated with the covariance operator Q if:*

$$E\left\langle k, S_Q^H(s)\right\rangle\left\langle k', S_Q^H(t)\right\rangle = \left\langle Qk, k'\right\rangle\left[s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right]\right] \text{ for all } s,t \in \mathbb{R}^+.$$

Definition 1.7. *[16] Let $Q \in \mathcal{L}(\mathcal{K}, U)$ be a non-negative, self- adjoint trace class operator* *on a separable Hilbert space $\mathcal{K}$, $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal basis in the Hilbert space $\mathcal{K}$ diagonalizing Q and the corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Let $\{S_n^H(t)\}_{n=1}^{\infty}$ be a sequence of real independent standard sub fractional Brownian motion, the process*

$$S_Q^H(t) = \sum_{n=1}^{\infty} S_n^H(t) Q^{\frac{1}{2}} e_n = \sum_{n=1}^{\infty} S_n^H(t) \sqrt{\lambda_n} e_n,$$

is called a $\mathcal{K}$-valued Q sub fractional Brownian motion.

1.5 Time-space white noise

Let n be a fixed natural number. Later we will set $n = 1 + d$. Define $\Omega = \mathcal{S}'(\mathbb{R}^n)$, equipped with the weak-star topology.

As events we will use the family $\mathcal{F} = \mathcal{B}(\mathcal{S}'(\mathbb{R}^n))$ of Borel subsets of $\mathcal{S}'(\mathbb{R}^d)$, and our probability measure $\mathbb{P}$ is defined by the following result:

Theorem 1.7. *[53]***(The Bochner–Minlos theorem)**
There exists a unique probability measure $\mathbb{P}$ on $\mathcal{B}(\mathcal{S}'(\mathbb{R}^n))$ with the following property:

$$\mathbb{E}[e^{i\langle \cdot, \phi \rangle}] := \int_{\mathcal{S}'} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2}; \quad i = \sqrt{-1}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, where $\|\phi\|^2 = \|\phi\|^2_{L^2(\mathbb{R}^n)}$, $\langle \omega, \phi \rangle = \omega(\phi)$ is the action of $\omega \in \mathcal{S}'(\mathbb{R}^n)$ on $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ denotes the expectation with respect to $\mathbb{P}$.

We will call the triplet $(\mathcal{S}'(\mathbb{R}^n), \mathcal{B}(\mathcal{S}'(\mathbb{R}^n)), \mathbb{P})$ the *white noise probability space*, and $\mathbb{P}$ is called the *white noise probability measure*.

The measure $\mathbb{P}$ is also often called the (normalised) *Gaussian measure* on $\mathcal{S}'(\mathbb{R}^n)$. It is not difficult to prove that if $\phi \in L^2(\mathbb{R}^n)$ and we choose $\phi_k \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi_k \to \phi$ in $L^2(\mathbb{R}^n)$, then

$$\langle \omega, \phi \rangle := \lim_{k\to\infty} \langle \omega, \phi_k \rangle \quad \text{exists in} \quad L^2(\mathbb{P})$$

and is independent of the choice of $\{\phi_k\}$. In particular, if we define

$$\widetilde{B}(x) := \widetilde{B}(x_1, \cdots, x_n, \omega) = \langle \omega, \chi_{[0,x_1]\times\cdots\times[0,x_n]} \rangle; \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n,$$

where $[0, x_i]$ is interpreted as $[x_i, 0]$ if $x_i < 0$, then $\widetilde{B}(x, \omega)$ has an x-continuous version $B(x, \omega)$, which becomes an *n-parameter Brownian motion*, in the following sense:

By an *n-parameter Brownian motion* we mean a family $\{B(x,\cdot)\}_{x\in\mathbb{R}^n}$ of random variables on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that

- $B(0,\cdot)=0$ almost surely with respect to $\mathbb{P}$,
- $\{B(x,\omega)\}$ is a continuous and Gaussian stochastic process.
- For all $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)\in\mathbb{R}^n_+$, $B(x,\cdot)$, $B(y,\cdot)$ have the covariance $\prod_{i=1}^n x_i\wedge y_i$. For general $x,y\in\mathbb{R}^n$ the covariance is $\prod_{i=1}^n\int_{\mathbb{R}}\theta_{x_i}(s)\theta_{y_i}(s)ds$, where $\theta_x(t_1,\dots,t_n)=\theta_{x_1}(t_1)\cdots\theta_{x_n}(t_n)$, with

$$\theta_{x_j}(s)=\begin{cases}1 & \text{if } 0<s\le x_j\\ -1 & \text{if } x_j<s\le 0\\ 0 & \text{otherwise.}\end{cases}$$

It can be proved that the process $\widetilde{B}(x,\omega)$ defined above has a modification $B(x,\omega)$ which satisfies all these properties. This process $B(x,\omega)$ then becomes an *n-parameter Brownian motion*.

We remark that for $n=1$ we get the classical (1-parameter) Brownian motion $B(t)$ if we restrict ourselves to $t\ge 0$. For $n\ge 2$ we get what is often called the *Brownian sheet*.

With this definition of Brownian motion, it is natural to define the n-parameter Wiener–Itô integral of $\phi\in L^2(\mathbb{R}^n)$ by

$$\int_{\mathbb{R}^n}\phi(x)dB(x,\omega):=\langle\omega,\phi\rangle;\quad \omega\in\mathcal{S}'(\mathbb{R}^d).$$

We see that by using the Bochner–Minlos theorem we have obtained an easy construction of n-parameter Brownian motion that works for any parameter dimension n. Moreover, we get a representation of the space Ω as the Fréchet space $\mathcal{S}'(\mathbb{R}^d)$. This is an advantage in many situations, for example in the construction of the Hida-Malliavin derivative, which can be regarded as a stochastic gradient on Ω.

1.6 Stochastic integration

In this section, we recall to the stochastic integration with respect to fractional Brownian motion, to cylindrical fractional Brownian motion and to Q-sub fractional Brownian motion.

1.6.1 Wiener integration with respect to fractional Brownian motion

Let $(\Omega, \mathcal{F}, P)$ be an arbitrary complete probability space.
Consider the space
$L_2^H(\mathbb{R}) := \left\{f : M^H.f \in L^2(\mathbb{R})\right\}$ equipped with the norm $\| f \|_{L_2^H(\mathbb{R})} = \| M_-^H f \|_{L^2(\mathbb{R})}$.

Definition 1.8. *[9] Let $f \in L_2^H(\mathbb{R})$. Then the Wiener integral with respect to f.B.m is defined as*

$$I_H(f) := \int_R f(s)dB_s^H := \int_{\mathbb{R}} (M_-^H f)(s)dW_s.$$

As a particular case, consider the step function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = \sum_{k=1}^n a_k \mathbf{1}_{[t_{k-1},t_k)}(t)$, where $t_0 < t_1 < ... < t_n \in \mathbb{R}$ and $a_k \in \mathbb{R}$, $1 \le k \le n$. Then from the linearity of the operator M_-^H we have that

$$I_H(f) = \sum_{k=1}^n a_k \int_{\mathbb{R}} M_-^H \mathbf{I}_{[t_{k-1},t_k)}(s)dw_s = \sum_{k=1}^n a_k \left(B_{t_k}^H - B_{t_{k-1}}^H\right)$$

which coincides with the usual Riemann-Stieltjes sum.
Note, that for a step function, it holds that

$$\begin{aligned}\| I_H(f) \|_{L^2(\Omega)}^2 &= \sum_{i,k=1}^n a_i a_k \int_{\mathbb{R}} M_-^H \mathbf{I}_{[t_{k-1},t_k)}(x) M_-^H \mathbf{I}_{[t_{i-1},t_i)}(x)dx \\ &= \| M_-^H f \|_{L_2(\mathbb{R})}^2 \\ &= 2\alpha H \int_{\mathbb{R}^2} f(u)f(v) \mid u-v \mid^{2\alpha-1} dudv,\end{aligned}$$

where the last equality holds for $H \in (\frac{1}{2}, 1)$ but not for $H \in (0, \frac{1}{2})$. For any $0 < H < 1$ we have the following.

Lemma 1.1. *[9] For $0 < H < 1$, it holds that the linear span of the set $\left\{M_-^H \mathbf{1}_{(u,v)},\ u, v \in \mathbb{R}\right\}$ is dense in $L^2(\mathbb{R})$.*

Theorem 1.3. *[72] The space L_2^H is incomplete for $H \in (\frac{1}{2}, 1)$, due to lemma (1.1), we can approximate any $f \in L_2^H(\mathbb{R})$ by step functions f_n in $L_2^H(\mathbb{R})$. Then $M_-^H f_n \to M_-^H f$ in $L_2(\mathbb{R})$, and we have that*

$$\begin{aligned}I_H(f) &:= \int_{\mathbb{R}} f(x)dB_s^H \\ &= \int_{\mathbb{R}} (M_-^H f)(s)dW_s \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n(s)dB_s^H,\end{aligned}$$

where the convergence is in $L_2(\Omega)$. Furthermore for $H \in (\frac{1}{2}, 1)$, we have that

$$\mathbb{E} \mid I(f) \mid^2 = \int_{\mathbb{R}} \mid (M_-^H f)(x) \mid^2 dx$$

for $f \in L_2^H(\mathbb{R})$; however in general it does not holds.

1.6.2 Stochastic Integral with respect to the cylindrical fractional Brownian motion

In this section we introduce the stochastic integral

$$\int \varphi(s) dB(s),$$

as a V-valued random variable for deterministic, operator-valued functions

$\varphi : [0,T] \to \mathcal{L}(U,V)$, where V is a separable Banach space.

To define the cylindrical integral, we recall the representation of a cylindrical f.B.m $(B^H(t) : \ t \geq 0)$ with Hurst parameter $H \in (0,1)$ in the Banach space U, according to theorem (1.2)

$$B^H(t)u^* = \sum_{k=1}^{\infty} < ie_k, u^* > b_k(t) \; for \; all \; u^* \in U^*, \, t \geq 0. \tag{1.7}$$

Here, X is a Hilbert space with an orthonormal basis $(e_k)_{k \in \mathbb{N}}$, $i : X \to U$ is a linear, continuous mapping and $(b_k)_{k \in \mathbb{N}}$ is a sequence of independent, real valued standard f.B.m. If we assume momentarily that we have already introduced a stochastic integral $\int_0^T \Psi(t) dB(t)$ as a V-valued random variable, then the representation 1.7 of B^H naturally results in

$$\sum_{k=1}^{\infty} \int_0^T < \Psi(t) ie_k, v^* > db_k(t) \; for \; all \; v^* \in V^*.$$

The integrals can be considered as the Fourier coefficients of the X-valued integral

$$\int_0^T i^* \Psi^*(t) v^* db_k(t).$$

The function $t \mapsto i^* \Psi^*(t) v^*$ must be integrable with respect to the real valued standard f.B.m, b_k for every $v^* \in V^*$ and $k \in \mathbb{N}$, that is the function Ψ must be in the linear space $I := \{\Phi : [0,T] \to \mathcal{L}(U,V) : i^* \Phi^*(.) v^* \in \hat{M} \;$ for all $v^* \in V^*\}$. Here, $\hat{M} = \hat{M}_X$ denotes the Banach space of functions $f : [0,T] \to X$. For this class of integrands we have the following property.

Proposition 1.5. *[48] For each $\Psi \in I$ the mapping*

$$L_\Psi : V^* \to \hat{M}, \quad L_{\Psi v^*} = i\Psi^*(.)v^*$$

is linear and continuous.

Lemma 1.2. *[48] For every $\Psi \in I$ we define*

$$< \Gamma_\Psi f, v^* >= \int_0^T [K^*(i^*\Psi^*(.)v^*)(t), f(t)]dt \ for\ all\ f \in L^2([0,T];X),\ v^* \in V^*.$$

In this way, one obtains a linear, bounded operator $\Gamma_\Psi : L^2([0,T];X) \to V^{**}$.

Proposition 1.6. *[48] let the f.B.m B^H be represented in the form 1.7. Then for each $\Psi \in I$ the mapping*

$$Z_\Psi : V^* \to L_p^2(\Omega, \mathbb{R}),\ Z_{\Psi v^*} := \sum_{k=1}^{\infty} \int_0^T < \Psi(t) i e_k, v^* > db_k(t), \tag{1.8}$$

defines a Gaussian cylindrical random variable in V with covariance operator $Q_\Psi : V^* \to V^{**}$, factorized by $Q_\Psi = \Gamma_\Psi \Gamma_\Psi^*$. Furthermore, the cylindrical random variable Z_Ψ is independent of the representation 1.7.

Definition 1.9. *A function $\Psi \in I$ is called stochastically integrable is there exists a random variable $I_\Psi : \Omega \to V$ such that*
$Z_\Psi v^* =< I_\Psi, v^* >$ for all $v^* \in V^*$ where Z_Ψ denotes the cylindrical integral of Ψ. We use the notation

$$I_\Psi := \int_0^T \Psi(t) dB^H(t).$$

Theorem 1.8. *[48] For $\Psi \in I$ the following are equivalent:*

(a) Ψ *is stochastically integrable,*

(b) the operator Γ_Ψ is V-valued and γ-radonifying.

Let $K_H(t,s)$ be the kernel function, for $0 \leq s \leq t \leq T$,

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\Big(\frac{1}{2} - H\Big) \int_s^t (u-s)^{H-\frac{3}{2}}\Big(1 - \big(\frac{s}{u}\big)^{\frac{1}{2}-H}\Big);$$

where $c_H = \big[\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\big]^{\frac{1}{2}}$ and $H \in (0,1)$.

If $H \in (\frac{1}{2}, 1)$, then K_H has a simple form as

$$K_H(t,s) = c_H(H - \frac{1}{2})s^{\frac{1}{2}-H} \int\limits_s^t \left(u-s\right)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du.$$

A definition of stochastic integral of deterministic $\mathcal{K}$-valued function with respect to a scalar fractional Brownian motion $(B(t), t \geq 0)$ is described.
Let $K_H^* : \varepsilon \longrightarrow L^2([0,T], \mathcal{K})$ be the linear operator given by

$$K_H^* \varphi(t) = \varphi(t) K_H(T,t) + \int_t^T \left(\varphi(s) - \varphi(t)\right) \frac{\partial K_H(s,t)}{\partial s} ds; \tag{1.9}$$

for $\varphi \in \varepsilon$, where ε is the linear space of $\mathcal{K}$-valued step function on $[0,T]$.
For $\varphi \in \varepsilon$,

$$\varphi(t) = \sum_{i=1}^{n-1} x_i \mathbb{I}_{[t_i, t_{i+1}]}(t),$$

where $x_i \in K$, $i \in \{1,...,n-1\}$ and $0 = t_1 < t_2 < ... < t_n = T$.
We define

$$\int_0^T \varphi dB = \sum_{i=1}^{n-1} x_i (B_{t_{i+1}} - B_{t_i}). \tag{1.10}$$

It follows directly that

$$E\left\| \int_0^T \varphi dB \right\|^2 = \mid K_H^* \varphi \mid^2_{L^2([0,T],\mathcal{K})}. \tag{1.11}$$

Let $(\mathcal{H}, \| \, . \, \|_{\mathcal{H}}, \langle ., . \rangle_{\mathcal{H}})$ be the Hilbert space obtained by the completion of the pre-Hilbert space ε with the inner product $\langle \varphi, \psi \rangle_{\mathcal{H}} := \langle K_H^* \varphi, K_H^* \psi \rangle_{L^2([0,T],\mathcal{K})}$, for $\varphi, \psi \in \varepsilon$. The stochastic integral 1.10 is extended to $\varphi \in \mathcal{H}$ by the isometry 1.11 .
Thus $\mathcal{H}$ is the space of integrable functions. If $H \in (\frac{1}{2}, 1)$ then it is easily verified that $\widetilde{\mathcal{H}} \subset \mathcal{H}$, where $\widetilde{\mathcal{H}}$ is the Banach space of Borel measurable functions with the norm $\| \, . \, \|_{\widetilde{\mathcal{H}}}$ given by

$$\| \varphi \|^2_{\widetilde{\mathcal{H}}} = \int_0^T \int_0^T |\varphi(u)||\varphi(v)| \phi(u-v) du dv,$$

where $\phi(u) = H(2H-1)|u|^{2H-2}$ and it is elementary to verify that $L^p([0,t], \mathcal{K}) \subset \widetilde{\mathcal{H}}$ for $p > \frac{1}{H}$ then,

$$E\|\int_0^T \varphi dB\|^2 = \int_0^T \int_0^T \langle \varphi(u), \varphi(v) \rangle \phi(u-v) du dv.$$

If $H \in (0, \frac{1}{2})$, then the space of integral functions is smaller than for $H \in (\frac{1}{2}, 1)$.
Associated with $(B(t), t \geq 0)$ is a standard cylindrical Wiener process $(W(t), t \geq 0)$ in $\mathcal{K}$

such that formally $B(t) = K_H(W(t))$.
For $x \in K \setminus \{0\}$, let $B_x(t) = \langle B(t), x \rangle$, it is elementary to verify from 1.10 that there is a scalar Wiener process $(w_x(t), t \geq 0)$ such that

$$B_x(t) = \langle B(t), x \rangle = \int_0^t K_H(t,s) dw_x(s);$$

for $t \in \mathbb{R}^+$. Furthermore, $w_x(t) = B_x\left((K_H^*)^{-1}\mathbf{1}_{[0,t]}\right)$, where K_H^* is given by 1.9.
Now we define the stochastic integral $\int_0^T GdB$ for an operator-valued function $G : [0,T] \longrightarrow \mathcal{L}(\mathcal{K})$ is a $\mathcal{K}$-valued random variable.

Definition 1.9. [*109*] *Let $G : [0,T] \longrightarrow \mathcal{L}(\mathcal{K})$, $(e_n, n \in \mathbb{N})$ be a complete orthonormal basis in K, $Ge_n(t) = G(t)e_n$, $Ge_n \in \mathcal{H}$ for $n \in \mathbb{N}$ and B is a standard cylindrical fractional Brownian motion. Define*

$$\int_0^T GdB := \sum_{n=1}^{\infty} \int_0^T Ge_n dB_n; \tag{1.12}$$

provided the infinite series converges in $L^2(\Omega)$.

Proposition 1.4. [*109*] *Let $G : [0,T] \longrightarrow \mathcal{L}(\mathcal{K})$ and $G(.)x \in \mathcal{H}$ for each $x \in V$. Let $\Gamma_T : \mathcal{K} \longrightarrow L^2([0,T], \mathcal{K})$ be given as*

$$(\Gamma_T(x))(t) = (K_H^* Gx)(t),$$

for $t \in [0,T]$ and $x \in \mathcal{K}$. If $\Gamma_T \in \mathcal{L}_2(\mathcal{K}, L^2([0,T], \mathcal{K}))$ is a Hilbert Schmidt operator then the stochastic integral (1.12) is a well-defined centered Gaussian $\mathcal{K}$-valued random variable with covariance operator $\tilde{Q}_T$ given by

$$\tilde{Q}_T x = \int_0^T \sum_{n=1}^{\infty} \langle (\Gamma_T e_n)(s), x \rangle (\Gamma_T e_n)(s) ds. \tag{1.13}$$

This integral does not depend on the choice of the complete orthonormal basis $(e_n, n \in \mathbb{N})$.

Remark 1.1. *Since $\Gamma_T \in \mathcal{L}_2(\mathcal{K}, L^2([0,T], \mathcal{K}))$, it follows that the map $x \longrightarrow (\Gamma_T x)(t)$ is the Hilbert -Schmidt on $\mathcal{K}$ for almost all $t \in [0,T]$. Let Γ_T^* be the adjoint of Γ_T. Then Γ_T^* is also Hilbert-Schmidt and $\tilde{Q}_T$ can be expressed as*

$$\tilde{Q}_T x = \int_0^T (\Gamma_T(\Gamma_T^* x))(t) dt, \tag{1.14}$$

for $x \in \mathcal{K}$.

If $H \in (\frac{1}{2}, 1)$ and G satisfies

$$\|G\|^2_{\tilde{\mathcal{H}}} = \int_0^T \int_0^T |G(u)|_{\mathcal{L}_2(\mathcal{K})} |G(v)|_{\mathcal{L}_2(\mathcal{K})} \phi(u-v) du dv < \infty;$$

then

$$\tilde{Q}_T = \int_0^T \int_0^T G(u) G^*(v) \phi(u-v) du dv;$$

where $\phi(u-v) = H(2H-1) \mid u-v \mid^{2H-2}$.

Proposition 1.5. [109] *If $\tilde{A} : Dom(\tilde{A}) \longrightarrow \mathcal{K}$ is closed linear operator, $G : [0,T] \longrightarrow \mathcal{K}$ satisfies $G([0,T]) \subset Dom(\tilde{A})$ and both G and $\tilde{A}G$ satisfy the conditions for G in property 1.4, then*

$$\int_0^T G dB \subset Dom(\tilde{A}) \quad \mathbb{P}.a.s,$$

and

$$\tilde{A} \int_0^T G dB = \int_0^T \tilde{A} G dB \quad \mathbb{P}.a.s.$$

1.6.3 Stochastic integral with respect to Q-sub fractional Brownian motion

Let ε the linear space of $\mathbb{R}$-valued step functions on $[0,T]$. For $\varphi \in \varepsilon$, we define its wiener integral with respect to one-dimensional sub fractional Brownian motion $\{S^H(t)\}_{t\geq 0}$ as follows

$$\int_0^T \varphi(s) dS^H(s) = \sum_{n=1}^{\infty} x_i (S^H_{t_i+1} - S^H_{t_i}).$$

Let $\mathcal{H}_{S^H}$ be the canonical Hilbert space associated to the sub-fBm S^H. That is $\mathcal{H}_{S^H}$ is the cloture of the linear span ε with respect to the scalar product,

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}_{S^H}} = cov\left(S^H(t), S^H(s)\right).$$

We know that the covariance of sub-fBm can be written as

$$\mathbb{E}\left[S^H(t) S^H(s)\right] = \int_0^t \int_0^s \phi_H(u,v) du dv = C_H(t,s), \tag{1.15}$$

where

$$\phi_H(u,v) = H(2H-1)\left(|\, u-v\,|^{2H-2} - (u+v)^{2H-2}\right).$$

Equation (1.15) implies that

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{SH}} = \int_0^t \int_0^t \varphi_u \psi_v \phi(u,v)\,du\,dv. \tag{1.16}$$

Now we consider the kernel

$$K_H(t,s) = \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-\frac{1}{2})} s^{3/2-H} \left(\int_s^t (x^2-s^2)^{H-3/2} dx \right) 1_{[0,t]}(s). \tag{1.17}$$

By Dzhaparidze and Van Zanten [47], we have

$$C_H(t,s) = c_H^2 \int_0^{t\wedge s} K_H(t,u) K_H(s,u)\,du, \tag{1.18}$$

where

$$c_H^2 = \frac{\Gamma(1+2H) sin(\pi H)}{\pi}.$$

Let K_H^* be the linear operator from ε to $L^2[0,T]$ defined by

$$\left(K_H^* \varphi\right)(s) = c_H \int_s^r \varphi_r \frac{\partial K_H}{\partial r}(r,s)\,dr.$$

By using the equalities (1.16) (1.18), we obtain

$$\begin{aligned}
\langle K_H^*\varphi, K_H^* \rangle_{L^2([0,T])} &= c_H^2 \int_0^T \left(\int_s^T \varphi_r \frac{\partial K_H}{\partial r}(r,s)\,dr \right) \left(\int_s^T \psi_u \frac{\partial K_H}{\partial u}(u,s)\,du \right) ds \\
&= c_H^2 \int_0^T \int_0^T \left(\int_0^{r\wedge u} \frac{\partial K_H}{\partial r}(r,s) \frac{\partial K_H}{\partial u}(u,s)\,ds \right) \varphi_r \psi_u\,dr\,du \\
&= c_H^2 \int_0^T \int_0^T \frac{\partial^2 K_H}{\partial r \partial u}(u,s) \varphi_r \psi_u\,dr\,du \\
&= H(2H-1) \int_0^T \int_0^T \left(|\, u-r\,|^{2H-2} - (u+r)^{2H-2}\right) \varphi_r \psi_u\,dr\,du \\
&= \langle \varphi, \psi \rangle_{\mathcal{H}_{SH}}.
\end{aligned} \tag{1.19}$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert space $\mathcal{H}_{S^H}$ and $L^2([0,T])$.
Hence, the process W defined by $W(t) := S^H\left((K_H^*)^{-1}1_{[0,t]}\right)$ is a Wiener process, and S^H has the following Wiener integral representation:

$$S^H(t) = c_H \int_0^t K_H(t,s)dW(s),$$

because $(K_H^*)(1_{[0,t]})(s) = c_H K_H(t,s)$.
By Dzhapridze and Van Zanten [47], we have

$$W(t) = \int_0^t \psi_H(t,s)dS^H(s),$$

where

$$\psi_H(t,s) = \frac{s^{H-1/2}}{\Gamma(3/2-H)}\left[t^{H-3/2}(t^2-s^2)^{1/2-H} - (H-3/2)\int_s^t (x^2-s^2)^{1/2-H}x^{H-3/2}dx\right]1_{[0,t]}(s).$$

In addition, for any $\varphi \in \mathcal{H}_{S^H}$,

$$\int_0^t \varphi(s)dS^H(s) = \int_0^t (K_H^*\varphi)(t)dW(t),$$

if and only if $K_H^*\varphi \in L^2([0,T])$.
Also, denoting $L^2_{\mathcal{H}_{S^H}}([0,T]) = \left\{\varphi \in \mathcal{H}_{S^H}, K_H^*\varphi \in L^2([0,T])\right\}$.
Since $H > \frac{1}{2}$, we have by (1.19) and lemma 1.2 of [94],

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset L^2_{\mathcal{H}_{S^H}}([0,T]). \tag{1.20}$$

Lemma 1.2. *([103]) For $\varphi \in L^{\frac{1}{H}}([0,T])$,*

$$H(2H-1)\int_0^T \int_0^T |\varphi_r||\varphi_u||u-r|^{2H-2}\,drdu \le C_H \|\varphi\|_{L^{\frac{1}{H}}([0,T])},$$

where $C_H = \left(\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}\right)^{1/2}$, with β denoting the beta function.

To define the stochastic integral with respect to Q-sub-fractional Brownian motion we proceed as follows: Let $\mathcal{L}_Q^0(\mathcal{K},U)$ be the space of all $\xi \in \mathcal{L}(\mathcal{K},U)$ such that $\xi Q^{\frac{1}{2}}$ is a

Hilbert-Schmidt operator. The norm is given by

$$\|\xi\|^2_{L^0_Q(\mathcal{K},U)} = \|\xi Q^{\frac{1}{2}}\|^2_{HS} = tr(\xi Q \xi^*).$$

Then ξ is called a Q-Hilbert Schmidt operator from $\mathcal{K}$ to U.
Let $\varphi : [0,T] \longrightarrow L^0_Q(\mathcal{K},U)$ such that

$$\sum_{n=1}^{\infty} \|K^*_H(\varphi Q^{\frac{1}{2}} e_n)\|_{L^2([0,T],U)} < \infty. \tag{1.21}$$

Theorem 1.4. *[16] Let $\varphi : [0,T] \longrightarrow L^0_Q(\mathcal{K},U)$ satisfy 1.21. Then its stochastic integral with respect to the sub-fBm S^H_Q is defined, for $t \geq 0$, as follows*

$$\begin{aligned}\int_0^t \varphi(s) dS^H_Q(s) &:= \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{\frac{1}{2}} e_n dS^H_n(s), \\ &= \sum_{n=1}^{\infty} \int_0^t K^*(\varphi Q^{\frac{1}{2}} e_n) dW(s).\end{aligned}$$

Notice that if

$$\sum_{n=1}^{\infty} \|\varphi(s) Q^{\frac{1}{2}} e_n\|_{L^{\frac{1}{H}}([0,T],U)} < \infty, \tag{1.22}$$

then in particular (1.21) holds, which follows immediately form (1.20).

The following lemma is obtained as a simple application of lemma 1.2.

Lemma 1.3. *([103]) For any $\varphi : [0,T] \longrightarrow L^0_Q(\mathcal{K},U)$ such that 1.22 holds, and for any $u, v \in [0,T]$ with $u > v$,*

$$\mathbb{E}\Big\| \int_v^u \varphi(s) dS^H_Q(s) \Big\|^2_U \leq C_H (u-v)^{2H-1} \sum_{n=1}^{\infty} \int_v^u \|\varphi(s) Q^{\frac{1}{2}} e_n\|^2_U ds.$$

If, in addition,

$$\sum_{n=1}^{\infty} \|\varphi(s) Q^{\frac{1}{2}} e_n\|^2_U \quad \text{is uniformly convergent for } t \in [0,T], \tag{1.23}$$

then

$$\mathbb{E}\Big\| \int_v^u \varphi(s) dS^H_Q(s) \Big\|^2_U \leq C_H (u-v)^{2H-1} \int_v^u \|\varphi(s)\|^2_{L^0_Q(\mathcal{K},U)} ds.$$

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be the complete orthonormal basis of $\mathcal{K}$ introduced above. Applying lemma 1.2, we obtain

$$\begin{aligned}
E\|\int_v^u \varphi(s)dS_Q^H(s)\|_U^2 &= E\|\sum_{n=1}^{\infty}\int_v^u \varphi(s)Q^{\frac{1}{2}}e_n dS^H(s)\|_U^2 \\
&= \sum_{n=1}^{\infty} E\|\int_v^u \varphi(s)Q^{\frac{1}{2}} dS^H(s)\|_U^2 \\
&= \sum_{n=1}^{\infty} H(2H-1)\int_v^u\int_v^u \|\varphi(t)Q^{\frac{1}{2}}e_n\|_U\|\varphi(s)Q^{\frac{1}{2}}e_n\|_U|t-s|^{2H-2}dtds \\
&\le c_H\sum_{n=1}^{\infty}\left(\int_v^u \|\varphi(s)Q^{\frac{1}{2}}e_n\|_U^{\frac{1}{H}}\right)^{2H} \\
&\le c_H(u-v)\sum_{n=1}^{\infty}\int_v^u \|\varphi(s)Q^{\frac{1}{2}}e_n\|_U^2 ds.
\end{aligned}$$

□

Remark 1.2. *If $\{\lambda_n\}_{n=1}^{\infty}$ is bounded sequence of non-negative real numbers such that the nuclear operator Q satisfies $Qe_n = \lambda_n e_n$, assuming that there exists a positive constant K_φ such that*

$$\|\varphi(t)\|_{\mathcal{L}_Q^2(K,U)} \le K_\varphi \text{ uniformly in [0, T],}$$

then 1.23 holds automatically.

1.7 The space of tempered distributions

For the convenience of the reader we recall some of the basic properties of the Schwartz space $\mathcal{S}$ of rapidly decreasing smooth functions and its dual, the space $\mathcal{S}'$ of tempered distributions.

1.7.1 The space of tempered distributions

Let n be a given natural number. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the space of rapidly decreasing smooth real functions f on $\mathbb{R}^n$ equipped with the family of seminorms:

$$\|f\|_{k,\alpha} := \sup_{y\in\mathbb{R}^n} \left\{(1+|y|^k)|\partial^\alpha f(y)|\right\} < \infty,$$

where $k = 0, 1, ...$, $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index with $\alpha_j = 0, 1, ...$ $(j = 1, ..., n)$ and

$$\partial^\alpha f := \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} \cdots \partial y_n^{\alpha_n}} f$$

for $|\alpha| = \alpha_1 + ... + \alpha_n$.

Then $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is a Fréchet space.

Let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ be its dual, called the space of *tempered distributions*. Let $\mathcal{B}$ denote the family of all Borel subsets of $\mathcal{S}'(\mathbb{R}^n)$ equipped with the weak* topology. If $\Phi \in \mathcal{S}'$ and $f \in \mathcal{S}$ we let

$$\Phi(f) \text{ or } \langle \Phi, f\rangle \tag{1.24}$$

denote the action of Φ on f.

Example 1.1.
- *(Evaluations) For $y \in \mathbb{R}$ define the function δ_y on $\mathcal{S}(\mathbb{R})$ by $\delta_y(\phi) = \phi(y)$. Then δ_y is a tempered distribution.*

- *(Derivatives) Consider the function D, defined for $\phi \in \mathcal{S}(\mathbb{R})$ by $D[\phi] = \phi'(y)$. Then D is a tempered distribution.*

- *(Distributional derivative)*
  Let T be a tempered distribution, i.e. $T \in \mathcal{S}'(\mathbb{R})$. We define the distributional derivative T' of T by
  $$T'[\phi] = -T[\phi']; \quad \phi \in \mathcal{S}.$$

Then $T^{'}$ is again a tempered distribution.

1.8 Theory of semigroup

In this part, we give some definitions and preliminaries results of semigroup theory that will be needed in the sequel.

Definition 1.10. *Let X be a Banach space. A family $(T(t))_{t\geq 0}$ of bounded linear operators from X to X is called a strongly continuous semigroup of bounded linear operators if the following three conditions are satisfied*

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s)$,

(iii) $\forall x \in X$, *the map* $\mathbb{R} \ni t \to T(t)x \in X$ *defined from* $[0, +\infty =$ *into* X *is continuous at the right of* 0.

A strongly continuous semigroup of bounded linear operators on X will be called a C_0-semigroup.

Remark 1.4. *A semigroup of bounded linear operators $(T(t))_{t\geq 0}$ is uniformly continuous if*

$$\lim_{t\to 0} \| T(t) - I \| = 0.$$

Examples of semigroups

Infinitesimal generator of a C_0-semigroup

Definition 1.11. *The linear operator A defined by*
$D(A) = lim_{t\to 0^+} \frac{T(t)x-x}{t} = \frac{d^+T(t)x}{dt}|_{t=0}$ *for $x \in D(A)$ is the infinitesimal generator of the semigroup $(T(t))_{t\geq 0}$; $D(A)$ is the domain of A.*

Theorem 1.9. *[70] Let $(T(t))_{t\geq 0}$ be a C_0-semigroup then there exist constants $w \in \mathbb{R}$ and $M \geq 1$, such that*

$$\| T(t) \| \leq Me^{wt} \ \ for \ \ 0 \leq t < +\infty.$$

Theorem 1.10. *[70] If $(T(t))_{t\geq 0}$ is a C_0-semigroup then $\forall x \in X$, $t \to T(t)x$ is continuous from $\mathbb{R}^+$ into X.*

Theorem 1.11. *[51] Let $(T(t))_{t\geq 0}$ be a C_0-semigroup and A be its infinitesimal generator. Then* *(a). For $x \in X$,*

$$\lim_{h \to 0} \int_t^{t+h} T(s)xds = T(t)x.$$

(b). For $x \in X$,

$$\int_0^t T(s)xds \in D(A) \text{ and } A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

(c). For $x \in D(A)$,

$$T(t)x \in D(A), \text{ and } \frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

(b). For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau.$$

Corollary 1.3. *[70] If A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ then $D(A)$ the domain of A, is dense in X and A is closed linear operator.*

Theorem 1.12. *[70] A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Theorem 1.13. *[70] Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two C_0-semigroup on X, generated respectively by A and B. If A=B then $T(t) = S(t)$, $t \geq 0$.*

Definition 1.12. *$(T(t))_{t\geq 0}$ is a C_0 semigroup of contraction if and only if*
$\| T(t) \| \leq 1$, $\forall t \geq 0$.

Integrated semigroups

Definition 1.13. *[68] Let U be a Banach space. An integrated semigroup is a family of operators $(S(t)_{t\geq 0})$ of bounded linear operators $S(t)$ on U with the following properties:*

(i) $S(0) = 0$;

(ii) $t \to S(t)$ *is strongly continuous;*

(iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$, $\forall s,t \geq 0$.

Definition 1.14. *[68] An operator is called a generator of an integrated semigroup if there exists $\eta \in \mathbb{R}$ such that $(\eta, +\infty) \subset \rho(A)$ (the resolvent set of A), and there exists a* *strongly continuous exponentially bounded family* $(S(t))_{t\geq 0}$ *of bounded linear operators such that*

$$S(0)=0, \quad (\lambda I-A)^{-1}=\lambda\int_0^{+\infty}e^{-\lambda t}S(t)dt, \ \forall\lambda>\eta.$$

Definition 1.15. *[68] An integrated semigroup* $S(t))_{t\geq 0}$ *is called exponentially bounded if there exists constants* $M\geq 0$ *and* $\beta\in\mathbb{R}$ *such that*

$$\| S^2(t) \| \leq Me^{\beta t}, \ \forall t\geq 0.$$

Definition 1.16. *[134] We say that the linear operator* A *satisfies the Hill-Yosida condition if there exists constant* $M\geq 0$ *and* $\eta\in\mathbb{R}$ *such that* $(\eta,+\infty)\subset\rho(A)$ *and*

$$sup\left\{(\lambda-\eta)^n \mid (\lambda I-A)^{-n} \mid : n\in\mathbb{N},\, \lambda>\eta\right\}\leq\sqrt{M}.$$

Definition 1.17. *[68] An integrated semigroup* $(S(t))_{t\geq 0}$ *is called locally Lipschitz continuous if, for all* $\delta>0$*, there exist a constant* $\Lambda\geq 0$ *such that;*

$$\| S(t)-S(s) \| \leq \Lambda \mid t-s \mid, \ t,s\in[0,\delta].$$

Chapter 2

Fractional Calculus

2.1 Some historical facts on fractional calculus

We begin to call for the history of the fractional calculus given by [79], so the Fractional Calculus (FC) is a generalization of classical calculus concerned with operations of integration and differentiation of non-integer (fractional) order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The first known reference can be found in the correspondence of G. W. Leibniz and Marquis de l'Hospital in 1695 where the question of meaning of the semi-derivative has been raised. This question consequently attracted the interest of many well- known mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, Letnikov and many others. Since the 19th century, the theory of fractional calculus developed rapidly, mostly as a foundation for a number of applied disciplines, including fractional geometry, fractional differential equations (FDE) and fractional dynamics. The applications of FC are very wide nowadays. It is safe to say that almost no discipline of modern engineering and science in general, remains untouched by the tools and techniques of fractional calculus. For example, wide and fruitful applications can be found in rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, bio-engineering, etc...In fact, one could argue that real world processes are fractional order systems in general. The main reason for the success of FC applications is that these new fractional-order models are often more accurate than integer-order ones, i.e. there are more degrees of freedom in the fractional order model than in the corresponding classical one. One of the intriguing beauties of the subject is that fractional derivatives (and integrals) are not a local (or point) quantities. All fractional operators consider the entire history of the process being considered, thus being able to model the non-local and distributed effects often encountered in natural and technical phenomena. Fractional calculus is therefore an excellent set of tools for describing the memory and hereditary properties of various materials and processes.

2.2 Special functions of fractional calculus

We will recall in this section some results of the special functions of fractional calculus which are important for other parts of this work.

2.2.1 Gamma function

Definition 2.1. *[73] The gamma function $\Gamma(z)$ is defined by the integral:*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,$$

where $t^{z-1} = e^{(z-1)log(t)}$. This integral is convergent for all complex $z \in \mathbb{C}$.

Properties 2.1. *[73] The gamma function satisfies the following functional equation:*

$$\Gamma(z+1) = z\Gamma(z). \tag{2.1}$$

Another important property can be represented also by the following limit:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^2}{z(z+1)\ldots(z+n)}, \tag{2.2}$$

where we initially suppose that $Re(z) > 0$.

2.2.2 Beta function

Definition 2.2. *[73] The Beta function is defined by the following integral:*

$$B(z,w) = \int_0^1 \tau^{z-1} (1-\tau)^{w-1} d\tau\,, \ \ (Re(z) > 0, Re(w) > 0). \tag{2.3}$$

Properties 2.2. *[73] The principal property of the function Beta is:*

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \tag{2.4}$$

from which it follows that:

$$B(z,w) = B(w,z).$$

2.2.3 Wright function

Definition 2.3. *The Wright function is defined by the following some by [49]*
The series representation, valid in the whole complex plane

$$W(z;\alpha,\beta) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k+\beta)},\ \alpha > -1,\ \beta \in \mathbb{C}, \tag{2.5}$$

it is an entire function of order $\frac{1}{1+\alpha}$*, which has known also as generalized Bessel functions cited by Podlubny [111] and Kiryakova [75].*

Properties 2.3. *The Wright function can be represented by the following integral given by [49]*

$$W(z;\alpha,\beta) = \frac{1}{2\pi i}\int_{H_a} \tau^{-\beta} e^{\tau+z\tau^{-\alpha}} d\tau,$$

where H_a *denotes Hankel's contour. It follows from (2.5) that*

$$W(z,0,1) = e^z.$$

2.2.4 The Mittag-Leffler functions

Definition 2.4. *[102] The Mittag-Leffler function of two parameters* α, β *is denoted by* $E_{\alpha,\beta}(z)$ *and defined by:*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{2.6}$$

where z, α, $\beta \in \mathbb{C}$, $Re(\alpha) > 0$ *and* $Re(\beta) > 0$, *and* Γ *is the Gamma function.*

For $\beta = 1$ *we obtain the Mittag-Leffler function of one parameter* α *denoted by* $E_\alpha(z)$ *and defined as:*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \tag{2.7}$$

where z, $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$.

Remark 2.1. *Note that* $E_\alpha(z) = E_{\alpha,1}(z)$ *and that*

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{2.8}$$

Now, we need to introduce the following space.

The space *AC*

Definition 2.1. *[73] Let* $[a,b]$ $(-\infty < a < b < \infty)$ *be a finite interval and let* $AC[a,b]$ *be the space of functions of which are absolutely continuous on* $[a,b]$. $AC[a,b]$ *coincides with the space of primitives of Lebesgue summable function [see Kolmogorov and Fomin[77]]*

$$f(x) \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt,$$

with $(\varphi(t) \in L(a,b))$*, and therefore an absolutely continuous function has a summable derivative* $f^{'}(x) = \varphi(x)$ *almost everywhere on* $[a,b]$.

Definition 2.2. *[73] For* $n \in \mathbb{N} = \{1,2,3,...\}$ *we denote by* $AC^n[a,b]$ *the space of complex-valued functions* $f(x)$ *which have continuous derivatives up to order* $n-1$ *on* $[a,b]$ *such that* $f^{(n-1)}(x) \in AC[a,b]$*:*
$AC^n[a,b] = \left\{f : [a,b] \to \mathbb{C} \text{ and } \left(D^{n-1}f\right)(x) \in AC[a,b] \ \ (D = \frac{d}{dx})\right\}$. $\mathbb{C}$ *being the set of complex numbers. In particular,* $AC^1[a,b] = AC[a,b]$.

Let us define now the space C^n_γ.

Definition 2.3. *[73] Let* $n \in \mathbb{N}_0 = \{0,1,...\}$ *and* $\gamma \in \mathbb{C}$ $(0 \leq R(\gamma) < 1)$.
The space $C^n_\gamma[a,b]$ *consists of those and only those functions of which are represented in the form*

$$f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt + \sum_{k=0}^{n-1} c_k(x-a)^k, \tag{2.9}$$

where $\varphi(t) \in C_\gamma[a,b]$ *and* c_k $(k = 0,1,...,n-1)$ *are arbitrary constants. Moreover,*

$$\varphi(t) = f^{(n)}(t), \ \ c_k = \frac{f^{(k)}(a)}{k!} \ (k = 0,1,...,n-1). \tag{2.10}$$

In particular, when $\gamma = 0$*, the space* $C^n[a,b]$ *consists of those functions f which are represented in the form 2.9, where* $\varphi(t) \in C[a,b]$ *and* c_k $(k = 0,1,...)$ *are arbitrary constants. Moreover, the relations in 2.10 holds.*

2.3 Fractional derivatives and integrals

A fractional differential equation is an equation which contains fractional derivatives; a fractional integral equation is an integral equation containing fractional integrals.

In this section we need to recall the definitions and the usefuls theorems and lemmas for fractional derivatives and integrals.

2.3.1 The Riemann-Liouville left-and right-sided fractional integrals

We first define the fractional integral operator according to Riemann-Liouville, which is the most widely used definition in fractional calculus.

Definition 2.5. *[101]Riemann-Liouville fractional integral on the real line*
The Riemann-Liouville fractional integral on $\mathbb{R}$ are defined as

$$\left(I_{+}^{\alpha}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}f(t)dt = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)_{+}^{\alpha-1}f(t)dt, \tag{2.11}$$

and

$$\left(I_{-}^{\alpha}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}f(t)dt = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)_{-}^{\alpha-1}f(t)dt. \tag{2.12}$$

Remark 2.2. *The function $f \in D\left(I_{+,-}^{\alpha}\right)$ if the corresponding integrals converge for a.a $x \in \mathbb{R}$.*

Proposition 2.1. *[101]*

i. Fractional integration admits the following composition formulas for fractional integrals:

$$I_{+,-}^{\alpha}I_{+,-}^{\beta}f = I_{+,-}^{\alpha+\beta}f \tag{2.13}$$

for $f \in L^{p}(\mathbb{R})$, $\alpha, \beta > 0$ and $\alpha+\beta < \frac{1}{p}$.

ii. We consider $f \in L^{p}(\mathbb{R})$, $g \in L^{q}(\mathbb{R})$, $p > 1$, $q > 1$, and $\frac{1}{p}+\frac{1}{q} = 1$, then we obtain the following integration by parts formula

$$\int_{\mathbb{R}} g(x)\left(I_{+}^{\alpha}f\right)(x)dx = \int_{\mathbb{R}} f(x)\left(I_{-}^{\alpha}g\right)(x)dx. \tag{2.14}$$

iii. (Inclusion property)
Let $C^{\lambda}(\mathbb{T})$ be the set of Hölder continuous functions $f : \mathbb{T} \to \mathbb{R}$ of order λ i.e,

$$C^{\lambda}(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathbb{R} \mid \| f \|_{\lambda} := \sup_{t\in\mathbb{T}} | f(t) | + \sup_{s,t\in\mathbb{T}} | f(s) - f(t) | (t-s)^{-\lambda} < \infty \right\}.$$

If $\alpha > 0$, and $\alpha p > 1$, then,

$$I_{+,-}^{\alpha}\left(L^{p}(\mathbb{R})\right) \subset C^{\lambda}[a,b]$$

for any $-\infty < a < b < \infty$ *and* $0 < \lambda < \alpha - \frac{1}{p}$.

2.3.2 The Riemann-Liouville left-and right-sided fractional derivatives

In this part we present the definitions and some properties of the Liouville fractional derivatives on the whole axis $\mathbb{R} = (-\infty, \infty)$.

Definition 2.6. *[73]**Liouville fractional derivatives on the real axis.***
The Liouville fractional derivatives on $\mathbb{R}$ *are defined by the following formulas:*

$$\left(D_{+}^{\alpha}y\right)(x) := \left(\frac{d}{dx}\right)^{n}\left(I_{+}^{n-\alpha}y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{-\infty}^{x}\frac{y(t)dt}{(x-t)^{\alpha-n+1}}, \tag{2.15}$$

and

$$\left(D_{-}^{\alpha}y\right)(x) := \left(-\frac{d}{dx}\right)^{n}\left(I_{-}^{n-\alpha}y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n}\int_{x}^{+\infty}\frac{y(t)dt}{(t-x)^{\alpha-n+1}}, \tag{2.16}$$

where $n = [Re(\alpha)] + 1$, $Re(\alpha) \geq 0$ *and* $x \in \mathbb{R}$, *respectively.*

Lemma 2.1. *[73] If* $\alpha > 0$, *then, for "sufficiently good" functions* $f(x)$ *the relations*

$$\left(D_{+}^{\alpha}I_{+}^{\alpha}f\right)(x) = f(x), \tag{2.17}$$

$$\left(D_{-}^{\alpha}I_{-}^{\alpha}f\right)(x) = f(x), \tag{2.18}$$

are true. In particular, these formulas holds for $f(x) \in L_1(\mathbb{R})$.

Properties 2.4. *[73] Let* $\alpha > 0$, $m \in \mathbb{N}$ *and* $D = \frac{d}{dx}$.

i. If the fractional derivatives $\left(D_{+}^{\alpha}y\right)(x)$ *and* $\left(D_{+}^{\alpha+m}y\right)(x)$ *exists, then*

$$\left(D^{m}D_{+}^{\alpha}y\right)(x) = \left(D_{+}^{\alpha+m}y\right)(x). \tag{2.19}$$

ii. If the fractional derivatives $\left(D_{-}^{\alpha}y\right)(x)$ *and* $\left(D_{-}^{\alpha+m}y\right)(x)$ *exist, then*

$$\left(D^{m}D_{-}^{\alpha}y\right)(x) = (-1)^{m}\left(D_{-}^{\alpha+m}y\right)(x). \tag{2.20}$$

2.3.3 Caputo fractional derivative

In this section we present the definitions and some properties of the Caputo derivatives.

Definition 2.7. *[101] The fractional derivatives* $\left({}^cD^{\alpha}_{a+}f\right)(x)$ *and* $\left({}^cD^{\alpha}_{b-}f\right)(x)$ *of order* $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$ *on* $[a,b]$ *are defined via the above Riemann-Liouville fractional derivatives by*

$$\left({}^cD^{\alpha}_{a+}f\right)(x) := \left(D^{\alpha}_{a+}\left[f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]\right)(x), \tag{2.21}$$

$$\left({}^cD^{\alpha}_{b-}f\right)(x) := \left(D^{\alpha}_{b-}\left[f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(b-t)^k\right]\right)(x). \tag{2.22}$$

Respectively, where

$$n = [R(\alpha)] + 1 \;\; for\, \alpha \notin \mathbb{N}_0;\; n = \alpha \;\; for\, \alpha \in \mathbb{N}_0. \tag{2.23}$$

These derivatives are called left-sided and right-sided Caputo fractional derivatives of order α*.*
In particular, when $0 < Re(\alpha) < 1$*, the relations (2.21) and (2.22) take the following forms:*

$$\left({}^cD^{\alpha}_{a+}f\right)(x) = \left(D^{\alpha}_{a+}\left[f(t) - f(a)\right]\right)(x) = (D^{\alpha}_{a_+}f)(x) - \frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha}, \tag{2.24}$$

$$\left({}^cD^{\alpha}_{b-}f\right)(x) = \left(D^{\alpha}_{b-}\left[f(t) - f(b)\right]\right)(x) = (D^{\alpha}_{b_-}f)(x) - \frac{f(b)}{\Gamma(1-\alpha)}(b-x)^{-\alpha}. \tag{2.25}$$

The Caputo fractional derivatives are defined for functions $f(x)$ *belonging to the space* $AC^n[a,b]$ *of absolutely continuous functions.*
Now we discuss the following cases of α*.*

(1)- If $\alpha \neq \mathbb{N}_0$*, then the Caputo fractional derivatives (2.21) and (2.22) coincide with the Riemann-Liouville fractional derivatives (2.15), (2.16) in the following statements:*

$$\left({}^cD^{\alpha}_{a+}f\right)(x) = \left(D^{\alpha}_{a+}f\right)(x), \tag{2.26}$$

if $f(a) = f^{'}(a) = ... = f^{n-1}(a) = 0 \;\; (n = [R(\alpha)] + 1)$*; and*

$$\left({}^cD^{\alpha}_{b-}f\right)(x) = \left(D^{\alpha}_{b-}f\right)(x), \tag{2.27}$$

if $f(b) = f^{'}(b) = ... = f^{n-1}(b) = 0 \;\; (n = [R(\alpha)] + 1)$*.*
In particular, when $0 < R(\alpha) < 1$*, we have*

$$\left({}^cD^{\alpha}_{a+}f\right)(x) = \left(D^{\alpha}_{a+}f\right)(x), \; when \; f(a) = 0, \tag{2.28}$$

$$\left({}^{c}D^{\alpha}_{b-}f\right)(x) = \left(D^{\alpha}_{b-}f\right)(x), \textit{ when } f(b)=0. \tag{2.29}$$

(2)- if $\alpha = n \in \mathbb{N}_0$ *and the usual derivative* $f^{n}(x)$ *exists, then*

$$\left({}^{c}D^{n}_{a+}f\right)(x) = f^{(n)}(x), \tag{2.30}$$

and

$$\left({}^{c}D^{n}_{b-}f\right)(x) = (-1)^{n} f^{(n)}(x) \;\; (n \in \mathbb{N}). \tag{2.31}$$

Theorem 2.1. *[73] Let* $R(\alpha) \geq 0$ *and let n be given by (2.23).*
If $f(x) \in AC^{n}[a,b]$*, then the Caputo fractional derivatives* $\left({}^{c}D^{\alpha}_{a+}f\right)(x)$ *and* $\left({}^{c}D^{\alpha}_{b-}f\right)(x)$ *exist almost every where on* $[a,b]$*.*

(a-) If $\alpha \notin \mathbb{N}_0$*,* $\left({}^{c}D^{\alpha}_{a+}f\right)(x)$ *and* $\left({}^{c}D^{\alpha}_{b-}f\right)(x)$ *are represented by*

$$\left({}^{c}D^{\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \frac{f^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} =: \left(I^{n-\alpha}_{a+} D^{n} f\right)(x), \tag{2.32}$$

and

$$\left({}^{c}D^{\alpha}_{b-}f\right)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} \frac{f^{(n)}(t)dt}{(t-x)^{\alpha-n+1}} =: \left(I^{n-\alpha}_{b-} D^{n} f\right)(x) \tag{2.33}$$

respectively, where $D = \frac{d}{dx}$ *and* $n = [R(\alpha)] + 1$*.*
In particular, when $0 < R(\alpha) < 1$ *and* $f(x) \in AC[a,b]$*.*

$$\left({}^{c}D^{\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} \frac{f^{'}(t)dt}{(x-t)^{\alpha}} =: \left(I^{1-\alpha}_{a+} Df\right)(x), \tag{2.34}$$

and

$$\left({}^{c}D^{\alpha}_{b-}f\right)(x) = -\frac{1}{\Gamma(n-\alpha)} \int_{x}^{b} \frac{f^{'}(t)dt}{(t-x)^{\alpha}} =: \left(I^{1-\alpha}_{b-} Df\right)(x). \tag{2.35}$$

(b)- If $\alpha = n \in \mathbb{N}_0$*, then* $\left({}^{c}D^{n}_{a+}f\right)(x)$ *and* $\left({}^{c}D^{n}_{b-}f\right)(x)$ *are represented by (2.30) and (2.31). In particular*

$$\left({}^{c}D^{0}_{a+}f\right)(x) = \left({}^{c}D^{0}_{b-}f\right)(x) = f(x). \tag{2.36}$$

Theorem 2.2. *[73] Let* $R(\alpha) \geq 0$ *and let n be given by (2.23), also let*
$f(x) \in C^{n}[a,b]$*. Then the Caputo fractional derivatives* $\left({}^{c}D^{\alpha}_{a+}f\right)(x)$*and* $\left({}^{c}D^{\alpha}_{b-}f\right)(x)$ *are continuous on* $[a,b]$*:*
$\left({}^{c}D^{\alpha}_{a+}f\right)(x) \in C[a,b]$ *and* $\left({}^{c}D^{\alpha}_{b-}f\right)(x) \in C[a,b]$*.*

(a) *If* $\alpha \notin \mathbb{N}_0$, *then* $\left({}^cD^{\alpha}_{a+}f\right)(x)$ *and* $\left({}^cD^{\alpha}_{b-}f\right)(x)$ *are represented by (2.32) and (2.33) respectively. Moreover*

$$\left({}^cD^{\alpha}_{a+}f\right)(a) = \left({}^cD^{\alpha}_{b-}f\right)(b) = 0. \tag{2.37}$$

In particular, they have respectively the forms (2.34) and (2.35) for $0 < Re(\alpha) < 1$.

(b) *If* $\alpha = n \in \mathbb{N}_0$, *then the fractional derivatives* $\left({}^cD^{n}_{a+}f\right)(x)$ *and* $\left({}^cD^{n}_{b-}f\right)(x)$ *have representations given in (2.30) and (2.31). In particular, the relations in (2.36).*

Lemma 2.2. *[117] Let* $R(\alpha) > 0$ *and let* $f(x) \in L_\infty(a,b)$ *or* $f(x) \in C[a,b]$

(a) *If* $R(\alpha) \notin \mathbb{N}$ *or* $\alpha \in \mathbb{N}$, *then*

$$\left({}^cD^{\alpha}_{a+}I^{\alpha}_{a+}f\right)(x) = f(x),$$

and

$$\left({}^cD^{\alpha}_{b-}I^{\alpha}_{b-}f\right)(x) = f(x).$$

(b) *If* $R(\alpha) \in \mathbb{N}$ *and* $Im(\alpha) \neq 0$, *then*

$$({}^cD^{\alpha}_{a^+}I^{\alpha}_{a^+}f)(x) = f(x) - \frac{(I^{\alpha+1-n}_{a^+}f)(a^+)}{\Gamma(n-\alpha)}(x-a)^{n-\alpha}, \tag{2.38}$$

$$({}^cD^{\alpha}_{b^-}I^{\alpha}_{b^-}f)(x) = f(x) - \frac{(I^{\alpha+1-n}_{a^+}f)(b^-)}{\Gamma(n-\alpha)}(b-x)^{n-\alpha}. \tag{2.39}$$

2.3.4 Laplace transform of Caputo derivatives

Recall that the Laplace transform L is defined by

$$Lf(s) = \int_0^\infty e^{-st}f(t)dt =: \widetilde{f}(s) \tag{2.40}$$

for all f such that the integral converges.

Some of the properties of the Laplace transform that we will need are:

$$L[\frac{\partial^\alpha}{\partial t^\alpha}f(t)](s) = s^\alpha (Lf)(s) - s^{\alpha-1}f(0), \tag{2.41}$$

$$L[E_\alpha(bx^\alpha)](s) = \frac{s^{\alpha-1}}{s^\alpha - b}, \tag{2.42}$$

$$L[x^{\alpha-1}E_{\alpha,\alpha}(-bx^\alpha)](s) = \frac{1}{s^\alpha + b}. \tag{2.43}$$

Recall that the convolution $f*g$ of two functions $f,g:[0,\infty)\mapsto\mathbb{R}$ is defined by

$$(f*g)(t) = \int_0^t f(t-r)g(r)dr; \quad t \geq 0. \tag{2.44}$$

The convolution rule for Laplace transform states that

$$L\left(\int_0^t f(t-r)g(r)dr\right)(s) = Lf(s)Lg(s),$$

or

$$\int_0^t f(t-w)g(w)dw = L^{-1}\left(Lf(s)Lg(s)\right)(t). \tag{2.45}$$

2.3.5 Hilfer fractional derivative

Hilfer [59] proposed a general operator for fractional derivative, called "Hilfer fractional derivative", which combines Caputo and Riemann-Liouville fractional derivatives.

Definition 2.8. [59] *The Hilfer fractional derivative of order* $0 \leq \alpha \leq 1$ *and* $0 < \beta < 1$ *for a function f is defined by*

$$D_{0^+}^{\alpha,\beta}f(t) = I_{0^+}^{\alpha(1-\beta)}\frac{d}{dt}I_{0^+}^{(1-\alpha)(1-\beta)}f(t).$$

Remark 2.3. *When* $\alpha = 0,\ 0 < \beta < 1,$ *the Hilfer fractional derivative coincides with classical Riemann-Liouville farctional derivative*

$$D_{0^+}^{0,\beta}f(t) = \frac{d}{dt}I_{0^+}^{1-\beta}f(t) = ^L D_{0^+}^{\beta}f(t),$$

when $\alpha = 1,\ 0 < \beta < 1,$ *the Hilfer fractional derivative coincides with classical Caputo fractional derivative*

$$D_{0^+}^{1,\beta}f(t) = I_{0^+}^{1-\beta}\frac{d}{dt}f(t) = ^c D_{0^+}^{\beta}f(t).$$

Now, we introduce the next spaces:

$$C_{1-\gamma}^{\alpha,\beta}[a,b] = \{f \in C_{1-\gamma}[a,b] : D_{a^+}^{\alpha,\beta} f \in C_{1-\gamma}[a,b]\},$$

and

$$C_{1-\gamma}^{\gamma}[a,b] = \{f \in C_{1-\gamma}[a,b] : D_{a^+}^{\gamma} f \in C_{1-\gamma}[a,b]\}.$$

Since $C_{1-\gamma}^{\gamma}[a,b] \subset C_{1-\gamma}^{\alpha,\beta}[a,b]$. The following lemmas follows directly from the semigroup property.

Lemma 2.3. *[50] Let* $0 \le \alpha \le 1$, $0 \le \beta \le 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $f \in C_{1-\gamma}^{\gamma}[a,b]$, *then*

$$I_{a^+}^{\gamma} D_{a^+}^{\gamma} f = I_{a^+}^{\alpha} D_{a^+}^{\alpha,\beta} f,$$

and

$$D_{a^+}^{\gamma} I_{a^+}^{\alpha} f = D_{a^+}^{\beta(1-\alpha)} f.$$

Lemma 2.4. *[50] Let* $f \in L^1[a,b]$. *If* $D_{a^+}^{\beta(1-\alpha)} f$ *exists and in* $L^1[a,b]$ *then*

$$D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} f = I_{a^+}^{\beta(1-\alpha)} D_{a^+}^{\beta(1-\alpha)} f.$$

Lemma 2.5. *[50] Let* $0 \le \alpha \le 1$, $0 \le \beta \le 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $f \in C_{1-\gamma}[a,b]$ *and* $I_{a^+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma}^{1}[a,b]$ *then* $D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} f$ *exist in* $(a,b]$ *and*

$$D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} f(x) = f(x) \quad x \in (a,b].$$

Chapter 3

Differential inclusions

Differential inclusion plays an important role as a tool in the study of various dynamical processes such a study of dynamics of economical, social and biological, macrosystems, they also are very useful in proving existence theorems in control theory. A differential inclusion is a relation of the form

$$x^{'} \in F(x),$$

where F is a multivalued map associating any point $x \in \mathbb{R}^n$ with a set $F(x) \subset \mathbb{R}^n$. The notion of a differential inclusion generalizes the notion of an ordinary differential equation of the form

$$x^{'} = F(x).$$

The key question is how to define the solution of such systems.
First of all it is important to introduce the basic definitions of a multivalued maps which will be used in the sequel of this chapter.
Differential equations or inclusions have recentely proved to be strong tools in modeling of many phenomena in various fields of engineering, physics and economics, see [139], [69], [3], and [121], As well as other researchers have shown important results on differential inclusion problems and their applications with mechanical modeling [35], [28] and a serie of books of Bressan [24],[25] and [23].

3.1 Multi-valued mapps

Multivalued maps play a significant role in the description of processes in control theory, in this section we introduce some basic definitions and results of multivalued maps. For more details on multivalued maps, see the books of Deimling [40], Hu and Papageorgiou

[61].
Let X be a Banach space and $P(X)$ denote the class of all subsets of X

$$P_f(\mathcal{H}) = \left\{A \subset \mathcal{H}/A \; is \; non-empty \; and \; has \; a \; property \; f\right\}.$$

Thus $P_{bd}(\mathcal{H}), P_{cl}(\mathcal{H}), P_{cv}(\mathcal{H}), P_{cp}(\mathcal{H}), P_{cl,bd}(\mathcal{H}), P_{cp,cv}(\mathcal{H})$ denote the classes of bounded, closed, convex,compact,closed-bounded and compact-convex subsets of X respectively.
Similary $P_{cl,cv,bd}(\mathcal{H})$ and $P_{cp,cv}(\mathcal{H})$ denote the classes of closed convex bounded and compact,convex subsets of $\mathcal{H}$ respectively.
$T : \mathcal{H} \to P_f(\mathcal{H})$ is called a multivalued operator or a multivalued mapping on $\mathcal{H}$.
A point $u \in \mathcal{H}$ is called a fixed point of T if $u \in T_u$.

Definition 3.1. *[83] A multivalued map $G : \mathcal{H} \to 2^{\mathcal{H}} \setminus \emptyset$ is convex(closed) valued if $G(x)$ is convex (closed) for all $x \in \mathcal{H}$. G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in $\mathcal{H}$ for any bounded set B of $\mathcal{H}$, i.e.,*

$$\sup_{x \in B}\{sup \parallel y \parallel : y \in G(x)\} < \infty.$$

Definition 3.2. *[83] A multivalued map G is called upper semi-continuous (u.s.c) on $\mathcal{H}$ if for each $x_0 \in \mathcal{H}$ the set $G(x_0)$ is non empty closed subset of $\mathcal{H}$, and if for each open set N of $\mathcal{H}$ containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq N$.*

Definition 3.3. *[83] The multi-valued map G is called lower semi continuous (l.w.c) if U is an open subset of $\mathcal{H}$, then*

$$G^{-1}(U) = \left\{x \in \mathcal{H}/G(x) \cap U \neq \emptyset\right\}$$

is an open subset of $\mathcal{H}$.

Definition 3.4. *[83] The multivalued operator G is called compact if $\overline{G(\mathcal{H})}$ is a compact subset of $\mathcal{H}$. G is said to be completely continuous if G(B) is relatively compact for every bounded subset B of $\mathcal{H}$.*
If the multivalued map G is completely continuous with nonempty values, then G is u.s.c, if and only if G has a closed graph, i.e.,

$$x_n \to x_*\ ,\ y_n \to y_*\ ,\ y_n \in G(x_n)\ imply\ \ y_* \in G(x_*).$$

Let us denote by $BCC(\mathcal{H})$ the set of all nonempty bounded, closed and convex subset of $\mathcal{H}$.

Definition 3.5. *[83] A multi-valued map $G :\to BCC(\mathcal{H})$ is said to be measurable if for each $x \in \mathcal{H}$, the function $U : J \to \mathbb{R}$, defined by*

$$U(t) = d(x, G(t)) = inf\{\| x - z \| : z \in G(t)\}$$

belongs to $L^1(J, \mathbb{R})$.

Definition 3.6. *[83] A multi-valued function $G : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is called carathéodory if*

(i). $t \to G(t, x)$ is measurable for each $x \in \mathbb{R}$ and

(ii). $x \to G(t, x)$ is an upper semi-continuous almost everywhere for $t \in J$.

Definition 3.7. *[83] A carathéodory multi-valued map $G(t, x)$ is called L^1-carathéodory if there exists a function $h \in L^1(J, \mathbb{R})$ such that $\| G(t, x) \| < h(t)$ a.e. $t \in J$ for all $x \in \mathbb{R}$, and the function h is called a growth function of G on $J \times \mathbb{R}$.*

Definition 3.8. *[83] The multi-valued map $G : J \times \mathcal{H} \to BCC(\mathcal{H})$ is said to be L^2-carathéodory if*

(i). $t \to G(t, x)$ is measurable for each $x \in \mathcal{H}$

(ii). $x \to G(t, x)$ is u.s.c. for almost all $t \in J$

(iii). for each $r > 0$, there exists $l_r \in L^1(J, \mathbb{R})$ such that

$$\| G(t, x) \|^2 := \sup_{\sigma \in G(t,x)} \mathbb{E} \| \sigma \|^2 \leq l_r(t)$$

for all $\| x \|_b^2 \leq r$ and for a.e. $t \in J$.

Lasota and Opial gives the following results

Lemma 3.1. *[78] Let J be a compact real interval , $BCC(\mathcal{H})$ be the set of all non-empty, bounded, closed and convex subset of $\mathcal{H}$ and G be a L^2-carathéodory multi-valued map, $S_{G,x} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^2(J, \mathcal{H})$ to $C(J, \mathcal{H})$. Then the operator*

$$\Gamma \circ S_G : C(J, \mathcal{H}) \to BCC(C(J, \mathcal{H}) \,, \ x \to (\Gamma \circ S_G) := \Gamma(S_G)$$

is a closed graph operator in $C(J, \mathcal{H})$, where $S_{G,x}$ is known as the selections set from G, is given by

$$\sigma \in S_{G,x} = \left\{ \sigma \in L^2(L(K, \mathcal{H})) : \sigma(t) \in G(t, x), \ for\, a.e.\, t \in J \right\}$$

Definition 3.9. *[135]*

i. A subset A of a normed space X is said to be weakly (relatively) compact if (the weak closure of) A is compact in the weak topology of X.

ii. A subset A of a Banach space X is weakly sequentially compact if any sequence in A has a subsequence which converges weakly to an element of X.

Definition 3.10. *[135]*

Suppose that X and Y are Banach spaces. A linear operator T from X into Y is weakly compact if $T(B)$ is a relatively weakly compact subset of Y whenever B is a bounded subset of X.

Theorem 3.1. *[67] Let Ω be a subset of a Banach space X. The following statements are equivalent:*

i. Ω is relatively weakly compact.

ii. Ω is relatively weakly sequentially compact.

Theorem 3.2. *[67] Let Ω be a subset of a Banach space X. The following statements are equivalent:*

i. Ω is weakly compact.

ii. Ω is weakly sequentially compact.

Definition 3.11. *[135] Let D be a nonempty subset of Banach space Y and $\varphi : D \to P(Y)$ be a multivalued map:*

1. *φ is said to have weakly sequentially closed graph if for every sequence $\{x_n\} \subset D$ with $x_n \rightharpoonup x$ in D and for every sequence $\{y_n\}$ with $y_n \in \varphi(x_n)$ $\forall n \in \mathbb{N}$ $y_n \rightharpoonup y \in Y$ implies $y \in \varphi(x)$.*

2. *φ is called weakly upper semi continuous if $\varphi^{-1}(A)$ is closed for all weakly closed $A \subset Y$.*

3. *β is $\varepsilon - \delta$ upper semi continuous if for every $w_0 \in Y$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\beta(y) \subset \beta(w_0) + B_\varepsilon(0)$ for all $y \in B_\delta(w_0) \bigcap D$.*

Now we define the set of selections of Σ by: Given $x \in C_r\left([-r,T], L^2(\Omega, \mathcal{H})\right)$

$$Sel_{\Sigma(x)} = \left\{\sigma \in C_r\left([-r,T]; L^2(\Omega, \mathcal{H})\right) : \sigma(t-r) \in \Sigma(t-r, x(t-r))\right\}.$$

Lemma 3.2. *[17] Let $\varphi : D \subset Y \to P(Z)$ be a multivalued map with weakly compact values, then*

1. *φ is weakly u.s.c if φ is $\varepsilon - \delta$ u.s.c.*
2. *suppose further that φ has convex values and Z is reflexive then φ is weakly u.s.c if and only if $\{x_n\} \subset D$ with $x_n \to x_0 \in D$ and $y_n \in \varphi(x_n)$ implies $y_n \rightharpoonup y_0 \in \varphi(x_0)$ up to a subsequence.*

Lemma 3.3. *[130] Let X be reflexive and $1 < p < \infty$. A subset $\mathcal{K} \subset L^p([a,b],X)$ is relatively weakly sequentially compact in $L^p([a,b],X)$ if and only if $\mathcal{K}$ is bounded in $L^p([a,b],X)$.*

Theorem 3.3. *[43] The convex hull of a weakly compact set in a Banach space X is weakly compact.*

Theorem 3.4. *[105] Let X be a metrisable locally convex linear topological space and let D be a weakly compact convex subset of X.*
Suppose, $\varphi : D \to P_{cl,cv}(D)$ has weakly sequentially closed graph then φ has a fixed point.

3.2 Differential inclusion

In this section, we give an example of differential inclusion in deterministic case, for more details see [37]. We consider the following differential inclusion

$$\begin{cases} \left(\frac{x(t)}{f(t,x(t))}\right)' \in G(t,x) \quad \text{a.e } t \in J \\ \qquad x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous and $G : J \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$. Find a solution of 3.1 is to find a function $x \in AC(J,\mathbb{R})$ that satisfies:

(i). the function $t \to \frac{x(t)}{f(t,x(t))}$ is differentiable, and

(ii). $\left(\frac{x(t)}{f(t,x(t))}\right)' = v(t)$, $t \in J$ for some $v \in L^1(J,\mathbb{R})$, satisfying $v(t) \in G(t,x(t))$ a.e $t \in J$ defined in (2.1).

If $f(t,x) = 1$, then the DI reduces to

$$\begin{cases} x(t)' \in G(t,x) \qquad \text{a.e } t \in J \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{3.2}$$

in the case when $G(t,x) = g(t,x)$, we obtain the differential equation

$$\begin{cases} \left(\frac{x(t)}{f(t,x(t))}\right)' = g(t,x) & \text{a.e } t \in J \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{3.3}$$

in this section we shall prove the existence of solution of (3.1) under Lipschitz and carathéodory conditions.

Define a norm $\| \, . \, \|$ in $C(J,\mathbb{R})$ by $\| x \| = \sup_{t\in J} | x(t) |$.

Remark 3.1. *It is known that if $G : J \to P_{cp}(\mathbb{R})$ is an integrably bounded multivalued operator, then the set S^1_G of all Lebesgue integrable selections of G is closed and non-empty,*

where

$$S^1_G(x) = \left\{ v \in L^1 (J,E) / v(t) \in G(t,x(t)) \; a.e. \; t \in J \right\},$$

then we have the following lemmas by Lasota and Opial.

Lemma 3.4. *[78] Let E be a Banach space, If dim$(E) < \infty$ and $G : J \times E \to P_{cp}(E)$ is L^1-carathéodory, then $S^1_G(x) \neq \emptyset$ for each $x \in E$.*

Lemma 3.5. *[78] Let E be a Banach space, G a carathéodory multi-valued operator with $S^1_G \neq \emptyset$ and let*
$\mathcal{L} \circ S^1_G : C(J,E) \to P_{bd,cl}(C(J,E))$ be a closed graph operator on $C(J,E) \times C(J,E)$.

We need to suppose some hypotheses in the sequel.

$(H_1)-$ The function f is bounded on $J \times \mathbb{R} \to \mathbb{R}$ with bound k.

$(H_2)-$ The function $f : J \times \mathbb{R} \to \mathbb{R} \setminus 0$ is continuous and there exists a bounded function $l : J \to \mathbb{R}$ with bound $\| l \|$ satisfying

$$| f(t,x) - f(t,y) | \leq l(t) \, | x - y | \;\; a.e \; t \in J \text{for all} x, y \in \mathbb{R}$$

$(H_3)-$ The multivalued operator $G : J \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ is L^1_X- carathéodory with growth function h.

Theorem 3.5. *[37] Assume that the hypotheses $(H_1) - (H_3)$ hold. Further if*

$$\| l \| \left(| \frac{x_0}{f(0,x_0} | + \| h \|_{L^1} \right) < 1,$$

the DI (3.1) *has a solution on J.*

Example 3.1. *Let $J = [0,1]$ and define a function $f : J \times \mathbb{R} \to \mathbb{R}$ by*

$$f(t,x) = \begin{cases} 1 & \textit{if } -\infty < x \leq 0 \\ 1+x & \textit{if } \ 0 \leq x \leq 1 \\ 2 & \textit{if } \ x \geq 1, \end{cases} \tag{3.4}$$

for all $t \in J$.
Now we consider the DI

$$\begin{cases} \left(\frac{x(t)}{f(t,x(t))}\right)' \in G(t,x(t)) \quad a.e\ t \in J \\ \qquad x(0) = \frac{1}{2} \in \mathbb{R}, \end{cases} \tag{3.5}$$

where $p : J \to \mathbb{R}$ is Lebesgue integrable, and $G : J \times \mathbb{R} \to P_f(\mathbb{R})$ is given by

$$G(t,x) = \begin{cases} p(t) & Si\ x < 0 \\ [\exp(-x)\, p(t), p(t)] & Si\ x \leq 0. \end{cases} \tag{3.6}$$

The function $f(t,x)$ is continuous and bounded on $J \times \mathbb{R}$ with bound with Lipshitz constant 1. also it follows that G is L^1_X−carathéodory with $h(t) = p(t)$, $t \in J$. Therefore if $\| p \|_{L^1} < \frac{1}{2}$, then the DI (3.5) has a solution on J.

Chapter 4

Stochastic fractional differential inclusion driven by fractional Brownian motion

Many systems in physics, mechanic, biology and medecine use the concept of differential inclusions to modelise there phenomenas. Also the fractional differential inclusions plays an important role in description of the memory and genetic properties, for this reason many researches have been dedicated to the existence of mild solution for fractional differential equations, please see Zhou [139]; Boudaoui and Caraballo[20], Kilbas[73], Øksendal[53], Boudaoui and Ouahab [19].

In this part we aim to study the existence of the mild solution for the stochastic fractional differential inclusion driven by cylindrical fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ with finite delay of the form

$$\begin{cases} {}^{c}D_t^q x(t) \in Ax(t) + f(x(t-r)) + \Sigma(t-r, x(t-r)) \frac{dB_Q^H(t)}{dt} & \text{for } t \in [0,T] \\ x(t) = \varphi(t) & \text{for } t \in [-r,0], \end{cases} \tag{4.1}$$

where ${}^{c}D_t^q$ is the Caputo fractional derivative of order $q \in \left(\frac{1}{2}, 1\right)$ takes a values in a Hilbert space $\mathcal{H}$, x(.) which takes its values in $\mathcal{H}$, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ in a Hilbert space, $f : \mathcal{H} \to \mathcal{H}$ is an appropriate function.

$\Sigma : [-r,T] \times \mathcal{H} \to \mathcal{H}$ is a non empty bounded closed and convex multivalued map, $r \geq 0$ represent a finite delay.

$\left\{B_Q^H(t), t \geq 0\right\}$ is a cylindrical fractional Brownian motion on space $\mathcal{K}$ with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$.

φ is a contionuous function such that $\varphi \in C_r\left([-r,T], L^2(\Omega, \mathcal{H})\right)$.

Let $(\mathcal{H}, \|\,.\,\|_{\mathcal{H}})$ and $(\mathcal{K}, \|\,.\,\|_{\mathcal{K}})$ denote two real separable Hilbert spaces, where $\|\,.\,\|$

denote the norms in $\mathcal{H}$, $\mathcal{K}$, and (.,.) denote the inner product.

Suppose that $B_Q^H(t)$ is a cylindrical $\mathcal{K}$-valued fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$, T is a fixed real number.

Let (Ω, F, P) be a complete probability space furnished with a family of right continuous and increasing σ-algebras $\{F_t, t \in [0,T]\}$ satisfying $F_t \subset F$ and for $t \geq 0$ F_t is generated by $\left\{B_Q^H(s), s \in [0,t]\right\}$ and the P-null sets.

$L^2(\Omega, \mathcal{H})$ stands of the space of all $\mathcal{H}$ valued random variables x such that

$$\mathbb{E} \parallel x \parallel^2 = \int_{\Omega} \parallel x \parallel^2 dP < \infty,$$

for $x \in L^2(\Omega, \mathcal{H})$,

$$\parallel x \parallel_2 = \left(\int_{\Omega} | x |^2 dP \right)^{\frac{1}{2}}.$$

$L^2(\Omega, \mathcal{H})$ is a Hilbert space equipped with the norm $\parallel . \parallel_2$.

Let $L(\mathcal{K}, \mathcal{H})$ denote the space of all bounded linear operators from $\mathcal{K}$ to $\mathcal{H}$ and $Q \in L(\mathcal{K}, \mathcal{H})$ represents a non negative self-adjoint operator.

Let $C_r\left([-r,T], L^2(\Omega, \mathcal{H})\right)$ denote the Banach space of the continuous functions $\{x(t-r), t \in [0,T]\}$ from $[-r,T]$ to $L^2(\Omega, \mathcal{H})$ such that

$$\sup_{t \in [-r,T]} \mathbb{E} \parallel x(t) \parallel^2 < \infty.$$

Let $\mathcal{H}_w$ denote the space $\mathcal{H}$ endowed with the weak topology, for $D \in \mathcal{H}$; $\overline{D}^w$ denotes the weak closure of D. Let $\mathcal{K}_0$ be an arbitrary separable Hilbert space and let $L^2 = L^2(\mathcal{K}_0, \mathcal{H})$ be a separable Hilbert space with respect to Hilbert Schmidt norm $\parallel . \parallel_{L_2^0}$, let $L_Q^2(\mathcal{K}, \mathcal{H})$ be a space of all $\psi \in L(\mathcal{K}, \mathcal{H})$ such that $\psi Q^{\frac{1}{2}}$ is a Hilbert -Schmidt operator. The norm is given by

$$\parallel \psi \parallel_{L_Q^0}^2 = \parallel \psi Q^{\frac{1}{2}} \parallel^2 = tr(\psi Q \psi^*).$$

Then ψ is called a Q Hilbert Schmidt operator from $\mathcal{K}$ to $\mathcal{H}$. $L_2^0(\Omega, \mathcal{H})$ denotes the space of F_0-measurable $\mathcal{H}$ valued and square integrable stochastic process.

Consider a time interval $J = [0,T]$ with arbitrary fixed horizon T and let $\left\{B^H(t), t \in J\right\}$ be a one dimensional f.B.m with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. We need to cite the Mazur's lemma.

Lemma 4.1. *[95] Let $(X, \parallel . \parallel)$ be a Banach space and let $(u_n)_n \in \mathbb{N}$ be a sequence in X*

that converges weakly to some u_0 in X, $u_n \rightharpoonup u_0$ as $n \to \infty$.
That is for every continuous linear function f in X^* the continuous dual space of X, $f(u_n) \to f(u_0)$ as $n \to \infty$ then there exist a function $N : \mathbb{N} \to \mathbb{N}$ and a sequence of sets of real numbers $\{\alpha(n)_k / k = n, ..., N(n)\}$ such that $\alpha(n)_k \geq 0$ and $\sum_{k=n}^{N(n)} \alpha(n)_k = 1$ such that the sequence (v_n) defined by the convex combination $v_n = \sum_{k=n}^{N(n)} \alpha(n)_k u_k$ converges strongly in X to u_0 i.e

$$\| v_n - u_0 \| \to 0 \text{ as } n \to \infty.$$

4.1 Existence of mild solution

In this section we study the existence of mild solution for the system (4.1).

Definition 4.1. *A stochastic process $x \in C_r\left([-r,T], L^2(\Omega, \mathcal{H})\right)$ is mild solution of inclusion if:*

i. *$x(t)$ is measurable and F_t adapted for each $t \geq -r$ and for each fixed $r \geq 0$.*

ii. *$x(t) \in L^2(\Omega, \mathcal{H})$ has cadlag paths on $[-r,T]$ and there exist $\sigma(t-r) \in Sel_{\Sigma(x(t-r))}$ for each $t \in [0,T]$ and $r \geq 0$ satisfying the following integral equation:*

$$\begin{cases} x(t) = S_q(t)\varphi(t) + \int_0^t (t-s)^{q-1} K_q(t-s) f(x(s-r))\, ds \\ \qquad\qquad + \int_0^t (t-s)^{q-1} K_q(t-s)\sigma(s-r)\, dB_Q^H(s) & for\ t \in]0,T], \\ x(t) = \varphi(t) & for\ t \in [-r,0]. \end{cases} \tag{4.2}$$

We introduce two families of operators on $\mathcal{H}$

$$S_q(t) = \int_0^\infty \psi_q(\theta) T(t^q\theta)\, d\theta \ \ for\ t \geq 0,$$

$$K_q(t) = \int_0^\infty q^\theta \psi_q(\theta) T(t^q\theta)\, d\theta\ for\ t \geq 0,$$

where ψ_q is the Wright function.

Lemma 4.2. *[139]*

The operators $S_q(t), K_q(t)$ have the following properties :

1. *For each fixed $t \geq 0$, $S_q(t)$ and $K_q(t)$ are bounded operators, i.e. for any $x \in C_r\left([-r,T], L^2(\Omega,\mathcal{H})\right)$*
$$S_q(t) \leq M_1 \mid x \mid,$$
$$K_q(t) \leq \frac{M_1 \mid x \mid}{\Gamma(q)}.$$

2. *$\left\{S_q(t)\right\}_{t\geq 0}$ are strongly continuous .*

3. *$\left\{S_q(t)\right\}_{t\geq 0}$ is compact, if $\{T(t)\}_{t\geq 0}$ is compact.*

Conditions and assumptions

We need to impose the following assumptions:

(H1): The operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operator $\{S(t), t > 0\}$ in $\mathcal{H}$ such that $\| S(t) \|^2 \leq M$ for some $M \geq 0$ and for each $t \in [0,T]$.

(H2): The function $f : \mathcal{H} \to \mathcal{H}$ is weakly sequentially continuous.
For every fixed $r \geq 0$ we suppose that:
The multivalued map $\Sigma : [-r,T] \times \mathcal{H} \to \mathcal{H}$ has a closed bounded and convex values and satisfies the following conditions.

(H3): $\Sigma(.,x) : [-r,T] \to \mathcal{H}$ has a measurable selection.

(H4): $\Sigma(t-r,.) : \mathcal{H} \to \mathcal{H}$ is weakly sequentially closed for each $t \in [0,T]$ and for any each fixed $r \geq 0$ i.e it has a weakly sequentially closed graph.

(H5): $\Sigma(t-r,.) : \mathcal{H} \to \mathcal{H}$ is weakly u.s.c.

(H6): For every $s > 0$ there exists a function $\mu_s \in L^1([-r,T], \mathbb{R}^+)$ such that
$$\| \Sigma(t-r,x) \|^2_{L_2^0} = sup\left\{\| \sigma(t) \|_{L_2^0} : \sigma \in \Sigma(t-r,x)\right\} \leq \mu_s(t);\ \forall \mid x \mid^2 \leq s,\ \forall t \in [-r,T].$$

(H7): $\varphi \in C_r\left([-r,T], L^2(\Omega,\mathcal{H})\right)$.

Now we need to prove that the set of selections of Σ is non empty.

Lemma 4.3. *Assume that Σ satisfies conditions (H3-H6) then the $Sel_{\Sigma(x)}$ is non empty.*

Proof. Let $x \in C_r\left([-r,T], L^2(\Omega,\mathcal{H})\right)$ we have that x is uniformly continuous, so there exists a sequence $\{x_n\}$ of step functions
$$x_n : [-r,T] \to L^2(\Omega,\mathcal{H}),$$

such that

$$\sup_{t\in[-r,T]} \| x_n(t) - x(t) \|^2 \to 0, \; as \;\; n \to \infty, \tag{4.3}$$

by (H3) there exists a sequence of measurable functions $\{\sigma_n\}$ such that
$\sigma_n(t-r) \in \Sigma(t-r, x_n(t-r))$ for any $t \in [0,T]$ and for each fixed $r \geq 0$, by equation 4.3 there exists a bounded set $E \subset L^2(\Omega, \mathcal{H})$ such that :
$x_n(t-r), x(t-r) \in E$ for any $t \in [0,T]$ and $n \in \mathbb{N}$, by (H6) there exists $\mu_s \in L^1([-r,T], \mathbb{R}^+)$ such that:

$$\| \sigma_n(t-r) \|^2_{L^0_2} \leq \| \Sigma(t-r) \|^2_{L^0_2} \leq \mu_s(t) \; \forall n \in \mathbb{N} \; \text{for a.e.} \;\; t \in [0,T].$$

Therefore $\{\sigma_n\} \subset L^2\left([-r,T], L^0_2\right)$ is bounded and uniformly integrable and $\{\sigma_n(t)\}$ is bounded in L^0_2 for any $t \in [-r,T]$, there exist a subsequence denoted as the sequence such that

$$\sigma_n \rightharpoonup \sigma \in L^2\left([-r,T], L^2_0\right),$$

by lemma (4.1), we obtain a sequence $\tilde{\sigma} = \sum\limits_{i=0}^{k_n} \lambda_{n,i}\sigma_{n+i}$, $\lambda_{n,i} \geq 0$, $\sum\limits_{i=0}^{k_n} \lambda_{n,i} = 1$,
such that $\tilde{\sigma}_n \longrightarrow \sigma$ in $L^2\left([-r,T], L^0_2\right)$ and

$$\tilde{\sigma}_n(t) \longrightarrow \sigma(t),$$

by $(H4)$ the multivalued map $\Sigma(t-r,.)$ is locally weakly compact for a.e. $t \in [0,T]$ and $r \geq 0$. Therefore by (H4) and the locally weak compactness, we get that

$$\Sigma(t-r,.) : \mathcal{H} \longrightarrow \mathcal{H}_w$$

is u.s.c for a e $t \in [0,T]$ and $r \geq 0$.
Now we need to prove that $\sigma(t-r) \in \Sigma(t-r, x(t-r))$ for a.e. $t \in [0,T]$ *and* $r \geq 0$ we consider the lebesgue measure of N_0 be zero such that:

$$\Sigma(t-r,.) : \mathcal{H} \longrightarrow \mathcal{H}_w \;\;\; \textit{is u.s.c.}$$

We denote by $\tilde{t} = t - r$ for any $t \in [0,T]$ and for each fixed $r \geq 0$.
$\sigma_n(\tilde{t}) \in \Sigma(\tilde{t}, x_n(\tilde{t}))$ and $\tilde{\sigma}_n(\tilde{t}) \to \sigma(\tilde{t})$ for all $\tilde{t} \in [-r,T] \setminus N_0$ and $n \in \mathbb{N}$.
Now we fix $\tilde{t_0} \notin N_0$ and we suppose by contradiction that $\sigma(\tilde{t_0}) \notin \Sigma(\tilde{t_0}, x(\tilde{t_0}))$ but $\Sigma(\tilde{t_0}, x(\tilde{t_0}))$

is closed and convex, by Hahn Banach theorem there is a weakly open convex set

$$V \supset \Sigma(\tilde{t_0}, x(\tilde{t_0}))$$

satisfying $\sigma(\tilde{t_0}) \in V$ since, $\Sigma(\tilde{t_0}, .) : \mathcal{H} \longrightarrow \mathcal{H}_w$ is u.s.c, there exist a neighborhood U of $x(\tilde{t_0})$ such that, $\Sigma(\tilde{t_0}, x) \subset V$ for all $x \in U$, the convergence $x_n(\tilde{t_0}) \rightharpoonup x(t_0)$, as $n \longrightarrow \infty$ implies the existence of $n_0 \in \mathbb{N}$ such that $x_n(\tilde{t_0}) \in U$ for all $n > n_0$ therefore $\sigma_n(\tilde{t_0}) \in \Sigma(\tilde{t_0}, x_n(\tilde{t_0})) \subset V$ for all $n > n_0$ and by the convergence we obtain contradiction conclusion about $\sigma(\tilde{t_0}) \in \bar{V}$. We arrive to the desired result; $\sigma(\tilde{t}) \in \Sigma(\tilde{t}, x(\tilde{t}))$ for a.e $\tilde{t} \in [-r, T]$. □

Lemma 4.4. *Let conditions (H3),(H5) and (H6) be satisfied, the $Sel_{\Sigma(x)}$ is weakly u.s.c with non empty convex and weakly compact values.*

Proof. Let $x \in C_r\left([-r,T], L^2(\Omega, \mathcal{H})\right)$, by the uniform continuity of x there exists a sequence $\{x_n\}$ of step functions.
$x_n : [-r, T] \longrightarrow L^2(\Omega, \mathcal{H})$ such that:

$$\sup_{t\in[0,T]} \parallel x_n(t-r) - x(t-r) \parallel^2 \rightarrow 0,\ as\ n \rightarrow \infty.$$

By (H3) there exists a sequence of functions $\{\sigma_n\}$ such that

$$\sigma_n(\tilde{t}) \in \Sigma(\tilde{t}, x_n(\tilde{t})),$$

where $\tilde{t} = t - r$ for a.e. $t \in [0, T]$ and for each fixed $r \geq 0$, note that $\sigma_n : [-r, T] \rightarrow L_2^0$ is measurable for any $n \in \mathbb{N}$.
By (H6) we have that $\{\sigma_n\} \subset L^2\left([-r,T], L_2^0\right)$ is bounded and uniformly integrable and $\{\sigma_n(\tilde{t})\}$ is bounded in L_2^0 for a.e $\tilde{t} \in [-r, T]$ by using the same method as lemma 4.3 we obtain a sequence $\tilde{\sigma}_n \in co\{\sigma_k\,;\, k \geq n\}$ such that $\tilde{\sigma}_n \rightarrow \sigma$ *in* $L^2\left([-r,T], L_2^0\right)$ and up to subsequence $\tilde{\sigma}_n(\tilde{t}) \rightarrow \sigma(\tilde{t})$ for a.e $\tilde{t} \in [-r, T]$ and $\sigma_n(\tilde{t}) \in \Sigma(\tilde{t}, x_n(\tilde{t}))$ for all $n \geq 1$.

Let **N** be the set of all $\tilde{t} \in [-r, T]$ such that $\sigma_n(\tilde{t}) \longrightarrow \sigma(\tilde{t})$ in L_2^0 and $\sigma_n(\tilde{t}) \in \Sigma_n(\tilde{t}, x_n(\tilde{t}))$ for all $n \geq 1$, let $\tilde{x} \in L_2^0$, $\varepsilon > 0$ by (H5) it follows that;
$< \tilde{x}, \Sigma(\tilde{t}\,,\, .) > :\ \mathcal{H} \rightarrow P(\mathbb{R})$ is u.s.c with compact convex values so $\varepsilon - \delta$ u.s.c with compact convex values and we have that:

$$< \tilde{x}\,, \tilde{\sigma}_n(\tilde{t}) > \in\ co\{< \tilde{x}, \sigma_k(\tilde{t}) >\,;\, k \geq n\} \subset < \tilde{x}\,,\, \Sigma(\tilde{t}\,, x_n(\tilde{t})) > \tag{4.4}$$

$$\subset < \tilde{x}\,,\, \Sigma(\tilde{t}\,, x_n(\tilde{t})) > + (-\varepsilon\,,\, \varepsilon)\,. \tag{4.5}$$

Seeing that Σ has a convex and closed values we obtain that $\sigma(\tilde{t}) \in\ \Sigma(\tilde{t}\,,\, x(\tilde{t}))$ for each

$\tilde{t} \in \mathbf{N}$, by consequence, $\sigma \in Sel_{\Sigma(x)}$.
Finally by using lemma 3.2 we obtain that $Sel_{\Sigma(x)}$ is weakly u.s.c with convex and weakly compact values, completing the proof. □

For abreviation we will denote by C_r the space $C_r\left([-r,T], L^2(\Omega, \mathcal{H})\right)$. For any $x \in C_r$, we define the solution multioperator : $\mathfrak{F} : C_r \to P(C_r)$

$$\mathfrak{F} = S \circ Sel_\Sigma,$$

where

$$S(\sigma) = S_q(t)\varphi(t) + \int_0^t (t-s)^{q-1} K_q(t-s) f(x(s-r))\, ds \\ + \int_0^t (t-s)^{q-1} K_q(t-s)\sigma(s-r)\, dB_Q^H(s).$$

We verify that the fixed points of the multioperator $\mathfrak{F}$ are mild solutions of our inclusion, we fix $n \in \mathbb{N}$ and we consider the space Q_n such that $Q_n = \left\{x \in C_r : \| x \|_{C_r}^2 \leq n\right\}$.

Let be $\mathfrak{F}_\mathfrak{n} = \mathfrak{F}\mid_{Q_n} : Q_n \to P(C_r)$.

Lemma 4.5. *The multi-operator $\mathfrak{F}_\mathfrak{n}$ has a weakly sequentially closed graph.*

Proof. Let $\{x_m\} \subset Q_n$ and $y_m \in \mathfrak{F}_\mathfrak{n}(x_m)$ $\forall m \in \mathbb{N}$ and $x_m \rightharpoonup x$ in Q_n, $y_m \longrightarrow y$ in C_r, we need to prove that $y \in \mathfrak{F}_\mathfrak{n}(x)$, but $x_m \in Q_n$ $\forall m \in \mathbf{N}$ and $x_m(t) \rightharpoonup x(t)$ $\forall t \in [-r,T]$, So $\| x(t-r) \| \leq \liminf_{m\to\infty} \| x_m(t-r) \| \leq n^{\frac{1}{2}}$, $\forall t \in [0,T]$ therefore $y_m \in \mathfrak{F}(x_m)$ so there exists a sequence σ_m; $\sigma_m \in sel_{\Sigma(x_m)}$ such that for every $t \in [0,T]$ and for each fixed $r \geq 0$

$$y_m(t) = S_q(t)\varphi(t) + \int_0^t (t-s) K_q(t-s) f(x_m(s-r))\, ds \tag{4.6}$$

$$+ \int_0^t (t-s)^{q-1} K_q(t-s)\sigma_m(s-r)\, dB_Q^H(s). \tag{4.7}$$

By (H6) we obtain that :

$$\| \sigma_m(t-r) \|_{L_2^0}^2 \leq \mu_s(t) \;\; \forall t \in [-r,T] \;\; and \; \forall\, m \geq 0,$$

i.e. σ_m is bounded and uniformly integrable and $\{\sigma_m(t)\}$ is bounded in L_2^0 for a.e $t \in [-r,T]$ by using the reflexivity of L_2^0 and the lemma we get the existence of a subsequence denoted as a sequence such that $\sigma_m \rightharpoonup \sigma$ in $L^2\left([-r,T], L_2^0\right)$.

Moreover, we have :

$$\int_0^t (t-s)^{q-1} K_q(t-s)\,\sigma_m(s)\,dB_Q^H \rightharpoonup \int_0^t (t-s)^{q-1} K_q(t-s)\,\sigma(s)\,dB_Q^H.$$

Let $\tilde{x} : L^2(\Omega, \mathcal{H}) \longrightarrow \mathbb{R}$ be a linear continuous operator. We must prove that the operator $R \longrightarrow \int_0^t (t-s)^{q-1} K_q(t-s) R(s-r)\,dB_Q^H(s)$ is linear and continuous from $L^2\left([-r,T]\,L_2^0\right)$ to $L^2(\Omega, H)$ for any $R_m, R \in L^2\left([-r,T]\,L_2^0\right)$ and $R_m \longrightarrow R\,(m \to \infty)$ by (H6), we get that for each $t \in [0,T]$ and for each fixed $r \geq 0$;

$$\mathbb{E} \mid \int_0^t (t-s)^{q-1} K_q(t-s)\left[R_m(s-r) - R(s-r)\right] dB_Q^H(s) \mid^2 \leq C_2 N(q)$$

$$\int_0^t \mathbb{E} \parallel (t-s)^{q-1} \left(R_m(s-r) - R(s-r)\right) \parallel_{L_2^0}^2 ds$$

$$\leq C_2 N(q) \left(\int_0^t (t-s)^{4(q-1)} ds\right) \tfrac{1}{2} \left(\int_0^t \mathbb{E} \parallel R_m(s-r) - R(s-r) \parallel^4 ds\right) \tfrac{1}{2}$$

$$\leq C_2 N(q)\, b^{\frac{4q-4}{2}} \int_0^t \mathbb{E} \parallel R_m(s-r) - R(s-r) \parallel^4 ds \to 0 \text{ when } m \to 0,$$

where $N(q) = \left(\frac{M_1}{\Gamma(q)}\right)^2$, it follows that the operator

$$R \longrightarrow \int_0^t (t-s)^{q-1} K_q(t-s) R(s-r)\,dB_Q^H(s),$$

is continuous, consequentially we have that the operator. $R \to \tilde{x} \circ \int_0^t (t-s)^{q-1} K_q(t-s) R(s-r)\,dB_Q^H(s)$ is linear and continuous from $L^2\left([-r,T]\,L_2^0\right) \to \mathbb{R}\ \forall t \in [-r,T]$. By the weak convergence of σ_m we get that for every $t \in [0,T]$ and for each fixed $r \geq 0$,

$$\tilde{x} \circ \int_0^t (t-s)^{q-1} K_q(t-s)\,\sigma_m(s-r)\,dB_Q^H(s) \longrightarrow \tilde{x} \circ \int_0^t (t-s)^{q-1} K_q(t-s)\,\sigma(s-r)\,dB_Q^H(s)$$

in the other part due to the hypothesis $f(x_m(s-r)) \rightharpoonup f(x(s-r))$, by the same method we prove that the operator

$$g \to \int_0^t (t-s)^{q-1} K_q(t-s)\, g(s-r)\,ds$$

is linear and continuous operator from $L^2\left([-r,T], L^2(\Omega,\mathcal{H})\right)$to $L^2(\Omega,\mathcal{H})$ thus

$$\int_0^t (t-s)^{q-1} K_q(t-s)\, f(x_m(s-r))\,ds \rightharpoonup \int_0^t (t-s)^{q-1} K_q(t-s)\, f(x(s-r))\,ds.$$

Finally we get that

$$y_m(t) \rightharpoonup S_q(t)\varphi(t) + \int_0^t (t-s)^{q-1} K_q(t-s) f(x(s-r))\,ds \tag{4.8}$$
$$+ \int_0^t (t-s)^{q-1} K_q(t-s)\,\sigma(s-r)\,dB_Q^H(s) = \tilde{y}(t) \quad \forall t \in [0,T]. \tag{4.9}$$

By the uniqueness of the weak limit in $L^2(\Omega, \mathcal{H})$, we obtain that,

$$\tilde{y}(t) = y(t)$$

for all $t \in [0,T]$ and for each fixed $r \geq 0$.

□

Lemma 4.6. *The multi-operator $\mathfrak{F}_n$ is weakly compact.*

Proof. We prove as first that $\mathfrak{F}_n(Q_n)$ is relatively weakly sequentially compact.
Let $x_m \subset Q_n$ and $y_m \subset C_r$ satisfy $y_m \in \mathfrak{F}_n(x_m)$ for all $m \geq 0$. There exist a sequence $\{\sigma_m\}$, $\sigma_m \in Sel_{\Sigma(x_m)}$ such that for all $t \in [0,T]$ and for each fixed $r \geq 0$.

$$y_m(t) = S_q(t)\varphi(t) + \int_0^t (t-s)^{q-1} K_q(t-s) f(x_m(s-r))\,ds$$
$$+ \int_0^t (t-s)^{q-1} K_q(t-s)\,\sigma_m(s-r)\,dB_Q^H(s).$$

By lemma (4.3) we have that there exists a subsequence denoted as the sequence, and a function σ such that $\sigma_m \rightharpoonup \sigma$ in $L^2\left([-r,T], L_2^0\right)$, since the operator f maps bounded and $f(x_m(s-r)) \rightharpoonup f(x(s-r)) \in H$ up to subsequence, in addition

$$y_m(t) \rightharpoonup l(t) = S_q(t)\varphi(t) + \int_0^t (t-s)^{q-1} K_q(t-s) f(x(s-r))\,ds$$
$$+ \int_0^t (t-s)^{q-1} K_q(t-s)\,\sigma(s-r)\,dB_Q^H(s), \quad \forall t \in [0,T]$$

by $(H1), (H2), (H6)$ we have :

$$\mathbb{E} \mid y_m(t) \mid^2 \leq 3\mathbb{E} \mid S_q(t)\varphi(t) \mid +3\mathbb{E} \mid \int_0^t (t-s)^{q-1} K_q(t-s) f(x_m(s-r))\, ds \mid^2$$

$$+3\mathbb{E} \mid \int_0^t (t-s)^{q-1} K_q(t-s) \sigma_m(s-r)\, dB_Q^H(s) \mid$$

$$\leq 3M_1^2 \mathbb{E} \mid \varphi(t) \mid^2 +3N(q)\mathbb{E} \mid \int_0^t (t-s)^{q-1} K_q(t-s) f(x_m(s-r))\, ds \mid^2$$

$$+C_p N(q)\, 3\int_0^t (t-s)^{2(q-1)} \mathbb{E} \parallel \sigma_m(s-r) \parallel_{L_2^0}^2 ds$$

$$\leq 3M_1^2 \mathbb{E} \mid \varphi(t) \mid^2 +3N(q)\mathbb{E} \mid \int_0^t (t-s)^{q-1} K_q(t-s) f(x_m(s-r))\, ds \mid^2$$

$$+C_p N(q)\, 3\int_0^t (t-s)^{2(q-1)} \mu_n(s-r)\, ds$$

$$\leq 3M_1^2 \parallel \varphi(t) \parallel^2 +3N(q) \tfrac{1}{2q-1} T^{2q} l^* + 3C_2 N(q)$$

$$\leq N,$$

$for\ all\ m \in \mathbb{N}\ \ for\ a\ .e.\ t \in [0,T]$.
By the weak convergence of C_r we have that $y_m \rightharpoonup l$ in C_r so $\mathfrak{F}_n(Q_n)$ is relatively weakly sequentially compact, by theorem (3.1) is weakly compact.

□

Lemma 4.7. *The multi-operator $\mathfrak{F}_n$ has convex and weakly compact values.*

Proof. for $x \in Q_n$; By the convexity of the multivalued map Σ and the linearity of the integral; it follows that the set $\mathfrak{F}_n(x)$ is convex, The weak compactness follows by the previous lemmas. □

Theorem 4.1. *Assume that (H1) (H2) and (H6) hold, Moreover*

$$\lim_{n\to\infty} inf \frac{1}{n}\int_0^t \mu_n(s-r)\, ds = 0,$$

then the inclusion 4.1 has at least a mild solution.

Proof. We indicate that there exists $n \in \mathbb{N}$ such that the operator $\mathfrak{F}_\mathrm{n}$ maps the ball Q_n into itself.

Suppose on the contrary that there exist sequences $\{z_n\}$, $\{y_n\}$ such that $z_n \in Q_n$, $y_n \in \mathfrak{F}_n(z_n)$ and $y_n \notin Q_n$, $\forall n \in \mathbb{N}$. Then there exist a sequence $\{\sigma_n\} \subset L^2\left([-r,T], L_2^0\right)$ $\sigma_n(s-r) \in \Sigma(s-r, z_n(s-r))$ $\forall n \in \mathbb{N}$ and $a.e\ t \in [0,T]$ for each fixed $r \geq 0$ such that,
$y_n(t) = S_q(t)\varphi(t) + \int_0^t (t-s)^{q-1} K_q(t-s) f(x_n(s-r))\, ds + \int_0^t (t-s)^{q-1} K_q(t-s)\sigma_n(s-r)\, dB_Q^H(s)$

$\forall t \in [0,T]$ and $r \geq 0$, as reason of the lemma (4.4), we get that

$$1 < \frac{\| y \|_{C_r}}{n} \leq \frac{1}{n} 3M_1^2 \| \varphi(t) \|_{C_r}^2 + 3\frac{N(q)}{2q-1} b^{2q} l \ n \in \mathbb{N},$$

we get a contradiction with lemma (4.3).
Now, we fix $n \in \mathbb{N}$ such that $\mathfrak{F}_{\mathfrak{n}}(Q_n) \subseteq Q_n$, by lemma (4.4) $\mathfrak{F}_{\mathfrak{n}}(Q_n)$ is weakly compact. Let $U_n = \overline{\mathfrak{F}_{\mathfrak{n}}}^w$, we consider $\tilde{U}_n = \overline{co}(U_n)$, where $\overline{co}(U_n)$ is the closed convex hull of U_n by theorem (3.3) $\tilde{U}_n$ is a weakly compact set. Additionally, we have that $\mathfrak{F}_{\mathfrak{n}}(Q_n) \subset Q_n$ and Q_n is a convex closed set; we get that $\tilde{U}_n \subset Q_n$ for this reason

$$\mathfrak{F}_{\mathfrak{n}}\left(\tilde{U}_n\right) = \mathfrak{F}_{\mathfrak{n}}(co(\mathfrak{F}_{\mathfrak{n}}(Q_n))) \subseteq \mathfrak{F}_{\mathfrak{n}}(Q_n) \subseteq \overline{\mathfrak{F}_{\mathfrak{n}}(Q_n)}^{\omega} = U_n \subset \tilde{U}_n.$$

By lemma (4.4) $\mathfrak{F}_{\mathfrak{n}}$ has a weakly sequentially closed graph, thus from theorem (3.4) inclusion (4.1) has a solution. □

4.2 Numerical application

We consider

$$\begin{cases} {}^cD_t^{\frac{1}{2}} y(t,\xi) \in \frac{\partial^2 y(t,\xi)}{\partial \xi^2} + f(y(t-2,\xi)) + G(t-2, y(t-2,\xi)) \frac{dB_Q^H(t)}{dt} & \text{for } t \in [0,1] \\ y(t,\xi) = \varphi(t)\xi & \text{for } t \in [-2,0], \end{cases} \tag{4.10}$$

where
${}^cD^{\frac{1}{2}}$ is the Caputo derivative of order $q = \frac{1}{2}$ we pose $\mathcal{H} = L^2([0,\pi],\mathbb{R})$, $f : \mathcal{H} \to \mathcal{H}$ is a continuous function and satisfy a condition (H_2), $G(.) : [-2,1] \times \mathcal{H} \to \mathcal{H}$ is a multivalued map, φ is a continuous function such that $\varphi \in C_r\left([-2,1], L^2([0,\pi], \mathcal{H})\right)$.
The operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$D(A) = \left\{ y \in \mathcal{H}/y, y' \ are\ absolutely\ continious, y'' \in \mathcal{H}/y(0) = y(\pi) = 0 \right\}.$$

$\left\{ B_Q^H(t), t \geq 0 \right\}$ is a cylindrical fractional Brownian motion on space $\mathcal{K}$ with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$.
such that :

$$Ay = y'' \ then\ Ay = \sum_{n=1}^{\infty} n^2 (y, y_n) y_n,$$

where $y_n(t) = \sqrt{\frac{2}{\pi}} sin(nt)$, $n = 1,2,...$ we see that A generates a compact analytic semigroup $\{T(t)\}_{t\geq 0}$ in $\mathcal{H}$.
Now we assume that $g_i : [-2,1] \times \mathcal{H} \to \mathcal{H}$ $i = 1,2$ such that :

i. g_1 and g_2 are u.s.c.

ii. $g_1 < g_2$.

iii. For every $s > 0$ there exists a function $\mu_s \in L^1([-2,1], \mathbb{R}^+)$ such that $\| g_i(t) \|_{L_2^0} \leq \mu_s(t)$ $\forall \mid y \mid^2 \leq s$ $\forall t \in [-2,1]$ for $i = 1,2$.

We take $G(t-2, y(t-2,\xi)) = [g_1(t-2,\xi)\ ,\ g_2(t-2,\xi)]$, so we obtain the following form:

$$\begin{cases} {}^cD_t^q x(t) \in Ax(t) + f(x(t-r)) + \Sigma(t-r, x(t-r))\frac{dB_Q^H(t)}{dt} & \text{for } t \in [0,T] \\ x(t) = \varphi(t) & \text{for } t \in [-r,0], \end{cases} \tag{4.11}$$

where $x(t)\xi = y(t,\xi)$, $\Sigma(t,x(t))(\xi) = G(t,y(t,\xi))$, from our assumptions (i)-(iii) it follows that the multivalued map $\Sigma(.) : [-2,1] \times \mathcal{H} \to \mathcal{H}$ satisfies the conditions (H_3)-(H_6). So all the assumptions in theorem 4.1 are verified thus this inclusion 4.10 has a mild solution.

Chapter 5

The fractional stochastic heat equation driven by time-space white noise

The *fractional derivative* of a function was first introduced by Niels Henrik Abel in 1823 [1], in connection with his solution of the tautochrone (isochrone) problem in mechanics.

The *Mittag-Leffler function* $E_\alpha(z)$ was introduced by Gösta Magnus Mittag-Leffler in 1903 [102]. He showed that this function has a connection to the fractional derivative introduced by Abel.

The fractional derivative turns out to be useful in many situations, e.g. in the study of waves, including ocean waves around an oil platform in the North Sea, and ultrasound in bodies. In particular, the fractional heat equation may be used to describe anomalous heat diffusion, and it is related to power law attenuation. This and many other applications of fractional derivatives can be found in the book by S. Holm [60].

In this chapter, we study the following fractional stochastic heat equation

$$\frac{\partial^\alpha}{\partial t^\alpha} Y(t,x) = \lambda \Delta Y(t,x) + \sigma W(t,x);\ (t,x) \in (0,\infty) \times \mathbb{R}^d \tag{5.1}$$

where $d \in \mathbb{N} = \{1,2,...\}$ and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo derivative of order $\alpha \in (0,2)$, and $\lambda > 0$ and $\sigma \in \mathbb{R}$ are given constants,

$$\Delta Y = \sum_{j=1}^{d} \frac{\partial^2 Y}{\partial x_j^2}(t,x) \tag{5.2}$$

is the Laplacian operator and

$$W(t,x) = W(t,x,\omega) = \frac{\partial}{\partial t} \frac{\partial^d B(t,x)}{\partial x_1 ... \partial x_d} \tag{5.3}$$

is time-space white noise,

$$B(t,x) = B(t,x,\omega); t \geq 0, x \in \mathbb{R}^d, \omega \in \Omega$$

is time-space Brownian sheet with probability law $\mathbb{P}$. The boundary conditions are:

$$Y(0,x) = \delta_0(x) \text{ (the point mass at 0)}, \tag{5.4}$$

$$\lim_{x \to +/- \infty} Y(t,x) = 0. \tag{5.5}$$

In the classical case, when $\alpha = 1$, this equation models the normal diffusion of heat in a random or noisy medium, the noise being represented by the time-space white noise $W(t,x)$.

- When $\alpha > 1$ the equation models *superdiffusion or enhanced diffusion*, where the particles spread faster than in regular diffusion. This occurs for example in some biological systems.

- When $\alpha < 1$ the equation models *subdiffusion*, in which travel times of the particles are longer than in the standard case. Such situation may occur in transport systems.

For more information about super- and subdiffusions, see Cherstvy et al. [34].

We consider the equation (5.1) in the sense of distribution, and in theorem (5.1) we find an explicit expression for the $\mathcal{S}'$-valued solution $Y(t,x)$, where $\mathcal{S}'$ is the space of tempered distributions.

Following the terminology of Y. Hu [62], we say that the solution is *mild* if $Y(t,x) \in L^2(\mathbb{P})$ for all t,x. It is well-known that in the classical case with $\alpha = 1$, the solution is mild if and only if the space dimension $d = 1$. See e.g.Y. Hu [62].
We show that if $\alpha \in (1,2)$ the solution is mild if $d = 1$ or $d = 2$.
Then we show that if $\alpha < 1$ then the solution is not mild for any space dimension d.

There are many papers dealing with various forms of stochastic fractional differential equations. Some papers which are related to ours are:

- In the paper by Kochubel et al. [76] the fractional heat equation corresponding to random time change in Brownian motion is studied.

-The papers by Bock et al. [11], [14] are considering stochastic equations driven by grey Brownian motion.

-The paper by Röckner et al. [88] proves the existence and uniqueness of general time-fractional linear evolution equations in the Gelfand triple setting.

-The paper which is closest to our work is Chen et al. [33], where a comprehensive discussion is given of a general fractional stochastic heat equations with *multiplicative* noise, and with fractional derivatives in both time and space, is given. In that paper, the authors prove the existence and uniqueness results as well as the regularity results of the solution, and they give sufficient conditions on the coefficients and the space dimension d, for the solution to be a random field.

Our work, however, is dealing with *additive* noise and a more special class of fractional heat equations. As in [33] we find explicit solution formulae in the sense of distributions and give conditions under which the solution is a random field in $L^2(\mathbb{P})$.

We refer to Holm [60], Ibe [64], Kilbas et al. [73] and Samko et al. [117] for more information about fractional calculus and their applications.

5.1 The solution of the fractional stochastic heat equation

We now state and prove the first main result of this work:

Theorem 5.1. *The unique solution $Y(t,x) \in \mathcal{S}'$ of the fractional stochastic heat equation* (5.1) - (5.5) *is given by*

$$Y(t,x) = I_1 + I_2, \tag{5.6}$$

where

$$I_1 = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ixy} E_\alpha(-\lambda t^\alpha |y|^2) dy = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ixy} \sum_{k=0}^{\infty} \frac{(-\lambda t^\alpha |y|^2)^k}{\Gamma(\alpha k + 1)} dy, \tag{5.7}$$

and

$$\begin{aligned} I_2 &= \sigma(2\pi)^{-d} \int_0^t (t-r)^{\alpha-1} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{i(x-z)y} E_{\alpha,\alpha}(-\lambda(t-r)^\alpha |y|^2) dy \right) B(dr,dz) \\ &= \sigma(2\pi)^{-d} \int_0^t (t-r)^{\alpha-1} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{i(x-z)y} \sum_{k=0}^{\infty} \frac{(-\lambda(t-r)^\alpha |y|^2)^k}{\Gamma(\alpha k + \alpha)} dy \right) B(dr,dz), \end{aligned} \tag{5.8}$$

where $|y|^2 = y^2 = \sum_{j=1}^{d} y_j^2$.

Proof. a) First assume that $Y(t,x)$ is a solution of (5.1). We apply the Laplace transform L to both sides of (5.1) and obtain (see (2.41)):

$$s^{\alpha}\widetilde{Y}(s,x)-s^{\alpha-1}Y(0,x)=\lambda\widetilde{\Delta Y}(s,x)+\sigma\widetilde{W}(s,x). \tag{5.9}$$

Applying the Fourier transform F, defined by

$$Fg(y)=\int_{\mathbb{R}}e^{-ixy}g(x)dx=:\widehat{g}(y);\ g\in L^{1}(\mathbb{R}^{d}), \tag{5.10}$$

we get, since $\widehat{Y}(0,y)=1$,

$$s^{\alpha}\widehat{\widetilde{Y}}(s,y)-s^{\alpha-1}=\lambda\sum_{j=1}^{d}y_j^2\widehat{\widetilde{Y}}(s,y)+\sigma\widehat{\widetilde{W}}(s,y), \tag{5.11}$$

or,

$$\left(s^{\alpha}+\lambda|y|^{2}\right)\widehat{\widetilde{Y}}(s,y)=s^{\alpha-1}\widehat{Y}(0^{+},y)+\sigma\widehat{\widetilde{W}}(s,y). \tag{5.12}$$

Hence

$$\widehat{\widetilde{Y}}(s,y)=\frac{s^{\alpha-1}}{s^{\alpha}+\lambda|y|^{2}}+\frac{\sigma\widehat{\widetilde{W}}(s,y)}{s^{\alpha}+\lambda|y|^{2}}. \tag{5.13}$$

Since the Laplace transform and the Fourier transform commute, this can be written

$$\widetilde{\widehat{Y}}(s,y)=\frac{s^{\alpha-1}}{s^{\alpha}+\lambda|y|^{2}}+\frac{\sigma\widetilde{\widehat{W}}(s,y)}{s^{\alpha}+\lambda|y|^{2}}. \tag{5.14}$$

Applying the inverse Laplace operator L^{-1} to this equation we get

$$\begin{aligned}\widehat{Y}(t,y)&=L^{-1}\Big(\frac{s^{\alpha-1}}{s^{\alpha}+\lambda|y|^{2}}\Big)(t,y)+L^{-1}\Big(\frac{\sigma\widetilde{\widehat{W}}(s,y)}{s^{\alpha}+\lambda|y|^{2}}\Big)(t,y)\\&=E_{\alpha,1}(-\lambda|y|^{2}t^{\alpha})+L^{-1}\Big(\frac{\sigma\widetilde{\widehat{W}}(s,y)}{s^{\alpha}+\lambda|y|^{2}}\Big)(t,y),\end{aligned} \tag{5.15}$$

where we recall that

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)} \tag{5.16}$$

is the Mittag-Leffler function.

It remains to find $L^{-1}\left(\frac{\sigma\widehat{\widetilde{W}}(s,y)}{s^{\alpha}+\lambda|y|^{2}}\right)$:

Recall that the convolution $f*g$ of two functions $f,g:[0,\infty)\mapsto\mathbb{R}$ is defined by

$$(f*g)(t)=\int_0^t f(t-r)g(r)dr;\quad t\geq 0. \tag{5.17}$$

The convolution rule for Laplace transform states that

$$L\left(\int_0^t f(t-r)g(r)dr\right)(s)=Lf(s)Lg(s),$$

or

$$\int_0^t f(t-w)g(w)dw=L^{-1}\left(Lf(s)Lg(s)\right)(t). \tag{5.18}$$

By (2.43) we have

$$\begin{aligned}L^{-1}\left(\frac{1}{s^\alpha+\lambda|y|^2}\right)(t)&=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha|y|^2)\\&=\sum_{k=0}^{\infty}\frac{t^{\alpha-1}(-\lambda t^\alpha|y|^2)^k}{\Gamma(\alpha k+\alpha)}\\&=\sum_{k=0}^{\infty}\frac{(-\lambda|y|^2)^k t^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))}\\&=\sum_{k=0}^{\infty}\frac{(-\lambda t^\alpha|y|^2)^k t^{\alpha-1}}{\Gamma(\alpha(k+1))}\\&=:\Lambda(t,y).\end{aligned} \tag{5.19}$$

In other words,

$$\frac{\sigma}{s^\alpha+\lambda|y|^2}=\sigma L\Lambda(s,y), \tag{5.20}$$

combining with (5.18) we get

$$L^{-1}\left(\frac{\sigma}{s^\alpha+\lambda|y|^2}\widehat{\widehat{W}}(s,y)\right)(t)=L^{-1}\left(L\left(\sigma\Lambda(s,y)\right)\widetilde{\widehat{W}}(s,y)\right)(t) \tag{5.21}$$

$$=\sigma\int_0^t\Lambda(t-r,y)\widehat{W}(r,y)dr. \tag{5.22}$$

Substituting this into (5.15) we get

$$\widehat{Y}(t,y)=E_{\alpha,1}\left(-\lambda t^\alpha|y|^2\right)+\sigma\int_0^t\Lambda(t-r,y)\widehat{W}(r,y)dr. \tag{5.23}$$

Taking inverse Fourier transform we end up with

$$Y(t,x)=F^{-1}\left(E_{\alpha,1}\left(-\lambda t^\alpha|y|^2\right)\right)(x)+\sigma F^{-1}\left(\int_0^t\Lambda(t-r,y)\widehat{W}(r,y)dr\right)(x). \tag{5.24}$$

Now we use that

$$F\left(\int_{\mathbb{R}} f(x-z)g(z)dz\right)(y) = Ff(y)Fg(y),$$

or

$$\int_{\mathbb{R}} f(x-z)g(z)dz = F^{-1}\Big(Ff(y)Fg(y)\Big)(x). \tag{5.25}$$

This gives

$$\begin{aligned}
F^{-1}\left(\int_0^t \Lambda(t-r,y)\widehat{W}(r,y)dr\right)(x) &= \int_0^t F^{-1}\left(\Lambda(t-r,y)\widehat{W}(r,y)\right)(x)dr \\
&= \int_0^t F^{-1}\Big(F\left(F^{-1}\Lambda(t-r,y)\right)(y)FW(r,x)(y)\Big)(x)dr \\
&= \int_0^t \int_{\mathbb{R}^d} \left(F^{-1}\Lambda(t-r,y)(x-z)\right)W(r,z)dzdr \\
&= \int_0^t \int_{\mathbb{R}^d} \left((2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x-z)y}\Lambda(t-r,y)dy\right)W(r,z)dzdr \\
&= (2\pi)^{-d}\int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{i(x-z)y}\Lambda(t-r,y)dy\right)B(dr,dz).
\end{aligned}$$

Combining this with (5.24), (5.16) and (5.19) we get

$$\begin{aligned}
Y(t,x) &= F^{-1}(\sum_{k=0}^{\infty}\frac{(-\lambda t^{\alpha}|y|^2)^k}{\Gamma(\alpha k+1)}) \\
&+\sigma(2\pi)^{-d}\int_0^t \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e^{i(x-z)y}\Lambda(t-r,y)dy\right)B(dr,dz) \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ixy}\sum_{k=0}^{\infty}\frac{(-\lambda t^{\alpha}|y|^2)^k}{\Gamma(\alpha k+1)}dy \\
&+\sigma(2\pi)^{-d}\int_0^t (t-r)^{\alpha-1} \\
&\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e^{i(x-z)y}\sum_{k=0}^{\infty}\frac{(-\lambda(t-r)^{\alpha}|y|^2)^k}{\Gamma(\alpha(k+1))}dy\right)B(dr,dz).
\end{aligned}$$

This proves uniqueness and also that the unique solution (if it exists) is given by the above formula.

b) Next, define $Y(t,x)$ by the above formula. Then we can prove that $Y(t,x)$ satisfies (5.1) by reversing the argument above. We skip the details. □

5.2 The classical case (α = 1)

It is interesting to compare the above result with the classical case when α=1:
If $\alpha = 1$, we get $Y(t,x) = I_1 + I_2$, where

$$I_1 = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ixy} \sum_{k=0}^{\infty} \frac{\left(-\lambda t|y|^2\right)^k}{k!} dy$$

and

$$I_2 = \sigma(2\pi)^{-d} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-z)y} \sum_{k=0}^{\infty} \frac{\left(-\lambda(t-r)|y|^2\right)^k}{k!} dy B(dr,dz),$$

where we have used that $\Gamma(k+1) = k!$

By the Taylor expansion of the exponential function, we get

$$\begin{aligned} I_1 &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ixy} e^{-\lambda t|y|^2} dy \\ &= (2\pi)^{-d} \left(\frac{\pi}{\lambda t}\right)^{\frac{d}{2}} e^{-\frac{|x|^2}{4\lambda t}} \\ &= (4\pi\lambda t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4\lambda t}}, \end{aligned}$$

where we used the general formula

$$\int_{\mathbb{R}^d} e^{-\left(a|y|^2+2by\right)} dy = \left(\frac{\pi}{a}\right)^{\frac{d}{2}} e^{\frac{b^2}{a}};\ a > 0;\ b \in \mathbb{C}^d. \tag{5.26}$$

Similarly,

$$\begin{aligned} I_2 &= \sigma(2\pi)^{-d} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-z)y} \sum_{k=0}^{\infty} \frac{\left(-\lambda(t-r)|y|^2\right)^k}{k!} dy B(dr,dz) \\ &= \sigma(2\pi)^{-d} \int_0^t \int_{\mathbb{R}^d} \left(\frac{\pi}{\lambda(t-r)}\right)^{\frac{d}{2}} e^{-\frac{|x-z|^2}{4\lambda(t-r)}} B(dr,dz) \\ &= \sigma(4\pi\lambda)^{-\frac{d}{2}} \int_0^t \int_{\mathbb{R}^d} (t-r)^{-\frac{d}{2}} e^{-\frac{|x-z|^2}{4\lambda(t-r)}} B(dr,dz). \end{aligned}$$

Summarising the above, we get, for $\alpha = 1$,

$$\begin{aligned} Y(t,x) &= (4\pi\lambda t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4\lambda t}} \\ &\quad + \sigma(4\pi\lambda)^{-\frac{d}{2}} \int_0^t \int_{\mathbb{R}^d} (t-r)^{-\frac{d}{2}} e^{-\frac{|x-z|^2}{4\lambda(t-r)}} B(dr,dz). \end{aligned} \tag{5.27}$$

This is in agreement with a well-known classical result. See e.g. Section 4.1 in Y.Hu [62].

5.3 When is $Y(t,x)$ a mild solution?

It was pointed out already in 1984 by John Walsh [131] that (classical) SPDEs driven by time-space white noise $W(t,x); (t,x) \in [0,\infty) \times \mathbb{R}^d$ may have only distribution valued solutions if $d \geq 2$. Indeed, the solution $Y(t,x)$ that we found in the previous section is in general distribution valued. But in some cases the solution can be represented as an element of $L^2(\mathbb{P})$. Following Y. Hu [62] we make the following definition:

Definition 5.1. *The solution $Y(t,x)$ is called* mild *if $Y(t,x) \in L^2(\mathbb{P})$ for all $t > 0, x \in \mathbb{R}^d$.*

The second main issue of this chapter is the following:

Problem For what values of $\alpha \in (0,2)$ and what dimensions $d = 1,2,...$ is $Y(t,x)$ mild?

Before we discuss this problem, we prove some auxiliary results:

Lemma 5.1. *(Abel's test)*
Suppose $\sum_{n=1}^{\infty} b_n$ is convergent and put $M = \sup_n |b_n|$. Let $\{\rho_n\}$ be a bounded monotone sequence, and put $R = \sup_n |\rho_n|$. Then $\sum_{n=1}^{\infty} b_n\rho_n$ is convergent, and $|\sum_{n=1}^{\infty} b_n\rho_n| \leq MR + R|\sum_{n=1}^{\infty} b_n|$.

Proof. By summation by parts we have with
$B_N = \sum_{k=1}^{N} b_k$; $N = 1,2,...,$

$$\sum_{k=1}^{N} b_k\rho_k = \sum_{k=0}^{N} \rho_k(B_k - B_{k-1}) \tag{5.28}$$
$$= \sum_{k=1}^{N-1} B_k(\rho_k - \rho_{k+1}) + \rho_N B_N. \tag{5.29}$$

Note that

$$|\sum_{k=0}^{N-1} B_k(\rho_k - \rho_{k+1})| \leq M|\sum_{k=0}^{N-1} \rho_k - \rho_{k+1}| = M(\rho_1 - \rho_n) \tag{5.30}$$
$$\leq MR. \tag{5.31}$$

Hence

$$|\sum_{k=1}^{N} b_k\rho_k| \leq MR + R|B_N|. \tag{5.32}$$

□

Lemma 5.2. *Suppose* $\alpha > 1$. *Define*

$$\rho_k = \frac{\Gamma(k+1)}{\Gamma(\alpha k+1)}; k = 1,2,... \tag{5.33}$$

Then $\{\rho_k\}_k$ *is a decreasing sequence.*

Proof. Consider

$$\begin{aligned}\frac{\rho_{k+1}}{\rho_k} &= \frac{\Gamma(k+2)\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)\Gamma(k+1)} = \frac{(k+1)\Gamma(k+1)\Gamma(\alpha k+1)}{\alpha(k+1)\Gamma(\alpha(k+1))\Gamma(k+1)} \\ &= \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\alpha)} < 1,\end{aligned} \tag{5.34}$$

since $\alpha > 1$. □

Lemma 5.3. *Suppose* $\alpha > 1$. *Define*

$$r_k = \frac{\Gamma(k+1)}{\Gamma(\alpha k+\alpha))}; \quad k = 1,2,... \tag{5.35}$$

Then $\{r_k\}_k$ *is a decreasing sequence.*

Proof. Consider

$$\begin{aligned}\frac{r_{k+1}}{r_k} &= \frac{\Gamma(k+2)\Gamma(\alpha(k+1))}{\Gamma(\alpha(k+2))\Gamma(k+1)} = \frac{(k+1)\Gamma(k+1)\Gamma(\alpha(k+1))}{(\alpha k+2\alpha-1)\Gamma(\alpha k+2\alpha-1)\Gamma(k+1)} \\ &= \frac{k+1}{\alpha k+2\alpha-1}\cdot\frac{\Gamma(\alpha k+\alpha)}{\Gamma(\alpha k+2\alpha-1)} < 1.\end{aligned}$$

□

We now return to the question about mildness:
A partial answer is given in the following:

Theorem 5.2. *Let* $Y(t,x)$ *be the solution of the* α*-fractional stochastic heat equation. Then the following holds:*

- *a) If* $\alpha = 1$, *then* $Y(t,x)$ *is mild if and only if* $d = 1$.
- *b) If* $\alpha > 1$ *then* $Y(t,x)$ *is mild if* $d = 1$ *or* $d = 2$.
- *c) If* $\alpha < 1$ *then* $Y(t,x)$ *is not mild for any* d.

Proof. Recall that $Y(t,x) = I_1 + I_2$, with

$$I_1 = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ixy} \sum_{k=0}^{\infty} \frac{(-\lambda t^{\alpha} |y|^2)^k}{\Gamma(\alpha k+1)} dy, \tag{5.36}$$

$$I_2 = \sigma(2\pi)^{-d} \int_0^t (t-r)^{\alpha-1} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{i(x-z)y} \sum_{k=0}^{\infty} \frac{(-\lambda(t-r)^{\alpha}|y|^2)^k}{\Gamma(\alpha(k+1))} dy \right) B(dr,dz). \tag{5.37}$$

a) The case $\alpha = 1$:

This case is well-known, but for the sake of completeness we prove this by our method:
By (5.27) and the Ito isometry we get

$$\mathbb{E}[Y^2(t,x)] = J_1 + J_2, \tag{5.38}$$

where

$$J_1 = I_1^2 = (4\pi\lambda t)^{-d} e^{-\frac{\|x\|^2}{2\lambda t}} \tag{5.39}$$

and, by using (5.26),

$$\begin{aligned} J_2 &= \sigma^2 (4\pi\lambda)^{-d} \int_0^t (t-r)^{-d} (2\pi\lambda(t-r))^{\frac{d}{2}} dr \\ &= \sigma^2 2^{-d} (2\pi\lambda)^{-\frac{d}{2}} \int_0^t (t-r)^{-\frac{d}{2}} dr, \end{aligned} \tag{5.40}$$

which is finite if and only if $d = 1$.

b) The case $\alpha > 1$

By the Itô isometry we have $\mathbb{E}\left[Y^2(t,x)\right] = J_1 + J_2$, where

$$\begin{aligned} J_1 &= (2\pi)^{-2d} \left(\int_{\mathbb{R}^d} e^{ixy} \sum_{k=0}^{\infty} \frac{\left(-\lambda t^{\alpha}|y|^2\right)^k}{\Gamma(\alpha k+1)} dy \right)^2 \\ &= (2\pi)^{-2d} \left(\int_{\mathbb{R}^d} e^{ixy} E_{\alpha}(-\lambda t^{\alpha}|y|^2) dy \right)^2 \end{aligned} \tag{5.41}$$

and

$$J_2 = \sigma^2(2\pi)^{-2d} \int_0^t \int_{\mathbb{R}^d} (t-r)^{2\alpha-2} \left(\int_{\mathbb{R}^d} e^{i(x-z)y} \sum_{k=0}^{\infty} \frac{\left(-\lambda(t-r)^{\alpha}|y|^2\right)^k}{\Gamma(\alpha k+\alpha))} dy \right)^2 dzdr$$
$$= \sigma^2(2\pi)^{-2d} \int_0^t \int_{\mathbb{R}^d} (t-r)^{2\alpha-2} \left(\int_{\mathbb{R}^d} e^{i(x-z)y} E_{\alpha,\alpha}(-\lambda(t-r)^{\alpha}|y|^2) dy \right)^2 dzdr. \tag{5.42}$$

By Abel's test and Lemma 5.2 and (5.26) we get

$$J_1 = (2\pi)^{-2d} \Big(\int_{\mathbb{R}^d} \left(\sum_{k=0}^{\infty} \frac{e^{ixy}\left(-\lambda t^{\alpha}|y|^2\right)^k}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(\alpha k+1)} \right) dy \Big)^2$$
$$\leq C_1 \Big(\int_{\mathbb{R}^d} e^{ixy} \sum_{k=0}^{\infty} \frac{\left(-\lambda t^{\alpha}|y|^2\right)^k}{\Gamma(k+1)} dy \Big)^2$$
$$= C_1 \Big(\int_{\mathbb{R}^d} e^{ixy} e^{-\lambda t^{\alpha}|y|^2} dy \Big)^2$$
$$= C_1 \left(\frac{\pi}{\lambda t^{\alpha}} \right)^d e^{-\frac{2|x|^2}{\lambda t^{\alpha}}} < \infty \text{ for all } t > 0, x \in \mathbb{R}^d \text{ and for all } d.$$

By the Plancherel theorem, Lemma 5.3 and (5.26) we get

$$J_2 = \sigma^2(2\pi)^{-2d} \int_0^t (t-r)^{2\alpha-2} \int_{\mathbb{R}^d} \left(\sum_{k=0}^{\infty} \frac{(-\lambda(t-r)^{\alpha}|x-z|^2)^k}{\Gamma(\alpha k+\alpha)} \right)^2 dzdr$$
$$= \sigma^2(2\pi)^{-2d} \int_0^t \int_{\mathbb{R}^d} (t-r)^{2\alpha-2}$$
$$\int_{\mathbb{R}^d} \left(\sum_{k=0}^{\infty} \frac{(-\lambda(t-r)^{\alpha}|x-z|^2)^k}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(\alpha k+\alpha)} \right)^2 dzdr$$
$$\leq C_2 \int_0^t (t-r)^{2\alpha-2} \int_{\mathbb{R}^d} \left(\sum_{k=0}^{\infty} \frac{(-\lambda(t-r)^{\alpha}|x-z|^2)^k}{\Gamma(k+1)} \right)^2 dzdr$$
$$= C_2 \int_0^t (t-r)^{2\alpha-2} \int_{\mathbb{R}^d} \left(e^{-\lambda(t-r)^{\alpha}|x-z|^2} \right)^2 dzdr$$
$$= C_2 \int_0^t (t-r)^{2\alpha-2} \int_{\mathbb{R}^d} \left(e^{-2\lambda(t-r)^{\alpha}|x-z|^2} \right) dzdr$$
$$= C_2 \int_0^t (t-r)^{2\alpha-2} \left(\frac{\pi}{2\lambda(t-r)^{\alpha}} \right)^{\frac{d}{2}} dr$$
$$= C_3 \int_0^t (t-r)^{2\alpha-2} (t-r)^{-\frac{\alpha d}{2}} dr$$
$$= C_3 \int_0^t (t-r)^{2\alpha-2-\frac{\alpha d}{2}} dr.$$

This is finite if and only if $2\alpha - 2 - \frac{\alpha d}{2} > -1$, i.e. $d < 4 - \frac{2}{\alpha}$
If $\alpha = 1+\varepsilon$, then $4 - \frac{2}{\alpha} = 2 + \frac{2\varepsilon}{1+\varepsilon} > 2$ for all $\varepsilon > 0$.
Therefore $J_2 < \infty$ for $d = 1$ or $d = 2$, as claimed. □

c) The case $\alpha < 1$

Proof. By (5.37) we see that

$$J_2 = \sigma^2(2\pi)^{-2d}\int_0^t (t-r)^{2\alpha-2}\int_{\mathbb{R}^d}\left(E_{\alpha,\alpha}(-\lambda(t-r)^{\alpha}|x-z|^2)\right)^2 dzdr$$
$$= \sigma^2(2\pi)^{-2d}\int_0^t (t-r)^{2\alpha-2}\int_{\mathbb{R}^d}\left(E_{\alpha,\alpha}(-\lambda(t-r)^{\alpha}|y|^2)\right)^2 dydr$$

Choose β such that $0 < \alpha \leq \beta \leq 1$.
A result of Pollard [113], as extended by Schneider [118], states that the map

$$x \mapsto h(x) := E_{\alpha,\beta}(-x);\ x \in \mathbb{R}^d \tag{5.43}$$

is completely monotone, i.e,

$$(-1)^n \frac{d^n}{dx^n} h(x) \geq 0\ for\ all\ n = 0,1,2,...;\ x \in \mathbb{R}^d. \tag{5.44}$$

Therefore by Bernstein's theorem there exists a positive, σ-finite measure μ on $\mathbb{R}^+$ such that

$$E_{\alpha,\beta}(-x) = \int_0^{\infty} e^{-xs}\mu(ds). \tag{5.45}$$

In fact, it is known that μ is absolutely continuous with respect to Lebesgue measure and

$$t^{\beta-1}E_{\alpha,\beta}(-t^{\alpha}) = \int_0^{\infty} e^{-st}K_{\alpha,\beta}(s)ds \tag{5.46}$$

where

$$K_{\alpha,\beta}(s) = \frac{s^{\alpha-\beta}\left[sin((\beta-\alpha)\pi) + s^{\alpha}sin(\beta\pi)\right]}{\pi\left[s^{2\alpha} + 2s^{\alpha}cos(\alpha\pi) + 1\right]} \tag{5.47}$$

See Capelas de Oliveira et al. [29], Section 2.3.

Putting $t^{\alpha} = x$ this can be written

$$E_{\alpha,\beta}(-x) = x^{\frac{1-\beta}{\alpha}}\int_0^{\infty} e^{-sx^{\frac{1}{\alpha}}}K_{\alpha,\beta}(s)ds;\ x > 0. \tag{5.48}$$

This gives

$$E_{\alpha,\beta}(-\rho|y|^2) = \rho^{\frac{1-\beta}{\alpha}}|y|^{\frac{2(1-\beta)}{\alpha}}\int_0^{\infty} e^{-s\rho^{\frac{1}{\alpha}}|y|^{\frac{2}{\alpha}}}K_{\alpha,\beta}(s)ds. \tag{5.49}$$

It follows that

$$\left(E_{\alpha,\beta}(-\rho|y|^2)\right)^2 \sim \left(\rho^{\frac{1-\beta}{\alpha}}|y|^{\frac{2(1-\beta)}{\alpha}}\rho^{\frac{-1}{\alpha}}|y|^{\frac{-2}{\alpha}}\right)^2$$
$$= \rho^{-\frac{2\beta}{\alpha}}|y|^{-\frac{4\beta}{\alpha}} \tag{5.50}$$

Hence, by using polar coordinates we see that

$$\int_{\mathbb{R}^d} \left(E_{\alpha,\beta}(-\rho|y|^2)\right)^2 dy \sim \int_0^\infty R^{-\frac{4\beta}{\alpha}} R^{d-1} dR = \infty, \tag{5.51}$$

for all d.

Therefore $J_2 = \infty$ for all d. □

Remark 5.1.
- *See Y. Hu [62], Proposition 4.1 for a generalization of the above result in the case* $\alpha = 1$.
- *In the cases* $\alpha > 1,\ d \geq 3$ *we do not know if the solution* $Y(t,x)$ *is mild or not. This is a topic for future research.*

5.4 Numerical examples

Example 1

Let us consider the following heat equation where $\alpha < 1$. In this case our equation models *subdiffusion*, in which travel times of the particles are longer than in the standard case. Such situation may occur in transport systems. For $\alpha = \frac{1}{2}$ and $d = 2$ we get

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} Y(t,x) = \lambda \Delta Y(t,x) + \sigma W(t,x);\ (t,x) \in (0,\infty) \times \mathbb{R}^2 \tag{5.52}$$

The solution is given by:

$$Y(t,x) = I_1 + I_2, \tag{5.53}$$

where

$$I_1 = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ixy} E_{\frac{1}{2}}(-\lambda t^{\frac{1}{2}}|y|^2) dy = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ixy} erfc(-\lambda t^{\frac{1}{2}}|y|^2)^{\frac{1}{2}} dy, \tag{5.54}$$

(with $erfc(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt$) and

$$I_2 = \sigma(2\pi)^{-2} \int_0^t (t-r)^{\frac{1}{2}-1} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} e^{i(x-z)y} E_{\frac{1}{2},\frac{1}{2}}(-\lambda(t-r)^{\frac{1}{2}}|y|^2) dy \right) B(dr,dz) \tag{5.55}$$

By the Theorem 5.2 this solution is not mild.

Example 2

Next, let us consider the heat equation for $\alpha = \frac{3}{2}$. In this case the equation models *superdiffusion or enhanced diffusion*, where the particles spread faster than in regular diffusion. This occurs for example in some biological systems. Now the equation gets the form

$$\frac{\partial^{\frac{3}{2}}}{\partial t^{\frac{3}{2}}} Y(t,x) = \lambda \Delta Y(t,x) + \sigma W(t,x);\ (t,x) \in (0,\infty) \times \mathbb{R}^2 \tag{5.56}$$

By Theorem 5.1 the solution is

$$Y(t,x) = I_1 + I_2, \tag{5.57}$$

where

$$I_1 = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ixy} E_{\frac{3}{2}}(-\lambda t^{\frac{3}{2}} |y|^2) dy = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ixy} \sum_{k=0}^{\infty} \frac{(-\lambda t^{\frac{3}{2}} |y|^2)^k}{\Gamma(\frac{3}{2}k+1)} dy, \tag{5.58}$$

and

$$\begin{aligned} I_2 &= \sigma(2\pi)^{-2} \int_0^t (t-r)^{\frac{3}{2}-1} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} e^{i(x-z)y} E_{\frac{3}{2},\frac{3}{2}}(-\lambda(t-r)^{\frac{3}{2}} |y|^2) dy \right) B(dr,dz) \\ &= \sigma(2\pi)^{-2} \int_0^t (t-r)^{\frac{1}{2}} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^d} e^{i(x-z)y} \sum_{k=0}^{\infty} \frac{(-\lambda(t-r)^{\frac{3}{2}} |y|^2)^k}{\Gamma(\frac{3}{2}k+\frac{3}{2}))} dy \right) B(dr,dz) \end{aligned} \tag{5.59}$$

By Theorem 5.2 this solution is mild.

Chapter 6

Impulsive stochastic differential equations involving Hilfer fractional derivatives

Differential equations and inclusions with fractional derivatives have recently proved to be strong tools in the modeling of many phenomena in various fields of engineering, economics, physics, biology, ecology, aerodynamics and fluid dynamic traffic models [6, 92, 114, 122]. For some fundamental results in the theory of differential equations involving Caputo and Riemann-Liouville fractional derivatives, please see [4, 5, 82, 126, 128, 129, 138] and the references therein.

Since Hilfer [59] proposed the generalized Riemann-Liouville fractional derivative, there has been shown some interest in studying differential equations involving Hilfer fractional derivatives (see [59] and the references therein).

The two-parameter family of Hilfer fractional derivative $D_{a^+}^{\alpha,\beta}$ of order α and type β permits to combine between the Caputo and Riemann derivatives and give an extra degree of freedom on the initial conditions and produce more types of stationary states. Models with Hilfer fractional derivatives are discussed in [52][126]. We prove the existence of integral solutions for stochastic differential equation with impulses driven by sub-fractional Brownian motion with Hilfer fractional derivative of the form

$$\begin{cases} D_{0^+}^{\alpha,\beta} X(t,x_t) = A(t)X(t,x_t) + f(t,x_t) + \sigma(t,x_t)\frac{dS_Q^H(t)}{dt} & \text{for } t \in [s_k, t_{k+1}],\ k = 0, \ldots m \\ x(t) = h_k(t,x_t), & \text{for } t \in (t_k, s_k],\ k = 1, 2, \ldots m \\ (I_0^{1-\gamma}x)(t)|_{t=0} = \phi \in \mathcal{D}_{\mathcal{F}_0}((-\infty,0],U]. \end{cases} \tag{6.1}$$

$D_{0^+}^{\alpha,\beta}$ is the generalized Hilfer fractional derivative of orders $\alpha \in (0,1)$ and type $\beta \in [0,1]$. x(.) takes value in a real separable Hilbert space U, with inner product (.,.) and norm $\|\ .\ \|$, and A is the infinitesimal generator of strongly continuous semigroup of bounded linear operator $\{T(t)\}_{t\geq 0}$. S_Q^H is an Q-sub-fBm with Hurst parameter $H \in (\frac{1}{2},1)$, $I_0^{1-\gamma}$ is the fractional integral of orders $1-\gamma$ $(\gamma = \alpha + \beta - \alpha\beta)$.

The impulses times satisfy:

$$0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq \ldots < t_m \leq s_m < t_{m+1} = T, \text{ for } t > 0.$$

x_t mean a segment solution which is defined by

$$x(.,.) : (-\infty, T] \times \Omega \to U,$$

then for any $t \geq 0$, $x_t(.,.) : (-\infty, 0) \times \Omega \to U$ is given by
$x_t(\theta, \omega) = x(t+\theta, \omega)$, for $\theta \in (-\infty, 0]$, $\omega \in \Omega$ with is valued in $\mathcal{D}^{\gamma}_{\mathcal{F}_T}$,
where

$$\mathcal{D}^{\gamma}_{\mathcal{F}_T} = \left\{ x : (-\infty, T] \times \Omega \to U;\, x\,|_{J_k} \in C\left(J_k; U\right),\, t^{1-\gamma} x(t) \in \mathcal{D}_{\mathcal{F}_T},\, \right\} k = 1, \ldots, m.$$

With the norm

$$\| x \|_{\mathcal{D}^{\gamma}_{\mathcal{F}_T}} = \| \phi \|_{\mathcal{D}_{\mathcal{F}_0}} + \left(\sup_{0 \leq t \leq T} \mathbb{E} \| t^{1-\gamma} x(t) \|^2 \right)^{\frac{1}{2}},$$

and $\phi \in \mathcal{D}_{\mathcal{F}_0}$, where, $J_k = (s_k, t_{k+1}]$, $k = 1, \ldots, m$.

The space $\mathcal{D}_{\mathcal{F}_t}$ is the space formed by all $\mathcal{F}_t$-adapted measurable square integrable $\mathcal{H}$-valued stochastic process $\{x(t) : t \in [0, T]\}$ with norm $\|x\|^2_{\mathcal{D}_{\mathcal{F}_t}} = \sup_{t \in [0,T]} \mathbb{E}\|x(t)\|^2$, then $(\mathcal{D}_{\mathcal{F}_t}, \|.\|_{\mathcal{D}_{\mathcal{F}_t}})$ is a Banach space.
$\mathcal{D}_{\mathcal{F}_0}$ denote the family of all almost surely bounded $\mathcal{F}_0$-measurable, and $\tilde{D}$-valued random variables. $\tilde{D} = D((-\infty, 0], U)$ denotes the family of all right piecewise continuous functions with left-hand limit ϕ from $(-\infty, 0]$ to U, with the norm

$$\| \phi \|_t = \sup_{-\infty < \theta \leq t} \| \phi(\theta) \|.$$

We assume in the sequel that $X(t, x_t) : J \times U \longrightarrow U$, such that $X(t, x_t) = \phi(0) - g(t, x_t)$,
$g : J \times \mathcal{D}^{\gamma}_{\mathcal{F}_T} \to U$ and $f : J \times \mathcal{D}^{\gamma}_{\mathcal{F}_T} \to U$, $h_k \in (t_k, s_k] \times \mathcal{D}^{\gamma}_{\mathcal{F}_T} \longrightarrow U$ for all $k = 1, \ldots, m$.
$\sigma : J \times \mathcal{D}^{\gamma}_{\mathcal{F}_T} \to L^0_Q(K, H)$.

In the next we mention an axiomatic definition of the phase space $\mathcal{D}_{\mathcal{F}_0}$ introduced by Hale and Kato [58].

Definition 6.1. *$\mathcal{D}_{\mathcal{F}_0}$ is a linear space of family of $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into U endowed with a norm $\| \, . \, \|_{\mathcal{D}_{\mathcal{F}_0}}$, which satisfies the following axioms:*

(A-1) If $x : (-\infty, T] \to U$, $T > 0$ is such that $y_0 \in \mathcal{D}_{\mathcal{F}_0}$, then for every $t \in [0, T)$ the following conditions hold

(i) $y_t \in \mathcal{D}_{\mathcal{F}_0}$

(ii) $\| y(t) \| \leq \mathcal{L} \| y_t \|_{\mathcal{D}_{\mathcal{F}_0}}$

(iii) $\| y_t \|_{\mathcal{D}_{\mathcal{F}_0}} \leq K(t) sup\{\| y(s) \| : \ 0 \leq s \leq t\} + N(t) \| y(0) \|_{\mathcal{D}_{\mathcal{F}_0}}$,
where $\mathcal{L} > 0$ *is a constant;* $K, N : [0,\infty) \longrightarrow [0,\infty)$, *K is continuous, N is locally bounded and K, N are independent of* $y(.)$.

(A-2) : For the function $y(.)$ *in (A-1),* y_t *is a* $\mathcal{D}_{\mathcal{F}_0}$*-valued function for* $t \in [0,T)$.

(A-3) : The space $\mathcal{D}_{\mathcal{F}_0}$ *is complete.*

Denote

$$\tilde{K} = sup\{K(t) : \ t \in J\} \text{ and } \tilde{N} = sup\{N(t) : \ t \in J\}.$$

Theorem 6.1. *[37](Banach's Fixed Point Theorem)*
Let (X,d) *be a complete metric space and let* $T : X \longrightarrow X$ *be a contraction on X. Then T has a unique fixed point* $x \in X$ *(such that* $T(x) = x$*).*

Let us define the operators $\{S_{\alpha,\beta}(t) : t \geq 0\}$ and $\{P_\beta(t) : t \geq 0\}$ by

$$S_{\alpha,\beta}(t) = I_{0^+}^{\alpha(1-\beta)} P_\beta(t),$$

$$P_\beta(t) = t^{\beta-1} T_\beta(t),$$

$$T_\beta(t) = \int_0^\infty \beta\theta\Psi_\beta(\theta) T(t^\beta\theta) d\theta;$$

where

$$\Psi_\beta(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-n\beta)}, 0 < \beta < 1, \theta \in (0,\infty)$$

is a function of wright type which satisfies

$$\int_0^\infty \theta^\xi \Psi_\beta(\theta) d\theta = \frac{\Gamma(1+\xi)}{\Gamma(1+\beta\xi)}, \ \ \xi \in (-1,\infty).$$

Lemma 6.1. *[56] The operator* $S_{\alpha,\beta}$ *and* P_β *have the following properties*

i) For any fixed $t \geq 0$, $S_{\alpha,\beta}(t)$ *and* $P_\beta(t)$ *are bounded linear operators, and*

$$\|P_\beta(t)x\|^2 \leq M \frac{t^{2(\beta-1)}}{(\Gamma(\beta))^2} \|x\|^2 \ \ \textit{and}$$

$$\|S_{\alpha,\beta}(t)x\|^2 \leq M \frac{t^{2(\alpha-1)(1-\beta)}}{(\Gamma(\alpha(1-\beta)+\beta))^2} \|x\|^2.$$

ii) $\{P_\beta(t) : t \geq 0\}$ *is compact if* $\{T(t) : t \geq 0\}$ *is compact.*

Remark 6.1. $D_{0^+}^{\alpha(1-\beta)} S_{\alpha,\beta}(t) = P_\beta(t)$.

6.1 Existence of mild solution

In this section, we first establish the existence of mild solutions to stochastic differential equations with non-instantaneous impulses driven by a Q-sub-fractional Brownian motions. More precisely we will formulate and prove sufficient conditions for the existence of solutions to 6.1. In order to establish the results, we will need to impose some of the following conditions.

(H1) The operator A is the infinitesimal generator of a strongly continuous of bounded linear operators $\{S(t)\}_{t\geq 0}$ which is compact for $t>0$ in $\mathcal{H}$ such that $\| S(t) \|^2 \leq M$ for each $t \in J$, where $J=[0,T]$.

(H2) The operators $S_{\alpha,\beta}, P_\beta \in D(A)$.

(H3) The function $f : J \times \mathcal{D}^\gamma_{\mathcal{F}_T} \to U$ satisfies that:
$\mathbb{E} \| f(t,\phi_1) - f(t,\phi_2) \|^2 \leq L_f \| \phi_1 - \phi_2 \|^2_{\mathcal{D}^\gamma_{\mathcal{F}_T}}$,
$for\ all\ \phi_1,\phi_2 \in \mathcal{D}^\gamma_{\mathcal{F}_T},\ t \in (s_k,t_{k+1}]\ and\ k=1,\dots,m.$

(H4) The function The function $g : J \times \mathcal{D}^\gamma_{\mathcal{F}_T} \to U$ and there exist a positive number K_g. For $t \in J$, we have
$\mathbb{E} \| g(t,\phi_1) - g(t,\phi_2) \|^2 \leq K_g \| \phi_1 - \phi_2 \|^2_{\mathcal{D}^\gamma_{\mathcal{F}_T}}$, $for\ all\ \phi_1,\phi_2 \in \mathcal{D}^\gamma_{\mathcal{F}_T},\ t \in J.$

(H5) The function $\sigma : J \times \mathcal{D}^\gamma_{\mathcal{F}_T} \to L^0_Q$ satisfies that there exists a positive constant L_σ such that
$\mathbb{E} \| \sigma(t,\phi_1) - \sigma(t,\phi_2) \|^2_{L^0_2} \leq L_\sigma \| \phi_1 - \phi_2 \|^2_{\mathcal{D}^\gamma_{\mathcal{F}_T}}$, $for\ all\ \phi_1,\phi_2 \in \mathcal{D}^\gamma_{\mathcal{F}_T},\ t \in (s_k,t_{k+1}]$
$and\quad k=1,\dots,m.$

(H6) There exist constants $L_{h_k} > 0$, for all $\phi_1,\ \phi_2 \in \mathcal{D}^\gamma_{\mathcal{F}_T}$, $t \in (t_k,s_k]$ and $k=1,\dots,m$ such that

$$\mathbb{E} \| h_k(t,\phi_1) - h_k(t,\phi_2) \|^2 \leq L_{h_k} \| \phi_1 - \phi_2 \|^2_{\mathcal{D}^\gamma_{\mathcal{F}_T}}$$

and $h_k \in C((t_k,s_k] \times \mathcal{D}^\gamma_{\mathcal{F}_T}, U)$, for all $k=1,\dots,m$.

Now, we give the definition of mild solutions to our problem.

Definition 6.2. *An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty,T] \to U$ is said to be an mild solution of (1) if $x_0 = \phi \in \mathcal{D}_{\mathcal{F}_0}$ and*

(i) $\{x_t,\ t \in J\} \in \mathcal{D}^\gamma_{\mathcal{F}_T}$.

(ii) $\int_0^t [x_s + g(s,x_s)]\, ds \in D(A)$ $,t \in [0,T]$.

(iii) for each $t > 0$

$$x(t)=\begin{cases} S_{\alpha,\beta}(t)\left[\phi(0)-g(0,\phi)\right]+g(t,x_t)+\int_0^t P_\beta(t-s)f(s,x(s))ds+\int_0^t P_\beta(t-s)\sigma(s,x_s)dS_Q^H(s) \\ \hfill for\, t\in[0,t_1] \\ h_k(t,x_t), \\ \hfill for\, t\in(t_k,s_k];\; k=1,...,m. \\ S_{\alpha,\beta}(t-s_k)h_k(s_k,x_{s_k})+g(s_k,x_{s_k})+\int_{s_k}^t P_\beta(t-s)f(s,x_s)ds+\int_{s_k}^t P_\beta(t-s)\sigma(s,x_s)dS_Q^H(s) \\ \hfill for\, t\in[s_k,t_{k+1}];\; k=1,...m. \end{cases} \tag{6.2}$$

To establish the existence and uniqueness theorem of the mild solution for system 6.1.

We use a Banach fixed point to investigate the existence and uniqueness of solution for impulsive stochastic differential equations.

Theorem 6.2. *Let (H1)-(H6) hold with $\phi(0)-g(0,\phi)\in\overline{D(A)}$. and*

$$L_0=max(\mu_1,\,\mu_2,\,\mu_3)<1,$$

where

$$\mu_1=3t_1^{2(\alpha\beta+1)}\left(t_1^{1-\alpha-\beta}K_g+\frac{t_1^{-\alpha}L_fM}{(\Gamma(\beta))^2}+\frac{ML_\sigma t_1^{H-\alpha}}{(\Gamma(\beta))^2}\right)$$

$$\mu_2=\max_{k=1,\dots,m}2L_{h_k}T^{2(1-\gamma)},$$

and

$$\mu_3=\max_{k=1,\dots,m}\left[\frac{4ML_{h_k}}{\Gamma(\alpha(1-\alpha)+\beta)^2}+4t^{2(1-\gamma)}K_g+\frac{4t^{2(1-\gamma)}M(t_{k+1}-s_k)^{2(\beta-1)}L_f}{(\Gamma(\beta))^2}+\frac{C_HML_\sigma t^{2(H-\gamma)+1}(t_{k+1}-s_k)^{2(\beta-1)}}{(\Gamma(\beta))^2}\right].$$

Then for every initial function $\phi\in\mathcal{D}_{\mathcal{F}_0}$ there exists a unique associated mild solution $x\in\mathcal{D}^\gamma_{\mathcal{F}_T}$ of the problem (6.1).

Proof. The proof is given in several steps. Consider the problem (6.1)

$$\begin{cases} D_{0^+}^{\alpha,\beta}X\left(t,x_t\right)=A\left(t\right)X\left(t,x_t\right)+f\left(t,x_t\right)+\sigma\left(t,x_t\right)\frac{dS_Q^H(t)}{dt}, & \text{for } t\in[s_k,t_{k+1}],\; k=0,...m \\ x\left(t\right)=h_k\left(t,x_t\right), & \text{for } t\in(t_k,s_k],\; k=1,2,...m \\ (I_0^{1-\gamma}x)(t)|_{t=0}=\phi\in\mathcal{D}_{\mathcal{F}_0}((-\infty,0],U]. \end{cases}$$

We transform our problem into a fixed point one. Consider the operator $\Phi:\mathcal{D}^\gamma_{\mathcal{F}_T}\longrightarrow\mathcal{D}^\gamma_{\mathcal{F}_T}$ defined by

$$\Phi(x)(t)=\begin{cases}\phi(t);\ t\in(-\infty,0]\\ S_{\alpha,\beta}(t)\left[\phi(0)-g(0,\phi)\right]+g(t,x_t)+\int_0^t P_\beta(t-s)f(s,x_s)ds+\int_0^t P_\beta(t-s)\sigma(s,x_s)dS_Q^H(s)\\ \qquad\qquad\qquad\qquad\qquad\qquad if\ t\in(0,t_1].\\ h_k(t,x_t),\\ \qquad\qquad\qquad\qquad\qquad\qquad if\ t\in(t_k,s_{k+1}],\ k=1,...m,\\ S_{\alpha,\beta}(t-s_k)h_k(s_k,x_{s_k})+g(s_k,x_{s_k})+\int_{s_k}^t P_\beta(t-s)f(s,x_s)ds+\int_{s_k}^t P_\beta(t-s)\sigma(s,x_s)dS_Q^H(s)\\ \qquad\qquad\qquad\qquad\qquad\qquad if\ t\in(s_k,t_{k+1}],\ k=1,...m.\end{cases}$$

For $\phi\in\mathcal{D}_{\mathcal{F}_0}$, we define $\tilde{\phi}$ by

$$\tilde{\phi}(t)=\begin{cases}\phi(t), & t\in(-\infty,0]\\ S_{\alpha,\beta}(t)\left[\phi(0)-g(0,\phi)\right], & t\in(0,t_1].\end{cases}$$

It is clear that $\tilde{\phi}\in\mathcal{D}^{\gamma}_{\mathcal{F}_T}$. Let $x(t)=z(t)+\tilde{\phi}(t);\ t\in(-\infty,T]$, $z(t)$ satisfy that

$$z(t)=\begin{cases}0,\ for\, t\in(-\infty,0]\\ g(t,z_t+\tilde{\phi}_t)+\int_0^t S_{\alpha,\beta}(t-s)f(s,z_s+\tilde{\phi}_s)ds+\int_0^t P_\beta(t-s)\sigma(s,z_s+\tilde{\phi}_s)dS_Q^H(s)\\ \qquad\qquad\qquad\qquad\qquad\qquad for\, t\in(0,t_1]\\ h_k(t,z_t+\tilde{\phi}_t)\\ \qquad\qquad\qquad\qquad\qquad\qquad for\, t\in(t_k,s_k]\\ S_{\alpha,\beta}(t-s_k)h_k(s_k,z_{s_k}+\tilde{\phi}_{s_k})+g(s_k,z_{s_k}+\tilde{\phi}_{s_k})+\int_{s_k}^t P_\beta(t-s)f(s,z_s+\tilde{\phi}_s)ds\\ \qquad\qquad\qquad\qquad +\int_{s_k}^t P_\beta(t-s)\sigma(s,z_s+\tilde{\phi}_s)dS_Q^H(s)\\ \qquad\qquad\qquad\qquad\qquad\qquad for\, t\in(s_k,t_{k+1}],\ k=1,...,m.\end{cases}$$

So
$\mathcal{D}'_{\mathcal{F}_T}=\left\{z\in\mathcal{D}^{\gamma}_{\mathcal{F}_T},\ such\,that\ z(0)=0\right\}$, and for any $z\in\mathcal{D}'_{\mathcal{F}_T}$, we have then $\left(\mathcal{D}'_{\mathcal{F}_T},\|\,.\,\|_{\mathcal{F}_T}\right)$ is a Banach space.
Let the operator $\tilde{\Phi}:\mathcal{D}'_{\mathcal{F}_T}\longrightarrow\mathcal{D}'_{\mathcal{F}_T}$ defined by

$$\tilde{\Phi}(x)(t)=\begin{cases}0,\ for\, t\in(-\infty,0],\\ g(t,z_t+\tilde{\phi}_t)+\int_0^t P_\beta(t-s)f(s,z_s+\tilde{\phi}_s)ds+\int_0^t P_\beta(t-s)\sigma(s,z_s+\tilde{\phi}_s)dS_Q^H(s)\\ \qquad\qquad\qquad\qquad\qquad\qquad for\, t\in(0,t_1],\\ h_k(t,z_t+\tilde{\phi}_t),\\ \qquad\qquad\qquad\qquad\qquad\qquad for\, t\in(t_k,s_k]\\ S_{\alpha,\beta}(t-s_k)h_k(s_k,z_{s_k}+\tilde{\phi}_{s_k})+g(s_k,z_{s_k}+\tilde{\phi}_{s_k})+\int_{s_k}^t P_\beta(t-s)\sigma(s,z_s+\tilde{\phi}_s)dS_Q^H(s),\\ \qquad\qquad\qquad\qquad\qquad\qquad for\, t\in(s_k,t_{k+1}],\ k=1,...,m.\end{cases}$$

From the assumptions, it is clear that $\tilde{\Phi}$ is well defined. Now we need only to show that $\tilde{\Phi}$ is a contraction mapping.

Case 1:

For $u,v \in \mathcal{D}'_{\mathcal{F}_T}$, and for $t \in [0,t_1]$, by using Lemma 6.1 we have

$$\begin{aligned}
\mathbb{E} \parallel t^{1-\gamma}\left[\tilde{\Phi}(u(t)) - \tilde{\Phi}(v(t))\right] \parallel^2 &\leq 3t^{2(1-\gamma)}\mathbb{E} \parallel g(t,u_t+\tilde{\phi}_t) - g(t,v_t+\tilde{\phi}_t) \parallel^2 \\
&+ 3t^{2(1-\gamma)}\mathbb{E} \parallel \int_0^t P_\beta(t-s)\left[f(s,u_s+\tilde{\phi}_s) - f(s,v_s+\tilde{\phi}_s)\right] \parallel^2 ds \\
&+ 3t^{2(1-\gamma)}\mathbb{E} \parallel \int_0^t P_\beta(t-s)\left[\sigma(s,u_s+\tilde{\phi}_s) - \sigma(s,v(s)+\tilde{\phi}_s)\right] dS_I^H(s) \parallel^2 \\
&\leq I_1 + I_2 + I_3,
\end{aligned} \tag{6.3}$$

where

$$\begin{aligned}
I_1 :&= 3t^{2(1-\gamma)}\mathbb{E} \parallel g(t,u_t+\tilde{\phi}_t) - g(t,v_t+\tilde{\phi}_t) \parallel^2 \\
&\leq 3t^{2(1-\gamma)}K_g \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}}, \\
I_2 :&= 3t^{2(1-\gamma)}\mathbb{E} \parallel \int_0^t P_\beta(t-s)\left[f(s,u_s+\tilde{\phi}_s) - f(s,v_s+\tilde{\phi}_s)\right] \parallel^2 ds \\
&\leq 3\frac{t^{2(1-\gamma)}M}{\Gamma^2(\beta)}\mathbb{E}\int_0^t (t-s)^{2(\beta-1)} \parallel f(s,u_s+\tilde{\phi}_s) - f(s,v_s+\tilde{\phi}_s) \parallel^2 ds, \\
&\leq 3\frac{t^{2\alpha(\beta-1)}L_f M}{(\Gamma(\beta))^2} \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}}, \text{ and} \\
I_3 :&= 3t^{2(1-\gamma)}\mathbb{E} \parallel \int_0^t P_\beta(t-s)\left[\sigma(s,u_s+\tilde{\phi}_s) - \sigma(s,v(s)+\tilde{\phi}_s)\right] dS_Q^H(s) \parallel^2 \\
&\leq 3t^{2(H-\gamma)+1}C_H \int_0^t \parallel P_\beta(t-s)\left[\sigma(s,u_s+\tilde{\phi}_s) - \sigma(s,v(s)+\tilde{\phi}_s)\right] ds \parallel^2_{L_Q^0} \\
&\leq \frac{3t^{2(H-\gamma)+1}C_H M}{(\Gamma(\beta))^2}\int_0^t (t-s)^{2(\beta-1)} \parallel \sigma(s,u_s+\tilde{\phi}_s) - \sigma(s,v(s)+\tilde{\phi}_s) \parallel_{L_Q^0} \\
&\leq \frac{ML_\sigma t^{2(H-\alpha+\alpha\beta)}}{(\Gamma(\beta))^2} \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}}.
\end{aligned}$$

By taking the supremum over t, we obtain

$$\begin{aligned}
\parallel \tilde{\Phi}(u)(t) - \tilde{\Phi}(v)(t) \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} &= \sup_{t\in[0,t_1]} \mathbb{E} \parallel t^{1-\gamma}\left[\tilde{\Phi}(u(t)) - \tilde{\Phi}(v(t))\right] \parallel^2 \\
&\leq 3t_1^{2(\alpha\beta+1)}\left(t_1^{1-\alpha-\beta}K_g + \frac{t_1^{-\alpha}L_f M}{(\Gamma(\beta))^2} + \frac{ML_\sigma t_1^{H-\alpha}}{(\Gamma(\beta))^2}\right) \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}}.
\end{aligned}$$

Case 2:

For $u,v \in \mathcal{D}'_{\mathcal{F}_T}$, $t \in (t_k,s_k]$, $k = 1,...,m$

$$\begin{aligned}\mathbb{E} \parallel t^{1-\gamma}\left[\tilde{\Phi}(u)(t)-\tilde{\Phi}(v)(t)\right] \parallel^2 &\leq L_{h_k} \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} \\ &\leq 2\tilde{K}t^{2(1-\gamma)}L_{h_k} \parallel u_t-v_t \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} .\end{aligned}$$

By taking the supremum over t, we obtain

$$\begin{aligned}\parallel \tilde{\Phi}(u)(t)-\tilde{\Phi}(v)(t) \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} &= \sup_{t\in[t_k,s_k],k=1,\ldots,m} \mathbb{E} \parallel t^{1-\gamma}\left[\tilde{\Phi}(u(t))-\tilde{\Phi}(v(t))\right] \parallel^2 \\ &\leq 2L_{h_k}T^{2(1-\gamma)} \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} .\end{aligned}$$

Case 3:
For $u,v\in \mathcal{D}'_{\mathcal{F}_T}$ and for $t\in(s_k,t_{k+1}]$, $k=1,\ldots,m$. we have

$$\begin{aligned}\mathbb{E} \parallel t^{1-\gamma}\left(\tilde{\Phi}(u)(t)-\tilde{\Phi}(v)(t)\right) \parallel^2 &\leq 4t^{2(1-\gamma)}\mathbb{E} \parallel S_{\alpha,\beta}(t-s_k)\left[h(s_k,u_{s_k}+\tilde{\phi}_{s_k})-h(s_k,v_{s_k}+\tilde{\phi}_{s_k})\right] \parallel^2 \\ &\quad +4t^{2(1-\gamma)}\mathbb{E} \parallel \left(g(t,u_t+\tilde{\phi}_t)-g(t,v_t+\tilde{\phi}_t)\right) \parallel^2 \\ &\quad +4t^{2(1-\gamma)}\mathbb{E} \parallel \int_{s_k}^{t_{k+1}} P_\beta(t-s)\left[f(s,u_s+\tilde{\phi_s})-f(s,v_s+\tilde{\phi_s})\right] \parallel^2 ds \\ &\quad +4t^{2(1-\gamma)}\mathbb{E} \parallel \int_{s_k}^{t_{k+1}} P_\beta(t-s)\left[\sigma(s,u_s+\tilde{\phi_s})-\sigma(s,v_s+\tilde{\phi_s})\right] dS_Q^H(s) \parallel^2 \\ &\leq I_1+I_2+I_3+I_4,\end{aligned}$$

where

$$
\begin{aligned}
I_1 &= 4t^{2(1-\gamma)}\mathbb{E} \parallel S_{\alpha,\beta}(t-s_k)\left[h(s_k,u_{s_k}+\tilde{\phi}_{s_k})-h(s_k,v_{s_k}+\tilde{\phi}_{s_k})\right] \parallel^2 \\
&\leq 4t^{2(1-\gamma)}\frac{Mt^{2(\alpha-1)(\beta-1)}}{\Gamma(\alpha(1-\alpha)+\beta)^2}\mathbb{E} \parallel h(s_k,u_{s_k}+\tilde{\phi}_{s_k})-h(s_k,v_{s_k}+\tilde{\phi}_{s_k}) \parallel^2 \\
&\leq \frac{4ML_{h_k}}{\Gamma(\alpha(1-\alpha)+\beta)^2} \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} . \\
I_2 &= 4t^{2(1-\gamma)}\mathbb{E} \parallel \left(g(t,u_t+\tilde{\phi}_t)-g(t,v_t+\tilde{\phi}_t)\right) \parallel^2 \\
&\leq 4t^{2(1-\gamma)}K_g \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} . \\
I_3 &= 4t^{2(1-\gamma)}\mathbb{E} \parallel \int_{s_k}^{t_{k+1}} P_\beta(t-s)\left[f(s,u_s+\tilde{\phi}_s)-f(s,v_s+\tilde{\phi}_s)\right]ds \parallel^2 \\
&\leq 4t^{2(1-\gamma)}\mathbb{E}\int_{s_k}^{t_{k+1}} \parallel P_\beta(t-s)\left[f(s,u_s+\tilde{\phi}_s)-f(s,v_s+\tilde{\phi}_s)\right]ds \parallel^2 \\
&\leq \frac{4t^{2(1-\gamma)}M(t_{k+1}-s_k)^{2(\beta-1)}}{(\Gamma(\beta))^2}\mathbb{E}\int_{s_k}^{t_{k+1}} \parallel f(s,u_s+\tilde{\phi}_s)-f(s,v_s+\tilde{\phi}_s) \parallel^2 ds \\
&\leq \frac{4t^{2(1-\gamma)}M(t_{k+1}-s_k)^{2(\beta-1)}L_f}{(\Gamma(\beta))^2} \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}}, \text{ and} \\
I_4 &= 4t^{2(1-\gamma)}\mathbb{E} \parallel \int_{s_k}^{t_{k+1}} P_\beta(t-s)\left[\sigma(s,u_s+\tilde{\phi}_s)-\sigma(s,v_s+\tilde{\phi}_s)\right]dS_Q^H(s) \parallel^2 \\
&\leq 3t^{2(H-\gamma)+1}C_H\mathbb{E}\int_{s_k}^{t_{k+1}} \parallel P_\beta(t-s)\left[\sigma(s,u_s+\tilde{\phi}_s)-\sigma(s,v(s)+\tilde{\phi}_s)\right] \parallel^2_{L_Q^0} ds \\
&\leq \frac{3t^{2(H-\gamma)+1}C_HM}{(\Gamma(\beta))^2}\mathbb{E}\int_{s_k}^{t_{k+1}} (t-s)^{2(\beta-1)} \parallel \sigma(s,u_s+\tilde{\phi}_s)-\sigma(s,v(s)+\tilde{\phi}_s) \parallel^2_{L_Q^0} \\
&\leq \frac{C_HML_\sigma t^{2(H-\gamma)+1}(t_{k+1}-s_k)^{2(\beta-1)}}{(\Gamma(\beta))^2} \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} .
\end{aligned}
$$

By taking the supremum over t, we obtain

$$
\begin{aligned}
\parallel \tilde{\Phi}(u)(t)-\tilde{\Phi}(v)(t) \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} \leq \Big(&\frac{4ML_{h_k}}{\Gamma(\alpha(1-\alpha)+\beta)^2}+4t^{2(1-\gamma)}K_g+\frac{4t^{2(1-\gamma)}M(t_{k+1}-s_k)^{2(\beta-1)}L_f}{(\Gamma(\beta))^2} \\
&+\frac{C_HML_\sigma t^{2(H-\gamma)+1}(t_{k+1}-s_k)^{2(\beta-1)}}{(\Gamma(\beta))^2}\Big) \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} .
\end{aligned}
$$

Which implies that $\tilde{\Phi}$ is a contraction and there exist a unique fixed point $z(t) \in \mathcal{D}'_{\mathcal{F}_T}$ so $x_t \in \mathcal{D}^\gamma_{\mathcal{F}_T}$ of Φ so is a mild solution of (6.1). The proof is completed.

□

6.2 Numerical application

$$\begin{cases} D_{0^+}^{\frac{1}{2},\frac{1}{4}}[v_t(.,\xi)-G(t,v_t(.,\xi))] = \frac{\partial^2}{\partial\xi^2}[v_t(.,\xi)-G(t,v_t(.,\xi))]\,dt + F(t,v_t(.,\xi)+\sigma(t,v_t(.,\xi))\frac{dS_Q^H(t)}{dt} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0\le\xi\le\pi,\ t\in[s_k,t_{k+1}],\ k=0,...m, \\ v(t,\xi)=H_k(t,v_t(.,\xi)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t\in(t_k,s_k],\ k=1,2,...m, \\ v_t(.,0)=v_t(.,\pi)=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t\in[0,2], \\ (I_0^{\frac{3}{8}}v_t(.,\xi))|_{t=0}=\phi(t,\xi) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t\in(-\infty,0], \end{cases} \tag{6.4}$$

where
$D_{0^+}^{\frac{1}{2},\frac{1}{4}}$ denotes the Hilfer fractional derivative. $S_Q^H(t)$ is an Q-sub-f.B.m with Hurst parameter $H\in(\frac{1}{2},1)$, defined on a complete probability space $(\Omega,\mathcal{F},P)$.
The impulses times satisfy:

$$0=t_0=s_0<t_1\le s_1<t_2\le ...<t_m\le s_m<t_{m+1}=T,\ \text{ for } t>0.$$

v_t mean a segment solution which is defined by

$$v(.,.):(-\infty,T]\times\Omega\to U,$$

then for any $t\ge0$, $v_t(.,.):(-\infty,0)\times\Omega\to U$ is given by:
$v_t(\theta,\omega)=x(t+\theta,\omega)$, for $\theta\in(-\infty,0]$, $\omega\in\Omega$ which is valued in $\mathcal{D}_{\mathcal{F}_T}^{\frac{5}{8}}$, and $U=L^2[0,\pi]$.
$F,G:[0,2]\times\mathcal{D}_{\mathcal{F}_T}^{\frac{5}{8}}\longrightarrow\mathbb{R}$ are continuous functions. $I_0^{\frac{3}{8}}$ is the fractional integral of order $\frac{3}{8}=1-\frac{5}{8}$, where $\gamma=\frac{5}{8}=\frac{1}{2}+\frac{1}{4}-\frac{1}{8}$.
Now let

$$y(t)(\xi)=u(t,\xi),\ t\in[0,2],\ \xi\in[0,\pi],$$

$H_k(t,\phi(\theta,\xi))=h_k(t,\phi)(\xi)$, $\theta\in(-\infty,0)$, $\xi\in[0,\pi]$ $k=1,...m$, and $\phi(\theta)(\xi)=\phi(\theta,\xi)$. We need now to define the operator $Q:K\longrightarrow K$, fot this we choose a sequence $\{\sigma_n\}_{n\ge1}\in\mathbb{R}^+$ such that $Qe_n=\sigma_ne_n$ and suppose that $tr(Q)=\sum_{n=1}^{\infty}\sqrt{\sigma_n}<\infty$.
The process $S_Q^H(s)$ will be defined by $S_Q^H(t)=\sum_{n=1}^{\infty}S_n^H(t)\sqrt{\sigma_n}e_n$, where $H\in(\frac{1}{2},1)$ and $\left\{S_n^H(t)\right\}_{n\in\mathbb{N}}$ is a sequence of one dimensional standard sub-fractional Brownian motions mutually independats over $(\omega,\mathcal{F},P)$.
Finally we assume that:

- For all $k=0,...,m$, the function $f:[s_k,t_{k+1}]\times\mathcal{D}_{\mathcal{F}_T}^{\frac{5}{8}}\longrightarrow U$ defined by $f(t,v)(.)=F(t,v(.))$ is continuous and we impose conditions on F to verify assumption (H_3). For example we take

$F(t,\phi) = t + \frac{2\phi}{1+\|\phi\|_{\mathcal{D}^{\frac{5}{8}}}}$; $t \in [s_k, t_{k+1}]$; $\phi \in \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T}$.

- For all $k = 0, ..., m$, the function $\sigma[s_k, t_{k+1}] \times \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T} \longrightarrow L^0_Q(K,U)$ is continuous, we impose conditions on σ to make assuptions (H_5) hold. We put: $\sigma(t,\phi) = t^3 + sin\phi$; $t \in [s_k, t_{k+1}]$; $\phi \in \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T}$.
- For all $k = 0, ..., m$, the function $h_k : [t_k, s_k] \times \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T} \longrightarrow U$ defined by $h_k(t,v)(.) = H_k(t, v(.))$ is continuous and we impose conditions on H_k to make assumption (H_6) hold. For example we take:
  $H_k(t,\phi) = R_k\phi$, $\xi \in \Omega$, $t \in [s_k, t_{k+1}]$, $\phi \in \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T}$.

Thus the problem 6.4 can be written in the abstract form

$$\begin{cases} D^{\alpha,\beta}_{0^+} X(t,x_t) = A(t) X(t,x_t) + f(t,x_t) + \sigma(t,x_t) \frac{dS^H_Q(t)}{dt}, & \text{for } t \in [s_k, t_{k+1}],\ k = 0, ...m, \\ x(t) = h_k(t,x_t), & \text{for } t \in (t_k, s_k],\ k = 1, 2, ...m, \\ (I^{1-\gamma}_0 x)(t)|_{t=0} = \phi \in \mathcal{D}_{\mathcal{F}_0}((-\infty, 0], U]. \end{cases} \tag{6.5}$$

Thanks to these assumptions, it is easy to check that (H1)-(H6) hold and thus assumptions in Theorem 6.2 are fulfilled, ensuring that system (6.4) possesses a mild solution on $(-\infty, T)$.

Chapter 7

Non-densely defined fractional stochastic evolution equations driven by fractional Brownian motion

The study of impulsive fractional differential and integro-differential systems is applicable to their efficacity in simulating processes and phenomena to short-time perturbations during their evolution. The non instanteneous and neutral impulsive stochastic functional differential equations have become an important object of investigation in recent years stimulated by their numerous applications in characterising many problems in physics, biology, mechanics, electrical engineering, medecine, we refer reader to [2] [27]. Stochastic differential equations play an important role in modeling many physical, biological, and engineering problems, see the monographs of Da-Prato and Zabczyk[36] and Sobczyk [122]. The notions of basic theory concerning differential equations are given in the monographs of Bharucha-Reid [7], Da Prato and Zabczyk [36] and tsokos and Padgett [124]. For more details, we give the reader to Liu [87] Mc Kibben[71] and [72]. Which, was very interesting to study a class of this type of equation by drawing inspiration from the work of [136] and that of [18] with the fractional derivation of Caputo, so this work is concerned with the existence of integral solutions for initial value problem with non-instantaneous impulses driven by a fractional Brownian motion of the form:

$$\begin{cases} {}^cD_t^q X(t,x_t) = A(t)X(t,x_t) + f(t,x_t) + \sigma(t)\frac{dB_Q^H(t)}{dt} & \text{for } t \in [s_k, t_{k+1}]\ k = 0,1,...m \\ x(t) = h_k(t,x_t) & \text{for } t \in (t_k, s_k]\ \ k = 1,2,...m \\ x(t) = \phi(t) \in D^B_{\mathcal{F}_0}((-\infty,0],U], \end{cases} \tag{7.1}$$

where

${}^cD_t^q$ is the Caputo fractional derivative of order $q \in \left(\frac{1}{2},1\right)$ takes a values in a Hilbert space U, x(.)takes value in a real separable Hilbert space U, with inner product (.,.) and norm $\| \ . \ \|$,

and $A : D(A) \subset U \to U$ is a family of closed operators of integral solutions for a class of first-order non-densely defined semilinear stochastic equations with non local initial conditions. The impulses times satisfy: $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq ... < t_m \leq s_m < t_{m+1} = T$, for $t > 0$;
the delay function $x_t : (-\infty, 0) \to 0,\ x_t(\theta) = x(t+\theta)$ with is valued in $\mathcal{D}_{\mathcal{F}_T}$, where

$$\mathcal{D}_{\mathcal{F}_T} = \{x : (-\infty, T] \times \Omega \to U;\ x\,|_{J_k} \in C\left((s_k, t_{k+1}]; U\right)\ \forall w \in \Omega\ for\ k = 0, ..., m.\},$$

with the norm

$$\| x \|_{\mathcal{D}_{\mathcal{F}_T}} = \left(\sup_{0 \leq t \leq T} \mathbb{E} \| x \|_t^2\right)^{\frac{1}{2}},$$

and $x_0 \in \mathcal{D}^B_{\mathcal{F}_0}$, where $\mathcal{D}^B_{\mathcal{F}_0}$ denote the family of all almost surely bounded $\mathcal{F}_0$-measurable, and $\tilde{D}$-valued random variables.
$\tilde{D} = D((-\infty, 0], U)$ denotes the family of all right piecewise continuous functions with left-hand limit φ from $(-\infty, 0]$ to U, with the norm

$$\| \varphi \|_t = \sup_{-\infty < \theta \leq t} \| \varphi(\theta) \|,$$

$\left\{B^H_Q(t), t \geq 0\right\}$ is a cylindrical fractional Brownian motion on space K with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. We assume in the sequel that $X(t, x_t) : J \times U \longrightarrow U$, such that

$$X(t, x_t) = \phi(0) - g(t, x_t),$$

$g : J \times \mathcal{D}_{\mathcal{F}_0} \to U$, and
$f : J \times \mathcal{D}_{\mathcal{F}_0} \to U$, $h_k \in C((t_k, s_k] \times \mathcal{D}_{\mathcal{F}_0}, U)$ for all $k = 1, ..., m$, $\sigma : J \to L^0_Q(K, H)$.

Theorem 7.1. *[13] Bihari inequality*
Let $T > 0$, $u_0 > 0$, and let $u(t)$, $v(t)$ be continuous functions on $[0, T]$.
let $k : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave continuous and non-decreasing function such that:
$k(r) \geq 0$ for $r > 0$. If

$$u(t) \leq u(0) + \int_0^t v(s) k(u(s)) ds\ \ \forall t \in [0, T],$$

then

$$u(t) \leq G^{-1}(G(u_0) + \int_0^t v(s) ds)$$

for all $t \in [0, T]$ such that

$$G(u_0) + \int_0^t v(s) ds \in Dom(G^{-1}),$$

where

$$G(r) = \int_1^r \frac{ds}{k(s)},\ \ r \geq 0,$$

and G^{-1} is the inverse function of G. In particular, if, moreover, $u_0 = 0$ and $\int_{0^+} \frac{ds}{k(s)} = +\infty$, then $u(t) = 0$ for all $t \in [0,T]$.

7.1 Main result

An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty,T] \to U$ is said to be an integral solution of (7.1) if $x_0 = \phi \in \mathcal{D}^B_{\mathcal{F}_0}$ and

(i) $\{x_t,\ t \in J\} \in \mathcal{D}_{\mathcal{F}_T}$.

(ii) $\int_0^t [x_s + g(s,x_s)]\, ds \in D(A),\ \ t \in [0,T]$.

(iii) for each $t > 0$

$$x(t) = \begin{cases} S'(t)\,[\phi(0) - g(0,\phi)] + g(t,x_t) \\ +\frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-1} S(t-s) f(s,x_s) ds \\ +\frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-1} S(t-s) \sigma(s) dB^H_Q(s), \quad for\, t \in [0,t_1] \\ h_k(t,x_t), \quad for\, t \in (t_k,s_k];\ k = 1,...,m. \\ S'(t-s_k) h_k(s_k,x_{s_k}) + g(s_k,x_{s_k}) \\ +\frac{1}{\Gamma(q-1)} \int_{s_k}^t (t-s)^{q-1} S(t-s) f(s,x_s) ds \\ +\frac{1}{\Gamma(q-1)} \int_{s_k}^t (t-s)^{q-1} S(t-s) \sigma(s) dB^H_Q(s). \quad for\, t \in [s_k,t_{k+1}]\ k = 1,...m. \end{cases} \tag{7.2}$$

7.2 Conditions and assumptions

We will work under the following assumptions.

(H1) The operator A satisfies the Hille- Yosida condition, $S'(t)$ is compact for $t > 0$, and there exist constant $M \geq 0$ and $\beta > 0$ such that $\| S'(t) \|^2 \leq Me^{\beta t}$, $\forall t \geq 0$.

(H2) The function $f : J \times \mathcal{D}_{\mathcal{F}_T} \to U$ satisfies the following conditions:

(i) $\| f(t,x_t) - f(t,y_t) \|^2 \leq H(\| x-y \|_t^2)$,
$x,y \in \mathcal{D}_{\mathcal{F}_T}$, $t \in J$, where: $H(0) = 0$, $H(s) > 0$ *for* $s > 0$ and $\int_{0^+} \frac{ds}{H(s)} = +\infty$.

(ii) $\| f(t,0) \|^2 \leq M_1\ \forall t \in J$, *where* M_1 is a positive constant.

(H3) The function $g : J \times \mathcal{D}_{\mathcal{F}_T} \to U$ and there exist a positive number K_g such that for $t \in [0,T]$ we have
$\| g(t,x_t) - g(t,y_t) \|^2 \leq K_g \| x-y \|_t^2\ \ x,y \in \mathcal{D}_{\mathcal{F}_T}$ $,t \in J$,
and $\| g(t,0) \|^2 \leq M_2$, where M_2 is a positive constant.

(H4) The function $\sigma : J \to L^0_Q$ satisfies that there exists a positive constant L such that

$$\| \sigma(s) \|^2_{L^0_2} \leq L.$$

(H5) The functions $h_i \in C((t_i,s_i] \times U, U)$ and there exist a positive constant L_h such that $\| h_i(t,x_t) - h_i(t,y_t) \|^2 \leq L_h \| x-y \|^2$, for all $x,y \in \mathcal{D}_{\mathcal{F}_T}$ $, t \in (t_i,s_i]$; $i = 1,2,...m$.
In addition $\| h_i(t,0) \|^2 \leq M_3$ $\forall t \in J$, *and* $\forall i \geq 1$, where M_3 is a positive constant.

7.3 Existence and uniqueness of integral solution

In this section we establish the existence and uniqueness theorem of the integral solution for system 7.1. We construct the sequence of successive approximations defined as follows:

$$\begin{cases} x^0(t) = S'(t)\phi(0) \\ x^n(t) = S'(t)\,[\phi(0) - g(0,\phi)] + g(t,x_t^n) + \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-1} S(t-s) f(s,x_s^{n-1}) ds \\ + \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-1} S(t-s) \sigma(s) dB_Q^H(s),\ t \in [0,t_1];\ n \geq 1. \\ x^n(t) = h_k(t,x_t),\ t \in (t_k,s_k];\ k = 1,...,m. \\ x^n(t) = S'(t-s_k) h_k(s_k,x_{s_k}) + g(s_k,x_{s_k}) + \frac{1}{\Gamma(q-1)} \int_{s_k}^t (t-s)^{q-1} S(t-s) f(s,x_s^{n-1}) ds \\ + \frac{1}{\Gamma(q-1)} \int_{s_k}^t (t-s)^{q-1} S(t-s) \sigma(s) dB_Q^H(s),\ t \in (s_k,t_{k+1}],\ k = 1,...,m. \\ x^n(t) = \phi(t), \quad -\infty < t \leq 0,\ n \geq 1. \end{cases} \tag{7.3}$$

Theorem 7.2. *Let (H1)-(H5) hold and* $\phi(0) - g(0,\phi) \in \overline{D(A)}$. *Then there exist a unique integral solution of 7.1 in the space* $\mathcal{D}_{\mathcal{F}_T}$ *if*

$$K = max\{8K_g, 2L_h, 3K_g + 3L_h\} < 1.$$

Proof. The proof is composed by several steps.
<u>**Step 1**</u>:
for all $t \in (-\infty, T]$, the sequence $x^n(t)$ $(n \geq 1) \in \mathcal{D}_{\mathcal{F}_T}$ is bounded.
Case 1: for $t \in (0,t_1]$ we have

$$\begin{aligned} \mathbb{E} \| x^n(t) \|^2 \leq 4\tilde{M}\mathbb{E} \| \phi(0) + g(0,\phi) \|^2 &+ 8\mathbb{E}\left[\| g(t,x_t^n) - g(t,0) \|^2 + \| g(t,0) \|^2\right] \\ + \frac{8}{\Gamma^2(q-1}\tilde{M}t_1^{2q-1}\mathbb{E}\int_0^t e^{\beta s}&\left[\| f(s,x_s^{n-1}) - f(s,0) \|^2 + \| f(s,0) \|^2\right] ds \\ &+ \frac{8\tilde{M}t_1^{2H-2q-3}H}{\Gamma^2(q-1)}\mathbb{E}\int_0^t e^{\beta s} \| \sigma(s) \|^2_{L_Q^2}\, ds. \end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E} \parallel x^n(t) \parallel^2 \leq & \frac{C_1}{1-8K_g} + \frac{8\tilde{M}M_1 t_1^{2q-1} e^{\beta t_1}}{(1-8K_g)\Gamma^2(q-1)} + \frac{8\tilde{M}Ht_1^{2q+2H-3} Le^{\beta t_1}}{(1-8K_g)\Gamma^2(q-1)} \\
& + \frac{8\tilde{M}t_1^{2q-1}}{(1-8K_g)\Gamma^2(q-1)} \int_0^t e^{\beta s} \mathbb{E} \parallel f(s, x_s^{n-1}) - f(s,0) \parallel^2 ds \\
& \leq C_2 + \frac{8\tilde{M}t_1^{2q-1}}{(1-8K_g)\Gamma^2(q-1)} \int_0^t e^{\beta s} \mathbb{E} H(\parallel x_s^{n-1} \parallel^2) ds,
\end{aligned}$$

where,

$$C_1 = 4\tilde{M}\mathbb{E} \parallel \phi(0) + g(0,\phi) \parallel^2 + 8M_2,$$

and

$$C_2 = \frac{C_1}{1-8K_g} + \frac{8\tilde{M}M_1 t_1^{2q-1} e^{\beta t_1}}{(1-8K_g)\Gamma^2(q-1)} + \frac{8\tilde{M}Ht_1^{2q+2H-3} Le^{\beta t_1}}{(1-8K_g)\Gamma^2(q-1)},$$

also we have that $H(.)$ is concave and $H(0) = 0$; where there exist a positive constants a and b such that $H(t) \leq a + bt, \; t \geq 0$, in the sequel we get that for $n \geq 1$

$$\mathbb{E} \parallel x^n(t) \parallel^2 \leq C_2 + ae^{\beta t_1} + b \int_0^t e^{\beta s} \mathbb{E} \parallel x_s^{n-1} \parallel^2 ds.$$

Since;
$\mathbb{E} \parallel x^0(t) \parallel^2 \leq \tilde{M}\mathbb{E} \parallel \phi(0) \parallel^2 := C_3 < \infty$, we get that:

$$\mathbb{E} \parallel x^n \parallel_t^2 < \infty, \;\; \forall n \geq 1, \; t \in [0, t_1].$$

Case 2:
For $t \in (t_k, s_k], \; k = 1, ..., m$

$$\begin{aligned}
\mathbb{E} \parallel x^n(t) \parallel^2 &= \mathbb{E} \parallel h_k(t, x_t^n) \parallel^2 \\
&= \mathbb{E} \parallel [h_k(t, x_t^n) - h_k(t,0)] + h_k(t,0) \parallel^2 \\
&\leq 2\mathbb{E}\left[L_h \parallel x^n \parallel_t^2\right] + 2\mathbb{E} \parallel h_k(t,0) \parallel^2 \\
&\leq \frac{M_3}{1-2L_h} < \infty, \;\; 0 < L_h < \frac{1}{2}.
\end{aligned}$$

Case 3:
For $t \in (s_k, t_{k+1}] \;\; k = 1, ..., m$

$$\begin{aligned}
\mathbb{E} \parallel x^n(t) \parallel^2 = & \mathbb{E} \parallel S'(t-s_k) h_k(s_k, x_{s_k}^n) + g(s_k, x_{s_k}) \\
& + \frac{1}{\Gamma(q-1)} \int_{s_k}^t (t-s)^{q-1} S(t-s) f(s, x_s^{n-1}) ds \\
& + \frac{1}{\Gamma(q-1)} \int_{s_k}^t (t-s)^{q-1} S(t-s) \sigma(s) dB_Q^H(s) \parallel^2 .
\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E} \parallel x^n(t) \parallel^2 \leq & 8\tilde{M}L_h\mathbb{E} \parallel x^n(s_k) \parallel^2 + 8\tilde{M}M3 + 8K_g\mathbb{E} \parallel x^n(s_k) \parallel^2 + 8M_2 \\ & + \frac{1}{\Gamma^2(q-1)} t_{k+1}^{2q-1} \mathbb{E} \int_{s_k}^{t_{k+1}} e^{\beta s} \parallel f(s, x_s^{n-1}) - f(s,0) \parallel^2 ds \\ & + \frac{8}{\Gamma^2(q-1)} t_{k+1}^{2q-1} e^{\beta t_{k+1}} M_1 + \frac{8}{\Gamma^2(q-1)} t_{k+1}^{2q-1} e^{\beta t^{k+1}} L.\end{aligned}$$

We obtain

$$\mathbb{E} \parallel x^n(t) \parallel^2 \leq C_4\mathbb{E} \parallel x^n(s_k) \parallel^2 + C_5 + C_6 \int_{s_k}^{t} e^{\beta s}\mathbb{E} \parallel x^{n-1}(s) \parallel^2 ds,$$

where

$$C_4 = \frac{8bt_{k+1}e^{\beta t_{k+1}}}{\Gamma^2(q-1)},$$

$$C_5 = 8(\tilde{M}M_3 + M_2) + \max_{k=1,\dots m}, \left\{ \frac{8e^{\beta t_{k+1}} t_k^{2q-1}(M_1 + t_{k+1}^{2H-2}LH + a)}{\Gamma^2(q-1)} \right\},$$

and

$$C_6 = \max_{k=1,\dots,m} \frac{8bt_{k+1}e^{\beta t_{k+1}}}{\Gamma^2(q-1)}.$$

We have that

(1) $\mathbb{E} \parallel x^n(s_k) \parallel^2 \leq \sup_{k=1,\dots,m} \mathbb{E} \parallel x^n(s_k) \parallel^2 := C_7 < \infty$,

(2) by case 2 we get that $\mathbb{E} \parallel x^{n-1}(s) \parallel^2 := C_8 < \infty$.

We conclude that the sequence $x^n(t)$ $(n \geq 1)$ is bounded on the space $\mathcal{D}_{\mathcal{F}_T}$.

Step 2:

We show that the sequence $x^n(t)$ $(n \geq 1)$ is a Cauchy sequence.

Case 1:

For $t \in [0, t_1]$

$$\mathbb{E} \parallel x^{n+1} - x^n \parallel_t^2 \leq 2\mathbb{E} \parallel g(t, x_t^{n+1}) - g(t, x_t^n) \parallel^2 + \frac{2t_1^{2q-1} e^{\beta t^1}}{\Gamma^2(q-1)} \int_0^t H(\mathbb{E} \parallel x^n - x^{n-1} \parallel_s^2 ds \tag{7.4}$$

Let

$$\Phi_n(t) = \sup_{t\in[0,t_1]} \mathbb{E} \parallel x^{n+1} - x^n \parallel_t^2 .$$

Then for $t \in [0, t_1]$ we have

$$\Phi_n(t) \leq C_9 \int_0^t H(\Phi_{n-1}(s))ds,$$

where

$$C_9 = \frac{2t_1^{2q-1}e^{\beta t_1}}{(1-2K_g)\Gamma^2(q-1)}.$$

In the sequel we choose t such that

$$\Phi_n(t) \leq C_9 \int_0^t \Phi_{n-1}(s)ds.$$

Moereover,

$$\mathbb{E} \parallel x^1 - x^0 \parallel_t^2 = \mathbb{E} \parallel -S^{'}(t)g(0,\phi) + g(t,x_t^1) + \frac{1}{\Gamma^(q-1)} \int_0^t (t-s)^{q-1} S(t-s) f(s,x_s^0) + \frac{1}{\Gamma^(q-1)} \int_0^t (t-s)^{q-1} S(t-s)\sigma(s) dB_Q^H(s) \parallel^2$$

we obtain that

$$\mathbb{E} \parallel x^1 - x^0 \parallel_t^2 \leq C_{10},$$

where

$$C_{10} = \frac{4\tilde{M}\|\phi\parallel_0^2}{1-4K_g} + \frac{4K_g C_3}{1-4K_g} + \frac{4t_1^{2H+2q-2}e^{\beta t_1}LH}{(1-4K_g)\Gamma^2(q-1)}.$$

Taking the supremum over t, and using Φ_n, we have

$$\Phi_0(t) = \sup_{t\in[0,t_1]} \mathbb{E} \parallel x^1 - x^0 \parallel_t^2 \leq C_{10}.$$

For $n = 1$, we have

$$\begin{aligned} \Phi_1(t) &\leq C_9 \int_0^t \Phi_0(s)ds \\ &\leq C_9 C_{10} t. \end{aligned}$$

For $n = 2$

$$\begin{aligned} \Phi_2(t) C_9 &\int_0^t \Phi_1(s)ds \\ &\leq C_9^2 C_{10} \frac{t^2}{2}. \end{aligned}$$

By induction we obtain

$$\Phi_n(t) \leq C_9^n C_{10} \frac{t^n}{n!}.$$

So for any $m \geq n \geq 0$ we have

$$\sup_{t\in[0,t_1]} \mathbb{E} \parallel x^m - x^n \parallel_t^2 \leq \sum_{r=n}^{\infty} \sup_{t\in[0,t_1]} \mathbb{E} \parallel x^{r+1} - x^r \parallel_t^2 \tag{7.5}$$

$$\leq \sum_{r=n}^{\infty} C_9^r C_{10} \frac{t^r}{r!} \longrightarrow 0,\ n \to \infty. \tag{7.6}$$

Case 2:
For $t \in (t_k, s_k],\ k = 1, ..., m$

$$\mathbb{E} \parallel x^{n+1} - x^n \parallel_t^2 = \mathbb{E} \parallel h_k(t, x^{n+1}) - h_k(t, x^n) \parallel_t^2 \tag{7.7}$$

$$\leq L_h \mathbb{E} \parallel x^{n+1} - x^n \parallel_t^2, \tag{7.8}$$

$0 \leq \mathbb{E} \parallel x^{n+1} - x^n \parallel_t^2 - L_h \mathbb{E} \parallel x^{n+1} - x^n \parallel_t^2 \leq 0,\ L_h > 0$, this implies that;
$\mathbb{E} \parallel x^{n+1} - x^n \parallel_t^2 \to 0.$
For any $m \geq n \geq 0$ we have

$$\sup_{t\in(t_k,s_k]} \mathbb{E} \parallel x^m - x^n \parallel_t^2 \leq \sum_{r=n}^{\infty} \sup_{t\in[0,t_1]} \mathbb{E} \parallel x^{r+1} - x^r \parallel_t^2 \to 0 \ \ n \to \infty$$

Case 3:
For $t \in (s_k, t_{k+1}],\ k = 1, ..., m$
Using the same method we obtain that

$$\Phi_n(t) \leq C_{11} \int_{s_k}^{t} H(\Phi_{n-1}(s))ds$$
$$\leq C_{11} \int_{0}^{t} H(\Phi_{n-1}(s))ds$$

we choose $t \in (s_k, t_{k+1}]$ such that

$$\Phi_n(t) \leq C_{11} \int_{s_k}^{t} \Phi_{n-1}(s)ds,$$

where

$$C_{11} = \frac{t_{k+1}^{2q-1} e^{\beta t_{k+1}}}{(1 - 3L_h - 3K_g)\Gamma^2(q-1)}$$

and

$$\Phi_0(t) \leq C_{12},$$

where

$$C_{12} = \frac{t_{k+1}^{2q-1} e^{\beta t_{k+1}} C_3}{(1 - 3L_h - 3K_g)\Gamma^2(q-1)}.$$

For $n = 1$

$$\Phi_1(t) \leq C_{11}C_{12}t.$$

For $n=2$

$$\Phi_2(t) \leq C_{11}^2 C_{12}\frac{t^2}{2},$$

by applying the mathematical induction we have,

$$\Phi_n(t) \leq C_{11}^n C_{12}\frac{t^n}{n!}.$$

So for any $m \geq n \geq 0$ we have

$$\begin{aligned}\sup_{t\in(s_k,t_{k+1}]} \mathbb{E} \parallel x^m - x^n \parallel_t^2 &\leq \sum_{r=n}^{\infty} \sup_{t\in(s_k,t_{k+1}]} \mathbb{E} \parallel x^{r+1} - x^r \parallel_t^2 \\ &\leq \sum_{r=n}^{\infty} C_{11}^r C_{12}\frac{t^r}{r!} \longrightarrow 0,\ n \to \infty.\end{aligned}$$

<u>**Step 3**</u>:
The existence and uniqueness of the solution for 7.1.
The lemma of Borel Cantelli give that $x^n(t) \to x(t)$ uniformly on each interval. Now we prove the uniqueness on each interval of solution.
Case 1:
For $t \in [0,t_1]$, let $x_1, x_2 \in \mathcal{D}_{\mathcal{F}_T}$ be two solution on $[0,t_1]$. We have,

$$\begin{aligned}\mathbb{E} \parallel x_1 - x_2 \parallel_t^2 &\leq 2K_g\mathbb{E} \parallel x_1 - x_2 \parallel_t^2 + \frac{2t_1^{2q-1}e^{\beta t_1}}{\Gamma^2(q-1)}\int_0^t H(\mathbb{E} \parallel x_1 - x_2 \parallel_s^2 ds \\ &\leq \frac{2t_1^{2q-1}e^{\beta t_1}}{(1-2K_g)\Gamma^2(q-1)}\int_0^t H(\mathbb{E} \parallel x_1 - x_2 \parallel_s^2)ds.\end{aligned}$$

Thus the Bihari inequality affirm that

$$\sup_{t\in(0,t_1]} \mathbb{E} \parallel x_1 - x_2 \parallel_t^2 = 0 \Longleftrightarrow x_1 = x_2.$$

Case 2:
Let $x_1, x_2 \in \mathcal{D}_{\mathcal{F}_T}$ be two solution on $t \in (t_k, s_k],\ k = 1, ..., m$.
We have

$$\mathbb{E} \parallel x_1 - x_2 \parallel_t^2 = \mathbb{E} \parallel h_k(t,x_1) - h_k(t,x_2) \parallel_t^2 \tag{7.9}$$
$$\leq L_h\mathbb{E} \parallel x_1 - x_2 \parallel_t^2 \tag{7.10}$$

$0 \leq \mathbb{E} \parallel x_1 - x_2 \parallel_t^2 - L_h\mathbb{E} \parallel x_1 - x_2 \parallel_t^2 \leq 0,\ L_h > 0$, this implies that;
$\sup_{t\in(t_k,s_k],\ k=1,...,m} \mathbb{E} \parallel x_1 - x_2 \parallel_t^2 = 0 \Longleftrightarrow x_1 = x_2.$

Case 3:
Let $x_1, x_2 \in \mathcal{D}_{\mathcal{F}_T}$ be two solution on $[s_k, t_{k+1}]$. We have,

$$\mathbb{E} \parallel x_1 - x_2 \parallel_t^2 \leq (3\tilde{M}L_h + 3K_g)\mathbb{E} \parallel x_1 - x_2 \parallel_{s_k}^2 + \frac{3t_{k+1}^{2q-1} e^{\beta t_{k+1}}}{\Gamma^2(q-1)} \int_{s_k}^{t} H(\mathbb{E} \parallel x_1 - x_2 \parallel_s^2) ds.$$

By taking the supremum on the both side we obtain that

$$\sup_{t\in(s_k,t_{k+1}]} \mathbb{E} \parallel x_1 - x_2 \parallel_t^2 \leq \frac{3t_{k+1}^{2q-1} e^{\beta t_{k+1}}}{(1 - 3K_g - 3\tilde{M}L_h)} \int_{s_k}^{t} H(\sup_{t\in(s_k,t_{k+1}]} \mathbb{E} \parallel x_1 - x_2 \parallel_s^2 ds),$$

the Bihari inequality affirm that $\mathbb{E} \parallel x_1 - x_2 \parallel_t^2 \to 0 \Longleftrightarrow x_1 = x_2$.
The proof is completed. $\square$

7.4 Numerical application

$$\begin{cases} {}^cD^{\beta}\left[v(t,\xi) - G(t, v(t-h,\xi))\right] & = \frac{\partial^2}{\partial x^2}\left[v(t,\xi) - G(t, v(t-h,\xi))dt\right] + \sigma(t)dB_Q^H(t) \\ & \text{for } 0 \leq \xi \leq \pi,\ h > 0,\ t \in [s_k, t_{k+1}]\ k = 1, ...m; \\ & 0 < \beta < 1 \\ v(t,0) = v(t,\pi) = 0 & \\ v(t,\xi) = \int_0^{t_k} h_k(s-t_k)v(s,\xi)ds & \text{for } t \in (t_k, s_k]\ k = 1,2,...m \\ v(t,\xi) = \phi(t,\xi) & \text{for } t \in (-\infty, 0], \end{cases} \tag{7.11}$$

where $B_Q^H(t)$ is an f.B.m with Hurst parameter $H \in (\frac{1}{2}, 1)$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$.
$U = C[0,\pi]$ is a Banach space.
$Az = \frac{\partial^2}{\partial x^2} z$ with domain $D(A) = \left\{ z \in U,\ z(0) = z(\pi) = 0;\ \frac{\partial^2}{\partial x^2} z \in C([0,\pi]) \right\}$.
$\overline{D(A)} = C_0([0,\pi]) = \{z \in C([0,\pi]) :\ z(0) = z(\pi) = 0\}$.
A generates an integrated semigroup $(S(t))_{t\geq 0}$ and

$$\parallel S'(t) \parallel^2 \leq e^{\beta t},$$

and satisfy the Hille- Yosida condition.

$$g(t,v)(.) = G(t, v(.))$$

$$\sigma(t) = \int_{-\infty}^{0} \nu(t)dt.$$

and

$$h_k(t,v)(.) = \int_{-\infty}^{0} b_k(s)v(s,.)ds.$$

We suppose that

$$\| G \|^2 \leq M_g.$$

So under this definitions and assumptions, our system can be written as the form of the problem 7.1, moreover, all the conditions of Theorem 7.2 are hold, so we conclude that system (7.11) has a unique integral solution.

Chapter 8

Conclusion

The principal goal of my thesis is to develop the subject of fractional stochastic differential equations and inclusions in Hilbert space.

We studied some classes of stochastic differential equations and inclusions, when we proved the existence results of a mild solution of fractional stochastic evolution inclusion involving Caputo derivative in Hilbert space driven by the fractional Brownian motion.

Also we studied a class of non densely defined fractional stochastic differential equation with non-instantaneous impulses driven by fractional Brownian motion under some conditions to prove existence and unicity of integral solutions by using Bihary inequality.

Moreover, we have studied the time fractional stochastic heat equation dealing with additive noise and more special classes of fractional heat equations.

Finally, we studied the existence of mild solution of Hilfer fractional stochastic differential equation with impulses driven by sub-fractional Brownian motion, by using Banach's fixed point theorem.

Our future work will concentrate on Malliaivin calculus and how to join fractional stochastic differential inclusion with Mallaivin calculus and how to apply these important tools in finance theory with Lévy process.

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" الدراسة النظرية لبعض المعادلات العشوائية للتطور "

الملخص:

في هذه الأطروحة ، نأخذ في الاعتبار فئات معينة من المعادلات التفاضلية الكسرية العشوائية و الانتماءات ونثبت وجود وتفرد الحلول الخفيفة في فضاءات هيلبرت باستخدام نظرية النقطة الثابتة بالإضافة إلى طرق التقريب التي توضحها التطبيقات.

كلمات مفتاحية: المعادلات التفاضلية العشوائية، الانتماءات التفاضلية العشوائية، التأثير الاندفاعي، نظرية النقطة الثابتة، الحركة البورونية الكسرية.

« Contribution à l'étude théorique de certaines équations d'évolutions stochastiques »

Résumé :

Dans cette thèse nous considérons certaines classes d'équations et d'inclusions différentielles fractionnaires stochastiques et nous prouvons l'existence et l'unicité de solutions mild dans les espaces de Hilbert.

Nous avons prouvé l'existence des résultats d'une solution mild d'inclusion d'évolution stochastique fractionnaire dans l'espace de Hilbert dirigée par un mBf, les résultats ont été obtenus en utilisant le calcul fractionnaire ainsi que la théorie du point fixe.

D'autre part, nous avons étudié le résultat de l'existence d'une solution mild de l'équation différentielle stochastique fractionnaire de Hilfer avec des impulsions entraînées par sub-mBf, les résultats sont obtenus en utilisant le théorème du point fixe.

Ensuite, nous avons étudié l'équation stochastique fractionnaire de chaleur muni d'un bruit additif, nous avons trouvé une formule de solution explicite dans le sens de distributions sous laquelle la solution est un corps aléatoire dans $L^2(P)$.

Enfin, des conditions suffisantes sont données pour prouver l'existence et l'unicité de la solution intégrale d'une équation différentielle stochastique fractionnaire avec des impulsions non instantanées entraînées par mBf.

Mots clés : Equations différentielles stochastiques, Inclusions différentielles stochastiques, Effet impulsif, Théorie du point fixe, Mouvement Borownien fractionnaire (mBf) .

« Contribution to the theoretical study of certain stochastic evolution equations »

Abstract :

The research circulated in this thesis loads with the problem of fractional stochastic differential equations and inclusions in Hilbert space.

We proved the existence results of a mild solution of fractional stochastic evolution inclusion involving the Caputo derivative in Hilbert space driven by a fBm, our desired results were obtained by using different tools such as; fractional calculation, operator semigroups, and fixed point theory.

Also, we have studied the existence result of mild solution of Hilfer fractional stochastic differential equation with impulses driven by sub-fBm, the results are obtained by using fixed point theorem.

Then, we have studied the time fractional stochastic heat equation dealing with additive noise. we found explicit solution formula in the sense of distributions under which the solution is a random field in $L^2(P)$.

Finally, sufficient conditions are given to prove the existence and unicity of integral solution of non-densely defined fractional stochastic differential equation with non-instantaneous impulses driven by fBm.

Key words :Stochastic functional differential equations, Stochastic functional differential inclusions, Impulsive effect, fixed point theory, fractional Brownian motion (fBm).